Direct Methods in the Calculus of Variations

Second Edition

Bernard Dacorogna

Applied Mathematical Sciences

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Second Edition
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Preface

The present monograph is a revised and augmented edition to Direct Methods in the Calculus of Variations [179] which is now out of print. The core and the structure of the present book are essentially the one of [179], although it has now almost doubled its size. While writing the present volume, it clearly appeared to me that a new subject has emerged and that it deserves to be called “quasiconvex analysis”. This name, of course, refers to “convex analysis”, although the new subject is still in its infancy when compared with the classical one.

The calculus of variations is an immense and very active field. It is therefore, when writing a book, necessary to make a severe selection. This was already the case for [179] and is even more so for this new edition. Rather than superficially covering a lot of materials, I preferred to privilege only some aspects of the field. Here are some main features of the book. I strongly emphasized the resemblances between convex and quasiconvex analysis as well as the “algebraic” aspect of the field, notably through the determinants and singular values. Besides the classical results on lower semicontinuity and relaxation, an important feature of the monograph is the emphasis on the existence of minimizers for non convex problems.

In doing so I missed several important aspects of the calculus of variations such as regularity theory, study of stationary points, existence and relaxation in BV spaces, minimal surfaces, Young measures and the mathematical study of microstructures, Γ convergence and homogenization. However there are already several excellent books on these subjects, some of them very classical, such as: Almgren [18], Ambrosio-Fusco-Pallara [25], Braides-Defranceschi [101], Buttazzo [112], Buttazzo-Giaquinta-Hildebrandt [117], Dal Maso [217], Dierkes-Hildebrandt-Küster-Wohlrab [248], Dolzmann [249], Ekeland [263], Ekeland-Temam [264], Evans [271], Fonseca-Leoni [284], Giaquinta [307], Giaquinta-Hildebrandt [309], Giaquinta-Modica-Soucek [312], Gilbarg-Trudinger [313], Giusti [315], Ladyzhenskaya-Ural’tseva [388], Mawhin-Willem [440], Morrey [455], Müller [462], Nitsche [476], Pedregal [492], Roubíček [517] or Struwe [546], [547]. I have also added in the bibliography several articles which present important developments that I did not discuss in the present monograph, but are still closely related.

For a reader not very familiar with the calculus of variations, it might be advisable to start with an introductory book such as [180], which could be considered as a companion to the present one. Nevertheless, the present monograph,
which is essentially a reference book on the subject of quasiconvex analysis, can be used, as was [179], for an advanced course on the calculus of variations.

I would next like to reiterate my thanks to the people who helped me while writing the earlier version [179], namely J.M. Ball, L. Boccardo, P. Ciarlet, I. Ekeland, J.C. Evard, B. Kawohl, P. Marcellini, J. Moser, C.A. Stuart, E. Zehnder and B. Zwahlen.


My thanks also go to Mme. G. Rime, who typed the manuscript of [179], and to Mme. M.F. De Carmine, who typed an earlier version of the present monograph. Finally, M. Hágler and C. Hebeisen prepared for me all the figures included in the book.

During the past several years, I have benefited from grants from the Fonds National Suisse and the Troisième Cycle Romand. Of course, particular thanks go to the Section de Mathématiques of the Ecole Polytechnique Fédérale de Lausanne.
Chapter 1

Introduction

1.1 The direct methods of the calculus of variations

The main problem that we will be investigating throughout the present monograph is the following. Consider the functional

\[ I(u) := \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx \]

where

- \( \Omega \subset \mathbb{R}^n, \ n \geq 1 \), is a bounded open set and a point in \( \Omega \) is denoted by \( x = (x_1, \ldots, x_n) \);
- \( u : \Omega \to \mathbb{R}^N, \ N \geq 1, \ u = (u^1, \ldots, u^N) \), and hence
  \[ \nabla u = \left( \frac{\partial u^j}{\partial x_i} \right)_{1 \leq i \leq n}^{1 \leq j \leq N} \in \mathbb{R}^{N \times n}; \]
- \( f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \ f = f(x, u, \xi), \) is a given function.

We say that the problem under consideration is \textit{scalar} if either \( N = 1 \) or \( n = 1 \); otherwise we speak of the \textit{vectorial} case.

Associated to the functional \( I \) is the minimization problem

\[(P) \quad m := \inf \{ I(u) : u \in X \}, \]

meaning that we wish to find \( \pi \in X \) such that

\[ m = I(\pi) \leq I(u) \text{ for every } u \in X. \]

Here \( X \) is the space of admissible functions (in most parts, it is the Sobolev space \( u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N) \), where \( u_0 \) is a given function).
We now give several examples.

(1) The classical calculus of variations dealt essentially with the case \( n = N = 1 \), where the most celebrated examples are the \textit{Fermat principle} in geometrical optics, where

\[
f(x, u, \xi) := g(x, u) \sqrt{1 + \xi^2},
\]

the \textit{Newton problem}, where

\[
f(x, u, \xi) = f(u, \xi) := \frac{2\pi u \xi^3}{1 + \xi^2},
\]

or the \textit{brachistochrone} problem, where

\[
f(x, u, \xi) = f(u, \xi) := \frac{\sqrt{1 + \xi^2}}{\sqrt{2} gu}.
\]

(2) When turning our attention to the case \( n > N = 1 \) (in our terminology, it is still part of the scalar case), the \textit{Dirichlet integral} surely plays a central role; we have there

\[
f(x, u, \xi) = f(\xi) := \frac{1}{2} |\xi|^2.
\]

A natural generalization is when \( 1 < p < \infty \) and

\[
f(x, u, \xi) = f(\xi) := \frac{1}{p} |\xi|^p.
\]

The \textit{minimal surface in non-parametric form} enters also in this framework; we have in this case

\[
f(x, u, \xi) = f(\xi) := \sqrt{1 + |\xi|^2}.
\]

In geometrical terms, the integral represents the area of the surface given by \((x, u(x)) \in \mathbb{R}^{n+1}\) when \(x \in \Omega \subseteq \mathbb{R}^n\).

(3) In the vectorial case \( n, N \geq 2 \), the first example is the case of \textit{minimal surfaces in parametric form}, a geometrical framework more general than the preceding one. In this case, we have \( N = n + 1 \) and therefore the matrix \( \xi \in \mathbb{R}^{(n+1) \times n} \). We denote by \( \text{adj}_n \xi \in \mathbb{R}^{n+1} \) the vector formed by all the \( n \times n \) minors of the matrix \( \xi \). Finally, we let

\[
f(x, u, \xi) = f(\xi) := |\text{adj}_n \xi|,
\]

where \(|.|\) stands for the Euclidean norm. In geometrical terms, the integral represents the area of the surface given by \( u(x) \in \mathbb{R}^{n+1}\) when \( x \in \Omega \subseteq \mathbb{R}^n \); moreover, \( \text{adj}_n \nabla u \) represents the normal to the surface.

Other important examples in the vectorial case are motivated by non-linear elasticity. A particularly simple one is when \( N = n \) and

\[
f(x, u, \xi) = f(\xi) := g(\det \xi),
\]
where \( g : \mathbb{R} \to \mathbb{R} \) is a given function.

We do not discuss the history of the calculus of variations and we refer for this matter to the books of Dierkes-Hildebrandt-Küster-Wohlrab [248], Giaquinta-Hildebrandt [309], Goldstine [319] and Monna [449].

The first question that arises in conjunction with problem \((P)\) is, of course, the existence of minimizers. This strongly depends on the choice of admissible functions, which we denoted by \( X \). A natural choice would be a subspace of \( C^1(\Omega; \mathbb{R}^N) \), or even \( C^2(\Omega; \mathbb{R}^N) \), if we want to be able to write the differential equation naturally associated to the minimization problem and known as the Euler-Lagrange equation. This turns out to be a strategy too hard to implement in most problems, particularly those dealing with partial derivatives (i.e. \( n > 1 \)). The essence of the direct methods of the calculus of variations is to split the problem into two parts. First to enlarge the space of admissible functions, for example by considering spaces such as the Sobolev spaces \( W^{1,p} \) so as to get a general existence theorem and then to prove some regularity results that should satisfy any minimizer of \((P)\). In the present book, we are essentially concerned only with the first problem. In most cases, the space of admissible functions is

\[
X = u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N),
\]

where \( u_0 \) is a given function and the notation \( u \in X \) is a shortcut meaning that \( u = u_0 \) on \( \partial \Omega \) and \( u \in W^{1,p}(\Omega; \mathbb{R}^N) \).

The existence of minimizers in the above space relies on the fundamental property of (sequential) weak lower semicontinuity, meaning that

\[
u
u
\]

\[
\liminf_{\nu \to \infty} I(u_\nu) \geq I(\bar{u}) ,
\]

where \( \to \) stands for weak convergence. This property is thoroughly investigated, notably in Chapters 3 and 8.

It turns out that the property (1.1) is intimately related to the convexity of the function \( \xi \to f(x, u, \xi) \) in the scalar case where \( N = 1 \) or \( n = 1 \) and to the quasiconvexity (in the sense of Morrey) of the same function in the vectorial case.

This leads us to the study of convex analysis in Chapter 2 and quasiconvex analysis in Chapters 5, 6 and 7.

We now discuss in more details the content of the monograph and outline some of the main results in every chapter. We state them, most of the time, under slightly stronger hypotheses than needed, but we refer to the precise theorems at each step.

1.2 Convex analysis and the scalar case

We start with the scalar case where \( n = 1 \) or \( N = 1 \). The first one corresponds to the case of one single independent variable and is much easier to deal with, in particular from the point of view of regularity. It is discussed in the general
framework of the scalar case in Chapter 3 but also has a special treatment in
Chapter 4. The second case, \( n > N = 1 \), involves partial derivatives and is
considerably harder; it is discussed in Chapter 3. However, since both cases use
in a significant way many results of convex analysis, we start with the study of
this classical subject.

1.2.1 Convex analysis

In Chapter 2, we present the most important results of convex analysis. Even
though many excellent books exist on the subject, we have decided, for the
c convenience of the reader, to state and to prove all the results that we need.
Another motivation in the presentation of this chapter has been to stress both
the similarities and the differences with quasiconvex analysis, which is discussed
in Part II.

Traditionally, convex analysis starts with the notion of a convex set and then
continues with that of convex functions. This is also the path we have followed,
in contrast with the quasiconvex case.

We start by recalling the notion of a convex set. A set \( E \subset \mathbb{R}^N \) is said to be
convex if for every \( x, y \in E \) and every \( t \in [0, 1] \)
\[
 tx + (1 - t) y \in E.
\]

We then give several elementary properties concerning the interior, closure and
boundary of convex sets. We next turn to two of the most useful results for con-
vex sets, namely the separation theorems (see Corollary 2.11) and Carathéodory
theorem (see Theorem 2.13). A typical separation theorem is, for example, the
following.

**Theorem 1.1** Let \( E \subset \mathbb{R}^N \) be convex and \( \mathbf{v} \in \partial E \). Then there exists \( a \in \mathbb{R}^N \),
\( a \neq 0 \), so that
\[
 \langle \mathbf{v}; a \rangle \leq \langle x; a \rangle \quad \text{for every } x \in E,
\]
where \( \langle \cdot; \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^N \).

We also recall that the convex hull of a set \( E \subset \mathbb{R}^N \) is the smallest convex
set containing \( E \) and is denoted by \( \text{co}E \). Carathéodory theorem then states the
following.

**Theorem 1.2** Let \( E \subset \mathbb{R}^N \). Then
\[
 \text{co}E = \left\{ x \in \mathbb{R}^N : x = \sum_{i=1}^{N+1} \lambda_i x_i, \ x_i \in E, \ \lambda_i \geq 0 \text{ with } \sum_{i=1}^{N+1} \lambda_i = 1 \right\}.
\]

We then conclude this brief account on convex sets by recalling the notion of extreme points of a convex set and Minkowski theorem, ensuring that if \( E \) is
compact and \( E_{\text{ext}} \) denotes the set of extreme points of \( \text{co}E \), then
\[
 \text{co}E = \text{co}E_{\text{ext}}.
\]
We next discuss the concept of a convex function. We recall that a function \( f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \) is said to be convex if
\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]
for every \( x, y \in \mathbb{R}^N \) and every \( t \in [0,1] \). An important property of convex functions that take only finite values (i.e. \( f : \mathbb{R}^N \to \mathbb{R} \)) is that they are everywhere continuous (see Theorem 2.31).

The notions of convex set and function are related through the indicator function of a set \( E \) defined by
\[
\chi_E(x) = \begin{cases} 
0 & \text{if } x \in E \\
+\infty & \text{if } x \notin E. 
\end{cases}
\]
Indeed the function \( \chi_E \) is convex if and only if the set \( E \) is convex.

As we defined the notion of a convex hull for a set, a natural concept is the convex envelope of a given function \( f \), which is, by definition, the largest convex function below \( f \) and is denoted by \( Cf \). We can therefore write, for every \( x \in \mathbb{R}^N \),
\[
Cf(x) := \sup \{ g(x) : g \leq f \text{ and } g \text{ convex} \}.
\]

Of central importance in convex analysis is the concept of a conjugate function (or Legendre transform). The conjugate of a function \( f \) is a function \( f^* : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \) defined by
\[
f^*(x^*) := \sup_{x \in \mathbb{R}^N} \{ \langle x; x^* \rangle - f(x) \},
\]
which is a convex function, independently of the convexity of \( f \). Iterating the process, we define the biconjugate of \( f \) as \( f^{**} : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\} \), it is given by
\[
f^{**}(x) = \sup_{x^* \in \mathbb{R}^N} \{ \langle x; x^* \rangle - f^*(x^*) \}.
\]
It turns out that if \( f \) takes only finite values then (see Theorem 2.43)
\[
Cf = f^{**}.
\]

Finally, we also investigate the differentiability of convex functions, discussing, in particular, the notion of a subgradient.

### 1.2.2 Lower semicontinuity and existence results

The main result of Chapter 3 is the following (more general ones are found in Theorem 3.15 and Corollary 3.24).

**Theorem 1.3** Let \( n, N \in \mathbb{N}, p \geq 1, \Omega \subset \mathbb{R}^n \) be a bounded open set with a Lipschitz boundary, \( f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R} \) be a non-negative continuous function and
\[
I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx.
\]
Part 1. If the function $\xi \rightarrow f(x,u,\xi)$ is convex, then $I$ is (sequentially) weakly lower semicontinuous in $W^{1,p}$ (meaning that (1.1) is satisfied).

Part 2. Conversely, if either $N = 1$ or $n = 1$ and $I$ is (sequentially) weakly lower semicontinuous in $W^{1,p}$, then the function $\xi \rightarrow f(x,u,\xi)$ is convex.

We should emphasize that in the vectorial case, $n, N \geq 2$, Part 1 of the theorem is valid but the conclusion of Part 2 does not hold.

This theorem, in the scalar case, has as a first direct consequence that the functional is (sequentially) weakly continuous in $W^{1,p}$, meaning that

$$u_\nu \rightharpoonup u \text{ in } W^{1,p} \Rightarrow \lim_{\nu \to \infty} I(u_\nu) = I(u)$$

if and only if $\xi \rightarrow f(x,u,\xi)$ is affine. This result again strongly contrasts with the vectorial case.

The main implication of the lower semicontinuity theorem is on the existence of minimizers for the problem

$$(P) \inf \left\{ I(u) : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\} = m.$$ 

Indeed we have, as a special case of our general theorem (see Theorem 3.30), the following result.

**Theorem 1.4** Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ with a Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ be continuous and satisfying the coercivity condition

$$f(x,u,\xi) \geq \alpha_1 |\xi|^p - \alpha_2, \quad \forall (x,u,\xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n},$$

for some $\alpha_1 > 0$, $\alpha_2 \in \mathbb{R}$ and $p > 1$. Assume that $\xi \rightarrow f(x,u,\xi)$ is convex and that $I(u_0) < \infty$. Then $(P)$ has at least one minimizer.

This theorem is also valid in the vectorial case, but can then be improved a great deal.

As is well known, associated with any variational problem is the differential equation known as the *Euler-Lagrange equation*. Under appropriate regularity hypotheses on the function $f$ and on a minimizer $\overline{u}$ of $(P)$, we find that $\overline{u}$ should satisfy, for every $x \in \Omega$,

$$(E) \quad \sum_{\alpha=1}^n \frac{\partial}{\partial x_\alpha} \left[ \frac{\partial f}{\partial \xi_\alpha}(x,\overline{u},\nabla \overline{u}) \right] = \frac{\partial f}{\partial u^i}(x,\overline{u},\nabla \overline{u}), \quad i = 1, \ldots, N.$$ 

The differential equation is a second order ordinary differential equation if $n = N = 1$, a system of such equations if $N > n = 1$, a single second order partial differential equation if $n > N = 1$ and a system of such equations when $n, N \geq 2$. In any case, the convexity of the function $\xi \rightarrow f(x,u,\xi)$ ensures the *ellipticity* of the Euler-Lagrange equations. The prototype example is the Dirichlet integral where $n > N = 1$,

$$f(x,u,\xi) = f(\xi) := \frac{1}{2} |\xi|^2,$$
and the associated equation is nothing other than the \textit{Laplace equation}
\[\Delta u = 0.\]

\subsection*{1.2.3 The one dimensional case}

In Chapter 4, we specialize to the case where $N = n = 1$, although most of the results are also valid if $N > n = 1$. We are therefore considering the problem

\[(P) \quad \inf \left\{ I(u) = \int_a^b f(x, u(x), u'(x)) \, dx : u \in X \right\},\]

where $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $p \geq 1$ and

\[X = \{ u \in W^{1,p} (a, b) : u(a) = \alpha, u(b) = \beta \}.\]

The Euler-Lagrange equation that should satisfy any minimizer $\varpi$ of $(P)$ is then given by

\[(E) \quad \frac{d}{dx} [f_\xi(x, \varpi(x), \varpi'(x))] = f_u(x, \varpi(x), \varpi'(x)), x \in [a, b],\]

where $f_\xi = \partial f / \partial \xi$ and $f_u = \partial f / \partial u$. When the function $f$ does not depend explicitly on the variable $x$, one can find a first integral of $(E)$ that is known as the \textit{second form} of the Euler-Lagrange equation and can be written as

\[f(\varpi(x), \varpi'(x)) - \varpi'(x) f_\xi(\varpi(x), \varpi'(x)) = \text{constant}, x \in [a, b].\]

At this stage it might be enlightening to see some examples that show that, even when $n = N = 1$, the hypotheses of the existence theorem (see Theorem 1.4) are essentially optimal. Indeed \textit{non-existence} of minimizers in Sobolev spaces occurs in all the following cases.

1. Let (see Example 4.4) $f(\xi) = e^{-\xi^2}$ and

\[(P) \quad \inf \left\{ I(u) = \int_0^1 f(u'(x)) \, dx : u \in X \right\},\]

where $X = W_0^{1,1}(0, 1) = \{ u \in W^{1,1}(0, 1) : u(0) = u(1) = 0 \}$. Here both the convexity and coercivity hypotheses of the theorem are violated.

2. Consider (see Example 4.5) the case $f(x, u, \xi) = f(u, \xi) = \sqrt{u^2 + \xi^2}$ and

\[(P) \quad \inf \left\{ I(u) = \int_0^1 f(u(x), u'(x)) \, dx : u \in X \right\},\]

where $X = \{ u \in W^{1,1}(0, 1) : u(0) = 0, u(1) = 1 \}$. In this case, the coercivity condition holds with $p = 1$ (and not, as it should, with $p > 1$).
(3) The present example (see Example 4.6) is known as the \textit{Weierstrass example}. Let \( f(x, u, \xi) = f(x, \xi) = x\xi^2 \) and
\[
(P) \quad \inf \left\{ I(u) = \int_0^1 f(x, u'(x)) \, dx : u \in X \right\},
\]
where \( X = \{ u \in W^{1,2}(0,1) : u(0) = 1, u(1) = 0 \} \). The coercivity hypothesis is violated at just one point (namely at \( x = 0 \)).

(4) Let (the example is known as the \textit{Bolza example}, see Example 4.8)
\[
f(x, u, \xi) = f(u, \xi) = (\xi^2 - 1)^2 + u^4
\]
\[
(P) \quad \inf \left\{ I(u) = \int_0^1 f(u(x), u'(x)) \, dx : u \in W^{1,4}_0(0,1) \right\}.
\]
Here it is the convexity assumption on the function \( \xi \to f(x, u, \xi) \) that is not satisfied.

Another advantage of the case \( N = n = 1 \) is that, under appropriate conditions on \( f \), notably the convexity of \( \xi \to f(x, u, \xi) \), the solutions of (E) are also solutions and conversely (see Theorem 4.29) of the \textit{Hamiltonian system}
\[
(H) \quad \begin{cases}
u'(x) = H_v(x, u(x), v(x)) \\
v'(x) = -H_u(x, u(x), v(x))
\end{cases},
\]
where \( v(x) = f_\xi(x, u(x), u'(x)) \) and \( H \) is the Legendre transform of \( \xi \to f(x, u, \xi) \), namely
\[
H(x, u, v) = \sup_{\xi \in \mathbb{R}} \{ v\xi - f(x, u, \xi) \}.
\]
In classical mechanics, \( f \) is called the \textit{Lagrangian} and \( H \) the \textit{Hamiltonian}.

We conclude the study of Chapter 4 with a brief discussion on \textit{Lavrentiev phenomenon}. We just study the following example (see Theorem 4.41) exhibited by Mania. We let
\[
f(x, u, \xi) = (x - u^3)^2 \xi^6,
\]
\[
I(u) = \int_0^1 f(x, u(x), u'(x)) \, dx.
\]
Consider the two different Sobolev spaces
\[
W_\infty = \{ u \in W^{1,\infty}(0,1) : u(0) = 0, u(1) = 1 \},
\]
\[
W_1 = \{ u \in W^{1,1}(0,1) : u(0) = 0, u(1) = 1 \},
\]
and the corresponding minimization problems
\[
\inf \{ I(u) : u \in W_\infty \} = m_\infty \quad \text{and} \quad \inf \{ I(u) : u \in W_1 \} = m_1.
\]
We prove that

\[ m_\infty > m_1 = 0 \]

and that \( \mathbf{p}(x) = x^{1/3} \) is a minimizer of \( I \) over \( \mathcal{W}_1 \).

### 1.3 Quasiconvex analysis and the vectorial case

We next turn to the vectorial case \( n, N \geq 2 \), which is the heart of our book and deals with what we call *quasiconvex analysis*. The structure is similar to that of Part I; namely, we develop the quasiconvex analysis in Chapters 5, 6 and 7 and then discuss lower semicontinuity and existence results in Chapter 8.

A first striking difference between our presentations of convex and quasiconvex analyses is the order in which we deal with sets and functions. In convex analysis we first defined, as do essentially all other authors, the concept of convex sets and then that of convex functions. In the present context, we do exactly the reverse. This has some historical reasons. The notion of a quasiconvex function was introduced by Morrey in 1952, while the corresponding notion for sets appeared almost fifty years later and is, in some sense, in its infancy.

The main motivation for introducing the notion of quasiconvexity is to generalize Theorem 1.3 to the vectorial case.

#### 1.3.1 Quasiconvex functions

Unfortunately, when generalizing the notion of a convex function to the vectorial case, several different concepts arise naturally. The notion of a *quasiconvex* function arises, as already said, in conjunction with (sequential) weak lower semicontinuity of the corresponding integral. When dealing with the Euler-Lagrange equation, the right concept is the ellipticity and this leads to the definition of a *rank one convex* function. Finally, when one wants to generalize the separation theorems, Carathéodory theorem, or the notion of duality, one is driven to the concept of *polyconvexity*.

We now describe the content of Chapter 5 and we start with the following definitions.

**Definition 1.5** Let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \).

(i) The function \( f \) is said to be rank one convex if

\[
    f (\lambda \xi + (1 - \lambda) \eta) \leq \lambda f (\xi) + (1 - \lambda) f (\eta)
\]

for every \( \lambda \in [0, 1] \), \( \xi, \eta \in \mathbb{R}^{N \times n} \) with \( \text{rank} \{\xi - \eta\} \leq 1 \).

(ii) If \( f \) is Borel measurable and locally bounded, then it is said to be quasi-convex if

\[
    f (\xi) \leq \frac{1}{\operatorname{meas} D} \int_D f (\xi + \nabla \varphi (x)) \, dx
\]
for every bounded open set $D \subset \mathbb{R}^n$, for every $\xi \in \mathbb{R}^{N \times n}$ and for every $\varphi \in W_0^{1,\infty}(D;\mathbb{R}^N)$.

(iii) The function $f$ is said to be polyconvex if there exists $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R}$ convex, such that

$$f(\xi) = F(T(\xi)),$$

where $T : \mathbb{R}^{N \times n} \to \mathbb{R}^{\tau(n,N)}$ is such that

$$T(\xi) = (\xi, \text{adj}_2 \xi, \cdots, \text{adj}_{n\wedge N} \xi).$$

In the previous definition, $\text{adj}_s \xi$ stands for the matrix of all $s \times s$ minors of the matrix $\xi \in \mathbb{R}^{N \times n}$, $2 \leq s \leq n \wedge N = \min\{n, N\}$, and

$$\tau(n,N) = \sum_{s=1}^{n\wedge N} \sigma(s)$$

where

$$\sigma(s) = \binom{N}{s} \binom{n}{s} = \frac{N!n!}{(s!)^2 (N-s)! (n-s)!}.$$

(iv) A function $f$ is said to be rank one affine, quasiaffine or polyaffine if $f$ and $-f$ are rank one convex, quasiconvex or polyconvex respectively.

Remark 1.6 (i) Note that in the case $N = n = 2$, the notion of polyconvexity can be read as follows:

$$\{ \tau(n,N) = \tau(2,2) = 5 \text{ (since } \sigma(1) = 4, \sigma(2) = 1) \}
\begin{align*}
T(\xi) &= (\xi, \det \xi), 
 f(\xi) &= F(\xi, \det \xi).
\end{align*}$$

(ii) The first and third definitions extend in a straightforward manner to functions $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$. However this is not the case for quasiconvex functions. At the moment, no good definition of quasiconvexity for such functions is available. This fact is a strong source of difficulty when dealing with the definition of quasiconvex sets.

The main properties of these functions are now given (see Theorems 5.3 and 5.20).

Theorem 1.7 Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$.

(i) The following implications hold

$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ quasiconvex } \Rightarrow f \text{ rank one convex}.$$

(ii) If $N = 1$ or $n = 1$, then all these notions are equivalent.

(iii) If $f \in C^2(\mathbb{R}^{N \times n})$, then rank one convexity is equivalent to the Legendre-Hadamard condition (or ellipticity condition)

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f(\xi)}{\partial \xi_i^\alpha \partial \xi_j^\beta} \lambda_i^\alpha \lambda_j^\beta \mu_\alpha \mu_\beta \geq 0$$
for every \( \lambda \in \mathbb{R}^N, \mu \in \mathbb{R}^n, \xi = (\xi_i^\alpha)_{1 \leq i \leq N}^{1 \leq \alpha \leq n} \in \mathbb{R}^{N \times n} \).

(iv) The notions of rank one affine, quasiaffine and polyaffine are equivalent. Moreover, any quasiaffine function is of the form

\[
f (\xi) = \alpha + \langle \beta; T (\xi) \rangle,
\]

where \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R}^{\tau(n,N)} \) and \( \langle \cdot; \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^{\tau(n,N)} \).

We now give some significant examples. The first one (see Theorem 5.25) concerns quadratic forms and is one of the most important, since then the associated Euler-Lagrange equations are linear.

**Theorem 1.8** Let \( M \) be a symmetric matrix in \( \mathbb{R}^{(N \times n) \times (N \times n)} \). Let

\[
f (\xi) = \langle M \xi; \xi \rangle,
\]

where \( \xi \in \mathbb{R}^{N \times n} \) and \( \langle \cdot; \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^{N \times n} \). The following statements then hold.

(i) \( f \) is rank one convex if and only if \( f \) is quasiconvex.

(ii) If \( N = 2 \) or \( n = 2 \), then

\[
f \text{ polyconvex} \iff f \text{ quasiconvex} \iff f \text{ rank one convex}.
\]

(iii) If \( N, n \geq 3 \), then in general

\[
f \text{ rank one convex} \not\Rightarrow f \text{ polyconvex}.
\]

We next turn to some more examples.

1) Let \( N = n = 2 \). The function

\[
f (\xi) = \det \xi
\]

is quasiaffine and thus polyconvex, quasiconvex or rank one convex, but not convex.

2) When \( n \geq 2 \) and \( N \geq 3 \), Sverak (see Theorem 5.50) produced an example of a function that is rank one convex but not quasiconvex, answering a long standing conjecture of Morrey. It is still not known if there are rank one convex but not quasiconvex functions in the case \( N = n = 2 \), or more generally \( n \geq N = 2 \).

3) Let \( N = n = 2 \). The function studied by Alibert-Dacorogna-Marcellini (see Theorem 5.51) and given by \( f_\gamma : \mathbb{R}^{2 \times 2} \to \mathbb{R} \), for \( \gamma \in \mathbb{R} \), where

\[
f_\gamma (\xi) = |\xi|^2 \left( |\xi|^2 - 2\gamma \det \xi \right),
\]
is such that

\[
\begin{align*}
  f_{\gamma} \text{ is convex} & \iff |\gamma| \leq \gamma_c = 2\sqrt{2}/3, \\
  f_{\gamma} \text{ is polyconvex} & \iff |\gamma| \leq \gamma_p = 1, \\
  f_{\gamma} \text{ is quasiconvex} & \iff |\gamma| \leq \gamma_q \text{ and } \gamma_q > 1, \\
  f_{\gamma} \text{ is rank one convex} & \iff |\gamma| \leq \gamma_r = 2/\sqrt{3}.
\end{align*}
\]

It is not presently known if \( \gamma_q = 2/\sqrt{3} \).

### 1.3.2 Quasiconvex envelopes

In Chapter 6, we define the convex \( Cf \) (already defined in Section 1.2.1) polyconvex \( Pf \), quasiconvex \( Qf \) and rank one convex envelope \( Rf \), which are, respectively, defined as the largest convex, polyconvex, quasiconvex and rank one convex functions below \( f \). We therefore have, for every \( \xi \in \mathbb{R}^{N \times n} \),

\[
\begin{align*}
  Cf (\xi) &= \sup \left\{ g(\xi) : g \leq f \text{ and } g \text{ convex} \right\}, \\
  Pf (\xi) &= \sup \left\{ g(\xi) : g \leq f \text{ and } g \text{ polyconvex} \right\}, \\
  Qf (\xi) &= \sup \left\{ g(\xi) : g \leq f \text{ and } g \text{ quasiconvex} \right\}, \\
  Rf (\xi) &= \sup \left\{ g(\xi) : g \leq f \text{ and } g \text{ rank one convex} \right\}.
\end{align*}
\]

Observe that Theorem 1.7 immediately implies

\[
Cf \leq Pf \leq Qf \leq Rf \leq f.
\]

Several representation formulas exist for computing these envelopes, we just give a formula for the quasiconvex envelope (see Theorem 6.9).

**Theorem 1.9** Let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) be locally bounded, non-negative and Borel measurable. Then, for every \( \xi \in \mathbb{R}^{N \times n} \),

\[
Qf (\xi) = \inf \left\{ \frac{1}{\text{meas } D} \int_{D} f (\xi + \nabla \varphi (x)) \, dx : \varphi \in W_{0}^{1,\infty} (D; \mathbb{R}^{N}) \right\},
\]

where \( D \subset \mathbb{R}^{n} \) is a bounded open set. In particular, the infimum in the formula is independent of the choice of \( D \).

We now give some examples.

1. Let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \), \( \Phi : \mathbb{R}^{N \times n} \to \mathbb{R} \) be quasiaffine not identically constant and \( g : \mathbb{R} \to \mathbb{R} \) such that

\[
f (\xi) = g (\Phi (\xi)).
\]

Then (see Theorem 6.24)

\[
Pf = Qf = Rf = Cg \circ \Phi
\]

and in general \( Qf > Cf \).
(2) Recall the area type case, where \( N = n + 1 \). Let \( f : \mathbb{R}^{(n+1) \times n} \to \mathbb{R} \) and \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) be such that
\[
f(\xi) = g(\text{adj}_n \xi).
\]
Then (see Theorem 6.26)
\[
Pf = Qf = Rf = Cg \circ \text{adj}_n
\]
and in general \( Qf > Cf \).

(3) An interesting problem in optimal design is the following. Let \( N = n = 2 \) and, for \( \xi \in \mathbb{R}^{2 \times 2} \),
\[
f(\xi) = \begin{cases} 
1 + |\xi|^2 & \text{if } \xi \neq 0 \\
0 & \text{if } \xi = 0.
\end{cases}
\]
Then (see Theorem 6.28) \( Pf = Qf = Rf \) and
\[
Qf(\xi) = \begin{cases} 
1 + |\xi|^2 & \text{if } |\xi|^2 + 2|\det \xi| \geq 1 \\
2(|\xi|^2 + 2|\det \xi|)^{1/2} - 2|\det \xi| & \text{if } |\xi|^2 + 2|\det \xi| < 1.
\end{cases}
\]
We also have
\[
Cf(\xi) = \begin{cases} 
1 + |\xi|^2 & \text{if } |\xi| \geq 1 \\
2|\xi| & \text{if } |\xi| < 1.
\end{cases}
\]

1.3.3 Quasiconvex sets

We have seen in Section 1.2.1 that the connection between convex functions and sets is made via the indicator function. We recall that, for a set \( E \), the indicator function is defined by
\[
\chi_E(x) = \begin{cases} 
0 & \text{if } x \in E \\
+\infty & \text{if } x \notin E.
\end{cases}
\]
Moreover, the function \( \chi_E \) is convex if and only if the set \( E \) is convex.

The aim of Chapter 7 is to extend the definition of convexity for sets to polyconvexity, quasiconvexity and rank one convexity. A natural way to define polyconvex, quasiconvex or rank one convex set \( E \) would be by requiring that \( \chi_E \) be polyconvex, quasiconvex or rank one convex. This is indeed so (see Proposition 7.5) for the first and third cases but not for quasiconvex sets, since, as we already said, we lack a good definition of quasiconvexity for functions that are allowed to take the value \(+\infty\).

Before giving the definitions, let us introduce some notation. In this section we let \( O(n) \) be the set of \( n \times n \) orthogonal matrices,
\[
D := (0, 1)^n \subset \mathbb{R}^n
\]
and $W^{1,\infty}_{per}(D; \mathbb{R}^N)$ be the space of periodic functions in $W^{1,\infty}(D; \mathbb{R}^N)$, meaning that
\[ u(x) = u(x + e_i), \text{ for every } x \in D \text{ and } i = 1, \cdots, n, \]
where \{e_1, \cdots, e_n\} is the standard orthonormal basis of $\mathbb{R}^n$. Finally, $W_{per}$ denotes the subspace of functions in $W^{1,\infty}_{per}(D; \mathbb{R}^N)$, whose gradients take only a finite number of values.

We are now in a position to give the following definitions (see Definition 7.2).

**Definition 1.10** (i) We say that $E \subset \mathbb{R}^{N \times n}$ is polyconvex if there exists a convex set $K \subset \mathbb{R}^{\tau(N,n)}$ such that
\[ \{ \xi \in \mathbb{R}^{N \times n} : T(\xi) \in K \} = E. \]

(ii) We say that $E \subset \mathbb{R}^{N \times n}$ is quasiconvex if we have
\[ \{ \xi + \nabla \varphi(x)R \in E, \ a.e. \ x \in D, \text{ for some } R \in O(n) \text{ and some } \varphi \in W_{per} \} \Rightarrow \xi \in E. \]

(iii) We say that $E \subset \mathbb{R}^{N \times n}$ is rank one convex if for every $\lambda \in [0,1]$ and $\xi, \eta \in E$ such that $\text{rank} \{ \xi - \eta \} = 1$, then
\[ \lambda \xi + (1 - \lambda)\eta \in E. \]

The best definition for quasiconvex sets is unclear. Several definitions have already been considered by other authors. The one we propose here is consistent with known properties for functions and has most properties that are desirable as witnessed by the following theorem (see Theorem 7.7).

**Theorem 1.11** Let $E \subset \mathbb{R}^{N \times n}$. The following implications then hold:

$E$ convex $\Rightarrow$ $E$ polyconvex $\Rightarrow$ $E$ quasiconvex $\Rightarrow$ $E$ rank one convex.

All counter implications are false as soon as $N, n \geq 2$.

We should draw attention to the last statement of the theorem. Surprisingly it is better than the corresponding one for functions, where the example of Sverak provides a rank one convex function that is not quasiconvex only when $n \geq 2$ and $N \geq 3$.

Before continuing, one main difference between convex sets and generalized ones should be emphasized. A set can be polyconvex, and thus quasiconvex and rank one convex, and be disconnected. Indeed, if $\xi, \eta \in \mathbb{R}^{N \times n}$ are such that $\text{rank} \{ \xi - \eta \} \geq 2$, then $E = \{ \xi, \eta \}$ is polyconvex.

We next point out a fact (the second one in the next proposition) strikingly different from the equivalent one for convex sets (see Proposition 7.24).
Proposition 1.12 (i) Let $E \subset \mathbb{R}^{N \times n}$ be, respectively, a polyconvex, quasi-convex or rank one convex set. Then $\text{int } E$ is also, respectively, polyconvex, quasiconvex or rank one convex.

(ii) There exists a polyconvex and bounded set $E \subset \mathbb{R}^{2 \times 2}$ such that $E$ is not rank one convex (and hence neither quasiconvex nor polyconvex).

We next define the polyconvex, quasiconvex and rank one convex hulls of a set $E \subset \mathbb{R}^{N \times n}$ as the smallest polyconvex, quasiconvex and rank one convex sets containing $E$; they are respectively denoted by $P_{\text{co}} E$, $Q_{\text{co}} E$ and $R_{\text{co}} E$.

We clearly have

$$E \subset R_{\text{co}} E \subset Q_{\text{co}} E \subset P_{\text{co}} E \subset \text{co } E.$$ 

Other hulls are also defined in Chapter 7.

We finally conclude this section by giving an example. We first recall that the singular values of a given matrix $\xi \in \mathbb{R}^{n \times n}$, denoted by

$$0 \leq \lambda_1 (\xi) \leq \cdots \leq \lambda_n (\xi),$$

are the eigenvalues of $(\xi \xi^t)^{1/2}$. Let $0 \leq \gamma_1 \leq \cdots \leq \gamma_n$ and consider the set

$$E = \{ \xi \in \mathbb{R}^{n \times n} : \lambda_i (\xi) = \gamma_i, \ i = 1, \cdots , n \}.$$

We prove (see Theorem 7.43) that

$$\text{co } E = \{ \xi \in \mathbb{R}^{n \times n} : \sum_{i=1}^{n} \lambda_i (\xi) \leq \sum_{i=1}^{n} \gamma_i, \ \nu = 1, \cdots , n \}$$

$$P_{\text{co}} E = Q_{\text{co}} E = R_{\text{co}} E = \{ \xi \in \mathbb{R}^{n \times n} : \prod_{i=1}^{n} \lambda_i (\xi) \leq \prod_{i=1}^{n} \gamma_i, \ \nu = 1, \cdots , n \}.$$

1.3.4 Lower semicontinuity and existence theorems

In Chapter 8, we extend the lower semicontinuity results (see Theorem 1.3) to the vectorial context. This is a delicate matter and, in Chapter 8, we deal with it in several steps. We now gather Theorems 8.1 and 8.11 to obtain the following result.

Theorem 1.13 Let $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary and let

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \ f = f (x, u, \xi),$$

be a continuous function satisfying

$$0 \leq f (x, u, \xi) \leq g (x, u) (1 + |\xi|^p),$$
where
\[ g : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}, \quad g = g(x,u), \]
is a non-negative continuous function. Let
\[ I(u) = \int_{\Omega} f(x,u(x),\nabla u(x)) \, dx. \]
Then \( I \) is (sequentially) weakly lower semicontinuous in \( W^{1,p}(\Omega;\mathbb{R}^N) \) if and only if \( \xi \rightarrow f(x,u,\xi) \) is quasiconvex, i.e.
\[ \frac{1}{\text{meas } D} \int_D f(x_0,u_0,\xi_0 + \nabla \varphi(x)) \, dx \geq f(x_0,u_0,\xi_0) \]
for every \( (x_0,u_0,\xi_0) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \), for every bounded open set \( D \subset \mathbb{R}^n \) and for every \( \varphi \in W^{1,\infty}_0(D;\mathbb{R}^n) \).

This result has as an immediate corollary that \( I \) is (sequentially) weakly continuous in \( W^{1,p} \) if and only if \( \xi \rightarrow f(x,u,\xi) \) is quasiaffine, i.e. all minors of the matrix \( \xi \in \mathbb{R}^{N \times n} \) are weakly continuous. We now restate this result, in a more convenient and more general way, in the case where \( N = n = 2 \) (see Theorem 8.20, Lemma 8.24 and Corollary 8.26). Let us start with the simple but fundamental observation that Jacobian determinants can be written in divergence form. More precisely if \( u \in C^2(\Omega;\mathbb{R}^2) \), then letting
\[ \text{Det} \, \nabla u := \frac{\partial}{\partial x_1} (u_1 \frac{\partial u_2}{\partial x_2}) - \frac{\partial}{\partial x_2} (u_1 \frac{\partial u_2}{\partial x_1}), \]
we find that
\[ \text{Det} \, \nabla u(x) = \text{det} \, \nabla u(x), \quad \text{for every } x \in \Omega, \]
since we trivially have
\[ \text{det} \, \nabla u = \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \frac{\partial u_1}{\partial x_2} \]
\[ = \frac{\partial}{\partial x_1} (u_1 \frac{\partial u_2}{\partial x_2}) - \frac{\partial}{\partial x_2} (u_1 \frac{\partial u_2}{\partial x_1}) = \text{Det} \, \nabla u. \]
The quantity \( \text{Det} \, \nabla u \) is called the distributional Jacobian of \( u \). We can now state the theorem (see Theorem 8.20, Lemma 8.24, Corollary 8.26 and Example 8.28).

**Theorem 1.14** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set, \( 1 < p < \infty \), and let
\[ u_\nu \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^2). \]

Part 1. If \( p > 2 \), then
\[ \text{det} \, \nabla u_\nu \rightharpoonup \text{det} \, \nabla u \text{ in } L^{p/2}(\Omega). \]
If \( p = 2 \), the result is false, but the following convergence holds
\[
\det \nabla u_\nu \rightharpoonup \det \nabla u \text{ in } D'(\Omega).
\]

Part 2. If \( p \geq 4/3 \), then \( \text{Det} \nabla u \in D'(\Omega) \) and if \( p \geq 2 \), then
\[
\text{Det} \nabla u = \det \nabla u \text{ in } D'(\Omega).
\]

Part 3. If \( p > 4/3 \), then \( \text{Det} \nabla u_\nu \rightharpoonup \text{Det} \nabla u \text{ in } D'(\Omega) \).

If \( p \leq 4/3 \), the result is false.

Theorem 1.13 also has as a direct consequence the following existence theorem (see Theorem 8.29).

**Theorem 1.15** Let \( p > 1 \), \( \Omega \subset \mathbb{R}^n \) be a bounded open set with a Lipschitz boundary. Let \( f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, f = f(x, u, \xi) \), be a continuous function satisfying
\[
\xi \to f(x, u, \xi) \text{ is quasiconvex},
\]
\[
\alpha_1 |\xi|^p + \beta_1 \leq f(x, u, \xi) \leq \alpha_2 (|\xi|^p + 1),
\]
for every \( (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \), where \( \alpha_2 \geq \alpha_1 > 0, \beta_1 \in \mathbb{R} \). Let
\[
(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\}.
\]
Then \( P \) admits at least one minimizer.

Using Theorem 1.14, we can also prove some existence theorems for polyconvex functions (see Theorem 8.31).

### 1.4 Relaxation and non-convex problems

In Part III, we go back to the study of
\[
(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\},
\]
where
- \( \Omega \subset \mathbb{R}^n, n \geq 1 \), is a bounded open set;
- \( u : \Omega \to \mathbb{R}^N, N \geq 1 \) and \( u_0 \in W^{1,p}(\Omega; \mathbb{R}^N) \) is a given function;
- \( f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, f = f(x, u, \xi) \), is a given non-convex (non-quasiconvex in the vectorial case) function.

The direct methods (see Theorems 1.4 and 1.15) do not apply and the general rule is that \( P \) has no minimizers, as already pointed out in Section 1.2.3.
However, there is a way of defining generalized solutions of \((P)\) via the so called relaxed problem

\[
(P) \quad \inf \left\{ T(u) = \int_{\Omega} Qf(x,u(x),\nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega;\mathbb{R}^N) \right\},
\]

where \(Qf\) is the quasiconvex envelope of \(f\) (with respect to the last variable \(\nabla u\)), defined in Section 1.3.2.

The relaxed problem is useful not only to define generalized solutions of \((P)\), but also to show that in many cases, although the direct methods do not apply, the problem \((P)\) does have minimizers.

### 1.4.1 Relaxation theorems

In Chapter 9, we prove the relaxation theorem (see Theorems 9.1 and 9.8) and we state it here, as usual under stronger hypotheses, in the case where \(f\) does not depend on lower order terms.

**Theorem 1.16** Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set. Let \(f : \mathbb{R}^{N \times n} \to \mathbb{R}\) be a Borel measurable function satisfying, for \(1 \leq p < \infty\),

\[
0 \leq f(\xi) \leq \alpha (1 + |\xi|^p), \text{ for every } \xi \in \mathbb{R}^{N \times n},
\]

where \(\alpha > 0\) is a constant while for \(p = \infty\) it is assumed that \(f\) is locally bounded. For every \(\xi \in \mathbb{R}^{N \times n}\), let

\[
Qf(\xi) = \sup \{ g(\xi) : g \leq f \text{ and } g \text{ quasiconvex} \}
\]

be the quasiconvex envelope of \(f\).

Part 1. Then

\[
\inf (P) = \inf (QP).
\]

More precisely, for every \(u \in W^{1,p}(\Omega;\mathbb{R}^N)\), there exists a sequence \(\{u_\nu\}_{\nu=1}^{\infty} \subset u + W^{1,p}_0(\Omega;\mathbb{R}^N)\) such that

\[
u \to \infty, \quad \gamma \to \infty.
\]

Part 2. Assume, in addition to the hypotheses of Part 1, that, if \(1 < p < \infty\), there exist \(\alpha \geq \beta > 0\), \(\gamma \in \mathbb{R}\) such that

\[
\gamma + \beta |\xi|^p \leq f(\xi) \leq \alpha (1 + |\xi|^p) \text{ for every } \xi \in \mathbb{R}^{N \times n}.
\]

Then, in addition to the conclusions of Part 1, the following holds:

\[
u \to \infty, \quad \gamma \to \infty.
\]
1.4.2 Some existence theorems for differential inclusions

When we apply, in Section 1.4.3, the relaxation theorems to get existence of minimizers for the problem \((P)\), we need to find solutions of some differential inclusions. This is achieved in Chapter 10, where we deal with the problem of finding a map \(u \in W^{1,\infty}(\Omega; \mathbb{R}^N)\) that solves

\[
\begin{align*}
\nabla u(x) & \in E \quad \text{a.e. } x \in \Omega \\
u(x) & = u_0(x) \quad x \in \partial \Omega,
\end{align*}
\]

where \(u_0 \in W^{1,\infty}(\Omega; \mathbb{R}^N)\) is a given map and \(E \subset \mathbb{R}^{N\times n}\) is a given set.

In this introductory chapter, we do not give any general result but discuss only some significant examples. The first one concerns the scalar case, where the result takes an almost optimal form (see Theorem 10.18).

**Theorem 1.17** Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set and \(E \subset \mathbb{R}^n\). Let \(u_0 \in W^{1,\infty}(\Omega)\) satisfy

\[
\nabla u_0(x) \in E \cup \text{int co } E \quad \text{a.e. } x \in \Omega
\]

(1.2) (where \(\text{int co } E\) stands for the interior of the convex hull of \(E\)); then there exists \(u \in u_0 + W^{1,\infty}_0(\Omega)\) such that

\[
\nabla u(x) \in E \quad \text{a.e. } x \in \Omega.
\]

The theorem has as an immediate consequence the following result (see Corollary 10.20). If \(F : \mathbb{R}^n \to \mathbb{R}\) is continuous and such that

\[
\lim_{|\xi| \to \infty} F(\xi) = +\infty
\]

and \(u_0 \in W^{1,\infty}(\Omega)\) verifies

\[
F(\nabla u_0(x)) \leq 0 \quad \text{a.e. } x \in \Omega,
\]

then there exists \(u \in u_0 + W^{1,\infty}_0(\Omega)\) such that

\[
F(\nabla u(x)) = 0 \quad \text{a.e. } x \in \Omega.
\]

The condition (1.2) is also necessary when the boundary datum is affine (see Theorem 10.24).

**Theorem 1.18** Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set, \(E \subset \mathbb{R}^n\) and \(u_0\) be such that

\[
\nabla u_0 = \xi_0
\]

for some \(\xi_0 \in \mathbb{R}^n\). If \(u \in u_0 + W^{1,\infty}_0(\Omega)\) solves

\[
\nabla u(x) \in E \quad \text{a.e. } x \in \Omega,
\]
then
\[ \xi_0 \in E \cup \text{int co } E. \]

The next result (see Theorem 10.25), which is now a vectorial one, should be related to the example given in Section 1.3.3.

**Theorem 1.19** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set, \( 0 < a_1 \leq \cdots \leq a_n \) and \( \xi_0 \in \mathbb{R}^{n \times n} \) be such that
\[
\prod_{i=\nu}^{n} \lambda_i(\xi_0) < \prod_{i=\nu}^{n} a_i, \quad \nu = 1, \cdots, n.
\]
If \( u_0 \) is an affine map such that \( \nabla u_0 = \xi_0 \), then there exists \( u \in u_0 + W_{0,1}^{1,\infty}(\Omega; \mathbb{R}^n) \) so that, for almost every \( x \in \Omega \),
\[ \lambda_{\nu}(\nabla u(x)) = a_{\nu}, \quad \nu = 1, \cdots, n. \]

### 1.4.3 Some existence results for non-quasiconvex integrands

We now apply (see Chapter 11) the results of the two previous sections to prove the existence of minimizers for
\[
(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(\nabla u(x)) \, dx : u \in u_0 + W_{0,1}^{1,\infty}(\Omega; \mathbb{R}^N) \right\},
\]
where \( u_0 \) is an affine map such that \( \nabla u_0 = \xi_0 \in \mathbb{R}^{N \times n} \), but without assuming any convexity or quasiconvexity hypothesis on the integrand \( f \). We could also treat integrands depending on lower order terms as well as boundary data that are not affine, but then only very few general results can be given and, moreover, they are often restricted to the scalar case.

Clearly, if the integrand \( f \) were quasiconvex, because of the special form of the boundary datum, we would trivially have that \( u_0 \) is a minimizer of \((P)\).

From the relaxation theorem, the following theorem easily follows (see Theorem 11.1).

**Theorem 1.20** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set, \( f \) a non-negative, locally bounded and lower semicontinuous function and \( u_0 \) be as above, in particular \( \nabla u_0 = \xi_0 \). The problem \((P)\) has a solution if and only if there exists \( \pi \in u_0 + W_{0,1}^{1,\infty}(\Omega; \mathbb{R}^N) \) such that
\[ f(\nabla \pi(x)) = Qf(\nabla \pi(x)) \text{ a.e. } x \in \Omega, \]
\[ \int_{\Omega} Qf(\nabla \pi(x)) \, dx = Qf(\xi_0) \text{ meas } \Omega. \]

We do not continue here with general necessary and sufficient conditions for the existence of minimizers for \((P)\), but we rather give several examples.

We start with the very elementary case where \( n = N = 1 \) (see Theorem 11.24). The result adapts in a straightforward manner to the case \( N > n = 1 \).
Theorem 1.21 Let \( f : \mathbb{R} \to \mathbb{R} \) be non-negative, locally bounded and lower semicontinuous. Let \( a < b, \alpha, \beta \in \mathbb{R} \) and

\[
\inf \left\{ I(u) = \int_a^b f(u'(x)) \, dx : u \in X \right\},
\]

where

\[
X = \{ u \in W^{1,\infty}((a,b): u(a) = \alpha, u(b) = \beta \}.
\]

The following two statements are then equivalent.

(i) Problem \((P)\) has a minimizer.

(ii) There exist \(0 \leq \lambda \leq 1\) and \(\gamma, \delta \in \mathbb{R}\) such that

\[
Cf\left(\frac{\beta - \alpha}{b - a}\right) = \lambda f(\gamma) + (1 - \lambda) f(\delta) \quad \text{and} \quad \frac{\beta - \alpha}{b - a} = \lambda \gamma + (1 - \lambda) \delta,
\]

where \(Cf = \sup \{ g \leq f : g \text{ convex} \} \).

Furthermore, if (1.3) is satisfied, then

\[
\pi(x) = \begin{cases} 
\gamma(x - a) + \alpha & \text{if } x \in [a, a + \lambda (b - a)] \\
\delta(x - a) + \lambda (\gamma - \delta)(b - a) + \alpha & \text{if } x \in (a + \lambda (b - a), b]
\end{cases}
\]

is a minimizer of \((P)\).

Note that by Carathéodory theorem we always have

\[
Cf\left(\frac{\beta - \alpha}{b - a}\right) = \inf \left\{ \lambda f(\gamma) + (1 - \lambda) f(\delta) : \frac{\beta - \alpha}{b - a} = \lambda \gamma + (1 - \lambda) \delta \right\}. \quad (1.4)
\]

Therefore (1.3) states that a necessary and sufficient condition for the existence of solutions is that the infimum in (1.4) be attained. Note also that if \(f\) is convex or coercive (in the sense that \(f(\xi) \geq a |\xi|^p + b\) with \(p > 1, a > 0\)), then the infimum in (1.4) is always attained. Hence, if \(f(x, u, \xi) = f(\xi)\), counterexamples to existence must be non-convex and non-coercive, as in the example already considered in Section 1.2.3, where \(f(\xi) = e^{-\xi^2}\).

Of course, if \(f\) depends explicitly on \(u\), the example of Bolza (given in Section 1.2.3) shows that the theorem is then false.

We now give three examples in the vectorial case.

(1) The first one (see Theorem 11.32) deals with the minimization problem

\[
\inf \left\{ \int_{\Omega} g(\Phi(\nabla u(x))) \, dx : u \in u_0 + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \right\},
\]

where:

- \(g : \mathbb{R} \to \mathbb{R}\) is a lower semicontinuous, locally bounded and non-negative function,
- \(\Phi : \mathbb{R}^{N \times n} \to \mathbb{R}\) is quasiaffine and non-constant (in particular, we can have, when \(N = n\), \(\Phi(\xi) = \det \xi\)).
The relaxed problem is then (see Section 1.3.2)

\[
(P) \quad \inf \left\{ \int_\Omega Cg(\Phi(\nabla u(x))) \, dx : u \in u_0 + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \right\},
\]
where \( Cg \) is the convex envelope of \( g \). The existence result is the following.

**Theorem 1.22** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set, \( g : \mathbb{R} \to \mathbb{R} \) as above and satisfying

\[
\lim_{|t| \to +\infty} \frac{g(t)}{|t|} = +\infty
\]

and \( u_0(x) = \xi_0 x \) with \( \xi_0 \in \mathbb{R}^{N \times n} \). Then there exists \( \bar{u} \in u_0 + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \) a minimizer of \((P)\).

(2) The second example deals with integrands of area type (see Section 1.3.2), where \( N = n + 1 \) and

\[
f(\xi) = g(\text{adj}_n \xi).
\]

The minimization problem is then

\[
(P) \quad \inf \left\{ \int_\Omega g(\text{adj}_n(\nabla u(x))) \, dx : u \in u_0 + W^{1,\infty}_0(\Omega; \mathbb{R}^{n+1}) \right\},
\]
where \( \Omega \) is a bounded open set of \( \mathbb{R}^n \), \( \nabla u_0 = \xi_0 \) and \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) is a non-negative, lower semicontinuous and locally bounded non-convex function.

From Section 1.3.2, we have

\[
Qf(\xi) = Cg(\text{adj}_n \xi).
\]

We next set

\[
S = \{ y \in \mathbb{R}^{n+1} : Cg(y) < g(y) \}
\]

and assume, in order to avoid the trivial situation, that \( \text{adj}_n \xi_0 \in S \).

The existence result (see Theorem 11.36) is then given by the following.

**Theorem 1.23** If \( S \) is bounded, \( Cg \) is affine in \( S \) and rank \( \xi_0 \geq n - 1 \), then \((P)\) has a solution.

(3) The third problem is that of optimal design, already discussed in Section 1.3.2, where

\[
f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases}
\]

Consider the problem

\[
(P) \quad \inf \left\{ \int_\Omega f(\nabla u(x)) \, dx : u \in u_0 + W^{1,\infty}_0(\Omega; \mathbb{R}^2) \right\},
\]
where \( \Omega \) is a bounded open set of \( \mathbb{R}^2 \) and \( \nabla u_0 = \xi_0 \).

We then have the following (see Theorem 11.35).
Theorem 1.24 Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, $f : \mathbb{R}^{2 \times 2} \to \mathbb{R}$ be as above and $\xi_0 \in \mathbb{R}^{2 \times 2}$. Then a necessary and sufficient condition for (P) to have a solution is that one of the following conditions hold:

(i) $\xi_0 = 0$ or $|\xi_0|^2 + 2|\det \xi_0| \geq 1$ (i.e. $f(\xi_0) = Qf(\xi_0)$)
(ii) $\det \xi_0 \neq 0$.

1.5 Miscellaneous

In Part IV, we gather some notations and standard results on function spaces and on singular values. We also devote the last two chapters to results that play only a marginal role in our analysis, but have some interest on their own.

1.5.1 Hölder and Sobolev spaces

In Chapter 12, we only fix the notation concerning the main function spaces that we use, namely the Hölder spaces $C^{m,\alpha} (\Omega; \mathbb{R}^N)$ and the Sobolev spaces $W^{m,p} (\Omega; \mathbb{R}^N)$, where $m$ is an integer, $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$. We recall without proofs the most important results, for example the Sobolev imbedding theorem, that we use throughout the book.

1.5.2 Singular values

We recall in Chapter 13 the definition and some elementary properties of the singular values of a matrix $\xi \in \mathbb{R}^{n \times n}$ (in the present introduction, we discuss only the case $N = n$, but in Chapter 13 we consider general matrices in $\mathbb{R}^{N \times n}$). We denote by

$$0 \leq \lambda_1 (\xi) \leq \cdots \leq \lambda_n (\xi)$$

the eigenvalues of $(\xi \xi^t)^{1/2}$. The main result (see Theorem 13.3) is the following.

Theorem 1.25 Let $\xi \in \mathbb{R}^{n \times n}$ and $0 \leq \lambda_1 (\xi) \leq \cdots \leq \lambda_n (\xi)$ be its singular values. Then there exist $R, Q \in O(n)$ (the set of orthogonal matrices $R \in \mathbb{R}^{n \times n}$ so that $R^t R = I$) such that

$$R \xi Q = \text{diag} (\lambda_1 (\xi), \cdots, \lambda_n (\xi)) := \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}.$$ 

In some applications, it might be better to replace the singular values $\lambda_1 (\xi) \leq \cdots \leq \lambda_n (\xi)$ of a given matrix $\xi \in \mathbb{R}^{n \times n}$ by its signed singular values

$$0 \leq |\mu_1 (\xi)| \leq \cdots \leq \mu_n (\xi)$$
defined by
\[ \mu_1(\xi) = \lambda_1(\xi) \text{sign}(\det \xi) \quad \text{and} \quad \mu_j(\xi) = \lambda_j(\xi), \quad j = 2, \ldots, n. \]

We then have the following inequality (see Theorem 13.10).

**Theorem 1.26** Let \( \xi, \eta \in \mathbb{R}^{n \times n} \). Then
\[
\max_{Q, R \in \text{SO}(n)} \{\text{trace}(Q \xi R^t \eta^t)\} = \sum_{j=1}^{n} \mu_j(\xi) \mu_j(\eta)
\]
and consequently
\[
\text{trace}(\xi \eta^t) \leq \sum_{j=1}^{n} \mu_j(\xi) \mu_j(\eta).
\]

**1.5.3 Some underdetermined partial differential equations**

In Chapter 14, we prove the existence of solutions for three types of underdetermined partial differential equations that are encountered in mechanics. All three problems bear in common that, in general, there are infinitely many solutions. The first one concerns the divergence operator (see Theorem 14.2) as stated in the next theorem.

**Theorem 1.27** Let \( m \geq 0 \) be an integer and \( 0 < \alpha < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a smooth bounded connected open set. The following conditions are then equivalent.

(i) \( f \in C^{m,\alpha}(\overline{\Omega}) \) satisfies
\[
\int_{\Omega} f(x) \, dx = 0.
\]

(ii) There exists \( u \in C^{m+1,\alpha}(\overline{\Omega}; \mathbb{R}^n) \) verifying
\[
\begin{cases}
\div u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
where \( \div u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} \).

The second problem is related to the curl operator (see Theorem 14.4).

**Theorem 1.28** Let \( m \geq 1 \) be an integer and \( 0 < \alpha < 1 \). Let \( \Omega \subset \mathbb{R}^3 \) be a smooth bounded convex set and \( \nu \) denote the outward unit normal. The following conditions are then equivalent.

(i) \( f \in C^{m,\alpha}(\overline{\Omega}; \mathbb{R}^3) \) verifies
\[
\div f = 0 \text{ in } \Omega \text{ and } \langle f; \nu \rangle = 0 \text{ on } \partial \Omega.
\]
(ii) There exists \( u \in C^{m+1,\alpha}(\overline{\Omega};\mathbb{R}^3) \) satisfying
\[
\begin{aligned}
\text{curl } u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]
where if \( u = (u^1, u^2, u^3) \), then
\[
\text{curl } u = \left( \frac{\partial u^3}{\partial x_2} - \frac{\partial u^2}{\partial x_3}, \frac{\partial u^1}{\partial x_3} - \frac{\partial u^3}{\partial x_1}, \frac{\partial u^2}{\partial x_1} - \frac{\partial u^1}{\partial x_2} \right).
\]

Both cases are examples of a study of Dirichlet problems associated to \( du = f \), where \( u \) is a \( k \) form and \( d \) is the exterior derivative.

The next theorem (see Theorem 14.6) is the nonlinear version of Theorem 1.27. In terms of fluid mechanics, the first one is in Eulerian coordinates, while the second one is in Lagrangian coordinates.

For \( m \geq 1 \) an integer, \( 0 < \alpha < 1 \) and \( \Omega \subset \mathbb{R}^n \) a bounded open set with a sufficiently regular boundary, we denote by \( \text{Diff}^{m,\alpha}(\overline{\Omega}) \) the set of diffeomorphisms \( u : \overline{\Omega} \rightarrow \overline{\Omega} \) such that \( u, u^{-1} \in C^{m,\alpha}(\overline{\Omega};\mathbb{R}^n) \).

**Theorem 1.29** Let \( m \geq 0 \) be an integer and \( 0 < \alpha < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded connected open set with a sufficiently smooth boundary. Let \( f \in C^{m,\alpha}(\overline{\Omega}) \), \( f > 0 \) in \( \Omega \), and
\[
\int_{\Omega} f(x) \, dx = \text{meas } \Omega.
\]
Then there exists \( u \in \text{Diff}^{m+1,\alpha}(\overline{\Omega}) \) satisfying
\[
\begin{aligned}
det \nabla u(x) &= f(x) \quad x \in \Omega \\
u(x) &= x \quad x \in \partial \Omega.
\end{aligned}
\]

The theorem can be applied in a straightforward manner to the minimization problem (see Corollary 14.9)
\[
(P) \quad \inf \left\{ I(u) = \int_{\Omega} g(\det \nabla u(x)) \, dx : u \in u_0 + W_0^{1,\infty}(\Omega;\mathbb{R}^n) \right\},
\]
where \( g : \mathbb{R} \rightarrow \mathbb{R} \) is convex and \( \Omega \subset \mathbb{R}^n \) and \( u_0 \) satisfy appropriate smoothness conditions.

### 1.5.4 Extension of Lipschitz maps

In Chapter 15, we consider the problem of extending a Lipschitz map defined on a set to the full space so as to preserve the Lipschitz constant. This is a classical problem and is of particular importance in the calculus of variations, where it is known as MacShane lemma in the scalar case and Kirszbraun theorem in the vectorial case.
Let us describe the problem more precisely. We consider two Banach spaces $(E, \| \cdot \|_E)$ and $(F, \| \cdot \|_F)$. We ask when a map $u : D \subset E \to F$ satisfying
\[ \| u(x) - u(y) \|_F \leq \| x - y \|_E, \quad x, y \in D, \]
can be extended to the whole of $E$ so as to preserve the inequality.

We need the following three definitions.

**Definition 1.30** (i) We say that $u : E \to F$ is a contraction on $D$ or $u$ is 1–Lipschitz on $D$ if
\[ \| u(x) - u(y) \|_F \leq \| x - y \|_E \quad \text{for all } x, y \in D. \]

In this case, we write that $u \in \text{Lip}_1(D, F)$.

(ii) When $u \in \text{Lip}_1(E, F)$, we simply say that $u$ is a contraction.

**Definition 1.31** (i) We say that $[E; F]$ has the extension property for contractions on $D$ if every $u \in \text{Lip}_1(D, F)$ has an extension $\tilde{u} \in \text{Lip}_1(E, F)$.

(ii) If $[E; F]$ has the extension property for contractions for every $D \subset E$, we simply say that $[E; F]$ has the extension property for contractions.

**Definition 1.32** The unit sphere $S_F$ (i.e. the set of $x \in F$ such that $\| x \|_F = 1$) is said to be strictly convex if it has no flat part, meaning that
\[ \| (1 - t)x + ty \|_F < (1 - t) \| x \|_F + t \| y \|_F = 1 \]
for all $t \in (0, 1)$ and all $x, y \in S_F$ such that $x \neq y$.

A particularly interesting example is the Hölder norms $|x|_p$ over $\mathbb{R}^n$, $1 \leq p \leq \infty$, and they are defined as
\[ |x|_p := \begin{cases} \left[ \sum_{i=1}^{n} |x_i|^p \right]^{1/p} & \text{if } 1 \leq p < \infty \\ \max_{1 \leq i \leq n} \{|x_i|\} & \text{if } p = \infty. \end{cases} \]
When $n \geq 2$, the unit sphere for $| \cdot |_p$ is strictly convex if and only if $1 < p < \infty$.

We can now state our main theorems (see Theorems 15.11 and 15.12). Part (i) of Theorem 1.33 is known as MacShane lemma and the implication (i) ⇒ (ii) in Theorem 1.34 is known as Kirszbraun theorem.

**Theorem 1.33** (i) Let $(E, \| \cdot \|_E)$ be a normed space. Then $[E; \mathbb{R}]$ has the extension property for contractions.

(ii) Let $(F, \| \cdot \|_F)$ be a Banach space. Then $[\mathbb{R}; F]$ has the extension property for contractions.

We now turn our attention to the case where both $E$ and $F$ have dimension at least 2 and we give a theorem that characterizes the Banach spaces for which $[E, F]$ has the extension property for contractions.
Theorem 1.34 Assume that \((E, \| \cdot \|_E)\) and \((F, \| \cdot \|_F)\) are Banach spaces such that \(\dim E, \dim F \geq 2\) and that the unit sphere in \(F\) is strictly convex. Assume also that every closed set \(D \subset E\) contains a countable set \(D_c \subset D\) whose closure is \(D\). Then, the following three properties are equivalent:

(i) \(\| \cdot \|_E\) and \(\| \cdot \|_F\) are induced by an inner product;
(ii) \([E; F]\) has the extension property for contractions;
(iii) for every \(\bar{x} \in E\) and every \(S := \{x_1, x_2, x_3\} \subset E\), every \(u \in \text{Lip}_1(S, F)\) has an extension \(\tilde{u} \in \text{Lip}_1(S \cup \{\bar{x}\}, F)\).

Several comments are in order.

(i) If \(S\) consists of only two points \(x, y \in E, x \neq y\), then the extension to any third point is always possible.

(ii) If one drops the assumption that \(S^F\) is strictly convex, \([E; F]\) may have the extension property for contractions even if none of the norms is induced by an inner product; for example, if \(F = \mathbb{R}^N, N \geq 2, \) and \(\| \cdot \|_F = | \cdot |_{\infty}\), then \([E; F]\) has the extension property for any \(E\).

(iii) In the case of Hölder norms with \((E = \mathbb{R}^n, | \cdot |_p)\) and \((F = \mathbb{R}^N, | \cdot |_q)\) with \(n, N \geq 2, 1 < q < \infty\) and \(1 \leq p \leq \infty\), then \([E; F]\) has the extension property if and only if \(p = q = 2\).
Chapter 2

Convex sets and convex functions

2.1 Introduction

We now give a brief introduction to convex analysis. The chapter is divided into two sections.

In Section 2.2, we give some of the most important theorems, namely the separation theorems (sometimes also called Hahn-Banach theorem which is their infinite dimensional version), Carathéodory theorem and Minkowski theorem, also usually better known as Krein-Milman theorem, which is its infinite dimensional version.

In Section 2.3, we list some properties of convex functions such as Jensen inequality, the continuity of such functions, the notion of duality and of subdifferential.

The reference book on convex analysis is Rockafellar [514]. However one can also consult Brézis [105], Ekeland-Temam [264], Fenchel [277], Hiriart Urruty-Lemaréchal [342], Hörmander [344], Ioffe-Tihomirov [351], Moreau [452] or Webster [597] for further references.

We adopt throughout this chapter the following notations.

- $F$ is a set $E \subset \mathbb{R}^N$, $\overline{E}$, $\partial E$, $\text{int} E$ and $E^c$ respectively stand for the closure, the boundary, the interior and the complement of $E$ respectively.

- $\langle ; , \rangle$ stands for the scalar product in $\mathbb{R}^N$ and, unless explicitly specified, $|.|$ denotes the Euclidean norm in $\mathbb{R}^N$.

- The ball centered at $x \in \mathbb{R}^N$ and of radius $r > 0$ is denoted by

$$B_r(x) := \left\{ y \in \mathbb{R}^N : |y - x| < r \right\}.$$
2.2 Convex sets

2.2.1 Basic definitions and properties

We recall the following definition.

**Definition 2.1** (i) A set \( E \subset \mathbb{R}^N \) is said to be *convex* if for every \( x, y \in E \) and every \( t \in [0, 1] \)

\[
tx + (1 - t)y \in E.
\]

(ii) A set \( E \subset \mathbb{R}^N \) is said to be *affine* if for every \( x, y \in E \) and every \( t \in \mathbb{R} \)

\[
tx + (1 - t)y \in E.
\]

(iii) The *affine hull* of a set \( E \subset \mathbb{R}^N \) is the smallest affine set containing \( E \). It is denoted by \( \text{aff} \, E \).

(iv) A hyperplane \( H \subset \mathbb{R}^N \) is a set of the form

\[
H = \{ x \in \mathbb{R}^N : \langle x; a \rangle = \alpha \}
\]

where \( a \in \mathbb{R}^N, a \neq 0, \) and \( \alpha \in \mathbb{R} \).

The next proposition is elementary.

**Proposition 2.2** (i) The intersection of an arbitrary collection of convex sets is convex.

(ii) The intersection of an arbitrary collection of affine sets is affine.

Important concepts in convex analysis are the notions of relative interior and relative boundary.

**Definition 2.3** Let \( E \subset \mathbb{R}^N \) be convex.

(i) The relative interior of \( E \), denoted by \( \text{ri} \, E \), is the interior of the set relative to its affine hull \( \text{aff} \, E \).

(ii) The relative boundary of \( E \), denoted by \( \text{rbd} \, E \), is the set of points in \( E \) but not in \( \text{ri} \, E \).

The following proposition is easily proved.

**Proposition 2.4** Let \( E \subset \mathbb{R}^N \) be convex. Then \( \overline{E} \), \( \text{ri} \, E \) and \( \text{int} \, E \) are convex. Moreover \( \text{int} \, E \) is empty if and only if \( E \) is contained in a hyperplane.

In a straightforward manner, we also deduce the next result (see Corollary 1.4.1 in Rockafellar [514]).

**Proposition 2.5** Every affine subset of \( \mathbb{R}^N \) is the intersection of a finite collection of hyperplanes, where, by convention, the intersection of the empty family is equal to \( \mathbb{R}^N \).
Finally we have the following relations between the interior and closures of convex sets.

**Theorem 2.6** Let $E \subset \mathbb{R}^N$ be convex.

(i) $\text{int } E = \text{int } \overline{E}$.

(ii) If $\text{int } E \neq \emptyset$, then $\text{int } \overline{E} = \overline{E}$.

(iii) $\partial E = \partial \overline{E}$.

**Remark 2.7** The results in the theorem remain valid if we replace the interior by the relative interior and the boundary by the relative boundary (see Theorem 6.3 in Rockafellar [514]).

**Proof.** We divide the proof into five steps.

**Step 1.** We first show that if $x \in \text{int } E$ and $y \in E$, then

$$z := \lambda x + (1 - \lambda) y \in \text{int } E, \text{ for every } \lambda \in (0, 1].$$

(2.1)

Since $x \in \text{int } E$, we can find $\epsilon > 0$ such that

$$B_\epsilon (x) := \{ b \in \mathbb{R}^N : |b - x| < \epsilon \} \subset E.$$  

(2.2)

To prove (2.1) we show that

$$B_{\lambda \epsilon} (z) \subset E, \text{ for every } \lambda \in (0, 1].$$

(2.3)

So we choose $a \in B_{\lambda \epsilon} (z)$ and we let

$$b := x + \frac{1}{\lambda} (a - z) = \frac{1}{\lambda} a + (1 - \frac{1}{\lambda}) y.$$  

(2.4)

Since $a \in B_{\lambda \epsilon} (z)$, we find that $b \in B_\epsilon (x)$ and from (2.2) we deduce that $b \in E$. Consequently from (2.4) we obtain that

$$a = \lambda b + (1 - \lambda) y.$$  

Since $b, y \in E$, we have that $a \in E$ and hence (2.3) is satisfied. This achieves the proof of Step 1.

**Step 2.** Let us now show that, in fact, the result of Step 1 holds even if $y \in \overline{E}$. If $\lambda = 1$, nothing is to be proved; so assume that $\lambda \in (0, 1)$ and $\epsilon > 0$ is so that $B_\epsilon (x) \subset \text{int } E$. We set

$$z := \lambda x + (1 - \lambda) y.$$  

Let us prove that $z \in \text{int } E$. Since $y \in \overline{E}$, we can find $\tilde{y} \in E$ so that

$$|y - \tilde{y}| < \frac{\lambda \epsilon}{1 - \lambda}.$$
Then set
\[ \tilde{x} := \frac{1}{\lambda} [z - (1 - \lambda) \tilde{y}] = x + \frac{1 - \lambda}{\lambda} (y - \tilde{y}). \tag{2.5} \]
We therefore have \( \tilde{x} \in B_\epsilon (x) \subset E \) and hence \( \tilde{x} \in \text{int} E \). From (2.5) we deduce that
\[ z = \lambda \tilde{x} + (1 - \lambda) \tilde{y} \]
and hence we apply Step 1 to get the claim, namely \( z \in \text{int} E \).

\textbf{Step 3.} We now prove the first claim: \( \text{int} E = \text{int} \overline{E} \). We have to consider two cases.

Case 1. If \( \text{int} E = \emptyset \), then from Proposition 2.4 we find that \( E \) is contained in a hyperplane and thus \( \overline{E} \) is contained in a hyperplane and hence \( \text{int} \overline{E} = \emptyset \).

Case 2. Consider now the case where \( \text{int} E \neq \emptyset \). We then show that \( \text{int} \overline{E} \subset \text{int} E \), the reverse inclusion being obvious. Let \( x \in \text{int} \overline{E} \); we want to prove that \( x \in \text{int} E \).

We choose \( z \in \text{int} E \) with \( z \neq x \) (if \( z = x \), then the claim is established). Since \( x \in \text{int} \overline{E} \) and \( z \in \text{int} E \subset \text{int} \overline{E} \), we can find \( \mu > 1 \) with \( \mu - 1 \) sufficiently small so that
\[ y = \mu x + (1 - \mu) z = x + (1 - \mu) (z - x) \in \text{int} \overline{E} \subset \overline{E}. \]
We therefore get that
\[ x = \frac{1}{\mu} y + (1 - \frac{1}{\mu}) z \]
with \( y \in \overline{E} \) and \( z \in \text{int} E \). Applying Step 2, we have the claim, namely \( x \in \text{int} E \).

\textbf{Step 4.} Let us prove that \( \overline{E} \subset \text{int} \overline{E} \), the reverse inclusion being trivial. So let \( x \in \overline{E} \) and \( y \in \text{int} E \) (\( \text{int} E \) is assumed to be non empty). From Step 2 we have that
\[ x_\lambda := \lambda y + (1 - \lambda) x \in \text{int} E, \quad \text{for every } \lambda \in (0, 1]. \]
Since \( x_\lambda \to x \) as \( \lambda \to 0 \), we have indeed proved that \( x \in \text{int} \overline{E} \).

\textbf{Step 5.} We now show the last claim of the theorem. This follows at once from Step 3, since
\[ \partial \overline{E} = \overline{E} - \text{int} \overline{E} = \overline{E} - \text{int} E = \partial E. \]

\[ \blacksquare \]

\subsection*{2.2.2 Separation theorems}

In this section, we present different separation theorems, that in the infinite dimensional case are called Hahn-Banach theorem. We recall that given two sets \( E, F \subset \mathbb{R}^N \), we let
\[ E + F := \{ x \in \mathbb{R}^N : x = y + z, \ y \in E \text{ and } z \in F \}. \]
We now have some definitions.

**Definition 2.8** (i) A hyperplane $H$, defined by $\langle x; a \rangle = \alpha$ with $a \in \mathbb{R}^N$, $a \neq 0$, and $\alpha \in \mathbb{R}$, is said to separate the sets $E, F \subset \mathbb{R}^N$ if either

$$\langle x; a \rangle \leq \alpha \leq \langle y; a \rangle$$

for every $x \in E$ and $y \in F$

or the same inequalities hold for every $x \in F$ and $y \in E$.

(ii) The hyperplane is said to separate properly the sets $E, F \subset \mathbb{R}^N$ if they are separated by $H$ and at least one of them is not contained in $H$ itself.

(iii) A hyperplane $H$ is said to separate strictly the sets $E, F \subset \mathbb{R}^N$ if there exists $\epsilon > 0$ such that $H$ separates $E + \epsilon B_1$ and $F + \epsilon B_1$, where $B_1$ is the unit ball of $\mathbb{R}^N$.

Before proving the main theorem of this section we need to define the projection onto a convex set.

**Theorem 2.9** Let $E \subset \mathbb{R}^N$ be a closed convex set, $x \in \mathbb{R}^N$ and denote by $|.|$ the Euclidean norm.

(i) There exists a unique $x_\infty \in E$ minimizing $z \rightarrow |x - z|$ over $E$. Moreover, if $x \notin \text{int} E$, then $x_\infty \in \partial E$. The map

$$x \rightarrow p_E(x) := x_\infty$$

is well defined and is referred to as the projection map onto $E$.

(ii) Furthermore, the following inequalities hold for every $x, y \in \mathbb{R}^N$ and $z \in E$

$$\langle x - p_E(x); z - p_E(x) \rangle \leq 0; \quad (2.6)$$

$$|p_E(x) - p_E(y)|^2 \leq \langle p_E(x) - p_E(y); x - y \rangle; \quad (2.7)$$

$$|p_E(x) - p_E(y)| \leq |x - y|. \quad (2.8)$$

**Proof.** (i) Let $x \in \mathbb{R}^N$ and let $\{z_\nu\}_{\nu=1}^\infty \subset E$ be such

$$\lim_{\nu \rightarrow +\infty} |x - z_\nu| = \inf_{z \in E} |x - z|.$$  

The set $\{z_\nu\}_{\nu=1}^\infty$ being bounded, we can extract a subsequence, which we still label $\{z_\nu\}_{\nu=1}^\infty$, converging to some $x_\infty \in E$ and hence

$$|x - x_\infty| = \lim_{\nu \rightarrow +\infty} |x - z_\nu| = \inf_{z \in E} |x - z|.$$  

Let us now show that if $x \notin \text{int} E$, then $x_\infty \in \partial E$. By contradiction if $x_\infty \in \text{int} E$, we would have for $t \in (0, 1)$ small enough that

$$x_t := (1 - t) x_\infty + tx \in E$$
and thus
\[ |x - x_t| = (1 - t) |x - x_\infty| < |x - x_\infty| \]

contradicting the definition of \( x_\infty \).

We now prove that the minimizer is unique. Assume for the sake of contradiction that for a certain \( x \notin E \) there exist two distinct minimizers \( x_\infty, x_\infty' \in E \) of \( |x - z| \) over \( E \). Since \( x_\infty, x_\infty' \in \partial E \), we find that
\[
x_0 := \frac{x_\infty + x_\infty'}{2} \in E
\]
is another minimizer of \( |x - z| \). It is easily seen that since \( |x - x_\infty| = |x - x_\infty'| > 0 \), then (using the fact that \( \xi \to |\xi|^2 \) is strictly convex) \( |x - x_0| < |x - x_\infty| \), which yields the desired contradiction. This proves that the minimizer of \( |x - z| \) over \( E \) is unique.

(ii) Since for every \( t \in [0,1] \) and \( z \in E \), we have
\[
|x - p_E(x)|^2 \leq g(t) := |x - [(1 - t) p_E(x) + tz]|^2,
\]
we find, from the fact that \( g'(0) \geq 0 \), that \( p_E(x) \) should satisfy (2.6).

If \( x, y \in \mathbb{R}^N \), we use (2.6), once with \( z = p_E(y) \) and once with \( z = p_E(x) \), to obtain that
\[
\langle x - p_E(x); p_E(y) - p_E(x) \rangle \leq 0 \text{ and } \langle y - p_E(y); p_E(x) - p_E(y) \rangle \leq 0.
\]
Adding up these two inequalities yields (2.7), namely
\[
|p_E(x) - p_E(y)|^2 \leq \langle p_E(x) - p_E(y); x - y \rangle.
\]
This, together with Cauchy-Schwarz inequality, leads to (2.8).

We are now in a position to state the theorem.

**Theorem 2.10 (Separation theorems)**  
(i) Let \( E \subset \mathbb{R}^N \) be closed and convex and \( \overline{x} \notin E \). Then there exists \( a \in \mathbb{R}^N, a \neq 0 \), such that
\[
\langle \overline{x}; a \rangle < \inf \{ \langle x; a \rangle : x \in E \}.
\]

(ii) Let \( E \subset \mathbb{R}^N \) be closed and convex and \( \overline{x} \in \partial E \). Then there exists \( a \in \mathbb{R}^N, a \neq 0 \), so that
\[
\langle \overline{x}; a \rangle \leq \langle x; a \rangle \text{ for every } x \in E.
\]

(iii) Let \( E, F \subset \mathbb{R}^N \) be non-empty, disjoint and convex. Let \( E \) be closed and \( F \) compact. Then there exists a hyperplane that separates \( E \) and \( F \) strictly.

(iv) Every closed convex set is the intersection of the closed half spaces that contain it.
Proof. (i) We choose \( a := p_E(\overline{x}) - \overline{x} \) and \( \alpha := \langle \overline{x}; a \rangle \). Observe that since \( \overline{x} \notin E \), then \( a \neq 0 \). We therefore get, for any \( x \in E \), that, using (2.6),
\[
\langle x - \overline{x}; a \rangle = - \langle x - p_E(\overline{x}); \overline{x} - p_E(\overline{x}) \rangle + |p_E(\overline{x}) - \overline{x}|^2 \geq |a|^2
\]
and the claim follows.

(ii) Since \( \overline{x} \in \partial E \), we can find a sequence \( x_\nu \notin E \) with \( x_\nu \to \overline{x} \). Applying the above result we can find \( a_\nu \in \mathbb{R}^N, a_\nu \neq 0 \) (and hence we can assume without loss of generality that \( |a_\nu| = 1 \)), such that
\[
\langle x_\nu; a_\nu \rangle < \inf \{ \langle x; a_\nu \rangle : x \in E \}.
\]
Extracting a subsequence from \( \{a_\nu\} \) we have the result by passing to the limit.

(iii) Define the set
\[
G = E - F := \{ z \in \mathbb{R}^N : z = x - y \text{ with } x \in E \text{ and } y \in F \}.
\]
It is clearly convex and closed, since \( E \) is closed and \( F \) is compact. Moreover since \( E \cap F = \emptyset \), we have that \( 0 \notin G \). We may then apply (i) of the theorem to find \( a \in \mathbb{R}^N, a \neq 0 \), so that
\[
0 < \inf \{ \langle z; a \rangle : z \in G \} = \inf \{ \langle x; a \rangle : x \in E \} - \sup \{ \langle y; a \rangle : y \in F \}
\]
which is the desired statement.

(iv) Let \( E \subset \mathbb{R}^N \) (\( E \neq \emptyset \) and \( E \neq \mathbb{R}^N \)) be closed and convex. For any \( \overline{x} \notin E \) we can find, from (i), \( a \in \mathbb{R}^N, a \neq 0 \), and \( \alpha \in \mathbb{R} \), so that
\[
\langle \overline{x}; a \rangle < \alpha < \inf \{ \langle x; a \rangle : x \in E \}.
\]
Therefore the closed half space
\[
H = \{ x \in \mathbb{R}^N : \langle x; a \rangle \geq \alpha \}
\]
contains \( E \) but does not contain \( \overline{x} \). Therefore the intersection of the closed half spaces containing \( E \) does not contain any other point. \( \blacksquare \)

We next show that in the statement of (ii) of Theorem 2.10 we can remove the assumption on the closedness of \( E \).

Corollary 2.11 Let \( E \subset \mathbb{R}^N \) be convex and \( \overline{x} \in \partial E \). Then there exists \( a \in \mathbb{R}^N, a \neq 0 \), such that
\[
\langle \overline{x}; a \rangle \leq \langle x; a \rangle, \text{ for every } x \in E.
\]

Proof. From Theorem 2.6, we have that \( \overline{x} \in \partial E \). Therefore applying Theorem 2.10 (ii) to \( E \), we have the claim. \( \blacksquare \)
2.2.3 Convex hull and Carathéodory theorem

We start with the following definition.

**Definition 2.12** The **convex hull** of a set $E \subset \mathbb{R}^N$, denoted by $\text{co } E$, is the smallest convex set containing $E$.

According to Proposition 2.2, it is equivalent to say that $\text{co } E$ is the intersection of all the convex sets that contain $E$. In the sequel we denote for any integer $s$

\[
\Lambda_s := \{ \lambda = (\lambda_1, \cdots, \lambda_s) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^s \lambda_i = 1 \}.
\]

One of the most important characterizations of the convex hull is Carathéodory theorem.

**Theorem 2.13 (Carathéodory theorem)** Let $E \subset \mathbb{R}^N$. Then

\[
\text{co } E = \left\{ x \in \mathbb{R}^N : x = \sum_{i=1}^{N+1} \lambda_i x_i, \; x_i \in E, \; \lambda \in \Lambda_{N+1} \right\}.
\]

**Proof.** We decompose the proof into two steps.

**Step 1.** Observe first that if $I$ is an integer,

\[
F_I := \left\{ x \in \mathbb{R}^N : x = \sum_{i=1}^I \lambda_i x_i, \; x_i \in E, \; \lambda \in \Lambda_I \right\}
\]

and

\[
F := \bigcup_{I \in \mathbb{N}} F_I
\]

then obviously $F$ is convex and $E \subset F$ and therefore $\text{co } E \subset F$. Conversely let $E \subset A$, $A$ convex, then $F \subset A$ and therefore $F \subset \text{co } E$ and thus $F = \text{co } E$.

**Step 2.** We now show that in fact we have

\[
F = \bigcup_{I=1}^{N+1} F_I.
\]

Let $m \in F = \text{co } E$, then trivially $(1, m) \in \{1\} \times \text{co } E = \text{co } (\{1\} \times E)$. Applying Step 1 to $\text{co } (\{1\} \times E)$ we have that there exist $I$, an integer, $\lambda \in \Lambda_I$, $m_i \in E$ such that

\[
\sum_{i=1}^I \lambda_i (1, m_i) = (1, m).
\]

We wish to show that we can take $I \leq N + 1$ in (2.9). Assume that $I > N + 1$, then there exist $\gamma_i \in \mathbb{R}$ not all zero such that

\[
\sum_{i=1}^I \gamma_i (1, m_i) = 0,
\]
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since \((1, m_i) \in \mathbb{R}^{N+1}\) and \(I > N + 1\).

Let \(T := \{i \in \{1, \cdots, I\} : \gamma_i > 0\}\). We may assume without loss of generality that \(T \neq \emptyset\), otherwise replace \(\gamma_i\) by \(-\gamma_i\). Let

\[
\beta := \min_{i \in T} \{\lambda_i / \gamma_i\} 
\]  
(2.11)

\[
\mu_i := \lambda_i - \beta \gamma_i, \ i = 1, \cdots, I.
\]  
(2.12)

We then deduce that

\[
\mu_i \geq 0, \ i = 1, \cdots, I.
\]  
(2.13)

\[
\sum_{i=1}^{I} \mu_i = 1
\]  
(2.14)

at least one of the \(\mu_i = 0\);  
(2.15)

where (2.13) has been obtained trivially if \(\gamma_i \leq 0\) and by (2.11) and (2.12) if \(\gamma_i > 0\); similarly (2.14) follows from (2.9) and (2.10). Finally (2.15) holds if one takes the index \(i \in T\) which corresponds to the minimum in (2.11). We moreover have

\[
\sum_{i=1}^{I} \lambda_i (1, m_i) = \sum_{i=1}^{I} \mu_i (1, m_i) = (1, m).
\]  
(2.16)

In view of (2.13), (2.14), (2.15) and (2.16), we have therefore reduced the number \(I\) to \((I - 1)\). Continuing this process up to \(I = N + 1\) we have indeed established the theorem. \(\blacksquare\)

An important consequence of Carathéodory theorem is the following.

**Theorem 2.14** Let \(E \subset \mathbb{R}^N\).

(i) If \(E\) is compact, then \(\text{co} \ E\) is compact.

(ii) If \(E\) is open, then \(\text{co} \ E\) is open.

Before proceeding with the proof, let us mention that the analogous statement for closed sets is false.

**Example 2.15** Let \(N = 2\),

\[ E = E_1 \cup E_2, \]

where

\[ E_1 = \{(x_1, 0) : x_1 \in [0, 1]\} \quad \text{and} \quad E_2 = \{(0, x_2) : x_2 \geq 0\}. \]

Clearly \(E\) is closed, while

\[ \text{co} \ E = \{(x_1, x_2) : x_1 \in [0, 1), \ x_2 \geq 0\} \cup \{(1, 0)\} \]

is not closed. \(\diamondsuit\)
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Proof. (i) We need only show that \( \text{co } E \) is closed. So let \( \{x^\nu\} \subset \text{co } E \) be a sequence converging to an element \( x \in \mathbb{R}^N \) and let us show that \( x \in \text{co } E \). Appealing to Theorem 2.13 we can find \( \lambda^\nu \in \Lambda_{N+1} \) and \( x^\nu_i \in E, \ 1 \leq i \leq N+1, \) such that

\[
x^\nu = \sum_{i=1}^{N+1} \lambda^\nu_i x^\nu_i.
\]

Since \( \Lambda_{N+1} \) and \( E \) are compact, we can find subsequences, still denoted \( \{\lambda^\nu\} \) and \( \{x^\nu_i\} \), such that

\[
\lambda^\nu \to \lambda \in \Lambda_{N+1} \quad \text{and} \quad x^\nu_i \to x_i \in E, \ i = 1, \cdots, N+1.
\]

We therefore have

\[
x = \sum_{i=1}^{N+1} \lambda_i x_i
\]

and thus, by Theorem 2.13, \( x \in \text{co } E \), as wished.

(ii) The proof is very similar in spirit to the preceding one. Let \( x \in \text{co } E \) and let us show that we can find \( \epsilon > 0 \) such that

\[
B_\epsilon(x) := \{y \in \mathbb{R}^N : |y - x| < \epsilon\} \subset \text{co } E.
\]

From Theorem 2.13, we can find \( \lambda \in \Lambda_{N+1} \) and \( x_i \in E, \ i = 1, \cdots, N+1, \) such that

\[
x = \sum_{i=1}^{N+1} \lambda_i x_i.
\]

Since \( E \) is open, we can find \( \epsilon > 0 \) so that \( B_\epsilon(x_i) \subset E, \ i = 1, \cdots, N+1. \) Let \( y \in B_\epsilon(x) \) and let us show that \( y \in \text{co } E \). Letting

\[
y_i := x_i + y - x, \ i = 1, \cdots, N+1,
\]

we find that \( y_i \in B_\epsilon(x_i) \subset E, \ i = 1, \cdots, N+1, \) and

\[
y = \sum_{i=1}^{N+1} \lambda_i y_i.
\]

Using again Theorem 2.13, we find that \( y \in \text{co } E \), as claimed. ■

Another direct application is the following corollary (see Lemma 2.11 in [202] and Theorem 20.4 in [514]).

**Corollary 2.16** Let \( E \subset \mathbb{R}^N \) and \( x \in \text{int co } E \). Then there exist

\[
x_1, x_2, \cdots, x_m \in E, \ m \geq N+1,
\]

such that \( \{x_1 - x, x_2 - x, \cdots, x_m - x\} \) spans the whole of \( \mathbb{R}^N \),

\[
x \in \text{int co } \{x_1, x_2, \cdots, x_m\}, \quad (2.17)
\]
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and there exist \( s_i > 0, \ i = 1, 2, \cdots, m \), with \( \sum_{i=1}^{m} s_i = 1 \) such that

\[
x = \sum_{i=1}^{m} s_i x_i.
\]  

(2.18)

**Proof.** Replacing \( E \) by \(-x + E\), we can assume that \( x = 0\). Since \( 0 \in \text{int} \co E \) we can find a cube \( C_\epsilon \) of side \( 2\epsilon \), with \( \epsilon > 0 \) sufficiently small, so that

\[
C_\epsilon = \{ x = (a_1, a_2, \cdots, a_N) \in \mathbb{R}^N : |a_i| < \epsilon, \ i = 1, 2, \cdots, N \} \subset \text{int} \co E.
\]

We denote by \( y_1, \cdots, y_{2^N} \) the vertices of the cube \( C_\epsilon \). Then

\[
0 \in \text{int} \co \{ y_1, \cdots, y_{2^N} \} = C_\epsilon \subset \text{int} \co E.
\]

(2.19)

Note that, since \( \sum_{i=1}^{2^N} y_i = 0 \), then the 0-vector can be expressed by the following convex combination

\[
0 = \sum_{i=1}^{2^N} \frac{1}{2^N} y_i.
\]

(2.20)

We next appeal to Carathéodory theorem (restricting attention only to strictly positive coefficients) to find, for every \( i = 1, 2, \cdots, 2^N \), integers \( N_i \leq N \) so that

\[
y_i = \sum_{k=1}^{N_i+1} t_{i,k} x_{i,k}, \ x_{i,k} \in E, \ t_{i,k} > 0 \quad \text{and} \quad \sum_{k=1}^{N_i+1} t_{i,k} = 1.
\]

Since the corresponding set of vectors of \( \mathbb{R}^N \)

\[
\{ x_{i,k}, \ k = 1, \cdots, N_i + 1, \ i = 1, \cdots, 2^N \}
\]

generate by convex combinations at least the whole cube \( C_\epsilon \), then the span of the set in (2.21) is \( \mathbb{R}^N \). By (2.20) we obtain

\[
0 = \sum_{i=1}^{2^N} \frac{1}{2^N} y_i = \sum_{i=1}^{2^N} \sum_{k=1}^{N_i+1} \frac{1}{2^N} t_{i,k} x_{i,k} = \sum_{i,k} s_{i,k} x_{i,k},
\]

where

\[
s_{i,k} = \frac{t_{i,k}}{2^N} > 0, \ \forall \ i, k, \quad \text{and} \quad \sum_{i,k} s_{i,k} = 1,
\]

which proves (2.18). By combining (2.19) with

\[
\co \{ y_1, y_2, \cdots, y_{2^N} \} \subset \co \{ x_{i,k}, \ k = 1, \cdots, N_i + 1, \ i = 1, \cdots, 2^N \}
\]

we also have (2.17).

It remains to prove that \( m \geq N + 1 \). From (2.18) namely

\[
0 = \sum_{i=1}^{m} s_i x_i
\]

it follows that \( \{ x_1, x_2, \cdots, x_m \} \) are linearly dependent and since it spans the whole of \( \mathbb{R}^N \), we deduce that \( m \geq N + 1 \). This achieves the proof of the corollary.■
2.2.4 Extreme points and Minkowski theorem

**Definition 2.17** Let $E \subset \mathbb{R}^N$ be convex. We say that $x \in E$ is an extreme point of $E$ if

$$x = ta + (1 - t)b \quad \text{for} \quad 0 < t < 1, \quad a, b \in E \Rightarrow a = b = x.$$ 

We denote the set of extreme points of $E$ by $E_{ext}$.

Note that the set of extreme points may be empty. This is indeed the case, for example, when $E$ is an open convex set.

In dimension $N = 2$, the set of extreme points of a closed convex set is closed (see Exercise 2.6 in Webster [597]); however in higher dimensions this is not so, as the following classical example shows.

**Example 2.18** Let $E, E_1, E_2 \subset \mathbb{R}^3$ be defined by

$$E_1 = \left\{ x = (x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1 \right\},$$

$$E_2 = \left\{ x = (1, 0, x_3) \in \mathbb{R}^3 : |x_3| \leq 1 \right\},$$

$$E = \text{co}(E_1 \cup E_2).$$

It is easy to see that

$$E_{ext} = (\partial E_1 \setminus \{(1,0,0)\}) \cup \{(1,0,1)\} \cup \{(1,0,-1)\}$$

and hence it is not closed since $(1,0,0)$ is not an extreme point.

The important fact about extreme points is given in the theorem below. This result is often known as Krein-Milman theorem, which is the infinite dimensional version of the result due to Minkowski. Before proving this theorem, we give a proposition, whose proof is straightforward.

**Proposition 2.19** Let $E \subset \mathbb{R}^N$.

(i) Let $E$ be convex. Then

$$e \in E_{ext} \Leftrightarrow E - \{e\} \text{ is convex}. $$

(ii) Let $K = \text{co} E$. Then $K_{ext} \subset E$.

We now have the main result.

**Theorem 2.20 (Minkowski theorem)** Let $E \subset \mathbb{R}^N$ be compact and let $E_{ext}$ denote the set of extreme points of $\text{co} E$. Then

$$\text{co} E = \text{co} E_{ext}.$$
Proof. The inclusion $co \ E_{ext} \subset co \ E$, follows at once from the above proposition. To prove the reverse inclusion we proceed by induction on the dimension $N$. If $N = 1$, the result is trivial since $co \ E$ is a closed interval and its extreme points are the end points of the interval.

So we now assume that the result has been established for any $(N - 1)$ dimensional space. Moreover we can assume that $int \ co \ E$ is non-empty otherwise the hypothesis of induction applies.

In view of Theorem 2.13, it is enough to show that any $e \in co \ E$ can be written as

$$ e = \sum_{i=1}^{N+1} \lambda_i x_i, \ x_i \in E_{ext}, \ \lambda \in \Lambda_{N+1}. $$

We consider two cases.

Case 1. We first assume that $e \in \partial (co \ E)$. Since $co \ E$ is closed by Theorem 2.14, we can apply Theorem 2.10 (ii) to get $a \in \mathbb{R}^N, \ a \neq 0, \ \text{and} \ \alpha \in \mathbb{R}$ so that

$$ \langle e; a \rangle = \alpha \leq \langle x; a \rangle, \ \text{for every} \ x \in co \ E. $$

Next define

$$ K := \{ x \in co \ E : \langle x; a \rangle = \alpha \} $$

and observe that it is a non empty (since $e \in K$) convex and compact set that lies in a subspace of dimension $(N - 1)$. We may therefore apply the induction hypothesis and write

$$ e = \sum_{i=1}^{N} \lambda_i x_i, \ x_i \in K_{ext}, \ \lambda \in \Lambda_N. $$

Since, obviously, $K_{ext} \subset E_{ext}$, we have the claim.

Case 2. We then consider the case where $e \in int \ co \ E$. Note first that by Case 1 we have that the set of extreme points is non empty. So choose any $x_{N+1} \in E_{ext}$ and consider the line containing the segment $[e, x_{N+1}]$. Since $co \ E$ is compact, we can find $y \in \partial (co \ E), \ y \neq e$, so that $e \in (x_{N+1}, y)$ and hence we can find $\mu \in (0, 1)$ so that

$$ e = \mu x_{N+1} + (1 - \mu) y. $$

Applying Case 1 to $y$ we find that

$$ y = \sum_{i=1}^{N} \nu_i x_i, \ x_i \in E_{ext}, \ \nu \in \Lambda_N. $$

Writing $\lambda_i = (1 - \mu) \nu_i, \ i = 1, \cdots, N$ and $\lambda_{N+1} = \mu$ we have indeed obtained the claim, namely

$$ e = \sum_{i=1}^{N+1} \lambda_i x_i, \ x_i \in E_{ext}, \ \lambda \in \Lambda_{N+1}. $$

This concludes the proof of the theorem. ■
2.3 Convex functions

2.3.1 Basic definitions and properties

While dealing with convex functions, it is convenient to allow the functions to take infinite values, so we always consider functions
\[ f : \mathbb{R}^N \to \mathbb{R} \cup \{\pm \infty\}. \]

However, since convex functions taking the value \(-\infty\) are rather special and lead to difficulties of notation, such as \(\infty - \infty\), we mostly consider functions of the form
\[ f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}. \]

But functions taking the value \(-\infty\) arise naturally in the next sections.

We start with the definition of convexity.

**Definition 2.21** (i) A function \( f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \) is said to be convex if
\[ f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \]
for every \( x, y \in \mathbb{R}^N \) and every \( t \in [0, 1] \).

(ii) A function \( f : E \subset \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \) is said to be strictly convex on a convex set \( E \) if
\[ f(tx + (1 - t)y) < tf(x) + (1 - t)f(y) \]
for every \( x, y \in E, x \neq y, \) and every \( t \in (0, 1) \).

**Remark 2.22** (i) We also adopt the convention that
\[ 0. (\pm \infty) = 0. \]

(ii) For functions \( f : \mathbb{R}^N \to \mathbb{R} \cup \{\pm \infty\} \), we adopt the convention that \( f \) is convex if and only if
\[ f(tx + (1 - t)y) < ta + (1 - t)b \]
for every \( x, y \in \mathbb{R}^N \), with \( f(x) < a \) and \( f(y) < b \), and every \( t \in (0, 1) \). But we constantly avoid being in the situation where we have to refer to this definition.

We now give some important examples of convex functions.

**Example 2.23** (i) *Indicator function.* The indicator function of a set \( E \subset \mathbb{R}^N \) is defined by
\[ \chi_E(x) := \begin{cases} 0 & \text{if } x \in E \\ +\infty & \text{if } x \notin E. \end{cases} \]
The function $\chi_E$ is convex if and only if the set $E$ is convex.

(ii) **Support function**. For a given convex set $E \subset \mathbb{R}^N$, we define the support function of $E$ as

$$\chi^*_E(x^*) := \sup_{x \in E} \{\langle x; x^* \rangle \}.$$ 

(iii) **Gauge**. For a given convex set $E \subset \mathbb{R}^N$, we define the gauge of $E$ (see Section 2.3.7) as

$$\rho_E(x) := \inf \{\lambda \geq 0 : x \in \lambda E \}.$$ 

(iv) **Distance function**. Given $E \subset \mathbb{R}^N$, we define

$$d_E(x) := \inf \{|x - e| : e \in E\}.$$ 

It is easily seen that if $E$ is convex, then $d_E$ is convex. If $E$ is closed, then

$$d_E(x) = 0 \iff x \in E.$$ 

We now recall some definitions and notations.

**Definition 2.24** Let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$.

(i) $f$ is said to be lower semicontinuous if

$$\liminf_{x_\nu \to x} f(x_\nu) \geq f(x).$$

(ii) The domain of $f$ is defined as

$$\text{dom } f := \{x \in \mathbb{R}^N : f(x) < +\infty\}.$$ 

(iii) The epigraph of $f$ is defined as

$$\text{epi } f := \{(x, \alpha) \in \mathbb{R}^N \times \mathbb{R} : f(x) \leq \alpha\}.$$ 

(iv) The level set of height $\alpha$, $\alpha \in \mathbb{R}$, of $f$ is defined as

$$\text{level} \alpha f := \{x \in \mathbb{R}^N : f(x) \leq \alpha\}.$$ 

**Example 2.25** For the indicator function of a set $E \subset \mathbb{R}^N$, we have

$$\begin{align*}
\text{dom } \chi_E &= E, \\
\text{epi } \chi_E &= E \times [0, +\infty) = E \times \mathbb{R}_+, \\
\text{level}_\alpha \chi_E &= \begin{cases} \\
\emptyset & \text{if } \alpha < 0 \\
E & \text{if } \alpha \geq 0.
\end{cases}
\end{align*}$$

The proof of the following theorem is easy and we do not discuss the details.
Theorem 2.26 Let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$.

Part 1. The following three conditions are equivalent:
(i) $f$ is lower semicontinuous;
(ii) $\text{epi}f$ is closed;
(iii) $\text{level}_\alpha f$ is closed for every $\alpha \in \mathbb{R}$.

Part 2. The following two conditions are equivalent:
(i) $f$ is convex;
(ii) $\text{epi}f$ is convex.
Furthermore, if $f$ is convex, then $\text{level}_\alpha f$ is convex for every $\alpha \in \mathbb{R}$.

Part 3. Let $f_\nu : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}, \nu \in I$, be an arbitrary family of convex (respectively lower semicontinuous) functions. Then

$$f = \sup_{\nu \in I} f_\nu$$

is a convex (respectively lower semicontinuous) function.

Remark 2.27 Note that in general the convexity of $\text{level}_\alpha f$ for every $\alpha \in \mathbb{R}$ does not imply the convexity of $f$, as the following example indicates. Let

$$f(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
1 & \text{if } x > 0
\end{cases}$$

then

$$\text{level}_\alpha f = \begin{cases} 
\emptyset & \text{if } \alpha < 0 \\
(-\infty, 0] & \text{if } 0 \leq \alpha < 1 \\
\mathbb{R} & \text{if } \alpha \geq 1
\end{cases}$$

is convex for every $\alpha \in \mathbb{R}$, while $f$ is not convex. In the context of optimization, functions whose level sets are convex are sometimes called quasiconvex; we will not be concerned with such functions and when we will use the notion of quasiconvexity in Part II it will have a different meaning.

We close this very brief introduction by recalling Jensen inequality.

Theorem 2.28 (Jensen inequality) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $u \in L^1(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ be convex. Then

$$f\left(\frac{1}{\text{meas } \Omega} \int_\Omega u(x) \, dx\right) \leq \frac{1}{\text{meas } \Omega} \int_\Omega f(u(x)) \, dx.$$ 

2.3.2 Continuity of convex functions

We now turn our attention to continuity of convex functions. Since the results of this section are available for separately convex functions, we start with the following definition.
**Definition 2.29** A function $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is said to be separately convex, or convex in each variable, if, letting $x = (x_1, \ldots, x_N)$, the function

$$x_i \to f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_N)$$

is convex for every $i = 1, \cdots, N$,

for every fixed $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N) \in \mathbb{R}^{N-1}$.

Obviously any convex function is separately convex, but the converse is false as seen in the next example.

**Example 2.30** The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = x_1 x_2$$

is separately convex but not convex. $\diamond$

The next theorem is usually stated for convex functions and not for separately convex functions, but in Part II and Part III we will need this stronger version. The proof is however a straightforward adaptation of the classical results.

**Theorem 2.31** Let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be separately convex and $f \neq +\infty$. Then $f$ is locally Lipschitz, and hence continuous, on int (dom $f$).

**Proof.** We divide the proof into three steps. In the sequel, we define

$$|x|_\infty := \max \{|x_i| : i = 1, \cdots, N\}.$$

**Step 1.** We first prove that if $x \in \text{int (dom } f\text{)}$, then $f$ is bounded above in a neighborhood of $x$.

There is no loss of generality, if we suppose that $x = 0$. Therefore, since $0 \in \text{int (dom } f\text{)}$, there exists $\epsilon > 0$ such that

$$\{x = (x_1, \cdots, x_N) \in \mathbb{R}^N : |x|_\infty \leq \epsilon\} \subset \text{dom } f. \quad (2.22)$$

Letting

$$a := \max \{f(\epsilon_1, \epsilon_2, \ldots, \epsilon_N) : \epsilon_i = -\epsilon, 0, \epsilon, \text{ for every } i = 1, \cdots, N\}$$

we deduce from (2.22) that $a < +\infty$. We now claim that

$$|x|_\infty \leq \epsilon \Rightarrow f(x) \leq a. \quad (2.23)$$

In order to prove (2.23), observe that if $0 \leq x_N \leq \epsilon$ and $\epsilon_i = -\epsilon, 0, \epsilon$, then the separate convexity of $f$ with respect to the last variable implies that

$$f(\epsilon_1, \epsilon_2, \cdots, \epsilon_{N-1}, x_N) \leq \frac{x_N}{\epsilon} f(\epsilon_1, \cdots, \epsilon_{N-1}, \epsilon) + (1 - \frac{x_N}{\epsilon}) f(\epsilon_1, \cdots, \epsilon_{N-1}, 0)$$

$$\leq \frac{x_N}{\epsilon} a + (1 - \frac{x_N}{\epsilon}) a = a.$$
Using the above inequality and the separate convexity of $f$ with respect to $x_{N-1}$ we have, if $0 \leq x_{N-1} \leq \epsilon$, that
\[
 f(\epsilon_1, \cdots, \epsilon_{N-2}, x_{N-1}, x_N) 
 \leq \frac{x_{N-1}}{\epsilon} f(\epsilon_1, \cdots, \epsilon_{N-2}, \epsilon, x_N) + \left(1 - \frac{x_{N-1}}{\epsilon}\right) f(\epsilon_1, \cdots, \epsilon_{N-2}, 0, x_N) \leq a.
\]

Iterating the process with respect to all the variables we have immediately (2.23) for $0 \leq x_i \leq \epsilon$. A similar argument applies if some of the $x_i$ are negative.

The inequality (2.23) implies that if $x \in \text{int}(\text{dom} f)$, then $f$ is bounded above in a neighborhood of $x$, as claimed.

**Step 2.** We next show that if $x \in \text{int}(\text{dom} f)$, then $f$ is continuous at $x$. There is no loss of generality if we assume that $x = 0$ and $f(0) = 0$. Since $f$ is bounded above in a neighborhood of $x = 0$, there exist $\lambda > 0$ and $a > 0$ such that
\[
 |x|_\infty \leq \lambda \Rightarrow f(x) \leq a.
\]

Fix $\epsilon > 0$ and without loss of generality assume that $\epsilon \leq aN2^N$ (otherwise choose $a$ even larger). We now show that
\[
 |x|_\infty \leq \frac{\epsilon}{aN2^N} \lambda \Rightarrow |f(x)| \leq \epsilon.
\]

We let
\[
 \delta := \frac{\epsilon}{aN2^N} \leq 1.
\]

Using the separate convexity of $f$, we have
\[
 f(x) = f(x_1, \cdots, x_N) = f(\delta(x_1/\delta, x_2, \cdots, x_N) + (1 - \delta)(0, x_2, \cdots, x_N)) 
 \leq \delta f(x_1/\delta, x_2, \cdots, x_N) + (1 - \delta) f(0, x_2, \cdots, x_N).
\]

Repeating the process with the second variable we have
\[
 f(x) \leq \delta f(x_1/\delta, x_2, \cdots, x_N) + (1 - \delta) \delta f(0, x_2/\delta, \cdots, x_N) 
 + (1 - \delta)^2 f(0, 0, x_3, \cdots, x_N).
\]

Iterating the process, we obtain
\[
 f(x) \leq \delta \sum_{i=1}^N (1 - \delta)^{i-1} f(0, \cdots, 0, x_i/\delta, x_{i+1}, \cdots, x_N) + (1 - \delta)^N f(0, \cdots, 0).
\]

If we now assume that
\[
 |x|_\infty \leq \delta \lambda = \frac{\epsilon \lambda}{aN2^N} \leq \lambda,
\]
we deduce immediately from the fact that \( f(0) = 0 \) and from (2.24) that

\[
f(x) \leq \delta a \sum_{i=1}^{N} (1 - \delta)^{i-1} \leq \delta a N \leq \epsilon
\]

which is one of the inequalities in (2.25).

In order to obtain (2.25), we still need to show that \( f(x) \geq -\epsilon \) and this is done similarly. We have

\[
0 = f(0, \ldots, 0) = f\left( \frac{1}{1 + \delta}(0, \ldots, 0, x_N) + \frac{\delta}{1 + \delta}(0, \ldots, 0, -\frac{x_N}{\delta}) \right)
\leq \frac{1}{1 + \delta} [f(0, \ldots, 0, x_N) + \delta f(0, \ldots, 0, -\frac{x_N}{\delta})].
\]

Proceeding similarly with the \( x_{N-1} \) variable, we get

\[
f(0, \ldots, 0, x_N) = f\left( \frac{1}{1 + \delta}(0, \ldots, 0, x_{N-1}, x_N) + \frac{\delta}{1 + \delta}(0, \ldots, 0, -\frac{x_{N-1}}{\delta}, x_N) \right)
\leq \frac{1}{1 + \delta} f(0, \ldots, 0, x_{N-1}, x_N) + \frac{\delta}{1 + \delta} f(0, \ldots, 0, -\frac{x_{N-1}}{\delta}, x_N)
\]

and thus combining the two estimates, we obtain

\[
0 \leq \frac{1}{(1 + \delta)^2} f(0, \ldots, 0, x_{N-1}, x_N) + \frac{\delta}{(1 + \delta)^2} f(0, \ldots, 0, -\frac{x_{N-1}}{\delta}, x_N)
+ \frac{\delta}{1 + \delta} f(0, \ldots, 0, -\frac{x_N}{\delta}).
\]

Iterating the process as above we deduce that

\[
0 \leq \frac{1}{(1 + \delta)^N} f(x_1, \ldots, x_N) + \sum_{i=1}^{N} \frac{\delta}{(1 + \delta)^{N-i+1}} f(0, \ldots, 0, -\frac{x_i}{\delta}, x_{i+1}, \ldots, x_N)
\]

and hence, if

\[
|x|_\infty \leq \delta \lambda = \frac{\epsilon \lambda}{a N 2^N} \leq \lambda,
\]

we find, from (2.24), that

\[
f(x_1, \ldots, x_N) \geq -\delta \sum_{i=1}^{N} (1 + \delta)^{i-1} f(0, \ldots, 0, -\frac{x_i}{\delta}, x_{i+1}, \ldots, x_N)
\geq -\delta a \sum_{i=1}^{N} (1 + \delta)^{i-1} \geq -\delta a N 2^N = -\epsilon.
\]

From the above inequality, we thus infer that

\[
|x|_\infty \leq \frac{\epsilon}{a N 2^N \lambda} \Rightarrow f(x) \geq -\epsilon.
\]
Thus (2.25) holds and the continuity of \( f \) at \( x = 0 \) follows.

**Step 3.** It therefore remains to show that \( f \) is locally Lipschitz in the interior of the domain of \( f \). Let \( x \in \text{int}(\text{dom } f) \). By continuity of \( f \) at \( x \), there exist \( \alpha, \beta > 0 \) such that

\[
|y - x|_\infty \leq 2\beta \Rightarrow |f(y)| \leq \alpha < +\infty. \tag{2.26}
\]

Let \( z \) and \( z_1 \) be such that

\[
|z_1 - z|_\infty, |z_1 - x|_\infty \leq \beta. \tag{2.27}
\]

Observe that (2.27) implies that \( |z - x|_\infty \leq 2\beta \). Therefore (2.26) and (2.27) lead to

\[
|z_1 - z|_\infty, |z_1 - x|_\infty \leq \beta \Rightarrow f(z) - f(z_1) \leq 2\alpha. \tag{2.28}
\]

Let \( \epsilon > 0 \) be chosen later. Combining (2.25) and (2.28), we have immediately

\[
|z_1 - z|_\infty, |z_1 - x|_\infty \leq \frac{\beta \epsilon}{2\alpha N 2^N} \Rightarrow |f(z) - f(z_1)| \leq \epsilon. \tag{2.29}
\]

Choosing

\[
\epsilon := \frac{2\alpha N 2^N}{\beta} |z_1 - z|_\infty
\]

we have from (2.27) and (2.29) that

\[
|z_1 - z|_\infty, |z_1 - x|_\infty \leq \beta \Rightarrow |f(z) - f(z_1)| \leq \frac{2\alpha N 2^N}{\beta} |z_1 - z|_\infty. \tag{2.30}
\]

Now let \( z_2 \) be such that \( |z_2 - x|_\infty \leq \beta \). Let

\[
u_1, u_2, \ldots, u_M \in [z_1, z_2]
\]

(the segment in \( \mathbb{R}^N \) with endpoints \( z_1 \) and \( z_2 \)) be such that

\[
u_1 = z_1, u_2, \ldots, u_M = z_2 \quad \text{and} \quad |u_m - u_{m+1}|_\infty \leq \beta, \quad m = 1, \ldots, M - 1.
\]

Note that, since \( |z_1 - x|_\infty, |z_2 - x|_\infty \leq \beta \), then

\[
|u_m - x|_\infty \leq \beta, \quad m = 1, \ldots, M.
\]

Using (2.30), we immediately get

\[
|u_m - u_{m+1}|_\infty \leq \beta \Rightarrow |f(u_m) - f(u_{m+1})| \leq \frac{2\alpha N 2^N}{\beta} |u_m - u_{m+1}|_\infty.
\]

Summing the above inequalities, we obtain

\[
|z_1 - x|_\infty, |z_2 - x|_\infty \leq \beta \Rightarrow |f(z_1) - f(z_2)| \leq \frac{2\alpha N 2^N}{\beta} |z_1 - z_2|_\infty
\]
and whence the result.

We terminate this subsection with an elementary proposition concerning convex functions (see Fusco [292], Marcellini [423], Morrey [455]) that should be related to Theorem 2.31.

**Proposition 2.32** Let $f : \mathbb{R}^N \to \mathbb{R}$ be separately convex such that
\[ |f(x)| \leq \alpha (1 + |x|^p) \]
for every $x \in \mathbb{R}^N$, where $\alpha \geq 0$, $p \geq 1$. Then there exists $\beta \geq 0$ such that
\[ |f(x) - f(y)| \leq \beta(1 + |x|^{p-1} + |y|^{p-1}) |x - y| \]
for every $x, y \in \mathbb{R}^N$.

**Proof.** We divide the proof into three steps.

*Step 1.* We first prove that if $g : \mathbb{R} \to \mathbb{R}$ is convex, then, for every $\lambda > \mu > 0$ and every $t \in \mathbb{R}$,
\[
\frac{g(t + \mu) - g(t)}{\mu} \leq \frac{g(t + \lambda) - g(t)}{\lambda}.
\]
(2.31)
This follows at once from the convexity of $g$. Indeed write
\[
g(t + \mu) = g\left(\frac{\mu}{\lambda}(t + \lambda) + (1 - \frac{\mu}{\lambda})t\right)
\]
\[
\leq \frac{\mu}{\lambda} g(t + \lambda) + (1 - \frac{\mu}{\lambda}) g(t)
\]
and (2.31) follows.

*Step 2.* Fix $\tilde{x}_1 = (x_2, \cdots, x_N) \in \mathbb{R}^{N-1}$ and define for $t \in \mathbb{R}$
\[ g(t) := f(t, \tilde{x}_1) \]
and let us prove that there exists $\beta_1 \geq 0$ such that
\[ |g(x_1) - g(y_1)| \leq \beta_1 (1 + |x|^{p-1} + |y|^{p-1}) |x_1 - y_1|, \]  
(2.32)
Assume, without loss of generality that $x_1 < y_1$. Choose then in (2.31)
\[ \lambda := 1 + |x| + |y| \text{ and } \mu := y_1 - x_1 \]
to get
\[ g(y_1) - g(x_1) = g(x_1 + (y_1 - x_1)) - g(x_1)
\]
\[ \leq (y_1 - x_1) \frac{g(x_1 + 1 + |x| + |y|) - g(x_1)}{1 + |x| + |y|} \]
and
\[
g(x_1) - g(y_1) = g(y_1 - (y_1 - x_1)) - g(y_1)
\]
\[
\leq (y_1 - x_1) \frac{g(y_1 - (1 + |x| + |y|)) - g(y_1)}{1 + |x| + |y|}.
\]
Using the hypothesis on \( f \), we have indeed obtained (2.32).

**Step 3.** Writing

\[
\begin{align*}
f(x) - f(y) &= [f(x_1, x_2, \ldots, x_N) - f(y_1, x_2, \ldots, x_N)] \\
&+ \sum_{i=1}^{N-2} [f(y_1, \ldots, y_i, x_{i+1}, x_{i+2}, \ldots, x_N) - f(y_1, \ldots, y_i, y_{i+1}, x_{i+2}, \ldots, x_N)] \\
&+ [f(y_1, \ldots, y_{N-1}, x_N) - f(y_1, \ldots, y_{N-1}, y_N)]
\end{align*}
\]

and applying the proper adaptation of (2.32) to each term of the above identity, we have indeed obtained the proposition. ■

### 2.3.3 Convex envelope

We start with a definition to which we already alluded earlier.

**Definition 2.33** Let \( f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \). Then the **convex envelope** of \( f \), denoted by \( C f \), is the largest convex function below \( f \).

**Remark 2.34**

(i) The definition can be equivalently written, for every \( x \in \mathbb{R}^N \), as

\[
Cf(x) = \sup \{g(x) : g \leq f \text{ and } g \text{ convex} \}.
\]

(ii) It might be (see Example 2.45 below) that \( Cf \) takes the value \(-\infty\), even though \( f > -\infty \). An easy way to avoid this situation is to assume that there exist \( a \in \mathbb{R}^N \) and \( \alpha \in \mathbb{R} \) such that

\[
f(x) \geq \langle a; x \rangle + \alpha \text{ for every } x \in \mathbb{R}^N.
\]

An immediate consequence of Carathéodory theorem is the following characterization of the envelope.

**Theorem 2.35** Let \( f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \) and, for every \( x \in \mathbb{R}^N \),

\[
Cf(x) = \sup \{g(x) : g \leq f \text{ and } g \text{ convex} \}.
\]

Assume that \( Cf > -\infty \). Then

\[
Cf(x) = \inf \{\sum_{i=1}^{N+1} \alpha_i f(x_i) : \sum_{i=1}^{N+1} \alpha_i x_i = x, \ \alpha_i \geq 0 \text{ with } \sum_{i=1}^{N+1} \alpha_i = 1\}.
\]

**Proof.** We first define

\[
C'f(x) := \inf \{\sum_{i=1}^{I} \alpha_i f(x_i) : I \in \mathbb{N}, \ \sum_{i=1}^{I} \alpha_i x_i = x, \ \alpha \in \Lambda_I\}
\]

(2.33)

where

\[
\Lambda_s = \{\lambda = (\lambda_1, \cdots, \lambda_s) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^{s} \lambda_i = 1\}.
\]
Note immediately that since $f \geq Cf$ and $Cf$ is convex, then $C'f \geq Cf > -\infty$.

**Step 1.** Let us show that $C'f$ is convex. So let $y, z \in \mathbb{R}^N$ and $t \in [0, 1]$, we have to prove that

$$C'f (tz + (1 - t) y) \leq tC'f (z) + (1 - t) C'f (y).$$

(2.34)

Fix $\epsilon > 0$ and find $I, J \geq N + 1, \lambda \in \Lambda_I, \mu \in \Lambda_J$ and $y_i, z_i \in \mathbb{R}^N$ such that

$$\left\{ \begin{array}{l}
\epsilon + C'f (z) \geq \sum_{i=1}^{I} \lambda_i f (z_i), \quad \text{with} \quad \sum_{i=1}^{I} \lambda_i z_i = z \\
\epsilon + C'f (y) \geq \sum_{i=1}^{J} \mu_i f (y_i), \quad \text{with} \quad \sum_{i=1}^{J} \mu_i y_i = y.
\end{array} \right.$$

Writing

$$\alpha_i := t\lambda_i, \quad x_i := z_i, \quad i = 1, \ldots, I$$

$$\alpha_{I+i} := (1 - t) \mu_i, \quad x_{I+i} := y_i, \quad i = 1, \ldots, J,$$

we find that $\alpha \in \Lambda_{I+J}$ and we get that

$$\epsilon + tC'f (z) + (1 - t) C'f (y) \geq \sum_{i=1}^{I+J} \alpha_i f (x_i) \quad \text{with} \quad \sum_{i=1}^{I+J} \alpha_i x_i = tz + (1 - t) y.$$

Using (2.33) in the right hand side of the above inequality and the arbitrariness of $\epsilon$, we have indeed obtained (2.34). We have therefore shown that $C'f$ is convex.

**Step 2.** We next prove that $C'f = Cf$. We first observe that (see the beginning of the proof)

$$Cf \leq C'f \leq f.$$

Since $C'f$ is convex, we thus deduce that $C'f = Cf$.

**Step 3.** Finally we show that we can take $I = N + 1$. We first prove that we can restrict attention to $I \leq N + 2$. Consider the set

$$F := \{(x_i, f (x_i))_{i=1}^{I} \subset \mathbb{R}^{N+1}$$

and note that for any $\alpha \in \Lambda_I$ we have

$$(\sum_{i=1}^{I} \alpha_i x_i, \sum_{i=1}^{I} \alpha_i f (x_i)) \in \text{co} \ F.$$ 

Appealing to Theorem 2.13 we can find $\beta \in \Lambda_{N+2}$ and $y_i \in \{x_1, \ldots, x_I\}$, $i = 1, \ldots, N + 2$, so that

$$(\sum_{i=1}^{I} \alpha_i x_i, \sum_{i=1}^{I} \alpha_i f (x_i)) = (\sum_{i=1}^{N+2} \beta_i y_i, \sum_{i=1}^{N+2} \beta_i f (y_i)).$$

We can therefore choose $I \leq N + 2$. 

We finally further reduce $I$ to $N + 1$. We show that we can find $\gamma \in \Lambda_{N+2}$, with at least one of the $\gamma_i = 0$ (this implies in particular that, removing this $\gamma_i$, we can assume, in fact, $\gamma \in \Lambda_{N+1}$), and $y_i, \ i = 1, \cdots, N + 2$, so that

$$\sum_{i=1}^{N+2} \gamma_i f(y_i) \leq \sum_{i=1}^{N+2} \beta_i f(y_i).$$

(2.35)

Since we have the convention that $0.\ (+\infty) = 0$, the claim, $I \leq N + 1$, will follow from (2.35). Assume that $\beta \in \Lambda_{N+2}$ is so that $\beta_i > 0, \ i = 1, \cdots, N + 2$, otherwise nothing is to be proved. Let us first denote by

$$x := \sum_{i=1}^{N+2} \beta_i y_i$$

which implies that $x \in \text{co}\{y_1, \cdots, y_{N+2}\} \subset \mathbb{R}^N$. Applying Carathéodory theorem, once more, we find

$$\tilde{\beta}_i \geq 0, \text{ for every } 1 \leq i \leq N + 2 \text{ and } \sum_{i=1}^{N+2} \tilde{\beta}_i = 1$$

and at least one of the $\tilde{\beta}_i = 0$ such that

$$\sum_{i=1}^{N+2} \tilde{\beta}_i y_i = x.$$

We may assume without loss of generality that

$$\sum_{i=1}^{N+2} \tilde{\beta}_i f(y_i) > \sum_{i=1}^{N+2} \beta_i f(y_i),$$

(2.36)

otherwise choosing $\gamma_i = \tilde{\beta}_i$ we would immediately obtain (2.35). We then let

$$J := \{ i \in \{1, \cdots, N + 2\} : \beta_i - \tilde{\beta}_i < 0 \}.$$

Observe that $J \neq \emptyset$, since otherwise $\beta_i \geq \tilde{\beta}_i \geq 0$ for every $1 \leq i \leq N + 2$ and since at least one of the $\tilde{\beta}_i = 0$, we would have a contradiction with $\sum_{i=1}^{N+2} \beta_i = \sum_{i=1}^{N+2} \tilde{\beta}_i = 1$ and the fact that $\beta_i > 0$ for every $i$. We then define

$$\lambda := \min_{i \in J} \{ \beta_i / (\tilde{\beta}_i - \beta_i) \}$$

and we have clearly $\lambda > 0$. Finally let

$$\gamma_i := \beta_i + \lambda \left( \beta_i - \tilde{\beta}_i \right), \ 1 \leq i \leq N + 2.$$

We therefore have

$$\gamma_i \geq 0, \sum_{i=1}^{N+2} \gamma_i = 1, \text{ at least one of the } \gamma_i = 0.$$
and from (2.36)
\[
\sum_{i=1}^{N+2} \gamma_i f(y_i) = \sum_{i=1}^{N+2} \beta_i f(y_i) + \lambda \sum_{i=1}^{N+2} \left( \beta_i - \tilde{\beta}_i \right) f(y_i) \\
\leq \sum_{i=1}^{N+2} \beta_i f(y_i).
\]

We have therefore obtained (2.35) and this concludes Step 3 and thus the theorem. ■

We will see several examples of convex envelopes in Section 2.3.5, but before that we want to make more precise the connection between the convex hull of a set and the convex envelope of its indicator function.

**Proposition 2.36** Let \( E \subset \mathbb{R}^N \) and \( \chi_E \) be the indicator function of \( E \), namely

\[
\chi_E(x) = \begin{cases} 
0 & \text{if } x \in E \\
+\infty & \text{if } x \notin E.
\end{cases}
\]

Then

\[ C\chi_E = \chi_{\text{co} E}. \]

Moreover, if

\[
\mathcal{F}_\infty^E := \{ f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} : f|_E \leq 0 \},
\]

\[
\mathcal{F}^E := \{ f : \mathbb{R}^N \to \mathbb{R} : f|_E \leq 0 \},
\]

then

\[
\text{co } E = \{ x \in \mathbb{R}^N : f(x) \leq 0, \text{ for every convex } f \in \mathcal{F}_\infty^E \}, \\
\overline{\text{co } E} = \{ x \in \mathbb{R}^N : f(x) \leq 0, \text{ for every convex } f \in \mathcal{F}^E \}.
\]

**Remark 2.37** Anticipating the results of Section 2.3.5, we also have

\[ \chi_{E}^{**} = \chi_{\text{co } E}. \]

**Proof.** (i) Let us start by showing \( C\chi_E = \chi_{\text{co } E} \). Since \( \chi_{\text{co } E} \leq \chi_E \) and \( \chi_{\text{co } E} \) is convex, we deduce immediately that

\[ \chi_{\text{co } E} \leq C\chi_E. \]

In order to show the reverse inequality, it is sufficient to show that

\[ \chi_{\text{co } E}(x) = 0 \Rightarrow C\chi_E(x) = 0. \]
This follows from Theorem 2.13. Indeed if \( \chi_{\mathrm{co} \ E} (x) = 0 \), this means \( x \in \mathrm{co} \ E \) and hence from Carathéodory theorem we can find \( x_i \in E, \lambda \in \Lambda_{N+1} \) so that
\[
x = \sum_{i=1}^{N+1} \lambda_i x_i.
\]
We therefore obtain
\[
\sum_{i=1}^{N+1} \lambda_i \chi_E (x_i) = 0
\]
and hence, from Theorem 2.35, \( C \chi_E (x) = 0 \), as wished.

(ii) Since we obviously have
\[
f \in \mathcal{F}_\infty^E \iff f \leq \chi_E,
\]
we deduce that any convex \( f \in \mathcal{F}_\infty^E \) must be such that
\[
f \leq C \chi_E = \chi_{\mathrm{co} \ E}
\]
and hence
\[
\mathrm{co} \ E \subset \{ x \in \mathbb{R}^N : f (x) \leq 0, \text{ for every convex } f \in \mathcal{F}_\infty^E \}.
\]
The reverse inclusion follows from the fact that \( \chi_{\mathrm{co} \ E} \in \mathcal{F}_\infty^E \) and is convex.

(iii) The set
\[
X := \{ x \in \mathbb{R}^N : f (x) \leq 0, \text{ for every convex } f \in \mathcal{F}^E \}
\]
is clearly convex and closed, since any convex function in \( \mathcal{F}^E \) is continuous; furthermore \( E \subset X \). We can therefore infer that
\[
\overline{\mathrm{co} \ E} \subset X.
\]
Since the distance function (see Example 2.23) \( d_{\mathrm{co} \ E} \) is convex and belongs to \( \mathcal{F}^E \) we deduce the reverse inclusion \( \overline{\mathrm{co} \ E} \supset X \).

### 2.3.4 Lower semicontinuous envelope

An important concept that we will encounter again in all the processes of relaxation is the following.

**Definition 2.38** Let \( f : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{ +\infty \} \). The lower semicontinuous envelope of \( f \), denoted by \( \overline{f} \), is the largest lower semicontinuous function that is below \( f \).

**Remark 2.39** (i) In view of Theorem 2.26, we can write, for every \( x \in \mathbb{R}^N \),
\[
\overline{f} (x) = \sup \{ g (x) : g \leq f \text{ and } g \text{ lower semicontinuous} \}.
\]
(ii) Another way of rewriting the function $\mathbf{f}$ is

$$\mathbf{f} (x) = \inf_{\{x_\nu\}} \liminf_{x_\nu \to x} f (x_\nu).$$

We have the following easy result that we state without proof.

**Proposition 2.40** Let $E \subset \mathbb{R}^N$ and $\chi_E$ be its indicator function. Then

$$\overline{\chi_E} = \chi_{\overline{E}}.$$

### 2.3.5 Legendre transform and duality

We now introduce the notions of duality and dual maps, following Fenchel [276] and Moreau [452]. These notions play a central role in convex analysis.

**Definition 2.41** Let $\langle \cdot ; \cdot \rangle$ denote the scalar product in $\mathbb{R}^N$ and

$$f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$$

with $f \not\equiv +\infty$.

(i) The function $f^* : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^* (x^*) := \sup_{x \in \mathbb{R}^N} \{\langle x; x^* \rangle - f (x)\}$$

is called the conjugate, or dual, function of $f$.

(ii) The function $f^{**} : \mathbb{R}^N \to \mathbb{R} \cup \{\pm \infty\}$ defined by

$$f^{**} (x) := \sup_{x^* \in \mathbb{R}^N} \{\langle x; x^* \rangle - f^* (x^*)\}$$

is called the biconjugate, or bidual, function of $f$.

**Remark 2.42** (i) The notion of duality is closely related to the concept of Legendre transform and we will, by abuse of language, often use both notions as equivalent.

(ii) In general, the bidual $f^{**}$, as well as $Cf$, may take the value $-\infty$ even though $f$ is never $-\infty$.

(iii) If $f \equiv +\infty$, then $f^* \equiv -\infty$ and $f^{**} \equiv +\infty$.  

Before giving some examples, we state some important properties of these functions, as established by Fenchel [276] and Moreau [452].

**Theorem 2.43** Let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$. Then the following statements hold.

(i) $f^*$ is convex and lower semicontinuous.

(ii) If $f$ is convex and lower semicontinuous, then $f^* \not\equiv +\infty$. 

$\diamondsuit$
(iii) The following inequalities hold:

\[ f^{**} \leq Cf \leq f. \]

Moreover, if \( Cf \) is lower semicontinuous and \( Cf > -\infty \), then

\[ f^{**} = Cf, \]

so that, in particular, if \( f \) is convex and lower semicontinuous, then

\[ f^{**} = Cf = f. \]

(iv) The identity \( f^{***} = f^* \) is always valid.

Remark 2.44 We therefore see that \( f^{**} \) is at the same time the convex and the lower semicontinuous envelope of the function \( f \), while \( Cf \) (respectively \( \overline{f} \)) is only the convex envelope (respectively the lower semicontinuous envelope) of \( f \).

We now discuss some examples.

Example 2.45 (i) We recall that the indicator function of a set \( E \subset \mathbb{R}^N \) is given by

\[ \chi_E = \begin{cases} 0 & \text{if } x \in E \\ +\infty & \text{if } x \notin E. \end{cases} \]

We then have

\[ \chi_E^*(x^*) = \sup_{x \in E} \{ \langle x; x^* \rangle \}, \]

which is nothing but the support function of \( E \). Again applying the duality, we obtain

\[ \chi_E^{**}(x) = \chi_{\text{co } E}(x) \quad \text{and} \quad C\chi_E(x) = \chi_{\text{co } E}(x), \]

where \( \text{co } E \) (respectively \( \overline{\text{co } E} \)) denotes the convex hull (respectively the closed convex hull) of \( E \). In particular, if \( E = (0, 1) \subset \mathbb{R} \), we get

\[ \chi(0,1) = C\chi(0,1) \quad \text{and} \quad \chi[0,1] = \chi^{**}(0,1). \]

The second identity, \( C\chi_E = \chi_{\text{co } E} \), has been shown in Proposition 2.36. The identity \( \chi^{**} = \chi_{\text{co } E} \) follows from the following observations.

- \( \chi_{\text{co } E} \) is convex and lower semicontinuous and hence \( \chi_{\text{co } E} = \chi_{\text{co } E} \). We therefore deduce, recalling the trivial fact \( \chi_{\text{co } E} \leq \chi_E \), that

\[ \chi_{\text{co } E} = \chi_{\text{co } E} \leq \chi_E^{**}. \]

- Appealing to Proposition 2.40 we see that \( \overline{\chi_{\text{co } E}} = \chi_{\text{co } E} \) and, since \( \chi_E^{**} \) is lower semicontinuous and \( \chi_E^{**} \leq C\chi_E = \chi_{\text{co } E} \), we deduce that

\[ \chi_E^{**} \leq \chi_{\text{co } E} \]

as wished.
(ii) The difference between $Cf$ and $f^{**}$ is even more striking if we consider

$$f(x) = \begin{cases} (x^2 - 1)^{-1} & \text{if } |x| < 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Then $f^* \equiv +\infty$ and hence $f^{**} \equiv -\infty$, while

$$Cf(x) = \begin{cases} -\infty & \text{if } |x| < 1 \\ +\infty & \text{otherwise.} \end{cases}$$

(iii) Define for $x \in \mathbb{R}^N$ and $1 \leq p \leq \infty$ the Hölder norm

$$|x|^p := \begin{cases} \left[ \sum_{i=1}^{N} |x_i|^p \right]^{1/p} & \text{if } 1 \leq p < \infty \\ \max_{1 \leq i \leq N} \{|x_i|\} & \text{if } p = \infty. \end{cases}$$

For $1 < p < \infty$, let

$$f(x) := \frac{1}{p} |x|^p.$$

Then, if $p' = p/(p-1)$,

$$f^*(x^*) = \frac{1}{p'} |x^*|^{p'}. $$

(iv) Let $A \in \mathbb{R}^{n^2}$ (the set of $n \times n$ matrices identified with $\mathbb{R}^{n^2}$) and $f(A) = \det A$. Then

$$f^*(A^*) \equiv +\infty \text{ and } f^{**}(A) = Cf(A) \equiv -\infty. \quad \Diamond$$

We now turn to the proof of Theorem 2.43.

**Proof.** (i) Since

$$x^* \rightarrow \langle x; x^* \rangle - f(x)$$

is convex (in fact affine) and lower semicontinuous (in fact continuous) then $f^*$ is convex and lower semicontinuous.

(ii) Note first that if $f \equiv +\infty$, then $f^* \equiv -\infty$ and the result is proved. So we may assume that there exists $x_0 \in \text{dom } f$. We next let $a_0 < f(x_0)$ and we apply Theorem 2.10 (iii) to

$$A = \text{epi } f \text{ and } B = \{(x_0, a_0)\}.$$ 

We then obtain that there exists a hyperplane over $\mathbb{R}^N \times \mathbb{R}$ defined by

$$\langle (x, a); (x^*; a^*) \rangle = \langle x; x^* \rangle + aa^* = \alpha$$

which separates strictly $A$ and $B$, i.e.

$$\langle x_0; x^* \rangle + a_0 a^* < \alpha < \langle x; x^* \rangle + f(x) a^*, \text{ for every } x \in \mathbb{R}^N. \quad (2.37)$$
Taking \( x = x_0 \) in (2.37) we immediately get
\[
\langle x_0; x^* \rangle + a_0 a^* < \alpha < \langle x_0; x^* \rangle + f(x_0) a^*
\]
and hence \( a^* > 0 \). We therefore deduce immediately from (2.37) that
\[
\langle x; -\frac{1}{a^*}x^* \rangle - f(x) < -\frac{\alpha}{a^*}
\]
and thus taking the supremum in (2.38) we obtain the result, i.e. \( f^* \neq +\infty \).

(iii) We proceed in three steps.

Step 1. Observe first that \( f^{**} \) is convex and lower semicontinuous and that, by definition, \( f(x) \geq \langle x; x^* \rangle - f^*(x^*) \), hence \( f^{**} \leq f \). The first inequality, \( f^{**} \leq Cf \leq f \), follows then immediately.

Step 2. There is no loss of generality if we assume \( Cf = f \). We next reduce the problem to the case where \( f \geq 0 \). We may assume without loss of generality that \( f \neq +\infty \). Choosing \( x^* \in \text{dom } f^* \), which is non-empty as seen in (ii), and defining
\[
g(x) := f(x) - \langle x; x^* \rangle + f^*(x^*)
\]
we obtain that \( g \geq 0 \), convex, lower semicontinuous and \( g \neq +\infty \). Observe also that
\[
g^{**}(x) = f^{**}(x) - \langle x; x^* \rangle + f^*(x^*)
\]
Therefore the result, \( f = f^{**} \), will follow from the corresponding result for \( g \).

Step 3. We may then assume that \( f \geq 0 \) (and thus \( f^{**} \geq 0 \)), convex, lower semicontinuous and \( f \neq +\infty \). In view of Step 1, we only need to show that \( f^{**} \geq f \). We proceed by contradiction and assume that there exists \( x_0 \in \mathbb{R}^N \) such that
\[
0 \leq f^{**}(x_0) < f(x_0).
\]
Applying Theorem 2.10 (iii) to
\[A = \text{epi } f \text{ and } B = \{(x_0, f^{**}(x_0))\}\]
we have that there exists a hyperplane \( \langle x; x^* \rangle + aa^* = \alpha \) which separates strictly \( A \) and \( B \), i.e.,
\[
\langle x; x^* \rangle + aa^* > \alpha \text{ for every } (x, a) \in \text{epi } f
\]
\[
\langle x_0; x^* \rangle + f^{**}(x_0) a^* < \alpha.
\]
Since in (2.40) \( x \in \text{dom } f \), letting \( a \to +\infty \) immediately yields \( a^* \geq 0 \). We then let \( \epsilon > 0 \) and use the fact that \( f \geq 0 \) and (2.40) to get
\[
\langle x; x^* \rangle + f(x)(a^* + \epsilon) > \alpha \text{ for every } x \in \text{dom } f
\]
and hence
\[
\langle x; -\frac{x^*}{a^* + \epsilon} \rangle - f(x) < -\frac{\alpha}{a^* + \epsilon}
\]
for every \(x \in \text{dom } f\).

The last inequality implies that
\[
f^*(\frac{-x^*}{a^* + \epsilon}) \leq -\frac{\alpha}{a^* + \epsilon}.
\]
Using the definition of \(f^{**}\) we therefore have
\[
f^{**}(x_0) \geq \langle x_0; -\frac{x^*}{a^* + \epsilon} \rangle - f^*(-\frac{x^*}{a^* + \epsilon}) \geq \langle x_0; -\frac{x^*}{a^* + \epsilon} \rangle + \frac{\alpha}{a^* + \epsilon}.
\]
Thus
\[
\langle x_0; x^* \rangle + f^{**}(x_0)(a^* + \epsilon) \geq \alpha.
\]
Using the arbitrariness of \(\epsilon\) and (2.41) we have a contradiction and this terminates Step 3.

(iv) We now want to show that \(f^{***} = f^*\). Since we always have \(f^{**} \leq f\), we deduce that \(f^{***} \geq f^*\). Furthermore from the definition of duality we have for every \(x \in \mathbb{R}^N, x^* \in \mathbb{R}^N\),
\[
\langle x; x^* \rangle - f^{**}(x) \leq f^*(x^*)
\]
and hence, taking the supremum in the left hand side, we obtain \(f^{***} \leq f^*\).

2.3.6 Subgradients and differentiability of convex functions

In this section we will always assume that
\[
f : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}
\]
is convex and \(f \neq +\infty\).

**Definition 2.46** We say that \(x^* \in \mathbb{R}^N\) is a subgradient of \(f\) at \(x\) if
\[
f(z) \geq f(x) + \langle x^*; z - x \rangle, \quad \forall z \in \mathbb{R}^N.
\]
The set of all subgradients of \(f\) at \(x\) is called the subdifferential of \(f\) at \(x\) and is denoted by \(\partial f(x)\).

We now give an elementary example and show a simple characterization of the subdifferential.

**Example 2.47** Let \(f : \mathbb{R} \rightarrow \mathbb{R}\) be defined by
\[
f(x) = |x|.
\]
Then

\[ \partial f(x) = \begin{cases} 
  \{1\} & \text{if } x > 0 \\
  [-1, 1] & \text{if } x = 0 \\
  \{-1\} & \text{if } x < 0.
\end{cases} \]

**Theorem 2.48**  Let \( f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \) be convex, lower semicontinuous and \( f \neq +\infty \). The following conditions are then equivalent:

(i) \( x^* \in \partial f(x) \);

(ii) \( \langle x^* ; z \rangle - f(z) \) achieves its maximum at \( z = x \);

(iii) \( f^*(x^*) + f(x) = \langle x^* ; x \rangle \);

(iv) \( x \in \partial f^*(x^*) \).

**Proof.** (i) \( \iff \) (ii) We have that \( x^* \in \partial f(x) \) is equivalent to

\[ \langle x^* ; x \rangle - f(x) \geq \langle x^* ; z \rangle - f(z), \quad \forall z \in \mathbb{R}^N \quad (2.42) \]

and therefore is equivalent to the fact that \( \langle x^* ; z \rangle - f(z) \) achieves its maximum at \( z = x \).

(ii) \( \iff \) (iii) Using the definition of the conjugate function

\[ f^*(x^*) = \sup_{x \in \mathbb{R}^N} \{ \langle x^* ; x \rangle - f(x) \} , \]

combined with (2.42) we have the equivalence.

(iii) \( \iff \) (iv) Appealing to Theorem 2.43 we find that (iii) is equivalent to

\[ f^*(x^*) + f^{**}(x) = \langle x^* ; x \rangle . \]

Using then the equivalence (i) \( \iff \) (iii) applied to \( f^{**} \) we get the result. ■

The notion of subdifferential is intimately related to the notion of directional derivative, notion that we now define.

**Definition 2.49**  Let \( f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \) and let \( x \in \text{dom } f \).

(i) The one sided directional derivative of \( f \) at \( x \) in the direction \( y \) is the limit, if it exists,

\[ f'(x, y) := \lim_{\lambda \to 0^+} \frac{f(x + \lambda y) - f(x)}{\lambda} . \]

(ii) The directional derivative of \( f \) at \( x \) in the direction \( y \) is \( f'(x, y) \), provided both \( f'(x, y) \) and \( f'(x, -y) \) exist and

\[ f'(x, -y) = -f'(x, y) . \]

We now have the following theorem.
Theorem 2.50 Let \( f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \) be convex and lower semicontinuous, \( f \neq +\infty \) and \( x \in \operatorname{int}(\operatorname{dom} f) \). The following conclusions then hold.

(i) \( f' (x, y) \) exists and

\[
 f' (x, y) \equiv \inf_{\lambda > 0} \frac{f(x + \lambda y) - f(x)}{\lambda}.
\]

Moreover the function \( y \to f' (x, y) \) is convex and

\[
 f'(x, ty) = tf'(x, y) \quad \text{for every} \ t \geq 0 \\
 f'(x, -y) = -f'(x, y) \quad \text{for every} \ y \in \mathbb{R}^N.
\]

(ii) \( x^* \in \partial f (x) \) if and only if

\[
 f' (x, y) \geq \langle x^* ; y \rangle \quad \text{for every} \ y \in \mathbb{R}^N.
\]

(iii) \( \partial f (x) \) is non-empty, convex and compact. Moreover, \( f' (x, y) \) is finite for every \( y \in \mathbb{R}^N \).

(iv) The function \( y \to f' (x, y) \) is lower semicontinuous, convex and

\[
 f'(x, y) = \sup \{ \langle x^* ; y \rangle : x^* \in \partial f (x) \}.
\]

(v) If \( f \) is differentiable at \( x \), then

\[
 \partial f (x) = \{ \nabla f (x) \}
\]

and

\[
 f (x) + f^* (\nabla f (x)) = \langle x ; \nabla f (x) \rangle.
\]

(vi) If \( f \) has a unique subgradient at \( x \), then \( f \) is differentiable at \( x \).

(vii) The set \( D \) where \( f \) is differentiable is dense in \( \operatorname{int}(\operatorname{dom} f) \) and its complement in \( \operatorname{int}(\operatorname{dom} f) \) has zero measure. Furthermore, the usual gradient map \( \nabla f : x \to \nabla f (x) \) is continuous on \( D \).

Proof. (i) Let us first show that since \( f \) is convex, then the function

\[
 \lambda \to \frac{f(x + \lambda y) - f(x)}{\lambda}
\]

is an increasing function of \( \lambda > 0 \). Observe that if \( \lambda \geq \mu > 0 \), we have

\[
 f(x + \mu y) = f\left(\frac{\mu}{\lambda} (x + \lambda y) + \frac{\lambda - \mu}{\lambda} x\right)
\]

\[
 \leq \frac{\mu}{\lambda} f(x + \lambda y) + \frac{\lambda - \mu}{\lambda} f(x)
\]

which implies that

\[
 \frac{f(x + \mu y) - f(x)}{\mu} \leq \frac{f(x + \lambda y) - f(x)}{\lambda}
\]
as claimed. It then follows that \( f' (x, y) \) exists and
\[
f'(x, y) = \inf_{\lambda > 0} \frac{f(x + \lambda y) - f(x)}{\lambda}.
\]
The other properties follow in an immediate way from the convexity of \( f \) and from the above formula.

(ii) From the definition of \( \partial f (x) \) and from (i), we get, for every \( y \in \mathbb{R}^N \) and every \( \lambda > 0 \),
\[
x^* \in \partial f (x) \iff \frac{f(x + \lambda y) - f(x)}{\lambda} \geq \langle x^*; y \rangle \iff f'(x, y) \geq \langle x^*; y \rangle,
\]
as wished.

(iii) Since \( f \) is lower semicontinuous and convex, it follows from Theorem 2.26 that \( \text{epi } f \subset \mathbb{R}^{N+1} \) is closed and convex. Since \( (x, f(x)) \in \partial (\text{epi } f) \), we may therefore use Theorem 2.10 (ii) to get that there exist \( a^* = (a_1^*, a_2^*) \in \mathbb{R}^N \times \mathbb{R} \), \( a^* \neq 0 \), and \( \alpha \in \mathbb{R} \) so that, for every \( (y, a) \in \text{epi } f \),
\[
\langle x; a_1^* \rangle + f(x) a_2^* = \alpha \leq \langle y; a_1^* \rangle + a a_2^*
\]
(2.43)
Note next that \( a_2^* \geq 0 \), since \( (x, f(x) + 1) \in \text{epi } f \). Moreover \( a_2^* \neq 0 \), otherwise \( \langle y - x; a_1^* \rangle \geq 0 \) for every \( y \) in the neighborhood of \( x \) (since \( x \in \text{int } (\text{dom } f) \)) and this would imply that \( a_1^* = 0 \), as well as \( a_2^* = 0 \), which is absurd. Therefore \( a_2^* > 0 \) and we deduce from (2.43) and from the fact that \( (y, f(y)) \in \text{epi } f \),
\[
\langle y; a_1^*/a_2^* \rangle + f(y) \geq \langle x; a_1^*/a_2^* \rangle + f(x).
\]
(2.44)
Letting \( x^* = -a_1^*/a_2^* \) in (2.44) we have that \( x^* \in \partial f (x) \) and hence \( \partial f (x) \neq \emptyset \).

We next prove that \( \partial f (x) \) is convex and compact; since it is clearly closed and convex, we only need to show that it is bounded. From Theorem 2.31 and from the fact that \( x \in \text{int } (\text{dom } f) \), we deduce that there exists \( L = L(x) > 0 \) so that for every \( y \) in a neighborhood of \( x \)
\[
|f(y) - f(x)| \leq L |y - x|.
\]
(2.45)
So let \( x^* \in \partial f (x) \) and use the definition of the subgradient to write
\[
f(y) \geq f(x) + \langle x^*; y - x \rangle
\]
and hence
\[
\frac{1}{|y - x|} \langle x^*; y - x \rangle \leq \frac{f(y) - f(x)}{|y - x|} \leq L.
\]
We therefore have
\[
|x^*| = \sup_{|z| = 1} \{\langle x^*; z \rangle \} \leq L
\]
and hence \( \partial f (x) \) is bounded.
It remains to show that $f'(x, y)$ is finite for every $y \in \mathbb{R}^N$. From (2.45) we have that, for every $y \in \mathbb{R}^N$ and every $\lambda > 0$ sufficiently small,

$$-L |y| \leq \frac{f(x + \lambda y) - f(x)}{\lambda} \leq L |y|.$$ 

Invoking (i) we find the result, namely that for every $y \in \mathbb{R}^N$ the inequality

$$|f'(x, y)| \leq L |y|$$

is valid.

(iv) We refer for a proof of (iv) to Theorem 23.4 in Rockafellar [514].

(v) Assume that $f$ is differentiable at $x$, then

$$f'(x, y) = \langle \nabla f(x); y \rangle.$$

Applying (ii) we get

$$\langle \nabla f(x); y \rangle \geq \langle x^*; y \rangle,$$

for every $y \in \mathbb{R}^N$ and hence $\partial f(x) = \{\nabla f(x)\}$. Moreover the identity

$$f(x) + f^*(\nabla f(x)) = \langle x; \nabla f(x) \rangle$$

then follows from (iii) of Theorem 2.48.

(vi) For the converse part of (v) we refer to Theorem 25.1 in Rockafellar [514].

(vii) We will not prove this last fact and we refer to Theorem 25.5 in Rockafellar [514].

We have as an immediate corollary the following.

**Corollary 2.51** Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex. Then, for every $x \in \mathbb{R}^N$, there exists $x^* \in \partial f(x)$ and thus

$$f(z) \geq f(x) + \langle x^*; z - x \rangle, \quad \forall z \in \mathbb{R}^N.$$ 

Moreover, the following identity holds for every $x \in \mathbb{R}^N$

$$f(x) = \sup \{g(x) : g \leq f \text{ and } g \text{ affine} \}.$$ 

**Proof.** (i) Since $f$ takes only finite values, then $\text{int} (\text{dom} f) = \mathbb{R}^N$ and $f$ is continuous. Thus Theorem 2.50 applies and we find $x^* \in \partial f(x)$. The inequality follows then from Theorem 2.48.

(ii) Since for every $x \in \mathbb{R}^N$ there exists $x^* \in \partial f(x)$, we obtain from Theorem 2.48 that

$$\sup_{z \in \mathbb{R}^N} \{\langle x^*; z \rangle - f(z)\} = \langle x^*; x \rangle - f(x) = f^*(x^*).$$
We thus have, for every \( z \in \mathbb{R}^N \),
\[
\begin{align*}
\langle x^*; z \rangle - f^*(x^*) & \leq f(z) \\
\langle x^*; x \rangle - f^*(x^*) & = f(x)
\end{align*}
\]
which completes the proof. \( \blacksquare \)

We now give some classical criteria equivalent to the convexity.

**Theorem 2.52** Let \( f : \mathbb{R}^N \to \mathbb{R} \), \( f \in C^1(\mathbb{R}^N) \) and \( \langle ; ; \rangle \) denote the scalar product in \( \mathbb{R}^N \).

**Part 1.** The following conditions are then equivalent:

(i) \( f \) is convex;

(ii) for every \( x, y \in \mathbb{R}^N \),
\[
f(y) \geq f(x) + \langle y - x; \nabla f(x) \rangle;
\]

(iii) for every \( x, y \in \mathbb{R}^N \),
\[
\langle y - x; \nabla f(y) - \nabla f(x) \rangle \geq 0.
\]

**Part 2.** If \( f \in C^2(\mathbb{R}^N) \), then \( f \) is convex if and only if its Hessian, \( \nabla^2 f \), is positive semi definite.

**Proof.** Part 1. (i) \( \Rightarrow \) (ii). Let \( \lambda > 0 \), we have from the convexity of \( f \) that
\[
\frac{1}{\lambda} [f(x + \lambda(y - x)) - f(x)] \leq f(y) - f(x).
\]
Letting \( \lambda \to 0 \), we have immediately (ii).

(ii) \( \Rightarrow \) (i). We have from the inequality (ii) that, for \( \lambda \in [0,1] \),
\[
\begin{align*}
f(x) & \geq f(\lambda x + (1 - \lambda)y) + \langle x - (\lambda x + (1 - \lambda)y); \nabla f(\lambda x + (1 - \lambda)y) \rangle \\
f(y) & \geq f(\lambda x + (1 - \lambda)y) + \langle y - (\lambda x + (1 - \lambda)y); \nabla f(\lambda x + (1 - \lambda)y) \rangle.
\end{align*}
\]
Multiplying the first equation by \( \lambda \) and the second by \( (1 - \lambda) \) and adding them, yields the convexity of \( f \).

(ii) \( \Rightarrow \) (iii). Using the inequality (ii) we have
\[
\begin{align*}
f(y) & \geq f(x) + \langle y - x; \nabla f(x) \rangle \\
f(x) & \geq f(y) + \langle x - y; \nabla f(y) \rangle.
\end{align*}
\]
Combining these two inequalities we have
\[
\langle y - x; \nabla f(y) \rangle \geq f(y) - f(x) \geq \langle y - x; \nabla f(x) \rangle
\]
and thus the result.
(iii) ⇒ (ii). Let \( \lambda \in [0, 1] \) and consider
\[
\phi(\lambda) := f(x + \lambda(y - x)).
\]
Observe that
\[
\phi'(\lambda) - \phi'(0) = \langle y - x; \nabla f(x + \lambda(y - x)) - \nabla f(x) \rangle
\]
\[
= \frac{1}{\lambda} [(x + \lambda(y - x) - x; \nabla f(x + \lambda(y - x)) - \nabla f(x))] \geq 0
\]
where we have used (iii). Therefore integrating the inequality we obtain
\[
\phi(\lambda) \geq \phi(0) + \lambda \phi'(0)
\]
and thus letting \( z = x + \lambda(y - x) \), we have
\[
f(z) \geq f(x) + \langle z - x; \nabla f(x) \rangle.
\]

**Part 2.** The monotonicity of the gradient of convex functions is ensured by (iii), which in turn is classically equivalent for \( C^2 \) functions to the fact that the Hessian, \( \nabla^2 f \), is positive semi-definite.

We end the section with the following corollary.

**Corollary 2.53** Let \( Q \in \mathbb{R}^{N \times N} \) be a symmetric positive semi definite matrix. Then \( f: \mathbb{R}^N \to \mathbb{R} \) defined by
\[
f(x) := \left( \langle Qx; x \rangle \right)^{1/2}
\]
is convex.

**Proof.** As a consequence of Theorem 13.3 (see also Theorem 2 in Section 4.7 of Bellman [74]) we can find \( U \in SO(N) \) and
\[
\Lambda = \text{diag}(\lambda_1^2, \ldots, \lambda_N^2) \in \mathbb{R}^{N \times N}
\]
so that
\[
UQU^t = \Lambda.
\]
Observe then that the function
\[
g(x) := \left( \langle \Lambda x; x \rangle \right)^{1/2} = \left( \sum_{i=1}^N \lambda_i^2 x_i^2 \right)^{1/2}
\]
is convex. Since
\[
f(x) = g(Ux),
\]
we get the claim.
2.3.7 Gauges and their polars

We now recall some facts about gauges and their polars.

Definition 2.54 (i) Let $E \subset \mathbb{R}^N$ be a convex set. Then the gauge (sometimes also called Minkowski function) associated to $E$ is defined as

$$\rho (x) := \inf \{ \lambda \geq 0 : x \in \lambda E \}.$$  

(ii) The polar of a gauge $\rho$ is defined as

$$\rho^0 (x^*) := \inf \{ \lambda^* \geq 0 : \langle x^*; x \rangle \leq \lambda^* \rho (x), \ \forall x \in \mathbb{R}^N \}.$$  

The main properties of gauges and polars are summarized in the following proposition.

Proposition 2.55 Let $E \subset \mathbb{R}^N$ be a compact and convex set with $0 \in \text{int} \ E$. The following properties then hold.

(i) The gauge $\rho$ associated to $E$ is finite everywhere, convex and satisfies

- $(a)$ $\rho (x) > 0, \ \forall x \neq 0$
- $(b)$ $\rho (tx) = t \rho (x), \ \forall x \in \mathbb{R}^N, \ \forall t > 0.$

(ii) One has $E = \{ x \in \mathbb{R}^N : \rho (x) \leq 1 \}.$

(iii) Another characterization of $\rho^0$ is given by

$$\rho^0 (x^*) = \sup_{x \neq 0} \{ \frac{\langle x^*; x \rangle}{\rho (x)} \}.$$  

(iv) The following identity holds: $\rho^{00} = \rho$.  

(v) Let $x \neq 0$ and $x^* \in \partial \rho (x).$ Then  

$$\rho^0 (x^*) = 1.$$  

Remark 2.56 (i) Note that if $0 \notin \text{int} \ E$, then, in general, $\rho$ is not finite everywhere. Similarly if $E$ is unbounded, then we may have $\rho (x) = 0$ for some $x \neq 0$.

(ii) The notions of a gauge and its polar are aimed at generalizing Cauchy-Schwarz inequality; namely, we have that

$$\langle x^*; x \rangle \leq \rho (x) \rho^0 (x^*),$$

in a similar manner as

$$\rho^* (x^*) = \sup_{x \in \mathbb{R}^N} \{ \langle x^*; x \rangle - \rho (x) \}$$
is the best possible inequality of the form

\[ \langle x^*; x \rangle \leq \rho(x) + \rho^*(x^*). \]

(iii) Note that in general we do not have \( \rho(x) = \rho(-x) \).

(iv) The typical examples are the ones involving Hölder norms; namely if \( 1 \leq p \leq \infty \), if \( 1/p + 1/p' = 1 \) and

\[
\rho(x) = |x|_p := \begin{cases} 
\left[ \sum_{i=1}^{N} |x_i|^p \right]^{1/p} & \text{if } 1 \leq p < \infty \\
\max_{1 \leq i \leq N} \{|x_i|\} & \text{if } p = \infty.
\end{cases}
\]

then

\[ E = \{ x \in \mathbb{R}^N : |x|_p \leq 1 \} \]

and \( \rho^0(x^*) = |x^*|_{p'} \).

(v) If we compare the definition of the polar of a gauge with the usual dual function, defined as

\[ \rho^*(x^*) = \sup_{x \in \mathbb{R}^N} \{ \langle x^*; x \rangle - \rho(x) \}, \]

we get, under the hypotheses of the proposition,

\[ \rho^*(x^*) = \begin{cases} 
0 & \text{if } \rho^0(x^*) \leq 1 \\
+\infty & \text{otherwise}.
\end{cases} \]

\[ \diamond \]

**Proof.** The proposition easily follows from the definitions and we do not discuss the details; we only, for the sake of illustration, prove (v).

From Theorem 2.50, we have that \( \partial \rho(x) \) is non empty and therefore

\[ \rho(y) \geq \rho(x) + \langle x^*; y - x \rangle, \text{ for every } y \in \mathbb{R}^N. \] (2.46)

We first choose \( y = 0 \) and get from (i) and (iii) that

\[ \frac{\langle x^*; x \rangle}{\rho(x)} \geq 1 \Rightarrow \rho^0(x^*) \geq 1. \]

Moreover choosing \( y = 2x \), we obtain that

\[ \rho(x) - \langle x^*; x \rangle \geq 0 \]

and thus returning to (2.46), we deduce that

\[ \rho(y) - \langle x^*; y \rangle \geq \rho(x) - \langle x^*; x \rangle \geq 0, \text{ for every } y \in \mathbb{R}^N. \]

This implies that

\[ \frac{\langle x^*; y \rangle}{\rho(y)} \leq 1, \forall y \in \mathbb{R}^N - \{0\} \overset{(iii)}{\Rightarrow} \rho^0(x^*) \leq 1 \]

as claimed. ■
2.3.8 Choquet function

Extreme points of a convex set can be characterized through the Choquet function and for more details we refer to Choquet [151] (see also Pianigiani [495]).

**Theorem 2.57** Let $E \subset \mathbb{R}^N$ be a non-empty compact convex set and $E_{ext}$ be the set of its extreme points. Then there exists $\varphi_E : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ (called the Choquet function) a convex function, strictly convex on $E$, so that

$$E_{ext} = \{x \in E : \varphi_E (x) = 0\}$$

$$\varphi_E (x) \leq 0 \iff x \in E.$$

**Proof.** We first define

$$f(x) := \begin{cases} -|x|^2 & \text{if } x \in E \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\varphi_E (x) := \begin{cases} Cf(x) - f(x) & \text{if } x \in E \\ +\infty & \text{otherwise.} \end{cases}$$

Observe that $\varphi_E : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex, since, letting $\chi_E$ be the indicator function of the set $E$, we have

$$\varphi_E (x) = Cf(x) + |x|^2 + \chi_E(x), \forall x \in \mathbb{R}^N.$$  

Furthermore $\varphi_E$ is strictly convex on $E$, since $Cf$ is convex and $-f (x) = |x|^2$ is strictly convex. Moreover we obtain that

$$\varphi_E (x) \leq 0 \text{ if } x \in E.$$  

Indeed the inequality is clear since on $E$ the function $f$ is finite and, by definition, $Cf$ is always not larger than $f$. We now show that

$$\varphi_E (x) = 0 \iff x \in E_{ext}.$$  

Note that if $x \in E$, then, applying Theorem 2.13, we have

$$\varphi_E (x) = |x|^2 + \inf_{x_i \in E} \left\{ - \sum_{i=1}^{N+1} \lambda_i |x_i|^2 : x = \sum_{i=1}^{N+1} \lambda_i x_i, \lambda \in \Lambda_{N+1} \right\}$$

where

$$\Lambda_s := \{\lambda = (\lambda_1, \ldots, \lambda_s) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^{s} \lambda_i = 1\}.$$  

Therefore if $x \in E_{ext}$, we deduce, by definition, that in the infimum the only admissible $x_i$ are $x_i = x$ (or if $x_i \neq x$ then the corresponding $\lambda_i = 0$, which in any case leads to the same result) and hence we have $\varphi_E (x) = 0$.
- We now show the reverse implication, namely

\[ \varphi_E(x) = 0 \Rightarrow x \in E_{ext}. \]

From the above representation formula we obtain, since \( \varphi_E(x) = 0 \) and \( x \in E \), that

\[ |x|^2 = \sup_{x_i \in E} \left\{ \sum_{i=1}^{N+1} \lambda_i |x_i|^2 : x = \sum_{i=1}^{N+1} \lambda_i x_i, \lambda \in \Lambda_{N+1} \right\}. \]

Combining the above with the convexity of the function \( x \to |x|^2 \) we get that

\[ |x|^2 \geq \sum_{i=1}^{N+1} \lambda_i |x_i|^2 \geq \left| \sum_{i=1}^{N+1} \lambda_i x_i \right|^2 = |x|^2; \]

the strict convexity of \( x \to |x|^2 \) implies then that \( x_i = x \) (or if \( x_i \neq x \) then the corresponding \( \lambda_i = 0 \), which is then an irrelevant index). Thus \( x \in E_{ext}. \)
Chapter 3

Lower semicontinuity and existence theorems

3.1 Introduction

In the present chapter, we deal with the minimization problem

\[
(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\},
\]

where

- \( \Omega \subset \mathbb{R}^n \) is an open set;
- \( u : \Omega \to \mathbb{R}^N \) and hence \( \nabla u \in \mathbb{R}^{N \times n} \);
- \( f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, f = f(x, u, \xi), \) is a Carathéodory function (see below for a definition).

Most of the results are concerned with the scalar case \( N = 1 \) or \( n = 1 \), although, in some cases, it is convenient to consider the general case \( N, n \geq 1 \).

The main result of this chapter, investigated in Section 3.2, deals with the (sequential) weak lower semicontinuity of \( I \), meaning that

\[
\lim_{\nu \to \infty} \inf \left\{ I(u_\nu) \right\} \geq I(u)
\]

for every sequence \( u_\nu \rightharpoonup u \) in \( W^{1,p} \). We show, roughly speaking, that the functional \( I \) is (sequentially) weakly lower semicontinuous if and only if \( \xi \to f(x, u, \xi) \) is convex (see Theorem 3.15 and Corollary 3.24).

Since the presence of the lower order terms \( (x, u) \) induces many technical difficulties, we first prove both the necessary and the sufficient parts when the function \( f \) depends only on the term \( \xi \), i.e. \( f = f(\xi) \).
In Section 3.3.1, we obtain as a direct consequence of the results of the preceding section that $I$ is (sequentially) weakly continuous, meaning that
\[ \lim_{\nu \to \infty} I(u_{\nu}) = I(u) \]
for every sequence $u_{\nu} \rightharpoonup u$ in $W^{1,p}$, if and only if $\xi \to f(x,u,\xi)$ is affine.

In Section 3.4, we apply the above mentioned results to prove the existence of minimizers for problem $(P)$. We then derive the necessary condition that should satisfy any minimizer namely the Euler-Lagrange equation.

For further references on this chapter we recommend Ambrosio-Fusco-Pallara [25], Buttazzo [112], Buttazzo-Giaquinta-Hildebrandt [117], Cesari [143], Dacorogna [179], Giaquinta [307], Giusti [316] and Morrey [455].

### 3.2 Weak lower semicontinuity

We now recall the following definition.

**Definition 3.1** Let $p \geq 1$ and $\Omega$, $u$, $f$ be as above. We say that $I$ is (sequentially) weakly lower semicontinuous in $W^{1,p}(\Omega;\mathbb{R}^N)$ if for every sequence $u_{\nu} \rightharpoonup u$ in $W^{1,p}$, then
\[ \liminf_{\nu \to \infty} I(u_{\nu}) \geq I(u). \]

If $p = \infty$, we say that $I$ is (sequentially) weak $\ast$ lower semicontinuous in $W^{1,\infty}(\Omega;\mathbb{R}^N)$ if the same inequality holds for every sequence $u_{\nu} \rightharpoonup u$ in $W^{1,\infty}$.

**Remark 3.2** (i) In the remaining part of the book we usually drop the word “sequentially” when referring to lower semicontinuity.

(ii) If $\Omega$ is bounded, it is clear that the first notion implies the second one, since any sequence $u_{\nu} \rightharpoonup u$ in $W^{1,\infty}$ is such that $u_{\nu} \to u$ in $W^{1,p}$ for every $p \geq 1$.

### 3.2.1 Preliminaries

We start with some definitions.

**Definition 3.3** Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a Borel measurable function. Then $f$ is said to be a normal integrand if $\xi \to f(x,\xi)$ is lower semicontinuous for almost every $x \in \Omega$.

**Remark 3.4** The fact that we require $f$ to be Borel measurable is just to ensure that if $u : \Omega \to \mathbb{R}^N$ is a measurable function, then the function $g : \Omega \to \mathbb{R} \cup \{+\infty\}$ defined by
\[ g(x) := f(x,u(x)) \]
is measurable. This property is ensured if, for example, $f$ is (globally, meaning as a function on $\Omega \times \mathbb{R}^N$) lower semicontinuous. However it is not, in general, true if we only assume that $\xi \to f(x,\xi)$ is lower semicontinuous and $x \to f(x,\xi)$ is measurable.
The most important example of normal integrands is the following (we do not fully prove this fact and we refer to Proposition VIII.1.1 in Ekeland-Temam [264], however we prove the most important part of it in the proposition below).

**Definition 3.5** Let \( \Omega \subset \mathbb{R}^n \) be an open set and let \( f : \Omega \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \). Then \( f \) is said to be a Carathéodory function if

(i) \( \xi \to f(x, \xi) \) is continuous for almost every \( x \in \Omega \),

(ii) \( x \to f(x, \xi) \) is measurable for every \( \xi \in \mathbb{R}^N \).

**Remark 3.6** In most of the uses of the above notion, we will apply it to functions \( f : \Omega \times \mathbb{R}^m \times \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\} \), \( f = f(x, u, \xi) \). When we speak of Carathéodory functions in this context, we will consider the variable \( \xi \) as playing the role of \((u, \xi)\) and \( \mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^M \).

The first result concerns the composition of these functions with measurable ones.

**Proposition 3.7** Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( f : \Omega \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} \) be a Carathéodory function and \( u : \Omega \to \mathbb{R}^N \) be a measurable function. Then the function \( g : \Omega \to \mathbb{R} \cup \{+\infty\} \) defined by

\[
g(x) := f(x, u(x))
\]

is measurable.

**Proof.** We start by proving the result for simple functions of the form

\[
u(x) = \sum_{i=1}^{m} \alpha_i 1_{A_i}(x)
\]

where \( \alpha_i \in \mathbb{R} \), \( A_i \) are measurable disjoint sets whose union is \( \Omega \) and \( 1_{A_i} \) is the characteristic function of the set \( A_i \), namely

\[
1_{A_i}(x) := \begin{cases} 1 & \text{if } x \in A_i \\ 0 & \text{if } x \notin A_i. \end{cases}
\]

Let \( a \in \mathbb{R} \) and observe that

\[
\{x \in \Omega : g(x) < a\} = \bigcup_{i=1}^{m} \{x \in A_i : f(x, \alpha_i) < a\}.
\]

Since \( x \to f(x, \xi) \) is measurable, we deduce that the set on the right hand side is measurable and hence \( g \) is measurable.

Since any measurable function \( u \) is a limit of simple functions \( u_\nu \) and \( \xi \to f(x, \xi) \) is continuous, we deduce that for almost every \( x \in \Omega \), we have

\[
g(x) = f(x, u(x)) = \lim_{\nu \to \infty} f(x, u_\nu(x))
\]

and thus \( g \) is measurable. \( \blacksquare \)
The next theorem is a generalization of the classical Lusin theorem to Carathéodory functions (for a proof, see, for example, Ambrosio-Fusco-Pallara [25], Cesari [143], Ekeland-Temam [264] or Giusti [316], and we here follow this last proof).

**Theorem 3.8 (Scorza-Dragoni theorem)** Let $\Omega \subset \mathbb{R}^n$ be bounded and measurable, $S \subset \mathbb{R}^N$ be compact and $f : \Omega \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a Carathéodory function. Then for every $\epsilon > 0$ there exists a compact set $K_\epsilon \subset \Omega$ such that $\text{meas}(\Omega - K_\epsilon) \leq \epsilon$ and $f$ restricted to $K_\epsilon \times S$ is continuous.

**Proof.** We first define for $i \in \mathbb{N}$

$$
\omega_i(x) := \sup \{|f(x,u) - f(x,v)| : u,v \in S, |u-v| < 1/i\}.
$$

By hypothesis we know that $\omega_i \to 0$ a.e. in $\Omega$ and hence, appealing to Egorov theorem, we deduce that for every $\epsilon > 0$ we can find a compact set $K_1^\epsilon \subset \Omega$ so that

$$
\omega_i \to 0 \text{ uniformly in } K_1^\epsilon \text{ and } \text{meas}(\Omega - K_1^\epsilon) \leq \epsilon/2.
$$

In other words we can find for every $\eta > 0$ and $u \in S$, $\delta_1 = \delta_1(u,\eta) > 0$ so that for every $x \in K_1^\epsilon$ and $v \in S$ we have

$$
|u-v| < \delta_1 \Rightarrow |f(x,u) - f(x,v)| < \eta/4. \tag{3.1}
$$

We next choose a sequence $\{u_i\}_{i=1}^{\infty}$ dense in $S$. Applying Lusin theorem, we can find for every fixed $i \in \mathbb{N}$ a compact set $K_i \subset \Omega$ such that

$$
x \to f(x,u_i) \text{ is continuous in } K_i \text{ and } \text{meas}(\Omega - K_i) \leq \epsilon/2^{i+1}.
$$

Letting $K_\epsilon^2 = \bigcap K_i$ we have that, for all $i \in \mathbb{N}$,

$$
x \to f(x,u_i) \text{ is continuous in } K_\epsilon^2 \text{ and } \text{meas}(\Omega - K_\epsilon^2) \leq \epsilon/2.
$$

In other words we can find for every $\eta > 0$, $x \in K_\epsilon^2$ and $u_i$, $\delta_2 = \delta_2(x,u_i,\eta) > 0$ so that for every $y \in K_\epsilon^2$ the following implication holds

$$
|x-y| < \delta_2 \Rightarrow |f(x,u_i) - f(y,u_i)| < \eta/4. \tag{3.2}
$$

We finally let $K_\epsilon = K_1^\epsilon \cap K_\epsilon^2$. It remains to show that $f$ restricted to $K_\epsilon \times S$ is continuous. So let $\eta > 0$, $x \in K_\epsilon$ and $u \in S$. We first choose $\delta_1 = \delta_1(u,\eta)$ as in (3.1) and then $u_i$ so that

$$
|u-u_i| < \delta_1.
$$

This choice implies that if $u, v \in S$ are such that

$$
|u-v| < \delta_1
$$
and \( x, y \in K_\epsilon \), then
\[
|f(x, u) - f(x, u_i)|, |f(y, u) - f(y, u_i)|, |f(y, u) - f(y, v)| < \eta/4. \tag{3.3}
\]
We then find \( \delta_2 = \delta_2(x, u_i, \eta) \) as in (3.2) so that for every \( y \in K_\epsilon^2 \)
\[
|x - y| < \delta_2 \Rightarrow |f(x, u_i) - f(y, u_i)| < \eta/4. \tag{3.4}
\]
Finally let
\[
\delta = \delta(x, u, \eta) := \min \{ \delta_1(u, \eta), \delta_2(x, u_i, \eta) \}.
\]
Combining (3.3) and (3.4) we obtain that for every \( y \in K_\epsilon \) and \( v \in S \)
\[
|x - y| + |u - v| < \delta \Rightarrow |f(x, u) - f(y, v)| < \eta.
\]
This concludes the proof of the theorem. \( \blacksquare \)

We finally point out an important result that allows a passage from weak to strong convergence (see, for example, Theorem 3.13 in Rudin [519] or Theorem V.1.2 in Yosida [605]). It is a direct consequence of Hahn-Banach theorem.

**Theorem 3.9 (Mazur theorem)** Let \((X, \|\cdot\|)\) be a normed space and let
\[
x_\nu \rightharpoonup x \text{ in } X.
\]
Then there exists a sequence \( \{y_\mu\}_{\mu=1}^{\infty} \subset \text{co} \{x_\nu\}_{\nu=1}^{\infty} \) such that
\[
y_\mu \to x \text{ in } X.
\]
More precisely, for every \( \mu \) there exist an integer \( m_\mu \) and
\[
\alpha_\mu^i > 0 \text{ with } \sum_{i=1}^{m_\mu} \alpha_\mu^i = 1
\]
such that
\[
y_\mu = \sum_{i=1}^{m_\mu} \alpha_\mu^i x_i
\]
and
\[
\|y_\mu - x\| \to 0 \text{ as } \mu \to \infty.
\]

### 3.2.2 Some approximation lemmas

On several occasions, we will have to construct functions whose gradient essentially takes only two values. The scalar version of Lemma 3.11 will be used in Section 3.2.4, while the vectorial version will be used in Theorem 5.3. In Chapter 10, we will have some more results in the same spirit.

We start with the case where \( n = 1 \) and we recall that by \( \text{Aff}_{\text{piec}} \) we mean the set of piecewise affine functions (see Chapter 12 for details).
Lemma 3.10 \((N \geq n = 1)\) Let \(a < b, \lambda, \mu \in \mathbb{R}^N, t \in [0, 1]\),
\[\xi = t\lambda + (1 - t)\mu\]
and \(u_\xi : \mathbb{R} \to \mathbb{R}^N\) defined by
\[u_\xi(x) = \xi x.\]

For every \(\epsilon > 0\), there exist \(u \in \text{Aff}_{\text{piec}}([a, b]; \mathbb{R}^N)\) and disjoint open sets \(I_\lambda, I_\mu \subset (a, b)\) such that
\[
\text{meas } I_\lambda = t(b - a), \quad \text{meas } I_\mu = (1 - t)(b - a),
\]
\[u(a) = u_\xi(a), \quad u(b) = u_\xi(b),\]
\[\|u - u_\xi\|_{L^\infty} \leq \epsilon,
\]
\[u'(x) = \begin{cases} 
\lambda & \text{if } x \in I_\lambda \\
\mu & \text{if } x \in I_\mu.
\end{cases}
\]

\textbf{Proof.} We easily reduce the problem to the case where \(a = 0\) and \(b = 1\). We then let \(\nu \in \mathbb{N}\) and we divide the interval \((0, 1)\) in disjoint intervals of length \(2^{-\nu}\). Each of these subintervals is then further subdivided into two disjoint intervals of respective length \(t2^{-\nu}\) and \((1 - t)2^{-\nu}\). More precisely we let for \(s = 0, \ldots, 2^\nu - 1\), an integer,
\[I_{s, \nu} := \left(\frac{s}{2^\nu}, \frac{s + t}{2^\nu}\right) \quad \text{and} \quad J_{s, \nu} := \left(\frac{s + t}{2^\nu}, \frac{s + 1}{2^\nu}\right)\]
and we let
\[I_\lambda := \bigcup_{s=0}^{2^\nu - 1} I_{s, \nu}, \quad I_\mu := \bigcup_{s=0}^{2^\nu - 1} J_{s, \nu}.
\]
We next define \( \varphi : [0,1] \to \mathbb{R}^N \) (see Figure 3.1) by
\[
\varphi(x) := \begin{cases} 
(1-t)(\lambda - \mu)(x - \frac{1}{2^t}) & \text{if } x \in I_{s,\nu} \\
-t(\lambda - \mu)(x - \frac{s+1}{2t}) & \text{if } x \in J_{s,\nu}
\end{cases}
\]
we have constructed a function \( \varphi \in W^{1,\infty}_0((0,1); \mathbb{R}^N) \), in fact \( \varphi \in \text{Aff}_{\text{piece}}([0,1]; \mathbb{R}^N) \), which satisfies
\[
\varphi'(x) = \begin{cases} 
(1-t)(\lambda - \mu) & \text{if } x \in I_{\lambda} \\
-t(\lambda - \mu) & \text{if } x \in I_{\mu}.
\end{cases}
\]
Choosing \( \nu \) so that
\[
|\lambda - \mu| 2^{-\nu} \leq \epsilon
\]
and setting \( u = u_\xi + \varphi \) we have indeed established the lemma. ■

We next consider the case with several variables.

**Lemma 3.11 \((N, n \geq 1)\)** Let \( \Omega \subset \mathbb{R}^n \) be an open set with finite measure. Let \( t \in [0,1] \) and \( \alpha, \beta \in \mathbb{R}^{N \times n} \) with rank \( \{\alpha - \beta\} = 1 \). Let \( u_\xi \) be such that
\[
\nabla u_\xi(x) = \xi = t\alpha + (1-t)\beta, \quad \forall x \in \Omega.
\]
Then, for every \( \epsilon > 0 \), there exist \( u \in \text{Aff}_{\text{piece}}(\overline{\Omega}; \mathbb{R}^N) \) and disjoint open sets \( \Omega_\alpha, \Omega_\beta \subset \Omega \), so that
\[
\left\{ \begin{array}{l}
|\text{meas } \Omega_\alpha - t \cdot \text{meas } \Omega|, |\text{meas } \Omega_\beta - (1-t) \cdot \text{meas } \Omega| \leq \epsilon, \\
u \equiv u_\xi \text{ near } \partial \Omega, \|u - u_\xi\|_{L^\infty} \leq \epsilon,
\end{array} \right.
\]
\[
\nabla u(x) = \begin{cases} 
\alpha & \text{in } \Omega_\alpha \\
\beta & \text{in } \Omega_\beta
\end{cases},
\]
\[
\text{dist (}\nabla u(x), \text{co } \{\alpha, \beta\}\text{)} \leq \epsilon \quad \text{a.e. in } \Omega,
\]
where \( \text{co } \{\alpha, \beta\} = [\alpha, \beta] \) is the closed segment joining \( \alpha \) to \( \beta \).

**Remark 3.12** If \( N = 1 \) or \( n = 1 \), then the hypothesis rank \( \{\alpha - \beta\} = 1 \) is not a restriction. We can also note that, when \( n = 1 \), Lemma 3.10 gives a sharper result than the present lemma. ◊

**Proof.** We divide the proof into two steps.

*Step 1.* Let us first assume that the matrix has the form
\[
\alpha - \beta = a \otimes e_1
\]
where \( e_1 = (1,0,\ldots,0) \in \mathbb{R}^n \) and \( a \in \mathbb{R}^N \), or equivalently
\[
\alpha - \beta = (a,0,\ldots,0) = \begin{pmatrix} 
a^1 & 0 & \cdots & 0 \\
a^2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a^N & 0 & \cdots & 0
\end{pmatrix} \in \mathbb{R}^{N \times n}.
\]
We can express $\Omega$ as union of cubes with faces parallel to the coordinate axes and a set of small measure. Then, by posing $u \equiv u_\xi$ on the set of small measure, and by homotheties and translations, we can reduce ourselves to work with $\Omega$ equal to the unit cube.

Let $\Omega_\epsilon$ be an open set compactly contained in $\Omega$ and let $\eta \in \text{Aff}_{\text{piec}}(\overline{\Omega})$ and $L > 0$ be such that

$$\begin{cases}
\text{meas } (\Omega - \Omega_\epsilon) \leq \epsilon, \ \text{supp } \eta \subset \Omega \\
0 \leq \eta(x) \leq 1, \ \forall x \in \Omega \\
\eta(x) = 1, \ \forall x \in \Omega_\epsilon \\
|\nabla \eta(x)| \leq \frac{L}{\epsilon}, \ \text{a.e. } x \in \Omega - \Omega_\epsilon.
\end{cases} \quad (3.5)$$

Let us define a function $\varphi : [0, 1] \to \mathbb{R}^N$, as in Lemma 3.10 (where $\xi = 0$, $\lambda = (1 - t)a$ and $\mu = -ta$), i.e. we can find for every $\delta > 0$, $I$, $J$ disjoint open sets such that

$$\begin{cases}
\overline{I} \cup \overline{J} = [0, 1], \ I \cap J = \emptyset \\
\text{meas } I = t, \ \text{meas } J = 1 - t \\
\varphi (0) = \varphi (1) = 0, \ |\varphi(x_1)| \leq \delta, \ \forall x_1 \in (0, 1) \\
\varphi'(x_1) = \begin{cases}
(1 - t)a & \text{if } x_1 \in I \\
-ta & \text{if } x_1 \in J.
\end{cases}
\end{cases}$$

We next let $\psi : \mathbb{R}^n \to \mathbb{R}^N$ be such that

$$\psi (x) = \psi (x_1, \cdots, x_n) := \varphi (x_1)$$

which implies in particular that

$$\nabla \psi (x) = \varphi'(x_1) \otimes e_1 = \begin{cases}
(1 - t)(\alpha - \beta) & \text{in } I \times \mathbb{R}^{n-1} \\
-t(\alpha - \beta) & \text{in } J \times \mathbb{R}^{n-1}.
\end{cases}$$

We then define $u$ as a convex combination of $\psi + u_\xi$ and $u_\xi$ in the following way

$$u := \eta(\psi + u_\xi) + (1 - \eta)u_\xi = \eta \psi + u_\xi.$$ 

Choosing $\delta > 0$ sufficiently small, namely

$$\delta := \min \{ \epsilon^2, \frac{\epsilon^2}{L} \},$$

we find that $u$ satisfies the conclusions of the lemma, with

$$\Omega_\alpha := \{ x \in \Omega_\epsilon : x_1 \in I \} \ \text{and} \ \Omega_\beta := \{ x \in \Omega_\epsilon : x_1 \in J \}.$$

In fact $u \equiv u_\xi$ near $\partial \Omega$ and we have for every $x \in \Omega$

$$\| u - u_\xi \|_{L^\infty} \leq \epsilon.$$
Since in $\Omega_\varepsilon$ we have $\eta \equiv 1$ we deduce that
\[
\nabla u = \nabla \psi + \nabla u_\xi = \nabla \psi + t\alpha + (1 - t)\beta = \begin{cases} 
\alpha & \text{in } \Omega_\alpha \\
\beta & \text{in } \Omega_\beta.
\end{cases}
\]
Finally it remains to show that
\[
\text{dist} (\nabla u(x), \text{co } \{\alpha, \beta\}) \leq \varepsilon \quad \text{a.e. in } \Omega.
\]
We have that
\[
\nabla u = \eta \nabla \psi + \nabla u_\xi + \psi \otimes \nabla \eta
\]
where, by definition of $\delta$,
\[
|\psi \otimes \nabla \eta| \leq \varepsilon.
\]
Since both $\nabla \psi + \nabla u_\xi$ ($= \alpha$ or $\beta$) and $\nabla u_\xi = t\alpha + (1 - t)\beta$ belong to $\text{co } \{\alpha, \beta\}$ we obtain that
\[
\eta \nabla \psi + \nabla u_\xi = \eta (\nabla \psi + \nabla u_\xi) + (1 - \eta) \nabla u_\xi \in \text{co } \{\alpha, \beta\};
\]
since the remaining term is arbitrarily small we deduce the result i.e.,
\[
\text{dist} (\nabla u; \text{co } \{\alpha, \beta\}) \leq \varepsilon.
\]

Step 2. Let us now assume that $\alpha - \beta$ is any matrix of rank one of $\mathbb{R}^{N \times n}$ and therefore it can be written as $\alpha - \beta = a \otimes \nu$, namely
\[
(\alpha - \beta)^i_j = a^i \nu_j
\]
for a certain $a \in \mathbb{R}^N$ and $\nu \in \mathbb{R}^n$ ($\nu$ not necessarily $e_1$ as in Step 1). Replacing $a$ by $|\nu| a$ we can assume that $|\nu| = 1$. We can then find
\[
R = (r^i_j)^{1 \leq i \leq n}_{1 \leq j \leq n} \in O (n) \subset \mathbb{R}^{n \times n}
\]
(the set of orthogonal matrices, see Chapter 13) so that $\nu = e_1 R$ and hence $e_1 = \nu R^t$. We then set $\tilde{\Omega} = R \Omega$ and for $1 \leq i \leq N, 1 \leq j \leq n$ we let
\[
\tilde{\alpha}^i_j = \sum_{k=1}^n \alpha^i_k r^j_k \quad \text{and} \quad \tilde{\beta}^i_j = \sum_{k=1}^n \beta^i_k r^j_k.
\]
\[
i.e., \quad \tilde{\alpha} = \alpha R^t \quad \text{and} \quad \tilde{\beta} = \beta R^t.
\]
We observe that by construction
\[
\tilde{\alpha} - \tilde{\beta} = a \otimes e_1.
\]
Indeed, since $e_1 = \nu R^t$, we have

$$(\tilde{\alpha} - \tilde{\beta})_i = \sum_{k=1}^{n} a^i \nu_k r^j_k = a^i \sum_{k=1}^{n} \nu_k r^j_k = a^i (e_1)_j.$$  

We can therefore apply Step 1 to $\tilde{\Omega}$ and to $\tilde{u}(\xi) = u_\xi (R^t y)$ and find $\tilde{\Omega}_\alpha, \tilde{\Omega}_\beta$ and $\tilde{u} \in \text{Aff}_{\text{piece}}(\tilde{\Omega}; \mathbb{R}^N)$ with the claimed properties. By setting

$$\left\{ \begin{array}{ll}
u (x) = \tilde{u}(Rx), & x \in \Omega \\
\Omega_\alpha = R^t \tilde{\Omega}_\alpha, & \Omega_\beta = R^t \tilde{\Omega}_\beta 
\end{array} \right.$$  

we get the result, since

$$\nabla u (x) = \nabla \tilde{u}(Rx)R.$$  

3.2.3 Necessary condition: the case without lower order terms

We start with a simpler version of Theorem 3.15.

**Theorem 3.13** Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ be continuous and

$$I (u) := \int_{\Omega} f (\nabla u (x)) \, dx.$$  

Assume that there exists $u_0 \in W^{1,\infty} (\Omega; \mathbb{R}^N)$ such that

$$|I (u_0)| < \infty. \quad (3.6)$$  

If $I$ is weak $\ast$ lower semicontinuous in $W^{1,\infty} (\Omega; \mathbb{R}^N)$, then the following two results hold.

(i) For every bounded open set $D \subset \mathbb{R}^n$, for every $\xi_0 \in \mathbb{R}^{N \times n}$ and for every $\varphi \in W_0^{1,\infty} (D; \mathbb{R}^N)$,

$$\frac{1}{\text{meas } D} \int_D f (\xi_0 + \nabla \varphi (y)) \, dy \geq f (\xi_0). \quad (3.7)$$  

(ii) If either $N = 1$ or $n = 1$, then $f$ is convex.

**Remark 3.14** (i) The condition (3.6) is automatically satisfied if $\Omega$ is bounded, just choose $u_0 \equiv 0$.

(ii) In Chapter 5, we will call a function $f$ satisfying the inequality in (i) *quasiconvex*. Note that the statement (i) is valid for any $N, n \geq 1$, while (ii) is only valid in the scalar case.
Proof. (i) We divide the proof into two steps.

Step 1. In view of Lemma 3.17, we can assume that $\Omega$ is bounded. We also note that if the inequality (3.7) holds for one bounded open set $D \subset \Omega$, it holds for any such bounded open set, see Proposition 5.11. We therefore only show the result for a particular cube defined below.

Let $D$ be a cube, whose faces are parallel to the axes, contained in $\Omega$ and $\xi_0 \in \mathbb{R}^{N \times n}$. Let $\varphi \in W^{1,\infty}_0 (D; \mathbb{R}^N)$ be extended by periodicity from $D$ to $\mathbb{R}^n$, meaning that, if the edge length of $D$ is $d$,

$$\varphi(x + dz) = \varphi(x), \quad \text{for every } x \in D \text{ and } z \in \mathbb{Z}^n.$$ 

Let $\nu$ be an integer and define 

$$\varphi_{\nu}(x) := \frac{1}{\nu} \varphi(\nu x).$$ 

Since $\varphi = 0$ on $\partial D$, we have that 

$$\varphi_{\nu} \rightharpoonup 0 \text{ in } W^{1,\infty}_0 (D; \mathbb{R}^N).$$ 

Defining $\overline{u} := u_{\xi_0}$, where $u_{\xi_0}(x) := \xi_0 x$, and letting

$$u_{\nu}(x) := \begin{cases} 
\xi_0(x) & \text{if } x \in \Omega - D \\
\xi_0(x) + \frac{1}{\nu} \varphi(\nu x) & \text{if } x \in D 
\end{cases}$$

we have 

$$u_{\nu} \rightharpoonup u_{\xi_0} \text{ in } W^{1,\infty}(\Omega; \mathbb{R}^N).$$

Observe also that

$$I(u_{\nu}) = \int_{\Omega} f(\nabla u_{\nu}(x)) \, dx = \int_{\Omega - D} f(\xi_0) \, dx + \int_{D} f(\xi_0 + \nabla \varphi(\nu x)) \, dx$$

$$= f(\xi_0) \text{ meas}(\Omega - D) + \frac{1}{\nu^n} \int_{\nu D} f(\xi_0 + \nabla \varphi(y)) \, dy$$

$$= f(\xi_0) \text{ meas}(\Omega - D) + \int_{D} f(\xi_0 + \nabla \varphi(y)) \, dy,$$

where we have used in the last equality the periodicity of $\varphi$. Taking the limit in the above identity and using the weak * lower semicontinuity of $I$ we have indeed obtained

$$\frac{1}{\text{meas } D} \int_{D} f(\xi_0 + \nabla \varphi(y)) \, dy \geq f(\xi_0).$$

(ii) We want to show that

$$f(\lambda \alpha + (1 - \lambda) \beta) \leq \lambda f(\alpha) + (1 - \lambda) f(\beta) \quad (3.8)$$
for every $\alpha, \beta \in \mathbb{R}^{N \times n}$ and $\lambda \in [0, 1]$. Recall also that we are now assuming that either $N = 1$ or $n = 1$.

**Step 1.** We first construct for every $\epsilon > 0$, using Lemma 3.11 (writing $\varphi_\epsilon = u - u_\xi$), a function $\varphi_\epsilon \in \text{Aff}_{\text{piece}}(\overline{D}; \mathbb{R}^N) \subset W^{1,\infty}(D; \mathbb{R}^N)$ and disjoint open sets $D_\alpha, D_\beta \subset D$, so that

$$
\begin{cases}
|\text{meas } D_\alpha - \lambda \text{ meas } D|, |\text{meas } D_\beta - (1 - \lambda) \text{ meas } D| \leq \epsilon, \\
\varphi_\epsilon \equiv 0 \text{ near } \partial D, \|\varphi_\epsilon\|_{L^\infty} \leq \epsilon, \|\nabla \varphi_\epsilon\|_{L^\infty} \leq \gamma, \\
\nabla \varphi_\epsilon(x) = \begin{cases}
(1 - \lambda)(\alpha - \beta) & \text{in } D_\alpha \\
-\lambda(\alpha - \beta) & \text{in } D_\beta
\end{cases}
\end{cases}
$$

where $\gamma > 0$ is a constant independent of $\epsilon$.

**Step 2.** We are now in a position to show (3.8), i.e. that $f$ is convex. In view of (i) we have

$$
\frac{1}{\text{meas } D} \int_D f(\lambda \alpha + (1 - \lambda) \beta + \nabla \varphi_\epsilon(x)) \, dx \geq f(\lambda \alpha + (1 - \lambda) \beta)
$$

where $\varphi_\epsilon \in W^{1,\infty}(D; \mathbb{R}^N)$ is as in Step 1. Evaluating the integral we find

$$
\int_D f(\lambda \alpha + (1 - \lambda) \beta + \nabla \varphi_\epsilon(x)) \, dx = \int_{D_\alpha} f(\alpha) \, dx + \int_{D_\beta} f(\beta) \, dx + \int_{D - D_\alpha \cup D_\beta} f(\lambda \alpha + (1 - \lambda) \beta + \nabla \varphi_\epsilon(x)) \, dx.
$$

Letting $\epsilon \to 0$ we have indeed obtained (3.8) and thus the theorem.

### 3.2.4 Necessary condition: the general case

We now discuss the necessary condition in the general context.

**Theorem 3.15** Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ be a Carathéodory function satisfying, for almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$,

$$
|f(x, u, \xi)| \leq a(x) + b(u, \xi),
$$

where $a, b \geq 0, a \in L^1(\mathbb{R}^n)$ and $b \in C(\mathbb{R}^N \times \mathbb{R}^{N \times n})$. Let

$$
I(u) = I(u, \Omega) := \int_\Omega f(x, u(x), \nabla u(x)) \, dx,
$$

and assume that there exists $u_0 \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ such that

$$
|I(u_0, \Omega)| < \infty. \quad (3.9)
$$

If $I$ is weak * lower semicontinuous in $W^{1,\infty}(\Omega; \mathbb{R}^N)$ and if either $N = 1$ or $n = 1$, then $\xi \to f(x, u, \xi)$ is convex for almost every $x \in \Omega$ and for every $u \in \mathbb{R}^N$. 

Remark 3.16 (i) The condition (3.9) is a restriction only if $\Omega$ is of infinite measure.

(ii) If $\Omega \subset \mathbb{R}^n$ is bounded, then any sequence $u_\nu \rightharpoonup^* u$ in $W^{1,\infty}$ automatically verifies $u_\nu \rightharpoonup u$ in $W^{1,p}$, $p \geq 1$. Therefore the convexity of $f$ is also necessary for $W^{1,p}$ weak lower semicontinuity of $I$.

(iii) The theorem remains valid if the functional $I$ is lower semicontinuous for every sequence

$$u_\nu \rightharpoonup^* \pi \text{ in } W^{1,\infty}(\Omega; \mathbb{R}^N)$$

and $u_\nu \in \overline{u} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$, since in the proof of Theorem 3.15 we use such sequences.

(iv) A theorem of the above type has been proved under various kinds of hypotheses. The first to notice the importance of convexity was Tonelli [579]. Then important contributions were made by Berkowitz [79], [80], Buttazzo [112], Cacciopoli-Scorza-Dragoni [120], Cesari [139], [140], [142], [143], Ioffe [347], [348], [349], MacShane [409], Marcellini-Sbordone [428], Morrey [455] and Olech [481], [482].

The proof that we are now about to give is neither the most direct nor the easiest one, but it has the advantage of giving an important result (see Lemma 3.18) that is also valid in the vectorial case. We start with two lemmas that hold for any $n, N \geq 1$; only the last part of the proof of Theorem 3.15 will require $N = 1$ or $n = 1$.

Lemma 3.17 Let $\Omega$, $f$ and $I$ be as in the theorem. Let $O \subset \subset \Omega$ be a bounded open set and

$$I(u, O) := \int_O f(x, u(x), \nabla u(x)) \, dx.$$ 

Let $I$ be weak $*$ lower semicontinuous in $W^{1,\infty}(\Omega; \mathbb{R}^N)$. Let $u \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ and $\{u_\nu\} \subset W^{1,\infty}(O; \mathbb{R}^N)$ be such that

$$u_\nu \rightharpoonup^* u \text{ in } W^{1,\infty}(O; \mathbb{R}^N).$$ 

Then

$$\liminf_{\nu \to \infty} I(u_\nu, O) \geq I(u, O).$$

Proof. We divide the proof into three steps.

Step 1. Let $U \subset \subset O$ be open and

$$u_\nu \rightharpoonup^* u \text{ in } W^{1,\infty}(O; \mathbb{R}^N).$$

We claim that, up to the extraction of a subsequence that we do not relabel, we can find a sequence $v_\nu \in u + W_0^{1,\infty}(U; \mathbb{R}^N)$ such that

$$v_\nu \rightharpoonup^* u \text{ in } W^{1,\infty}(U; \mathbb{R}^N)$$ 

$$|I(v_\nu, U) - I(u_\nu, U)| \leq \frac{1}{\nu}.$$
Let us construct such a sequence. Since \( u_\nu \rightharpoonup u \) in \( W^{1,\infty}(O;\mathbb{R}^N) \), we can assume, up to the extraction of a subsequence (that we do not relabel), that
\[
\|u_\nu - u\|_{L^\infty} \leq 1/\nu^2.
\] (3.10)

We next choose \( U_\nu \subset \subset U \) and \( \eta \in C_0^\infty(U) \) with
\[
\left\{ \begin{array}{l}
\text{meas}(U - U_\nu) \leq 1/\nu \\
0 \leq \eta(x) \leq 1, \quad \forall x \in U \\
\eta(x) = 1, \quad \forall x \in U_\nu \\
|\nabla \eta(x)| \leq L\nu, \quad \forall x \in U - U_\nu.
\end{array} \right.
\]
where \( L > 0 \) is a constant.

We then set
\[
v_\nu := (1-\eta)u + \eta u_\nu.
\]
Note that \( v_\nu \in u + W^{1,\infty}_0(U;\mathbb{R}^N) \) and
\[
\|v_\nu - u\|_{L^\infty} \leq 1/\nu^2.
\]

Let \( \gamma > 0 \) be such that
\[
\|u_\nu\|_{W^{1,\infty}}, \|u\|_{W^{1,\infty}} \leq \gamma
\]
and observe that
\[
\|v_\nu\|_{W^{1,\infty}} \leq \gamma + \frac{L}{\nu},
\]
since (3.10) holds and
\[
\nabla v_\nu = (1-\eta)\nabla u + \eta \nabla u_\nu + \nabla \eta \otimes (u_\nu - u).
\]

We can then extract a subsequence, still denoted \( \{v_\nu\} \), with the claimed properties.

**Step 2.** Let \( O \subset \Omega \),
\[
u_\nu \rightharpoonup u \) in \( W^{1,\infty}(O;\mathbb{R}^N) \) and \( u \in W^{1,\infty}(\Omega;\mathbb{R}^N) \).

Let \( U \subset \subset O \) and let us show that
\[
\liminf_{\nu \to \infty} I(u_\nu, U) \geq I(u, U).
\]

Replacing, if necessary, \( \{u_\nu\} \) by a subsequence, we can use Step 1 to construct \( v_\nu \in u + W^{1,\infty}_0(U;\mathbb{R}^N) \) such that
\[
\nabla v_\nu \rightharpoonup u \) in \( W^{1,\infty}(U;\mathbb{R}^N)
\]
\[
|I(v_\nu, U) - I(u_\nu, U)| \leq \frac{1}{\nu}.
\] (3.11)
Next let \( \theta \in C_0^\infty(\Omega) \) be such that \( \theta \equiv 1 \) in \( U \) and define

\[
w := \theta u + (1 - \theta) u_0.
\]

Observe that \( |I(w, \Omega)| < \infty \), since (3.9) holds. Finally let

\[
w_\nu := \begin{cases} v_\nu & \text{in } U \\ w & \text{in } \Omega - U \end{cases}
\]

and note that

\[
w_\nu \rightharpoonup^* w \text{ in } W^{1,\infty}(\Omega; \mathbb{R}^N).
\]

Moreover, since

\[
I(v_\nu, U) = I(w_\nu, \Omega) - I(w, \Omega - U)
\]

we get from (3.11) and the fact that \( I(u, \Omega) \) is weak * lower semicontinuous in \( W^{1,\infty}(\Omega; \mathbb{R}^N) \),

\[
\liminf_{\nu \to \infty} I(u_\nu, U) = \liminf_{\nu \to \infty} I(v_\nu, U) = \liminf_{\nu \to \infty} [I(w_\nu, \Omega)] - I(w, \Omega - U) 
\geq I(w, \Omega) - I(w, \Omega - U) = I(u, U)
\]

which is the claim.

**Step 3.** Since \( O \) is bounded, we can choose a sequence of open sets \( O_h \subset \subset O \subset \Omega \) so that

\[
\text{meas}(O - O_h) \to 0, \text{ as } h \to \infty.
\]

We claim that for every \( \epsilon > 0 \), we can find \( h_0 = h_0(\epsilon) \), independent of \( \nu \), so that, for every \( h \geq h_0 \),

\[
|I(u_\nu, O) - I(u_\nu, O_h)| \leq \epsilon \quad \text{and} \quad |I(u, O) - I(u, O_h)| \leq \epsilon. \tag{3.12}
\]

Indeed use the bound on \( f \) to write

\[
|I(u_\nu, O) - I(u_\nu, O_h)| \leq \int_{O - O_h} |f(x, u_\nu(x), \nabla u_\nu(x))| \, dx 
\leq \int_{O - O_h} a(x) \, dx + \int_{O - O_h} b(u_\nu(x), \nabla u_\nu(x)) \, dx.
\]

Since the sequence \( u_\nu \rightharpoonup^* u \) in \( W^{1,\infty}(O; \mathbb{R}^N) \), we have, using the hypotheses on \( a \) and \( b \), the first claim in (3.12). The second one follows in the same way.

We therefore have, using Step 2 and (3.12),

\[
\liminf_{\nu \to \infty} I(u_\nu, O) \geq -\epsilon + \liminf_{\nu \to \infty} I(u_\nu, O_h) \geq -\epsilon + I(u, O_h) 
\geq -2\epsilon + I(u, O).
\]

Since \( \epsilon \) is arbitrary, we have indeed proved the lemma. \( \blacksquare \)

We continue with the following.
Lemma 3.18 Let $\Omega$, $f$ and $I$ be as in the theorem. Assume that $I$ is weak $^*$ lower semicontinuous in $W^{1,\infty}(\Omega; \mathbb{R}^N)$. Then

$$\frac{1}{\text{meas } D} \int_D f(x_0, u_0, \xi_0 + \nabla \varphi(y)) \, dy \geq f(x_0, u_0, \xi_0)$$

for every bounded open set $D \subset \mathbb{R}^n$, for almost every $x_0 \in \Omega$, for every $(u_0, \xi_0) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$ and for every $\varphi \in W^{1,\infty}_0(D; \mathbb{R}^N)$.

Remark 3.19 (i) Note once more that, contrary to Theorem 3.15, we do not assume that either $N = 1$ or $n = 1$. The lemma is therefore valid also in the vectorial case, i.e. $N, n \geq 1$. In Chapter 5, we will call a function $f$ satisfying the above inequality, quasiconvex.

(ii) The lemma remains valid if we assume the functional $I$ to be lower semicontinuous for every sequence $u_{\nu} \rightharpoonup u$ in $W^{1,\infty}(\Omega; \mathbb{R}^N)$ and $u_{\nu} \in u + W^{1,\infty}_0(\Omega; \mathbb{R}^N)$, since in the proof of Lemma 3.18, we use such sequences.

(iii) The above lemma is essentially due to Morrey [453], [455] and has been refined by Meyers [442] and Silvermann [537] for the case of continuous functions $f$ and by Acerbi-Fusco [3] for the case of Carathéodory ones. We follow this last proof.

♦

Proof. (Lemma 3.18). We start by observing that, in view of Lemma 3.17, there is no loss of generality in assuming that $\Omega$ is bounded. In fact this is possible since all limit functions $u$, that we will consider in the proof, will be affine and therefore can be extended from any $O \subset \subset \Omega$ as $v \in W^{1,\infty}(\Omega; \mathbb{R}^N)$.

As in the proof of Theorem 3.13, it is sufficient to prove the lemma only for the unit cube $D$. We proceed in three steps.

Step 1. We first fix the notations.

- Let $\varphi \in W^{1,\infty}_0(D; \mathbb{R}^N)$, $(u_0, \xi_0) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$ be given and define

$$\lambda = \lambda(u_0, \xi_0, \varphi) := \|\varphi\|_{W^{1,\infty}(D; \mathbb{R}^N)} + |\xi_0| + \sup_{x,y \in \Omega} \{ |u_0 + \xi_0 (x-y)| \}.$$ 

- We then define

$$S_\lambda := \{(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n} : |u| + |\xi| \leq \lambda \}$$

and we let

$$\gamma := \max \{ b(u, \xi) : (u, \xi) \in S_\lambda \}.$$ 

- For any $\mu \in \mathbb{N}$, we can find, by Theorem 3.8, a compact set $K_\mu \subset \Omega$ such that $\text{meas } (\Omega - K_\mu) \leq 1/\mu$ and $f$ restricted to $K_\mu \times S_\lambda$ is continuous. We next define, using Tietze extension theorem (see Rudin [518]), a continuous function

$$f_\mu : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$$
Weak lower semicontinuity

which coincide with \( f \) on \( K_\mu \times S_\lambda \) and such that

\[
|f_\mu| \leq \max \{|f(x,u,\xi)| : (x,u,\xi) \in K_\mu \times S_\lambda\}.
\]

We can also assume that for every \( \epsilon > 0 \), we have

\[
\int_{\Omega - K_\mu} |f_\mu(x,u(x)\nabla u(x))| \, dx \leq \epsilon \tag{3.13}
\]

for any \( u \in W^{1,\infty}(\Omega;\mathbb{R}^N) \). This is indeed possible by replacing, if necessary, \( f_\mu \) by \( \eta_\mu f_\mu \), where \( \eta_\mu \in C_0(\Omega) \), \( 0 \leq \eta_\mu \leq 1 \), \( \eta_\mu \equiv 1 \) on \( K_\mu \) and

\[
\int_{\Omega - K_\mu} \eta_\mu(x) \, dx \leq \epsilon / \max \{|f(x,u,\xi)| : (x,u,\xi) \in K_\mu \times S_\lambda\}.
\]

- We, in addition, can find for every \( \epsilon > 0 \), \( \mu \in \mathbb{N} \), so that for any \( \mu \geq \overline{\mu} \) the following holds

\[
\int_{\Omega - K_\mu} [a(x) + \gamma] \, dx \leq \epsilon. \tag{3.14}
\]

- We next define \( \Omega_0 \subset \Omega \) to be the set of points \( x \in \Omega \) so that

\[
x \in \bigcup_{\mu \in \mathbb{N}} K_\mu
\]

and \( x \) is a Lebesgue point of \( 1_{K_\mu} \) and \( a.1_{\Omega - K_\mu} \), for every \( \mu \in \mathbb{N} \), where

\[
1_{K_\mu}(x) := \begin{cases} 1 & \text{if } x \in K_\mu \\ 0 & \text{if } x \in \Omega - K_\mu. \end{cases}
\]

Observe that \( \text{meas}(\Omega - \Omega_0) = 0 \).

- From now on \( x_0 \) will be a given element of \( \Omega_0 \).

- For \( h \) an integer we let

\[
Q_h := x_0 + \frac{1}{h}D = \{x \in \mathbb{R}^n : (x_0)_i < x_i < (x_0)_i + 1/h, \; i = 1, \ldots, n\}.
\]

We choose \( h \) sufficiently large so that \( Q_h \subset \Omega \).

- Let \( \varphi \in W^{1,\infty}_0(D;\mathbb{R}^N) \) be fixed as above. Extend \( \varphi \) by periodicity from \( D \) to \( \mathbb{R}^n \) and define

\[
\varphi_{\nu,h}(x) := \begin{cases} \frac{1}{\nu h} \varphi(\nu h(x - x_0)) & \text{if } x \in Q_h \\ 0 & \text{if } x \notin Q_h. \end{cases} \tag{3.15}
\]

Fixing \( h \) we clearly have

\[
\varphi_{\nu,h} \rightharpoonup 0 \text{ in } W^{1,\infty}(\Omega;\mathbb{R}^N), \text{ as } \nu \to \infty.
\]
Observe that if
\[ \pi(x) := u_0 + \zeta_0(x - x_0) \quad \text{and} \quad u_\nu(x) := \pi(x) + \varphi_{\nu,h}(x) \]
we get
\[ u_\nu \rightharpoonup \pi \quad \text{in} \quad W^{1,\infty}(\Omega; \mathbb{R}^N) \]
and, for almost every \( x \in \Omega \),
\[ (\pi(x), \nabla \pi(x)), (u_\nu(x), \nabla u_\nu(x)) \in S_\lambda. \]

We then split \( Q_h \) into cubes \( Q_{h,j}^\nu \) of edge length \( 1/\nu h \) (see Figure 3.2) and denote by \( x_j, 0 \leq j \leq \nu^n - 1 \), the corner of \( Q_{h,j}^\nu \) closest to \( x_0 \). Therefore
\[ Q_h = \bigcup_{j=0}^{\nu^n-1} Q_{h,j}^\nu \quad \text{and} \quad Q_{h,j}^\nu = x_j + \frac{1}{\nu h} D. \]

Figure 3.2: Cubes \( Q_{h,j}^\nu \) and points \( x_j, 0 \leq j \leq \nu^n - 1 \)

**Step 2.** We now consider
\[
I(u_\nu) = \int_{\Omega} f(x, u_\nu(x), \nabla u_\nu(x)) \, dx
\]
\[
= \int_{\Omega - Q_h} f(x, \pi(x), \nabla \pi(x)) \, dx + \int_{Q_h} f(x, u_\nu(x), \nabla u_\nu(x)) \, dx
\]
\[
= \int_{\Omega - Q_h} f(x, \pi(x), \nabla \pi(x)) \, dx + \int_{Q_h} f_\mu(x, u_\nu(x), \nabla u_\nu(x)) \, dx
\]
\[
+ \int_{Q_h} [f(x, u_\nu(x), \nabla u_\nu(x)) - f_\mu(x, u_\nu(x), \nabla u_\nu(x))] \, dx.
\]
We can rewrite it as

$$I(\nu) = \int_{\Omega - Q_h} f(x, \overline{u}(x), \nabla \overline{u}(x)) \, dx$$

$$+ \sum_{j=0}^{\nu^n - 1} \int_{Q^*_{h,j}} f_\mu(x, u_\nu(x), \nabla u_\nu(x)) \, dx + I'_3$$

$$= \int_{\Omega - Q_h} f(x, \overline{u}(x), \nabla \overline{u}(x)) \, dx + I'_1 + I'_2 + I'_3$$

where

$$I'_1 = \sum_{j=0}^{\nu^n - 1} \int_{Q^*_{h,j}} f_\mu(x, \overline{u}(x_j), \nabla u_\nu(x)) \, dx$$

$$I'_2 = \sum_{j=0}^{\nu^n - 1} \int_{Q^*_{h,j}} [f_\mu(x, u_\nu(x), \nabla u_\nu(x)) - f_\mu(x, \overline{u}(x_j), \nabla u_\nu(x))] \, dx$$

$$I'_3 = \int_{Q_h} [f(x, u_\nu(x), \nabla u_\nu(x)) - f_\mu(x, u_\nu(x), \nabla u_\nu(x))] \, dx.$$ 

Let us estimate the last term

$$|I'_3| \leq \int_{Q_h} |f(x, u_\nu(x), \nabla u_\nu(x)) - f_\mu(x, u_\nu(x), \nabla u_\nu(x))| \, dx$$

$$\leq \int_{\Omega} |f(x, u_\nu(x), \nabla u_\nu(x)) - f_\mu(x, u_\nu(x), \nabla u_\nu(x))| \, dx.$$ 

Using the definition of $f$, $f_\mu$ and $K_\mu$ as well as (3.18) and our hypotheses we find that

$$|I'_3| \leq \int_{\Omega - K_\mu} |f(x, u_\nu(x), \nabla u_\nu(x))| \, dx + \int_{\Omega - K_\mu} |f_\mu(x, u_\nu(x), \nabla u_\nu(x))| \, dx$$

$$\leq \int_{\Omega - K_\mu} [a(x) + \gamma] \, dx + \int_{\Omega - K_\mu} |f_\mu(x, u_\nu(x), \nabla u_\nu(x))| \, dx.$$ 

This, combined with (3.13) and (3.14), finally yields that, for every $\epsilon > 0, \nu \in \mathbb{N}$ and $\mu \geq \overline{\mu}$,

$$|I'_3| \leq 2\epsilon.$$ 

(3.21)

The uniform continuity of $f_\mu$ on $\overline{Q_h} \times S_\lambda$, (3.16), (3.17) and (3.18) lead, for $h$ and $\mu$ fixed, to

$$\lim_ {\nu \to \infty} I'_2 = 0.$$ 

(3.22)
It then remains to estimate the first term in (3.20), i.e.

\[ I'_1 = \sum_{j=0}^{n-1} \int_{Q_{h,j}} f_\mu (x_j, \overline{u}(x_j), \nabla u_\nu (x)) \, dx \]

\[ = \sum_{j=0}^{n-1} \int_{x_j + \frac{1}{n} D} f_\mu (x_j, u_0 + \xi_0 (x_j - x_0), \xi_0 + \nabla \varphi (\nu h (x - x_0))) \, dx \]

\[ = \sum_{j=0}^{n-1} \frac{1}{(\nu h)^n} \int_{D} f_\mu (x_j, u_0 + \xi_0 (x_j - x_0), \xi_0 + \nabla \varphi (y + \nu h (x_j - x_0))) \, dy \]

where we have used (3.16), (3.19) and performed a change of variables \( y = \nu h (x - x_j) \). Using finally the periodicity of \( \varphi \) we find that for \( h \) and \( \mu \) fixed

\[ I'_1 = \sum_{j=0}^{n-1} \frac{1}{(\nu h)^n} \int_{D} f_\mu (x_j, u_0 + \xi_0 (x_j - x_0), \xi_0 + \nabla \varphi (y)) \, dy. \]

We then immediately deduce that, for \( h \) and \( \mu \) fixed,

\[ \lim_{\nu \to \infty} I'_1 = \int_{Q_h} \int_{D} f_\mu (x, u_0 + \xi_0 (x - x_0), \xi_0 + \nabla \varphi (y)) \, dy \, dx \]

\[ = \int_{Q_h} \int_{D} f_\mu (x, \overline{u}(x), \xi_0 + \nabla \varphi (y)) \, dy \, dx. \]  

(3.23)

Collecting (3.20), (3.22) and (3.23) and using the weak * lower semicontinuity of \( I \) we have

\[ \liminf_{\nu \to \infty} I (u_\nu) = \int_{\Omega - Q_h} f (x, \overline{u}(x), \nabla \overline{u}(x)) \, dx \]

\[ + \int_{Q_h} \int_{D} f_\mu (x, \overline{u}(x), \xi_0 + \nabla \varphi (y)) \, dy \, dx + \liminf_{\nu \to \infty} I'_3 \]

\[ \geq I (\overline{u}) = \int_{\Omega} f (x, \overline{u}(x), \nabla \overline{u}(x)) \, dx. \]

Hence letting \( \mu \to \infty \) and since \( \epsilon \) is arbitrary, we find, using (3.21),

\[ \frac{1}{\text{meas} Q_h} \int_{Q_h} \int_{D} f (x, u_0 + \xi_0 (x - x_0), \xi_0 + \nabla \varphi (y)) \, dy \, dx \]

\[ \geq \frac{1}{\text{meas} Q_h} \int_{Q_h} f (x, u_0 + \xi_0 (x - x_0), \xi_0) \, dx. \]  

(3.24)

Step 3. Denote by

\[ F (x) := \int_{D} f (x, u_0 + \xi_0 (x - x_0), \xi_0 + \nabla \varphi (y)) \, dy - f (x, u_0 + \xi_0 (x - x_0), \xi_0). \]
Therefore (3.24) is equivalent to
\[ \frac{1}{\text{meas } Q_h} \int_{Q_h} F(x) \, dx \geq 0. \] (3.25)
Since \( f \) is continuous on \( K_\mu \times S_\lambda \), we deduce that
\[ \lim_{h \to \infty} \frac{1}{\text{meas } (Q_h \cap K_\mu)} \int_{Q_h \cap K_\mu} F(x) \, dx = F(x_0). \] (3.26)
We next write
\[ \frac{1}{\text{meas } Q_h} \int_{Q_h \cap K_\mu} F = \frac{\text{meas } (Q_h \cap K_\mu)}{\text{meas } Q_h} \frac{1}{\text{meas } (Q_h \cap K_\mu)} \int_{Q_h \cap K_\mu} F = \frac{1}{\text{meas } Q_h} \int_{Q_h} 1_{K_\mu} \frac{1}{\text{meas } (Q_h \cap K_\mu)} \int_{Q_h \cap K_\mu} F. \]
Recalling that \( x_0 \in K_\mu \) is a Lebesgue point of \( 1_{K_\mu} \) and that (3.26) holds, we find, letting \( h \to \infty \),
\[ \lim_{h \to \infty} \frac{1}{\text{meas } Q_h} \int_{Q_h \cap K_\mu} F(x) \, dx = 1_{K_\mu}(x_0) F(x_0) = F(x_0). \] (3.27)
On the other hand, we have
\[ | \frac{1}{\text{meas } Q_h} \int_{Q_h \cap (Q_h \cap K_\mu)} F | = \left| \frac{1}{\text{meas } Q_h} \int_{Q_h} F(x) \, dx \right| \leq \frac{2}{\text{meas } Q_h} \int_{Q_h} [a(x) + \gamma] 1_{\Omega - K_\mu}(x) \, dx. \]
We therefore get
\[ \lim_{h \to \infty} | \frac{1}{\text{meas } Q_h} \int_{Q_h \cap (Q_h \cap K_\mu)} F \, dx | \leq 2 [a(x_0) + \gamma] 1_{\Omega - K_\mu}(x_0) = 0. \]
Combining the above estimate with (3.25) and (3.27), we have indeed obtained the claim, namely
\[ F(x_0) = \int_D f(x, u_0, \xi_0 + \nabla \varphi(y)) \, dy - f(x, u_0, \xi_0) \geq 0. \]
This finishes the proof of the lemma. \( \blacksquare \)

We now continue with the proof of the main theorem.

**Proof.** (Theorem 3.15). We want to show that
\[ f(x_0, u_0, \lambda \alpha + (1 - \lambda) \beta) \leq \lambda f(x_0, u_0, \alpha) + (1 - \lambda) f(x_0, u_0, \beta) \] (3.28)
for almost every \( x_0 \in \Omega \), every \( u_0 \in \mathbb{R}^N \), \( \alpha, \beta \in \mathbb{R}^{N \times n} \) and \( \lambda \in [0, 1] \). Recall also that we are now assuming that either \( N = 1 \) or \( n = 1 \).

We then proceed exactly as in the proof of (ii) of Theorem 3.13, using Lemma 3.18, instead of (i) of Theorem 3.13. \( \blacksquare \)
3.2.5 Sufficient condition: a particular case

We start with a simpler version of the general theorem that we will prove below (see Theorem 3.23); this simpler result will be used in the general one.

**Theorem 3.20** Let $\Omega$ be an open set of $\mathbb{R}^n$ and $q \geq 1$. Let $f : \Omega \times \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$ be a normal integrand satisfying

\[
f(x, \xi) \geq \langle a(x); \xi \rangle + b(x)
\]

for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^M$, for some $a \in L^{q'}(\Omega; \mathbb{R}^M)$, $1/q + 1/q' = 1$, $b \in L^1(\Omega)$, and where $\langle ; ; \rangle$ denotes the scalar product in $\mathbb{R}^M$. Let

\[
J(\xi) := \int_\Omega f(x, \xi(x)) \, dx.
\]

Assume that $\xi \to f(x, \xi)$ is convex and that

\[
\xi_\nu \rightharpoonup \xi \text{ in } L^q(\Omega; \mathbb{R}^M).
\]

Then

\[
\liminf_{\nu \to \infty} J(\xi_\nu) \geq J(\xi).
\]

**Remark 3.21** Since Carathéodory functions are normal integrands, the theorem applies also to such functions.

We then have as a direct consequence the following corollary that applies to the setting of the calculus of variations.

**Corollary 3.22** Let $p \geq 1$, $\Omega \subset \mathbb{R}^n$ be an open set and let

\[
f : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}, f = f(x, \xi),
\]

be a Carathéodory function satisfying

\[
f(x, \xi) \geq \langle a(x); \xi \rangle + b(x)
\]

for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^{N \times n}$, for some $a \in L^{p'}(\Omega; \mathbb{R}^{N \times n})$, $1/p + 1/p' = 1$, $b \in L^1(\Omega)$ and where $\langle ; ; \rangle$ denotes the scalar product in $\mathbb{R}^{N \times n}$. Let

\[
I(u) := \int_\Omega f(x, \nabla u(x)) \, dx.
\]

Assume that $\xi \to f(x, \xi)$ is convex and that

\[
u
\]

Then

\[
liminf_{\nu \to \infty} I(u_\nu) \geq I(\bar{u}).
\]
We now continue with the proof of Theorem 3.20.

**Proof.** We divide the proof into three steps.

*Step 1.* Observe first that if we let

\[ h(x, \xi) := \langle a(x); \xi \rangle + b(x) \]

we have, by definition of weak convergence, that

\[ \lim_{\nu \to \infty} \int_\Omega h(x, \xi_\nu(x)) \, dx = \int_\Omega h(x, \xi(x)) \, dx \]

since \( a \in L^{q'} \) and \( b \in L^1 \) and

\[ \xi_\nu \to \xi \quad \text{in} \quad L^q. \]

Writing

\[ g(x, \xi) = f(x, \xi) - h(x, \xi) \]

we have that \( g \geq 0 \) and therefore if we can show that

\[ \liminf_{\nu \to \infty} \int_\Omega g(x, \xi_\nu(x)) \, dx \geq \int_\Omega g(x, \xi(x)) \, dx \]

the theorem will follow. Therefore from now on we assume, without loss of generality, that \( f \geq 0 \).

*Step 2.* We next show that \( J \) is (strongly) lower semicontinuous. Assume that

\[ \xi_\nu \to \xi \quad \text{in} \quad L^q \]

and extract a sequence that we still label \( \{ \xi_\nu \} \) so that

\[ \xi_\nu \to \xi \quad \text{a.e.} \quad \text{.} \]

Since \( f \geq 0 \), we can apply Fatou lemma and obtain that

\[ \liminf_{\nu \to \infty} \int_\Omega f(x, \xi_\nu(x)) \, dx \geq \int_\Omega \liminf_{\nu \to \infty} f(x, \xi_\nu(x)) \, dx. \]

Combining the above inequality with the lower semicontinuity of \( f \), we have the claim.

*Step 3.* We now have to pass from (strong) lower semicontinuity to weak lower semicontinuity. First let

\[ L := \liminf_{\nu \to \infty} J(\xi_\nu) \]

and observe that \( L \geq 0 \), since \( f \geq 0 \). We may also assume that \( L < +\infty \), otherwise the theorem is trivial. Restricting our attention, if necessary, to a subsequence, we may furthermore assume that

\[ L = \lim_{\nu \to \infty} J(\xi_\nu) \]
and therefore for every $\epsilon > 0$ we can find $\nu_\epsilon$ so that for every $\nu \geq \nu_\epsilon$ the following inequality holds
\[ J (\xi_\nu) \leq L + \epsilon. \tag{3.29} \]

We next fix $\epsilon > 0$ and apply Mazur Theorem (Theorem 3.9) to find a sequence $\{\eta_\mu\}_{\mu=1}^\infty \subset \text{co} \{\xi_\nu\}_{\nu=\nu_\epsilon}$ as in the theorem and in particular so that
\[ \eta_\mu \to \xi \text{ in } L^q \tag{3.30} \]

and such that for every $\mu$ there exist an integer $\nu_\mu \geq \nu_\epsilon$ and $\alpha^i_\mu > 0$ with $\sum_{i=\nu_\epsilon}^{\nu_\mu} \alpha^i_\mu = 1$ and
\[ \eta_\mu = \sum_{i=\nu_\epsilon}^{\nu_\mu} \alpha^i_\mu \xi_i. \]

Appealing to the convexity of $\xi \to f (x, \xi)$ and to (3.29) we deduce that
\[ J (\eta_\mu) \leq \sum_{i=\nu_\epsilon}^{\nu_\mu} \alpha^i_\mu J (\xi_i) \leq L + \epsilon. \]

The above inequality combined with (3.30) and Step 2, lead to
\[ J (\xi) \leq L + \epsilon. \]

Since $\epsilon$ is arbitrary, we have the theorem. 

### 3.2.6 Sufficient condition: the general case

We now have our main lower semicontinuity result.

**Theorem 3.23** Let $\Omega$ be an open set of $\mathbb{R}^n$ and $p, q \geq 1$. Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$ be a Carathéodory function satisfying
\[ f (x, u, \xi) \geq \langle a (x) ; \xi \rangle + b (x) + c |u|^p \]

for almost every $x \in \Omega$, for every $(u, \xi) \in \mathbb{R}^m \times \mathbb{R}^M$, for some $a \in L^{q'} (\Omega; \mathbb{R}^M)$, $1/q + 1/q' = 1$, $b \in L^1 (\Omega)$, $c \in \mathbb{R}$ and where $\langle ; , ; \rangle$ denotes the scalar product in $\mathbb{R}^M$. Let
\[ J (u, \xi) := \int_\Omega f (x, u (x), \xi (x)) \, dx. \]

Assume that $\xi \to f (x, u, \xi)$ is convex and that
\[ u_\nu \to \bar{u} \text{ in } L^p (\Omega; \mathbb{R}^m) \text{ and } \xi_\nu \to \bar{\xi} \text{ in } L^q (\Omega; \mathbb{R}^M). \]

Then
\[ \liminf_{\nu \to \infty} J (u_\nu, \xi_\nu) \geq J (\bar{u}, \bar{\xi}). \]
Before making some remarks, we have the following corollary that applies to the calculus of variations.

**Corollary 3.24** Let $p \geq 1$, $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and 
$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}, \ f = f(x,u,\xi),$$
be a Carathéodory function satisfying
$$f(x,u,\xi) \geq \langle a(x); \xi \rangle + b(x) + c|u|^r$$
for almost every $x \in \Omega$, for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$, for some $a \in L^{p'}(\Omega; \mathbb{R}^{N \times n})$, $1/p + 1/p' = 1$, $b \in L^1(\Omega)$, $c \in \mathbb{R}$, $1 \leq r < np/(n-p)$ if $p < n$ and $1 \leq r < \infty$ if $p \geq n$ and where $\langle \cdot; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{N \times n}$. Let
$$I(u) := \int_{\Omega} f(x,u(x),\nabla u(x)) \, dx.$$ 
Assume that $\xi \to f(x,u,\xi)$ is convex and that 
$$u_\nu \rightharpoonup \overline{u} \ \text{in} \ W^{1,p}(\Omega; \mathbb{R}^N).$$
Then
$$\liminf_{\nu \to \infty} I(u_\nu) \geq I(\overline{u}).$$

**Remark 3.25** (i) Summarizing the results of Theorem 3.15 and Corollary 3.24 we find that a necessary and sufficient condition for $I$ to be weakly lower semicontinuous in $W^{1,p}$ is that $\xi \to f(x,u,\xi)$ be convex.

(ii) Of course both the theorem and the corollary remain valid if we replace respectively weak convergence in $L^q$ or $W^{1,p}$ by weak* convergence in $L^\infty$ or $W^{1,\infty}$. Therefore the convexity of $\xi \to f(x,u,\xi)$ implies that $I$ is weak* lower semicontinuous.

(iii) There are some advantages in proving Theorem 3.23 as stated and not restricting the functional to the case of the calculus of variations; one of the reasons will be clearer in Part II (Theorem 8.16).

(iv) The hypotheses of the theorem are nearly optimal. As mentioned above this theorem has a long history and we quote here only a few of the contributors starting with Tonelli [579]. Important contributions follow from the work of Berkowitz [79], [80], Buttazzo [112], Cesari [139], [140], [142], [143], De Giorgi [239], De Giorgi-Buttazzo-Dal Maso [242], Eisen [259], Ekeland-Temam [264], Ioffe [348], [349], MacShane [409], Marcellini-Sbordone [428], Marcus-Mizel [430], [431], Morrey [455], Olech [481], [482], Rockafellar [515], Sbordone [523] and Serrin [532], [533]. This theorem has also been generalized in many respects, and we refer to the bibliography for more details. ♦

We first prove Corollary 3.24.
Proof. It follows from Rellich theorem (see Theorem 12.12) that
\[ u_\nu \to \overline{u} \text{ in } L^r(\Omega;\mathbb{R}^N). \]

We therefore can apply Theorem 3.23 with \( q = p \) and \( p = r \) and the corollary follows.

We now proceed with the proof of Theorem 3.23 and we follow here that of
De Giorgi [239].

Proof. We decompose the proof into four steps.

Step 1. Replacing if necessary \( f \) by \( \tilde{f} \) where
\[ \tilde{f}(x,u,\xi) := f(x,u,\xi) - \langle a(x);\xi \rangle - b(x) - c \|u\|^p \]
we may assume, without loss of generality, that
\[ f(x,u,\xi) \geq 0, \quad (x,u,\xi) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^M. \]

Indeed note that
\[ N(u,\xi) := \int_\Omega \left[ \langle a(x);\xi(x) \rangle + b(x) + c \|u(x)\|^p \right] dx \]
is continuous with respect to the weak convergence of \( \xi_\nu \to \overline{\xi} \) in \( L^q \) and strong convergence of \( u_\nu \to \overline{u} \) in \( L^p \).

Step 2. Observe that if
\[ L := \liminf_{\nu \to \infty} J(u_\nu,\xi_\nu) \]
then \( L \geq 0 \), since \( f \geq 0 \). We may also assume that \( L < +\infty \), otherwise the theorem is trivial. Restricting our attention to a subsequence, if necessary, we may furthermore consider that
\[ L = \lim_{\nu \to \infty} J(u_\nu,\xi_\nu). \]

We next show that there is no loss of generality in assuming that \( \Omega \) is bounded. To emphasize the dependence on the domain let us write
\[ J(u,\xi,\Omega) := \int_\Omega f(x,u(x),\xi(x)) dx. \]

From the above consideration we have
\[ L = \lim_{\nu \to \infty} J(u_\nu,\xi_\nu,\Omega) < +\infty. \]

We next suppose that we have shown the desired lower semicontinuity result for any bounded open set \( \Omega_\mu \subset \Omega \), meaning that
\[ J(\overline{u},\overline{\xi},\Omega_\mu) \leq \liminf_{\nu \to \infty} J(u_\nu,\xi_\nu,\Omega_\mu). \]
Since \( f \geq 0 \), we obtain that
\[
J(u_\nu, \xi_\nu, \Omega_\mu) \leq J(u_\nu, \xi_\nu, \Omega)
\]
and hence
\[
J(\pi, \xi, \Omega_\mu) \leq L.
\]
Choosing then a sequence of increasing bounded open sets \( \Omega_\mu \subset \Omega \) so that \( \Omega_\mu \not\to \Omega \) and applying Lebesgue monotone convergence theorem, we get the result.

**Step 3.** So from now on we assume that \( \Omega \) is bounded, \( f \geq 0 \) and \( \lim_{\nu \to \infty} J(u_\nu, \xi_\nu) = L < +\infty \).

(Note that in the present step we will not use either the convexity of \( \xi \to f(x, u, \xi) \) or the fact that \( f \geq 0 \).)

We next fix \( \epsilon > 0 \) and we wish to show that there exists a measurable set \( \Omega_\epsilon \subset \Omega \) and a subsequence \( \nu_j \), with \( \nu_j \to \infty \), such that
\[
\text{meas} (\Omega - \Omega_\epsilon) < \epsilon \quad \text{(3.31)}
\]
We now construct \( \Omega_\epsilon \) with the property (3.31). Note first that since \( u_\nu \to \pi \) in \( L^p(\Omega) \) and \( \xi_\nu \to \xi \) in \( L^q(\Omega) \), we have that for every \( \epsilon > 0 \), there exists \( M_\epsilon > 0 \), which is independent of \( \nu \), such that if
\[
K_{1,\nu}^1 := \{ x \in \Omega : |\pi(x)| \text{ or } |u_\nu(x)| \geq M_\epsilon \}
\]
\[
K_{2,\nu}^1 := \{ x \in \Omega : |\xi_\nu(x)| \geq M_\epsilon \}
\]
then
\[
\text{meas } K_{1,\nu}^1, \text{meas } K_{2,\nu}^1 < \frac{\epsilon}{6}
\]
for every \( \nu \). Hence if
\[
\Omega_{1,\nu}^1 := \Omega - (K_{1,\nu}^1 \cup K_{2,\nu}^1)
\]
then
\[
\text{meas} (\Omega - \Omega_{1,\nu}^1) < \frac{\epsilon}{3} \quad \text{(3.32)}
\]
Since \( f \) is a Carathéodory function, there exists (see Scorza-Dragoni theorem, Theorem 3.8) \( \Omega_{2,\nu}^2 \subset \Omega_{1,\nu}^1 \) a compact set with
\[
\text{meas} (\Omega_{1,\nu}^1 - \Omega_{2,\nu}^2) < \frac{\epsilon}{3} \quad \text{(3.33)}
\]
and such that \( f \) restricted to \( \Omega_{2,\nu}^2 \times S_\epsilon \) is continuous where
\[
S_\epsilon := \{(u, \xi) \in \mathbb{R}^m \times \mathbb{R}^M : |u| < M_\epsilon \text{ and } |\xi| < M_\epsilon \}.
\]
We therefore have that there exists $\delta(\epsilon) > 0$ such that
\[
|u - v| < \delta(\epsilon) \Rightarrow |f(x, u, \xi) - f(x, v, \xi)| < \epsilon,
\]
for every $x \in \Omega^2_{\epsilon, \nu}$, every $|u|, |v| < M_\epsilon$ and $|\xi| < M_\epsilon$.

Having fixed $\delta(\epsilon)$ in this way and using the fact that $u_\nu \rightharpoonup u$ in $L^p$ we can find $\nu_\epsilon = \nu_{\epsilon, \delta(\epsilon)}$ such that if
\[
\Omega^3_{\epsilon, \nu} := \{ x \in \Omega : |u_\nu(x) - \bar{u}(x)| < \delta(\epsilon) \}
\]
then
\[
\text{meas} (\Omega - \Omega^3_{\epsilon, \nu}) < \frac{\epsilon}{3}, \text{ for every } \nu \geq \nu_\epsilon.
\]
Therefore letting
\[
\Omega_{\epsilon, \nu} := \Omega^2_{\epsilon, \nu} \cap \Omega^3_{\epsilon, \nu}
\]
we have from (3.32), (3.33), (3.34) and (3.35)
\[
\text{meas}(\Omega - \Omega_{\epsilon, \nu}) < \epsilon
\]
for every $\nu \geq \nu_\epsilon$. We now choose $\epsilon_j = \epsilon/2^j$, $j \in \mathbb{N}$. We therefore have that (3.36) holds with $\epsilon$ and $\nu_\epsilon$ replaced by $\epsilon_j, \nu_{\epsilon_j}$. We then choose any $\nu_j \geq \nu_{\epsilon_j}$ with $\lim \nu_j = \infty$ and we let
\[
\Omega_\epsilon := \bigcap_{j=1}^{\infty} \Omega_{\epsilon_j, \nu_j}.
\]
We immediately deduce (3.31) and this concludes Step 3.

**Step 4.** We are finally in a position to show the theorem. Let
\[
1_{\Omega_\epsilon}(x) := \begin{cases} 
1 & \text{if } x \in \Omega_\epsilon \\
0 & \text{if } x \in \Omega - \Omega_\epsilon.
\end{cases}
\]
Let
\[
g(x, \xi) := 1_{\Omega_\epsilon}(x) f(x, \bar{u}(x), \xi)
\]
then $g : \Omega \times \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$ is a Carathéodory function and $\xi \to g(x, \xi)$ is convex for almost every $x \in \Omega$. Applying Theorem 3.20 to
\[
G(\xi) := \int_{\Omega} g(x, \xi(x)) \, dx
\]
and to $\xi_{\nu_j} \rightharpoonup \bar{\xi}$ in $L^q(\Omega; \mathbb{R}^M)$ we get
\[
\liminf_{\nu_j \to \infty} G(\xi_{\nu_j}) = \liminf_{\nu_j \to \infty} \int_{\Omega} 1_{\Omega_\epsilon}(x) f(x, \bar{u}(x), \xi_{\nu_j}(x)) \, dx
\]
\[
\geq G(\bar{\xi}) = \int_{\Omega} 1_{\Omega_\epsilon}(x) f(x, \bar{u}(x), \bar{\xi}(x)) \, dx.
\]
(3.37)
Therefore, using (3.31), we have, for $\nu_j$ sufficiently large, that

$$\int_{\Omega_{\epsilon}} f(x, u_{\nu_j}(x), \xi_{\nu_j}(x)) \, dx$$

$$\geq \int_{\Omega_{\epsilon}} f(x, \overline{u}(x), \xi_{\nu_j}(x)) \, dx$$

$$- \int_{\Omega_{\epsilon}} \left| f(x, u_{\nu_j}(x), \xi_{\nu_j}(x)) - f(x, \overline{u}(x), \xi_{\nu_j}(x)) \right| \, dx$$

$$\geq \int_{\Omega_{\epsilon}} f(x, \overline{u}(x), \xi_{\nu_j}(x)) \, dx - \epsilon \text{meas } \Omega.$$

Combining the above inequality and the fact that $f \geq 0$, we find that

$$\int_{\Omega} f(x, u_{\nu_j}(x), \xi_{\nu_j}(x)) \, dx \geq \int_{\Omega_{\epsilon}} f(x, \overline{u}(x), \xi_{\nu_j}(x)) \, dx - \epsilon \text{meas } \Omega$$

$$= G(\xi_{\nu_j}) - \epsilon \text{meas } \Omega.$$ 

Letting $\nu_j \to \infty$ and using (3.37) we have

$$L = \liminf_{\nu_j \to \infty} \int_{\Omega} f(x, u_{\nu_j}(x), \xi_{\nu_j}(x)) \, dx$$

$$\geq \int_{\Omega} 1_{\Omega_{\epsilon}}(x) f(x, \overline{u}(x), \overline{\xi}(x)) \, dx - \epsilon \text{meas } \Omega.$$

Letting $\epsilon \to 0$, using the fact that $\text{meas } (\Omega - \Omega_{\epsilon}) \to 0$ and Lebesgue monotone convergence theorem in the right hand side of the above inequality, we have indeed obtained the theorem. ■

### 3.3 Weak continuity and invariant integrals

We show in this section that if

$$I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx$$

then $I$ is weakly continuous in $W^{1,p}$ if and only if $\xi \to f(x, u, \xi)$ is affine. In the second part of this section we show that invariant integrals (i.e. integrals that are constant whenever the boundary condition is fixed) can be fully characterized as those that are in divergence form.

#### 3.3.1 Weak continuity

Combining Theorem 3.15 and Corollary 3.24 we immediately have the following.

**Theorem 3.26** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary, $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ be a Carathéodory function satisfying, for almost every
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\[ x \in \Omega \text{ and for every } (u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}, \]

\[ |f(x, u, \xi)| \leq a(x) + b(u, \xi), \]

where \( a, b \geq 0, a \in L^1(\mathbb{R}^n) \) and \( b \in C(\mathbb{R}^N \times \mathbb{R}^{N \times n}) \). Assume that either \( N = 1 \) or \( n = 1 \) and let

\[ I(u) := \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx. \]

Then \( I \) is weak * continuous in \( W^{1,\infty}(\Omega; \mathbb{R}^N) \) if and only if \( \xi \to f(x, u, \xi) \) is affine, i.e. there exist Carathéodory functions \( g : \Omega \times \mathbb{R}^N \to \mathbb{R}^{N \times n} \) and \( h : \Omega \times \mathbb{R}^N \to \mathbb{R} \) such that

\[ f(x, u, \xi) = \langle g(x, u); \xi \rangle + h(x, u), \]

where \( \langle ; ; \rangle \) denotes the scalar product in \( \mathbb{R}^{N \times n} \).

**Remark 3.27**

(i) Similar results hold in \( W^{1,p} \), provided one imposes some restrictions on \( g \) and \( h \), in particular, that \( g(x, u) \in L^{p'}(\Omega) \) whenever \( u \in W^{1,p}(\Omega) \).

(ii) Note also that the result is strictly restricted to the scalar case. It is false in the vectorial case \( (N, n > 1) \). We will see in Chapter 5 that if \( N = n \) and

\[ I(u) = \int_{\Omega} \det \nabla u(x) \, dx \]

then \( I \) is weakly continuous although \( f(\xi) = \det \xi \) is not affine in \( \xi \).

(iii) The necessary part of the theorem remains valid if the function \( I \) is continuous for every sequence

\[ u_\nu \rightharpoonup \overline{u} \text{ in } W^{1,\infty}(\Omega; \mathbb{R}^N) \]

and \( u_\nu \in \mathbb{R}^{1,\infty}(\Omega; \mathbb{R}^N) \), since the proof is a direct consequence of Theorem 3.15.

**Proof.** The necessity follows immediately from Theorem 3.15 applied to \( f, I \) and then to \(-f, -I\) and we find

\[ f(x, u, \xi) = \langle g(x, u); \xi \rangle + h(x, u). \]

The fact that \( h \) is a Carathéodory function follows by setting \( \xi = 0 \) and use the fact that \( (x, u) \to f(x, u, 0) \) is itself a Carathéodory function. A similar argument applies to \( g \).

The sufficiency is also obvious since, if \( u_\nu \rightharpoonup \overline{u} \), in \( W^{1,\infty} \), then \( u_\nu \to \overline{u} \) in \( L^\infty \) and the conclusion follows from the fact that \( g \) and \( h \) are Carathéodory functions.
3.3.2 Invariant integrals

We now turn our attention to invariant integrals, which are important in the field theories of the calculus of variations. Following Carathéodory and Weyl [599], we give here a complete characterization of such integrals.

**Theorem 3.28** Let \( \Omega \subset \mathbb{R}^n \) be a bounded connected open set sufficiently regular so that the divergence theorem holds. Let \( f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R} \) be a \( C^\infty \) function satisfying, for every \( (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \),

\[
|f(x, u, \xi)| \leq a(x) + b(u, \xi)
\]

where \( a, b \geq 0, a \in C^\infty(\mathbb{R}^n) \) and \( b \in C^\infty(\mathbb{R}^N \times \mathbb{R}^{N \times n}) \). Let \( N = 1 \) or \( n = 1 \) and

\[
I(u) := \int_\Omega f(x, u(x), \nabla u(x)) \, dx.
\]

The following two conditions are then equivalent.

(i) \( I \) is invariant, meaning that \( I(u) = \text{constant} \), for every \( u \in u_0 + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \).

(ii) There exist \( C^\infty \) functions \( \varphi : \Omega \times \mathbb{R}^N \to \mathbb{R}^n \) and \( \beta : \Omega \to \mathbb{R} \) such that

\[
f(x, u, \xi) = \langle \varphi_u(x, u) ; \xi \rangle + \text{div}_x \varphi(x, u) + \beta(x)
\]

for every \( (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \); where \( \langle ; , \rangle \) denotes the scalar product in \( \mathbb{R}^{N \times n} \) and, writing \( \varphi_x = \frac{\partial \varphi}{\partial x} \) and \( \varphi_u = \frac{\partial \varphi}{\partial u} \),

\[
\varphi_u(x, u) = \begin{cases} 
(\varphi_1^1, \ldots, \varphi_n^1) & \text{if } N = 1 \\
(\varphi_1^2, \ldots, \varphi_n^2) & \text{if } n = 1;
\end{cases}
\]

\[
\text{div}_x \varphi(x, u) = \begin{cases} 
\sum_{i=1}^n \varphi_i^i(x, u) & \text{if } N = 1 \\
\varphi_x(x, u) & \text{if } n = 1.
\end{cases}
\]

In particular if \( \xi = \nabla u \), then

\[
f(x, u, \nabla u) = \text{div} [\varphi(x, u(x))] + \beta(x).
\]

**Remark 3.29** As mentioned above this result is strictly restricted to the scalar case; for the vectorial case, see Chapter 5 and Ericksen [265], Rund [520] and Sivaloganathan [541].

**Proof.** \((ii) \Rightarrow (i)\) Let \( f \) be as above then

\[
I(u) = \int_\Omega \beta(x) \, dx + \int_\Omega \text{div} [\varphi(x, u(x))] \, dx
\]

and since \( u = u_0 \) on \( \partial \Omega \), we have after an integration by parts that \( I \) is constant.
(i) ⇒ (ii) Following Theorem 3.26 we have that if $I$ is constant, then it is weak * continuous and therefore there exist $g$ and $h$ such that

$$f(x, u, \xi) = \langle g(x, u); \xi \rangle + h(x, u).$$

Note that since $f \in C^\infty$, then so are $g$ and $h$. We now study separately the cases $N = 1$ and $n = 1$.

**Case 1: $N = 1$.** By hypothesis we have, denoting partial derivatives by indices as for example $\partial u/\partial x_i = u_{x_i}$,

$$I(u) = \int_\Omega \left[ \sum_{i=1}^n g_i(x, u) u_{x_i} + h(x, u) \right] dx = \text{constant}.$$

Choosing $u \in u_0 + W^{1,\infty}_0(\Omega)$ and $v \in C^\infty_0(\Omega)$ we have that

$$\left. \frac{d}{d\epsilon} I(u + \epsilon v) \right|_{\epsilon = 0} = \int_\Omega \left[ \sum_{i=1}^n \left[ g_i(x, u) u_{x_i} v + g_i(x, u) v_{x_i} \right] + h_u(x, u) v \right] dx$$

Choosing $u \in u_0 + W^{1,\infty}_0(\Omega)$ and $v \in C^\infty_0(\Omega)$ we have that

$$\left. \frac{d}{d\epsilon} I(u + \epsilon v) \right|_{\epsilon = 0} = \int_\Omega \left[ \sum_{i=1}^n \left[ g_i(x, u) u_{x_i} - g_i(x, u) v_{x_i} \right] + h_u(x, u) \right] v dx$$

$$= \int_\Omega \left[ - \sum_{i=1}^n g_{x_i}^i(x, u) + h_u(x, u) \right] v dx \equiv 0.$$

Applying Theorem 3.40, we obtain that, for every $x \in \Omega$

$$h_u(x, u(x)) = \text{div}_x g(x, u(x)) = \sum_{i=1}^n g_{x_i}^i(x, u(x)). \tag{3.38}$$

Since this holds for every $u \in u_0 + W^{1,\infty}_0(\Omega)$, we deduce that the identity holds for every $x \in \Omega$ and $u \in \mathbb{R}$, namely

$$h_u(x, u) = \text{div}_x g(x, u) = \sum_{i=1}^n g_{x_i}^i(x, u).$$

Let

$$\varphi^i(x_1, \cdots, x_n, u) = \int_0^u g^i(x_1, \cdots, x_n, s) ds, \quad i = 1, \cdots, n.$$ We have that if $\varphi = (\varphi^1, \cdots, \varphi^n)$, then

$$h_u(x, u) = \text{div}_x (\varphi_u(x, u)) = [\text{div}_x \varphi(x, u)]_u$$

and thus

$$\begin{cases} h(x, u) = \beta(x) + \text{div}_x \varphi(x, u) \\ g(x, u) = \varphi_u(x, u). \end{cases}$$

This concludes the proof in the case $N = 1$.

**Case 2: $n = 1$.** We now have $\Omega = (a, b)$. The same argument leading to (3.38) gives, for every $x \in (a, b)$, every $j = 1, \cdots, N$ and every $u \in u_0 +
$W^{1,\infty}_0 ((a, b); \mathbb{R}^N)$,

$$h_{ij}(x, u(x)) - g^j_x (x, u(x)) + \sum_{i=1}^n [g^j_{ui} (x, u(x)) - g^j_{u_i} (x, u(x))](u^i(x))' = 0.$$  

This implies that, for every $(x, u) \in (a, b) \times \mathbb{R}^N$ and every $i, j = 1, \ldots, N$,

$$h_{ij}(x, u) = g^j_x (x, u) \quad \text{and} \quad g^i_{u_j} (x, u) = g^i_{u_j} (x, u). \quad (3.39)$$

Letting

$$\psi (x, u) := \int_a^x h(s, u) \, ds$$

we find from the first set of equations in (3.39) that

$$(\psi_{ij} (x, u) - g^j (x, u))_x = 0$$

and thus there exists $\gamma^j \in C^\infty (\mathbb{R}^N)$ such that

$$g^j (x, u) = \psi_{ij} (x, u) + \gamma^j (u).$$

From the second set of equations in (3.39) we find

$$\gamma^i_{u_j} (u) = \gamma^j_{u_i} (u)$$

and hence there exists $\gamma \in C^\infty (\mathbb{R}^N)$ such that

$$\gamma^j (u) = \gamma_{u^j} (u).$$

Setting

$$\varphi (x, u) := \psi (x, u) + \gamma (u) \quad \text{and} \quad \beta (x) \equiv 0$$

we have the claim. \[ \square \]

### 3.4 Existence theorems and Euler-Lagrange equations

In this section, we first show how to apply the above results to the existence of minima. We also derive the Euler-Lagrange equations under various types of conditions. We then mention some regularity results, but we omit their proofs.

#### 3.4.1 Existence theorems

We are now in a position to show the existence of minimizers for our problem.
Theorem 3.30 Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ with a Lipschitz boundary. Let \( f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\} \) be a Carathéodory function satisfying the coercivity condition
\[
f (x, u, \xi) \geq \alpha_1 |\xi|^p + \alpha_2 |u|^q + \alpha_3 (x) \tag{3.40}
\]
for almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$ and for some $\alpha_3 \in L^1(\Omega)$, $\alpha_2 \in \mathbb{R}$, $\alpha_1 > 0$ and $p > q \geq 1$. Assume that $\xi \to f (x, u, \xi)$ is convex. Let
\[
I (u) := \int_{\Omega} f (x, u(x), \nabla u(x)) \, dx.
\]
Assume that $I (u_0) < \infty$, then
\[
(P) \quad \inf \left\{ I (u) : u \in u_0 + W^{1,p}_0 (\Omega; \mathbb{R}^N) \right\}
\]
attains its minimum.

Furthermore, if $(u, \xi) \to f (x, u, \xi)$ is strictly convex for almost every $x \in \Omega$, then the minimizer is unique.

Remark 3.31 (i) The theorem is also valid in the vectorial case $N, n > 1$, but it can be extended a great deal in this case; see Chapter 8.

(ii) Of course the theorem applies to the Dirichlet integral; indeed we have that
\[
f (x, u, \xi) = f (\xi) = \frac{1}{2} |\xi|^2
\]
satisfies all the hypotheses of the theorem with $p = 2$. The natural generalization of the preceding example is
\[
f (x, u, \xi) = \frac{1}{p} |\xi|^p + g (x, u)
\]
where $g$ is continuous and non-negative and $p > 1$.

(iii) Note that the minimal surface case where
\[
f (x, u, \xi) = \sqrt{1 + |\xi|^2}
\]
is not contained in the above theorem although $f$ is convex, since the coercivity condition holds only for $p = 1$ and then $W^{1,1}$ is not a reflexive space. For the treatment of this problem we refer to Almgren [17], [19], De Giorgi [237], [238], Dierkes-Hildebrandt-Küster-Wohlrab [248], Ekeland-Temam [264], Federer [275], Giusti [315], Morrey [455], Nitsche [476] and the references quoted there.

(iv) The hypothesis $I (u_0) < \infty$ can be ensured if for example we impose a growth condition on the function $f$, such as
\[
f (x, u, \xi) \leq \beta_1 (x) + \beta_2 (|u|^p + |\xi|^p),
\]
where $\beta_1 \in L^1(\Omega)$, $\beta_2 > 0$ and $p^* = np/n - p$ if $1 < p < n$ and no condition on $p^*$ if $p \geq n$.

(v) Note that, in general, neither the convexity of $f$, nor the coercivity condition (3.40) can be weakened. Examples of non-existence of solutions in these cases occur already when $n = N = 1$ and are given in Chapter 4. ♦

**Proof.** Step 1 (Existence). Write

$$\inf \left\{ I(u) : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\} = m$$

and observe that since $I(u_0) < \infty$, we have that $m < +\infty$. Note also that because of the lower bound on $f$, $m > -\infty$.

Let $\{u_\nu\}$ be a minimizing sequence, i.e. $I(u_\nu) \to m$. We have from (3.40) that for $\nu$ sufficiently large

$$m + 1 \geq I(u_\nu) \geq \alpha_1 \|\nabla u_\nu\|_{L^p}^p - |\alpha_2| \|u_\nu\|_{L^q}^q - \int_\Omega |\alpha_3(x)| \, dx.$$

From now on we will denote by $\gamma_k > 0$ constants that are independent of $\nu$. Since by H"older inequality we have

$$\|u_\nu\|_{L^q}^q = \int_\Omega |u_\nu|^q \leq \left( \int_\Omega |u_\nu|^p \right)^{q/p} \left( \int_\Omega |dx| \right)^{(p-q)/p} = (\text{meas } \Omega)^{(p-q)/p} \|u_\nu\|_{L^p}^q,$$

we deduce that we can find constants $\gamma_1$ and $\gamma_2$ such that

$$m + 1 \geq \alpha_1 \|\nabla u_\nu\|_{L^p}^p - |\alpha_2| \|u_\nu\|_{L^q}^q - \gamma_2.$$

Invoking Poincaré inequality, we can find $\gamma_3 \ , \gamma_4 \ , \gamma_5 \ , \gamma_6$ so that

$$m + 1 \geq \gamma_3 \|u_\nu\|_{W^{1,p}}^p - \gamma_4 \|u_0\|_{W^{1,p}}^p - \gamma_1 \|u_\nu\|_{W^{1,p}}^q - \gamma_5$$

and hence, $\gamma_6$ being a constant,

$$m + 1 \geq \gamma_3 \|u_\nu\|_{W^{1,p}}^p - \gamma_1 \|u_\nu\|_{W^{1,p}}^q - \gamma_6.$$

Since $1 \leq q < p$, we can find $\gamma_7 \ , \gamma_8$ so that

$$m + 1 \geq \gamma_7 \|u_\nu\|_{W^{1,p}}^p - \gamma_8$$

which, combined with the fact that $m < \infty$, leads to the claim, namely

$$\|u_\nu\|_{W^{1,p}} \leq \gamma_9.$$

We may therefore extract a subsequence, that we still denote $\{u_\nu\}$, and find $u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N)$ so that

$$u_\nu \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N).$$
Appealing to Corollary 3.24 we get
\[ \liminf_{\nu \to \infty} I(u_\nu) \geq I(u) \]
and hence \( u \) is a minimizer of \((P)\).

Step 2 (Uniqueness). Assume that there exist \( u, v \in u_0 + W^{1,p}_0(\Omega) \) so that
\[ I(u) = I(v) = m \]
and let us prove that this implies \( u = v \). Denote by \( w = \frac{u + v}{2} \) and observe that \( w \in u_0 + W^{1,p}_0(\Omega) \). The function \( (u, \xi) \to f(x, u, \xi) \) being convex, we can infer that \( w \) is also a minimizer since
\[ m \leq I(w) \leq \frac{1}{2} I(u) + \frac{1}{2} I(v) = m, \]
which readily implies that
\[ \int_{\Omega} \left[ \frac{1}{2} f(x, u, \nabla u) + \frac{1}{2} f(x, v, \nabla v) - f(x, \frac{u + v}{2}, \frac{\nabla u + \nabla v}{2}) \right] dx = 0. \]
The convexity of \( (u, \xi) \to f(x, u, \xi) \) implies that the integrand is non-negative, while the integral is 0. This is possible only if
\[ \frac{1}{2} f(x, u, \nabla u) + \frac{1}{2} f(x, v, \nabla v) - f(x, \frac{u + v}{2}, \frac{\nabla u + \nabla v}{2}) = 0 \text{ a.e. in } \Omega. \]
We now use the strict convexity of \( (u, \xi) \to f(x, u, \xi) \) to obtain that \( u = v \) and \( \nabla u = \nabla v \text{ a.e. in } \Omega \), which implies the desired uniqueness, namely \( u = v \text{ a.e. in } \Omega \). ■

3.4.2 Euler-Lagrange equations
We now compute \( I'(u) \), the Gâteaux derivative of
\[ I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) dx. \]
We first recall some notation. We will write, for \( f = f(x, u, \xi) \),
\[ D_u f = (f_{u_1}, \ldots, f_{u_N}) \in \mathbb{R}^N, \text{ where } f_{u_i} = \partial f / \partial u^i, \ i = 1, \ldots, N \]
\[ D_\xi f = (f_{\xi_\alpha}^{i})_{1 \leq i \leq N} \in \mathbb{R}^{N \times n}, \text{ where } f_{\xi_\alpha}^{i} = \partial f / \partial \xi_\alpha^i, \ i = 1, \ldots, N, \ \alpha = 1, \ldots, n. \]
We have to consider several restrictions on the growth of \( f \) and its derivatives and we list them below.

Condition 3.32 (Growth condition on \( f \)) The function
\[ f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \ f = f(x, u, \xi), \]
is a Carathéodory function satisfying, for almost every \( x \in \Omega \), for every \((u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}\),
\[
|f(x, u, \xi)| \leq \alpha_1(x) + \beta(|u|^p + |\xi|^p)
\]
where \( \alpha_1 \in L^1(\Omega) \) and \( \beta \geq 0 \).

With the use of Sobolev imbedding theorem, we can further improve the growth condition as follows (here, for simplicity, we consider the case of bounded \( \Omega \) with a Lipschitz boundary).

- If \( p > n \), then we assume that for every \( R > 0 \), there exist \( \alpha_1 \in L^1(\Omega) \) and \( \beta = \beta(R) \geq 0 \) such that, for almost every \( x \in \Omega \) and for every \((u, \xi) \in B_R \times \mathbb{R}^{N \times n}\),
\[
|f(x, u, \xi)| \leq \alpha_1(x) + \beta |\xi|^p
\]
where \( B_R := \{u \in \mathbb{R}^N : |u| \leq R\} \).

- If \( p = n \), then there exist \( \alpha_1 \in L^1(\Omega) \) and \( \beta \geq 0 \) such that, for almost every \( x \in \Omega \) and for every \((u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}\),
\[
|f(x, u, \xi)| \leq \alpha_1(x) + \beta (|u|^q + |\xi|^p)
\]
and where \( q \geq 1 \).

- If \( 1 \leq p < n \), then there exist \( \alpha_1 \in L^1(\Omega) \) and \( \beta \geq 0 \) such that, for almost every \( x \in \Omega \) and for every \((u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}\),
\[
|f(x, u, \xi)| \leq \alpha_1(x) + \beta(|u|^{p^*} + |\xi|^p)
\]
and where \( p^* = np/(n - p) \).

We now turn our attention to the conditions on the derivatives.

**Condition 3.33 (Growth condition (I))** The functions \( f_{u_i}, f_{\xi_i} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \) are Carathéodory functions for every \( i = 1, \cdots, N, \alpha = 1, \cdots, n \). Moreover, they satisfy, for almost every \( x \in \Omega \) and for every \((u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}\),
\[
|D_u f(x, u, \xi)|, |D_\xi f(x, u, \xi)| \leq \alpha_1(x) + \beta(|u|^p + |\xi|^p)
\]
where \( \alpha_1 \in L^1(\Omega) \) and \( \beta \geq 0 \).

As before, the condition can be improved (we consider here only the case of bounded \( \Omega \) with a Lipschitz boundary) as follows.

- If \( p > n \), then we assume that for every \( R > 0 \), there exist \( \alpha_1 \in L^1(\Omega) \) and \( \beta = \beta(R) \geq 0 \) such that, for almost every \( x \in \Omega \) and for every \((u, \xi) \in B_R \times \mathbb{R}^{N \times n}\),
\[
|D_u f(x, u, \xi)|, |D_\xi f(x, u, \xi)| \leq \alpha_1(x) + \beta |\xi|^p
\]
where \( B_R := \{u \in \mathbb{R}^N : |u| \leq R\} \).
- If $p = n$, then there exist $\alpha_1 \in L^1(\Omega)$ and $\beta \geq 0$ such that, for almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$,

$$|D_u f(x, u, \xi)|, |D_\xi f(x, u, \xi)| \leq \alpha_1(x) + \beta(|u|^q + |\xi|^p)$$

and where $q \geq 1$.

- If $1 \leq p < n$, then there exist $\alpha_1 \in L^1(\Omega)$ and $\beta \geq 0$ such that, for almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$,

$$|D_u f(x, u, \xi)|, |D_\xi f(x, u, \xi)| \leq \alpha_1(x) + \beta(|u|^{p^*} + |\xi|^p)$$

and where $p^* = np/(n - p)$.

**Condition 3.34 (Growth condition (II))** The functions $f_{u^i}, f_{\xi^i} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ are Carathéodory functions for every $i = 1, \ldots, N$, $\alpha = 1, \ldots, n$. Moreover for every $R > 0$, there exist $\alpha_1 \in L^1(\Omega)$, $\alpha_2 \in L^{p/(p-1)}(\Omega)$ and $\beta = \beta(R) \geq 0$ such that, for almost every $x \in \Omega$ and for every $(u, \xi) \in B_R^N \times \mathbb{R}^{N \times n}$,

$$|D_u f(x, u, \xi)| \leq \alpha_1(x) + \beta |\xi|^p,$$

$$|D_\xi f(x, u, \xi)| \leq \alpha_2(x) + \beta |\xi|^{p-1},$$

where $B_R^N := \{u \in \mathbb{R}^N : |u| \leq R\}$. \hfill \(\diamondsuit\)

**Condition 3.35 (Growth condition (III))** The functions $f_{u^i}, f_{\xi^i} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ are Carathéodory functions for every $i = 1, \ldots, N$, $\alpha = 1, \ldots, n$. Moreover they satisfy, for almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$,

$$|D_u f(x, u, \xi)| \leq \alpha_1(x) + \beta(|u|^{p-1} + |\xi|^{p-1}),$$

$$|D_\xi f(x, u, \xi)| \leq \alpha_2(x) + \beta(|u|^{p-1} + |\xi|^{p-1}),$$

where $\alpha_1, \alpha_2 \in L^{p/(p-1)}(\Omega)$ and $\beta \geq 0$. \hfill \(\diamondsuit\)

As before, the condition can be improved (we consider here only the case of bounded $\Omega$ with a Lipschitz boundary) as follows.

- If $p > n$, then we assume that for every $R > 0$, there exist $\alpha_1 \in L^1(\Omega)$, $\alpha_2 \in L^{p/(p-1)}(\Omega)$ and $\beta = \beta(R) \geq 0$ such that, for almost every $x \in \Omega$ and for every $(u, \xi) \in B_R^N \times \mathbb{R}^{N \times n}$,

$$|D_u f(x, u, \xi)| \leq \alpha_1(x) + \beta |\xi|^p,$$

$$|D_\xi f(x, u, \xi)| \leq \alpha_2(x) + \beta |\xi|^{p-1},$$

where $B_R^N := \{u \in \mathbb{R}^N : |u| \leq R\}$. 


- If \( p = n \), then there exist \( \alpha_1 \in L^s(\Omega) \), \( \alpha_2 \in L^{p/(p-1)}(\Omega) \) and \( \beta \geq 0 \) such that, for almost every \( x \in \Omega \) and for every \( (u, \xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{N} \),

\[
|D_u f(x, u, \xi)| \leq \alpha_1(x) + \beta(|u|^{r_1} + |\xi|^{r_2})
\]

\[
|D_\xi f(x, u, \xi)| \leq \alpha_2(x) + \beta(|u|^q + |\xi|^{p-1})
\]

where \( s > 1, q, r_1 \geq 1 \) and \( 1 \leq r_2 < p \).

- If \( 1 \leq p < n \), then there exist \( \alpha_1 \in L^{np/(np-n+p)}(\Omega) \), \( \alpha_2 \in L^{p/(p-1)}(\Omega) \) and \( \beta \geq 0 \) such that, for almost every \( x \in \Omega \) and for every \( (u, \xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{N} \),

\[
|D_u f(x, u, \xi)| \leq \alpha_1(x) + \beta(|u|^{r_1} + |\xi|^{r_2})
\]

\[
|D_\xi f(x, u, \xi)| \leq \alpha_2(x) + \beta(|u|^q + |\xi|^{p-1})
\]

where \( 1 \leq q \leq (np - n) / (n - p) \), \( 1 \leq r_1 \leq (np - n + p) / (n - p) \) and \( 1 \leq r_2 \leq (np - n + p) / n \).

**Remark 3.36** (i) The conditions are more and more restrictive in the sense that

\[(III) \Rightarrow (II) \Rightarrow (I)\,.
\]

For example if \( 1 \leq p \leq n \) then (III) is a stronger hypothesis than (II), since we have only \( 1 \leq r_2 < p \) for the growth condition on \( D_u f \) for (III) while \( r_2 = p \) is allowed in (II).

Another example is the case where \( N = 1 \),

\[ f(x, u, \xi) = f(u, \xi) = a(u)|\xi|^2 \]

where \( 0 < a_1 \leq a(u), a'(u) \leq a_2 < \infty \) and \( n \geq p = 2 \). Then

\[ |D_u f(u, \xi)| \leq a_2 |\xi|^2 \]

and therefore (II) is satisfied while (III) is not.

(ii) Growth condition (III) is sometimes called *controllable growth condition* and (II) *natural growth condition*, see Giaquinta [307], Giusti [316], Ladyzhenskaya-Uraltseva [388] and Morrey [455].

We now prove the main theorem of this section which gives the weak form of the Euler-Lagrange equation. It is only based on several applications of Hölder inequality and Sobolev imbedding theorem.

**Theorem 3.37 (Weak form of Euler-Lagrange equation)** Let \( f \) be as in Condition 3.32 and for \( \varphi : \Omega \to \mathbb{R}^N \) let

\[
L(u, \varphi) := \int_\Omega \sum_{i=1}^N \{ \sum_{a=1}^n \frac{\partial f}{\partial x_a} (x, u, \nabla u) \frac{\partial \varphi^i}{\partial x_a} + \frac{\partial f}{\partial u} (x, u, \nabla u) \varphi^i \} dx
\]

\[
= \int_\Omega \{ \langle D_\xi f(x, u, \nabla u); \nabla \varphi \rangle + \langle D_u f(x, u, \nabla u); \varphi \rangle \} dx.
\]
Assume that $\bar{u} \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N)$ is a minimizer of (P), where

$$(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\}.$$

(I) If Growth condition (I) holds, then

$$(E_w) \quad L(u, \varphi) = 0 \text{ for every } \varphi \in C_0^\infty(\Omega; \mathbb{R}^N).$$

(II) If Growth condition (II) holds and in addition $u \in L^\infty(\Omega; \mathbb{R}^N)$, then

$$(E_w) \quad L(u, \varphi) = 0 \text{ for every } \varphi \in W^{1,p}_0(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N).$$

(III) If Growth condition (III) holds, then

$$(E_w) \quad L(u, \varphi) = 0 \text{ for every } \varphi \in W^{1,p}_0(\Omega; \mathbb{R}^N).$$

Conversely, if $\bar{u}$ satisfies $(E_w)$ and if $(u, \xi) \to f(x, u, \xi)$ is convex for almost every $x \in \Omega$, then $\bar{u}$ is a minimizer of $(P)$.

**Proof.** Step 1. Note first that because of the growth condition on $f$ itself we have, for every $\epsilon \in \mathbb{R}$, and every $\varphi \in W^{1,p}$, that $I(\bar{u} + \epsilon \varphi)$ is well defined.

Since $\bar{u}$ is a minimizer of $(P)$ then

$$I(\bar{u} + \epsilon \varphi) \geq I(\bar{u}),$$

for every $\varphi \in C_0^\infty(\Omega; \mathbb{R}^N)$ in (I), $\varphi \in W^{1,p}_0(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ in (II) and $\varphi \in W^{1,p}_0(\Omega; \mathbb{R}^N)$ in (III).

We thus have, if the limit exists, that (cf. below)

$$L(\bar{u}, \varphi) = \lim_{\epsilon \to 0} \frac{I(\bar{u} + \epsilon \varphi) - I(\bar{u})}{\epsilon} = 0$$

which leads to $(E_w)$, as wished.

Indeed let us show that

$$L(\bar{u}, \varphi) = \lim_{\epsilon \to 0} \frac{I(\bar{u} + \epsilon \varphi) - I(\bar{u})}{\epsilon}. \quad (3.41)$$

We first introduce the following notation

$$g(x, \epsilon) := \int_0^1 \{ \langle D_u f(x, \bar{u} + t\epsilon \varphi, \nabla \bar{u} + t\epsilon \nabla \varphi) ; \varphi \rangle + \langle D_\xi f(x, \bar{u} + t\epsilon \varphi, \nabla \bar{u} + t\epsilon \nabla \varphi) ; \nabla \varphi \rangle \} \, dt.$$

We therefore find that

$$\frac{I(\bar{u} + \epsilon \varphi) - I(\bar{u})}{\epsilon} = \frac{1}{\epsilon} \int_{\Omega} dx \int_0^1 \frac{d}{dt} \left[ f(x, \bar{u}(x) + t\epsilon \varphi(x), \nabla \bar{u}(x) + t\epsilon \nabla \varphi(x)) \right] \, dt$$

$$= \int_{\Omega} g(x, \epsilon) \, dx.$$
If we can show that there exists \( \gamma \in L^1(\Omega) \) such that for every \( \epsilon \) sufficiently small
\[
|g(x, \epsilon)| \leq \gamma(x), \text{ a.e. } x \in \Omega
\]  
we will have (3.41) by applying Lebesgue dominated convergence theorem.

So let us show (3.42). We have to consider the three cases.

**Growth condition (I).** We find, since \( \varphi \in C_0^\infty(\Omega; \mathbb{R}^N) \), that
\[
|\langle D_u f(x, \mathbf{u} + t\epsilon \varphi, \nabla \mathbf{u} + t\epsilon \nabla \varphi); \varphi \rangle| \\
\leq [\alpha_1(x) + \beta(|\mathbf{u} + t\epsilon \varphi|^p + |\nabla \mathbf{u} + t\epsilon \nabla \varphi|^p)] |\varphi| \\
|\langle D_\xi f(x, \mathbf{u} + t\epsilon \varphi, \nabla \mathbf{u} + t\epsilon \nabla \varphi); \nabla \varphi \rangle| \\
\leq [\alpha_1(x) + \beta(|\mathbf{u} + t\epsilon \varphi|^p + |\nabla \mathbf{u} + t\epsilon \nabla \varphi|^p)] |\nabla \varphi|.
\]

Summing up the two inequalities and taking the supremum in \((t, \epsilon) \in [0, 1] \times [-1, 1]\), we have indeed obtained (3.42).

**Growth condition (II).** Since \( \mathbf{u}, \varphi \in L^\infty(\Omega; \mathbb{R}^N) \), we can find \( R > 0 \) so that, for every \((t, \epsilon) \in [0, 1] \times [-1, 1]\),
\[
|\mathbf{u} + t\epsilon \varphi|, |\varphi| \leq R, \text{ a.e. } x \in \Omega.
\]

We therefore find
\[
|\langle D_u f(x, \mathbf{u} + t\epsilon \varphi, \nabla \mathbf{u} + t\epsilon \nabla \varphi); \varphi \rangle| \leq [\alpha_1(x) + \beta(|\mathbf{u} + t\epsilon \varphi|^p + |\nabla \mathbf{u} + t\epsilon \nabla \varphi|^p)] |\varphi| \\
|\langle D_\xi f(x, \mathbf{u} + t\epsilon \varphi, \nabla \mathbf{u} + t\epsilon \nabla \varphi); \nabla \varphi \rangle| \leq [\alpha_2(x) + \beta(|\mathbf{u} + t\epsilon \varphi|^p + |\nabla \mathbf{u} + t\epsilon \nabla \varphi|^p)] |\nabla \varphi|.
\]

Noting that, since \( \mathbf{u}, \varphi \in W^{1,p}(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N) \), we have by Hölder inequality
\[
\alpha_1 |\varphi|, |\nabla \mathbf{u} + t\epsilon \nabla \varphi|^p |\varphi| \in L^1(\Omega) \\
\alpha_2 |\nabla \varphi|, |\nabla \mathbf{u} + t\epsilon \nabla \varphi|^{p-1} |\nabla \varphi| \in L^1(\Omega).
\]

Summing up the two inequalities and taking the supremum in \((t, \epsilon) \in [0, 1] \times [-1, 1]\), we have indeed obtained (3.42).

**Growth condition (III).** We find
\[
|\langle D_u f(x, \mathbf{u} + t\epsilon \varphi, \nabla \mathbf{u} + t\epsilon \nabla \varphi); \varphi \rangle| \\
\leq [\alpha_1(x) + \beta(|\mathbf{u} + t\epsilon \varphi|^{p-1} + |\nabla \mathbf{u} + t\epsilon \nabla \varphi|^{p-1})] |\varphi| \\
|\langle D_\xi f(x, \mathbf{u} + t\epsilon \varphi, \nabla \mathbf{u} + t\epsilon \nabla \varphi); \nabla \varphi \rangle| \\
\leq [\alpha_2(x) + \beta(|\mathbf{u} + t\epsilon \varphi|^{p-1} + |\nabla \mathbf{u} + t\epsilon \nabla \varphi|^{p-1})] |\nabla \varphi|.
\]

Noting that, since \( \mathbf{u}, \varphi \in W^{1,p}(\Omega; \mathbb{R}^N) \), we have by Hölder inequality
\[
\alpha_1 |\varphi|, |\mathbf{u} + t\epsilon \varphi|^{p-1} |\varphi|, |\nabla \mathbf{u} + t\epsilon \nabla \varphi|^{p-1} |\varphi| \in L^1(\Omega)
\]
\[ \alpha_2 |\nabla \varphi|, |\nabla + t \epsilon \varphi|^{p-1} |\nabla \varphi|, |\nabla \pi + t \epsilon \nabla \varphi|^{p-1} |\nabla \varphi| \in L^1(\Omega). \]

Summing up the two inequalities and taking the supremum in \((t, \epsilon) \in [0, 1] \times [-1, 1]\), we have indeed obtained (3.42).

The theorem is therefore proved. The use of Sobolev imbedding theorem allows to improve the exponents, but the proof is straightforward and we do not discuss the details.

Step 2. It remains to show that, provided \((u, \xi) \rightarrow f(x, u, \xi)\) is convex, then any solution \(\bar{u}\) of \((E_w)\) is a minimizer of \((P)\). From the hypotheses on \(f\) we deduce that, for almost every \(x \in \Omega\),

\[ f(x, u, \nabla u) \geq f(x, \bar{u}, \nabla \bar{u}) + \langle D_\xi f(x, \bar{u}, \nabla \bar{u}); \nabla (u - \bar{u}) \rangle + \langle D_u f(x, \bar{u}, \nabla \bar{u}); u - \bar{u} \rangle. \]

Therefore for any \(u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N)\), we have after integration and appealing to \((E_w)\) (since \(u - \bar{u} \in W^{1,p}_0(\Omega; \mathbb{R}^N)\)) that

\[ \int_\Omega f(x, u(x), \nabla u(x)) \, dx \geq \int_\Omega f(x, \bar{u}(x), \nabla \bar{u}(x)) \, dx \]

as claimed. \(\blacksquare\)

We get as a corollary the classical form of the following equation.

**Corollary 3.38 (Euler-Lagrange equation)** Let \(f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \rightarrow \mathbb{R}\) be a \(C^2\) function. Assume that \(\bar{u} \in C^2(\bar{\Omega}; \mathbb{R}^N)\) is a minimizer of

\[ (P) \quad \inf \left\{ I(u) = \int_\Omega f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\}. \]

Then \(\bar{u}\) satisfies, for every \(x \in \Omega\),

\[ (E) \quad \sum_{\alpha=1}^n \frac{\partial}{\partial x_\alpha} \left[ \frac{\partial f}{\partial \xi_\alpha} (x, \bar{u}, \nabla \bar{u}) \right] = \frac{\partial f}{\partial u^i} (x, \bar{u}, \nabla \bar{u}), \quad i = 1, \ldots, N. \]

**Remark 3.39** (i) Note that if \(n = 1\), then \((E)\) is reduced to a system of ordinary differential equations, namely

\[ (E) \quad \frac{d}{dx} \left( \frac{\partial f}{\partial \xi^i} (x, \bar{u}, \bar{u}) \right) = \frac{\partial f}{\partial u^i} (x, \bar{u}, \bar{u}), \quad i = 1, \ldots, N. \]

If \(N = 1\) it is reduced to a single partial differential equation, given by

\[ (E) \quad \sum_{\alpha=1}^n \frac{\partial}{\partial x_\alpha} \left( \frac{\partial f}{\partial \xi_\alpha} (x, \bar{u}, \nabla \bar{u}) \right) = \frac{\partial f}{\partial u} (x, \bar{u}, \nabla \bar{u}) \]

which can be rewritten as

\[ (E) \quad \text{div} \left( D_\xi f(x, \bar{u}, \nabla \bar{u}) \right) = f_u (x, \bar{u}, \nabla \bar{u}), \]

while if \(N, n > 1\), \((E)\) is a system of partial differential equations.
(ii) Note also that if \( N = 1 \) and if \( \xi \to f(x,u,\xi) \) is convex then we must have, provided \( f \) is \( C^2 \),
\[
\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} \lambda_i \lambda_j \geq 0
\]
for every \( \lambda \in \mathbb{R}^n \), which in the context of a single partial differential equation is the usual ellipticity condition. \( \diamond \)

**Proof.** It follows at once as in the theorem that if \( \varphi \in C^\infty_0 (\Omega; \mathbb{R}^N) \) then
\[
\int_\Omega \sum_{i=1}^{N} \left\{ \sum_{\alpha=1}^{n} \frac{\partial f}{\partial \xi_\alpha} (x,\bar{u},\nabla \bar{u}) \frac{\partial \varphi^i}{\partial x_\alpha} + \frac{\partial f}{\partial u^i} (x,\bar{u},\nabla \bar{u}) \varphi^i \right\} dx = 0.
\]
Integrating by parts we get that
\[
\int_\Omega \sum_{i=1}^{N} \left\{ \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha} \left[ \frac{\partial f}{\partial \xi_\alpha} (x,\bar{u},\nabla \bar{u}) \right] - \frac{\partial f}{\partial u^i} (x,\bar{u},\nabla \bar{u}) \right\} \varphi^i dx = 0.
\]
Applying Theorem 3.40 to each component of \( \varphi \), we get the result. \( \blacksquare \)

We have used in the proof of the corollary the following classical result.

**Theorem 3.40 (Fundamental lemma of the calculus of variations)** Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( u \in L^1_{\text{loc}}(\Omega) \) be such that
\[
\int_\Omega u(x) \psi(x) dx = 0, \quad \forall \psi \in C^\infty_0 (\Omega). \tag{3.43}
\]
Then \( u = 0 \) almost everywhere in \( \Omega \).

**Proof.** We divide the proof into two steps.

*Step 1.* By approximation, in the uniform norm, of any function \( \psi \in C_0 (\Omega) \) by \( \varphi \in C^\infty_0 (\Omega) \) it is sufficient to prove the theorem assuming that
\[
\int_\Omega u(x) \psi(x) dx = 0, \quad \forall \psi \in C_0 (\Omega). \tag{3.44}
\]

*Step 2.* It clearly suffices to show the result for \( u \in L^1(O) \) where \( O \subset \subset \Omega \). Fix \( \epsilon > 0 \) and find \( v \in C^\infty_0 (O) \subset C^\infty_0 (\Omega) \) such that
\[
\|u - v\|_{L^1(O)} \leq \epsilon. \tag{3.45}
\]
We then define
\[
K := K_+ \cup K_- \quad \text{where} \quad K_\pm := \{ x \in O : \pm v(x) \geq \epsilon \}.
\]
Since \( v \) is continuous, we find that \( K_+ \) and \( K_- \) are compact, disjoint and compactly contained in \( O \). Define next \( \eta \in C(K) \) by
\[
\eta(x) := \begin{cases} 
1 & \text{if } x \in K_+ \\
-1 & \text{if } x \in K_-.
\end{cases}
\]
Extend then $\eta$ as a function in $C_0(O) \subset C_0(\Omega)$ so that $|\eta(x)| \leq 1$ for every $x \in O$, this is possible by Tietze extension theorem (cf. Rudin [518]). Applying (3.44) and (3.45) we deduce that

$$|\int_O v\eta| \leq |\int_O (v-u)\eta + \int_O u\eta| \leq \epsilon \|\eta\|_{L^\infty(O)} = \epsilon. \quad (3.46)$$

Observe now that

$$\int_O |v| = \int_{O-K} |v| + \int_K v\eta = \int_{O-K} [|v| - v\eta] + \int_O v\eta \leq 2 \int_{O-K} |v| + \epsilon \leq \epsilon [2 \text{meas}(O-K) + 1] \leq \epsilon [2 \text{meas} O + 1],$$

where we have used (3.46) and the fact that $|v| \leq \epsilon$ in $O - K$.

We therefore deduce from (3.45) and from the above inequality that

$$\|u\|_{L^1(O)} \leq \|v\|_{L^1(O)} + \|u - v\|_{L^1(O)} \leq 2\epsilon [\text{meas} O + 1]$$

and since $\epsilon$ is arbitrary, we have indeed obtained the theorem.

### 3.4.3 Some regularity results

The question of knowing if the minimizers, that we found to exist in a Sobolev space, are in fact more regular, is the 19th of the famous problems of Hilbert and there is an extensive literature on this subject and refer to Giaquinta [307], Giusti [316], Ladyzhenskaya-Uraltseva [388] and Morrey [455].

Here we just mention some results without proof. We just consider the scalar case $n > N = 1$. The case $N \geq n = 1$ is simpler and will be dealt with in Chapter 4. For the vectorial case $N, n > 1$, we refer to the bibliography. The following theorem is Theorem IX.1.1 in Giaquinta [307].

**Theorem 3.41** Let

$$I(u) := \int_\Omega f(x, u(x), \nabla u(x)) \, dx$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying

$$\alpha_1 |\xi|^p - \beta |u|^q - \gamma(x) \leq f(x, u, \xi) \leq \alpha_2 |\xi|^p + \beta |u|^q + \gamma(x)$$

where $\gamma \in L^s(\Omega)$ with $s > n/p$, $\alpha_2 \geq \alpha_1 > 0$, $\beta \geq 0$ and either $1 \leq p < n$ and $1 \leq q < \frac{np}{n-p}$ or $p = n$ and $q \geq 1$. Assume that $u \in W^{1,p}(\Omega)$ is such that

$$I(u) \leq I(u + \varphi)$$

for every $\varphi \in W^{1,p}(\Omega)$ with $\text{supp } \varphi \subset \subset \Omega$. Then $u$ is (locally) Hölder continuous (in particular, $u$ is locally bounded).
Remark 3.42 (i) When \( p > n \), the above result is trivial by the Sobolev imbedding theorem. The case \( p = n \), is simpler than when \( p < n \), see Remark 6.2 in Giusti [316]. Note also that no convexity hypothesis is required on \( f \).

(ii) When looking at the global problem

\[
(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega) \right\}
\]

it is possible to extend the above result up to the boundary. Namely (see Theorem 7.8 in Giusti [316]) if \( \Omega \subset \mathbb{R}^n \) is a bounded open set with a Lipschitz boundary and \( u_0 \) is Hölder continuous in \( \Omega \), then any minimizer is also Hölder continuous in \( \Omega \).

The result of Theorem 3.41 can now be improved if one assumes some more properties on the function \( f \). We will not prove the following theorem and we refer to the above mentioned books (see in particular Theorem 1.10.4 in Morrey [455]). The improvements with respect to Theorem 3.41 are obtained using the Euler-Lagrange equations. We again recall that we are considering here only the scalar case \( \mathbb{N} = 1 \).

\[\text{Theorem 3.43} \]

Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and \( f \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^n), \]

\[ f = f(x, u, \xi) \]

Let \( f_x = (f_{x_1}, \ldots, f_{x_n}) \), \( f_{\xi} = (f_{\xi_1}, \ldots, f_{\xi_n}) \) and similarly for the higher derivatives. Let \( f \) satisfy, for every \((x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \) and \( \lambda \in \mathbb{R}^n \),

\[
\begin{align*}
&\alpha_1 V^p - \alpha_2 \leq f(x, u, \xi) \leq \alpha_3 V^p, \\
&|f_{\xi}|, |f_{x\xi}|, |f_u|, |f_{xu}| \leq \alpha_3 V^{p-1}, |f_{u\xi}|, |f_{uu}| \leq \alpha_3 V^{p-2}, \\
&\alpha_4 V^{p-2} |\lambda|^2 \leq \sum_{i,j=1}^n f_{\xi_i\xi_j}(x, u, \xi) \lambda_i \lambda_j \leq \alpha_5 V^{p-2} |\lambda|^2
\end{align*}
\]

where \( p \geq 2 \), \( V^2 = 1 + u^2 + |\xi|^2 \) and \( \alpha_i > 0 \), \( i = 1, \ldots, 5 \), are constants (if \( f(x, u, \xi) = f(x, \xi) \) then take \( V^2 = 1 + |\xi|^2 \) in \( A \)).

Then any minimizer of

\[
(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega) \right\}
\]

is in \( C^\infty(D) \), for every \( D \subset D \subset \Omega \) and is analytic if \( f \) is analytic.

Remark 3.44 (i) The above results are strictly restricted to the scalar case, i.e. \( u : \mathbb{R}^n \rightarrow \mathbb{R}^N \) with \( N = 1 \) and they are false if \( N, n > 1 \), see the bibliography.

(ii) Note that the last condition in \( A \) is a kind of uniform convexity of \( f \) with respect to the variable \( \xi \) and it ensures the ellipticity of the Euler-Lagrange equation.
Chapter 4

The one dimensional case

4.1 Introduction

In the one dimensional case, the results of Chapter 3, can be improved a great deal. The classical methods of the calculus of variations give some important qualitative properties. Moreover, the regularity results are at the same time easier to obtain and more general.

We recall that we are considering

\[ (P) \inf \left\{ I(u) = \int_a^b f(x, u(x), u'(x)) \, dx : u \in X \right\}, \]

where \( f : (a, b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function, \( p \geq 1 \) and

\[ X := \{ u \in W^{1,p}(a, b), u(a) = \alpha, u(b) = \beta \}. \]

Most of the results that are given in the present chapter immediately extend to the case where \( u : [a, b] \to \mathbb{R}^N \) with \( N \geq 1 \).

The chapter is organized as follows.

In Section 4.2, we restate, without proof, the general existence theorem obtained in Chapter 3. We then recall the weak form of the Euler-Lagrange equation.

In Section 4.3, we briefly discuss some aspects of the classical Euler-Lagrange equation

\[ (E) \quad \frac{d}{dx} [f_x(x, \bar{u}(x), \bar{u}'(x))] = f_u(x, \bar{u}(x), \bar{u}'(x)), \quad x \in [a, b] \]

and its integrated form

\[ \frac{d}{dx} [f(x, \bar{u}(x), \bar{u}'(x)) - \bar{u}'(x) f_x(x, \bar{u}(x), \bar{u}'(x))] = f_x(x, \bar{u}(x), \bar{u}'(x)). \]
This rewriting of the equation turns out to be particularly useful when $f$ does not depend explicitly on the variable $x$. Indeed we then have a first integral of $(E)$ that is

$$f(\overline{u}'(x), \overline{u}'(x)) - \overline{u}'(x) f_{\xi}(\overline{u}(x), \overline{u}'(x)) = \text{constant}, \forall x \in [a,b].$$

We then discuss some classical examples.

In Section 4.4, we study two important inequalities, namely Poincaré-Wirtinger inequality and its generalization, Wirtinger inequality, which is equivalent to the isoperimetric inequality.

In Section 4.5, we present the Hamiltonian formulation of the problem. Roughly speaking, the idea is that the solutions of $(E)$ are also solutions (and conversely) of

$$(H) \left\{ \begin{array}{l}
  u'(x) = H_v(x, u(x), v(x)) \\
  v'(x) = -H_u(x, u(x), v(x))
\end{array} \right.$$

where $v(x) = f_{\xi}(x, u(x), u'(x))$ and $H$ is the Legendre transform of $f$, namely

$$H(x, u, v) := \max_{\xi \in \mathbb{R}} \{ v \xi - f(x, u, \xi) \}.$$ 

In classical mechanics, $f$ is called the Lagrangian and $H$ the Hamiltonian.

In Section 4.6, we prove some simple and general regularity results.

Finally, in Section 4.7, we conclude with some remarks on Lavrentiev phenomenon. This phenomenon shows that substituting the space of admissible functions by a dense one may give different values for the infimum.

We refer for more developments to the following books: Akhiezer [8], Bliss [84], Bolza [90], Buttazzo-Giaquinta-Hildebrandt [117], Carathéodory [121], Cesari [143], Courant [163], Courant-Hilbert [164], Dacorogna [180], Gelfand-Fomin [304], Giaquinta-Hildebrandt [309], Hestenes [337], Pars [490], Rund [520], Troutman [581] or Weinstock [598]. Our presentation closely follows the one in [180].

### 4.2 An existence theorem

We recall, without proof, the main existence theorem of Chapter 3.

**Theorem 4.1** Let $a < b$ and let $f : (a, b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f = f(x, u, \xi)$, be a Carathéodory function satisfying

1. $(H1)$ $\xi \to f(x, u, \xi)$ is convex for almost every $x \in (a, b)$ and every $u \in \mathbb{R}$;
2. $(H2)$ there exist $p > q \geq 1$ and $\alpha_1 > 0$, $\alpha_2, \alpha_3 \in \mathbb{R}$, such that, for almost every $x \in (a, b)$ and every $(u, \xi) \in \mathbb{R} \times \mathbb{R}$,

$$f(x, u, \xi) \geq \alpha_1 |\xi|^p + \alpha_2 |u|^q + \alpha_3.$$
An existence theorem

Let
\[ (P) \quad m := \inf \left\{ I(u) = \int_a^b f(x,u(x),u'(x)) \, dx : u \in X \right\}, \]
\[ X := \left\{ u \in W^{1,p}(a,b), \, u(a) = \alpha, \, u(b) = \beta \right\}. \]

Assume that $m < \infty$. Then there exists $\bar{u} \in X$ a minimizer of $(P)$.

Furthermore, if $(u, \xi) \to f(x,u,\xi)$ is strictly convex for almost every $x \in (a,b)$, then the minimizer is unique.

**Remark 4.2**

(i) It is easy to see that uniqueness holds under a slightly weaker condition, namely that $(u, \xi) \to f(x,u,\xi)$ is convex and either $u \to f(x,u,\xi)$ is strictly convex or $\xi \to f(x,u,\xi)$ is strictly convex.

(ii) This theorem has a long history and we refer to Chapter 3 for details. ♦

We now discuss several examples showing that the theorem is nearly optimal, but let us first start with the prototype of examples where the theorem applies.

**Example 4.3** A typical example is
\[ f(x,u,\xi) = \frac{1}{p} |\xi|^p + g(x,u), \]
where $g$ is a non-negative Carathéodory function and $p > 1$. ♦

We now discuss several counterexamples showing that neither the coercivity nor the convexity of $f$ can be weakened.

**Example 4.4** We start with an example where $f$ is neither convex nor coercive. Consider $f(\xi) = e^{-\xi^2}$ and
\[ (P) \quad \inf_{u \in X} \left\{ I(u) = \int_0^1 f(u'(x)) \, dx \right\} = m \]
where $X = W^{1,1}_0(0,1) = \left\{ u \in W^{1,1}(0,1) : u(0) = u(1) = 0 \right\}$. We now show that $(P)$ has no minimizer. Assume for a moment that $m = 0$. Then, clearly, no function $u \in X$ can satisfy
\[ \int_0^1 e^{-(u'(x))^2} \, dx = 0 \]
and hence $(P)$ has no solution. Let us now show that $m = 0$. Let $\nu \in \mathbb{N}$ and define
\[ u_\nu(x) := \nu(x - 1/2)^2 - \frac{\nu}{4} \]
then $u_\nu \in X$ and
\[ I(u_\nu) = \int_0^1 e^{-4\nu^2(x-1/2)^2} \, dx = \frac{1}{2\nu} \int_{-\nu}^{\nu} e^{-y^2} \, dy \to 0 \text{ as } \nu \to \infty. \]
Thus $m = 0$, as claimed. ♦
Example 4.5 This example is of the area type but easier and it also shows that all the hypotheses of the theorem are satisfied except the coercivity (H2) which is true with \( p = 1 \). This weakening of (H2) leads to the following counterexample. Let \( f(x, u, \xi) = f(u, \xi) = \sqrt{u^2 + \xi^2} \) and

\[
(P) \quad \inf \left\{ I(u) = \int_0^1 f(u(x), u'(x)) \, dx : u \in X \right\} = m
\]

where \( X = \{ u \in W^{1,1}(0, 1) : u(0) = 0, u(1) = 1 \} \). Let us prove that \((P)\) has no solution. We first show that \( m = 1 \) and start by observing that \( m \geq 1 \) since \( I(u) \geq \int_0^1 |u'(x)| \, dx \geq \int_0^1 u'(x) \, dx = u(1) - u(0) = 1 \).

To establish that \( m = 1 \), we construct a minimizing sequence \( u_\nu \in X \) (\( \nu \) being an integer) as follows:

\[
u(x) := \begin{cases} 0 & \text{if } x \in \left[0, 1 - \frac{1}{\nu}\right] \\ 1 + \nu(x - 1) & \text{if } x \in \left(1 - \frac{1}{\nu}, 1\right) \end{cases}
\]

We therefore have \( m = 1 \) since

\[
1 \leq I(u_\nu) = \int_{1 - \frac{1}{\nu}}^1 \sqrt{(1 + \nu(x - 1))^2 + \nu^2} \, dx \\
\leq \frac{1}{\nu} \sqrt{1 + \nu^2} \to 1 \text{ as } \nu \to \infty.
\]

Assume now, for the sake of contradiction, that there exists \( \overline{u} \in X \) a minimizer of \((P)\). We should then have, as above,

\[
1 = I(\overline{u}) = \int_0^1 \sqrt{\overline{u}^2 + \overline{u}^2} \, dx \geq \int_0^1 |\overline{u}| \, dx \\
\geq \int_0^1 \overline{u} \, dx = \overline{u}(1) - \overline{u}(0) = 1.
\]

This implies that \( \overline{u} = 0 \) a.e. in \((0, 1)\). Since elements of \( X \) are continuous, we have that \( \overline{u} \equiv 0 \) and this is incompatible with the boundary data. Thus \((P)\) has no solution.

Example 4.6 (Weierstrass example) Let \( f(x, u, \xi) = f(x, \xi) = x\xi^2 \) and

\[
(P) \quad \inf \left\{ I(u) = \int_0^1 f(x, u'(x)) \, dx : u \in X \right\} = m,
\]

where \( X = \{ u \in W^{1,2}(0, 1) : u(0) = 1, u(1) = 0 \} \). All the hypotheses of the theorem are verified with the exception of (H2), which degenerates only at one point. This is enough to show that \((P)\) has no minimizer in \( X \). Let us first show that \( m = 0 \). Let \( \nu \in \mathbb{N} \) and consider the sequence

\[
u(x) := \begin{cases} 1 & \text{if } x \in \left[0, \frac{1}{\nu}\right] \\ \frac{\log x}{\log \nu} & \text{if } x \in \left(\frac{1}{\nu}, 1\right) \end{cases}
\]
We easily have
\[ I(u_ν) = \frac{1}{\log ν} \to 0 \text{ as } ν \to ∞, \]

and hence \( m = 0 \). Now assume, by absurd hypothesis, that \( (P) \) has a solution \( \overline{u} \in X \). We should then have \( I(\overline{u}) = 0 \), but since the integrand is non-negative we deduce that \( \overline{u}' = 0 \) a.e. in \((0,1)\). Since elements of \( X \) are continuous, we have that \( \overline{u} \) is constant, and this is incompatible with the boundary data. Hence \( (P) \) has no solution.

\[ \diamond \]

**Example 4.7 (Poincaré-Wirtinger inequality)** The present example shows that we cannot allow, in general, that \( q = p \) in \((H2)\). Let \( λ > π \) and
\[ f(x,u,ξ) = f(u,ξ) = \frac{1}{2}(ξ^2 - λ^2u^2) \]

and
\[ (P) \inf \left\{ I(u) = \int_0^1 f(u(x),u'(x)) \, dx : u \in W^{1,2}_0(0,1) \right\} = m. \]

Clearly \( m = -∞ \), since, letting \( u_α(x) = α \sin πx \) with \( α \in \mathbb{R} \), we have
\[ I(u_α) = α^2 \int_0^1 \left[ π^2 \cos^2(πx) - λ^2 \sin^2(πx) \right] \, dx \to -∞ \text{ as } α \to ∞. \]

This means that \( (P) \) has no solution.

\[ \diamond \]

**Example 4.8 (Bolza example)** We now show that, as a general rule, one cannot weaken \((H1)\) either. One such example was already seen above, where we had \( f(x,u,ξ) = f(ξ) = e^{-ξ^2} \) (which satisfied neither \((H1)\) nor \((H2)\)). Let
\[ f(x,u,ξ) = f(u,ξ) = (ξ^2 - 1)^2 + u^4 \]

and
\[ (P) \inf \left\{ I(u) = \int_0^1 f(u(x),u'(x)) \, dx : u \in W^{1,4}_0(0,1) \right\} = m. \]

Assume for a moment that we already proved that \( m = 0 \) and let us show that \( (P) \) has no solution, using an argument by contradiction. Let \( \overline{u} \in W^{1,4}_0(0,1) \) be a minimizer of \( (P) \); i.e. \( I(\overline{u}) = 0 \). This implies that \( \overline{u}' = 0 \) and \( |\overline{u}'| = 1 \) a.e. in \((0,1)\). Since the elements of \( W^{1,4} \) are continuous, we have that \( \overline{u} ≡ 0 \) and hence \( \overline{u}' ≡ 0 \) which is clearly absurd.

So let us show that \( m = 0 \) by constructing an appropriate minimizing sequence. Let \( u_ν \in W^{1,4}_0(ν \geq 2 \text{ being an integer}; \text{ see Figure 4.1}) \) defined on each interval \([k/ν,(k+1)/ν]\), \( 0 ≤ k ≤ ν - 1 \), as follows
\[ u_ν(x) := \begin{cases} \frac{x - k}{ν} & \text{if } x \in \left[\frac{2k}{2ν}, \frac{2k+1}{2ν}\right] \\ -x + \frac{k+1}{ν} & \text{if } x \in \left[\frac{2k+1}{2ν}, \frac{2k+2}{2ν}\right]. \end{cases} \]
The one dimensional case

Observe that \( |u'_\nu| = 1 \) a.e. and \( |u_\nu| \leq 1/(2\nu) \), therefore leading to the desired convergence, namely

\[
0 \leq I(u_\nu) \leq \frac{1}{(2\nu)} \to 0, \text{ as } \nu \to \infty. \quad \diamondsuit
\]

The following elementary example shows that in absence of strict convexity one cannot expect uniqueness of minimizers.

**Example 4.9** Let \( X = \{ u \in W^{1,1}(0,1) : u(0) = 0, u(1) = 1 \} \) and

\[
(P) \quad \inf_{u \in X} \left\{ I(u) = \int_0^1 |u'(x)| \, dx : u \in X \right\} = m.
\]

By Jensen inequality, we have \( m = 1 \). Clearly any function \( u \in X \) with \( u' \geq 0 \),

is a minimizer of \( (P) \).

\( \diamondsuit \)

We now give two examples showing that, in general, solutions of \( (P) \) are not smooth, even if the integrand is smooth.

**Example 4.10** Let \( f(\xi) = (\xi^2 - 1)^2 \)

\[
(P) \quad \inf_{u \in X} \left\{ I(u) = \int_0^1 f(u'(x)) \, dx : u \in X \right\} = m
\]

where \( X = W^{1,1}_0(0,1) \). Clearly

\[
v(x) := \begin{cases} 
    x & \text{if } x \in [0, 1/2] \\
    1 - x & \text{if } x \in (1/2, 1]
\end{cases}
\]

is a solution since \( v \) is piecewise \( C^1 \) and satisfies \( v(0) = v(1) = 0 \) and \( I(v) = 0 \);

thus \( m = 0 \). This readily implies that \( (P) \) has no \( C^1 \) solution. Indeed \( I(u) = 0 \)
implies that $|u'| = 1$ almost everywhere and no function $u \in C^1$ can satisfy $|u'| = 1$ (since by continuity of the derivative we should have either $u' = 1$ everywhere or $u' = -1$ everywhere and this is incompatible with the boundary data).

Example 4.11 This time the integrand is convex in the variable $\xi$. Let $f(u, \xi) = u^2 (1 - \xi)^2$ and

$$
(P) \quad \inf \left\{ I(u) = \int_{-1}^{1} f(u(x), u'(x)) \, dx : u \in X \right\} = m
$$

where $X = \{ u \in W^{1,1}(-1,1) : u(-1) = 0, u(1) = 1 \}$. Observe that

$$
\bar{u}(x) := \begin{cases} 
0 & \text{if } x \in [-1,0] \\
x & \text{if } x \in (0,1]
\end{cases}
$$

is a solution of $(P)$. However, it is easy to see that $(P)$ has no $C^1$ minimizer, since $m = 0$ and no $u \in C^1([-1,1])$ can satisfy $I(u) = 0$.

4.3 The Euler-Lagrange equation

4.3.1 The classical and the weak forms

We first recall Theorem 3.37 and Corollary 3.38 of Chapter 3 applied to the present context.

Theorem 4.12 Let $a < b$, $p \geq 1$ and $f, f_u, f_\xi : (a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f = f(x, u, \xi)$, be Carathéodory functions, where $f_\xi = \partial f/\partial \xi$ and $f_u = \partial f/\partial u$. Assume that at least one of the following two hypotheses hold.

(H3) For every $R > 0$, there exist $\alpha_1 \in L^1(a,b)$, $\alpha_2 \in L^{p/(p-1)}(a,b)$ and $\beta = \beta(R)$ such that for almost every $x \in (a,b)$ and every $(u, \xi) \in [-R, R] \times \mathbb{R}$, the following inequalities hold

$$
|f(x, u, \xi)|, |f_u(x, u, \xi)| \leq \alpha_1(x) + \beta |\xi|^p,
|f_\xi(x, u, \xi)| \leq \alpha_2(x) + \beta |\xi|^{p-1}.
$$

(H3') For every $R > 0$, there exist $\alpha_1 \in L^1(a,b)$ and $\beta = \beta(R)$ such that for almost every $x \in (a,b)$ and every $(u, \xi) \in [-R, R] \times \mathbb{R}$, the following inequalities hold:

$$
|f(x, u, \xi)|, |f_u(x, u, \xi)|, |f_\xi(x, u, \xi)| \leq \alpha_1(x) + \beta |\xi|^p.
$$

Let $u \in X$ be a minimizer of

$$
(P) \quad \inf \left\{ I(u) = \int_{a}^{b} f(x, u(x), u'(x)) \, dx : u \in X \right\} = m,
$$

$$
X := \{ u \in W^{1,p}(a,b) : u(a) = \alpha, u(b) = \beta \}.
$$
If (H3) holds, then \( \mathbf{u} \) satisfies the weak form of the Euler-Lagrange equation

\[
(E_w) \quad \int_a^b \left[ f_u(x, \mathbf{u}, \mathbf{u}') \phi + f_\xi(x, \mathbf{u}, \mathbf{u}') \phi' \right] dx = 0, \quad \forall \phi \in W^{1,p}_0(a, b).
\]

If (H3') holds, then \( \mathbf{u} \) satisfies the even weaker form of the Euler-Lagrange equation

\[
(E'_w) \quad \int_a^b \left[ f_u(x, \mathbf{u}, \mathbf{u}') \phi + f_\xi(x, \mathbf{u}, \mathbf{u}') \phi' \right] dx = 0, \quad \forall \phi \in C^\infty_0(a, b).
\]

Moreover if \( f \in C^2([a, b] \times \mathbb{R} \times \mathbb{R}^n) \) and \( \mathbf{u} \in C^2([a, b]) \) then \( \mathbf{u} \) satisfies the Euler-Lagrange equation

\[
(E) \quad \frac{d}{dx} [f_\xi(x, \mathbf{u}, \mathbf{u}')] = f_u(x, \mathbf{u}, \mathbf{u}'), \quad \forall x \in [a, b].
\]

Conversely, if \((u, \xi) \to f(x, u, \xi)\) is convex for almost every \(x \in (a, b)\) and if \(\mathbf{u}\) is a solution of either \((E_w), (E'_w), \) or \((E)\) then it is a minimizer of \((P)\).

**Remark 4.13** (i) The hypothesis (H3) corresponds to Growth condition (II) or (III) in Theorem 3.37, while (H3') is just Growth condition (I). They both ensure that \(f_u\phi, f_\xi\phi' \in L^1(a, b)\). We also recall that hypothesis (H3) implies (H3').

(ii) In the statement of the theorem, we do not need hypothesis (H1) or (H2) of Theorem 4.1. Therefore we do not use the convexity of \(f\) (naturally for the converse we need the convexity of \(f\)). However, we require that a minimizer of \((P)\) does exist.

We now discuss some simple examples.

**Example 4.14** Consider the case where

\[
f(x, u, \xi) = f(\xi) = \frac{1}{p} |\xi|^p + g(x, u).
\]

The equation \((E)\) becomes

\[
\frac{d}{dx} [||\mathbf{u}'||^{p-2} \mathbf{u}'] = g_u(x, \mathbf{u}), \quad \text{in } (a, b).
\]

**Example 4.15** Let \(f(x, u, \xi) = f(\xi)\). Then the Euler-Lagrange equation is

\[
\frac{d}{dx} [f'(u')] = 0, \quad \text{i.e. } f'(u') = \text{constant}.
\]

Note that

\[
\mathbf{u}(x) = \frac{\beta - \alpha}{b - a} (x - a) + \alpha \quad (4.1)
\]

is a solution of the equation and furthermore satisfies the boundary conditions \(\mathbf{u}(a) = \alpha, \mathbf{u}(b) = \beta\). It is not, however, always a minimizer of \((P)\), as was seen in Example 4.4.
If $f$ is convex the above $\pi$ is indeed a minimizer. This follows from the theorem but it can be seen in a more elementary way (which is also valid even if $f \in C^0(\mathbb{R})$). From Jensen inequality, it follows that for any $u \in W^{1,\infty}(a,b)$ with $u(a) = \alpha$, $u(b) = \beta$

$$\frac{1}{b-a} \int_a^b f(u'(x)) \, dx \geq f\left( \frac{1}{b-a} \int_a^b u'(x) \, dx \right) = f\left( \frac{u(b) - u(a)}{b-a} \right)$$

$$= f\left( \frac{\beta - \alpha}{b-a} \right) = f\left( \frac{\pi}{b-a} \right)$$

$$= \frac{1}{b-a} \int_a^b f\left( \frac{\pi}{b-a} \right) \, dx,$$

which is the claim.

(ii) Let $f(x, u, \xi) = f(x, \xi)$. The Euler-Lagrange equation is then

$$\frac{d}{dx} [f_\xi(x, u')] = 0, \text{ i.e. } f_\xi(x, u') = \text{constant}.$$

The equation is already harder to solve than the preceding one and, in general, it does not have as simple a solution as in (4.1), see Example 4.33.

We continue with two classical examples.

**Example 4.16 (Brachistochrone)** The function under consideration is

$$f(u, \xi) = \frac{\sqrt{1 + \xi^2}}{u}$$

and

$$\inf_{u \in X} \left\{ I(u) = \int_0^b f(u(x), u'(x)) \, dx \right\} = m$$

where

$$X := \left\{ u \in W^{1,1}(0,b) : u(0) = 0, u(b) = \beta \text{ and } u(x) > 0, \forall x \in (0,b) \right\}.$$ 

The Euler-Lagrange equation and its first integral (see Theorem 4.20 below) are

$$\left[ \frac{u'}{\sqrt{u\sqrt{1 + u'^2}}} \right]' = -\frac{\sqrt{1 + u'^2}}{2u^3},$$

$$\sqrt{1 + u'^2} = u'\left[ \frac{u'}{\sqrt{u\sqrt{1 + u'^2}}} \right] = \text{constant}.$$

This leads ($\mu$ being a positive constant) to

$$u(1 + u'^2) = 2\mu.$$
The solution is a cycloid and is given in implicit form by
\[ u(x) = \mu \left( 1 - \cos \theta^{-1}(x) \right) \]
where
\[ \theta(t) = \mu(t - \sin t) . \]
Note that \( u(0) = 0 \). It therefore remains to choose \( \mu \) so that \( u(b) = \beta \).

\[ \text{Example 4.17 (Minimal surfaces of revolution)} \]
The function under consideration is \( f(u, \xi) = 2\pi u \sqrt{1 + \xi^2} \) and the minimization problem (which corresponds to minimization of the area of a surface of revolution) is
\[
(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(u(x), u'(x)) \, dx \right\} = m,
\]
where, for \( \alpha, \beta > 0 \), we set
\[ X := \{ u \in W^{1,1}(a, b) : u(a) = \alpha, \; u(b) = \beta, \; u > 0 \} \].

The Euler-Lagrange equation and its first integral (see Theorem 4.20 below) are
\[
\left[ \frac{u'u}{\sqrt{1 + u'^2}} \right]' = \sqrt{1 + u'^2} \iff u''u = 1 + u'^2,
\]
\[ u \sqrt{1 + u'^2} - u' \frac{u'u}{\sqrt{1 + u'^2}} = \lambda = \text{constant}. \]
This leads to
\[ u'^2 = \frac{u^2}{\lambda^2} - 1. \]
The solutions, if they exist (this depends on \( a, b, \alpha \) and \( \beta \)), are of the form (\( \mu \) being a constant)
\[ u(x) = \lambda \cosh\left( \frac{x}{\lambda} + \mu \right). \]

It is clear that if the function \( (u, \xi) \to f(x, u, \xi) \) is not convex for every \( x \in [a, b] \), then, in general, a solution of the Euler-Lagrange equation \( (E) \) is not a minimizer of \( (P) \). However, an important part of the classical calculus of variations is devoted to the fields theories, which sometimes allows us in the absence of the convexity of \( (u, \xi) \to f(x, u, \xi) \) to prove that a solution of the Euler-Lagrange equation is a minimizer of \( (P) \). We do not discuss this approach here, but we try to explain the nature of the theory with a particularly simple case that is given in the next theorem and that turns out to be useful in Section 4.4.1.

\[ \text{Theorem 4.18} \]
Let \( f \in C^2([a, b] \times \mathbb{R} \times \mathbb{R}) \). If there exists \( \Phi \in C^3([a, b] \times \mathbb{R}) \) with \( \Phi(a, \alpha) = \Phi(b, \beta) \) such that
\[ (u, \xi) \to \tilde{f}(x, u, \xi) \text{ is convex for every } x \in [a, b], \]
The Euler-Lagrange equation

where
\[ \tilde{f}(x, u, \xi) := f(x, u, \xi) + \Phi_u(x, u) \xi + \Phi_x(x, u); \]

then any solution \( \tilde{\pi} \) of \( (E) \) is a minimizer of \( (P) \).

Remark 4.19 We should immediately point out that in order to have \((u, \xi) \to \tilde{f}(x, u, \xi) \) convex for every \( x \in [a, b] \), we should, at least, have that \( \xi \to f(x, u, \xi) \) is convex for every \((x, u) \in [a, b] \times \mathbb{R} \). If \((u, \xi) \to f(x, u, \xi) \) is already convex, then choose \( \Phi \equiv 0 \) and apply Theorem 4.12.

Proof. Define
\[ \varphi(x, u, \xi) := \Phi_u(x, u) \xi + \Phi_x(x, u). \]

Observe that the two following identities (the first one uses that \( \Phi(a, \alpha) = \Phi(b, \beta) \) and the second one is just straight differentiation)
\[
\int_a^b \frac{d}{dx} \left[ \Phi(x, u(x)) \right] \, dx = \Phi(b, \beta) - \Phi(a, \alpha) = 0
\]
\[
\frac{d}{dx} \left[ \varphi_x(x, u, u') \right] = \varphi_u(x, u, u'), \quad x \in [a, b]
\]
hold for any \( u \in X = \{ u \in C^1([a, b]) : u(a) = \alpha, u(b) = \beta \} \). The first identity expresses that the integral is invariant, while the second one says that \( \varphi(x, u, u') \) satisfies the Euler-Lagrange equation identically (it is then called a null Lagrangian).

With the help of the above observations we immediately obtain the result by applying Theorem 4.12 to \( \tilde{f} \). Indeed we have that \((u, \xi) \to \tilde{f}(x, u, \xi) \) is convex,
\[
I(u) = \int_a^b \tilde{f}(x, u(x), u'(x)) \, dx = \int_a^b f(x, u(x), u'(x)) \, dx
\]
for every \( u \in X \) and any solution \( \tilde{\pi} \) of \( (E) \) satisfies also
\[
(\tilde{E}) \quad \frac{d}{dx} \left[ \tilde{f}_x(x, \tilde{\pi}, \tilde{\pi}') \right] = \tilde{f}_u(x, \tilde{\pi}, \tilde{\pi}'), \quad x \in (a, b).
\]

This concludes the proof. \( \blacksquare \)

### 4.3.2 Second form of the Euler-Lagrange equation

The next theorem gives a different way of expressing the Euler-Lagrange equation, this new equation is sometimes called the DuBois-Reymond equation. It turns out to be an important help when \( f \) does not depend explicitly on \( x \), as already seen in some of the above examples.

Theorem 4.20 Let \( f \in C^2([a, b] \times \mathbb{R} \times \mathbb{R}) \), \( f = f(x, u, \xi) \), and
\[
(P) \quad \inf_{u \in X} \left\{ I(u) = \int_a^b f(x, u(x), u'(x)) \, dx \right\} = m,
\]
where \( X = \{ u \in C^1([a, b]) : u(a) = \alpha, u(b) = \beta \} \). Let \( u \in X \cap C^2([a, b]) \) be a minimizer of \( (P) \). Then, for every \( x \in [a, b] \), the following equation holds:
\[
\frac{d}{dx} [f(x, u(x), u'(x)) - u'(x) f_\xi(x, u(x), u'(x))] = f_x(x, u(x), u'(x)). \tag{4.2}
\]

**Remark 4.21** The theorem is particularly interesting when \( f \) does not depend explicitly on \( x \), namely \( f = f(u, \xi) \). We then have
\[
f(u(x), u'(x)) - u'(x) f_\xi(u(x), u'(x)) = \text{constant}, \quad x \in (a, b). \quad \Diamond
\]

**Proof.** We will give two different proofs of the theorem. The first one is very elementary and uses the Euler-Lagrange equation. The second one is more involved but has several advantages, notably that it can be derived under weaker regularity hypotheses on the minimizer \( u \).

**Proof 1.** Observe first that for any \( u \in C^2([a, b]) \) we have, by straight differentiation,
\[
\frac{d}{dx} [f(x, u, u') - u' f_\xi(x, u, u')] = f_x(x, u, u') + u'[f_u(x, u, u') - \frac{d}{dx} [f_\xi(x, u, u')]].
\]
By Theorem 4.12 we know that any solution \( u \) of \( (P) \) satisfies the Euler-Lagrange equation
\[
\frac{d}{dx} [f_\xi(x, u(x), u'(x))] = f_u(x, u(x), u'(x))
\]
and hence combining the two identities we have the result.

**Proof 2.** We will use a technique known as **variations of the independent variables**; the classical derivation of Euler-Lagrange equation can be seen as a technique of **variations of the dependent variables**.

Let \( \epsilon \in \mathbb{R}, \varphi \in C^\infty_0(a, b), \lambda = (2 \|\varphi'\|_{L^\infty})^{-1} \) and
\[
\xi(x, \epsilon) = x + \epsilon \lambda \varphi(x) = y.
\]
Observe that for \( |\epsilon| \leq 1 \), then \( \xi(., \epsilon) : [a, b] \to [a, b] \) is a diffeomorphism with \( \xi(a, \epsilon) = a, \xi(b, \epsilon) = b \) and \( \xi_x(x, \epsilon) > 0 \). Let \( \eta(., \epsilon) : [a, b] \to [a, b] \) be its inverse, i.e.
\[
\xi(\eta(y, \epsilon), \epsilon) = y.
\]
Since
\[
\xi_x(\eta(y, \epsilon), \epsilon) \eta_y(y, \epsilon) = 1 \quad \text{and} \quad \xi_x(\eta(y, \epsilon), \epsilon) \eta_x(y, \epsilon) + \xi_\epsilon(\eta(y, \epsilon), \epsilon) = 0,
\]
we find (\( O(t) \) stands for a function \( f \) so that \( |f(t) / t| \) is bounded in a neighborhood of \( t = 0)\)
\[
\eta_y(y, \epsilon) = 1 - \epsilon \lambda \varphi'(y) + O(\epsilon^2) \quad \eta_x(y, \epsilon) = -\lambda \varphi(y) + O(\epsilon).
\]
Set for \( u \) a minimizer of \((P)\)

\[ u^\epsilon (x) = u (\xi (x, \epsilon)) . \]

Note that, performing also a change of variables \( y = \xi (x, \epsilon) \),

\[ I (u^\epsilon) = \int_a^b f (x, u^\epsilon (x), (u^\epsilon)' (x)) \, dx \]

\[ = \int_a^b f (x, u (\xi (x, \epsilon)), u' (\xi (x, \epsilon)) \xi_x (x, \epsilon)) \, dx \]

\[ = \int_a^b f (\eta (y, \epsilon), u(y), u' (y) / \eta_y (y, \epsilon)) \eta_y (y, \epsilon) \, dy . \]

Denoting by \( g (\epsilon) \) the last integrand, we get

\[ g'(\epsilon) = \eta_y \epsilon f + \left[ f_x \eta_e - \frac{\eta_y \epsilon u' f_{\xi}}{\eta_y} \right] \eta_y \]

which leads to

\[ g' (0) = \lambda \left[ - f_x \phi + (u' f_{\xi} - f) \phi' \right] . \]

Since by hypothesis \( u \) is a minimizer of \((P)\) and \( u^\epsilon \in X \), we have \( I (u^\epsilon) \geq I (u) \) and hence

\[ 0 = \frac{d}{d\epsilon} I (u^\epsilon) \bigg|_{\epsilon=0} = \lambda \int_a^b \left\{ - f_x (x, u(x), u'(x)) \phi (x) \right. \]

\[ + \left[ u'(x) f_{\xi} (x, u(x), u'(x)) - f (x, u(x), u'(x)) \right] \phi' (x) \} \, dx \]

\[ = \lambda \int_a^b \phi (x) \left\{ - f_x (x, u(x), u'(x)) \right. \]

\[ + \left. \frac{d}{dx} \left[ -u'(x) f_{\xi} (x, u(x), u'(x)) + f (x, u(x), u'(x)) \right] \right\} \, dx . \]

Invoking Theorem 3.40, we have indeed obtained the claim. \( \blacksquare \)

One should note, as seen in the following example, that it might happen that a solution of \((4.2)\) is not necessarily a solution of the Euler-Lagrange equation \((E)\).

**Example 4.22** Let

\[ f (x, u, \xi) = f (u, \xi) = \frac{1}{2} \xi^2 - u . \]

The second form of the Euler-Lagrange equation is

\[ 0 = \frac{d}{dx} \left[ f (u(x), u'(x)) - u'(x) f_{\xi} (u(x), u'(x)) \right] = -u'(x) [u'' (x) + 1] , \]

and it is satisfied by \( u \equiv 1 \). However, \( u \equiv 1 \) does not verify the Euler-Lagrange equation, which is in the present case

\[ u'' (x) = -1 . \] \( \diamondsuit \)
4.4 Some inequalities

4.4.1 Poincaré-Wirtinger inequality

Theorem 4.23 (Poincaré-Wirtinger inequality) For every $u \in W^{1,2}_0(a, b)$, the following inequality holds

$$\int_a^b u'^2 \, dx \geq (\frac{\pi}{b-a})^2 \int_a^b u^2 \, dx.$$ 

Proof. By a change of variable we immediately reduce the study to the case $a = 0$ and $b = 1$ and we therefore have to prove that

$$\int_0^1 u'^2 \, dx \geq \pi^2 \int_0^1 u^2 \, dx, \text{ for every } u \in W^{1,2}_0(0, 1).$$

We will in fact prove that, for every $0 \leq \lambda < \pi$,

$$\int_0^1 u'^2 \, dx \geq \lambda^2 \int_0^1 u^2 \, dx, \text{ for every } u \in W^{1,2}_0(0, 1).$$

An elementary passage to the limit leads to Poincaré-Wirtinger inequality. For a different proof of a slightly more general form of Poincaré-Wirtinger inequality see Theorem 4.24.

We first let $f_{\lambda}(u, \xi) := (\xi^2 - \lambda^2 u^2) / 2$ and

$$I_{\lambda}(u) := \int_0^1 f_{\lambda}(u(x), u'(x)) \, dx.$$ 

We then apply Theorem 4.18, with

$$\Phi(x, u) := \frac{\lambda}{2} \tan[\lambda(x - \frac{1}{2})]u^2,$$

$$\tilde{f}(x, u, \xi) := \frac{1}{2} \xi^2 + \lambda \tan[\lambda(x - \frac{1}{2})]u \xi + \frac{\lambda^2}{2} \tan^2[\lambda(x - \frac{1}{2})]u^2$$

and observe that $\Phi$ satisfies all the properties of Theorem 4.18. It is easy to see that $(u, \xi) \to \tilde{f}(x, u, \xi)$ is convex and therefore applying Theorem 4.18 we have that, for every $0 \leq \lambda < \pi$,

$$I_{\lambda}(u) \geq I_{\lambda}(0), \forall u \in X,$$

which is the claim. ■

4.4.2 Wirtinger inequality

The Wirtinger inequality is a generalization of the Poincaré-Wirtinger one. It turns out to be equivalent to the isoperimetric inequality; this will be briefly discussed below.
We first introduce the following notation, for any $p \geq 1$,

$$W_{\text{per}}^{1,p}(a,b) := \{ u \in W^{1,p}(a,b) : u(a) = u(b) \}.$$ 

**Theorem 4.24 (Wirtinger inequality)** Let

$$X := \left\{ u \in W_{\text{per}}^{1,2}(a,b) : \int_a^b u(x) \, dx = 0 \right\}.$$ 

Then

$$\int_a^b u'^2 \, dx \geq \left( \frac{2\pi}{b-a} \right)^2 \int_a^b u^2 \, dx, \quad \forall u \in X.$$

Furthermore, equality holds if and only if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$u(x) = \alpha \cos \frac{2\pi x}{b-a} + \beta \sin \frac{2\pi x}{b-a}.$$

**Remark 4.25** (i) The inequality can also be generalized (see Croce-Dacorogna [168]) to

$$\left( \int_a^b |u'|^p \, dx \right)^{1/p} \geq \alpha(p,q,r) \left( \int_a^b |u'|^q \, dx \right)^{1/q}, \quad \forall u \in X$$

for some appropriate $\alpha(p,q,r)$ (in particular, $\alpha(2,2,2) = 2\pi/(b-a)$) and where

$$X := \left\{ u \in W_{\text{per}}^{1,p}(a,b) : \int_a^b |u(x)|^{r-2} u(x) \, dx = 0 \right\}.$$ 

(ii) The above inequality is a generalization of Theorem 4.23, namely

$$\int_c^d v'^2 \, dx \geq \left( \frac{\pi}{d-c} \right)^2 \int_c^d v^2 \, dx, \quad \forall v \in W_0^{1,2}(c,d).$$

The Poincaré-Wirtinger inequality can be inferred from the theorem by setting $b = d$, $a = 2c - d$ and

$$u(x) := \begin{cases} 
  v(x) & \text{if } x \in (c,d) \\
  -v(2c-x) & \text{if } x \in (2c-d,c)
\end{cases}.$$ 

**Proof.** By a change of variable, we immediately reduce the study to the case $a = -1$ and $b = 1$ and we therefore have to prove that if

$$X := \left\{ u \in W_{\text{per}}^{1,2}(-1,1) : \int_{-1}^1 u(x) \, dx = 0 \right\}$$

then

$$\int_{-1}^1 u'^2 \, dx \geq \pi^2 \int_{-1}^1 u^2 \, dx, \quad \forall u \in X.$$
We give here two proofs.

**Proof 1.** The first proof is the classical one of Hurwitz. We divide it into three steps.

**Step 1.** We start by proving the theorem under the further restriction that \( u \in X \cap C^2 [-1, 1] \). We express \( u \) in Fourier series

\[
u(x) = \sum_{n=1}^{\infty} [a_n \cos n \pi x + b_n \sin n \pi x].
\]

Note that there is no constant term since \( \int_{-1}^{1} u(x) \, dx = 0 \). We know at the same time that

\[
u'(x) = \pi \sum_{n=1}^{\infty} [-na_n \sin n \pi x + nb_n \cos n \pi x].
\]

We can now invoke Parseval formula to get

\[
\int_{-1}^{1} u^2(x) \, dx = \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \quad \text{and} \quad \int_{-1}^{1} u'^2(x) \, dx = \pi^2 \sum_{n=1}^{\infty} (a_n^2 + b_n^2) n^2.
\]

The desired inequality follows then at once

\[
\int_{-1}^{1} u'^2(x) \, dx \geq \pi^2 \int_{-1}^{1} u^2(x) \, dx, \quad \forall u \in X \cap C^2.
\]

Moreover equality holds if and only if \( a_n = b_n = 0 \), for every \( n \geq 2 \). This implies that equality holds if and only if \( u(x) = \alpha \cos \pi x + \beta \sin \pi x \), for any \( \alpha, \beta \in \mathbb{R} \), as claimed.

**Step 2.** We now show that we can remove the restriction \( u \in X \cap C^2 [-1, 1] \). By the usual density argument we can find for every \( u \in X \) a sequence \( u_\nu \in X \cap C^2 [-1, 1] \) so that

\[
u_\nu \to u \text{ in } W^{1,2} (-1, 1).
\]

Therefore, for every \( \epsilon > 0 \), we can find \( \nu \) sufficiently large so that

\[
\int_{-1}^{1} u'^2(x) \, dx \geq \int_{-1}^{1} u'^2_\nu(x) \, dx - \epsilon \quad \text{and} \quad \int_{-1}^{1} u^2(x) \, dx \geq \int_{-1}^{1} u^2_\nu(x) \, dx - \epsilon.
\]

Combining these inequalities with Step 1 we find

\[
\int_{-1}^{1} u'^2(x) \, dx \geq \pi^2 \int_{-1}^{1} u^2(x) \, dx - (\pi^2 + 1) \epsilon.
\]

Letting \( \epsilon \to 0 \) we have indeed obtained the inequality.

**Step 3.** We still need to see that equality in \( X \) holds if and only if \( u(x) = \alpha \cos \pi x + \beta \sin \pi x \), for any \( \alpha, \beta \in \mathbb{R} \). This has been proved in Step 1 only if \( u \in X \cap C^2 [-1, 1] \). Since the minimum in \((P)\) is attained by \( u \in X \), we have, for any \( v \in X \cap C^{\infty}_0 (-1, 1) \) and any \( \epsilon \in \mathbb{R} \), that

\[
I(u + \epsilon v) \geq I(u).
\]
Therefore the Euler-Lagrange equation is satisfied, namely
\[
\int_{-1}^{1} (u'v' - \pi^2 uv) \, dx = 0, \quad \forall v \in X \cap C_0^\infty (-1, 1).
\] (4.3)

Let us transform it in a more classical way and choose a function \( f \in C_0^\infty (-1, 1) \) with \( f(1) = 1 \) and let \( \varphi \in C_0^\infty (-1, 1) \) be arbitrary. Set
\[
v(x) := \varphi(x) - \left( \int_{-1}^{1} \varphi \, dx \right) f(x) \quad \text{and} \quad \lambda := -\frac{1}{\pi^2} \int_{-1}^{1} (u'f' - \pi^2 uf) \, dx.
\]
Observe that \( v \in X \cap C_0^\infty (-1, 1) \). Use (4.3), the fact that \( \int_{-1}^{1} f = 1 \), \( \int_{-1}^{1} v = 0 \) and the definition of \( \lambda \) to get, for every \( \varphi \in C_0^\infty (-1, 1) \),
\[
\int \left[ u' \varphi' - \pi^2 (u - \lambda) \varphi \right] = \int \left[ u'(v' + f') \, \varphi - \pi^2 u(v + f) \, \varphi \right] + \pi^2 \lambda \int \varphi = \int \left( u'v' - \pi^2 uv \right) + \left[ \int \varphi \right] \left[ \pi^2 \lambda + \int \left( u'f' - \pi^2 uf \right) \right] = 0.
\]

The regularity of \( u \) (which is a minimizer of \( P \) in \( X \)) then follows (as in Theorem 4.36) at once from the above equation. Since we know (from Step 1) that among smooth minimizers of \( P \) the only ones are of the form \( u(x) = \alpha \cos \pi x + \beta \sin \pi x \), we have the result.

**Proof 2.** An alternative proof, due to H. Lewy (cf. Hardy-Littlewood-Polya [334], page 185), more in the spirit of Section 4.4.1, is now discussed. Let \( u \in X \) where
\[
X := \left\{ u \in W^{1,2} (-1, 1) : u(-1) = u(1) \text{ with } \int_{-1}^{1} u = 0 \right\}.
\]

Define
\[
z(x) := u(x + 1) - u(x)
\]
and note that \( z(-1) = -z(0) \), since \( u(-1) = u(1) \). We deduce that we can find \( \alpha \in (-1, 0) \) so that \( z(\alpha) = 0 \), which means that \( u(\alpha + 1) = u(\alpha) \). We denote this common value by \( a \), namely
\[
a := u(\alpha + 1) = u(\alpha).
\]

Since \( u \in W^{1,2} (-1, 1) \) it is easy to see that the function
\[
v(x) := (u(x) - a)^2 \cot (\pi (x - \alpha))
\]
vanishes at \( x = \alpha \) and \( x = \alpha + 1 \) (this follows from Hölder inequality). We therefore have (recalling that \( u(-1) = u(1) \))
\[
\int_{-1}^{1} \left\{ u'^2 - \pi^2 (u-a)^2 - (u' - \pi (u-a) \cot \pi (x-\alpha))^2 \right\} \, dx = \pi \left[ (u(x) - a)^2 \cot (\pi (x-\alpha)) \right]_{-1}^{1} = 0.
\]
The one dimensional case

Since \( \int_{-1}^{1} u = 0 \), we get from the above identity that

\[
\int_{-1}^{1} \left( u'^2 - \pi^2 u^2 \right) \, dx = 2\pi^2 a^2 + \int_{-1}^{1} \left( u' - \pi \left( u - a \right) \cot \pi \left( x - \alpha \right) \right)^2 \, dx
\]

and hence Wirtinger inequality follows. Moreover we have equality in Wirtinger inequality if and only if \( a = 0 \) and

\[
u' = \pi u \cot \pi \left( x - \alpha \right) \iff u = c \sin \pi \left( x - \alpha \right)
\]

where \( c \) is a constant. \( \blacksquare \)

We get the following as a direct consequence of the theorem.

**Corollary 4.26** The following inequality holds

\[
\int_{-1}^{1} \left( u'^2 + v'^2 \right) \, dx \geq 2\pi \int_{-1}^{1} u v' \, dx, \ \forall u, v \in W^{1,2}_{\text{per}}(-1, 1).
\]

Furthermore equality holds if and only if

\[
(u (x) - r_1)^2 + (v (x) - r_2)^2 = r_3^2, \ \forall x \in [-1, 1]
\]

where \( r_1, r_2, r_3 \in \mathbb{R} \) are constants.

**Proof.** We first observe that if we replace \( u \) by \( u - r_1 \) and \( v \) by \( v - r_2 \) the inequality remains unchanged, therefore we can assume that

\[
\int_{-1}^{1} u \, dx =\int_{-1}^{1} v \, dx = 0
\]

and hence that

\[
u, v \in X := \left\{ u \in W^{1,2}_{\text{per}}(-1, 1) : \int_{-1}^{1} u (x) \, dx = 0 \right\}.
\]

We write the inequality in the equivalent form

\[
\int_{-1}^{1} \left( u'^2 + v'^2 - 2\pi uv' \right) \, dx = \int_{-1}^{1} \left( v' - \pi u \right)^2 \, dx + \int_{-1}^{1} \left( u'^2 - \pi^2 u^2 \right) \, dx \geq 0.
\]

From Theorem 4.24 we deduce that the second term in the above inequality is non negative while the first one is trivially non negative; thus the inequality is established.

We now discuss the equality case. If equality holds we should have

\[
v' = \pi u \quad \text{and} \quad \int_{-1}^{1} \left( u'^2 - \pi^2 u^2 \right) \, dx = 0
\]

which implies from Theorem 4.24 that

\[
u (x) = \alpha \cos \pi x + \beta \sin \pi x \quad \text{and} \quad v (x) = \alpha \sin \pi x - \beta \cos \pi x.
\]
Since we can replace \( u \) by \( u - r_1 \) and \( v \) by \( v - r_2 \), we have that
\[
(u(x) - r_1)^2 + (v(x) - r_2)^2 = r_3^2, \quad \forall x \in [-1, 1]
\]
as wished.

We now briefly discuss the implication of the Wirtinger inequality and its corollary. It is easily shown that they are equivalent (see below) to the isoperimetric inequality, which states that
\[
[L(\partial A)]^2 - 4\pi M(A) \geq 0,
\]
where for \( A \subset \mathbb{R}^2 \) a bounded open set whose boundary, \( \partial A \), is a sufficiently regular simple closed curve, \( L(\partial A) \) denotes the length of the boundary and \( M(A) \) the measure (the area) of \( A \). Furthermore, equality holds if and only if \( A \) is a disk (i.e., \( \partial A \) is a circle).

The sketch of the proof of the claim is as follows. We first parametrize the boundary \( \partial A \) by \( u, v \in W_{per}^{1,2}(-1, 1) \), so that the length and area are given by
\[
L(\partial A) = L(u,v) = \int_{-1}^{1} \sqrt{u'^2 + v'^2} \, dx
\]
\[
M(A) = M(u,v) = \frac{1}{2} \int_{-1}^{1} (uv' - vu') \, dx = \int_{-1}^{1} uv' \, dx.
\]
We next reparametrize the curve by a multiple of its arc length so that
\[
[L(u,v)]^2 = 2 \int_{-1}^{1} (u'^2 + v'^2) \, dx
\]
and then use the corollary to get the result.

There are several articles and books devoted to the isoperimetric inequality in any dimension, we recommend the review article of Osserman [487] and the books by Berger [77], Blaschke [83], Dacorogna [180], Federer [275], Hardy-Littlewood-Polya [334] (for the two dimensional case) and Webster [597].

\section{4.5 Hamiltonian formulation}

Recall that we are considering functions \( f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, f = f(x,u,\xi), \) and
\[
I(u) := \int_{a}^{b} f(x,u(x),u'(x)) \, dx.
\]
The Euler-Lagrange equation is
\[
(E) \quad \frac{d}{dx} [f_{\xi}(x,u,u')] = f_u(x,u,u'), \quad x \in [a,b].
\]
We have seen in the preceding sections that a minimizer of \( I \), if it is sufficiently regular, is also a solution of \( (E) \). The aim of this section is to show that, in
certain cases, solving \((E)\) is equivalent to finding stationary points of a different functional, namely

\[ J(u,v) := \int_a^b [u'(x) v(x) - H(x, u(x), v(x))] \, dx \]

whose Euler-Lagrange equations are

\[
\begin{align*}
H & \quad \begin{cases} 
    u'(x) = H_v(x, u(x), v(x)) \\
    v'(x) = -H_u(x, u(x), v(x)).
\end{cases}
\end{align*}
\]

The function \(H\) is called the Hamiltonian and is defined as the Legendre transform of \(f\), namely

\[ H(x, u, v) := \sup_{\xi \in \mathbb{R}} \{ v\xi - f(x, u, \xi) \}. \]

Sometimes the system \((H)\) is called the canonical form of the Euler-Lagrange equation.

We start our analysis with a lemma.

**Lemma 4.27** Let \(k \geq 2\), \(f \in C^k([a, b] \times \mathbb{R} \times \mathbb{R})\), \(f = f(x, u, \xi)\), such that

\[ f_{\xi\xi}(x, u, \xi) > 0, \text{ for every } (x, u, \xi) \in [a, b] \times \mathbb{R} \times \mathbb{R}, \tag{4.4} \]

\[ \lim_{|\xi| \to \infty} \frac{f(x, u, \xi)}{|\xi|} = +\infty, \text{ for every } (x, u) \in [a, b] \times \mathbb{R}. \tag{4.5} \]

Let

\[ H(x, u, v) := \sup_{\xi \in \mathbb{R}} \{ v\xi - f(x, u, \xi) \}. \tag{4.6} \]

Then \(H \in C^k([a, b] \times \mathbb{R} \times \mathbb{R})\) and

\[
\begin{align*}
H_x(x, u, v) &= -f_x(x, u, H_v(x, u, v)), \tag{4.7} \\
H_u(x, u, v) &= -f_u(x, u, H_v(x, u, v)), \tag{4.8} \\
H(x, u, v) &= v H_v(x, u, v) - f(x, u, H_v(x, u, v)), \tag{4.9} \\
v &= f_{\xi}(x, u, \xi) \iff \xi = H_v(x, u, v). \tag{4.10}
\end{align*}
\]

**Remark 4.28** (i) The lemma remains partially true if we replace the hypothesis \((4.4)\) by the weaker condition

\[ \xi \to f(x, u, \xi) \text{ is strictly convex.} \]

In general, however the function \(H\) is only \(C^1\), as the following simple example shows

\[ f(x, u, \xi) = \frac{1}{4} |\xi|^4 \text{ and } H(x, u, v) = \frac{3}{4} |v|^{4/3}. \]

(See also Example 4.31.)
(ii) The lemma also remains valid if the hypothesis (4.5) does not hold but then, in general, $H$ is no longer finite everywhere as the following simple example suggests. Consider the strictly convex function

$$f(x, u, \xi) = f(\xi) = \sqrt{1 + \xi^2}$$

and observe that

$$H(v) = \begin{cases} -\sqrt{1 - v^2} & \text{if } |v| \leq 1 \\ +\infty & \text{if } |v| > 1. \end{cases}$$

\[\Box\]

**Proof.** We only discuss the case $k = 2$, the general one, $k \geq 2$, being handled similarly. We divide the proof into several steps.

**Step 1.** Fix $(x, u) \in [a, b] \times \mathbb{R}$. From the definition of $H$ and from (4.5) we deduce that there exists $\xi = \xi(x, u, v)$ such that

$$\begin{align*}
H(x, u, v) &= v \xi - f(x, u, \xi) \\
v &= f\xi(x, u, \xi).
\end{align*}$$

(4.11)

**Step 2.** The function $H$ is easily seen to be continuous. Indeed let $(x, u, v), (x', u', v') \in [a, b] \times \mathbb{R} \times \mathbb{R}$, using (4.11) we find $\xi = \xi(x, u, v)$ such that

$$H(x, u, v) = v \xi - f(x, u, \xi).$$

Appealing to the definition of $H$ we also have

$$H(x', u', v') \geq v' \xi - f(x', u', \xi).$$

Combining the two facts we get

$$H(x, u, v) - H(x', u', v') \leq (v - v') \xi + f(x', u', \xi) - f(x, u, \xi),$$

since the reverse inequality is obtained similarly, we deduce the continuity of $H$ from the one of $f$.

**Step 3.** The inverse function theorem, the fact that $f \in C^2$ and the inequality (4.4) imply that $\xi \in C^1$. Let us however discuss it in details. First let us prove that $\xi$ is continuous (in fact locally Lipschitz). Let $R > 0$ be fixed. From (4.5) we deduce that we can find $R_1 > 0$ so that

$$|\xi(x, u, v)| \leq R_1, \text{ for every } x \in [a, b], |u|, |v| \leq R.$$

Since $f\xi$ is $C^1$, we can find $\gamma_1 > 0$ so that

$$|f\xi(x, u, \xi) - f\xi(x', u', \xi')| \leq \gamma_1 (|x - x'| + |u - u'| + |\xi - \xi'|)$$

(4.12)

for every $x, x' \in [a, b], |u|, |u'| \leq R, |\xi|, |\xi'| \leq R_1$. 
From (4.4), we find that there exists $\gamma_2 > 0$ so that

$$f_{\xi \xi} (x, u, \xi) \geq \gamma_2,$$

for every $x \in [a, b]$, $|u| \leq R$, $|\xi| \leq R_1$

and we thus have, for every $x \in [a, b]$, $|u| \leq R$, $|\xi|, |\xi'| \leq R_1$,

$$|f_{\xi} (x, u, \xi) - f_{\xi} (x, u, \xi')| \geq \gamma_2 |\xi - \xi'|.$$  \hfill (4.13)

Let $x, x' \in [a, b]$, $|u|, |u'| \leq R$, $|v|, |v'| \leq R$. By definition of $\xi$ we have

$$f_{\xi} (x, u, \xi (x, u, v)) = v \quad \text{and} \quad f_{\xi} (x', u', \xi (x', u', v')) = v',$$

which leads to

$$f_{\xi} (x, u, \xi (x', u', v')) - f_{\xi} (x, u, \xi (x, u, v)) = f_{\xi} (x, u, \xi (x', u', v')) - f_{\xi} (x', u', \xi (x', u', v')) + v' - v$$

Combining this identity with (4.12) and (4.13) we get

$$\gamma_2 |\xi (x, u, v) - \xi (x', u', v')| \leq \gamma_1 (|x - x'| + |u - u'|) + |v - v'|$$

which, indeed, establishes the continuity of $\xi$.

We now show that $\xi$ is in fact $C^1$. From the equation $v = f_{\xi} (x, u, \xi)$ we deduce that

\[
\begin{align*}
&f_{x\xi} (x, u, \xi) + f_{\xi \xi} (x, u, \xi) \xi_x = 0 \\
&f_{u\xi} (x, u, \xi) + f_{\xi \xi} (x, u, \xi) \xi_u = 0 \\
&f_{\xi \xi} (x, u, \xi) \xi_v = 1.
\end{align*}
\]

Since (4.4) holds and $f \in C^2$, we deduce that $\xi \in C^1 ([a, b] \times \mathbb{R} \times \mathbb{R})$.

**Step 4.** We therefore have that the functions

$$(x, u, v) \rightarrow \xi (x, u, v), \ f_x (x, u, \xi (x, u, v)), \ f_u (x, u, \xi (x, u, v))$$

are $C^1$. We then immediately obtain (4.7), (4.8), and thus $H \in C^2$. Indeed we have, differentiating (4.11),

\[
\begin{align*}
H_x &= v \xi_x - f_x - f_{\xi \xi} \xi_x = (v - f_{\xi}) \xi_x - f_x = -f_x \\
H_u &= v \xi_u - f_u - f_{\xi \xi} \xi_u = (v - f_{\xi}) \xi_u - f_u = -f_u \\
H_v &= \xi + v \xi_v - f_{\xi \xi} \xi_v = (v - f_{\xi}) \xi_v + \xi = \xi
\end{align*}
\]

and in particular

$$\xi = H_v (x, u, v).$$

This achieves the proof of the lemma.

The main theorem of the present section is the following.
Theorem 4.29 Let $f$ and $H$ be as in the above lemma. Let $(u, v) \in C^2([a, b]) \times C^2([a, b])$ satisfy for every $x \in [a, b]$

\[
(H) \begin{cases} 
  u'(x) = H_v(x, u(x), v(x)) \\
  v'(x) = -H_u(x, u(x), v(x)).
\end{cases}
\]

Then $u$ verifies

\[
(E) \quad \frac{d}{dx} [f_{\xi}(x, u(x), u'(x))] = f_u(x, u(x), u'(x)), \quad \forall x \in [a, b].
\]

Conversely, if $u \in C^2([a, b])$ satisfies $(E)$, then $(u, v)$ are solutions of $(H)$ where

$v(x) = f_{\xi}(x, u(x), u'(x)), \quad \forall x \in [a, b].$

Proof. Part 1. Let $(u, v)$ satisfy $(H)$. Using (4.10) and (4.8) we get

\[
  u' = H_v(x, u, v) \iff v = f_{\xi}(x, u, u')
\]

\[
  v' = -H_u(x, u, v) = f_u(x, u, u')
\]

and thus $u$ satisfies $(E)$.

Part 2. Conversely by (4.10) and since $v = f_{\xi}(x, u, u')$ we get the first equation

\[
u' = H_v(x, u, v).
\]

Moreover since $v = f_{\xi}(x, u, u')$ and $u$ satisfies $(E)$, we have

\[
v' = \frac{d}{dx} [v] = \frac{d}{dx} [f_{\xi}(x, u, u')] = f_u(x, u, u').
\]

The second equation follows then from the combination of the above identity and (4.8). ■

Example 4.30 Let $m > 0$, $g \in C^1([a, b])$ and

\[
f(x, u, \xi) = \frac{m}{2} \xi^2 - g(x) u.
\]

The integral under consideration is

\[
I(u) = \int_a^b f(x, u(x), u'(x)) \, dx
\]

and the associated Euler-Lagrange equation is

\[
mu''(x) = -g(x), \quad x \in (a, b).
\]
The Hamiltonian is then
\[ H(x, u, v) = \frac{v^2}{2m} + g(x)u \]
while the associated Hamiltonian system is
\[
\begin{align*}
  u'(x) &= v(x)/m \\
  v'(x) &= -g(x).
\end{align*}
\]

**Example 4.31** We now generalize the preceding example. Let \( p > 1 \) and \( p' = p/(p - 1) \),
\[ f(x, u, \xi) = \frac{1}{p} |\xi|^p - g(x, u) \quad \text{and} \quad H(x, u, v) = \frac{1}{p'} |v|^{p'} + g(x, u). \]
The Euler-Lagrange equation and the associated Hamiltonian system are
\[
\frac{d}{dx} \left[ |u'|^{p-2} u' \right] = -g_u(x, u)
\]
and
\[
\begin{align*}
  u' &= |v|^{p'-2} v \\
  v' &= -g_u(x, u).
\end{align*}
\]

**Example 4.32** Consider the simplest case, where \( f(x, u, \xi) = f(\xi) \) with \( f'' > 0 \) (or more generally \( f \) is strictly convex) and \( \lim_{|\xi| \to \infty} f(\xi)/\xi = +\infty \). The Euler-Lagrange equation and its integrated form are
\[
\frac{d}{dx} [f'(u')] = 0 \Rightarrow f'(u') = \lambda = \text{constant}.
\]
The Hamiltonian is given by
\[ H(v) = f^*(v) = \sup_{\xi} \{ v\xi - f(\xi) \}. \]
The associated Hamiltonian system is
\[
\begin{align*}
  u' &= f''(v) \\
  v' &= 0.
\end{align*}
\]
We find trivially that, denoting by \( \lambda \) and \( \mu \) some constants, \( v' = \lambda \) and hence
\[ u(x) = f''(\lambda) x + \mu. \]

**Example 4.33** We now look for the slightly more involved case where \( f(x, u, \xi) = f(x, \xi) \) with the appropriate hypotheses. The Euler-Lagrange equation and its integrated form are
\[
\frac{d}{dx} [f_{\xi}(x, u')] = 0 \Rightarrow f_{\xi}(x, u') = \lambda = \text{constant}.
\]
The Hamiltonian of \( f \) is given by
\[
H(x, v) = \sup_\xi \{ v\xi - f(x, \xi) \}.
\]
The associated Hamiltonian system is
\[
\begin{align*}
\dot{u}(x) &= H_v(x, v(x)) \\
\dot{v}(x) &= 0.
\end{align*}
\]
The solution is then given by \( v = \lambda = \text{constant} \) and \( \dot{u}(x) = H_v(x, \lambda) \).

**Example 4.34** We next consider the more difficult case where \( f(x, u, \xi) = f(u, \xi) \) with the hypotheses of the theorem. The Euler-Lagrange equation and its integrated form are
\[
\frac{d}{dx} [f_\xi(u, u')] = f_u(u, u') \Rightarrow f(u, u') - u' f_\xi(x, u') = \lambda = \text{constant}.
\]
The Hamiltonian of \( f \) is given by
\[
H(u, v) = \sup_\xi \{ v\xi - f(u, \xi) \} \text{ with } v = f_\xi(u, \xi).
\]
The associated Hamiltonian system is
\[
\begin{align*}
\dot{u}(x) &= H_v(u(x), v(x)) \\
\dot{v}(x) &= -H_u(u(x), v(x)).
\end{align*}
\]
The Hamiltonian system has also a first integral given by
\[
\frac{d}{dx} [H(u(x), v(x))] = H_u(u, v) \dot{u} + H_v(u, v) \dot{v} \equiv 0.
\]
In physical terms, we can say that if the Lagrangian \( f \) is independent of the variable \( x \) (which here is the time), the Hamiltonian \( H \) is constant along the trajectories.

4.6 Regularity

Let us restate the problem. We consider
\[
(P) \quad \inf \left\{ I(u) = \int_a^b f(x, u(x), u'(x)) \, dx : u \in X \right\} = m,
\]
where \( X := \{ u \in W^{1,p}(a, b) : u(a) = \alpha, \ u(b) = \beta \} \),
\[
f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \ f = f(x, u, \xi),
\]
is a Carathéodory function.

We have seen (see Theorems 4.1 and 4.12) that if \( f \) satisfies
(H1) \( \xi \to f(x,u,\xi) \) is convex for almost every \( x \in (a,b) \) and every \( u \in \mathbb{R} \);

(H2) there exist \( p > q \geq 1 \) and \( \alpha_1 > 0, \alpha_2, \alpha_3 \in \mathbb{R} \) such that, for almost every \( x \in (a,b) \) and every \( (u,\xi) \in \mathbb{R} \times \mathbb{R} \),

\[
f(x,u,\xi) \geq \alpha_1 |\xi|^p + \alpha_2 |u|^q + \alpha_3,
\]

then \((P)\) has a solution \( \overline{u} \in X \).

If, furthermore, \( f \in C^1([a,b] \times \mathbb{R} \times \mathbb{R}) \) and verifies that

(H3') for every \( R > 0 \), there exists \( \alpha_4 = \alpha_4(R) \) such that, for every \( (x,u,\xi) \in [a,b] \times [-R,R] \times \mathbb{R} \),

\[
|f(x,u,\xi)|, |f_u(x,u,\xi)|, |f_\xi(x,u,\xi)| \leq \alpha_4 (1 + |\xi|^p),
\]

then any minimizer \( \overline{u} \in X \) satisfies the weak form of the Euler-Lagrange equation

\[
(E_w) \quad \int_a^b \left[ f_u(x,\overline{u},\overline{u}') v + f_\xi(x,\overline{u},\overline{u}') v' \right] \, dx = 0, \forall v \in C_0^\infty(a,b).
\]

We will show that under some strengthening of the hypotheses, we have that if \( f \in C^\infty \), then \( \overline{u} \in C^\infty \). These results are, in part, also valid if \( u : [a,b] \to \mathbb{R}^N \) for \( N > 1 \).

We start with a lemma.

**Lemma 4.35** Let \( f \in C^1([a,b] \times \mathbb{R} \times \mathbb{R}) \) satisfy (H1), (H2) and (H3'). Then any minimizer \( \overline{u} \in W^{1,p}(a,b) \) of \((P)\) is in fact in \( W^{1,\infty}(a,b) \) and the Euler-Lagrange equation holds almost everywhere, i.e.

\[
\frac{d}{dx} [f_\xi(x,\overline{u},\overline{u}')] = f_u(x,\overline{u},\overline{u}'), \text{ a.e. } x \in (a,b).
\]

**Proof.** We know from Theorem 4.12 that the following equation holds

\[
(E_w) \quad \int_a^b \left[ f_u(x,\overline{u},\overline{u}') v + f_\xi(x,\overline{u},\overline{u}') v' \right] \, dx = 0, \forall v \in C_0^\infty(a,b). \quad (4.14)
\]

We then divide the proof into two steps.

**Step 1.** Define

\[
\varphi(x) := f_\xi(x,\overline{u}(x),\overline{u}'(x)) \quad \text{and} \quad \psi(x) := f_u(x,\overline{u}(x),\overline{u}'(x)).
\]

We easily see that \( \varphi \in W^{1,1}(a,b) \) and that \( \varphi'(x) = \psi(x) \), for almost every \( x \in (a,b) \), which means that

\[
\frac{d}{dx} [f_\xi(x,\overline{u},\overline{u}')] = f_u(x,\overline{u},\overline{u}'), \text{ a.e. } x \in (a,b). \quad (4.15)
\]
Indeed since $\mathbf{w} \in W^{1,p}(a,b)$, and hence $\mathbf{w} \in L^\infty(a,b)$, we deduce from (H3') that $\psi \in L^1(a,b)$. We also have from (4.14) that

$$\int_a^b \psi(x)v(x)\,dx = -\int_a^b \varphi(x)v'(x)\,dx, \quad \forall v \in C_0^\infty(a,b).$$

Since $\varphi \in L^1(a,b)$ (from (H3')), we have by definition of weak derivatives the claim, namely $\varphi \in W^{1,1}(a,b)$ and $\varphi' = \psi$ a.e.

**Step 2.** Since $\varphi \in W^{1,1}(a,b)$, we have that $\varphi \in C_0^\infty([a,b])$ which means that there exists a constant $\alpha_5 > 0$ so that

$$|\varphi(x)| = |f_\xi(x,\mathbf{w}(x),\mathbf{w}'(x))| \leq \alpha_5, \quad \forall x \in [a,b]. \quad (4.16)$$

Since $\mathbf{w}$ is bounded (and even continuous), let us say $|\mathbf{w}(x)| \leq R$ for every $x \in [a,b]$, we have from (H1) that

$$f(x,u,0) \geq f(x,u,\xi) - \xi f_\xi(x,u,\xi), \quad \forall (x,u,\xi) \in [a,b] \times [-R,R] \times \mathbb{R}.$$ 

Combining this inequality with (H2) we find that there exists $\alpha_6 \in \mathbb{R}$ such that, for every $(x,u,\xi) \in [a,b] \times [-R,R] \times \mathbb{R},$

$$\xi f_\xi(x,u,\xi) \geq f(x,u,\xi) - f(x,u,0) \geq \alpha_1 |\xi|^p + \alpha_6.$$ 

Using (4.16) and the above inequality we find

$$\alpha_1 |\mathbf{w}'|^p + \alpha_6 \leq |\mathbf{w}| f_\xi(x,\mathbf{w},\mathbf{w}') \leq |\mathbf{w}'| |f_\xi(x,\mathbf{w},\mathbf{w}')| \leq \alpha_5 |\mathbf{w}'|, \quad \text{a.e.} \quad x \in (a,b)$$

which implies, since $p > 1$, that $|\mathbf{w}'|$ is uniformly bounded. Thus the lemma. ■

**Theorem 4.36** Let $f \in C^\infty([a,b] \times \mathbb{R} \times \mathbb{R})$ satisfy (H2), (H3') and

(H1') \quad $f_{\xi\xi}(x,u,\xi) > 0, \quad \forall (x,u,\xi) \in [a,b] \times \mathbb{R} \times \mathbb{R}.$

Then any minimizer of (P) is in $C^\infty([a,b]).$

**Remark 4.37** (i) Note that (H1') is more restrictive than (H1). This stronger condition is usually, but not always, as will be seen in Theorem 4.38, necessary to get higher regularity.

(ii) The proof will show that if $f \in C^k, \, k \geq 2$, then the minimizer is also $C^k$.

(iii) Of course, the convexity of $f$ is essential for regularity, see Example 4.10. 

**Proof.** We divide the proof into two steps.

**Step 1.** We know from Lemma 4.35 that

$$x \rightarrow \varphi(x) := f_\xi(x,\mathbf{u}(x),\mathbf{u}'(x))$$
is in $W^{1,1} (a,b)$ and hence it is continuous. Appealing to Lemma 4.27 (and the remark following this lemma), we have that if

$$H (x, u, v) := \sup_{\xi \in \mathbb{R}} \{ v \xi - f (x, u, \xi) \}$$

then $H \in C^\infty ([a, b] \times \mathbb{R} \times \mathbb{R})$ and, for every $x \in [a, b]$, we have

$$\varphi (x) = f_{\xi} (x, \overline{u} (x), \overline{u}' (x)) \iff \overline{u}' (x) = H_u (x, \overline{u} (x), \varphi (x)).$$

Since $H_v, \overline{u}$ and $\varphi$ are continuous, we infer that $\overline{u}'$ is continuous and hence $\overline{u} \in C^1 ([a, b])$. We therefore deduce that $x \rightarrow f_u (x, \overline{u} (x), \overline{u}' (x))$ is continuous, which combined with the fact that (cf. (4.15))

$$\frac{d}{dx} \left[ |\overline{u}' (x)|^p - 2 \overline{u}' (x) \right] = g_u (x, \overline{u} (x)), \quad \forall x \in (a, b)$$

(or equivalently, by Lemma 4.27, $\varphi' = -H_u (x, \overline{u}, \varphi)$) leads to $\varphi \in C^1 ([a, b])$.

**Step 2.** Returning to our Hamiltonian system

$$\begin{align*}
\overline{u}' (x) &= H_v (x, \overline{u} (x), \varphi (x)) \\
\varphi' (x) &= -H_u (x, \overline{u} (x), \varphi (x))
\end{align*}$$

we can start our iteration. Indeed since $H$ is $C^\infty$ and $\overline{u}$ and $\varphi$ are $C^1$ we deduce from our system that, in fact, $\overline{u}$ and $\varphi$ are $C^2$. Returning to the system we get that $\overline{u}$ and $\varphi$ are $C^3$. Finally we get that $\overline{u}$ is $C^\infty$, as wished.

We next give an example where we can get further regularity without assuming the non-degeneracy condition $f_{\xi \xi} > 0$.

**Theorem 4.38** Let $g \in C^1 ([a, b] \times \mathbb{R})$ satisfy

(H2) there exist $p > q \geq 1$ and $\alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$g (x, u) \geq \alpha_2 |u|^q + \alpha_3, \quad \forall (x, u) \in [a, b] \times \mathbb{R}.$$ 

Let

$$f (x, u, \xi) = \frac{1}{p} |\xi|^p + g (x, u).$$

Then there exists $\overline{u} \in C^1 ([a, b])$, with $|\overline{u}'|^{p-2} \overline{u}' \in C^1 ([a, b])$, a minimizer of (P) and the Euler-Lagrange equation holds everywhere, i.e.

$$\frac{d}{dx} \left[ |\overline{u}' (x)|^{p-2} \overline{u}' (x) \right] = g_u (x, \overline{u} (x)), \quad \forall x \in [a, b].$$

Moreover, if $1 < p \leq 2$, then $\overline{u} \in C^2 ([a, b])$.

If, in addition, $u \rightarrow g (x, u)$ is convex for every $x \in [a, b]$, then the minimizer is unique.

**Proof.** The existence (and uniqueness, if $g$ is convex) of a solution $\overline{u} \in W^{1,p} (a,b)$ follows from Theorem 4.1. According to Lemma 4.35, we know that
\( \bar{\pi} \in W^{1,\infty}(a,b) \) and since \( x \to g_u(x, \bar{\pi}(x)) \) is continuous, we have that the Euler-Lagrange equation holds everywhere, i.e.

\[
\frac{d}{dx} [ |\bar{\pi}'(x)|^{p-2} \bar{\pi}'(x) ] = g_u(x, \bar{\pi}(x)), \ x \in [a,b].
\]

We thus have that \( |\bar{\pi}'|^{p-2} \bar{\pi}' \in C^1([a,b]) \). Call \( v := |\bar{\pi}'|^{p-2} \bar{\pi}' \). We may then infer that

\[
\bar{\pi}' = |v|^{\frac{2-p}{p-1}} v.
\]

Since the function \( t \to |t|^{\frac{2-p}{p-1}} t \) is continuous if \( p > 2 \) and \( C^1(1 < p \leq 2) \), we obtain, from the fact that \( v \in C^1([a,b]) \), the conclusions of the theorem. ■

The result cannot be improved in general, as the following example shows.

**Example 4.39** Let \( p > 2q > 2 \) and

\[
f(x,u,\xi) = f(u,\xi) = \frac{1}{p} |\xi|^p + \frac{\lambda}{q} |u|^q, \ \text{where} \ \lambda = \frac{q p^{q-1} (p-1)}{(p-q)^q}
\]

(note that if, for example, \( p = 6 \) and \( q = 2 \), then \( f \in C^\infty(\mathbb{R}^2) \)).

(i) It is easy to see that \( \bar{\pi} \in C^1([-1,1]) \) but \( \bar{\pi} \notin C^2([-1,1]) \); indeed, we have

\[
\bar{\pi}' = |x|^\frac{p}{p-q}-2 x \ \text{and} \ \bar{\pi}'' = \frac{q}{p-q} |x|^{\frac{2q-p}{p-q}}.
\]

(ii) Since

\[
|\bar{\pi}'|^{p-2} \bar{\pi}' = |x|^{\frac{p(q-1)}{p-q}} x \ \text{and} \ |\bar{\pi}'|^{q-2} \bar{\pi}' = ((p-q)/p)^{q-1} |x|^{\frac{p(q-1)}{p-q}}
\]

we find for instance that if \( \frac{p(q-1)}{p-q} = 4 \) (which is realized, for example, if \( p = 8 \) and \( q = 10/3 \)), then \( |\bar{\pi}'|^{p-2} \bar{\pi}' , |\bar{\pi}'|^{q-2} \bar{\pi}' \in C^\infty([-1,1]) \), although \( \bar{\pi} \notin C^2([-1,1]) \).

(iii) Let

\[
(P) \quad \inf_{u \in W^{1,p(-1,1)}} \left\{ I(u) = \int_{-1}^{1} f(u(x), u'(x)) \ dx : u(-1) = u(1) = \frac{p-q}{p} \right\}.
\]

Since the function \( (u,\xi) \to f(u,\xi) \) is strictly convex and satisfies all the hypotheses of Theorems 4.1, 4.12 and 4.38, we have that \( P \) has a unique minimizer and that it should be the solution of the Euler-Lagrange equation

\[
\left( |u'|^{p-2} u' \right)' = \lambda |u|^{q-2} u.
\]

A direct computation shows that, indeed, \( \bar{\pi} \) is a solution of this equation and therefore it is the unique minimizer of \( P \). \( \diamond \)

Finally, we conclude this section by giving a partial regularity result (for a proof see Buttazzo-Giaquinta-Hildebrandt [117]).
The one dimensional case

**Theorem 4.40 (Tonelli partial regularity theorem)** Let $f \in C^\infty ([a, b] \times \mathbb{R} \times \mathbb{R})$ satisfy

$$(H1') \quad f_{\xi \xi} (x, u, \xi) > 0, \forall (x, u, \xi) \in [a, b] \times \mathbb{R} \times \mathbb{R}.$$ 

Let $\overline{u} \in W^{1,1} (a, b)$ be a minimizer of

$$(P) \quad \inf \left\{ I (u) = \int_a^b f (x, u (x), u' (x)) \, dx : u \in X \right\} = m$$

where $X := \{ u \in W^{1,1} (a, b) : u (a) = \alpha, u (b) = \beta \}$. Then $\overline{u}$ has a classical derivative (possibly infinite) at every point in $[a, b]$ and $\overline{u}' : [a, b] \to \mathbb{R} \cup \{ \pm \infty \}$ is continuous. Furthermore, the singular set

$$E := \{ x \in [a, b] : |\overline{u}' (x)| = \infty \}$$

is closed and has zero measure and $\overline{u}$ is $C^\infty$ outside $E$.

### 4.7 Lavrentiev phenomenon

We conclude this chapter by presenting an example of the so-called *Lavrentiev phenomenon*. It illustrates that some of the hypotheses used in order to get existence results, to derive Euler-Lagrange equations or to obtain regularity results are optimal. In particular, a careful choice of the space of admissible functions is necessary.

**Theorem 4.41 (Mania example)** Let $f (x, u, \xi) := (x - u^3)^2 \xi^6$ and

$$I (u) := \int_0^1 f (x, u (x), u' (x)) \, dx.$$ 

Let

$$\mathcal{W}_\infty := \{ u \in W^{1,\infty} (0, 1) : u (0) = 0, u (1) = 1 \},$$

$$\mathcal{W}_1 := \{ u \in W^{1,1} (0, 1) : u (0) = 0, u (1) = 1 \}.$$ 

Then

$$\inf \{ I (u) : u \in \mathcal{W}_\infty \} > \inf \{ I (u) : u \in \mathcal{W}_1 \} = 0.$$ 

Moreover, $\overline{u} (x) = x^{1/3}$ is a minimizer of $I$ over $\mathcal{W}_1$.

**Remark 4.42**

(i) The first observation of this phenomenon was due to Lavrentiev [391]. The example presented here is essentially due to Mania [417]; see also Ball-Mizel [64] and Cesari [143]. For more on this phenomenon we refer to Belloni [75], Buttazzo-Mizel [119], Clarke-Vinter [159], Davie [222], Ferriero [278], Mizel [448] and Sarychev [522].

(ii) It is interesting to note that the usual finite element methods (by taking piecewise affine functions, which are in $W^{1,\infty}$) in numerical analysis will then
not be able to detect the minimum of some integrals such as the one in the theorem.

(iii) Note also that one can show (see Ball-Mizel [64]) a similar result to that of the theorem with a function such as

\[ f(x,u,\xi) = (x^4 - u^6) |\xi|^s + \epsilon |\xi|^2 \]

\( \epsilon > 0, \ s \geq 27. \) This last example has the advantage of leading to a coercive integral in \( W^{1,2} \), while this is not the case in the above theorem. ♦

Before proceeding with the proof, we establish a preliminary lemma.

**Lemma 4.43** Let \( 0 < \alpha < \beta < 1 \) and

\[ W_{\alpha\beta} := \{ u \in W^{1,\infty}(\alpha, \beta) : u(\alpha) = \frac{1}{4}\alpha^{1/3}, \ u(\beta) = \frac{1}{2}\beta^{1/3}; \]

\[ \frac{1}{4}\alpha^{1/3} \leq u(x) \leq \frac{1}{2}\beta^{1/3}, \ \text{for every} \ x \in [\alpha, \beta] \} \]

If \( f(x,u,\xi) = (x - u^3)^2 \xi^6 \) and

\[ I_{\alpha\beta}(u) := \int_{\alpha}^{\beta} f(x,u(x),u'(x)) \, dx, \]

then

\[ I_{\alpha\beta}(u) \geq \frac{c_0}{\beta} \]

for every \( u \in W_{\alpha\beta} \) and for \( c_0 = \frac{7}{2^3}3^52^{-18}5^{-5} \).

**Proof.** Since \( u(x) \leq \frac{1}{2}x^{1/3} \) we have

\[ 1 - \frac{u^3}{x} \geq 1 - \frac{1}{x} \left( \frac{x^{1/3}}{2} \right)^3 = \frac{7}{2^3}, \ \text{for every} \ x \in [\alpha, \beta]. \]

We thus obtain

\[ I_{\alpha\beta}(u) = \int_{\alpha}^{\beta} x^2 \left( 1 - \frac{u^3}{x} \right)^2 u'^6 \, dx \geq \frac{7^2}{2^6} \int_{\alpha}^{\beta} x^2 u'^6 \, dx. \]  \( \text{(4.17)} \)

We next let

\[ y := x^{3/5} \text{ and } u(x) := \tilde{u}(y) = \tilde{u}(x^{3/5}). \]

We immediately deduce that

\[ u'(x) = \tilde{u}'(y) \frac{dy}{dx} = \frac{3}{5} \tilde{u}'(y) x^{-2/5} = \frac{3}{5} \tilde{u}'(y) y^{-2/3}. \]

Returning to (4.17) we have

\[ I_{\alpha\beta}(u) \geq \frac{7^2}{2^6} \int_{\alpha^{3/5}}^{\beta^{3/5}} y^{10/3} \left( \frac{3}{5} \tilde{u}'(y) y^{-2/3} \right)^6 \left( \frac{5}{3} y^{2/3} \right) dy \]

\[ \geq \frac{7^23^5}{2^65^5} \int_{\alpha^{3/5}}^{\beta^{3/5}} (\tilde{u}'(y))^6 \, dy. \]
Applying Jensen inequality to the right hand side we obtain
\[
I_{\alpha\beta}(u) \geq \frac{7^2 3^5}{2^6 5^5} \frac{(\tilde{u}(\beta^{3/5}) - \tilde{u}(\alpha^{3/5}))^6}{(\beta^{3/5} - \alpha^{3/5})^5} = \frac{7^2 3^5}{2^6 5^5} \frac{(\frac{1}{2} \beta^{1/3} - \frac{1}{2} \alpha^{1/3})^6}{(\beta^{3/5} - \alpha^{3/5})^5}
\]
and thus
\[
I_{\alpha\beta}(u) \geq \frac{7^2 3^5}{2^{12} 5^5} \frac{\beta^2}{\beta^{3/5}} \left(1 - \frac{1}{2} (\alpha/\beta)^{1/3}\right)^6 \left(1 - (\alpha/\beta)^{3/5}\right)^5.
\] (4.18)

Observe finally that since 0 < \alpha < \beta, then
\[
\left(1 - \frac{1}{2} (\alpha/\beta)^{1/3}\right)^6 \geq \left(\frac{1}{2}\right)^6 \quad \text{and} \quad \left(1 - (\alpha/\beta)^{1/3}\right)^{-^5} \geq 1.
\] (4.19)

Combining (4.18) and (4.19) we have indeed obtained the lemma. \[\blacksquare\]

We now prove Theorem 4.41.

**Proof.** We divide the proof into three steps.

**Step 1.** We first prove that if \( u \in \mathcal{W}_{\infty} \), then there exist 0 < \( \alpha < \beta < 1 \) such that \( u \in \mathcal{W}_{\alpha\beta} \) (\( \mathcal{W}_{\alpha\beta} \) as in the lemma), namely
\[
\begin{cases} 
  u(\alpha) = \frac{1}{4} \alpha^{1/3}, & u(\beta) = \frac{1}{2} \beta^{1/3} \\
  \frac{1}{4} x^{1/3} \leq u(x) \leq \frac{1}{2} x^{1/3}, & \text{for every } x \in [\alpha, \beta].
\end{cases}
\] (4.20)

The existence of such \( \alpha \) and \( \beta \) is easily seen (see Figure 4.2). Let
\[
A := \left\{ a \in (0,1) : u(a) = \frac{1}{4} a^{1/3} \right\} \\
B := \left\{ b \in (0,1) : u(b) = \frac{1}{2} b^{1/3} \right\}.
\]

Since \( u \) is Lipschitz, \( u(0) = 0 \) and \( u(1) = 1 \), it follows that \( A \neq \emptyset \) and \( B \neq \emptyset \). Next choose
\[
\alpha := \max \{ a : a \in A \} \quad \text{and} \quad \beta := \min \{ b : b \in B \ \text{and} \ b > \alpha \}.
\]

It is then clear that \( \alpha \) and \( \beta \) satisfy (4.20).

**Step 2.** We may therefore use the lemma to deduce that, for every \( u \in \mathcal{W}_{\infty} \),
\[
I(u) = \int_0^1 (x - u^3)^2 u^6 \, dx \geq \int_\alpha^\beta (x - u^3)^2 u^6 \, dx \geq \frac{c_0}{\beta} > c_0 > 0
\]
and thus
\[
\inf \{ I(u) : u \in \mathcal{W}_{\infty} \} \geq c_0 > 0.
\]
Step 3. The fact that $\varpi(x) = x^{1/3}$ is a minimizer of $I$ over all $u \in W_1$ is trivial and hence
\[
\inf \{ I(u) : u \in W_1 \} = 0.
\]
This achieves the proof of the theorem. ■
Chapter 5

Polyconvex, quasiconvex and rank one convex functions

5.1 Introduction

We now turn our attention to the vectorial case. Recall that we are considering integrals of the form

$$I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx$$

where
- $\Omega \subset \mathbb{R}^n$ is an open set;
- $u : \Omega \to \mathbb{R}^N$ and hence $\nabla u \in \mathbb{R}^{N \times n}$;
- $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$, $f = f(x, u, \xi)$, is a Carathéodory function.

While in Part I we were essentially concerned with the scalar case ($N = 1$ or $n = 1$), we now deal with the vectorial case ($N, n > 1$). The convexity of $\xi \to f(x, u, \xi)$ played the central role in the scalar case ($N = 1$ or $n = 1$), see Chapter 3. In the vectorial case, it is still a sufficient condition to ensure weak lower semicontinuity of $I$ in $W^{1,p}(\Omega; \mathbb{R}^N)$; it is, however, far from being a necessary one. Such a condition is the so-called quasiconvexity introduced by Morrey. It turns out (see Chapter 8) that

$$f \text{ quasiconvex} \iff I \text{ weakly lower semicontinuous}.$$  

Since the notion of quasiconvexity is not a pointwise condition, it is hard to verify if a given function $f$ is quasiconvex. Therefore one is led to introduce a slightly weaker condition, known as rank one convexity, that is equivalent to the ellipticity of the Euler-Lagrange system of equations associated to the
functional $I$. We also define a stronger condition, called *polyconvexity*, that naturally arises when we try to generalize the notions of duality for convex functions to the vectorial context. One can relate all these definitions through the following diagram

$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ quasiconvex } \Rightarrow f \text{ rank one convex}.$$  

We should again emphasize that in the scalar case all these notions are equivalent to the usual convexity condition.

The definitions and main properties of these generalized notions of convexity are discussed in Section 5.2.

In Section 5.3, we give several examples. In particular we show that all the reverse implications are false.

Finally, in an appendix (Section 5.4), we gather certain elementary properties of determinants.

### 5.2 Definitions and main properties

#### 5.2.1 Definitions and notations

Recall that, if $\xi \in \mathbb{R}^{N \times n}$, we write

$$\xi = \begin{pmatrix} \xi_1^1 & \cdots & \xi_1^n \\ \vdots & \ddots & \vdots \\ \xi_N^1 & \cdots & \xi_N^n \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} = (\xi_1, \ldots, \xi_n) = (\xi_{\alpha}^i)_{1 \leq i \leq N, 1 \leq \alpha \leq n}.$$  

In particular if $u : \mathbb{R}^n \to \mathbb{R}^N$ we write

$$\nabla u = \begin{pmatrix} \frac{\partial u^1}{\partial x_1} & \cdots & \frac{\partial u^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial u^N}{\partial x_1} & \cdots & \frac{\partial u^N}{\partial x_n} \end{pmatrix}.$$  

We may now define all the notions introduced above.

**Definition 5.1** (i) A function $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ is said to be rank one convex if

$$f (\lambda \xi + (1 - \lambda) \eta) \leq \lambda f (\xi) + (1 - \lambda) f (\eta)$$  

for every $\lambda \in [0, 1]$, $\xi, \eta \in \mathbb{R}^{N \times n}$ with rank $\{\xi - \eta\} \leq 1$.

(ii) A Borel measurable and locally bounded function $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ is said to be quasiconvex if

$$f (\xi) \leq \frac{1}{\text{meas } D} \int_D f (\xi + \nabla \varphi (x)) \, dx$$
for every bounded open set $D \subset \mathbb{R}^n$, for every $\xi \in \mathbb{R}^{N \times n}$ and for every $\varphi \in W^{1,\infty}_0(D;\mathbb{R}^N)$.

(iii) A function $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ is said to be polyconvex if there exists $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \cup \{+\infty\}$ convex, such that

$$f(\xi) = F(T(\xi)),$$

where $T : \mathbb{R}^{N \times n} \to \mathbb{R}^{\tau(n,N)}$ is such that

$$T(\xi) := (\xi, \text{adj}_2\xi, \ldots, \text{adj}_{n \wedge N}\xi).$$

In the preceding definition, $\text{adj}_s\xi$ stands for the matrix of all $s \times s$ minors of the matrix $\xi \in \mathbb{R}^{N \times n}$, $2 \leq s \leq n \wedge N = \min\{n, N\}$ and

$$\tau(n,N) := \sum_{s=1}^{n \wedge N} \sigma(s), \text{ where } \sigma(s) := \binom{N}{s}\binom{n}{s} = \frac{N!n!}{(s!)^2 (N-s)! (n-s)!}.$$

(iv) A function $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is said to be separately convex, or convex in each variable, if the function

$$x_i \to f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m)$$

is convex for every $i = 1, \ldots, m$, for every fixed $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \in \mathbb{R}^{m-1}$.

(v) A function $f$ is called polyaffine, quasiaffine or rank one affine if $f$ and $-f$ are, respectively, polyconvex, quasiconvex or rank one convex.

Remark 5.2 (i) The concepts were introduced by Morrey [453], but the terminology is that of Ball [53]; note, however, that Ball calls quasiaffine functions null Lagrangians.

(ii) If we adopt the tensorial notation, the notion of rank one convexity can be read as follows: the function $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $\varphi = \varphi(t)$, defined by

$$\varphi(t) := f(\xi + ta \otimes b)$$

is convex for every $\xi \in \mathbb{R}^{N \times n}$ and for every $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$, where we have denoted by

$$a \otimes b = (a^i b_\alpha)_{1 \leq i \leq N, 1 \leq \alpha \leq n}.$$

(iii) It is easily seen that in the definition of quasiconvexity, one can replace the set of test functions $W^{1,\infty}_0(D;\mathbb{R}^N)$ by $C^\infty_0(D;\mathbb{R}^N)$.

(iv) We will see in Proposition 5.11 that if in the definition of quasiconvexity the inequality holds for one bounded open set $D$, it holds for any such set.

(v) We did not give a definition of quasiconvex functions $f$ that may take the value $+\infty$, contrary to polyconvexity and rank one convexity. There have been such definitions given, for example by Ball-Murat [65] and Dacorogna-Fusco [186] (see also Wagner [594]), in the case where $f$ is allowed to take the
value $+\infty$. However, although such definitions have been shown to be necessary for weak lower semicontinuity, it has not been proved that they were sufficient and this seems to be a difficult problem. The notion of quasiconvexity being useful only as an equivalent to weak lower semicontinuity we have disregarded the extension to the case $\mathbb{R} \cup \{+\infty\}$; while those of polyconvexity and rank one convexity will be shown to be useful.

(vi) We have gathered in Section 5.4 some elementary facts about determinants and adj_\_ of matrices. Note that in the case $N = n = 2$, the notion of polyconvexity can be read as follows

\[
\begin{align*}
\sigma (1) &= 4, \quad \sigma (2) = 1, \quad \tau (n, N) = \tau (2, 2) = 5, \\
T (\xi) &= (\xi, \det \xi), \quad f (\xi) = F (\xi, \det \xi).
\end{align*}
\]

(vii) In the definition of polyconvexity of a given function $f$, the associated function $F$ (i.e. $f (\xi) = F (T (\xi))$) in general is not unique. For example, let $N = n = 2$,

\[
\xi = \left(\begin{array}{cc}
\xi_1^1 & \xi_2^1 \\
\xi_1^2 & \xi_2^2
\end{array}\right)
\]

and

\[
f (\xi) = |\xi|^2 = (\xi_1^1)^2 + (\xi_2^1)^2 + (\xi_1^2)^2 + (\xi_2^2)^2 = (\xi_1^1 - \xi_2^2)^2 + (\xi_2^1 + \xi_2^2)^2 + 2 \det \xi.
\]

Let $F_1, F_2 : \mathbb{R}^5 \to \mathbb{R}$ be defined by

\[
F_1 (\xi, a) := |\xi|^2 \quad \text{and} \quad F_2 (\xi, a) := (\xi_1^1 - \xi_2^2)^2 + (\xi_2^1 + \xi_2^2)^2 + 2a.
\]

Then $F_1$ and $F_2$ are convex, $F_1 \neq F_2$ and

\[
f (\xi) = F_1 (T (\xi)) = F_1 (\xi, \det \xi) = F_2 (T (\xi)) = F_2 (\xi, \det \xi).
\]

We will see, after Theorem 5.6, that using either Carathéodory theorem or the separation theorem one can privilege one among the numerous functions $F$.

(viii) The notion of separate convexity does not play any direct role in the calculus of variations. However it can serve as a model for better understanding of the more difficult notion of rank one convexity.

(ix) We will see (see Theorem 5.20) that the notions of polyaffine, quasiaffine or rank one affine are equivalent. Therefore the first and third concepts will not be used anymore.

\[\diamondsuit\]

5.2.2 Main properties

In Section 5.3, we give several examples of polyconvex, quasiconvex and rank one convex functions, but before that we show the relationship between these notions. The following result is essentially due to Morrey [453], [455].
Theorem 5.3 (i) Let $f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$. Then

\[ f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex}. \]

If $f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \cup \{+\infty\}$, then

\[ f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ rank one convex}. \]

(ii) If $N = 1$ or $n = 1$, then all these notions are equivalent.

(iii) If $f \in C^2(\mathbb{R}^{N \times n})$, then rank one convexity is equivalent to Legendre-Hadamard condition (or ellipticity condition)

\[
\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f(\xi)}{\partial \xi^i_{\alpha} \partial \xi^j_{\beta}} \lambda^i \lambda^j \mu_\alpha \mu_\beta \geq 0
\]

for every $\lambda \in \mathbb{R}^N$, $\mu \in \mathbb{R}^n$, $\xi = (\xi^i_{\alpha})_{1 \leq i \leq N, 1 \leq \alpha \leq n} \in \mathbb{R}^{N \times n}$.

(iv) If $f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$ is convex, polyconvex, quasiconvex or rank one convex, then $f$ is locally Lipschitz.

Remark 5.4 (i) We will show later that all the counter implications are false.

- The fact that

\[ f \text{ polyconvex} \nRightarrow f \text{ convex} \]

is elementary. For example, when $N = n = 2$, the function

\[ f(\xi) := \det \xi \]

is polyconvex but not convex.

- We will see several examples (with $N, n \geq 2$), notably in Sections 5.3.2, 5.3.8 and 5.3.9, of quasiconvex functions that are not polyconvex so that we have

\[ f \text{ quasiconvex} \nRightarrow f \text{ polyconvex}. \]

However, there are no elementary examples of this fact.

- The result that

\[ f \text{ rank one convex} \nRightarrow f \text{ quasiconvex} \]

is the fundamental example of Sverak (see Section 5.3.7), which is valid for $n \geq 2$ and $N \geq 3$. However it is still an open problem to know whether $f$ rank one convex implies $f$ quasiconvex, when $N = 2$ (so, in particular, the case $N = n = 2$ is open).

(ii) The Legendre-Hadamard condition is the usual inequality required for the Euler-Lagrange system of equations and is known in this case as ellipticity (see Agmon-Douglis-Nirenberg [7]).
(iii) It is straightforward to see that

\[ f \text{ rank one convex } \Rightarrow f \text{ separately convex.} \]

However, the reverse implication is false, as the following example shows. Let \( N = n = 2 \) and

\[ f(\xi) := \xi_1^1 \xi_2^1. \]

This function is clearly separately convex but not rank one convex. \( \diamond \)

Before proceeding with the proof of the theorem, we give a lemma involving some elementary properties of the determinants.

**Lemma 5.5** Let \( \xi \in \mathbb{R}^{N \times n} \) and \( T(\xi) \) be defined as above.

(i) For every \( \xi, \eta \in \mathbb{R}^{N \times n} \) with \( \text{rank} \{\xi - \eta\} \leq 1 \) and for every \( \lambda \in [0, 1] \), the following identity holds:

\[ T(\lambda \xi + (1 - \lambda) \eta) = \lambda T(\xi) + (1 - \lambda) T(\eta). \]

(ii) For every \( D \subset \mathbb{R}^n \) a bounded open set, \( \xi \in \mathbb{R}^{N \times n} \), \( \varphi \in W^{1,\infty}_0(D; \mathbb{R}^N) \), the following result is valid:

\[ T(\xi) = \frac{1}{\text{meas } D} \int_D T(\xi + \nabla \varphi(x)) \, dx. \]

**Proof.** The proof is elementary and can be found in Proposition 5.65 and Theorem 8.35. We give here, for the sake of illustration, the proof in the case \( N = n = 2 \). We then have

\[ \xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix} \]

and

\[ T(\xi) = (\xi, \det \xi) = (\xi_1^1, \xi_2^1, \xi_1^2, \xi_2^2, \xi_1^1 \xi_2^2 - \xi_1^2 \xi_2^1). \]

(i) Since \( \text{rank} \{\xi - \eta\} \leq 1 \), there exist \( a, b \in \mathbb{R}^2 \) such that

\[ \eta = \xi + a \otimes b = \begin{pmatrix} \xi_1^1 + a^1 b_1 & \xi_1^2 + a^1 b_2 \\ \xi_2^1 + a^2 b_1 & \xi_2^2 + a^2 b_2 \end{pmatrix}. \]

We therefore get that

\[ \det (\lambda \xi + (1 - \lambda) \eta) = \det (\xi + (1 - \lambda) a \otimes b) = \lambda \det \xi + (1 - \lambda) \det \eta. \]

We then deduce that, whenever \( \text{rank} \{\xi - \eta\} \leq 1 \),

\[ T(\lambda \xi + (1 - \lambda) \eta) = (\lambda \xi + (1 - \lambda) \eta, \det (\lambda \xi + (1 - \lambda) \eta)) = \lambda T(\xi) + (1 - \lambda) T(\eta). \]
The proof is similar to the preceding one. Note first that if \( \varphi \in C^2(D; \mathbb{R}^2) \), then

\[
\det \nabla \varphi = \frac{\partial \varphi^1}{\partial x_1} \frac{\partial \varphi^2}{\partial x_2} - \frac{\partial \varphi^1}{\partial x_2} \frac{\partial \varphi^2}{\partial x_1} = \frac{\partial}{\partial x_1} (\varphi^1 \frac{\partial \varphi^2}{\partial x_2}) - \frac{\partial}{\partial x_2} (\varphi^1 \frac{\partial \varphi^2}{\partial x_1}).
\]

Integrating by part the above identity, we have that, if \( \varphi \in C^2_0(D; \mathbb{R}^2) \), then

\[
\det \varphi \text{ meas } D = \int_D [\det \varphi + \xi_1 \frac{\partial \varphi^2}{\partial x_2} + \xi_2 \frac{\partial \varphi^1}{\partial x_1} - \xi_2 \frac{\partial \varphi^1}{\partial x_2} - \xi_1 \frac{\partial \varphi^2}{\partial x_1} + \det \nabla \varphi] dx
\]

By density, the above identity holds also if \( \varphi \in W^{1,\infty}_0(D; \mathbb{R}^2) \). We then deduce that for every \( \varphi \in W^{1,\infty}_0(D; \mathbb{R}^2) \), we must have

\[
T(\xi) \text{ meas } D = \left( \int_D (\xi + \nabla \varphi(x)) dx, \int_D \det (\xi + \nabla \varphi(x)) dx \right)
\]

This concludes the proof of the lemma.

We may now proceed with the proof of Theorem 5.3.

**Proof.** Part 1: \( f \) convex \( \Rightarrow \) \( f \) polyconvex. This implication is trivial.

**Part 2:** \( f \) polyconvex \( \Rightarrow \) \( f \) quasiconvex. Since \( f \) is polyconvex, there exists \( F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \) convex, such that

\[
f(\xi) = F(T(\xi)).
\]

Using Lemma 5.5 and Jensen inequality we obtain

\[
\frac{1}{\text{meas } D} \int_D f(\xi + \nabla \varphi(x)) dx = \frac{1}{\text{meas } D} \int_D F(T(\xi + \nabla \varphi(x))) dx 
\]

\[
\geq F\left( \frac{1}{\text{meas } D} \int_D T(\xi + \nabla \varphi(x)) dx \right) = F(T(\xi)) = f(\xi),
\]

for every bounded open set \( D \subset \mathbb{R}^n \), for every \( \xi \in \mathbb{R}^{N\times n} \) and for every \( \varphi \in W^{1,\infty}_0(D; \mathbb{R}^N) \). The inequality is precisely the definition of quasiconvexity.

**Part 3:** \( f \) quasiconvex \( \Rightarrow \) \( f \) rank one convex. The proof is similar to that of Theorem 3.13 of Chapter 3. Recall that we want to show that

\[
f(\lambda \xi + (1 - \lambda) \eta) \leq \lambda f(\xi) + (1 - \lambda) f(\eta)
\]

for every \( \lambda \in [0,1] \), \( \xi, \eta \in \mathbb{R}^{N\times n} \) with rank \( \{\xi - \eta\} \leq 1 \). To achieve this goal we let \( \varepsilon > 0 \) and we apply Lemma 3.11. We therefore find disjoint open sets
D_\xi, D_\eta \subset D \text{ and } \varphi \in W^{1,\infty}_0 (D; \mathbb{R}^N) \text{ such that }
\begin{align*}
\nabla \varphi (x) = \begin{cases}
|\text{meas } D_\xi - \lambda \text{meas } D| \leq \epsilon, \\
|\text{meas } D_\eta - (1 - \lambda) \text{meas } D| \leq \epsilon
\end{cases}
\begin{align*}
(1 - \lambda) (\xi - \eta) & \quad \text{if } x \in D_\xi \\
- \lambda (\xi - \eta) & \quad \text{if } x \in D_\eta
\end{align*}
\n\|\nabla \varphi\|_{L^\infty} \leq \gamma
\end{align*}

where } \gamma = \gamma (\xi, \eta, D) \text{ is a constant independent of } \epsilon. \text{ We may then use the quasiconvexity of } f \text{ to get }

\begin{align*}
\int_D f (\lambda \xi + (1 - \lambda) \eta + \nabla \varphi (x)) \, dx \\
= \int_{D_\xi} f (\xi) \, dx + \int_{D_\eta} f (\eta) \, dx + \int_{D - (D_\xi \cup D_\eta)} f (\lambda \xi + (1 - \lambda) \eta + \nabla \varphi (x)) \, dx \\
\geq f (\lambda \xi + (1 - \lambda) \eta) \text{meas } D.
\end{align*}

Using the properties of the function } \varphi \text{ and the fact that } \epsilon \text{ is arbitrary, we have indeed obtained that } f \text{ is rank one convex.}

\textbf{Part 4.} \text{ If we now consider the case where } f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}, \text{ the first implication: } f \text{ convex } \Rightarrow f \text{ polyconvex is still trivial. The implication } f \text{ polyconvex } \Rightarrow f \text{ rank one convex is also elementary if we use Lemma 5.5. Indeed since } f \text{ is polyconvex, there exists } F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \cup \{+\infty\} \text{ convex so that }

f (\xi) = F(T(\xi)).

Let } \lambda \in [0,1], \xi, \eta \in \mathbb{R}^{N \times n} \text{ with rank } \{\xi - \eta\} \leq 1, \text{ then, using Lemma 5.5, we get }

\begin{align*}
\lambda f (\xi) + (1 - \lambda) f (\eta) & = F (\lambda T (\xi) + (1 - \lambda) T (\eta)) \\
\leq \lambda F (T (\xi)) + (1 - \lambda) F (T (\eta)) = \lambda f (\xi) + (1 - \lambda) f (\eta)
\end{align*}

which is precisely the rank one convexity of } f. \text{ (ii) The second statement of the theorem, asserting that if } N = 1 \text{ or } n = 1, \text{ then all the notions are equivalent, is trivial. (iii) We now assume that } f \text{ is } C^2 \text{ and rank one convex, that is }

\varphi (t) := f \left( \xi + t \lambda \otimes \mu \right)

\text{is convex in } t \in \mathbb{R} \text{ for every } \xi \in \mathbb{R}^{N \times n} \text{ and for every } \lambda \in \mathbb{R}^N, \mu \in \mathbb{R}^n. \text{ Since } \varphi \text{ is also } C^2, \text{ we obtain immediately Legendre-Hadamard condition, by computing } \varphi''(t) \text{ and using the convexity of } \varphi. \text{ (iv) The last part of Theorem 5.3 is an immediate consequence of Theorem 2.31 of Chapter 2, since a rank one convex function is evidently separately convex. }$
5.2.3 Further properties of polyconvex functions

We now give different characterizations of polyconvex functions that are based on Carathéodory theorem and separation theorems. The next result is due to Dacorogna [177] and [179], following earlier results of Ball [53].

We first recall the notation that for any integer $I$

$$\Lambda_I := \{ \lambda = (\lambda_1, \cdots, \lambda_I) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^I \lambda_i = 1 \}.$$

**Theorem 5.6** Part 1. Let $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$, then the following two statements are equivalent:

(i) $f$ is polyconvex;

(ii) the next two properties hold:

- there exists a convex function $c : \mathbb{R}^\tau \to \mathbb{R} \cup \{+\infty\}$, where $\tau = \tau(n, N)$, such that
  $$f(\xi) \geq c(T(\xi)) \text{ for every } \xi \in \mathbb{R}^{N \times n}; \quad (5.1)$$
- for every $\xi_i \in \mathbb{R}^{N \times n}$, $\lambda \in \Lambda_{\tau+1}$, satisfying
  $$\sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = T(\sum_{i=1}^{\tau+1} \lambda_i \xi_i), \quad (5.2)$$
then
  $$f(\sum_{i=1}^{\tau+1} \lambda_i \xi_i) \leq \sum_{i=1}^{\tau+1} \lambda_i f(\xi_i). \quad (5.3)$$

Part 2. If (ii) is satisfied and if $F : \mathbb{R}^\tau \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$F(X) := \inf \{ \sum_{i=1}^{\tau+1} \lambda_i f(\xi_i) : \lambda \in \Lambda_{\tau+1}, \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = X \}, \quad (5.4)$$
then $F$ is convex and

$$f(\xi) = F(T(\xi)) \text{ for every } \xi \in \mathbb{R}^{N \times n}. \quad (5.5)$$

Moreover, for every $X \in \mathbb{R}^\tau$,

$$F(X) = \sup \{ G(X) : G : \mathbb{R}^\tau \to \mathbb{R} \cup \{+\infty\} \text{ convex and } f(\xi) = G(T(\xi)), \forall \xi \in \mathbb{R}^{N \times n} \}. \quad (5.6)$$

Part 3. Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$, i.e. $f$ takes only finite values. Then the following conditions are equivalent:

(i) $f$ is polyconvex;

(iii) for every $\xi \in \mathbb{R}^{N \times n}$, there exists $\beta = \beta(\xi) \in \mathbb{R}^\tau$ such that

$$f(\eta) \geq f(\xi) + \langle \beta(\xi) ; T(\eta) - T(\xi) \rangle \quad (5.6)$$

for every $\eta \in \mathbb{R}^{N \times n}$ and where $\langle \cdot ; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^\tau$.

Part 4. If (iii) is satisfied, then the function

$$h(X) := \sup_{\xi \in \mathbb{R}^{N \times n}} \{ \langle \beta(\xi) ; X - T(\xi) \rangle + f(\xi) \} \quad (5.7)$$

is convex, takes only finite values and satisfies

$$f(\xi) = h(T(\xi)) \text{ for every } \xi \in \mathbb{R}^{N \times n}. \quad (5.8)$$
Example 5.7 Let \( N = n = 2 \). Then (5.3) and (5.2) become
\[
\begin{align*}
&f(\sum_{i=1}^{\tau} \lambda_i \xi_i) \leq \sum_{i=1}^{\tau} \lambda_i f(\xi_i), \\
&\sum_{i=1}^{\tau} \lambda_i \det(\xi_i) = \det(\sum_{i=1}^{\tau} \lambda_i \xi_i)
\end{align*}
\]
and (5.6) is read
\[
f(\eta) \geq f(\xi) + \langle \gamma(\xi); \eta - \xi \rangle + \delta(\xi)(\det \eta - \det \xi)
\]
where \( \gamma(\xi) \in \mathbb{R}^{2 \times 2} \) and \( \delta(\xi) \in \mathbb{R} \).

\[ \text{\ding{51}} \]

Remark 5.8 (i) The above theorem is a direct adaptation of Carathéodory theorem and the separation theorems for polyconvex functions.

(ii) The condition (5.1) in the theorem implies that \( F \) defined in (5.4) does not take the value \(-\infty\).

(iii) The theorem is important for the following reasons.

- It gives an \textit{intrinsic} definition of polyconvexity, in the sense that it is not given in terms of convexity properties of an associated function \( F \).

- As already mentioned in the definition of the polyconvexity of a given function \( f \), the associated convex function \( F \) is not unique. Equation (5.4) allows us to privilege one such function \( F \). A similar remark can be done using (5.7), as was also observed by Kohn and Strang [373], [374].

- If \( f: \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \) (i.e. \( f \) takes only finite values), then \( F \) defined by (5.4) also takes finite values.

(iv) In view of the above remark we can conclude that if \( f \) takes only finite values then (i), (ii) and (iii) of Theorem 5.6 are equivalent.

(v) Some other properties of polyconvex functions in the cases \( N = n = 2 \) or \( N = n = 3 \) are given by Aubert [39].

\[ \text{\ding{51}} \]

\textbf{Proof.} We follow here the proof of Dacorogna [177], [179], inspired by earlier considerations by Ball [53], which were based on results of Busemann-Ewald-Shephard [110] and Busemann-Shephard [111].

\textit{Parts 1 and 2. (i) \( \Rightarrow \) (ii).} Since \( f \) is polyconvex, there exists \( F: \mathbb{R}^\tau \rightarrow \mathbb{R} \cup \{+\infty\}, \tau = \tau(n,N) \), convex such that
\[
f(\xi) = F(T(\xi)).
\]
(5.9)
The existence of a function \( c \) is trivial, just choose \( c = F \). The convexity of \( F \) coupled with (5.2) gives immediately (5.3).

(ii) \( \Rightarrow \) (i). Assume that (5.3) holds for every \((\lambda_i, \xi_i), 1 \leq i \leq \tau + 1, \) satisfying (5.2). We wish to show that there exists \( F: \mathbb{R}^{\tau(n,N)} \rightarrow \mathbb{R} \cup \{+\infty\} \) convex satisfying (5.9). Let \( I \geq \tau + 1 (\tau = \tau(n,N)) \) be an integer and for \( X \in \mathbb{R}^\tau \) define
\[
F_I(X) := \inf\{\sum_{i=1}^{I} \lambda_i f(\xi_i) : \lambda \in \Lambda_I, \sum_{i=1}^{I} \lambda_i T(\xi_i) = X\}.
\]
(5.10)
We will show that $F_I$ satisfies (5.9) and that one can choose $I = \tau + 1$, without loss of generality, establishing hence (5.4). The proof is divided into four steps.

Step 1. We first show that $F_I$ is well defined.

Step 2. We next prove that $I$ can be taken to be \( \tau + 1 \) in (5.10) without loss of generality and we therefore denote $F_I$ by $F$ (satisfying then (5.4)).

Step 3. We then show that $F$ is convex.

Step 4. We finally establish that $F$ satisfies (5.5).

We now proceed with the details of these four steps.

Step 1. Let us start by showing that $F_I$ is well defined. To do this we must see that given $X \in \mathbb{R}^{\sigma(n)}$ and $I \geq \tau + 1$, then there exist $\lambda \in \Lambda_I$ and $\xi_i$ such that $\sum \lambda_i T(\xi_i) = X$. In view of Carathéodory theorem, this is equivalent to showing that $\text{co} T(\mathbb{R}^{N \times n}) = \mathbb{R}^{\tau(n,N)}$, (5.11)

where $\text{co} M$ denotes the convex hull of $M$ and

$$T(\mathbb{R}^{N \times n}) = \left\{ X \in \mathbb{R}^{\tau(n,N)} : \text{there exists } \xi \in \mathbb{R}^{N \times n} \text{ with } T(\xi) = X \right\}.$$  

In order to establish (5.11), we proceed by contradiction. Assume that $\text{co} (T(\mathbb{R}^{N \times n})) \neq \mathbb{R}^{\tau}$.

Then from the separation theorems (see Corollary 2.11), there exist $0 \neq \alpha \in \mathbb{R}^{\tau}$, $\beta \in \mathbb{R}$, such that

$$\text{co} (T(\mathbb{R}^{N \times n})) \subset V := \{ X \in \mathbb{R}^{\tau} : \langle \alpha; X \rangle \leq \beta \}$$  (5.12)

where $\langle \cdot; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{\tau}$, $\tau = \tau(n, N)$. Recall from the definition of polyconvexity that

$$\tau(n, N) = \sum_{s=1}^{n \land N} \sigma(s)$$

where $\sigma(s) = \binom{N}{s} \binom{n}{s}.$ We then let for $X \in \mathbb{R}^{\tau(n,N)}$

$$X = (X_1, X_2, \cdots, X_{n \land N}) \in \mathbb{R}^{\sigma(1)} \times \mathbb{R}^{\sigma(2)} \times \cdots \times \mathbb{R}^{\sigma(n \land N)} = \mathbb{R}^{\tau(n,N)}$$

and similarly for $\alpha \in \mathbb{R}^{\tau}$. We may then write

$$\langle \alpha; X \rangle = \sum_{s=1}^{n \land N} \langle \alpha_s; X_s \rangle.$$  

Since $\alpha \neq 0$, there exists $t \in \{1, \cdots, n \land N\}$ such that $\alpha_t \neq 0$ while $\alpha_s = 0$ if $s < t$ (if $\alpha_1 \neq 0$, then take $t = 1$). We now show that (5.12) leads to a contradiction and therefore (5.11) holds. Let $\xi \in \mathbb{R}^{N \times n}$ and therefore

$$T(\xi) = (\xi, \text{adj}_2 \xi, \cdots, \text{adj}_{n \land N} \xi) \in T(\mathbb{R}^{N \times n}) \subset \text{co} T(\mathbb{R}^{N \times n}).$$
We choose $\xi \in \mathbb{R}^{N \times n}$ such that
\[ \langle \alpha; T(\xi) \rangle = \langle \alpha_t; \text{adj}_t \xi \rangle \neq 0. \]

This is possible by choosing $(N - t)$ lines of $\xi$ to be zero vectors of $\mathbb{R}^n$ and choosing the other $t$ lines of $\xi$ so that $\langle \alpha_t; \text{adj}_t \xi \rangle$ is non-zero.

Let $\lambda \in \mathbb{R}$ be arbitrary and multiply any of the $t$ non-zero lines of $\xi$ by $\lambda$. Denote the obtained matrix by $\eta$. We then have $T(\eta) \in T(\mathbb{R}^{N \times n}) \subset \text{co} T(\mathbb{R}^{N \times n})$ and
\[ \langle \alpha; T(\eta) \rangle = \lambda \langle \alpha_t; \text{adj}_t \xi \rangle = \lambda \langle \alpha_t; \text{adj}_t \xi \rangle. \]

Using (5.12), we deduce that $T(\eta), T(\xi) \in V$, i.e.
\[ \begin{cases} 
\langle \alpha; T(\eta) \rangle \leq \beta \\
\langle \alpha; T(\xi) \rangle \leq \beta.
\end{cases} \]

The arbitrariness of $\lambda$ and the fact that $\langle \alpha; T(\xi) \rangle \neq 0$ lead immediately to a contradiction. This completes Step 1.

**Step 2.** We now want to show that in (5.10) we can take $I = \tau + 1$. This is done as in Theorem 2.13.

So let $X \in \mathbb{R}^\tau$, $\xi_i \in \mathbb{R}^{N \times n}$ and $\lambda \in \Lambda_I$ be such that
\[ X = \sum_{i=1}^I \lambda_i T(\xi_i). \]

We first prove that there is no loss of generality if we choose $I = \tau + 2$. Define
\[ T(\text{epi} f) := \{(T(\xi), a) \in \mathbb{R}^\tau \times \mathbb{R} : f(\xi) \leq a\} \subset \mathbb{R}^{\tau+1}. \]

We then trivially have that $(T(\xi), f(\xi)) \in T(\text{epi} f)$ and if $\lambda \in \Lambda_I$, we get
\[ (X, \sum_{i=1}^I \lambda_i f(\xi_i)) = \sum_{i=1}^I \lambda_i (T(\xi_i), f(\xi_i)) \in \text{co} T(\text{epi} f). \]

Using Carathéodory theorem, we find that in (5.10) we can take $I = \tau + 2$. It now remains to reduce $I$ from $\tau + 2$ to $\tau + 1$ and this is done as in Theorem 2.35. We show that given $X, T(\xi_i) \in \mathbb{R}^\tau$, $1 \leq i \leq \tau + 2$, $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ and $\alpha \in \Lambda_{\tau+2}$ with
\[ \sum_{i=1}^{\tau+2} \alpha_i T(\xi_i) = X, \]

then there exist $\beta \in \Lambda_{\tau+2}$ such that at least one of the $\beta_i = 0$ (meaning, upon relabeling, that $\beta \in \Lambda_{\tau+1}$) and
\[ \sum_{i=1}^{\tau+2} \beta_i f(\xi_i) \leq \sum_{i=1}^{\tau+2} \alpha_i f(\xi_i) \quad \text{with} \quad \sum_{i=1}^{\tau+2} \beta_i T(\xi_i) = X. \]
It is clear that (5.14) will imply Step 2. Assume all $\alpha_i > 0$ in (5.13) and (5.14), otherwise (5.14) would be trivial. Since from (5.13), we have

$$X \in \text{co}\{T(\xi_1), \cdots, T(\xi_{\tau+2})\} \subseteq \mathbb{R}^\tau,$$

it results, from Carathéodory theorem, that there exists $\tilde{\alpha} \in \Lambda_{\tau+2}$ with at least one of the $\tilde{\alpha}_i = 0$ such that

$$\sum_{i=1}^{\tau+2} \tilde{\alpha}_i T(\xi_i) = X.$$  

We may assume without loss of generality that

$$\sum_{i=1}^{\tau+2} \tilde{\alpha}_i f(\xi_i) > \sum_{i=1}^{\tau+2} \alpha_i f(\xi_i), \quad (5.15)$$

otherwise choosing $\beta_i = \tilde{\alpha}_i$ we would immediately obtain (5.14). We then let

$$J := \{i \in \{1, \cdots, \tau + 2\} : \alpha_i - \tilde{\alpha}_i < 0\}.$$ 

Observe that $J \neq \emptyset$, since otherwise $\alpha_i \geq \tilde{\alpha}_i \geq 0$ for every $1 \leq i \leq \tau + 2$ and since at least one of the $\tilde{\alpha}_i = 0$, we would have a contradiction with $\sum_{i=1}^{\tau+2} \alpha_i = \sum_{i=1}^{\tau+2} \tilde{\alpha}_i = 1$ and the fact that $\alpha_i > 0$ for every $i$. We then define

$$\lambda := \min_{i \in J} \left\{ \frac{\alpha_i}{\tilde{\alpha}_i - \alpha_i} \right\}$$

and we have clearly $\lambda > 0$. Finally let

$$\beta_i := \alpha_i + \lambda (\alpha_i - \tilde{\alpha}_i), \quad 1 \leq i \leq \tau + 2.$$ 

We therefore have

$$\beta_i \geq 0, \quad \sum_{i=1}^{\tau+2} \beta_i = 1, \quad \text{at least one of the } \beta_i = 0, \quad \sum_{i=1}^{\tau+2} \beta_i T(\xi_i) = X$$

and from (5.15)

$$\sum_{i=1}^{\tau+2} \beta_i f(\xi_i) = \sum_{i=1}^{\tau+2} \alpha_i f(\xi_i) + \lambda \sum_{i=1}^{\tau+2} (\alpha_i - \tilde{\alpha}_i) f(\xi_i) \leq \sum_{i=1}^{\tau+2} \alpha_i f(\xi_i).$$

We have therefore obtained (5.14) and this concludes Step 2. Since $I$ can be taken to be $\tau + 1$, we will then denote $F_I$ by $F$ (i.e. (5.10) can be replaced by (5.4)).
Step 3. We now show $F$ is convex. Let $\lambda \in [0, 1]$, $X, Y \in \mathbb{R}^\tau$. We want to prove that

$$\lambda F(X) + (1 - \lambda) F(Y) \geq F(\lambda X (1 - \lambda) Y).$$

Fix $\epsilon > 0$. From (5.4) we deduce that there exist $\lambda, \mu \in \Lambda_{\tau + 1}$ and $\xi_i, \eta_i \in \mathbb{R}^{N \times n}$ such that

$$\lambda F(X) + (1 - \lambda) F(Y) + \epsilon \geq \lambda \sum_{i=1}^{\tau+1} \lambda_i f(\xi_i) + (1 - \lambda) \sum_{i=1}^{\tau+1} \mu_i f(\eta_i), \quad (5.16)$$

with

$$\sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = X, \quad \sum_{i=1}^{\tau+1} \mu_i T(\eta_i) = Y. \quad (5.17)$$

For $1 \leq i \leq \tau + 1$, let

$$\left\{ \begin{array}{ll}
\tilde{\lambda}_i = \lambda \lambda_i & \quad C_i = \xi_i \\
\tilde{\lambda}_{i+\tau+1} = (1 - \lambda) \mu_i & \quad C_{i+\tau+1} = \eta_i.
\end{array} \right.$$

Then (5.16) and (5.17) can be rewritten as

$$\lambda F(X) + (1 - \lambda) F(Y) + \epsilon \geq \sum_{i=1}^{2\tau+2} \tilde{\lambda}_i f(C_i) \quad (5.18)$$

with $\tilde{\lambda} \in \Lambda_{2\tau + 2}$ and

$$\sum_{i=1}^{2\tau+2} \tilde{\lambda}_i T(C_i) = \lambda X + (1 - \lambda) Y. \quad (5.19)$$

Taking the infimum in the right hand side of (5.18) over all $\tilde{\lambda}_i, C_i$ satisfying (5.19), using (5.10) and Step 2 we have

$$\lambda F(X) + (1 - \lambda) F(Y) + \epsilon \geq F(\lambda X + (1 - \lambda) Y);$$

$\epsilon > 0$ being arbitrary, we have indeed established the convexity of $F$.

Step 4. It now remains to show (5.5), i.e.

$$f(\xi) = F(T(\xi))$$

where $F$ satisfies (5.4), namely

$$F(X) = \inf\{ \sum_{i=1}^{\tau+1} \lambda_i f(\xi_i) : \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = X \}.$$
hence (5.5) holds. The fact that $F$ is the supremum over all convex functions $G$ satisfying
\[ f(\xi) = G(T(\xi)) \text{ for every } \xi \in \mathbb{R}^{N \times n}, \]
follows at once from (5.4). This concludes Part 2.

Parts 3 and 4. \( \text{(i) } \Rightarrow \text{ (iii)} \). Since $f$ is polyconvex and finite we may use Parts 1 and 2 to find $F : \mathbb{R}^\tau \to \mathbb{R}$ convex and finite satisfying (see (5.4))
\[
\begin{align*}
  f(\xi) &= F(T(\xi)) \\
  F(X) &= \inf\{ \sum_{i=1}^{\tau+1} \lambda_i f(\xi_i) : \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = X \}.
\end{align*}
\]
Since $F$ is convex and finite, it is continuous and therefore (see Corollary 2.51 of Chapter 2), for each $X \in \mathbb{R}^\tau$, there exists $\gamma(X) \in \mathbb{R}^\tau$ such that
\[ F(Y) \geq F(X) + \langle \gamma(X) ; Y - X \rangle \]
for all $Y \in \mathbb{R}^\tau$. Choosing $Y = T(\eta)$, $X = T(\xi)$, $\beta(\xi) = \gamma(T(\xi))$, we get (5.6), namely
\[ f(\eta) \geq f(\xi) + \langle \beta(\xi) ; T(\eta) - T(\xi) \rangle. \]

\( \text{(iii) } \Rightarrow \text{ (i)} \). We define $h$ as in (5.7), namely
\[ h(X) := \sup_{\xi \in \mathbb{R}^{N \times n}} \{ \langle \beta(\xi) ; X - T(\xi) \rangle + f(\xi) \}. \]
The function $h$, being a supremum of affine functions, is convex. If $X = T(\eta)$ then (5.6) ensures that the supremum in (5.7) is attained by $f(\eta)$ and therefore we have
\[ f(\eta) = h(T(\eta)) \]
as claimed. Moreover, $h$ takes only finite values, since by Part 2 we have $h \leq F$, where $F$ is as in (5.4).

We now obtain as a corollary that a polyconvex function with subquadratic growth must be convex. This is in striking contrast with quasiconvex and rank one convex functions as was established by Sverak [549] (see Theorem 5.54) and later by Gangbo [300] in an indirect way; see also Section 5.3.10. We also prove that a polyconvex function cannot have an arbitrary bound from below, contrary to quasiconvex and rank one convex functions (see Section 5.3.8).

**Corollary 5.9** Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ be polyconvex.

(i) If there exist $\alpha \geq 0$ and $0 \leq p < 2$ such that
\[ f(\xi) \leq \alpha (1 + |\xi|^p) \text{ for every } \xi \in \mathbb{R}^{N \times n}, \]
then $f$ is convex.

(ii) There exists $\gamma \geq 0$ such that
\[ f(\xi) \geq -\gamma (1 + |\xi|^{nN}) \text{ for every } \xi \in \mathbb{R}^{N \times n}. \]
Proof. (i) Since $f$ is polyconvex and finite, we can find, for every $\xi \in \mathbb{R}^{N \times n}$, according to Theorem 5.6 (iii), $\beta = \beta(\xi) \in \mathbb{R}^T$ such that

$$f(\eta) \geq f(\xi) + \langle \beta(\xi); T(\eta) - T(\xi) \rangle,$$

for every $\eta \in \mathbb{R}^{N \times n}$.

(5.20)

Using the growth condition on $f$, we find that

$$f(\xi) + \langle \beta(\xi); T(\eta) - T(\xi) \rangle \leq f(\eta) \leq \alpha (1 + |\eta|^p),$$

for every $\eta \in \mathbb{R}^{N \times n}$.

(5.21)

We can also rewrite it as

$$f(\xi) + \langle \beta(\xi); T(\eta) - T(\xi) \rangle = f(\xi) + \langle \beta_1(\xi); \eta - \xi \rangle + \sum_{s=2}^{n \wedge N} \langle \beta_s(\xi); \text{adj}_s \eta - \text{adj}_s \xi \rangle$$

and hence, for every $\eta \in \mathbb{R}^{N \times n}$,

$$g(\xi) + \langle \beta_1(\xi); \eta \rangle + \sum_{s=2}^{n \wedge N} \langle \beta_s(\xi); \text{adj}_s \eta \rangle \leq \alpha (1 + |\eta|^p)$$

(5.22)

where

$$g(\xi) := f(\xi) - \langle \beta_1(\xi); \xi \rangle - \sum_{s=2}^{n \wedge N} \langle \beta_s(\xi); \text{adj}_s \xi \rangle.$$

Replacing $\eta$ by $t\eta$, with $t \in \mathbb{R}$, in (5.22) we get

$$g(\xi) + t \langle \beta_1(\xi); \eta \rangle + \sum_{s=2}^{n \wedge N} t^s \langle \beta_s(\xi); \text{adj}_s \eta \rangle \leq \alpha (1 + |t|^p |\eta|^p).$$

Letting $t \to \infty$, using the fact that $\eta$ is arbitrary and $p < 2$, we obtain that $\beta_s(\xi) = 0$ for every $s = 2, \ldots, n \wedge N$. Returning to (5.21) we find that, for every $\xi \in \mathbb{R}^{N \times n}$,

$$f(\xi) + \langle \beta_1(\xi); \eta - \xi \rangle \leq f(\eta),$$

for every $\eta \in \mathbb{R}^{N \times n}$

which implies that $f$ is convex. Indeed we have that, for $\lambda \in [0, 1]$,

$$f(\xi) \geq f(\lambda \xi + (1 - \lambda) \eta) + \langle \xi - (\lambda \xi + (1 - \lambda) \eta); \beta_1(\lambda \xi + (1 - \lambda) \eta) \rangle$$

$$f(\eta) \geq f(\lambda \xi + (1 - \lambda) \eta) + \langle \eta - (\lambda \xi + (1 - \lambda) \eta); \beta_1(\lambda \xi + (1 - \lambda) \eta) \rangle.$$

Multiplying the first equation by $\lambda$ and the second by $(1 - \lambda)$ and adding them we obtain the convexity of $f$.

(ii) The second part of the corollary follows at once from (5.20). More precisely, we have from (5.20) that, for every $\xi \in \mathbb{R}^{N \times n}$,

$$f(\xi) \geq f(0) + \langle \beta(0); T(\xi) \rangle \geq -\gamma (1 + |\xi|^{n \wedge N})$$

for an appropriate $\gamma = \gamma(f(0), \beta(0))$.
Another direct consequence of Theorem 5.6 is that we can easily construct (see Dacorogna [177]) rank one convex functions that are not polyconvex. We will see more sophisticated examples in the next sections.

Let $N = n = 2$, $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^{2 \times 2}$ and $\lambda_1, \lambda_2, \lambda_3 \in (0, 1)$ be such that
\[
\begin{cases}
\lambda_1 + \lambda_2 + \lambda_3 = 1, \\
\sum_{i=1}^3 \lambda_i \det \xi_i = \det(\sum_{i=1}^3 \lambda_i \xi_i) \\
\det (\xi_1 - \xi_2) \neq 0, \\
\det (\xi_1 - \xi_3) \neq 0, \\
\det (\xi_2 - \xi_3) \neq 0.
\end{cases}
\]
For example we can choose $\lambda_1 = \lambda_2 = \lambda_3 = 1/3$ and
\[
\xi_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}.
\]
We then define $f : \mathbb{R}^{2 \times 2} \to \mathbb{R} \cup \{+\infty\}$ as
\[
f(\xi) := \begin{cases}
0 & \text{if } \xi = \xi_1, \xi_2, \xi_3 \\
+\infty & \text{otherwise}.
\end{cases}
\]

**Proposition 5.10** $f$ is rank one convex but not polyconvex.

**Proof.** *Part 1.* To show that $f$ is rank one convex, we have to prove that
\[
f(\lambda \xi + (1 - \lambda) \eta) \leq \lambda f(\xi) + (1 - \lambda) f(\eta)
\] (5.23)
for every $\lambda \in [0, 1]$ and every $\xi, \eta \in \mathbb{R}^{2 \times 2}$ such that $\text{rank} \{\xi - \eta\} \leq 1$. Three cases can happen.

*Case 1.* $\xi \neq \xi_i$ or $\eta \neq \xi_i$ for every $i = 1, 2, 3$, then $f(\xi) = +\infty$ or $f(\eta) = +\infty$ and therefore (5.23) is trivially satisfied.

*Case 2.* $\xi = \xi_i$ and $\eta = \xi_j$ with $i \neq j$. This case is impossible, since by construction $\text{rank} \{\xi_i - \xi_j\} = 2$ if $i \neq j$.

*Case 3.* $\xi = \eta = \xi_i$, then (5.23) is trivially satisfied.

*Part 2.* It now remains to show that $f$ is not polyconvex. We proceed by contradiction. If $f$ were polyconvex, we should have, using Theorem 5.6 and the construction of $(\lambda_i, \xi_i)_{1 \leq i \leq 3}$, that
\[
f(\sum_{i=1}^3 \lambda_i \xi_i) \leq \sum_{i=1}^3 \lambda_i f(\xi_i).
\]
This is however impossible since the left hand side takes the value $+\infty$ while the right hand side is 0.

**5.2.4 Further properties of quasiconvex functions**

We first show that if in the definition of quasiconvexity the inequality holds for one bounded open set, it holds for any such set.
Proposition 5.11 Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ be Borel measurable and locally bounded. Let $D \subset \mathbb{R}^n$ be a bounded open set and let the inequality

$$f(\xi) \, \text{meas } D \leq \int_D f(\xi + \nabla \varphi(x)) \, dx$$  \hspace{1cm} (5.24)

hold for every $\xi \in \mathbb{R}^{N \times n}$ and for every $\varphi \in W^{1,\infty}_0(D; \mathbb{R}^N)$. Then the inequality

$$f(\xi) \, \text{meas } E \leq \int_E f(\xi + \nabla \psi(x)) \, dx$$  \hspace{1cm} (5.25)

holds for every bounded open set $E \subset \mathbb{R}^n$, for every $\xi \in \mathbb{R}^{N \times n}$ and for every $\psi \in W^{1,\infty}_0(E; \mathbb{R}^N)$.

Proof. We wish to show (5.25) assuming that (5.24) holds. So let $\psi \in W^{1,\infty}_0(E; \mathbb{R}^N)$ be given and choose first $a > 0$ sufficiently large so that

$$E \subset Q_a := (-a, a)^n.$$

Define next

$$v(x) := \begin{cases} \psi(x) & \text{if } x \in E \\ 0 & \text{if } x \in Q_a - E \end{cases}$$

so that $v \in W^{1,\infty}_0(Q_a; \mathbb{R}^N)$.

Let then $x_0 \in D$ and choose $\nu$ sufficiently large so that

$$x_0 + \frac{1}{\nu} Q_a = x_0 + (-\frac{a}{\nu}, \frac{a}{\nu})^n \subset D.$$

Define next

$$\varphi(x) := \begin{cases} \frac{1}{\nu} v(\nu(x - x_0)) & \text{if } x \in x_0 + \frac{1}{\nu} Q_a \\ 0 & \text{if } x \in D - [x_0 + \frac{1}{\nu} Q_a] \end{cases}$$

Observe that $\varphi \in W^{1,\infty}_0(D; \mathbb{R}^N)$ and

$$\int_D f(\xi + \nabla \varphi(x)) \, dx$$

$$= f(\xi) \, \text{meas}(D - [x_0 + \frac{1}{\nu} Q_a]) + \int_{x_0 + \frac{1}{\nu} Q_a} f(\xi + \nabla v(\nu(x - x_0))) \, dx$$

$$= f(\xi) \left[ \text{meas}(D) - \frac{\text{meas } Q_a}{\nu^n} \right] + \frac{1}{\nu^n} \int_{Q_a} f(\xi + \nabla v(y)) \, dy$$

$$= f(\xi) \left[ \text{meas}(D) - \frac{\text{meas } Q_a}{\nu^n} + \frac{\text{meas } (Q_a - E)}{\nu^n} \right] + \frac{1}{\nu^n} \int_{E} f(\xi + \nabla \psi(y)) \, dy.$$

Appealing to (5.24), we deduce that

$$f(\xi) \, \text{meas } D \leq f(\xi) \left[ \text{meas}(D) - \frac{\text{meas } E}{\nu^n} \right] + \frac{1}{\nu^n} \int_{E} f(\xi + \nabla \psi(y)) \, dy.$$
which is equivalent to the claim, namely (5.25). ■

In some examples (such as Sverak example in Section 5.3.7), it might be more convenient to replace the set of test functions $W^{1,\infty}_0$ by the set of periodic functions.

**Notation 5.12** For $D := (0,1)^n$, we let

$$W^{1,\infty}_{\text{per}}(D; \mathbb{R}^N) := \{ u \in W^{1,\infty}([0,1]^n; \mathbb{R}^N) : u(x + e_i) = u(x), \ i = 1, \cdots, n \}$$

where $\{e_1, \cdots, e_n\}$ is the standard orthonormal basis. ♦

We therefore have the following.

**Proposition 5.13** Let $f : \mathbb{R}^N \times [0,1]^n \to \mathbb{R}$ be Borel measurable and locally bounded. The following two statements are then equivalent:

(i) $f$ is quasiconvex;

(ii) for $D = (0,1)^n$, the inequality

$$f(\xi) \leq \int_D f(\xi + \nabla \psi(x)) \, dx$$

(5.26)

holds for every $\xi \in \mathbb{R}^{N \times n}$ and for every $\psi \in W^{1,\infty}_{\text{per}}(D; \mathbb{R}^N)$.

**Proof.** (ii) $\Rightarrow$ (i). This follows at once from Proposition 5.11 and the fact that

$$W^{1,\infty}_0(D; \mathbb{R}^N) \subset W^{1,\infty}_{\text{per}}(D; \mathbb{R}^N).$$

(i) $\Rightarrow$ (ii). Let $\psi \in W^{1,\infty}_{\text{per}}(D; \mathbb{R}^N)$ and observe first that if $\nu \in \mathbb{N}$ and if

$$\psi_{\nu}(x) := \frac{1}{\nu} \psi(\nu x)$$

then, from the periodicity of $\psi$, we get

$$\int_D f(\xi + \nabla \psi_{\nu}(x)) \, dx = \frac{1}{\nu^n} \int_{\nu D} f(\xi + \nabla \psi(y)) \, dy = \int_D f(\xi + \nabla \psi(x)) \, dx. \quad (5.27)$$

Choose then $\eta_{\nu} \in C^{\infty}_0(D)$ such that $0 \leq \eta_{\nu} \leq 1$ in $D$,

$$\eta_{\nu} \equiv 1 \text{ on } D_{\nu} := \left(\frac{1}{\nu}, 1 - \frac{1}{\nu}\right)^n \text{ and } \|\nabla \eta_{\nu}\|_{L^\infty} \leq c_1 \nu$$

where $c_1 > 0$ is a constant independent of $\nu$.

Let then

$$\varphi_{\nu}(x) := \eta_{\nu}(x) \psi_{\nu}(x)$$

and observe that $\varphi_{\nu} \in W^{1,\infty}_0(D; \mathbb{R}^N)$ and

$$\|\nabla \varphi_{\nu} - \nabla \psi_{\nu}\|_{L^\infty} = \|\eta_{\nu} - 1\| \nabla \psi_{\nu} + \nabla \eta_{\nu} \otimes \psi_{\nu}\|_{L^\infty} \leq c_2 \|\psi\|_{W^{1,\infty}}$$
where \( c_2 > 0 \) is a constant, independent of \( \nu \). Since the function \( f \) is locally bounded we can find \( c_3 > 0 \), independent of \( \nu \), so that

\[
\| f(\xi + \nabla \psi_\nu) - f(\xi + \nabla \varphi_\nu) \|_{L^\infty} \leq c_3.
\]

Appealing to the quasiconvexity of \( f \), to (5.27) and to the preceding observations, we find

\[
\int_D f(\xi + \nabla \psi(x)) \, dx = \int_D f(\xi + \nabla \varphi_\nu(x)) \, dx \\
+ \int_D [f(\xi + \nabla \psi_\nu(x)) - f(\xi + \nabla \varphi_\nu(x))] \, dx \\
= \int_D f(\xi + \nabla \varphi_\nu(x)) \, dx \\
+ \int_{D-D_\nu} [f(\xi + \nabla \psi_\nu(x)) - f(\xi + \nabla \varphi_\nu(x))] \, dx \\
\geq f(\xi) - c_3 \text{meas}(D-D_\nu).
\]

Letting \( \nu \to \infty \) we have indeed obtained (5.26), as wished.

### 5.2.5 Further properties of rank one convex functions

There is no known equivalent to Theorem 5.6 for rank one convex functions. We, nevertheless, give here a characterization of rank one convex functions that is in the same spirit as Part 1 of Theorem 5.6, but much weaker. It will turn out to be useful in Chapter 6.

To characterize rank one convex functions, we give a property of matrices \( \xi_i \in \mathbb{R}^{N \times n} \) that will play the same role as (5.2) of Theorem 5.6 for polyconvex functions. We follow here the presentation of Dacorogna [176] and [179].

We also recall that for any integer \( I \)

\[
\Lambda_I := \{ \lambda = (\lambda_1, \ldots, \lambda_I) : \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^I \lambda_i = 1 \}.
\]

**Definition 5.14** Let \( I \) be an integer and \( \lambda \in \Lambda_I \). Let \( \xi_i \in \mathbb{R}^{N \times n} \), \( 1 \leq i \leq I \). We say that \( (\lambda_i, \xi_i)_{1 \leq i \leq I} \) satisfy \((H_1)\) if

(i) when \( I = 2 \), then \( \text{rank} \{ \xi_1 - \xi_2 \} \leq 1 \);

(ii) when \( I > 2 \), then, up to a permutation, \( \text{rank} \{ \xi_1 - \xi_2 \} \leq 1 \) and if, for every \( 2 \leq i \leq I - 1 \), we define

\[
\begin{align*}
\mu_1 &= \lambda_1 + \lambda_2 \\
\eta_1 &= \frac{\lambda_1 \xi_1 + \lambda_2 \xi_2}{\lambda_1 + \lambda_2} \\
\mu_i &= \lambda_{i+1} \\
\eta_i &= \xi_{i+1}
\end{align*}
\]

then \( (\mu_i, \eta_i)_{1 \leq i \leq I-1} \) satisfy \((H_{I-1})\).
Example 5.15 (a) When $I = 2$, $\lambda \in \Lambda_2$, then $(\lambda_1, \xi_1), (\lambda_2, \xi_2)$ satisfy $(H_2)$ if and only if
\[ \text{rank} \{\xi_1 - \xi_2\} \leq 1. \]

(b) When $I = 3$, $\lambda \in \Lambda_3$, then $(\lambda_i, \xi_i)_{1 \leq i \leq 3}$ satisfy $(H_3)$ if, up to a permutation,
\[
\begin{align*}
\text{rank} \{\xi_1 - \xi_2\} &\leq 1 \\
\text{rank} \left\{ \frac{\lambda_1 \xi_1 + \lambda_2 \xi_2}{\lambda_1 + \lambda_2} \right\} &\leq 1.
\end{align*}
\]

(c) When $I = 4$, $\lambda \in \Lambda_4$, then $(\lambda_i, \xi_i)_{1 \leq i \leq 4}$ satisfy $(H_4)$ if, up to a permutation, either one of the conditions
\[
\begin{cases}
\text{rank} \{\xi_1 - \xi_2\} \leq 1, & \text{rank} \left\{ \frac{\lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_3}{\lambda_1 + \lambda_2 + \lambda_3} \right\} \leq 1 \\
\text{rank} \left\{ \frac{\lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_3 + \lambda_4 \xi_4}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4} \right\} \leq 1
\end{cases}
\]
holds. \(\diamondsuit\)

Proposition 5.16 Let $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$, then the following two conditions are equivalent.

(i) $f$ is rank one convex.

(ii) The expression
\[ f(\sum_{i=1}^{I} \lambda_i \xi_i) \leq \sum_{i=1}^{I} \lambda_i f(\xi_i) \] (5.28)
holds whenever $(\lambda_i, \xi_i)_{1 \leq i \leq I}$ satisfy $(H_I)$.\]

Proof. (ii) $\Rightarrow$ (i). This is trivial since it suffices to choose $I = 2$ in (5.28).

(i) $\Rightarrow$ (ii). We establish (5.28) by induction. By definition of rank one convexity, (5.28) holds for $I = 2$; assume therefore that the proposition is true for $I - 1$. Observe that
\[
\sum_{i=1}^{I} \lambda_i f(\xi_i) = (\lambda_1 + \lambda_2) \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} f(\xi_1) + \frac{\lambda_2}{\lambda_1 + \lambda_2} f(\xi_2) \right) + \sum_{i=3}^{I} \lambda_i f(\xi_i).
\]
If we now use the rank one convexity of $f$ and the hypothesis $(H_I)$ we get
\[
(\lambda_1 + \lambda_2) f\left( \frac{\lambda_1 \xi_1 + \lambda_2 \xi_2}{\lambda_1 + \lambda_2} \right) + \sum_{i=3}^{I} \lambda_i f(\xi_i) \leq \sum_{i=1}^{I} \lambda_i f(\xi_i).
\]
Using again the rank one convexity of $f$, hypothesis $(H_I)$ and the hypothesis of induction, we have indeed established (5.28). \(\blacksquare\)
The above result is much weaker than Theorem 5.6 in the sense that one cannot fix an upper bound on $I$. Two simple examples show that the situation is intrinsically more complicated for rank one convex functions. The first one has been established in Dacorogna [176], [179].

Example 5.17 Let $N = n = 2,$

\[
A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -2 \\ 1/2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -1/4 & 4 \\ 0 & 4 \end{pmatrix},
\]

and

\[
\begin{align*}
\lambda_1 &= \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1/5 \\
\xi_1 &= A, \quad \xi_2 = B, \quad \xi_3 = C, \quad \xi_4 = D, \quad \xi_5 = A.
\end{align*}
\]

It is then easy to see that $(\lambda_i, \xi_i)_{1 \leq i \leq 5}$ satisfy $(H_5)$ since

\[
\begin{align*}
\det (\xi_1 - \xi_2) &= 0 \\
\det \{\xi_3 - \frac{\lambda_1 \xi_1 + \lambda_2 \xi_2}{\lambda_1 + \lambda_2}\} &= 0 \\
\det \{\xi_4 - \frac{\lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_3}{\lambda_1 + \lambda_2 + \lambda_3}\} &= 0 \\
\det \{\xi_5 - \frac{\lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_3 + \lambda_4 \xi_4}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}\} &= 0.
\end{align*}
\]

However, if we combine together $\xi_1$ and $\xi_5$ and if we consider

\[
\begin{align*}
\mu_1 &= \lambda_1 + \lambda_5 = 2/5, \quad \mu_2 = \mu_3 = \mu_4 = 1/5 \\
\eta_1 &= A, \quad \eta_2 = B, \quad \eta_3 = C, \quad \eta_4 = D
\end{align*}
\]

then it is easy to see that $(\mu_i, \eta_i)_{1 \leq i \leq 4}$ do not satisfy $(H_4)$ . In other words, if we use Proposition 5.16, we have the surprising result that if $f : \mathbb{R}^{2 \times 2} \to \mathbb{R} \cup \{+\infty\}$ is rank one convex then

\[
f\left(\frac{2}{5}A + \frac{1}{5}B + \frac{1}{5}C + \frac{1}{5}D\right) \leq \frac{2}{5}f(A) + \frac{1}{5}f(B) + \frac{1}{5}f(C) + \frac{1}{5}f(D)
\]

i.e.

\[
f(\sum_{i=1}^{4} \mu_i \eta_i) \leq \sum_{i=1}^{4} \mu_i f(\eta_i) \tag{5.29}
\]

even though $(\mu_i, \eta_i)_{1 \leq i \leq 4}$ do not satisfy $(H_4)$ . In order to show (5.28), we have to write the inequality (with $(\lambda_i, \xi_i)_{1 \leq i \leq 5}$) as

\[
d \left(\frac{1}{5}A + \frac{1}{5}B + \frac{1}{5}C + \frac{1}{5}D + \frac{1}{5}A\right) \\
&\leq \frac{1}{5}f(A) + \frac{1}{5}f(B) + \frac{1}{5}f(C) + \frac{1}{5}f(D) + \frac{1}{5}f(A). \quad \diamond
\]

The next example is even more striking and has been given by Casadio Tarabusi [127]. A similar example has also been found by Aumann-Hart [50] and Tartar [571].
Example 5.18 Let $N = n = 2$ and (see Figure 5.1)

$$\begin{cases}
\xi_1 = \begin{pmatrix} -1 & 0 \\
0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 1 & 0 \\
0 & -1 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 2 & 0 \\
0 & 1 \end{pmatrix}, \quad \xi_4 = \begin{pmatrix} 0 & 0 \\
0 & 2 \end{pmatrix}, \\
\lambda_1 = \frac{8}{15}, \quad \lambda_2 = \frac{4}{15}, \quad \lambda_3 = \frac{2}{15}, \quad \lambda_4 = \frac{1}{15}.
\end{cases}$$

Observe that $\lambda \in \Lambda_4$ and

$$\begin{align*}
\eta_1 &= \xi_1, & \eta_2 &= \xi_2, & \eta_3 &= \xi_3, & \eta_4 &= \xi_4, & \eta_5 &= 0 = \begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix} = \sum_{i=1}^{4} \lambda_i \xi_i \\
\mu_1 &= \frac{8}{16}, & \mu_2 &= \frac{4}{16}, & \mu_3 &= \frac{2}{16}, & \mu_4 &= \frac{1}{16}, & \mu_5 &= \frac{1}{16}.
\end{align*}$$

Observe that $(\mu_i, \eta_i)_{1 \leq i \leq 5}$ satisfy $(H_5)$, since

$$\begin{cases}
\det (\eta_4 - \eta_5) = 0 \\
\det \left\{ \eta_3 - \frac{\mu_4 \eta_4 + \mu_5 \eta_5}{\mu_4 + \mu_5} \right\} = 0 \\
\det \left\{ \eta_2 - \frac{\mu_3 \eta_3 + \mu_4 \eta_4 + \mu_5 \eta_5}{\mu_3 + \mu_4 + \mu_5} \right\} = 0 \\
\det \left\{ \eta_1 - \frac{\mu_2 \eta_2 + \mu_3 \eta_3 + \mu_4 \eta_4 + \mu_5 \eta_5}{\mu_2 + \mu_3 + \mu_4 + \mu_5} \right\} = 0.
\end{cases}$$

rank $\{\xi_i - \xi_j\} = 2$, if $i \neq j$.

Figure 5.1: The matrices $\xi_1, \xi_2, \xi_3, \xi_4$
Therefore, using Proposition 5.16, we obtain for every \( f : \mathbb{R}^{2 \times 2} \to \mathbb{R} \)
\[
f(0) = f(\sum_{i=1}^{5} \mu_i \eta_i) \leq \sum_{i=1}^{5} \mu_i f(\eta_i);
\]
which means that
\[
16f(0) \leq 8f(\xi_1) + 4f(\xi_2) + 2f(\xi_3) + f(\xi_4) + f(\eta_5). \tag{5.30}
\]
Noting that \( \eta_5 = 0 \) and dividing the above inequality by 15, we have that
\[
f(0) = f(\sum_{i=1}^{4} \lambda_i \xi_i) \leq \sum_{i=1}^{4} \lambda_i f(\xi_i). \tag{5.31}
\]
We have therefore obtained the inequality (5.31) of rank one convexity even though none of the \( \xi_i - \xi_j \) differs by rank one. ♦

**Remark 5.19** An interesting point should be emphasized if one compares the two examples, namely the inequalities (5.29) and (5.31) of rank one convexity. The first one deals with any rank one convex function \( f : \mathbb{R}^{2 \times 2} \to \mathbb{R} \cup \{+\infty\} \), while in the second one we have to restrict our analysis to functions \( f : \mathbb{R}^{2 \times 2} \to \mathbb{R} \) (i.e. that are finite everywhere), since we subtract \( f(0) \) from both sides in the inequality (5.30).

Indeed, the inequality (5.31) does not hold if we allow the function \( f \) to take the value \(+\infty\) as the following example shows. Let
\[
f(\xi) = \chi_{\{\xi_1, \xi_2, \xi_3, \xi_4\}}(\xi) = \begin{cases} 
0 & \text{if } \xi \in \{\xi_1, \xi_2, \xi_3, \xi_4\} \\
+\infty & \text{otherwise.}
\end{cases}
\]
This function is clearly rank one convex, since \( \text{rank}\{\xi_i - \xi_j\} = 2 \) for \( i \neq j \). Therefore
\[
\sum_{i=1}^{4} \lambda_i f(\xi_i) = 0 < f(\sum_{i=1}^{4} \lambda_i \xi_i) = f(0) = +\infty. \tag*{\diamond}
\]

### 5.3 Examples
We have seen in Section 5.2 the definitions and the relations between the notions of convexity, polyconvexity, quasiconvexity and rank one convexity. We now discuss several examples, the most important being the following.

i) We start in Section 5.3.1 with the complete characterization of the *quasi-affine* functions (i.e. the functions \( f \) such that \( f \) and \(-f\) are quasiconvex) by showing that they are linear combinations of *minors* of the matrix \( \nabla u \).

ii) In Section 5.3.2 we study the case of *quadratic* functions \( f \). The main result being that rank one convexity and quasiconvexity are equivalent. Note that the quadratic case is important in the sense that it leads to associated *linear* Euler-Lagrange equations. Therefore, in the linear case, the ellipticity of the Euler-Lagrange equations corresponds exactly to the quasiconvexity of the integrand and thus, anticipating the results of Chapter 8, to the weak lower semicontinuity of the associated variational problem.
Examples

iii) The third important result is considered in Sections 5.3.3 and 5.3.4. We study functions invariant under rotations, notably those depending on singular values. We characterize their convexity and polyconvexity.

iv) In Section 5.3.7, we present the celebrated example of Sverak that provides, in dimensions $N \geq 3$ and $n \geq 2$, an example of a rank one convex function that is not quasiconvex.

v) In Section 5.3.8, we consider the example of Alibert-Dacorogna-Marcellini, which is valid when $N = n = 2$. It characterizes for a homogeneous polynomial of degree four the different notions of convexity encountered in Section 5.2.

5.3.1 Quasiaffine functions

We start with a result established by Ball [53], that is an extension of results of Edelen [255], Ericksen [265] and Rund [520]. It characterizes completely the quasiaffine functions (see also Anderson-Duchamp [27], Ball-Curie-Olver [59], Sivaloganathan [541] and Vasilenko [588]). We follow here the proof of Dacorogna [179].

**Theorem 5.20** Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$. The following conditions are equivalent.

(i) $f$ is quasiaffine.

(ii) $f$ is rank one affine, meaning that $f$ and $-f$ are rank one convex, i.e.

$$f(\lambda \xi + (1 - \lambda) \eta) = \lambda f(\xi) + (1 - \lambda) f(\eta)$$

for every $\lambda \in [0, 1]$, $\xi, \eta \in \mathbb{R}^{N \times n}$ with rank $\{\xi - \eta\} \leq 1$.

(ii') The function $f \in C^1$ and for every $\xi \in \mathbb{R}^{N \times n}$, $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$,

$$f(\xi + a \otimes b) = f(\xi) + \langle \nabla f(\xi); a \otimes b \rangle,$$

where $\langle \cdot; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{N \times n}$.

(iii) $f$ is polyaffine, i.e. $f$ and $-f$ are polyconvex.

(iii') There exists $\beta \in \mathbb{R}^{\tau(n,N)}$ such that

$$f(\xi) = f(0) + \langle \beta; T(\xi) \rangle$$

for every $\xi \in \mathbb{R}^{N \times n}$ and where $\langle \cdot; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{\tau(n,N)}$ and $T$ is as in Definition 5.1.

**Example 5.21** (i) If $N = n = 2$, then the theorem asserts that the only quasi-affine functions are of the type

$$f(\xi) = f(0) + \langle \beta; \xi \rangle + \gamma \det \xi.$$

In particular the only fully non-linear quasiaffine function is $\det \xi$.

(ii) More generally if $n, N > 1$, then the only non-linear quasiaffine functions are linear combinations of the $s \times s$ minors of the matrix $\xi \in \mathbb{R}^{N \times n}$, where $2 \leq s \leq n \land N = \min \{n, N\}$.
Before proceeding with the proof of the theorem, we mention two corollaries. The first one is a straightforward combination of Theorems 5.20 and 8.35.

**Corollary 5.22** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) be quasiaffine. Let \( v \in u + W_{0}^{1,p}(\Omega) \), with \( p \geq n \wedge N \), then

\[
\int_{\Omega} f(\nabla u(x)) \, dx = \int_{\Omega} f(\nabla v(x)) \, dx.
\]

The second one was established by Dacorogna-Ribeiro [212] and we will use it in Theorems 6.24 and 7.47.

**Corollary 5.23** Let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) be quasiaffine.

(i) If \( f \) is locally constant, then it is constant.

(ii) If \( f \) has a local extremum, then it is constant.

**Proof.** (Corollary 5.23). (i) We show that if \( f \) is locally constant around a point \( \xi \in \mathbb{R}^{N \times n} \) then \( f \) is constant everywhere, establishing the result. So assume that there exists \( \epsilon > 0 \) such that

\[
f(\xi + v) = f(\xi), \quad \forall \, v \in \mathbb{R}^{N \times n} \text{ with } |v_j^i| \leq \epsilon \quad (5.32)
\]

and let us show that

\[
f(\xi + w) = f(\xi), \quad \forall \, w \in \mathbb{R}^{N \times n}. \quad (5.33)
\]

The procedure consists in working component by component. We start to show that for every \( w_1^1 \in \mathbb{R} \) and \( |v_j^i| \leq \epsilon \) we have (denoting by \( \{e^1, \cdots , e^N\} \) and \( \{e_1, \cdots , e_n\} \) the standard basis of \( \mathbb{R}^N \) and \( \mathbb{R}^n \) respectively)

\[
f(\xi + w_1^1 e^1 \otimes e_1 + \sum_{(i,j)\neq (1,1)} v_j^i e^i \otimes e_j) = f(\xi + w_1^1 e^1 \otimes e_1) = f(\xi). \quad (5.34)
\]

Indeed if \( |w_1^1| \leq \epsilon \) this is nothing else than (5.32) so we may assume that \( |w_1^1| > \epsilon \) and use the fact that \( f \) is quasiaffine, to deduce that

\[
f(\xi + \frac{\epsilon w_1^1}{|w_1^1|} e^1 \otimes e_1 + \sum_{(i,j)\neq (1,1)} v_j^i e^i \otimes e_j)
\]

\[= \frac{\epsilon}{|w_1^1|} f(\xi + w_1^1 e^1 \otimes e_1 + \sum_{(i,j)\neq (1,1)} v_j^i e^i \otimes e_j)
\]

\[+ (1 - \frac{\epsilon}{|w_1^1|}) f(\xi + \sum_{(i,j)\neq (1,1)} v_j^i e^i \otimes e_j).
\]

Therefore appealing to (5.32) and to the preceding identity we have indeed established (5.34). Proceeding iteratively in a similar manner with the other components \( (w_2^1, w_3^1, \cdots ) \) we have indeed obtained (5.33) and thus the proof of (i) is complete.
(ii) We now show that if $\xi$ is a local extremum point of $f$, then $f$ is constant in a neighborhood of $\xi$ and thus applying (i) we have the result.

Assume that $\xi$ is a local minimum point of $f$ (the case of a local maximizer being handled similarly). We therefore have that there exists $\epsilon > 0$ so that
\[ f(\xi) \leq f(\xi + v), \text{ for every } v \in \mathbb{R}^{N \times n} \text{ so that } |v_j^i| \leq \epsilon. \] (5.35)
Let us show that this implies that
\[ f(\xi) = f(\xi + v), \text{ for every } v \in \mathbb{R}^{N \times n} \text{ so that } |v_j^i| \leq \epsilon. \] (5.36)
We write
\[ v = \sum_{i=1}^{N} \sum_{j=1}^{n} v_j^i e_i \otimes e_j \]
and observe that, since $f$ is quasiaffine,
\[ f(\xi) = \frac{1}{2} f(\xi + v_1^1 e_1 \otimes e_1) + \frac{1}{2} f(\xi - v_1^1 e_1 \otimes e_1) \]
and since (5.35) is satisfied we deduce that
\[ f(\xi \pm v_1^1 e_1 \otimes e_1) = f(\xi), |v_1^1| \leq \epsilon. \] (5.37)
We next write, using again the fact that $f$ is quasiaffine,
\[ f(\xi + v_1^1 e_1 \otimes e_1) = \frac{1}{2} f(\xi + v_1^1 e_1 \otimes e_1 + v_2^1 e_1 \otimes e_2) + \frac{1}{2} f(\xi + v_1^1 e_1 \otimes e_1 - v_2^1 e_1 \otimes e_2) \]
and since (5.35) and (5.37) hold, we deduce that
\[ f(\xi + v_1^1 e_1 \otimes e_1 \pm v_2^1 e_1 \otimes e_2) = f(\xi + v_1^1 e_1 \otimes e_1) = f(\xi), |v_1^1|, |v_2^1| \leq \epsilon. \]
Iterating the procedure we have indeed established (5.36). Appealing to (i), we have therefore proved the corollary.

We should mention that some of the results of Theorem 5.20 will be proved in a more straightforward way in Sections 5.4 and 8.5. Indeed, the implication $(iii') \Rightarrow (ii)$ can also be found in Proposition 5.65, while the implication $(iii') \Rightarrow (i)$ is also established in Theorem 8.35.

We now turn to the proof of Theorem 5.20.

**Proof.** (i) $\Rightarrow$ (ii). This implication follows immediately from Theorem 5.3.

(ii') $\Rightarrow$ (ii). This case is trivial.

(ii) $\Rightarrow$ (ii'). We fix $\xi \in \mathbb{R}^{N \times n}$, $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$ and let for $t \in [0,1]$
\[ \varphi (t) := f \left( \xi + ta \otimes b \right). \]
Since $f$ is rank one affine then $\varphi$ is affine and thus $\varphi \in C^1$ and
\[ \varphi (t) = \varphi (0) + t \varphi' (0). \]
Since \( \varphi \in C^1 \), then, obviously, \( f \in C^1 \) and the result immediately follows from the above identity.

(iii') \( \Rightarrow \) (iii). This implication follows from the definition of polyconvexity.

(iii) \( \Rightarrow \) (i). The result follows from Theorem 5.3.

(ii') \( \Rightarrow \) (iii'). This is the only non trivial implication. So recall that

\[
\xi = \begin{pmatrix}
\xi_1^1 & \cdots & \xi_n^1 \\
\vdots & \ddots & \vdots \\
\xi_1^N & \cdots & \xi_n^N
\end{pmatrix} = \begin{pmatrix}
\xi^1 \\
\vdots \\
\xi^N
\end{pmatrix} = (\xi_1, \cdots, \xi_n).
\]

Assume also that \( f \) is such that

\[
f(\xi + a \otimes b) - f(\xi) = \langle \nabla f(\xi); a \otimes b \rangle,
\]

for every \( \xi \in \mathbb{R}^{N \times n} \), \( a \in \mathbb{R}^N \), \( b \in \mathbb{R}^n \). We wish to show that there exists \( \beta \in \mathbb{R}^{\tau(n,N)} \) such that

\[
f(\xi) - f(0) = \langle \beta; T(\xi) \rangle, \quad \text{for every } \xi \in \mathbb{R}^{N \times n}.
\]

In the sequel we assume that \( n \geq N \), otherwise we reverse the roles of \( n \) and \( N \). We then proceed by induction on \( N \).

Step 1. \( N = 1 \). Since \( N = 1 \), (5.38) can be read as

\[
f(\xi + \eta) - f(\xi) = \langle \nabla f(\xi); \eta \rangle
\]

for every \( \xi, \eta \in \mathbb{R}^n \). It is then trivial to see that the above identity implies that \( f \) is affine and therefore if we choose \( \beta = \nabla f(0) \), we have immediately (5.39).

Step 2. \( N = 2 \). This step is unnecessary but we prove it for the sake of illustration. Let

\[
\xi = \begin{pmatrix}
\xi_1^1 & \cdots & \xi_n^1 \\
\xi_1^2 & \cdots & \xi_n^2
\end{pmatrix} = \begin{pmatrix}
\xi^1 \\
\xi^2
\end{pmatrix} = (\xi_1, \cdots, \xi_n)
\]

and for \( a \in \mathbb{R}^2 \), \( b \in \mathbb{R}^n \)

\[
a \otimes b = \begin{pmatrix}
a^1b \\
a^2b
\end{pmatrix} = \begin{pmatrix}
a^1b_1 & \cdots & a^1b_n \\
a^2b_1 & \cdots & a^2b_n
\end{pmatrix}.
\]

We want to show that if \( f \) is rank one affine, i.e.

\[
f(\xi + a \otimes b) - f(\xi) = \langle \nabla f(\xi); a \otimes b \rangle
\]

then there exists \( \beta \in \mathbb{R}^{\tau(n,2)} \) such that

\[
f(\xi) = f(0) + \langle \beta; T(\xi) \rangle
\]
where

\[ T(\xi) = (\xi, \text{adj}_2 \xi) \in \mathbb{R}^{2 \times n} \times \mathbb{R}^{n} = \mathbb{R}^{n \times 2}. \]

For the notations concerning \text{adj}_2 \xi, see Section 5.4. But note that, up to a sign and the ordering, an element of the matrix \text{adj}_2 \xi is essentially \( \det (\xi_k, \xi_l) \), \( 1 \leq k < l \leq n \). We then fix \( \xi^2 \) and choose \( a = e^1 = (1, 0) \) in (5.38) and define

\[ g(\xi^1) := f\left( \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \right). \]

Thus the function

\[ t \rightarrow g(\xi^1 + tb) = f\left( \begin{pmatrix} \xi^1 + tb \\ \xi^2 \end{pmatrix} \right) \]

is affine and we may then use Step 1 to find \( \gamma = \gamma(\xi^2) \in \mathbb{R}^n \) such that

\[ g(\xi^1) = g(0) + \langle \gamma(\xi^2); \xi^1 \rangle = f\left( \begin{pmatrix} 0 \\ \xi^2 \end{pmatrix} \right) + \langle \gamma(\xi^2); \xi^1 \rangle. \]

Repeating the argument when \( \xi^1 = 0 \) for \( f\left( \begin{pmatrix} 0 \\ \xi^2 \end{pmatrix} \right) \), we have

\[ f\left( \begin{pmatrix} 0 \\ \xi^2 \end{pmatrix} \right) = f(0) + \langle \beta^2; \xi^2 \rangle. \]

Combining the above two identities, we obtain

\[ f\left( \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \right) = f(0) + \langle \beta^2; \xi^2 \rangle + \langle \gamma(\xi^2); \xi^1 \rangle. \]  \( \text{(5.40)} \)

Since \( f \) is rank one affine, it is affine (when \( \xi^1 \) is fixed) with respect to \( \xi^2 \) and therefore \( \gamma(\xi^2) = (\gamma_1(\xi^2), \cdots, \gamma_n(\xi^2)) \) is affine and hence there exist \( \beta^1 = (\beta^1_1, \cdots, \beta^1_n) \in \mathbb{R}^n, \delta_1, \cdots, \delta_n \in \mathbb{R}^n \) such that

\[ \gamma_l(\xi^2) = \beta^1_l + \langle \delta_l; \xi^2 \rangle, \quad l = 1, \cdots, n. \]

Returning to (5.40), we therefore get

\[ f\left( \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \right) = f(0) + \langle \beta^1; \xi^1 \rangle + \langle \beta^2; \xi^2 \rangle + \sum_{l=1}^n \xi^1_l \langle \delta_l; \xi^2 \rangle \]

or in other words

\[ f\left( \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \right) = f(0) + \langle \beta^1; \xi^1 \rangle + \langle \beta^2; \xi^2 \rangle + \sum_{l=1}^n \sum_{\alpha=1}^n \delta_{l\alpha} \xi^1_l \xi^2_\alpha. \]  \( \text{(5.41)} \)

Since \( f \) is rank one affine we have from (5.41) that if

\[ h(\xi) := \sum_{l=1}^n \sum_{\alpha=1}^n \delta_{l\alpha} \xi^1_l \xi^2_\alpha \]
then $h$ is rank one affine and therefore using Lemma 5.24 we must have

$$\delta_{l \alpha} = -\delta_{\alpha l}.$$ 

Thus there exists $\epsilon \in \mathbb{R}^{(n)}$ such that

$$h(\xi) = \sum_{1 \leq l < \alpha \leq n} \delta_{l \alpha} (\xi^1 \xi^2_{\alpha} - \xi^1_{\alpha} \xi^2) = \langle \epsilon; \text{adj}_2 \xi \rangle.$$ 

Combining (5.41) with the above identity, we deduce (5.39) and this concludes Step 2.

**Step N.** We now proceed with the general case. Assume that we have proved the theorem for every $l < N$. Fixing $\xi^2, \ldots, \xi^N$ and using the fact that $f$ is rank one affine, then $f$ is affine in $\xi^1$, for $\xi^2, \ldots, \xi^N$ fixed. Therefore there exist

$$\psi \left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^N \end{array} \right) = (\psi_1, \ldots, \psi_n) \in \mathbb{R}^n \quad \text{and} \quad \chi \left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^N \end{array} \right) \in \mathbb{R},$$

such that

$$f(\xi) = \langle \psi \left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^N \end{array} \right); \xi^1 \rangle + \chi \left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^N \end{array} \right). \quad (5.42)$$

Using the hypothesis of induction and proceeding as in Step 2 we find that

$$\begin{cases} 
\chi \left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^N \end{array} \right) = f(0) + \langle \beta^0; T \left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^N \end{array} \right) \rangle \\
\psi_l \left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^N \end{array} \right) = \beta_l + \langle \gamma_l; T \left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^N \end{array} \right) \rangle, \ l = 1, \ldots, n \end{cases} \quad (5.43)$$

for some $\beta^0, \gamma_1, \ldots, \gamma_n \in \mathbb{R}^{T(n)}$ and $\beta^1 = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$. Combining (5.42) and (5.43) we have that

$$f(\xi) = f(0) + \langle \beta^0; T \left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^N \end{array} \right) \rangle + \langle \beta^1; \xi^1 \rangle + \sum_{l=1}^{n} \xi^1_l \langle \gamma_l; T \left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^N \end{array} \right) \rangle$$
which can be rewritten as

$$
f (\xi) = f (0) + \langle \beta^0; T \begin{pmatrix} \xi^2 \\
\vdots \\
\xi^N \end{pmatrix} \rangle + \langle \beta^1; \xi^1 \rangle \tag{5.44}$$

$$+ \sum_{s=1}^{N-1} n \sum_{l=1}^{(n_s)} \left( \frac{(N-1)}{s} \right) \sum_{i=1}^{\gamma_{i\alpha} \xi^1} \left( \text{adj}_s \begin{pmatrix} \xi^2 \\
\vdots \\
\xi^N \end{pmatrix} \right)_i^\alpha.$$

Letting

$$h (\xi) := \sum_{s=1}^{N-1} h_s (\xi) \quad \text{where} \quad h_s (\xi) := \sum_{l=1}^{n_s} \sum_{\alpha=1}^{(N-1)} \gamma_{i\alpha} \xi^1 \left( \text{adj}_s \begin{pmatrix} \xi^2 \\
\vdots \\
\xi^N \end{pmatrix} \right)_i^\alpha,$$

we deduce from the fact that $f$ is rank one affine and from (5.44) that $h$ is rank one affine. Since $h$ is rank one affine, we deduce that so is $h_s$. Indeed let us first show this for $h_1$. Write

$$h_1 (\xi) = \sum_{i=2}^N h_i^1 (\xi) \quad \text{where} \quad h_i^1 (\xi) := \sum_{l=1}^{n_s} \sum_{\alpha=1}^{(N-1)} \gamma_{i\alpha} \xi^i \left( \text{adj}_s \begin{pmatrix} \xi^2 \\
\vdots \\
\xi^N \end{pmatrix} \right)_i^\alpha.$$

By first choosing $\xi^3 = \cdots = \xi^N = 0$, we obtain that $h_1^2$ is rank one affine (since then $h = h_1^2$); iterating this process we find that all the $h_i^1$ are rank one affine and thus $h_1$ is rank one affine. We then infer that so is $h - h_1$. With the same reasoning, we get that all the $h_s$, $1 \leq s \leq N - 1$, are rank one affine.

We may then use Lemma 5.24 to deduce that there exist

$$\delta_{j\beta}^{i\alpha} \in \mathbb{R}, \quad 1 \leq s \leq N - 1, \quad 1 \leq \beta \leq (n_s), \quad 1 \leq j \leq \left( \frac{n}{s+1} \right)$$

such that

$$h (\xi) = \sum_{s=1}^{N-1} \left( \frac{n}{s+1} \right) \left( \frac{N}{s+1} \right) \sum_{\beta=1}^\beta \sum_{j=1}^{(n/s+1)} \delta_{j\beta}^{i\alpha} (\text{adj}_{s+1} \xi^i \beta).$$

Combining (5.44) and the above identity, we have indeed found $\beta \in \mathbb{R}^{\tau(n,N)}$ such that

$$f (\xi) = f (0) + \langle \beta; T (\xi) \rangle,$$

which is the claimed result. ■

In the above proof we have used the following lemma.
Lemma 5.24 Let \( n \geq N \) and \( \xi \in \mathbb{R}^{N \times n} \),

\[
\xi = \begin{pmatrix}
\xi_1^1 & \cdots & \xi_n^1 \\
\vdots & \ddots & \vdots \\
\xi_1^N & \cdots & \xi_n^N
\end{pmatrix} = \begin{pmatrix}
\xi_1^1 \\
\vdots \\
\xi_N^1
\end{pmatrix} = (\xi_1, \cdots, \xi_n).
\]

For \( 1 \leq s \leq N - 1 \), let

\[
g(\xi) := \sum_{l=1}^{n} \sum_{\alpha=1}^{(n-s)} \sum_{i=1}^{(N-s)} \gamma_{l,\alpha}^i \xi_l^1 \begin{pmatrix}
\text{adj}_s \xi_l^1 \\
\vdots \\
\xi_N^1
\end{pmatrix}^i_{\alpha}.
\]

If \( g \) is rank one affine, meaning that

\[
g(\xi + a \otimes b) = g(\xi) + \langle \nabla g(\xi) ; a \otimes b \rangle,
\]

then there exist \( \delta^i_{\beta} \in \mathbb{R} \), \( 1 \leq \beta \leq \binom{n}{s+1} \), \( 1 \leq j \leq \binom{N}{s+1} \) such that

\[
g(\xi) = \sum_{\beta=1}^{\binom{n}{s+1}} \sum_{j=1}^{\binom{N}{s+1}} \delta^i_{\beta} \begin{pmatrix}
\text{adj}_{s+1} \xi_l^1 \\
\vdots \\
\xi_N^1
\end{pmatrix}^j_{\beta} = \langle \delta ; \text{adj}_{s+1} \xi \rangle.
\]

Proof. Part 1. We start, for the sake of illustration, with the case \( N = 2 \), therefore \( s = 1 \) and

\[
g(\xi) = \sum_{l=1}^{n} \sum_{\alpha=1}^{n} \gamma_{l,\alpha} \xi_l^1 \xi_\alpha^2.
\]

Since \( g \) is rank one affine and quadratic then

\[
\frac{d^2}{dt^2} g(\xi + ta \otimes b) = g(\xi) + \langle \nabla g(\xi) ; a \otimes b \rangle = \sum_{l,\alpha=1}^{n} \gamma_{l,\alpha} a^1 b_l b_\alpha = 0,
\]

for every \( a = (a^1, a^2) \in \mathbb{R}^2 \), \( b = (b_1, \cdots, b_n) \in \mathbb{R}^n \). We therefore immediately deduce that \( \gamma_{l,\alpha} = -\gamma_{\alpha l} \) and hence

\[
g(\xi) = \sum_{1 \leq l < \alpha \leq n}^{n} \gamma_{l,\alpha} \left( \xi_l^1 \xi_\alpha^2 - \xi_l^1 \xi_\alpha^2 \right) = \sum_{1 \leq l < \alpha \leq n}^{n} \gamma_{l,\alpha} \det \begin{pmatrix}
\xi_l^1 & \xi_\alpha^1 \\
\xi_l^2 & \xi_\alpha^2
\end{pmatrix}
\]

\[
= \sum_{\beta=1}^{\binom{n}{2}} \delta^i_{\beta} (\text{adj}_2 \xi)_\beta = \langle \delta ; \text{adj}_2 \xi \rangle,
\]

since \( \text{adj}_2 \xi \) is a vector of \( \mathbb{R}^{\binom{n}{2}} \) composed of elements of the form \( \det (\xi_l, \xi_\alpha) \), \( 1 \leq l < \alpha \leq n \) and therefore \( \delta^i_{\beta} \) is essentially \( \gamma_{l,\alpha} \) with the appropriate sign.
Part 2. We now proceed with the general case. Let
\[ g(\xi) = \sum_{i=1}^{\binom{N-1}{s}} g^i(\xi) \text{ where } g^i(\xi) := \sum_{l=1}^{n_s} \sum_{\alpha=1}^{\binom{n}{s}} \gamma_{\alpha l}^i \xi_1^l \left( \adj_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \right)_\alpha. \]

As in the theorem, it is easy to see that \( g \) is rank one affine if and only if \( g^i \) is rank one affine. Therefore it is enough to prove, the stronger version, that for every \( i, 1 \leq i \leq \binom{N-1}{s} \) there exists \( j, 1 \leq j \leq \binom{N}{s+1} \), and \( \delta^j_\beta \in \mathbb{R} \), so that if
\[ g^i(\xi) := \sum_{l=1}^{n_s} \sum_{\alpha=1}^{\binom{n}{s}} \gamma_{\alpha l}^i \xi_1^l \left( \adj_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \right)_\alpha \]
is rank one affine, then
\[ g^i(\xi) = \sum_{\beta=1}^{\binom{n}{s+1}} \delta^j_\beta \left( \adj_{s+1} \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} \right)_\beta. \]

It is clear that the above identities imply the lemma. We should draw the attention that all the \( \delta^j_\beta \) corresponding to
\[ \left( \adj_{s+1} \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} \right)_\beta \]
which do not contain the row \( \xi^1 \) are chosen to be 0.

For notational convenience, we show the above result only when \( i = \binom{N-1}{s} \), the general case being handled similarly. So let \( i = \binom{N-1}{s} \), which corresponds to \( j = \binom{N}{s+1} \) and we therefore have
\[ \left( \adj_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} \right)_\alpha = (-1)^{i+1} \left( \adj_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^{s+1} \end{pmatrix} \right)_\alpha, \quad 1 \leq \alpha \leq \binom{n}{s}. \]

We also, from now on, drop the indices \( i \) and \( j \) and write, to simplify the notations, \( \gamma_{\alpha l}^i = (-1)^{i+1} \gamma_{\alpha l} \) in this case. We therefore have to show that if
\[ g(\xi) := \sum_{l=1}^{n_s} \sum_{\alpha=1}^{\binom{n}{s}} \gamma_{\alpha l} \xi_1^l \left( \adj_s \begin{pmatrix} \xi^2 \\ \vdots \\ \xi^{s+1} \end{pmatrix} \right)_\alpha \]
is rank one affine then there exists $\delta_\beta \in \mathbb{R}$, $1 \leq \beta \leq \binom{n}{s+1}$, such that

$$g(\xi) = \sum_{\beta = 1}^{\binom{n}{s+1}} \delta_\beta \left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^{s+1} \end{array} \right)_{\beta}. \quad (5.45)$$

Recall that for given $\alpha$, $1 \leq \alpha \leq \binom{n}{s}$, there exists a unique $s$-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ with $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_s \leq n$, such that

$$\left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^{s+1} \end{array} \right)_{\alpha} = (-1)^{1+\alpha} \det \left( \begin{array}{cccc} \xi^1_{\lambda_1} & \cdots & \xi^1_{\lambda_s} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_1} & \cdots & \xi^{s+1}_{\lambda_s} \end{array} \right). \quad (5.46)$$

We now fix an arbitrary $(s+1)$-tuple $(\lambda_1, \cdots, \lambda_{s+1})$, where $1 \leq \lambda_1 < \cdots < \lambda_{s+1} \leq n$ and we denote by $\beta$ the associate integer (as in (5.46)), more precisely

$$\left( \begin{array}{c} \xi^2 \\ \vdots \\ \xi^{s+1} \end{array} \right)_{\beta} = (-1)^{1+\beta} \det \left( \begin{array}{cccc} \xi^1_{\lambda_1} & \cdots & \xi^1_{\lambda_{s+1}} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_1} & \cdots & \xi^{s+1}_{\lambda_{s+1}} \end{array} \right).$$

Note that there are $\binom{n}{s+1}$ such $(s+1)$-tuples. Denote by $\alpha_1$ the integer corresponding (as in (5.46)) to the $s$-tuple $(\lambda_1, \cdots, \lambda_s)$, by $\alpha_k$ the integer corresponding to the $s$-tuple $(\lambda_1, \cdots, \lambda_{k-1}, \lambda_{k+1}, \cdots, \lambda_{s+1})$, $2 \leq k \leq s$ and by $\alpha_{s+1}$ the integer corresponding to the $s$-tuple $(\lambda_2, \cdots, \lambda_{s+1})$. Finally let

$$X_\beta(\xi) := \sum_{l_1 = 1}^{n} (-1)^{1+\alpha_1} \gamma_{l_1} \alpha_1 \xi^1_{l_1} \det \left( \begin{array}{cccc} \xi^2_{\lambda_1} & \cdots & \xi^2_{\lambda_s} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_1} & \cdots & \xi^{s+1}_{\lambda_s} \end{array} \right)$$

$$+ \sum_{k=2}^{s} \sum_{l_k = 1}^{n} (-1)^{1+\alpha_k} \gamma_{l_k} \alpha_k \xi^1_{l_k} \det \left( \begin{array}{cccc} \xi^2_{\lambda_1} & \cdots & \xi^2_{\lambda_{k-1}} & \xi^2_{\lambda_{k+1}} & \cdots & \xi^2_{\lambda_{s+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_1} & \cdots & \xi^{s+1}_{\lambda_{k-1}} & \xi^{s+1}_{\lambda_{k+1}} & \cdots & \xi^{s+1}_{\lambda_{s+1}} \end{array} \right)$$

$$+ \sum_{l_{s+1} = 1}^{n} (-1)^{1+\alpha_{s+1}} \gamma_{l_{s+1}} \alpha_{s+1} \xi^1_{l_{s+1}} \det \left( \begin{array}{cccc} \xi^2_{\lambda_2} & \cdots & \xi^2_{\lambda_{s+1}} \\ \vdots & \ddots & \vdots \\ \xi^{s+1}_{\lambda_2} & \cdots & \xi^{s+1}_{\lambda_{s+1}} \end{array} \right). \quad (5.47)$$
We then obviously have that
\[ g(\xi) = \sum_{\beta=1}^{\binom{n}{s+1}} X_\beta(\xi). \]

Since \( g \) is rank one affine, then so is \( X_\beta \). Therefore in order to show (5.45) it is then sufficient to find \( \delta_\beta \in \mathbb{R}, 1 \leq \beta \leq \binom{n}{s+1} \) such that
\[ X_\beta(\xi) = \delta_\beta \det \begin{pmatrix} \xi_{\lambda_1}^1 & \cdots & \xi_{\lambda_{s+1}}^1 \\ \vdots & \ddots & \vdots \\ \xi_{\lambda_1}^{s+1} & \cdots & \xi_{\lambda_{s+1}}^{s+1} \end{pmatrix}. \quad (5.48) \]

To deduce the claim we will use the fact that the function \( t \to X_\beta(\xi + ta \otimes b) \) is affine for every \( \xi \in \mathbb{R}^{N \times n}, a \in \mathbb{R}^N, b \in \mathbb{R}^n \). We will always choose
\[ a^1 = a^2 = 1 \quad \text{and} \quad a^3 = \cdots = a^N = 0 \]
and we will make several different choices of \( \xi \in \mathbb{R}^{N \times n} \) and \( b \in \mathbb{R}^n \).

1) We first choose \( \xi_{\lambda_1} = \xi_{\lambda_{s+1}} \), meaning that
\[ \xi_{\lambda_1} = \begin{pmatrix} \xi_{\lambda_1}^2 \\ \vdots \\ \xi_{\lambda_1}^{s+1} \end{pmatrix} = \xi_{\lambda_{s+1}} = \begin{pmatrix} \xi_{\lambda_{s+1}}^2 \\ \vdots \\ \xi_{\lambda_{s+1}}^{s+1} \end{pmatrix}. \quad (5.49) \]

For such a choice of \( \xi \), we have
\[
X_\beta(\xi) = \sum_{l_1=1}^{n} (-1)^{1+\alpha_1} \gamma_{l_1} \alpha_1 \xi_{l_1}^1 \det \begin{pmatrix} \xi_{\lambda_1}^2 & \cdots & \xi_{\lambda_{s+1}}^2 \\ \vdots & \ddots & \vdots \\ \xi_{\lambda_1}^{s+1} & \cdots & \xi_{\lambda_{s+1}}^{s+1} \end{pmatrix} \\
+ \sum_{l_{s+1}=1}^{n} (-1)^{1+\alpha_{s+1}} \gamma_{l_{s+1}} \alpha_{s+1} \xi_{l_{s+1}}^1 \det \begin{pmatrix} \xi_{\lambda_2}^2 & \cdots & \xi_{\lambda_{s+1}}^2 \\ \vdots & \ddots & \vdots \\ \xi_{\lambda_2}^{s+1} & \cdots & \xi_{\lambda_{s+1}}^{s+1} \end{pmatrix}.
\]

We then let
\[ b_l = 0 \quad \text{if} \quad l = \lambda_2, \cdots, \lambda_s. \quad (5.50) \]

Using the fact that the function \( t \to X_\beta(\xi + ta \otimes b) \) is affine, we deduce that the coefficient of the term in \( t^2 \) must be 0 for every above choices of \( \xi \) and \( b \).
We thus obtain that

\[
\left[ \sum_{l_1=1}^{n} (-1)^{1+\alpha_1} \gamma_{l_1 \alpha_1} b_{l_1} b_{\lambda_1} + \sum_{l_{s+1}=1}^{n} (-1)^{1+\alpha_{s+1}} (-1)^{s+1} \gamma_{l_{s+1} \alpha_{s+1}} b_{l_{s+1}} b_{\lambda_{s+1}} \right]
\]

\[
\begin{array}{cccc}
\xi_{\lambda_2}^3 & \cdots & \xi_{\lambda_s}^3 \\
\vdots & \ddots & \vdots \\
\xi_{\lambda_k}^{s+1} & \cdots & \xi_{\lambda_s}^{s+1}
\end{array}
\]

\[
=0.
\]

Since \( \xi \in \mathbb{R}^{N \times n} \) and \( b \in \mathbb{R}^n \) are arbitrary, letting aside (5.49) and (5.50), we find that

\[
\begin{array}{l}
\gamma_{l_1 \alpha_1} = 0 \text{ if } l_1 \neq \lambda_{s+1} \text{ and } \gamma_{l_{s+1} \alpha_{s+1}} = 0 \text{ if } l_{s+1} \neq \lambda_1 \\
(-1)^{1+\alpha_{s+1}} \gamma_{\lambda_1 \alpha_{s+1}} = (-1)^{s+1+\alpha_1} \gamma_{\lambda_{s+1} \alpha_1}.
\end{array}
\] (5.51)

2) We proceed in a similar manner with the other coefficients, namely we let, if \( 2 \leq k \leq s \),

\[
\xi_{\lambda_k} = \xi_{\lambda_{s+1}} \text{ and } b_l = 0 \text{ if } l = \lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_s.
\] (5.52)

We then use the fact that the function \( t \rightarrow X_\beta (\xi + ta \otimes b) \) is affine and thus the coefficient of the term in \( t^2 \) must be 0 for every \( \xi \) and \( b \) as in (5.52). We therefore get that

\[
\left[ \sum_{l_1=1}^{n} (-1)^{1+\alpha_1} \gamma_{l_1 \alpha_1} b_{l_1} (-1)^{k+1} b_{\lambda_k} + \sum_{l_k=1}^{n} (-1)^{1+\alpha_k} \gamma_{l_k \alpha_k} b_{l_k} (-1)^{s+1} b_{\lambda_{s+1}} \right]
\]

\[
\begin{array}{cccc}
\xi_{\lambda_1}^3 & \cdots & \xi_{\lambda_k}^3 & \cdots \\
\vdots & \ddots & \vdots & \ddots \\
\xi_{\lambda_k}^{s+1} & \cdots & \xi_{\lambda_k}^{s+1} & \cdots \\
\xi_{\lambda_k}^{s+1} & \cdots & \xi_{\lambda_s}^{s+1} & \cdots 
\end{array}
\]

\[
=0.
\]

As above we can then deduce that, for every \( 2 \leq k \leq s \),

\[
\begin{array}{l}
\gamma_{l_1 \alpha_1} = 0 \text{ if } l_1 \neq \lambda_{s+1} \text{ and } \gamma_{l_k \alpha_k} = 0 \text{ if } l_k \neq \lambda_k \\
(-1)^{1+\alpha_k} \gamma_{\lambda_k \alpha_k} = (-1)^{s+k+1+\alpha_1} \gamma_{\lambda_{s+1} \alpha_1}.
\end{array}
\] (5.53)
Combining (5.47), (5.51) and (5.53), we have

\[
X_\beta (\xi) = (-1)^{1+\alpha_1} \gamma_{\lambda_{s+1} \alpha_1} \xi_{\lambda_{s+1}}^1 \det \left( \begin{array}{cccc} \xi_{\lambda_1}^2 & \cdots & \xi_{\lambda_s}^2 \\ \vdots & \ddots & \vdots \\ \xi_{\lambda_1}^{s+1} & \cdots & \xi_{\lambda_s}^{s+1} \end{array} \right) \\
+ \sum_{k=2}^{s} (-1)^{1+\alpha_1} (-1)^{s+k+1} \gamma_{\lambda_{s+1} \alpha_1} \xi_{\lambda_k}^1 \\
\det \left( \begin{array}{cccc} \xi_{\lambda_1}^2 & \cdots & \xi_{\lambda_{k-1}}^2 & \xi_{\lambda_{k+1}}^2 & \cdots & \xi_{\lambda_{s+1}}^2 \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \xi_{\lambda_1}^{s+1} & \cdots & \xi_{\lambda_{k-1}}^{s+1} & \xi_{\lambda_{k+1}}^{s+1} & \cdots & \xi_{\lambda_{s+1}}^{s+1} \end{array} \right) \\
+ (-1)^{s} (-1)^{1+\alpha_1} \gamma_{\lambda_{s+1} \alpha_1} \xi_{\lambda_1}^1 \det \left( \begin{array}{cccc} \xi_{\lambda_2}^2 & \cdots & \xi_{\lambda_{s+1}}^2 \\ \vdots & \ddots & \vdots \\ \xi_{\lambda_2}^{s+1} & \cdots & \xi_{\lambda_{s+1}}^{s+1} \end{array} \right).
\]

Letting, in the above computation,

\[\delta_\beta := (-1)^{s+1+\alpha_1} \gamma_{\lambda_{s+1} \alpha_1}\]

we have indeed obtained (5.48). This completes the proof of the lemma. ■

5.3.2 Quadratic case

We now turn our attention to the case where \( f \) is quadratic. This case is of particular interest since the associated Euler-Lagrange equations are linear. It has therefore received much attention. Let us first mention the theorem.

**Theorem 5.25** Let \( M \) be a symmetric matrix in \( \mathbb{R}^{(N \times n) \times (N \times n)} \). Let

\[ f(\xi) := \langle M \xi; \xi \rangle, \]

where \( \xi \in \mathbb{R}^{N \times n} \) and \( \langle \cdot;\cdot \rangle \) denotes the scalar product in \( \mathbb{R}^{N \times n} \). The following statements then hold.

(i) \( f \) is rank one convex if and only if \( f \) is quasiconvex.

(ii) If \( N = 2 \) or \( n = 2 \), then

\[ f \text{ polyconvex} \iff f \text{ quasiconvex} \iff f \text{ rank one convex}. \]

(iii) If \( N, n \geq 3 \), then in general

\[ f \text{ rank one convex} \not\Rightarrow f \text{ polyconvex}. \]

**Remark 5.26** (i) The proof of (i) of Theorem 5.25 was given by Van Hove [585], [586], although it was implicitly known earlier.
(ii) The second part of the theorem has received considerable attention. The question was raised in 1937 by Bliss and received a progressive answer through the works of Albert [9], Hestenes-MacShane [338], MacShane [411], Marcellini [422], Reid [506], Serre [530] and Terpstra [575]. The proof of (ii) of Theorem 5.25 relies on an algebraic lemma whose importance is summarized in Uhlig [582].

(iii) A counterexample to the third part of the theorem was given by Terpstra [575] and later by Serre [530] (see also Ball [56]).

(iv) Note also that even if $N = n = 2$ and $f$ is quadratic, then in general $f$ polyconvex $\not\Rightarrow f$ convex,
as the trivial example $f(\xi) = \det \xi$ shows.

Before proceeding with the proof of the theorem we mention two simple facts that are summarized in the next lemmas.

**Lemma 5.27** Let $M$ be a symmetric matrix in $\mathbb{R}^{(N \times n) \times (N \times n)}$ and let

$$f(\xi) := \langle M\xi; \xi \rangle.$$  

Then the following results hold.

(i) $f$ is convex if and only if $f(\xi) \geq 0$ for every $\xi \in \mathbb{R}^{N \times n}$.

(ii) $f$ is polyconvex if and only if there exists $\alpha \in \mathbb{R}^{\sigma(2)}$ such that $f(\xi) \geq \langle \alpha; \text{adj}_2 \xi \rangle$ for every $\xi \in \mathbb{R}^{N \times n}$ and where $\langle \cdot; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{\sigma(2)}$ and $\sigma(2) = \binom{N}{2} \binom{n}{2}$.

(iii) $f$ is quasiconvex if and only if

$$\int_D f(\nabla \varphi(x)) \, dx \geq 0$$

for every bounded open set $D \subset \mathbb{R}^n$ and for every $\varphi \in W^{1,\infty}_0(D; \mathbb{R}^N)$.

(iv) $f$ is rank one convex if and only if $f(a \otimes b) \geq 0$ for every $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$. 

Proof. (Lemma 5.27). Parts (i), (iii) and (iv) are trivial. The fact that
\[ f(\xi) \geq \langle \alpha; \text{adj}_2 \xi \rangle \] (5.54)
implies that \( f \) is polyconvex follows immediately from the following observation.

Let
\[ g(\xi) := f(\xi) - \langle \alpha; \text{adj}_2 \xi \rangle \]
then by (5.54) and (i) of the lemma, we deduce that \( g \) is convex. Thus \( f(\xi) = g(\xi) + \langle \alpha; \text{adj}_2 \xi \rangle \) is polyconvex.

Assume now that \( f \) is polyconvex. We wish to show that (5.54) holds for some \( \alpha \in \mathbb{R}^{\sigma(2)} \). Using Theorem 5.6, bearing in mind that \( f(0) = 0 \), we find that there exists \( \beta = (\beta_{\sigma(1)}, \beta_{\sigma(2)}, \ldots, \beta_{\sigma(n \wedge N)}) \in \mathbb{R}^{\tau(n,N)} \) such that
\[ f(\xi) \geq \langle \beta; T(\xi) \rangle = \sum_{s=1}^{n \wedge N} \langle \beta_{\sigma(s)}; \text{adj}_s \xi \rangle. \]

Multiplying \( \xi \) by \( \epsilon > 0 \), we get
\[ f(\epsilon \xi) = \epsilon^2 f(\xi) \geq \epsilon \langle \beta_{\sigma(1)}; \xi \rangle + \epsilon^2 \langle \beta_{\sigma(2)}; \text{adj}_2 \xi \rangle + O(\epsilon^3). \] (5.55)

Dividing by \( \epsilon \) and letting \( \epsilon \to 0 \), we obtain
\[ \langle \beta_{\sigma(1)}; \xi \rangle \leq 0 \]
for every \( \xi \in \mathbb{R}^{n \times n} \), thus \( \beta_{\sigma(1)} = 0 \). Returning to (5.55), dividing by \( \epsilon^2 \) and letting \( \epsilon \to 0 \) we have indeed obtained (5.54) with \( \alpha = \beta_{\sigma(2)} \).

The second important point that we wish to mention is the following lemma concerning Fourier transforms for which the proof is straightforward.

Lemma 5.28 Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set. Let \( \varphi \in W^{1,\infty}_0(\Omega; \mathbb{R}^N) \) be extended by \( \varphi \equiv 0 \) outside of \( \Omega \). Define for \( \xi \in \mathbb{R}^n \)
\[ \hat{\varphi}^\alpha(\xi) := \int_{\mathbb{R}^n} \varphi^\alpha(x) e^{-2\pi i \langle \xi; x \rangle} dx, \; 1 \leq \alpha \leq N. \]
Then
\[ \hat{\nabla} \varphi = 2\pi i \left( \hat{\varphi}^\alpha \xi_j \right)_{1 \leq j \leq n} = 2\pi i \otimes \xi, \]
in particular \( \text{rank}\{\text{Re}(\hat{\nabla} \varphi)\}, \text{rank}\{\text{Im}(\hat{\nabla} \varphi)\} \leq 1. \)

Remark 5.29 Lemma 5.28 explains in a way other than that of Theorem 5.3 why matrices of rank one play such an important role in quasiconvex analysis.

We now proceed with the proof of Theorem 5.25.

Proof. (i) Recall that
\[ f(\xi) = \langle M\xi; \xi \rangle. \]
Theorem 5.3 implies that if \( f \) is quasiconvex then \( f \) is rank one convex. We now prove the converse. By Lemma 5.27 we have to show that

\[
\int \Omega \langle M \nabla \varphi (x) ; \nabla \varphi (x) \rangle \, dx \geq 0 \quad (5.56)
\]

for every bounded open set \( \Omega \), for every \( \varphi \in W^{1,\infty}_0(\Omega;\mathbb{R}^N) \) (we will set \( \varphi \equiv 0 \) outside of \( \Omega \)), knowing that

\[
f (a \otimes b) = \langle Ma \otimes b ; a \otimes b \rangle \geq 0. \quad (5.57)
\]

We then use Plancherel formula (we write \( \bar{\xi} \) for the complex conjugate of \( \xi \)) to get

\[
\int \Omega \langle M \nabla \varphi (x) ; \nabla \varphi (x) \rangle \, dx = \int_{\mathbb{R}^n} \langle M \nabla \varphi (\xi) ; \nabla \varphi (\xi) \rangle \, d\xi. \quad (5.58)
\]

Using Lemma 5.28 and (5.57) in (5.58), we obtain (5.56).

(ii) We do not prove this result and we refer to the above bibliography.

(iii) We now want to show that if \( N = n = 3 \), then there exists \( f \) rank one convex which is not polyconvex. We give here an example due to Serre [530]. Let

\[
\xi = \begin{pmatrix}
\xi_1^1 & \xi_2^1 & \xi_3^1 \\
\xi_1^2 & \xi_2^2 & \xi_3^2 \\
\xi_1^3 & \xi_2^3 & \xi_3^3
\end{pmatrix}
\]

and let

\[
f (\xi) := 
(\xi_1^1 - \xi_2^3 - \xi_3^2)^2 + (\xi_1^3 - \xi_2^1 + \xi_3^1)^2 \\
+ (\xi_1^2 - \xi_3^1 - \xi_3^3)^2 + (\xi_2^3)^2 + (\xi_3^3)^2.
\]

We divide the proof into two steps.

**Step 1.** We first show that there exists \( \epsilon > 0 \) such that

\[
f (a \otimes b) - \epsilon |a \otimes b|^2 \geq 0 \quad (5.59)
\]

for every \( a, b \in \mathbb{R}^3 \) and where \( |\xi|^2 := \langle \xi ; \bar{\xi} \rangle \) denotes the Euclidean norm. Lemma 5.27 will then ensure that

\[
g (\xi) = f (\xi) - \epsilon |\xi|^2 \quad (5.60)
\]

is rank one convex. In Step 2 we then prove that this \( g \) is not polyconvex and this will end the proof of the theorem. We first let

\[
\epsilon_0 := \inf \{ f (a \otimes b) : a, b \in \mathbb{R}^3, |a \otimes b| = 1 \}. \quad (5.61)
\]
Then, since \( f \geq 0 \), we have \( \epsilon_0 \geq 0 \). In order to prove (5.59) it is sufficient to prove that \( \epsilon_0 > 0 \). We proceed by contradiction and assume that \( \epsilon_0 = 0 \). Observe that in (5.61) the minimum is attained and therefore there exist \( a, b \in \mathbb{R}^3 \) such that

\[
f (a \otimes b) = \epsilon_0 = 0 \quad \text{and} \quad |a \otimes b| = 1.
\]

(5.62)

Recall that

\[
a \otimes b = \begin{pmatrix} a^1 b_1 & a^1 b_2 & a^1 b_3 \\ a^2 b_1 & a^2 b_2 & a^2 b_3 \\ a^3 b_1 & a^3 b_2 & a^3 b_3 \end{pmatrix},
\]

therefore the first equation of (5.62) becomes

\[
\begin{align*}
a^1 b_1 &= a^2 b_3 + a^3 b_2 \\
a^1 b_2 &= a^3 b_1 - a^1 b_3 \\
a^2 b_1 &= a^1 b_3 + a^3 b_1 \\
a^2 b_2 &= 0 \\
a^3 b_3 &= 0.
\end{align*}
\]

(5.63)

We then show that (5.63) is in contradiction with the fact that \( |a \otimes b| = 1 \). To do so, we carefully examine (5.63) and separate the discussion in several cases.

**Case 1.** \( a^2 = a^3 = 0 \) (cf. the two last equations of (5.63)), then (5.63) becomes

\[
\begin{align*}
a^2 &= a^3 = 0 \\
a^1 b_1 &= a^1 b_3 = 0 \\
a^1 b_2 &= -a^1 b_3.
\end{align*}
\]

(5.64)

**Case 1a.** \( a^1 = 0 \), therefore \( a^1 = a^2 = a^3 = 0 \) and hence \( |a \otimes b| = 0 \), contradiction.

**Case 1b.** \( b_1 = 0 \), hence from (5.64), \( a^1 b_3 = 0 \) and thus \( a^1 b_2 = 0 \). We then also conclude that \( |a \otimes b| = 0 \) and this is a contradiction.

**Case 2.** \( a^2 = b_3 = 0 \) (cf. the two last equations of (5.63)), then (5.63) becomes

\[
\begin{align*}
a^2 &= b_3 = 0 \\
a^1 b_1 &= a^3 b_2 \\
a^1 b_2 &= a^3 b_1 \\
a^3 b_1 &= 0.
\end{align*}
\]

**Case 2a.** \( a^3 = 0 \), then \( a^1 b_1 = a^1 b_2 = 0 \) and therefore \( |a \otimes b| = 0 \), contradiction.

**Case 2b.** \( b_1 = 0 \), then \( a^3 b_2 = a^3 b_2 = 0 \) and therefore \( |a \otimes b| = 0 \), contradiction.

Similarly for the case \( a^3 = b_2 = 0 \) and \( b_2 = b_3 = 0 \). Thus \( \epsilon_0 > 0 \) and hence Step 1, i.e. \( g \) defined by (5.60), is rank one convex for every \( 0 < \epsilon \leq \epsilon_0 \).
Step 2. We now show that \( g \) is not polyconvex. In view of Lemma 5.27 it is sufficient to show that for every \( \alpha \in \mathbb{R}^{3 \times 3} \), there exists \( \xi \in \mathbb{R}^{3 \times 3} \) such that

\[
g(\xi) + \langle \alpha; \text{adj}_2 \xi \rangle < 0.
\]

We prove that the above inequality holds for matrices \( \xi \) of the following form

\[
\xi := \begin{pmatrix}
    b + d & c - a & a \\
    c + a & 0 & b \\
    c & d & 0
\end{pmatrix}.
\]

For such matrices we have \( f(\xi) = 0 \) and therefore

\[
g(\xi) = -\varepsilon |\xi|^2 = -\varepsilon \left[(b + d)^2 + (c - a)^2 + a^2 + (c + a)^2 + b^2 + c^2 + d^2\right]
\]

and

\[
\text{adj}_2 \xi = \begin{pmatrix}
    -bd & bc & cd + ad \\
    ad & -ac & -(bd + d^2 - c^2 + ac) \\
    bc - ab & ac + a^2 - b^2 - bd & a^2 - c^2
\end{pmatrix}.
\]

Therefore

\[
\langle \alpha; \text{adj}_2 \xi \rangle = -\alpha_1 bd + \alpha_2 bc + \alpha_3 (cd + ad)
+ \alpha_4 ad - \alpha_5 ac - \alpha_6 (bd + d^2 - c^2 + ac)
+ \alpha_7 (bc - ab) + \alpha_8 (ac + a^2 - b^2 - bd) + \alpha_9 (a^2 - c^2).
\]

As in Step 1 we consider several cases.

Case 1. If \( \alpha_8 > 0 \), then take \( a = c = d = 0 \) and \( b \neq 0 \), to get

\[
g(\xi) + \langle \alpha; \text{adj}_2 \xi \rangle = -\varepsilon |\xi|^2 + \langle \alpha; \text{adj}_2 \xi \rangle
= -\varepsilon (2b^2) - \alpha_8 b^2 < 0.
\]

Case 2. If \( \alpha_6 > 0 \), then take \( a = b = c = 0 \) and \( d \neq 0 \), to get

\[
g(\xi) + \langle \alpha; \text{adj}_2 \xi \rangle = -\varepsilon (2d^2) - \alpha_6 d^2 < 0.
\]

We therefore can assume that \( \alpha_8 \leq 0 \) and \( \alpha_6 \leq 0 \).

Case 3. If \( \alpha_9 - \alpha_6 > 0 \) (\( \alpha_8 \leq 0 , \alpha_6 \leq 0 \)), then take \( a = b = d = 0 \) and \( c \neq 0 \) to get

\[
g(\xi) + \langle \alpha; \text{adj}_2 \xi \rangle = -\varepsilon (3c^2) + (\alpha_6 - \alpha_9) c^2 < 0.
\]

We therefore assume \( \alpha_8 \leq 0 , \alpha_6 \leq 0 \) and \( \alpha_9 - \alpha_6 \leq 0 \). From these three inequalities we deduce that \( \alpha_8 + \alpha_9 \leq 0 \), and then taking \( b = c = d = 0 \) and \( a \neq 0 \), we get

\[
g(\xi) + \langle \alpha; \text{adj}_2 \xi \rangle = -\varepsilon (3a^2) + (\alpha_8 + \alpha_9) a^2 < 0.
\]

And this concludes the proof of the theorem. \( \blacksquare \)
5.3.3 Convexity of $SO(n) \times SO(n)$ and $O(N) \times O(n)$ invariant functions

We now discuss the different notions of convexity for functions having some symmetries and follow the presentation of Dacorogna-Maréchal [204].

Let $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ and let $\Gamma_1 \subset \mathbb{R}^{N \times N}$ be a subgroup of $GL(N)$ (the set of invertible matrices) and $\Gamma_2 \subset \mathbb{R}^{n \times n}$ be a subgroup of $GL(n)$. Assume that $f$ is $\Gamma_1 \times \Gamma_2$-invariant, meaning that

$$f(U\xi V) = f(\xi), \quad \forall U \in \Gamma_1, \forall V \in \Gamma_2.$$

We will be concerned with groups $\Gamma$ that are either $O(n)$ (the set of orthogonal matrices) or $SO(n)$ (the set of special orthogonal matrices); see Chapter 13 for precise definitions.

We start with some notation and we refer to Chapter 13 for more details. In the whole of this section, we assume that $N \geq n$, but all the results can be carried in a straightforward way to the case where $N \leq n$.

**Notation 5.30**

(i) Let $N \geq n$ and $\xi \in \mathbb{R}^{N \times n}$. The *singular values* of $\xi$, denoted by

$$0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi),$$

are defined to be the square root of the eigenvalues of the symmetric and positive semidefinite matrix $\xi^t \xi \in \mathbb{R}^{n \times n}$. A similar definition holds when $N \leq n$. We let

$$\lambda(\xi) = (\lambda_1(\xi), \cdots, \lambda_n(\xi)).$$

(ii) When $N = n$, we denote by

$$0 \leq \mu_1(\xi) \leq \cdots \leq \mu_n(\xi),$$

the *signed singular values* of $\xi \in \mathbb{R}^{n \times n}$; they are defined as

$$\mu_1(\xi) = \lambda_1(\xi) \text{ sign}(\det \xi) \quad \text{and} \quad \mu_j(\xi) = \lambda_j(\xi), \quad j = 2, \cdots, n.$$

We let

$$\mu(\xi) = (\mu_1(\xi), \cdots, \mu_n(\xi)).$$

(iii) We denote, for every integer $m \geq 1$:

- $\Pi(m)$ the subgroup of $O(m)$ that consists of the matrices having exactly one nonzero entry per row and per column, moreover each entry belongs to $\{-1, 1\}$;
- $\Pi_e(m)$ the subgroup of $\Pi(m)$ that consists of the matrices having an even number of entries equal to $-1$;
- $S(m)$ the subgroup of $\Pi_e(m)$ of all permutation matrices.
We therefore have
\[ S(m) \subset \Pi_e(m) \subset \Pi(m) \subset O(m) \subset GL(m). \]

(iv) We let \( \mathbb{R}^{N \times n}_d \) be the subspace of \( \mathbb{R}^{N \times n} \) consisting of diagonal matrices, meaning that
\[ \xi \in \mathbb{R}^{N \times n}_d \Rightarrow \xi_{ij} = 0 \text{ if } i \neq j. \]

(v) For a vector \( x = (x_1, \cdots, x_n) \in \mathbb{R}^n \), we denote by \( \text{diag}_{N \times n} \) (when \( N = n \) we simply write \( \text{diag} \)) the matrix \( \xi \in \mathbb{R}^{N \times n}_d \) such that
\[ \xi_{ii} = x_i. \]

We start with some simple observations. The first proposition is an immediate consequence of the singular values decomposition theorem (see Theorem 13.3).

**Proposition 5.31** (i) Let \( f : \mathbb{R}^{n \times n} \to \mathbb{R} \cup \{+\infty\} \). Then \( f \) is \( SO(n) \times SO(n) \)-invariant if and only if \( f \) satisfies
\[ f = f \circ \text{diag} \circ \mu, \]
and
\[ g := f \circ \text{diag} \]
is then the unique \( \Pi_e(n) \)-invariant function such that \( f = g \circ \mu. \)

(ii) Let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\} \), where \( N \geq n \). Then \( f \) is \( O(N) \times O(n) \)-invariant if and only if \( f \) satisfies
\[ f = f \circ \text{diag}_{N \times n} \circ \lambda, \]
and
\[ g := f \circ \text{diag}_{N \times n} \]
is then the unique \( \Pi(n) \)-invariant function such that \( f = g \circ \lambda. \)

It is clear that, if \( N = n \), the notions of \( O(N) \times O(n) \), \( SO(N) \times O(n) \) and \( O(N) \times SO(n) \)-invariance coincide but differ from that of \( SO(N) \times SO(n) \)-invariance. However, if \( N \neq n \), all four notions coincide as we now show.

**Proposition 5.32** Let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\} \), where \( N > n \). Then the following are equivalent:

(i) \( f \) is \( O(N) \times O(n) \)-invariant;

(ii) \( f \) is \( SO(N) \times SO(n) \)-invariant.

**Proof.** Obviously, we need only prove that (ii) implies (i). We will see that, if \( f \) is \( SO(N) \times SO(n) \)-invariant, then
\[ f = f \circ \text{diag}_{N \times n} \circ \lambda. \] (5.65)
The conclusion will then follow from Proposition 5.31.

Let $\xi \in \mathbb{R}^{N \times n}$. By the singular values decomposition theorem (Theorem 13.3), there exist $U \in O(N)$, $V \in O(n)$ such that

$$\xi = U\Lambda V^t,$$

where $\Lambda := \text{diag}_{N \times n}(\lambda_1(\xi), \cdots, \lambda_n(\xi))$.

So we have to consider several cases. First of all let us introduce the following notation. If $m \geq 1$ is an integer, we let

$$H_m := \text{diag}(-1, 1, \cdots, 1) \in \mathbb{R}^{m \times m} \quad \text{and} \quad K_m := \text{diag}(1, \cdots, 1, -1) \in \mathbb{R}^{m \times m}.$$

- If $U \in SO(N)$ and $V \in SO(n)$, then, from (ii) the conclusion follows, namely

$$f(\xi) = f(\Lambda) = (f \circ \text{diag}_{N \times n} \circ \lambda)(\xi).$$

- If $U \in O(N) - SO(N)$ and $V \in O(n) - SO(n)$, we may write $\Lambda = H_N \Lambda H_n$, so that

$$U\Lambda V^t = (UH_N)\Lambda(VH_n)^t$$

with $UH_N \in SO(N)$ and $VH_n \in SO(n)$. Thus (5.65) holds by (ii).

- If $U \in O(N) - SO(N)$ and $V \in SO(n)$, we may write $\Lambda = K_N \Lambda$, so that

$$U\Lambda V^t = (UK_N)\Lambda V^t$$

with $UK_N \in SO(N)$. Equation (5.65) then follows from (ii).

- If $U \in SO(N)$ and $V \in O(n) - SO(n)$, we may write $\Lambda = H_N K_N \Lambda H_n$, so that

$$U\Lambda V^t = (UH_N K_N)\Lambda(VH_n)^t,$$

with $UH_N K_N \in SO(N)$ and $VH_n \in SO(n)$. Thus (5.65) holds.

We have therefore shown the claim, namely that $f = f \circ \text{diag}_{N \times n} \circ \lambda$. ■

The main result concerns the convexity of such functions.

**Theorem 5.33 (A) Let $f: \mathbb{R}^{n \times n} \to \mathbb{R} \cup \{+\infty\}$ be $SO(n) \times SO(n)$-invariant, $f \neq +\infty$, and let $g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the unique $\Pi_e(n)$-invariant function such that

$$f = g \circ \mu.$$**

Then the following are equivalent:

(i) $f$ is lower semicontinuous and convex;

(ii) the restriction of $f$ to $\mathbb{R}^{d \times n}$, the subspace of $\mathbb{R}^{n \times n}$ of diagonal matrices, is lower semicontinuous and convex;

(iii) $g$ is lower semicontinuous and convex.
Let $N > n$, let $f: \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ be $SO(N) \times SO(n)$-invariant or, equivalently, $O(N) \times O(n)$-invariant, $f \neq +\infty$, and let $g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ to be the unique $\Pi(n)$-invariant function such that $f = g \circ \lambda$.

Then the following are equivalent:

(i) $f$ is lower semicontinuous and convex;

(ii) the restriction of $f$ to $\mathbb{R}^d_{N \times n}$, the subspace of $\mathbb{R}^{N \times n}$ of diagonal matrices, is lower semicontinuous and convex;

(iii) $g$ is lower semicontinuous and convex.

Remark 5.34 (i) We discuss now the history of this theorem first in the case where $N = n$ and in the $O(n) \times O(n)$-invariant case. The result was established by Ball [53], Hill [341] and Thompson-Freede [577]; see also Dacorogna-Marcellini [202] and Le Dret [397]. In elasticity, an $O(n) \times O(n)$-invariant function is called isotropic.

(ii) The case $N = n$ and $SO(n) \times SO(n)$-invariant, was first established by Dacorogna-Koshigoe [192] in the case $n = 2$, and later by Vincent [589] when $n \geq 3$, as a consequence of the convexity theorem of Kostant [377]. A different proof, inspired by Rosakis [516] and based on the notion of signed singular values and a generalized Von Neumann inequality (see Theorem 13.10), was given by Dacorogna-Maréchal [204]. In this last paper, the case $N \neq n$ was also handled.

Proof. (A) The fact that (i) implies (ii) is clear. The fact that (ii) implies (iii) results immediately from the equality $g = f \circ \text{diag}$. Finally, suppose that (iii) holds. Then $g^{**} = g$, and Theorem 6.17 (i) implies that

$$f^{**} = g^{**} \circ \mu = g \circ \mu = f,$$

which shows that $f$ is lower semicontinuous and convex.

(B) The fact that (i) implies (ii) is clear. The fact that (ii) implies (iii) results immediately from the equality $g = f \circ \text{diag}_{N \times n}$. Finally, suppose that (iii) holds. Theorem 6.17 (ii) then implies that

$$f^{**} = g^{**} \circ \lambda = g \circ \lambda = f,$$

which shows that $f$ is lower semicontinuous and convex. 

In the case of $O(n) \times O(n)$-invariant functions, the analogous statement can be derived in several ways from the above results and we do not discuss the details.

Corollary 5.35 Let $f: \mathbb{R}^{n \times n} \to \mathbb{R} \cup \{+\infty\}$ be $O(n) \times O(n)$-invariant, $f \neq +\infty$, and let $g: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the unique $\Pi(n)$-invariant function such that $f = g \circ \lambda$. 

Then the following are equivalent:

(i) $f$ is lower semicontinuous and convex;

(ii) the restriction of $f$ to $\mathbb{R}^{n\times n}_d$ is lower semicontinuous and convex;

(iii) $g$ is lower semicontinuous and convex.

**Remark 5.36** As a convex $\Pi(n)$-invariant function, the function $g$ appearing in Theorem 5.33 (B) or in Corollary 5.35 must be such that each function $x_k \to g(x_1, \cdots, x_n), \quad k = 1, \cdots, n$

is non-decreasing on $\mathbb{R}_+$. We now prove this only when $k = 1$, the other cases being handled similarly. As a matter of fact, for all $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ with $x_1 \geq 0$,

$g(0, x_2, \cdots, x_n) \leq \frac{1}{2}g(-x_1, x_2, \cdots, x_n) + \frac{1}{2}g(x_1, x_2, \cdots, x_n) = g(x),$

and if $z > 0$, we see, using the above inequality, that

$g(x) \leq \frac{x_1}{x_1 + z}g(x_1 + z, x_2, \cdots, x_n) + \frac{z}{x_1 + z}g(0, x_2, \cdots, x_n)$

$\leq \frac{x_1}{x_1 + z}g(x_1 + z, x_2, \cdots, x_n) + \frac{z}{x_1 + z}g(x_1 + z, x_2, \cdots, x_n)$

$= g(x_1 + z, x_2, \cdots, x_n).$

Thus $x_1 \to g(x_1, \cdots, x_n)$ is non-decreasing on $\mathbb{R}_+$. ♦

We now give a simple corollary, which follows from Theorem 5.33 and in a more direct way from Theorem 13.10. It will be used in Theorems 5.39, 5.43 and 7.43.

**Corollary 5.37** Let $\xi \in \mathbb{R}^{n\times n}$ and

$0 \leq b_1 \leq \cdots \leq b_n.$

The functions

$f_\nu(\xi) = \sum_{i=\nu}^{n} b_i \lambda_i(\xi)$

are convex for every $\nu = 1, \cdots, n$.

If $|b_1| \leq b_2 \leq \cdots \leq b_n$, then the following functions are also convex

$g_\nu(\xi) = \sum_{i=\nu}^{n} b_i \mu_i(\xi), \quad \nu = 1, \cdots, n.$
5.3.4 Polyconvexity and rank one convexity of $SO(n) \times SO(n)$ and $O(N) \times O(n)$ invariant functions

We now discuss the polyconvexity and rank one convexity of functions having the symmetries considered in the previous section. We first discuss the case of a $O(N) \times O(n)$-invariant function and then the $SO(2) \times SO(2)$-invariant case. We also assume, as in the previous section, that $N \geq n$, but all the results immediately extend to the case where $N \leq n$.

We start with some notation.

**Notation 5.38** Let $N \geq n$.

(i) We let

$$
\mathbb{R}^n_+ := \{ x \in \mathbb{R}^n : x_i \geq 0, \ i = 1, \cdots, n \},
$$

$$
K^n_+ := \{ x \in \mathbb{R}^n : 0 \leq x_1 \leq \cdots \leq x_n \}.
$$

In particular, when $n = 1$, $K^1_+ = \mathbb{R}^1_+$.

(ii) For $X \in \mathbb{R}^{(N)} \times (n)$, $1 \leq s \leq n-1$, we denote by $\Lambda^s(X) \in K^{(n)}_+$ its singular values. In particular, when $s = 1$, we have

$$
\Lambda^1(\xi) = (\lambda_1(\xi), \cdots, \lambda_n(\xi)).
$$

In the notation of Section 5.3.3 we have $\Lambda^1(\xi) = \lambda(\xi)$.

(iii) For every $x \in K^n_+$, we adopt the following notation.

- If $s = 2$, we let

$$
\text{adj}_2 x \in K^{(n)}_+
$$

the vector in $\mathbb{R}^{(n)}_2$ composed of every $x_i x_j$ with $i < j$ rearranged in an increasing way (for example if $n = 3$ then $\text{adj}_2 x = (x_1 x_2, x_1 x_3, x_2 x_3)$). Note that, unless $n = 2, 3$, the ordering of $\text{adj}_2 x$ depends on $x$ itself. For example, if $n = 4$, then for some $x$ we can have

$$
\text{adj}_2 x = (x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, x_3 x_4)
$$

and for others

$$
\text{adj}_2 x = (x_1 x_2, x_1 x_3, x_2 x_3, x_1 x_4, x_2 x_4, x_3 x_4).
$$

- Similarly, if $2 < s < n$, we let

$$
\text{adj}_s x \in K^{(n)}_+
$$

to be the vector in $\mathbb{R}^{(n)}_s$ composed of every $x_{i_1} \cdots x_{i_s}$, $i_1 < \cdots < i_s$ rearranged in an increasing way.
- Finally, when $s = n$, we denote by either of the following symbols

$$\text{adj}_n x = \det x = \prod_{i=1}^{n} x_i.$$ 

Note that with these notations we have for every $\xi \in \mathbb{R}^{N \times n}$ and every $1 \leq s \leq n$ that

$$\Lambda^s (\text{adj}_s \xi) = \text{adj}_s \Lambda^1 (\xi).$$

The next theorem is stated, for the convenience of the reader, first when $N = n = 2$, then when $N = n = 3$ and finally in the general case $N \geq n$.

**Theorem 5.39** Let $N \geq n$,

$$0 \leq \lambda_1 (\xi) \leq \cdots \leq \lambda_n (\xi),$$

be the singular values of $\xi \in \mathbb{R}^{N \times n}$. Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ and $g : \mathbb{R}^{n}_+ \to \mathbb{R}$ be such that

$$f (\xi) = g (\lambda_1 (\xi), \cdots, \lambda_n (\xi)).$$

(i) Let $N = n = 2$. Assume that there exists

$$G : \mathbb{R}^{2}_+ \times \mathbb{R}^{2}_+ \to \mathbb{R}, \ G = G (x, \delta) = G (x_1, x_2, \delta),$$

convex, non-decreasing in each variable, symmetric with respect to the first two variables, meaning that

$$G (x_2, x_1, \delta) = G (x_1, x_2, \delta),$$

and such that

$$g (x_1, x_2) = G (x_1, x_2, x_1 x_2),$$

then $f$ is polyconvex.

(ii) Let $N = n = 3$. Assume that there exists

$$G : \mathbb{R}^{3}_+ \times \mathbb{R}^{3}_+ \times \mathbb{R}_+ \to \mathbb{R}$$

$$G = G (x, y, \delta) = G (x_1, x_2, x_3, y_1, y_2, y_3, \delta)$$

convex, non-decreasing in each variable and symmetric in the variables $x$ and $y$ separately, meaning that for every permutation $P$ and $P'$ of three elements

$$G (Px, P'y, \delta) = G (x, y, \delta),$$

and such that

$$g (x_1, x_2, x_3) = G (x_1, x_2, x_3, x_1 x_2, x_1 x_3, x_2 x_3, x_1 x_2 x_3).$$
Then \( f \) is polyconvex.

(iii) General case: \( N \geq n \). Assume that there exists

\[
G : \mathbb{R}_+^n \times \mathbb{R}_+^{(n)} \times \cdots \times \mathbb{R}_+^{(n-1)} \times \mathbb{R}_+ \to \mathbb{R}
\]

\[
G = G(z) = G(z^1, z^2, \ldots, z^{n-1}, z^n)
\]

convex, non-decreasing in each variable and symmetric in each of the variables \( z^i \) separately, i.e., for every permutation \( P_i \) of \( \binom{n}{i} \) elements

\[
G(P_1 \Lambda^1, P_2 \Lambda^2, \cdots, P_{n-1} \Lambda^{n-1}, \Lambda^n) = G(\Lambda^1, \Lambda^2, \cdots, \Lambda^{n-1}, \Lambda^n)
\]

and such that

\[
g(x) = G(x, \text{adj}_2 x, \cdots, \text{adj}_{n-1} x, \text{adj}_n x).
\]

Then \( f \) is polyconvex.

Remark 5.40 (i) The above result is due to Ball [53] when \( N = n = 2 \) and \( N = n = 3 \) and to Dacorogna-Marcellini [202] when \( N = n \). Here we follow this last proof. A different approach, more in the spirit of Section 5.3.3, has been given by Dacorogna-Maréchal [205]. One can also consult Mielke [443].

(ii) The above sufficient condition is in some sense also necessary, once we have taken care of the appropriate symmetries implied by the fact that \( f \) depends only on singular values. For example, since the function \( f \) does not see changes of signs of the determinant, then \( G \) should not see it either (and the function \( F \), defined in the proof, as well). This will be achieved in Theorem 5.43 when \( N = n = 2 \).

Proof. We first proceed, just for the sake of better understanding the proof, with the case \( N = n = 2 \).

Case: \( N = n = 2 \). We divide the proof into two steps.

Step 1. We start with the following preliminary observation. Since \( G \) is convex over \( \mathbb{R}_+^2 \times \mathbb{R}_+ \) we have (cf. Corollary 2.51)

\[
G(x, \delta) = \sup_{b_0, b_2 \in \mathbb{R}} \left\{ \begin{array}{l}
\left. b_0 + \langle b_1; x \rangle + b_2 \delta : \right. \\
\left. b_0 + \langle b_1; y \rangle + b_2 \epsilon \leq G(y, \epsilon), \ \forall (y, \epsilon) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \right\}
\right.
\]

It is easy to see (cf. below) that since \( x \in K_+^2 \) and \( \delta \geq 0 \) and since \( G \) is non decreasing in each variable and symmetric in the \( x \) variable, there is no loss of generality in considering the supremum only on \( b_2 \geq 0 \) and \( b_1 \in K_+^2 \). Hence,
for every \((x, \delta) \in K_+^2 \times \mathbb{R}_+\), we have

\[
G(x, \delta) = \sup_{\substack{b_0 \in \mathbb{R} \\ b_2 \geq 0 \\ b_1 \in K_+^2}} \left\{ \begin{array}{l}
\sum b_0 + \langle b_1; x \rangle + b_2 \delta : \\
b_0 + \langle b_1; y \rangle + b_2 \epsilon \leq G(y, \epsilon), \ \forall (y, \epsilon) \in K_+^2 \times \mathbb{R}_+ 
\end{array} \right\}.
\]

Let us now prove that we can indeed restrict the supremum to \((b_1, b_2) \in K_+^2 \times \mathbb{R}_+\). Define

\[
L(b_0, b_1, b_2, x, \delta) := b_0 + \langle b_1; x \rangle + b_2 \delta.
\]

1) Assume first that we have \(b_2 < 0\) and

\[
L(b_0, b_1, b_2, y, \epsilon) \leq G(y, \epsilon), \ \forall (y, \epsilon) \in K_+^2 \times \mathbb{R}_+
\]

and let us show that we can increase the value by considering \(b_2 = 0\). Indeed, since \(\delta \geq 0\), we surely have

\[
L(b_0, b_1, b_2, x, \delta) \leq L(b_0, b_1, 0, x, \delta)
\]

and moreover, since \(G\) is non decreasing in the variable \(\epsilon\),

\[
L(b_0, b_1, 0, y, \epsilon) = L(b_0, b_1, b_2, y, 0) \leq G(y, 0) \leq G(y, \epsilon), \ \forall (y, \epsilon) \in K_+^2 \times \mathbb{R}_+.
\]

We have therefore shown that the supremum can be restricted to \(b_2 \geq 0\).

2) A completely analogous argument shows that we can also restrict our attention to \(b_1 \in \mathbb{R}_+^2\). Once this is achieved, we can further consider only \(b_1 \in K_+^2\), since \(x\) itself belongs to \(K_+^2\) and \(G\) is symmetric with respect to the two first variables.

**Step 2.** Let \(F : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}\) be defined by

\[
F(\xi, \delta) := G(\Lambda^1(\xi), |\delta|) = G(\lambda_1(\xi), \lambda_2(\xi), |\delta|).
\]

Observe that

\[
F(\xi, \det \xi) = G(\lambda_1(\xi), \lambda_2(\xi), \lambda_1(\xi) \lambda_2(\xi)) = g(\lambda_1(\xi), \lambda_2(\xi)) = f(\xi).
\]

Hence if we prove that \(F\) is convex, we will have established that \(f\) is polyconvex.

We have by Step 1 that, for every \((x, \delta) \in K_+^2 \times \mathbb{R}_+\),

\[
G(x, \delta) = \sup_{\substack{b_0 \in \mathbb{R} \\ b_2 \geq 0 \\ b_1 \in K_+^2}} \left\{ \begin{array}{l}
\sum b_0 + \langle b_1; x \rangle + b_2 \delta : \\
b_0 + \langle b_1; y \rangle + b_2 \epsilon \leq G(y, \epsilon), \ \forall (y, \epsilon) \in K_+^2 \times \mathbb{R}_+ 
\end{array} \right\}.
\]
Since for every $y \in K^2_+$, we can find $\eta \in \mathbb{R}^{2 \times 2}$ so that
\[ \Lambda^1 (\eta) = y \]
(just choose $\eta = \text{diag}(y_1, y_2)$), we deduce that
\[
F (\xi, \delta) = \sup_{b_0 \in \mathbb{R}, \; b_2 \geq 0} \left\{ b_0 + \langle b_1; \Lambda^1 (\xi) \rangle + b_2 |\delta| : \begin{array}{l}
\quad b_0 + \langle b_1; \Lambda^1 (\eta) \rangle + b_2 |\epsilon| \leq F (\eta, \epsilon), \\
\quad \forall (\eta, \epsilon) \in \mathbb{R}^{2 \times 2} \times \mathbb{R}
\end{array} \right\}.
\]
Since the function $(\eta, \epsilon) \to b_0 + \langle b_1; \Lambda^1 (\eta) \rangle + b_2 |\epsilon|$ is convex (by Corollary 5.37 and since $b_2 \geq 0$ and $b_1 \in K^2_+$), we deduce that $F$ is convex. The proof, in the case $N = n = 2$, is therefore complete.

General case: $N \geq n$. Recall first the notations of Sections 5.2 and 5.4. Let
\[
\tau (n, N) := \sum_{s=1}^{n} \binom{N}{s} (\frac{n}{s})
\]
and $T : \mathbb{R}^{N \times n} \to \mathbb{R}^{\tau(n,N)}$ be such that
\[
T (\xi) := (\xi, \text{adj}_2 \xi, \ldots, \text{adj}_n \xi)
\]
where
\[
\mathbb{R}^{\tau(n,N)} := \mathbb{R}^{N \times n} \times \mathbb{R}^{(N^2)} \times \mathbb{R}^{(N)} \times \cdots \times \mathbb{R}^{(N^{n-1})} \times \mathbb{R}^{(N)}.
\]
For $X = (X^1, X^2, \ldots, X^{n-1}, X^n) \in \mathbb{R}^{\tau(n,N)}$ we denote by
\[
\Lambda (X) := (\Lambda^1 (X^1), \Lambda^2 (X^2), \ldots, \Lambda^{n-1} (X^{n-1}), \Lambda^n (X^n)) \in K^\theta(n)
\]
where
\[
K^\theta(n) := K^1_+ \times K^2_+ \times \cdots \times K^{n-1}_+ \times K_+.
\]
Finally define $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R}$ by
\[
F (X) := G (\Lambda (X)).
\]
Observe that, for $\xi \in \mathbb{R}^{N \times n}$,
\[
F (T (\xi)) = G (\Lambda (T (\xi)))
\]
\[
= G (\Lambda^1 (\xi), \Lambda^2 (\text{adj}_2 \xi), \ldots, \Lambda^{n-1} (\text{adj}_{n-1} \xi), \Lambda^n (\text{adj}_n \xi))
\]
\[
= G (\Lambda^1 (\xi), \text{adj}_2 \Lambda^1 (\xi), \ldots, \text{adj}_{n-1} \Lambda^1 (\xi), \text{adj}_n \Lambda^1 (\xi))
\]
\[
= g (\Lambda^1 (\xi)) = g (\lambda_1 (\xi), \ldots, \lambda_n (\xi)) = f (\xi).
\]
Hence to prove the polyconvexity of $f$ it remains only to prove the convexity of $F$. We then use the convexity of $G$ to deduce, for every $z = (z^1, \ldots, z^n) \in K^\theta(n)$,
that
\[ G(z) = \sup_{b_0, b_\nu \in \mathbb{R}^n} \left\{ b_0 + \sum_{\nu=1}^{n} \langle b_\nu; z_\nu \rangle : b_0 + \sum_{\nu=1}^{n} \langle b_\nu; y_\nu \rangle \leq G(y), \forall y \in \mathbb{R}^{\theta(n)} \right\}. \]

The facts that \( G \) is non-decreasing in each variable and symmetric in each of the variables but the last one, that \( z_\nu \in K_{+}^{(n)} \), for every \( \nu = 1, \cdots, n \), allow (as in Step 1 of the case where \( N = n = 2 \)) to restrict the above supremum to
\[ G(z) = \sup_{b_0 \in \mathbb{R}, b_\nu \in K_{+}^{(n)}} \left\{ b_0 + \sum_{\nu=1}^{n} \langle b_\nu; z_\nu \rangle : b_0 + \sum_{\nu=1}^{n} \langle b_\nu; y_\nu \rangle \leq G(y), \forall y \in \mathbb{K}^{\theta(n)} \right\}. \]

Since for every \( y_\nu \in K_{+}^{(n)} \) and every \( \nu = 1, \cdots, n \), we can find \( \eta_\nu \in \mathbb{R}^{N \times n} \) so that
\[ \Lambda_\nu(\eta_\nu) = y_\nu \] (just choose \( \eta_\nu \) a diagonal matrix with the appropriate entries), we obtain that for every \( X = (X^1, \cdots, X^n) \in \mathbb{R}^{(n,N)} \),
\[ F(X) = G(\Lambda(X)) \]
\[ = \sup_{b_0 \in \mathbb{R}, b_\nu \in K_{+}^{(n)}} \left\{ b_0 + \sum_{\nu=1}^{n} \langle b_\nu; \Lambda_\nu(X_\nu) \rangle : b_0 + \sum_{\nu=1}^{n} \langle b_\nu; \Lambda_\nu(\eta_\nu) \rangle \leq F(\eta), \forall \eta \in \mathbb{R}^{(n,N)} \right\}. \]

Observe that since \( b_\nu \in K_{+}^{(n)} \) for \( \nu = 1, \cdots, n \), we have that the function
\[ \eta = (\eta^1, \cdots, \eta^n) \in \mathbb{R}^{(n,N)} \to b_0 + \sum_{\nu=1}^{n} \langle b_\nu; \Lambda_\nu(\eta_\nu) \rangle \]
is convex (cf. Corollary 5.37) and hence \( F \) is convex. Thus the function \( f \) is polyconvex and this achieves the proof of the theorem. \( \blacksquare \)

The next example will turn out, in the subsequent chapters, to be useful.

**Example 5.41** Let \( \xi \in \mathbb{R}^{n \times n} \), then the functions
\[ f_\nu(\xi) := \prod_{i=\nu}^{n} \lambda_i(\xi) \]
are polyconvex for every \( \nu = 1, \cdots, n \). The proof follows from the theorem, but it can be seen in a more straightforward way from the following argument. For \( 1 \leq s \leq n \), the function
\[ X \in \mathbb{R}^{(n)} \times (n) \to \lambda_{(n)}(X) \]
is convex, according to Corollary 5.37. Hence the function
\[ \xi \to \lambda_{(s)}(\text{adj}_s \xi) \]
is polyconvex. Since
\[ \lambda(v)(\text{adj}_s \xi) = \prod_{i=n-s+1}^n \lambda_i(\xi), \]
we have the claim.  

We now turn our attention to the $SO(2) \times SO(2)$-invariant case and give here a theorem due to Dacorogna-Koshigoe [192], which shows, in particular, that at least when $N = n = 2$, the sufficient condition of Theorem 5.39 is also necessary. We here follow the proof of Dacorogna-Maréchal [205]; but let us first introduce the following definition of polyconvexity for vectors.

**Definition 5.42** A function $g : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ is said to be polyconvex if there exists $G : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ convex such that
\[ g(x_1, x_2) = G(x_1, x_2, x_1 x_2). \]

There is of course a similar definition for polyconvex functions over $\mathbb{R}^n$ (for details see [205]), but we will not need this extension here.

In the next theorem we use the notations of Section 5.3.3.

**Theorem 5.43** Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be $SO(2) \times SO(2)$-invariant and let $g : \mathbb{R}^2 \to \mathbb{R}$ be the unique $\Pi_e(2)$-invariant function such that
\[ f = g \circ \mu. \]

The following statements are all equivalent.

(i) $f$ is polyconvex.

(ii) $g$ is polyconvex.

(iii) For every $(a_i, b_i) \in \mathbb{R}^2$, $t_i \geq 0$, $i = 1, 2, 3, 4$ with
\[ \sum_{i=1}^4 t_i = 1 \quad \text{and} \quad \sum_{i=1}^4 t_i a_i b_i = (\sum_{i=1}^4 t_i a_i)(\sum_{i=1}^4 t_i b_i) \]
the following inequality holds
\[ g(\sum_{i=1}^4 t_i (a_i, b_i)) \leq \sum_{i=1}^4 t_i g(a_i, b_i). \]

In particular, if $G : \mathbb{R}^3 \to \mathbb{R}$ is defined by
\[ G(a, b, \delta) := \inf \left\{ \sum_{i=1}^4 t_i g(a_i, b_i) : \sum_{i=1}^4 t_i (a_i, b_i, a_i b_i) = (a, b, \delta) \quad \text{and} \quad \sum_{i=1}^4 t_i = 1 \right\}, \]
then $G$ is well defined. Moreover if $g$ satisfies the above condition, then $G$ is convex and
\[ g(a, b) = G(a, b, ab) \]
for every \((a, b) \in \mathbb{R}^2\).

(iv) For every \((a, b) \in \mathbb{R}^2\), there exists \(\beta = \beta (a, b) \in \mathbb{R}^3\) such that

\[
g(x, y) \geq g(a, b) + \langle \beta(a, b); (x, y, xy) - (a, b, ab) \rangle
\]

for every \((x, y) \in \mathbb{R}^2\) and where \(\langle \cdot; \cdot \rangle\) denotes the scalar product in \(\mathbb{R}^3\).

Remark 5.44 (i) The equivalence between (i) and (ii) can be restated as:

\[
f|_{\mathbb{R}^2_d} \text{ is polyconvex } \iff f \text{ is polyconvex},
\]

where \(\mathbb{R}^2_d\) is the subspace of diagonal matrices of \(\mathbb{R}^{2 \times 2}\) and \(f|_{\mathbb{R}^2_d}\) is the restriction of \(f\) to this subspace.

(ii) The same result holds if \(f: \mathbb{R}^{2 \times 2} \to \mathbb{R}\) is \(O(2) \times O(2)\)-invariant and \(g: \mathbb{R}^2 \to \mathbb{R}\) is the unique \(\Pi(2)\)-invariant function such that

\[
f = g \circ \lambda.
\]

(iii) The result can be, in part, extended to the case where \(f: \mathbb{R}^{2 \times 2} \to \mathbb{R} \cup \{+\infty\}\), see Dacorogna-Maréchal [205] for details.

(iv) We recall that when we say that a function \(g: \mathbb{R}^2 \to \mathbb{R}\) is \(\Pi_e(2)\)-invariant we mean that, for every \(x_1, x_2 \in \mathbb{R}\),

\[
g(x_1, x_2) = g(x_2, x_1) = g(-x_1, -x_2) = g(-x_2, -x_1).
\]

Proof. The equivalence between (ii), (iii) and (iv) is proved in exactly the same way as the one of Theorem 5.6 and we will therefore omit the proof.

(i) \(\Rightarrow\) (ii). Since \(f\) is polyconvex, we can find a convex function

\[
F: \mathbb{R}^{2 \times 2} \times \mathbb{R} \to \mathbb{R}
\]

so that

\[
f(\xi) = F(\xi, \det \xi).
\]

Let \((x_1, x_2, \delta) \in \mathbb{R}^3\) and let

\[
G(x_1, x_2, \delta) := F(\xi, \delta)
\]

where \(\xi = \text{diag}(x_1, x_2) \in \mathbb{R}^{2 \times 2}\). Observe that \(G: \mathbb{R}^3 \to \mathbb{R}\) is convex and, since \(g\) is \(\Pi_e(2)\)-invariant, we have

\[
g(x_1, x_2) = G(x_1, x_2, x_1 x_2).
\]

Thus \(g\) is polyconvex.

(ii) \(\Rightarrow\) (i). We divide the proof into two steps.
Step 1. Since $g$ is polyconvex, we can find $G : \mathbb{R}^3 \to \mathbb{R}$ convex such that

$$ g(x_1, x_2) = G(x_1, x_2, x_1 x_2) . $$

In general the function $(x_1, x_2) \to G(x_1, x_2, \delta)$ is not $\Pi_\epsilon (2)$-invariant, although $g$ is. To remedy to this difficulty, we let $H : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$ H(x_1, x_2, \delta) := \frac{1}{4} \left[ G(x_1, x_2, \delta) + G(x_2, x_1, \delta) + G(-x_1, -x_2, \delta) + G(-x_2, -x_1, \delta) \right] . $$

The function $H$ is convex and furthermore $(x_1, x_2) \to H(x_1, x_2, \delta)$ is $\Pi_\epsilon (2)$-invariant. Moreover, since $g$ is $\Pi_\epsilon (2)$-invariant we also have

$$ g(x_1, x_2) = H(x_1, x_2, x_1 x_2) . $$

We then define, for $\xi \in \mathbb{R}^{2 \times 2}$,

$$ F(\xi, \delta) := H(\mu_1(\xi), \mu_2(\xi), \delta) . $$

Since we clearly have

$$ f(\xi) = F(\xi, \det \xi) , $$

we will deduce the claim, namely that $f$ is polyconvex, once we will have shown that $F : \mathbb{R}^{2 \times 2} \times \mathbb{R} \to \mathbb{R}$ is convex.

This is done in a completely analogous manner to the one of Theorem 5.39. Indeed since $H$ is convex over $\mathbb{R}^3$ we have (cf. Corollary 2.51)

$$ H(x_1, x_2, \delta) = \sup_{b_0, b_1, b_2, b_3 \in \mathbb{R}} \left\{ \begin{array}{l}
 b_0 + b_1 x_1 + b_2 x_2 + b_3 \delta : \\
 b_0 + b_1 y_1 + b_2 y_2 + b_3 \epsilon \leq H(y_1, y_2, \epsilon), \\
 \forall (y_1, y_2, \epsilon) \in \mathbb{R}^3
\end{array} \right\} . $$

It is easy to see (cf. Step 2 below) that, if $|x_1| \leq x_2$, we have

$$ H(x_1, x_2, \delta) = \sup_{b_0, b_3 \in \mathbb{R}} \left\{ \begin{array}{l}
 b_0 + b_1 x_1 + b_2 x_2 + b_3 \delta : \\
 b_0 + b_1 y_1 + b_2 y_2 + b_3 \epsilon \leq H(y_1, y_2, \epsilon), \\
 \forall (y_1, y_2, \epsilon) \in \mathbb{R}^3, \\
 \text{for every } |y_1| \leq y_2 \text{ and } \epsilon \in \mathbb{R}
\end{array} \right\} . $$

since $(x_1, x_2) \to H(x_1, x_2, \delta)$ is $\Pi_\epsilon (2)$-invariant.

Since for every $|y_1| \leq y_2$, we can find $\eta \in \mathbb{R}^{2 \times 2}$ so that

$$ \mu_1(\eta) = y_1 \text{ and } \mu_2(\eta) = y_2 $$

(just choose $\eta = \text{diag}(y_1, y_2)$), we deduce that

$$ F(\xi, \delta) = \sup_{b_0, b_3 \in \mathbb{R}} \left\{ \begin{array}{l}
 b_0 + b_1 \mu_1(\xi) + b_2 \mu_2(\xi) + b_3 \delta : \\
 b_0 + b_1 \mu_1(\eta) + b_2 \mu_2(\eta) + b_3 \epsilon \leq F(\eta, \epsilon), \\
 \forall (\eta, \epsilon) \in \mathbb{R}^{2 \times 2} \times \mathbb{R}
\end{array} \right\} . $$
Since $|b_1| \leq b_2$, we find that the function

$$(\eta, \epsilon) \rightarrow b_0 + b_1 \mu_1(\eta) + b_2 \mu_2(\eta) + b_3 \epsilon$$

is convex (by Corollary 5.37) and we thus deduce that $F$ is convex. The proof is therefore complete.

**Step 2.** Let us now prove that (5.66) holds. So let, for $|x_1| \leq x_2$ and $b_0, b_1, b_2, b_3, \delta \in \mathbb{R}$,

$$L(b_1, b_2, x_1, x_2, \delta) := b_0 + b_1 x_1 + b_2 x_2 + b_3 \delta$$

(we do not denote in $L$ the dependence on $b_0, b_3$, since they will not change in the following computations) be such that

$$L(b_1, b_2, y_1, y_2, \epsilon) \leq H(y_1, y_2, \epsilon), \forall (y_1, y_2, \epsilon) \in \mathbb{R}^3. \quad (5.67)$$

The claim (5.66) will follow, if we can find $|c_1| \leq c_2$ so that

$$L(b_1, b_2, x_1, x_2, \delta) \leq L(c_1, c_2, x_1, x_2, \delta) \quad (5.68)$$

while

$$L(c_1, c_2, y_1, y_2, \epsilon) \leq H(y_1, y_2, \epsilon), \forall (y_1, y_2, \epsilon) \in \mathbb{R}^3. \quad (5.69)$$

This is done as follows. Let

$$\sigma(b_1, b_2) := \begin{cases} 
1 & \text{if } b_1 b_2 > 0 \\
0 & \text{if } b_1 b_2 = 0 \\
-1 & \text{if } b_1 b_2 < 0.
\end{cases}$$

Let $\tau$ be a permutation of $\{1, 2\}$ such that

$$|b_{\tau(1)}| \leq |b_{\tau(2)}|$$

and

$$c_1 := \sigma(b_1, b_2) |b_{\tau(1)}| \quad \text{and} \quad c_2 := |b_{\tau(2)}|.$$

According to Proposition 13.9, the inequality (5.68) is satisfied. Observe that, for every $y_1, y_2 \in \mathbb{R}$,

$$c_1 y_1 + c_2 y_2 = \begin{cases} 
b_1 y_1 + b_2 y_2 & \text{if } b_2 \geq |b_1| \\
-b_1 y_1 - b_2 y_2 & \text{if } -b_2 \geq |b_1| \\
b_2 y_1 + b_1 y_2 & \text{if } b_1 \geq |b_2| \\
-b_2 y_1 - b_1 y_2 & \text{if } -b_1 \geq |b_2|.
\end{cases}$$

This implies that

$$L(c_1, c_2, y_1, y_2, \epsilon) \leq \max\{L(b_1, b_2, y_1, y_2, \epsilon), L(b_1, b_2, -y_1, -y_2, \epsilon), L(b_1, b_2, y_2, y_1, \epsilon), L(b_1, b_2, -y_2, -y_1, \epsilon)\}.$$
Since (5.67) holds and \((x_1, x_2) \rightarrow H(x_1, x_2, \delta)\) is \(\Pi_e (2)\)-invariant, we get (5.69) and hence the claim (5.66) is established.

Having discussed the convexity and the polyconvexity of \(SO(2) \times SO(2)\) or \(O(N) \times O(n)\)-invariant functions, one would be tempted to think that similar results exist for rank one and quasiconvex functions. This is not the case as was first observed by Dacorogna-Koshigoe [192] (see Example 5.45) for rank one convex functions. Later Müller [463] showed the same result for quasiconvex functions.

**Example 5.45** The examples are based on computations of Dacorogna-Douchet-Gangbo-Rappaz in [185]. In both examples, \(N = n = 2\) and \(b \geq 0\).

(i) Let \(\alpha > 2 + \sqrt{2}\) and

\[
f_{\alpha,b}(\xi) = |\xi|^{2\alpha} - 2^{\alpha-1}b|\det \xi|^\alpha.
\]

(ii) Let \(\alpha > (9 + 5\sqrt{5})/4\) and

\[
f_{\alpha,b}(\xi) = |\xi|^{2\alpha} (|\xi|^2 - 2b \det \xi).
\]

Note that both functions are \(SO(2) \times SO(2)\)-invariant. In both cases, there exist \(b_2 < b_1\) (for the precise values of \(b_1, b_2\) see [185]) such that

\[
f_{\alpha,b} \text{ is rank one convex } \iff b \leq b_2,
\]

\[
f_{\alpha,b}|_{\mathbb{R}^2} \text{ is rank one convex } \iff b \leq b_1.
\]

We finally conclude this section by mentioning other results on rank one convexity of \(O(n) \times O(n)\)-invariant functions. As seen in Proposition 5.31, any such function is necessarily of the form

\[
f(\xi) = g(\lambda_1(\xi), \ldots, \lambda_n(\xi))
\]

where \(0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)\) are the singular values of the matrix \(\xi \in \mathbb{R}^{n \times n}\). Assuming that the function \(f\) is twice differentiable, it is therefore natural to ask conditions on the derivatives of \(g\) that ensure the rank one convexity of the function \(f\). This was achieved by Knowles-Sternberg [371] when \(n = 2\) and then in various different ways by Aubert [41], Aubert-Tahraoui [48], Ball [55], Dacorogna-Marcellini [202] and Davies [223]. When \(n = 3\), Aubert-Tahraoui in [47] gave also some necessary conditions and, although in a slightly different context, necessary and sufficient conditions were derived by Simpson-Spector [540] (see also Zee-Sternberg [613]). In the case of general \(n\), certain results exist but are less explicit; see Dacorogna [182] and Silhavy [536].

### 5.3.5 Functions depending on a quasiaffine function

The following theorem was established in Dacorogna [173].
Theorem 5.46 Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$, $\Phi : \mathbb{R}^{N \times n} \to \mathbb{R}$ be quasiaffine but not identically constant and $g : \mathbb{R} \to \mathbb{R}$ be such that

$$f (\xi) = g (\Phi (\xi))$$

(in particular, if $N = n$, one can take $\Phi (\xi) = \det \xi$). Then

$$f \text{ polyconvex} \iff f \text{ quasiconvex} \iff f \text{ rank one convex} \iff g \text{ convex}.$$ 

Proof. The implications

$$g \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex}$$

follow immediately from Theorem 5.3. It therefore remains to show that

$$f \text{ rank one convex} \Rightarrow g \text{ convex}.$$ 

We want to prove that for $t \in (0, 1)$, $\alpha, \beta \in \mathbb{R}$, then

$$g (t \alpha + (1 - t) \beta) \leq tg (\alpha) + (1 - t) g (\beta)$$

provided $f$ is rank one convex. Following Theorem 5.20, we have that

$$\Phi (\xi) = a_0 + \langle a; T (\xi) \rangle = a_0 + \langle a_1; \xi \rangle + \sum_{j=2}^{n \wedge N} \langle a_j; \adj_j \xi \rangle,$$

where $a_0 \in \mathbb{R}$, $a_1 \in \mathbb{R}^{N \times n}$ and $a_j \in \mathbb{R}^{\sigma (j)}$ where $\sigma (j) = \binom{N}{j} \binom{n}{j}$. Since $\Phi$ is not identically constant, then at least one of the $a_j$, $1 \leq j \leq n \wedge N$ is not zero. Let $s$ be such that $a_s \neq 0$ but $a_{s-1} = a_{s-2} = \cdots = a_1 = 0$ (if $a_1 \neq 0$, we then take $s = 1$). Since $a_s \neq 0$ ($\in \mathbb{R}^{\sigma (s)}$) we have that at least one of the components of $a_s = (a_0^s, \cdots, a_\sigma^s (s))$ is non-zero. For notational convenience, we take $a_s^\sigma (s) \neq 0$. 

First choose $\eta \in \mathbb{R}^{N \times n}$ in the following way

\[
\eta = \begin{pmatrix}
\eta_1^1 & \cdots & \eta_s^1 & \eta_{s+1}^1 & \cdots & \eta_n^1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\eta_1^s & \cdots & \eta_s^s & \eta_{s+1}^s & \cdots & \eta_n^s \\
\eta_1^{s+1} & \cdots & \eta_s^{s+1} & \eta_{s+1}^{s+1} & \cdots & \eta_n^{s+1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\eta_1^N & \cdots & \eta_s^N & \eta_{s+1}^N & \cdots & \eta_n^N \\
\end{pmatrix}
\]

More precisely, we take all components to be zero but the following ones:

\[\eta_1^1 = \frac{\alpha - a_0}{a_s^{\sigma(s)}}, \quad \eta_i^j = 1 \text{ for } 2 \leq i \leq s.\]

We next choose $\lambda \in \mathbb{R}^{N \times n}$ in exactly the same manner except that we replace the first component by $(\beta - a_0)/a_s^{\sigma(s)}$. We then immediately have

\[
\begin{cases}
\Phi(\eta) = \alpha, & \Phi(\lambda) = \beta \\
\text{rank } \{\eta - \lambda\} \leq 1
\end{cases}
\]

since $a_j = 0$ if $j < s$,

\[\text{adj}_s \eta = (0, \cdots, 0, \frac{\alpha - a_0}{a_s^{\sigma(s)}}) \quad \text{and} \quad \text{adj}_s \lambda = (0, \cdots, 0, \frac{\beta - a_0}{a_s^{\sigma(s)}})\]

and $\text{adj}_j \eta = \text{adj}_j \lambda = 0$ if $j \geq s + 1$.

We also clearly have from Theorem 5.20 that

\[\Phi(t\eta + (1-t)\lambda) = t\alpha + (1-t)\beta.\]

Using the rank one convexity of $f$ and the above construction we get

\[g(t\alpha + (1-t)\beta) = g(\Phi(t\eta + (1-t)\lambda)) = f(t\eta + (1-t)\lambda) \leq tf(\eta) + (1-t)f(\lambda) = tg(\Phi(\eta)) + (1-t)g(\Phi(\lambda)) = tg(\alpha) + (1-t)g(\beta)\]

which is the desired result.
5.3.6 The area type case

The next result is due to Morrey [453], but we follow a different proof, established in Dacorogna [171].

**Theorem 5.47** Let \( N = n + 1 \) and for \( \xi \in \mathbb{R}^{(n+1) \times n} \) let
\[
\text{adj}_n \xi = (\det \hat{\xi}^1, - \det \hat{\xi}^2, \ldots, (-1)^{k+1} \det \hat{\xi}^k, \ldots, (-1)^{n+2} \det \hat{\xi}^{n+1}),
\]
where \( \hat{\xi}^k \) is the \( n \times n \) matrix obtained from \( \xi \) by suppressing the \( k \)th row. Let \( f : \mathbb{R}^{(n+1) \times n} \to \mathbb{R} \) and \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) be such that
\[
f(\xi) = g(\text{adj}_n \xi).
\]
Then
\[
f \text{ polyconvex} \iff f \text{ quasiconvex} \iff f \text{ rank one convex} \iff g \text{ convex}.
\]

**Remark 5.48** It is clear that if \( u : \mathbb{R}^n \to \mathbb{R}^{n+1} \), then \( \text{adj}_n \nabla u \) represents the normal to the surface \( \{u(x) : x \in \mathbb{R}^n\} \).

In the case \( n = 2 \), \( u(x_1, x_2) = (u^1, u^2, u^3) \) we have
\[
\text{adj}_2 \nabla u = \begin{pmatrix}
\frac{\partial u^2}{\partial x_1} \frac{\partial u^3}{\partial x_2} - \frac{\partial u^2}{\partial x_2} \frac{\partial u^3}{\partial x_1} \\
\frac{\partial u^3}{\partial x_1} \frac{\partial u^1}{\partial x_2} - \frac{\partial u^3}{\partial x_2} \frac{\partial u^1}{\partial x_1} \\
\frac{\partial u^1}{\partial x_1} \frac{\partial u^2}{\partial x_2} - \frac{\partial u^1}{\partial x_2} \frac{\partial u^2}{\partial x_1}
\end{pmatrix}.
\]

Before proceeding with the proof of the theorem, we mention an algebraic lemma, stronger than needed, that will be fully used in Section 6.6.4. We will prove the lemma, established in Dacorogna [171], after the proof of Theorem 5.47.

**Lemma 5.49** Let \( 0 < t < 1 \), \( a, b \in \mathbb{R}^{n+1} \) and \( \xi \in \mathbb{R}^{(n+1) \times n} \) be such that
\[
\text{adj}_n \xi = ta + (1 - t)b \neq 0.
\]
Then there exist \( \alpha, \beta \in \mathbb{R}^{(n+1) \times n} \) such that
\[
\begin{align*}
\xi &= t\alpha + (1 - t)\beta \\
\text{adj}_n \alpha &= a, \quad \text{adj}_n \beta = b \\
\text{rank} \{\alpha - \beta\} &\leq 1.
\end{align*}
\]

**Proof.** (Theorem 5.47). The implications
\[
g \text{ convex} \implies f \text{ polyconvex} \implies f \text{ quasiconvex} \implies f \text{ rank one convex}
\]
follow immediately from Theorem 5.3.

It therefore remains to show that

\[ f \text{ rank one convex } \Rightarrow g \text{ convex}. \]

We let \( t \in (0, 1) \), \( a, b \in \mathbb{R}^{n+1} \) and we wish to show that

\[ g(ta + (1-t)b) \leq tg(a) + (1-t)g(b) \tag{5.70} \]

provided \( f \) is rank one convex and \( f(\xi) = g(\text{adj}_n \xi) \). We divide the proof into two cases.

**Case 1:** \( ta + (1-t)b \neq 0 \). We let

\[ c := ta + (1-t)b = (c^1, \ldots, c^{n+1}) \in \mathbb{R}^{n+1}. \]

Since \( c \neq 0 \), we assume, for notational convenience, that \( c^1 \neq 0 \) (the general case is handled similarly). We then let

\[
\xi := \begin{pmatrix}
\xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\
\xi_1^2 & \xi_2^2 & \cdots & \xi_n^2 \\
\xi_1^3 & \xi_2^3 & \cdots & \xi_n^3 \\
\vdots & \vdots & \ddots & \vdots \\
\xi_1^{n+1} & \xi_2^{n+1} & \cdots & \xi_n^{n+1}
\end{pmatrix}
= \begin{pmatrix}
-c^2 & -c^3 & \cdots & -c^{n+1} \\
c^1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
\]

It is then easy to see that

\[ \text{adj}_n \xi = c = ta + (1-t)b \neq 0. \]

We may now apply Lemma 5.49 to get \( \alpha, \beta \in \mathbb{R}^{(n+1) \times n} \) such that

\[
\begin{cases}
\xi = t\alpha + (1-t)\beta \\
\text{adj}_n \alpha = a, \quad \text{adj}_n \beta = b \\
\text{rank } \{\alpha - \beta\} \leq 1.
\end{cases}
\]

Returning to (5.70), using the rank one convexity of \( f \), we obtain

\[
g(ta + (1-t)b) = g(\text{adj}_n \xi) = f(\xi) = f(t\alpha + (1-t)\beta) \\
\leq tf(\alpha) + (1-t)f(\beta) = tg(a) + (1-t)g(b),
\]

which is precisely the result.

**Case 2:** \( ta + (1-t)b = 0 \). Observe first that the rank one convexity of \( f \) implies that \( f \) is continuous (cf. Theorem 5.3), thus from \( f(\xi) = g(\text{adj}_n \xi) \) we deduce that \( g \) is continuous. Therefore using Case 1 for \( \tilde{a} = a + (\epsilon,0,\cdots,0) \)

and \( \tilde{b} = b + (\epsilon,0,\cdots,0) \) where \( \epsilon > 0 \) is arbitrary, we deduce (5.70) by continuity of \( g \).

\[ \blacksquare \]
We now conclude this section by proving Lemma 5.49.

**Proof.** We give here a different proof than the one in Dacorogna [171] or [179]. We decompose the proof into two steps.

**Step 1.** We start by assuming that \( \xi \in \mathbb{R}^{(n+1) \times n} \) has the following special form

\[
\xi = \text{diag}_{(n+1) \times n} (x_1, \cdots, x_n) = \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \\ 0 & \cdots & 0 \end{pmatrix}
\]

with \( x_1, \cdots, x_n \in \mathbb{R} \), all different from 0, and thus

\[
\text{adj}_n \xi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^n x_1 \cdots x_n \end{pmatrix} = ta + (1 - t)b \neq 0.
\]

We next observe that for every \( \lambda \in \mathbb{R}^{n+1} \) and \( \mu \in \mathbb{R}^n \) we have

\[
\text{adj}_n (\xi + \lambda \otimes \mu) = \text{adj}_n \xi + \langle \text{adj}_{n-1} \xi; \lambda \otimes \mu \rangle
\]

where

\[
\langle \text{adj}_{n-1} \xi; \lambda \otimes \mu \rangle = (-1)^n \begin{pmatrix} -\lambda^{n+1} \mu_1 \prod_{j \neq 1} x_j \\ \vdots \\ -\lambda^{n+1} \mu_n \prod_{j \neq n} x_j \\ \sum_{s=1}^n [\lambda^s \mu_s \prod_{j \neq s} x_j] \end{pmatrix}.
\]

We then search for \( \alpha, \beta \in \mathbb{R}^{(n+1) \times n} \) of the form

\[
\begin{cases} 
\alpha = \xi + (1 - t) \lambda \otimes \mu \\
\beta = \xi - t \lambda \otimes \mu
\end{cases}
\]

where \( \lambda \in \mathbb{R}^{n+1} \) and \( \mu \in \mathbb{R}^n \) are to be determined. We therefore immediately deduce that

\[
\xi = ta + (1 - t) \beta \quad \text{and} \quad \text{rank} \{\alpha - \beta\} \leq 1.
\]

We next observe that

\[
\begin{align*}
\text{adj}_n \alpha &= \text{adj}_n \xi + (1 - t) \langle \text{adj}_{n-1} \xi; \lambda \otimes \mu \rangle \\
\text{adj}_n \beta &= \text{adj}_n \xi - t \langle \text{adj}_{n-1} \xi; \lambda \otimes \mu \rangle.
\end{align*}
\]

Thus the equations \( \text{adj}_n \alpha = a \) and \( \text{adj}_n \beta = b \) reduce to the single system of equations

\[
\langle \text{adj}_{n-1} \xi; \lambda \otimes \mu \rangle = a - b := c \quad (5.71)
\]

that we solve by considering two cases.
Case 1: $c^1 = \cdots = c^n = 0$. We then choose
\[ \lambda^1 = 1, \lambda^2 = \cdots = \lambda^{n+1} = 0, \mu_2 = \cdots = \mu_n = 0 \]
and
\[ \mu_1 = (-1)^n \frac{c^{n+1}}{\prod_{j=2}^n x_j} \]
so as to satisfy (5.71).

Case 2: there exists $k \in \{1, \cdots, n\}$ with $c^k \neq 0$. Equation (5.71) is then satisfied if we choose
\[ \mu_i = (-1)^{n+1} \frac{c^i}{\prod_{j \neq i} x_j}, \quad i = 1, \cdots, n \]
and $\lambda^i = 0$ whenever $i \neq k, n+1$, $\lambda^{n+1} = 1$ and
\[ \lambda^k = (-1)^n \frac{c^{n+1}}{\mu_k \prod_{j \neq k} x_j} = -\frac{c^{n+1}}{c^k}. \]

Step 2. We now reduce the general case $\xi \in \mathbb{R}^{(n+1) \times n}$ to the special form of the previous step by using the singular values decomposition theorem (cf. Theorem 13.3). We can indeed find $R \in O(n+1)$, $Q \in SO(n)$ and $x_1, \cdots, x_n \in \mathbb{R}$ so that
\[ \tilde{\xi} := R\xi Q = \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \\ 0 & \cdots & 0 \end{pmatrix}. \]
Using Proposition 5.66, and noting that $\text{adj}_n Q = \det Q = 1$, we find that
\[ \text{adj}_n \tilde{\xi} = \text{adj}_n R \text{adj}_n \xi \neq 0. \]
Observing that $\text{adj}_n R \in O(n+1)$ (by Proposition 5.66), we set
\[ \tilde{a} := \text{adj}_n R a \quad \text{and} \quad \tilde{b} := \text{adj}_n R b \]
and we can find, from Step 1, $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}^{(n+1) \times n}$ such that
\[ \begin{cases} \tilde{\xi} = t\tilde{\alpha} + (1-t)\tilde{\beta} \\ \text{adj}_n \tilde{\alpha} = \tilde{a}, \quad \text{adj}_n \tilde{\beta} = \tilde{b} \\ \text{rank}\{\tilde{\alpha} - \tilde{\beta}\} \leq 1. \end{cases} \]
Setting
\[ \alpha := R^t \tilde{\alpha} Q^t \quad \text{and} \quad \beta := R^t \tilde{\beta} Q^t \]
we have indeed obtained the claim of the lemma. $\blacksquare$
5.3.7 The example of Sverak

We now turn to an example of a rank one convex function that is not quasiconvex. This fundamental result was obtained by Sverak [551] when \( N \geq 3 \) and \( n \geq 2 \) and we follow his presentation here. The question of extending Sverak example to the case where \( n \geq N = 2 \) is still open.

**Theorem 5.50** Let \( N \geq 3 \) and \( n \geq 2 \). Then there exists \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) rank one convex but not quasiconvex.

**Proof.** The proof is divided into four steps.

**Step 1.** Assume that we have already constructed a rank one convex function \( g : \mathbb{R}^{3 \times 2} \to \mathbb{R} \), that is not quasiconvex. In particular (appealing to Proposition 5.13), there exists \( \eta \in \mathbb{R}^{3 \times 2} \) and \( \psi \in W_{\text{per}}^{1,\infty} (D_2; \mathbb{R}^3) \), where \( D_2 = (0,1)^2 \) such that

\[
\int_{D_2} g(\eta + \nabla \psi(x)) \, dx < g(\eta).
\]

Then define \( \pi : \mathbb{R}^{N \times n} \to \mathbb{R}^{3 \times 2} \) to be

\[
\pi(\xi) = \begin{pmatrix}
\xi_1^1 & \xi_2^1 \\
\xi_1^2 & \xi_2^2 \\
\xi_1^3 & \xi_2^3
\end{pmatrix}, \text{ for } \xi \in \mathbb{R}^{N \times n}.
\]

Finally, let

\[
f(\xi) = g(\pi(\xi)).
\]

This function is clearly rank one convex, since \( g \) is. It is also not quasiconvex, since choosing any \( \xi \in \mathbb{R}^{N \times n} \) so that \( \pi(\xi) = \eta \), \( D_n = (0,1)^n \) and

\[
\varphi^i(x_1, \cdots, x_n) := \begin{cases} 
\psi^i(x_1, x_2) & \text{if } i = 1, 2, 3 \\
0 & \text{if not}
\end{cases}
\]

we get that \( \varphi \in W_{\text{per}}^{1,\infty} (D_n; \mathbb{R}^N) \) and

\[
\int_{D_n} f(\xi + \nabla \varphi(x)) \, dx < f(\xi).
\]

**Step 2.** In view of Step 1, it is therefore sufficient to prove the theorem for functions \( f : \mathbb{R}^{3 \times 2} \to \mathbb{R} \). We first let

\[
L := \{ \xi \in \mathbb{R}^{3 \times 2} : \xi = \begin{pmatrix} x & 0 \\ 0 & y \\ z & z \end{pmatrix} \text{ where } x, y, z \in \mathbb{R} \}
\]

and \( P : \mathbb{R}^{3 \times 2} \to L \) be defined by

\[
P(\xi) := \begin{pmatrix} 
\xi_1^1 & 0 \\
0 & \xi_2^2 \\
(\xi_1^3 + \xi_2^3)/2 & (\xi_1^3 + \xi_2^3)/2
\end{pmatrix}.
\]
We next let \( g : L \to \mathbb{R} \) be defined by
\[
g \begin{pmatrix} x & 0 \\ 0 & y \\ z & z \end{pmatrix} = -xyz.
\]

Finally, for \( \epsilon, \gamma \geq 0 \) let the function \( f_{\epsilon, \gamma} : \mathbb{R}^{3 \times 2} \to \mathbb{R} \) be such that
\[
f_{\epsilon, \gamma} (\xi) := g(P(\xi)) + \epsilon |\xi|^2 + \epsilon |\xi|^4 + \gamma |\xi - P(\xi)|^2.
\]

We claim that we can find \( \epsilon \) and \( \gamma \) so that \( f_{\epsilon, \gamma} \) is rank one convex (see Step 4) but not quasiconvex (see Step 3), giving the desired claim.

**Step 3.** Choose \( \xi = 0 \) and
\[
\varphi(x_1, x_2) = \frac{1}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_2 \\ \sin 2\pi (x_1 + x_2) \end{pmatrix}.
\]

Observe that \( \varphi \in W^{1,\infty}_{per}(D; \mathbb{R}^3) \), where \( D = (0,1)^2 \) and \( \nabla \varphi \in L \) (hence \( P(\nabla \varphi) = \nabla \varphi \)). Moreover,
\[
\int_D g(\nabla \varphi(x)) \, dx = -\int_0^1 \int_0^1 (\cos 2\pi x_1)^2 (\cos 2\pi x_2)^2 \, dx_1 \, dx_2 < 0.
\]

Therefore (see Proposition 5.13), for every \( \epsilon \geq 0 \) sufficiently small and for every \( \gamma \geq 0 \), the function \( f_{\epsilon, \gamma} \) is not quasiconvex.

**Step 4.** We now show that for every \( \epsilon > 0 \), we can find \( \gamma = \gamma(\epsilon) > 0 \) so that \( f_{\epsilon, \gamma} \) is rank one convex. This is equivalent to showing that the Legendre-Hadamard condition is satisfied, namely
\[
L_f(\xi, \eta) := \left. \frac{d^2}{dt^2} f_{\epsilon, \gamma}(\xi + t\eta) \right|_{t=0} \geq 0, \quad \forall \xi, \eta \in \mathbb{R}^{3 \times 2} \text{ with rank } \{\eta\} = 1.
\]

Letting
\[
L_g(\xi, \eta) := \left. \frac{d^2}{dt^2} g(P(\xi + t\eta)) \right|_{t=0}
\]
we find
\[
L_f(\xi, \eta) = L_g(\xi, \eta) + 2\epsilon |\eta|^2 + 4\epsilon |\xi|^2 |\eta|^2 + 8\epsilon (|\xi; \eta|)^2 + 2\gamma |\xi - P(\eta)|^2.
\]

We show (5.72) in two substeps.

**Step 4’.** Observe that since \( g \) is a homogeneous polynomial of degree three, we can find \( c > 0 \) such that
\[
L_g(\xi, \eta) \geq -c |\xi| |\eta|^2.
\]
We therefore deduce that
\[ L_f (\xi, \eta) \geq (-c + 4\epsilon |\xi|) |\xi| |\eta|^2 \]
and thus (5.72) holds for every \( \eta \in \mathbb{R}^{3\times2} \) (independently of the fact that \( \text{rank} \{ \eta \} = 1 \)) and for every \( \xi \in \mathbb{R}^{3\times2} \) that satisfies
\[ |\xi| \geq \frac{c}{4\epsilon} . \]

**Step 4”**. It therefore remains to show (5.72) in the compact set
\[ K := \{ (\xi, \eta) \in \mathbb{R}^{3\times2} \times \mathbb{R}^{3\times2} : |\xi| \leq \frac{c}{4\epsilon}, |\eta| = 1, \text{ rank} \{ \eta \} = 1 \} \]
in view of Step 4’ and of the fact that \( L_f (\xi, \eta) \) is homogeneous of degree two in the variable \( \eta \).

Moreover, we also find that
\[ L_f (\xi, \eta) \geq H (\xi, \eta, \gamma) := L_g (\xi, \eta) + 2\epsilon |\eta|^2 + 2\gamma |\eta - P(\eta)|^2 \]
and therefore (5.72) will follow if we can show that for every \( \epsilon > 0 \) we can find \( \gamma = \gamma(\epsilon) \) so that \( H \geq 0 \) on \( K \).

Assume, for the sake of contradiction, that this is not the case. We can then find \( \gamma_{\nu} \to \infty, (\xi_{\nu}, \eta_{\nu}) \in K \) so that
\[ L_g (\xi_{\nu}, \eta_{\nu}) + 2\epsilon \leq L_g (\xi_{\nu}, \eta_{\nu}) + 2\epsilon + 2\gamma_{\nu} |\eta_{\nu} - P(\eta_{\nu})|^2 < 0. \]

Since \( K \) is compact, we have up to a subsequence (still labeled \( (\xi_{\nu}, \eta_{\nu}) \)) that
\[ (\xi_{\nu}, \eta_{\nu}) \to (\xi, \eta) \in K, L_g (\xi, \eta) + 2\epsilon \leq 0 \text{ and } P(\eta) = \eta. \]

However we have \( \epsilon > 0 \) and, by construction,
\[ L_g (\xi, \eta) \equiv 0, \forall \xi, \eta \in \mathbb{R}^{3\times2} \text{ with } P(\eta) = \eta \text{ and } \text{rank} \{ \eta \} = 1. \]

This leads to the desired contradiction and therefore the theorem holds. ■

### 5.3.8 The example of Alibert-Dacorogna-Marcellini

We now turn our attention to an example where \( N = n = 2 \). It involves a homogeneous polynomial of degree four. We characterize, with the help of one single real parameter, the different notions of convexity.

**Theorem 5.51** Let \( \gamma \in \mathbb{R} \) and let \( f_{\gamma} : \mathbb{R}^{2\times2} \to \mathbb{R} \) be defined as
\[ f_{\gamma} (\xi) = |\xi|^2 (|\xi|^2 - 2\gamma \det \xi). \]
Then

\[ f_\gamma \text{ is convex} \iff |\gamma| \leq \gamma_c = \frac{2}{3}\sqrt{2}, \]
\[ f_\gamma \text{ is polyconvex} \iff |\gamma| \leq \gamma_p = 1, \]
\[ f_\gamma \text{ is quasiconvex} \iff |\gamma| \leq \gamma_q \text{ and } \gamma_q > 1, \]
\[ f_\gamma \text{ is rank one convex} \iff |\gamma| \leq \gamma_r = \frac{2}{\sqrt{3}}. \]

We now make some comments about this theorem.

(i) The last result and the fact that if \( f_\gamma \) is polyconvex, then \(|\gamma| \leq 1\), were established by Dacorogna-Marcellini [193]. All the other results were first proved in Alibert-Dacorogna [14]. The most interesting fact is the third one.

(ii) The example also provides a quasiconvex function that is not polyconvex (such an example was already seen in Theorem 5.25 when \( N, n \geq 3 \); see also when \( n = N = 2 \), Theorem 5.54 and Sverak [552]).

(iii) The problem of knowing if \( \gamma_q = 2/\sqrt{3} \) is still open. If this is not the case (meaning that \( \gamma_q < 2/\sqrt{3} \)), then this would provide a rank one convex function that is not quasiconvex, giving a final answer to this long standing question. However many numerical evidences tend to indicate that \( \gamma_q = 2/\sqrt{3} \), see Dacorogna-Douchet-Gangbo-Rappaz [185], Dacorogna-Haeberly [191] and Gremaud [321].

(iv) The polyconvexity of the function \( f_1 (\xi) = |\xi|^2 (|\xi|^2 - 2 \det \xi) \) has, since the work of Alibert-Dacorogna [14], been reproved notably by Iwaniecz-Lutoborski [353]. Hartwig [335] also proved this fact exhibiting a convex function \( F : \mathbb{R}^{2 \times 2} \times \mathbb{R} \to \mathbb{R} \), namely

\[ F (\xi, \delta) = \begin{cases} \left( |\xi|^2 + 2 \det \xi - 2 \delta \right) \left[ |\xi|^2 + 2 \det \xi - 4 \delta \right] & \text{if } |\xi|^2 + 2 \det \xi \geq 4 \delta \\ 0 & \text{otherwise,} \end{cases} \]

so that

\[ f_1 (\xi) = F (\xi, \det \xi). \]

We now proceed with the proof of the theorem. But before that we want to observe that in all four statements we can restrict our attention to the case where \( \gamma \geq 0 \). Indeed, observe first that

\[ f_\gamma (Q \xi) = f_{-\gamma} (\xi) \text{ for every } \xi \in \mathbb{R}^{2 \times 2} \text{ and } Q \in O(2) \text{ with } \det Q = -1. \]

This easily implies that \( f_\gamma \) is convex (respectively polyconvex, quasiconvex, rank one convex) if and only if \( f_{-\gamma} \) is convex (respectively polyconvex, quasiconvex, rank one convex). Hence, we may assume throughout, without loss of generality, that \( \gamma \geq 0 \).
We first start with the statement on the convexity of $f_\gamma$.

**Proof.** (Theorem 5.51: Convexity). We have to show that

$$f_\gamma \text{ is convex } \iff \gamma \leq \gamma_c = \frac{2}{3} \sqrt{2}.$$

This result was first proved by Alibert-Dacorogna, but we give here the proof based on Dacorogna-Maréchal [204].

According to Theorem 5.33, it is sufficient to verify the claim only on diagonal matrices. So let

$$g(x, y) := (x^2 + y^2)[(x^2 + y^2) - 2\gamma xy].$$

The Hessian of $g$ is therefore given by

$$\nabla^2 g(x, y) = \begin{pmatrix} 4(x^2 + y^2) + 8x^2 - 12\gamma xy & 8xy - 6\gamma(x^2 + y^2) \\ 8xy - 6\gamma(x^2 + y^2) & 4(x^2 + y^2) + 8y^2 - 12\gamma xy \end{pmatrix}.$$

Setting

$$x = r \cos \frac{\theta}{2}, \ y = r \sin \frac{\theta}{2},$$

we find that

$$\nabla^2 g(x, y) = 2r^2 \begin{pmatrix} 4 + 2\cos \theta - 3\gamma \sin \theta & 2\sin \theta - 3\gamma \\ 2\sin \theta - 3\gamma & 4 - 2\cos \theta - 3\gamma \sin \theta \end{pmatrix}.$$

The function $g$ is therefore convex if and only if the trace and the determinant of $\nabla^2 g$ are non negative. This is true if and only if

$$4 - 3\gamma \sin \theta \geq 0,$$

$$12 - 9\gamma^2 - 12\gamma \sin \theta + 9\gamma^2 \sin^2 \theta \geq 0.$$

**Step 1:** ($\Leftarrow$). We first consider the case where $\gamma \leq \gamma_c = 2\sqrt{2}/3$. This immediately implies that the first inequality holds. Since the discriminant of the polynomial (in $\sin \theta$) that appears in the second inequality is given by

$$\Delta = 36\gamma^2(9\gamma^2 - 8) \leq 0,$$

we have indeed obtained the claim.

**Step 2:** ($\Rightarrow$). We now show that if $f_\gamma$ is convex, then $\gamma \leq \gamma_c$. We prove the result by contradiction and write for a certain $t > 1$

$$\gamma = t\gamma_c = \frac{2}{3} \sqrt{2} t.$$

The polynomial that appears in the second inequality is then transformed into

$$12 - 8t^2 - 8t \sqrt{2} t \sin \theta + 8t^2 \sin^2 \theta.$$
Observe that the minimum of this polynomial (in $\sin \theta$) is attained at

$$\sin \theta = \frac{1}{\sqrt{2} t}$$

and its value is then

$$8 (1 - t^2) < 0.$$
\begin{align*}
|\xi|^2 - 2 \det \xi = 2 |\xi^-|^2 \quad \text{and} \quad |\xi|^2 + 2 \det \xi = 2 |\xi^+|^2.
\end{align*}

**Second variation.** We next compute the second variation of \( f_\gamma \)

\[
\psi_\gamma (\xi, \eta) := \sum_{i,j=1}^{2} \sum_{\alpha, \beta=1}^{2} \frac{\partial^2 f_\gamma}{\partial \xi^i_\alpha \partial \xi^j_\beta} \eta^i_\alpha \eta^j_\beta.
\]

We first calculate \( \nabla f \) and find

\[
\frac{\partial f_\gamma}{\partial \xi^i_\alpha} = 4 |\xi|^2 \xi^i_\alpha - 4 \gamma (\det \xi) \xi^i_\alpha - 2 \gamma |\xi|^2 \tilde{\xi}^i_\alpha.
\]

We then deduce that, for \( 1 \leq i, j, \alpha, \beta \leq 2 \),

\[
\frac{\partial^2 f_\gamma}{\partial \xi^i_\alpha \partial \xi^j_\beta} = 8 \xi^i_\alpha \xi^j_\beta + 4 |\xi|^2 \delta^{ij} \delta_{\alpha \beta} - 4 \gamma \xi^i_\alpha \xi^j_\beta
\]

\[
-4 \gamma (\det \xi) \delta^{ij} \delta_{\alpha \beta} - 4 \gamma \tilde{\xi}^i_\alpha \tilde{\xi}^j_\beta - 2 \gamma |\xi|^2 \tilde{\delta}^{ij} \delta_{\alpha \beta},
\]

where

\[
\delta^{ij} = \begin{cases} 1 & \text{if } i = j \vspace{1mm} \\ 0 & \text{if } i \neq j \end{cases}, \quad \tilde{\delta}^{ij} = \begin{cases} (-1)^j & \text{if } i \neq j \\ 0 & \text{otherwise} \end{cases}
\]

and similarly for \( \delta_{\alpha \beta} \) and \( \tilde{\delta}_{\alpha \beta} \). We therefore have that, if

\[
\psi_\gamma (\xi, \eta) = \sum_{i,j,\alpha,\beta=1}^{2} \frac{\partial^2 f_\gamma}{\partial \xi^i_\alpha \partial \xi^j_\beta} \eta^i_\alpha \eta^j_\beta,
\]

then

\[
\psi_\gamma (\xi, \eta) = 8 \langle \xi; \eta \rangle^2 + 4 |\xi|^2 |\eta|^2 - 8 \gamma \langle \xi; \eta \rangle \langle \tilde{\xi}; \eta \rangle
\]

\[
-4 \gamma |\eta|^2 \det \xi - 4 \gamma |\xi|^2 \det \eta.
\]

(5.73)

In terms of the above decomposition we have

\[
\frac{1}{4} \psi_\gamma (\xi, \eta) = 2 (1 - \gamma) \langle \xi^+; \eta^+ \rangle^2 + 4 \langle \xi^+; \eta^+ \rangle \langle \xi^-; \eta^- \rangle
\]

\[
+2 (1 + \gamma) \langle \xi^-; \eta^- \rangle^2 + (1 - \gamma) |\xi^+|^2 |\eta^+|^2
\]

\[
+ |\xi^+|^2 |\eta^-|^2 + |\xi^-|^2 |\eta^+|^2 + (1 + \gamma) |\xi^-|^2 |\eta^-|^2.
\]

(5.74)

**Step 1.** \((\Leftarrow)\). We first show that if \( \gamma \leq 2/\sqrt{3} \), then \( f_\gamma \) is rank one convex. This is equivalent to showing (see Theorem 5.3) that the Legendre-Hadamard condition holds, i.e.,

\[
\psi_\gamma (\xi, \eta) \geq 0, \text{ for every } \xi, \eta \in \mathbb{R}^{2 \times 2} \text{ with } \det \eta = 0.
\]

(5.75)
Using (5.74) and the fact that $\det \eta = 0$ if and only if $|\eta^+|^2 = |\eta^-|^2$, we immediately obtain
\[
\frac{1}{4} \psi_\gamma (\xi, \eta) = [(4 - 3\gamma) \langle \xi^+; \eta^+ \rangle^2 + 4 \langle \xi^+; \eta^+ \rangle \langle \xi^-; \eta^- \rangle \\
+ (4 + 3\gamma) \langle \xi^-; \eta^- \rangle^2] \\
+[(2 - \gamma) (|\xi^+|^2 |\eta^+|^2 - \langle \xi^+; \eta^+ \rangle^2) \\
+ (2 + \gamma) (|\xi^-|^2 |\eta^-|^2 - \langle \xi^-; \eta^- \rangle^2)].
\]
Since $\gamma \leq 2/\sqrt{3} \leq 2$, we deduce that the term in the second bracket is non-negative. The discriminant of the term in the first bracket is
\[
\Delta = 4 [4 - (4 - 3\gamma) (4 + 3\gamma)]
\]
and is non-positive if $\gamma \leq 2/\sqrt{3}$. Therefore
\[
\psi_\gamma (\xi, \eta) \geq 0, \text{ for every } \gamma \leq \frac{2}{\sqrt{3}},
\]
as claimed and the proof of Step 1 is complete.

**Step 2**: $(\Rightarrow)$. We now prove that if $f_\gamma$ is rank one convex, then $\gamma \leq 2/\sqrt{3}$. In order to show the result, we prove that if $\gamma > 2/\sqrt{3}$, then $f_\gamma$ is not rank one convex, which is equivalent (see (5.75)) to showing that there exist $\xi_\gamma, \eta_\gamma \in \mathbb{R}^{2 \times 2}$ with $\det \eta_\gamma = 0$ such that $\psi_\gamma (\xi_\gamma, \eta_\gamma) < 0$. This is easily done. Choose
\[
\xi_\gamma = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad \eta_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]
with $a$ defined below. A direct computation gives
\[
\frac{1}{4} \psi_\gamma (\xi_\gamma, \eta_\gamma) = 3a^2 - 3\gamma a + 1.
\]
If the discriminant $\Delta = 9\gamma^2 - 12$ is positive, and this happens if $\gamma > 2/\sqrt{3}$, we can then choose $a$ so that $\psi_\gamma (\xi_\gamma, \eta_\gamma) < 0$, as wished.

This concludes the study of the rank one convexity of the function $f_\gamma$. \(\blacksquare\)

**Proof.** (Theorem 5.51: Polyconvexity). We have to prove that
\[
f_\gamma \text{ is polyconvex } \iff 0 \leq \gamma \leq \gamma_p = 1.
\]

**Step 1**: $(\Rightarrow)$. We first show that if $f_\gamma$ is polyconvex, then $0 \leq \gamma \leq 1$. Using Corollary 5.9, we can find $c \geq 0$ such that
\[
f_\gamma (\xi) \geq -c(1 + |\xi|^2) \text{ for every } \xi \in \mathbb{R}^{2 \times 2}.
\]
In particular the inequality holds for
\[
\xi = t I, \quad t \in \mathbb{R}.
\]
We therefore find that
\[ f_\gamma (\xi) = 4 (1 - \gamma) t^4 \geq -c(1 + 2t^2). \]

Dividing both sides by \( t^4 \) and letting \( t \to \infty \), we find that
\[ 1 - \gamma \geq 0, \]
as wished.

**Step 2:** (⇐). We start with a preliminary step.

**Step 2’.** We show that if \( f_\gamma \) is polyconvex, then \( f_\beta \) is polyconvex for every \( 0 \leq \beta \leq \gamma \). We have to prove, according to Theorem 5.6, that
\[ f_\beta (\xi) \leq \sum_{i=1}^{6} \lambda_i f_\beta (\xi_i) \]
whenever \( \xi, \xi_i \in \mathbb{R}^{2 \times 2}, \lambda \in \Lambda_6 \), satisfy
\[ \xi = \sum_{i=1}^{6} \lambda_i \xi_i, \quad \sum_{i=1}^{6} \lambda_i \det \xi_i = \det \xi. \]

We consider two cases.

Case 1. Assume that
\[ \sum_{i=1}^{6} \lambda_i |\xi_i|^2 \det \xi_i \leq |\xi|^2 \det \xi. \]

Then the claim is trivial since, recalling that \( \beta \geq 0 \) and observing that the function \( \xi \to |\xi|^4 \) is convex,
\[ f_\beta (\xi) = |\xi|^4 - 2\beta |\xi|^2 \det \xi \leq \sum_{i=1}^{6} \lambda_i [ |\xi_i|^4 - 2\beta |\xi_i|^2 \det \xi_i ] = \sum_{i=1}^{6} \lambda_i f_\beta (\xi_i) . \]

Case 2. Assume now that
\[ \sum_{i=1}^{6} \lambda_i |\xi_i|^2 \det \xi_i \geq |\xi|^2 \det \xi. \]

Then the claim follows from the observation
\[ f_\beta (\xi) = f_\gamma (\xi) - 2(\beta - \gamma) |\xi|^2 \det \xi, \]
from the hypothesis \( 0 \leq \beta \leq \gamma \) and from the polyconvexity of \( f_\gamma \).

This achieves the proof of Step 2’.

**Step 2”.** It therefore remains to show that
\[ f_1 (\xi) = |\xi|^2 ( |\xi|^2 - 2 \det \xi ) \]
is polyconvex and the proof will be complete. As we already mentioned, there are three proofs of the preceding fact: the original one of Alibert-Dacorogna, the one of Hartwig and that of Iwaniec-Lutoborski, which is in the same spirit as the one of Alibert-Dacorogna but slightly simpler, and we will follow here this last one. We will show that, for every \( \xi, \eta \in \mathbb{R}^{2 \times 2}, \)
\[
f_1(\eta) \geq f_1(\xi) + 4(|\xi|^2 - \det \xi) \langle \xi; \eta - \xi \rangle - 2 |\xi|^2 [\det \eta - \det \xi].
\]
This last inequality, combined with Theorem 5.6, gives that \( f_1 \) is polyconvex.

In order to show the inequality, it is sufficient (see Theorem 5.43 and the remark following it) to verify it on diagonal matrices, so we will set \( \xi = \text{diag} (a, b) \) and \( \eta = \text{diag} (x, y) \).

We therefore have to prove that
\[
(x - y)^2 (x^2 + y^2) \geq (a - b)^2 (a^2 + b^2) + 4 (a^2 + b^2 - ab) [a (x - a) + b (y - b)] - 2 (a^2 + b^2) (xy - ab).
\]
This can be rewritten, setting \( X = x - a \) and \( Y = y - b \), as
\[
\alpha X^2 - 2\beta XY + \gamma Y^2 \geq 0 \quad (5.76)
\]
where
\[
\alpha = (x - y + a)^2 + a^2 + (a - b)^2
\]
\[
\beta = (a - b) (x - y + a - b)
\]
\[
\gamma = (x - y - b)^2 + b^2 + (a - b)^2.
\]
The inequality (5.76), and thus the polyconvexity of \( f_1 \), follows from the fact that \( \alpha, \gamma \geq 0 \) and from
\[
\Delta = \alpha \gamma - \beta^2
\]
\[
= [a^2 + b^2 - (x - y) (a - b)]^2 + (x - y + a - b)^2 [(x - y)^2 + (a - b)^2]
\]
\[
\geq 0.
\]
This concludes the claim for the polyconvexity. ■

We finally show the statement on quasiconvexity. It is clearly the most difficult to prove and we will first start with the following result, proved by Alibert-Dacorogna [14], which is a consequence of regularity results for Laplace equation. We will use it twice: once when \( \xi = 0 \) and \( p = 4 \), in the proof of Theorem 5.51, and the second time when \( \xi = 0 \) and \( 1 < p < 2 \) in Theorem 5.54. The statement with \( \xi \neq 0 \) and \( p = 4 \) is just a curiosity.
Theorem 5.52 Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^2$ be a bounded open set. Then there exists $\epsilon = \epsilon (\Omega, p) > 0$ such that

$$
\int_{\Omega} \left[ |\nabla \varphi (x)|^2 \pm 2 \det (\nabla \varphi (x)) \right]^{p/2} dx \geq \epsilon \int \Omega |\nabla \varphi (x)|^p dx \quad (5.77)
$$

for every $\varphi \in W_0^{1,\infty} (\Omega; \mathbb{R}^2)$.

Moreover, when $p = 4$, the inequality

$$
\int_{\Omega} \left[ |\xi + \nabla \varphi (x)|^2 \pm 2 \det (\xi + \nabla \varphi (x)) \right]^2 dx 
\geq (|\xi|^2 \pm 2 \det \xi)^2 \meas \Omega + \epsilon \int \Omega |\nabla \varphi (x)|^4 dx \quad (5.78)
$$

holds for every $\xi \in \mathbb{R}^{2 \times 2}$ and every $\varphi \in W_0^{1,\infty} (\Omega; \mathbb{R}^2)$.

The result (5.77) is clearly non-trivial, except when $p = 2$ (in this case we can take $\epsilon = 1$ and equality, instead of inequality, holds). Observe also that the inequality (5.77) shows that the functional on the left-hand side of (5.77) is coercive in $W_0^{1,p} (\Omega, \mathbb{R}^2)$, even though the integrand is not coercive (not even up to a quasiaffine function, which here can be at most quadratic).

Proof. (Theorem 5.52). We prove (5.77) and (5.78) only for the minus sign, the proof being identical for the plus sign. For this purpose we adapt an idea of Sverak [552].

Step 1. We first prove the result for $\xi = 0$ and $1 < p < \infty$. We start with an algebraic relation. We clearly have that there exists a constant $\alpha = \alpha (p)$ such that for every $\xi \in \mathbb{R}^{2 \times 2}$

$$
\left[ |\xi|^2 - 2 \det \xi \right]^{p/2} = \left[ (\xi_1^1 - \xi_2^2)^2 + (\xi_1^2 + \xi_2^1)^2 \right]^{p/2} 
\geq \alpha \left[ |\xi_1^1 - \xi_2^2|^p + |\xi_1^2 + \xi_2^1|^p \right].
$$

We now turn to the claim and note that it is sufficient to prove the claim for $\varphi = (\varphi_1^1, \varphi_2^2) \in C_0^{\infty} (\Omega, \mathbb{R}^2)$, the general result being obtained by density. We also extend the function outside $\Omega$ by setting $\varphi \equiv 0$ there. Then denoting $\partial \varphi_j^i / \partial x_i$ by $\partial_i \varphi_j^i$, $i, j \in \{1, 2\}$, we have from the above algebraic relation

$$
\int_{\Omega} \left[ |\nabla \varphi (x)|^2 - 2 \det (\nabla \varphi (x)) \right]^{p/2} dx 
\geq \alpha \int_{\Omega} \left[ |\partial_1 \varphi_1^1 (x) - \partial_2 \varphi_2^2 (x)|^p + |\partial_2 \varphi_1^1 (x) + \partial_1 \varphi_2^2 (x)|^p \right] dx.
$$

The classical regularity results for Cauchy-Riemann equations (see, for example, Proposition 4 on page 60 in Stein [543]) leads to the existence of a constant $\beta > 0$ such that

$$
\|\nabla \varphi\|_{L^p}^p \leq \beta \int_{\Omega} \left[ |\partial_1 \varphi_1^1 (x) - \partial_2 \varphi_2^2 (x)|^p + |\partial_2 \varphi_1^1 (x) + \partial_1 \varphi_2^2 (x)|^p \right] dx.
$$
Choosing \(\epsilon \leq \alpha/\beta\), we have (5.77).

**Step 2.** We now prove the general case, where \(\xi\) is not necessarily 0 but \(p = 4\). We start with the following algebraic observation

\[
\langle \xi - \tilde{\xi}, \nabla \varphi \rangle^2 \leq |\xi - \tilde{\xi}|^2 |\eta|^2 = 2|\xi|^2 - 2\det \xi |\eta|^2.
\] (5.79)
We next compute

\[
|\xi + \nabla \varphi|^2 - 2\det (\xi + \nabla \varphi)|^2
\]
\[
= \left[|\xi|^2 - 2\det \xi + 2 \langle \xi - \tilde{\xi}, \nabla \varphi \rangle + |\nabla \varphi|^2 - 2\det (\nabla \varphi) \right]^2
\]
\[
= \left[|\xi|^2 - 2\det \xi + 4 \langle \xi - \tilde{\xi}, \nabla \varphi \rangle + |\nabla \varphi|^2 - 2\det (\nabla \varphi) \right]^2
\]
\[
+ 4\left|\xi|^2 - 2\det \xi \right| \langle \xi - \tilde{\xi}, \nabla \varphi \rangle + \det (\nabla \varphi) + 2 \left|\nabla \varphi|^2 - 2\det (\nabla \varphi) \right| \langle \xi - \tilde{\xi}, \nabla \varphi \rangle.
\]

Using (5.79), we obtain

\[
|\xi + \nabla \varphi|^2 - 2\det (\xi + \nabla \varphi)|^2
\]
\[
\geq \left[|\xi|^2 - 2\det \xi + 5 \langle \xi - \tilde{\xi}, \nabla \varphi \rangle \right]^2 + |\nabla \varphi|^2 - 2\det (\nabla \varphi)
\]
\[
+ 4\left|\xi|^2 - 2\det \xi \right| \langle \xi - \tilde{\xi}, \nabla \varphi \rangle - \det (\nabla \varphi)
\]
\[
+ 4\left|\nabla \varphi|^2 - 2\det (\nabla \varphi) \right| \langle \xi - \tilde{\xi}, \nabla \varphi \rangle.
\]
Noticing that

\[
0 \leq 5 \langle \xi - \tilde{\xi}, \nabla \varphi \rangle^2
\]
\[
+ 4\left|\nabla \varphi|^2 - 2\det (\nabla \varphi) \right| \langle \xi - \tilde{\xi}, \nabla \varphi \rangle + \frac{4}{5} \left[|\nabla \varphi|^2 - 2\det (\nabla \varphi) \right]^2
\]
we deduce that

\[
|\xi + \nabla \varphi|^2 - 2\det (\xi + \nabla \varphi)|^2 \geq \left[|\xi|^2 - 2\det \xi \right]^2 + \frac{1}{5} \left[|\nabla \varphi|^2 - 2\det (\nabla \varphi) \right]^2
\]
\[
+ 4\left|\xi|^2 - 2\det \xi \right| \langle \xi - \tilde{\xi}, \nabla \varphi \rangle - \det (\nabla \varphi).
\]
We then integrate the above inequality, bearing in mind that \(\varphi = 0\) on \(\partial \Omega\), and we find

\[
\int_{\Omega} \left[|\xi + \nabla \varphi|^2 - 2\det (\xi + \nabla \varphi) \right]^2 dx \geq \left[|\xi|^2 - 2\det \xi \right] \meas \Omega
\]
\[
+ \frac{1}{5} \int_{\Omega} \left[|\nabla \varphi|^2 - 2\det (\nabla \varphi) \right]^2 dx.
\]
Using Step 1, with \(p = 4\), we find that

\[
\int_{\Omega} \left[|\xi + \nabla \varphi|^2 - 2\det (\xi + \nabla \varphi) \right]^2 dx \geq \left[|\xi|^2 - 2\det \xi \right]^2 \meas \Omega + \frac{\alpha}{5\beta} \int_{\Omega} |\nabla \varphi|^4 dx.
\]
Choosing \(\epsilon = \alpha/5\beta\), we have indeed established (5.78) and thus the theorem is proved. \(\blacksquare\)
We now continue with the proof of the main theorem.

**Proof.** (Theorem 5.51: Quasiconvexity). We have to establish that

$$f_\gamma \text{ is quasiconvex } \iff \gamma \leq \gamma_q \text{ and } \gamma_q > 1.$$ 

In the first step, we prove the existence of a $\gamma_q$ with the above property; this is the easy part of the proof. The difficult part, which will be dealt with in Step 2, is to show that $\gamma_q > 1$.

**Step 1: Existence of $\gamma_q$.** We start by showing that if $f_\gamma$ is quasiconvex, then $f_\beta$ is quasiconvex for every $0 \leq \beta \leq \gamma$. Let

$$I_\gamma (\xi, \varphi) := \int_\Omega \left[ f_\gamma (\xi + \nabla \varphi (x)) - f_\gamma (\xi) \right] dx$$

for every $\xi \in \mathbb{R}^{2\times 2}$ and every $\varphi \in W_{0}^{1,\infty} (\Omega; \mathbb{R}^2)$. We have to show that $I_\gamma (\xi, \varphi) \geq 0$ implies $I_\beta (\xi, \varphi) \geq 0$. We have to deal with two cases.

Case 1. If

$$\int_\Omega \left[ \left| \xi + \nabla \varphi (x) \right|^2 \det (\xi + \nabla \varphi (x)) - \left| \xi \right|^2 \det \xi \right] dx \leq 0,$$

then the claim is trivial using the convexity of $\xi \rightarrow \left| \xi \right|^4$ and the fact that $\beta \geq 0$.

Case 2. If

$$\int_\Omega \left[ \left| \xi + \nabla \varphi (x) \right|^2 \det (\xi + \nabla \varphi (x)) - \left| \xi \right|^2 \det \xi \right] dx \geq 0,$$

we observe that

$$I_\beta (\xi, \varphi) - I_\gamma (\xi, \varphi) = 2 (\gamma - \beta) \int_\Omega \left[ \left| \xi + \nabla \varphi (x) \right|^2 \det (\xi + \nabla \varphi (x)) - \left| \xi \right|^2 \det \xi \right] dx \geq 0,$$

as wished.

We may now define $\gamma_q$ by taking the largest $\gamma$ such that $f_\gamma$ is quasiconvex. It exists because of the preceding observation and from the fact that

$$1 = \gamma_p \leq \gamma_q \leq \gamma_r = \frac{2}{\sqrt{3}}$$

and this completes Step 1.

**Step 2:** $\gamma_q > 1$. We therefore have to show that there exists $\alpha > 0$ small enough, so that if $\gamma = 1 + \alpha$, then $f_\gamma$ is quasiconvex. We start with a preliminary result.

**Step 2'.** We prove the quasiconvexity of $f_\gamma$ at 0 for $\gamma = 1 + \alpha$ with $\alpha > 0$ small enough. We have to prove that

$$\int_\Omega f_\gamma (\nabla \varphi (x)) dx \geq 0$$
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for every $\varphi \in W^{1,\infty}_0(\Omega;\mathbb{R}^2)$ and for some $\alpha > 0$. Observe first the following algebraic inequality (we use the fact that $|\xi|^2 \geq 2 \det \xi$), valid for any $\xi \in \mathbb{R}^{2 \times 2}$,

$$f_{\gamma}(\xi) = |\xi|^4 - 2(1 + \alpha)|\xi|^2 \det \xi$$

$$= \frac{1}{2} [|\xi|^4 - 4|\xi|^2 \det \xi + 4(\det \xi)^2]$$

$$+ \frac{1}{2} [|\xi|^4 - 4(\det \xi)^2] - 2\alpha |\xi|^2 \det \xi$$

$$\geq \frac{1}{2} [|\xi|^2 - 2 \det \xi]^2 - \alpha |\xi|^4 .$$

We then integrate and use Theorem 5.52 to get

$$\int_{\Omega} f_{\gamma}(\nabla \varphi(x)) \, dx \geq (\epsilon - \alpha) \int_{\Omega} |\nabla \varphi(x)|^4 \, dx . \quad (5.80)$$

Choosing $0 \leq \alpha \leq \epsilon$, we have indeed obtained the result.

**Step 2.** We now proceed with the general case. We already know that $\gamma_q \geq \gamma_p = 1$, so we will assume throughout this step that $\gamma \geq 1$ and we will set $\alpha = \gamma - 1$.

Expanding $f_{\gamma}$, keeping in mind its special structure, we find

$$f_{\gamma}(\xi + \eta) = f_{\gamma}(\xi) + \langle \nabla f_{\gamma}(\xi); \eta \rangle + \frac{1}{2} \langle \nabla^2 f_{\gamma}(\xi) \eta; \eta \rangle$$

$$+ \langle \nabla f_{\gamma}(\eta); \xi \rangle + f_{\gamma}(\eta) .$$

Recall that $\langle \nabla^2 f_{\gamma}(\xi) \eta; \eta \rangle$ is given by (5.73). We rewrite this as

$$f_{\gamma}(\xi + \eta) - f_{\gamma}(\xi) = A_{\gamma}(\xi, \eta) + B_{\gamma}(\xi, \eta) + C_{\gamma}(\xi, \eta) + D_{\gamma}(\eta) + E_{\gamma}(\eta) \quad (5.81)$$

where

$$A_{\gamma}(\xi, \eta) := \langle \nabla f_{\gamma}(\xi); \eta \rangle - 2\gamma |\xi|^2 \det \eta$$

$$B_{\gamma}(\xi, \eta) := \frac{1}{2} \langle \nabla^2 f_{\gamma}(\xi) \eta; \eta \rangle + 2\gamma |\xi|^2 \det \eta$$

$$= 4(\langle \xi; \eta \rangle)^2 + 2|\xi|^2 |\eta|^2 - 4\gamma \langle \xi; \eta \rangle \langle \tilde{\xi}; \eta \rangle - 2\gamma |\eta|^2 \det \xi$$

$$C_{\gamma}(\xi, \eta) := \langle \tilde{\nabla} f_{\gamma}(\eta); \xi \rangle$$

$$= 4 \langle \xi; \eta \rangle |\eta|^2 - 4\gamma \langle \xi; \eta \rangle \det \eta - 2\gamma \langle \tilde{\xi}; \eta \rangle |\eta|^2$$

$$D_{\gamma}(\eta) := (1 - \epsilon) f_1(\eta) + \frac{\epsilon^2}{2} |\eta|^4$$

$$E_{\gamma}(\eta) := \epsilon f_1(\eta) - 2(\gamma - 1) |\eta|^2 \det \eta - \frac{\epsilon^2}{2} |\eta|^4$$

$$\geq \epsilon f_1(\eta) - (\alpha + \frac{\epsilon^2}{2}) |\eta|^4 .$$

Observe that

$$D_{\gamma}(\eta) + E_{\gamma}(\eta) = f_{\gamma}(\eta) .$$
From Step 2’ (applying (5.80) with $\gamma = 1$ and hence $\alpha = 0$), we have that, for every $\varphi \in W^{1,\infty}_0(\Omega; \mathbb{R}^2)$,

$$\int_{\Omega} E_\gamma (\nabla \varphi (x)) \, dx \geq [\epsilon^2 - (\alpha + \frac{\epsilon^2}{2})] \int_{\Omega} |\nabla \varphi (x)|^4 \, dx$$

which for $\alpha > 0$ sufficiently small with respect to $\epsilon^2$ leads to

$$\int_{\Omega} E_\gamma (\nabla \varphi (x)) \, dx \geq 0. \quad (5.82)$$

We also have that for $\epsilon > 0$ and $\alpha > 0$ even smaller (see Lemma 5.53)

$$\sigma_{\epsilon, \alpha} (\xi, \eta) = B_\gamma (\xi, \eta) + C_\gamma (\xi, \eta) + D_\gamma (\eta) \geq 0 \quad (5.83)$$

for every $\xi, \eta \in \mathbb{R}^{2\times2}$.

We are now in a position to conclude by combining (5.81), (5.82) and (5.83). We therefore have, for every $\xi \in \mathbb{R}^{2\times2}$, $\varphi \in W^{1,\infty}_0(\Omega; \mathbb{R}^2)$,

$$\int_{\Omega} [f_\gamma (\xi + \nabla \varphi (x)) - f_\gamma (\xi)] \, dx \geq \int_{\Omega} A_\gamma (\xi, \nabla \varphi (x)) \, dx = 0.$$

This is the desired claim. $\blacksquare$

The above proof relied on the following algebraic lemma.

**Lemma 5.53** Let

$$\sigma_{\epsilon, \alpha} (\xi, \eta) = B_\gamma (\xi, \eta) + C_\gamma (\xi, \eta) + D_\gamma (\eta)$$

where $\gamma = 1 + \alpha$ and

$$B_\gamma (\xi, \eta) = 4 (\langle \xi; \eta \rangle)^2 + 2 |\xi|^2 |\eta|^2 - 4 \gamma \langle \xi; \eta \rangle \langle \tilde{\xi}; \eta \rangle - 2 \gamma |\eta|^2 \det \xi$$

$$C_\gamma (\xi, \eta) = 4 \langle \xi; \eta \rangle |\eta|^2 - 4 \gamma \langle \xi; \eta \rangle \det \eta - 2 \gamma \langle \tilde{\xi}; \eta \rangle |\eta|^2$$

$$D_\gamma (\eta) = (1 - \epsilon) [ |\eta|^4 - 2 |\eta|^2 \det \eta ] + \epsilon^2 |\eta|^4.$$

For every $\epsilon > 0$ sufficiently small, there exists $\alpha_0 = \alpha_0 (\epsilon) > 0$ such that if $0 \leq \alpha \leq \alpha_0$, then

$$\sigma_{\epsilon, \alpha} (\xi, \eta) \geq 0, \text{ for every } \xi, \eta \in \mathbb{R}^{2\times2}.$$

**Proof.** The idea of the proof is to show that, for every $\epsilon > 0$ sufficiently small, there exists $\alpha_0 = \alpha_0 (\epsilon) > 0$ such that if $0 \leq \alpha \leq \alpha_0$, then

$$\xi \to \sigma_{\epsilon, \alpha} (\xi, \eta)$$
is a strictly convex polynomial of degree two for every \( \eta \in \mathbb{R}^{2 \times 2} \). In Step 2 we prove that by choosing both \( \epsilon \) sufficiently small and \( \alpha_0 (\epsilon) \) even smaller (uniformly with respect to \( \eta \)), then

\[
\sigma_{\epsilon, \alpha} (\xi, \eta) \geq 0
\]

at the unique minimum point \( \xi = \xi (\eta) \).

**Step 1.** We first show that for \( \alpha = \gamma - 1 > 0 \) sufficiently small

\[
B_\gamma (\xi, \eta) \geq \frac{11 - 9 \gamma^2}{6} |\xi| |\eta| \quad \text{for every} \quad \xi, \eta \in \mathbb{R}^{2 \times 2}.
\]

The case \( \xi = 0 \) or \( \eta = 0 \) being trivial, we can assume because of the homogeneity of \( B_\gamma \) that

\[
|\xi| = |\eta| = 1.
\]

Moreover, since \( Q\tilde{\xi}R = Q\tilde{\xi}R \) for every \( Q, R \in SO (2) \), we have

\[
B_\gamma (\xi, Q\eta R) = B_\gamma (Q^t\xi R^t, \eta)
\]

and thus it is enough to prove (5.84) for matrices \( \xi \) and \( \eta \) of the form (according to Theorem 13.3)

\[
\xi = \begin{pmatrix}
\cos \theta \cos A & \sin A \cos B \\
\sin A \sin B & \sin \theta \cos A
\end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix}
\cos \varphi & 0 \\
0 & \sin \varphi
\end{pmatrix}.
\]

We therefore find

\[
B_\gamma (\xi, \eta) = 2 + \gamma \sin (2B) \sin^2 A + [4 \cos^2 (\theta - \varphi) - 4 \gamma \cos (\theta - \varphi) \sin (\theta + \varphi) - \gamma \sin (2\theta)] \cos^2 A.
\]

Since \( \sin (2B) \geq -1 \), we find that

\[
B_\gamma (\xi, \eta) \geq 2 - \gamma + [\gamma + 4 \cos^2 (\theta - \varphi) - 4 \gamma \cos (\theta - \varphi) \sin (\theta + \varphi) - \gamma \sin (2\theta)] \cos^2 A.
\]

Since \( \gamma > 1 \) is sufficiently close to 1 and we want to minimize \( B_\gamma (\xi, \eta) \), we have to choose \( \cos^2 A = 1 \). We can thus write

\[
B_\gamma (\xi, \eta) \geq 2 + 4 \cos^2 (\theta - \varphi) - 4 \gamma \cos (\theta - \varphi) \sin (\theta + \varphi) - \gamma \sin (2\theta)
\]

or, writing \( a = 2\theta \) and \( b = 2\varphi \),

\[
B_\gamma (\xi, \eta) \geq g (a, b) := 4 + 2 \cos (a - b) - 3 \gamma \sin a - 2 \gamma \sin b. \tag{5.85}
\]

We easily have that

\[
\nabla g (a, b) = 0 \iff \cos b = -\frac{3}{2} \cos a = \frac{1}{\gamma} \sin (a - b). \tag{5.86}
\]
We can next write that
\[ g(a, b) \geq \min \{ g(a, b) : \nabla g(a, b) = 0 \} \] (5.87)
and therefore two cases can happen.

Case 1: \( \cos a = \cos b = \sin (a - b) = 0 \). At such a point (recalling that \( \gamma \) is sufficiently close to 1) we have
\[ g(a, b) \geq 6 - 5\gamma. \] (5.88)

Case 2: \( \cos a \neq 0 \) and \( \cos b \neq 0 \). From (5.86), we find
\[ \cos b = -\frac{3}{2} \cos a \quad \text{and} \quad \sin b = \frac{3}{2} (\gamma - \sin a). \]
We hence deduce that
\[ \frac{4}{9} = \frac{4}{9} \cos^2 b + \frac{4}{9} \sin^2 b = \gamma^2 + 1 - 2\gamma \sin a. \]
Therefore at such a point \((a, b)\) we have
\[ g(a, b) = 4 + 2 \cos a \cos b + 2 \sin a \sin b - 3\gamma \sin a - 2\gamma \sin b \]
\[ = 1 - 3\gamma^2 + 3\gamma \sin a = \frac{11 - 9\gamma^2}{6}. \]
Combining (5.85), (5.87), (5.88) and the above identity, we have indeed obtained (5.84).

Step 2. We now prove that by choosing both \( \epsilon \) sufficiently small and \( \alpha_0(\epsilon) \) even smaller (uniformly with respect to \( \eta \)), then
\[ \sigma_{\epsilon, \alpha}(\xi, \eta) \geq 0 \] for every \( \xi, \eta \in \mathbb{R}^{2 \times 2} \).
We start by observing that
\[ \sigma_{\epsilon, \alpha}(\xi, 0) = 0 \] for every \( \xi \in \mathbb{R}^{2 \times 2} \).
So from now on we will assume that \( \eta \neq 0 \) and is fixed. From Step 1, we see that the function
\[ \xi \rightarrow \sigma_{\epsilon, \alpha}(\xi, \eta) \]
has a unique minimum, which satisfies
\[ \nabla_\xi \sigma_{\epsilon, \alpha}(\xi, \eta) = 0; \]
i.e.
\[ 4 |\eta|^2 \eta - 4\gamma (\det \eta) \eta - 2\gamma |\eta|^2 \tilde{\eta} \tilde{\eta} + 4 |\eta|^2 \xi - 2\gamma |\eta|^2 \tilde{\xi} \]
\[ + 8 \langle \xi; \eta \rangle \eta - 4\gamma \langle \xi; \eta \rangle \tilde{\eta} - 4\gamma \langle \tilde{\xi}; \eta \rangle \eta = 0. \] (5.89)
We now multiply (5.89) first by $\xi$, then by $\eta$ and finally by $\tilde{\eta}$ to get
\[
2 \langle \xi; \eta \rangle \left( |\eta|^2 - \gamma \det \eta \right) - \gamma \langle \tilde{\xi}; \eta \rangle |\eta|^2 \\
= 2 |\xi|^2 |\eta|^2 - 2 \gamma |\eta|^2 \det \xi + 4 \langle \xi; \eta \rangle^2 - 4 \gamma \langle \xi; \eta \rangle \langle \tilde{\xi}; \eta \rangle \\
+ 4 \langle \xi; \eta \rangle |\eta|^2 - 4 \gamma \langle \xi; \eta \rangle \det \eta - 2 \gamma \langle \tilde{\xi}; \eta \rangle |\eta|^2 \\
= -\gamma \langle \tilde{\xi}; \eta \rangle |\eta|^2 \\
= -\frac{2}{3} |\eta|^4 + \frac{4}{3} \gamma |\eta|^2 \det \eta - 2 \langle \xi; \eta \rangle |\eta|^2 + \frac{4}{3} \gamma \langle \xi; \eta \rangle \det \eta
\] (5.90)
\[
2 \langle \tilde{\xi}; \eta \rangle \left( |\eta|^2 - 2 \gamma \det \eta \right) \\
= \langle \xi; \eta \rangle (3 \gamma |\eta|^2 - 8 \det \eta) + \gamma |\eta|^4 - 4 |\eta|^2 \det \eta + 4 \gamma (\det \eta)^2.
\] (5.91)

We next combine (5.89) to (5.92) to show that $\sigma_{\epsilon, \alpha} \geq 0$ at a stationary point provided $\alpha = \gamma - 1$ and $\epsilon$ are small enough. Combining (5.91) and (5.92), so as to eliminate $\langle \tilde{\xi}; \eta \rangle$, we find that
\[
\langle \xi; \eta \rangle (3 \gamma |\eta|^2 - 8 \det \eta) + \gamma |\eta|^4 - 4 |\eta|^2 \det \eta + 4 \gamma (\det \eta)^2.
\] (5.93)

We now use (5.90), (5.91) and (5.93) to compute $\sigma_{\epsilon, \alpha}$ at the minimum point. First appeal to (5.90) to obtain
\[
\sigma_{\epsilon, \alpha} = 2 \langle \xi; \eta \rangle (|\eta|^2 - \gamma \det \eta) - \gamma \langle \tilde{\xi}; \eta \rangle |\eta|^2 \\
+ (1 - \epsilon + \frac{\epsilon^2}{2}) |\eta|^4 - 2 (1 - \epsilon) |\eta|^2 \det \eta.
\]

Replacing the second term, with the help of (5.91), we find
\[
\sigma_{\epsilon, \alpha} = -\frac{2}{3} \gamma \langle \xi; \eta \rangle \det \eta + \left( \frac{1}{3} - \epsilon + \frac{\epsilon^2}{2} \right) |\eta|^4 - 2 (1 - \epsilon - \frac{2}{3} \gamma) |\eta|^2 \det \eta.
\]

Inserting (5.93) in the above identity, we obtain
\[
\frac{3\sigma_{\epsilon, \alpha}}{|\eta|^2} [3 (4 - 3 \gamma^2) |\eta|^4 - 8 \gamma |\eta|^2 \det \eta + 16 \gamma^2 (\det \eta)^2] \\
= [(1 - 3 \epsilon + \frac{3 \epsilon^2}{2}) |\eta|^2 - 2 (3 - 3 \epsilon - 2 \gamma) \det \eta] \\
\times [3 (4 - 3 \gamma^2) |\eta|^4 - 8 \gamma |\eta|^2 \det \eta + 16 \gamma^2 (\det \eta)^2] \\
+ 2 \gamma \det \eta [(4 - 3 \gamma^2) |\eta|^4 - 4 \gamma |\eta|^2 \det \eta + 4 \gamma^2 (\det \eta)^2].
\]

Setting $t = |\eta|$ and $\delta = 2 \det (\eta/|\eta|)$, ($\Rightarrow |\delta| \leq 1$),
Examples

we get

\[ \frac{3 \sigma_{\epsilon, \alpha}}{t^4} \geq 3 \left[ 3 \left( 4 - 3 \gamma^2 \right) - 4 \gamma \delta + 4 \gamma^2 \delta^2 \right] \]

\[ = \left[ \left( 1 - 3 \epsilon + \frac{3}{2} \epsilon^2 \right) - (3 - 3 \epsilon - 2 \gamma) \delta \right] \left[ 3 \left( 4 - 3 \gamma^2 \right) - 4 \gamma \delta + 4 \gamma^2 \delta^2 \right] \]

\[ + \gamma \delta \left[ 4 - 3 \gamma^2 \right] - 2 \gamma \delta^2 + \gamma^2 \delta^2 \right].\]

Letting \( \alpha = \gamma - 1 \geq 0 \) and using the fact that \( |\delta| \leq 1 \), we find the following three estimates for \( \alpha \) small enough

\[ \left[ 3 \left( 4 - 3 \gamma^2 \right) - 4 \gamma \delta + 4 \gamma^2 \delta^2 \right] \]

\[ \leq \left[ 3 \left( 1 - 6 \alpha - 3 \alpha^2 \right) - 4 (1 + \alpha) \delta + 4 (1 + \alpha)^2 \delta^2 \right] \]

\[ \leq \left[ 3 - 4 \delta + 4 \delta^2 \right] + 1 \leq 12 \]

\[ \left[ (1 - 3 \epsilon + \frac{3}{2} \epsilon^2) - (3 - 3 \epsilon - 2 \gamma) \delta \right] \left[ 3 \left( 4 - 3 \gamma^2 \right) - 4 \gamma \delta + 4 \gamma^2 \delta^2 \right] \]

\[ = \frac{3}{2} \epsilon^2 \left[ 2 + (1 - 2 \delta)^2 \right] \]

\[ + (1 - 3 \epsilon) (1 - \delta) \left[ 3 - 4 \delta + 4 \delta^2 \right] + O_\delta (\alpha) \]

\[ = (1 + \alpha) \delta \left[ (1 - 6 \alpha - 3 \alpha^2) - 2 (1 + \alpha) \delta + (1 + \alpha)^2 \delta^2 \right] \]

\[ = \delta (1 - \delta)^2 + O_\delta (\alpha) \]

where \( O_\delta (\alpha) \) stands for a term that goes to 0 as \( \alpha \) tends to 0 uniformly for \( |\delta| \leq 1 \).

Combining these three estimates, we find for \( \epsilon \) sufficiently small, since \( |\delta| \leq 1 \),

\[ \frac{3 \sigma_{\epsilon, \alpha}}{t^4} \geq 3 \epsilon^2 + 3 \left[ 1 - \delta + \delta^2 - \epsilon (3 - 4 \delta + 4 \delta^2) \right] + O_\delta (\alpha) \geq 3 \epsilon^2 + O_\delta (\alpha). \]

Choosing \( \alpha << \epsilon \) (recalling that \( \epsilon \) is small), we get the result; i.e.

\[ \sigma_{\epsilon, \alpha} (\xi, \eta) \geq 0, \text{ for every } \xi, \eta \in \mathbb{R}^{2 \times 2}. \]

This concludes the proof of the lemma. ■

5.3.9 Quasiconvex functions with subquadratic growth.

We have seen in Corollary 5.9 that a polyconvex function having a subquadratic growth, must be convex. This, however, is not the case for quasiconvex and rank one convex functions. We now give such an example, following Sverak [549] (for the case \( p = 1 \), see Theorem 5.55).

Theorem 5.54 Let \( 1 < p < 2 \). Then there exists a function \( f : \mathbb{R}^{2 \times 2} \to \mathbb{R} \) quasiconvex, non-convex and satisfying

\[ 0 \leq f (\xi) \leq \gamma (1 + |\xi|^p), \text{ } \forall \xi \in \mathbb{R}^{2 \times 2} \]

and where \( \gamma \) is a positive constant.
Proof. We start with the following easily established algebraic inequality valid for any $\xi \in \mathbb{R}^{2 \times 2}$

$$\min\{|\xi - I|^2, |\xi + I|^2\} \geq \frac{1}{2}(|\xi|^2 - 2\det \xi) \geq 0. \quad (5.94)$$

We next define

$$g(\xi) := \min \{|\xi - I|^p, |\xi + I|^p\}.$$  

Anticipating on the definition and properties of the quasiconvex envelope given in Chapter 6 (see Theorem 6.9), we let

$$f := Qg$$

and we claim that $f$ has all the desired properties. By definition it is quasiconvex and satisfies the growth condition, we therefore only need to show that it is not convex. This will be proved, once shown that

$$f(0) = Qg(0) > 0, \quad (5.95)$$

since clearly

$$Cg(0) = 0$$

where $Cg$ denotes the convex envelope of $g$.

Assume for the sake of contradiction that $Qg(0) = 0$

and use Theorem 6.9 to find a sequence $\varphi^\nu \in W_0^{1,\infty}(D; \mathbb{R}^2)$, here $D \subset \mathbb{R}^2$ is a bounded open set with meas $D = 1$, such that

$$0 = Qg(0) \geq -\frac{1}{\nu} + \int_D g(\nabla \varphi^\nu(x)) \, dx. \quad (5.96)$$

Invoking (5.94), we can deduce from the above inequality that

$$\frac{1}{\nu} \geq 2^{-p/2} \int_D \left[|\nabla \varphi^\nu(x)|^2 - 2\det(\nabla \varphi^\nu(x))\right]^{p/2} \, dx.$$  

The estimate of Theorem 5.52 then implies that

$$\varphi^\nu \to 0 \text{ in } W^{1,p}(D; \mathbb{R}^2).$$

This therefore leads to

$$\lim_{\nu \to \infty} \int_D g(\nabla \varphi^\nu(x)) \, dx = 2^{p/2},$$

contradicting (5.96). We have therefore proved (5.95) and the theorem follows.
5.3.10 The case of homogeneous functions of degree one

We would now like to discuss the convexity properties of homogeneous functions of degree one, \( f : \mathbb{R}^{2 \times 2} \to \mathbb{R} \) and we have the following theorem.

**Theorem 5.55** Let \( f : \mathbb{R}^{2 \times 2} \to \mathbb{R} \) be positively homogeneous of degree one, namely

\[
f(t\xi) = tf(\xi) \quad \text{for every } t \geq 0 \text{ and every } \xi \in \mathbb{R}^{2 \times 2}. \tag{5.97}
\]

The following three properties hold.

(i) \( f \) is polyconvex if and only if it is convex.

(ii) If \( f \) is \( SO(2) \times SO(2) \)-invariant, in the sense that

\[
f(\xi) = f(Q\xi R) \quad \text{for every } Q, R \in SO(2),
\]

then \( f \) is rank one convex if and only if it is convex.

(iii) The function

\[
f(\xi) = \begin{cases} 
7|\xi| + \frac{3(\xi_1^4 + 2\xi_1^2\xi_2^2 + 3(\xi_2^2)^2 + 4\xi_1^2\xi_2^2)}{|\xi|} & \text{if } \xi \neq 0 \\
0 & \text{if } \xi = 0
\end{cases}
\]

is rank one convex but not convex.

**Remark 5.56**

(i) The first statement follows at once from Corollary 5.9.

(ii) The second assertion has been proved by Dacorogna [181] and the last one is a particular case of the study undertaken by Dacorogna-Haeberly [190].

(iii) Müller [461] (see also Zhang [618]) produced, in an indirect way similar to that of Theorem 5.54, an example of a quasiconvex function satisfying (5.97) and that is not convex.

(iv) It is not presently known if the function given in (iii) of the theorem is quasiconvex. Numerical evidences given in Dacorogna-Haeberly [191] tend to indicate that it is quasiconvex.

Before proceeding with the proof we need the following elementary lemma established in Dacorogna [181], for a different proof see Dacorogna-Maréchal [206]. The lemma is false if either the function is not everywhere finite or in dimensions 3 and higher, see [206] for details. Note that in dimension 4, the function given in Theorem 5.55, being rank one convex, is separately convex but not convex.

**Lemma 5.57** Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be positively homogeneous of degree one and separately convex (meaning that \( x \to g(x, y) \) and \( y \to g(x, y) \) are both convex). Then \( g \) is convex.

**Proof.** (Lemma 5.57). Since \( g \) is homogeneous of degree one, it is clear that \( g \) is convex if and only if

\[
g(x_1 + x_2, y_1 + y_2) \leq g(x_1, y_1) + g(x_2, y_2). \tag{5.98}
\]
We consider two cases.

Case 1: $x_1x_2 \geq 0$ or $y_1y_2 \geq 0$. Since the hypothesis $x_1x_2 \geq 0$ is handled similarly to $y_1y_2 \geq 0$, we will assume that this last one holds. Since $g$ is separately convex it is continuous (cf. Theorem 2.31) and hence it is enough to prove the result for $y_1y_2 > 0$. Observe then that

$$\sigma := \frac{y_1 + y_2}{|y_1 + y_2|} = \frac{y_1}{|y_1|} = \frac{y_2}{|y_2|} \in \{ \pm 1 \}.$$  

We therefore have, using the convexity of $g$ with respect to the first variable,

$$g(x_1 + x_2, y_1 + y_2) = |y_1 + y_2| g\left( \frac{|y_1|}{|y_1 + y_2|} x_1 + \frac{|y_2|}{|y_1 + y_2|} x_2, \sigma \right) \leq |y_1| g\left( \frac{x_1}{|y_1|}, \sigma \right) + |y_2| g\left( \frac{x_2}{|y_2|}, \sigma \right) = g(x_1, y_1) + g(x_2, y_2)$$

as wished.

Case 2: $x_1x_2 < 0$ and $y_1y_2 < 0$. This case is more involved than the previous one and we divide the proof into two steps.

**Step 1.** We first show that

$$g(x_1 + x_2, 0) \leq g(x_1, y) + g(x_2, -y), \quad \forall y \in \mathbb{R}. \quad (5.99)$$

Since $x_1x_2 < 0$, we have either

$$x_1(x_1 + x_2) \geq 0 \quad \text{or} \quad x_2(x_1 + x_2) \geq 0.$$  

Without loss of generality (otherwise exchange the roles of $(x_1, y)$ with that of $(x_2, -y)$), we will assume that

$$x_1(x_1 + x_2) \geq 0. \quad (5.100)$$

We then choose $\epsilon > 0$ sufficiently small and let

$$a := \frac{x_1 + (1 - 2\epsilon)x_2}{(1 - \epsilon)} \quad \text{and} \quad \mu := \frac{1 - 2\epsilon}{1 - \epsilon}.$$

Observe that

$$\left\{ \begin{array}{l}
-2\epsilon \mu + 2(1 - \mu)(1 - 2\epsilon) = 0 \\
\mu(x_1 + x_2) + 2(1 - \mu)x_1 = a \\
2\epsilon x_2 + (1 - \epsilon)a = x_1 + x_2.
\end{array} \right.$$  

Appealing to Case 1, since $(-y)0 \geq 0$, we find

$$g(x_1 + x_2, -2\epsilon y) = g(2\epsilon x_2 + (1 - \epsilon)a, 2\epsilon(-y) + (1 - \epsilon)0) \leq 2\epsilon g(x_2, -y) + (1 - \epsilon)g(a, 0).$$

Since (5.100) holds, we also have from Case 1

$$g(a, 0) = g(\mu(x_1 + x_2) + 2(1 - \mu)x_1, \mu(-2\epsilon y) + 2(1 - \mu)(1 - 2\epsilon)y) \leq \mu g(x_1 + x_2, -2\epsilon y) + (1 - \mu)g(2x_1, 2(1 - 2\epsilon)y) = \mu g(x_1 + x_2, -2\epsilon y) + 2(1 - \mu)g(x_1, (1 - 2\epsilon)y).$$
Combining the last two inequalities, we find
\[ g(x_1 + x_2, -2\epsilon y) \leq 2\epsilon g(x_2, -y) + (1 - 2\epsilon) g(x_1 + x_2, -2\epsilon y) + 2\epsilon g(x_1, (1 - 2\epsilon) y) \]
or, in other words,
\[ 2\epsilon g(x_1 + x_2, -2\epsilon y) \leq 2\epsilon g(x_2, -y) + 2\epsilon g(x_1, (1 - 2\epsilon) y). \]
Dividing by $2\epsilon$ and letting $\epsilon$ tend to 0, using the continuity of $g$, we have indeed obtained (5.99).

**Step 2.** We now prove (5.98). Observe that the hypothesis $y_1y_2 < 0$ implies
\[ \frac{y_1 + y_2}{y_1} \geq 0 \text{ or } \frac{y_1 + y_2}{y_2} \geq 0. \]
We will assume that the first possibility happens, the second one being handled similarly.

We can therefore write,
\[ g(x_1 + x_2, y_1 + y_2) = g\left(\frac{y_1 + y_2}{y_1}x_1 + x_2 - \frac{y_2}{y_1}x_1, \frac{y_1 + y_2}{y_1}y_1 + 0\right). \]
Since $(y_1 + y_2) \cdot 0 \geq 0$, we can apply Case 1 and get
\[ g(x_1 + x_2, y_1 + y_2) \leq \frac{y_1 + y_2}{y_1}g(x_1, y_1) + g(x_2 - \frac{y_2}{y_1}x_1, 0). \quad (5.101) \]
We also have, invoking Step 1,
\[ g(x_2 - \frac{y_2}{y_1}x_1, 0) \leq g(x_2, y_2) + g(-\frac{y_2}{y_1}x_1, -\frac{y_2}{y_1}y_1) = g(x_2, y_2) - \frac{y_2}{y_1}g(x_1, y_1). \]
Combining the above inequality and (5.101), we obtain (5.98) and thus the lemma.

We now proceed with the proof of the theorem.

**Proof.** (Theorem 5.55). (i) As already mentioned the proof of the first part immediately follows from Corollary 5.9.

(ii) The implication $f$ convex $\Rightarrow$ $f$ rank one convex, being always true, we need only prove the reverse one. According to Theorem 5.33, it is sufficient to show that $f$ is convex on diagonal matrices. Therefore let
\[ g(x_1, x_2) := f \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \]
and observe that the rank one convexity of $f$ implies the separate convexity of $g$. Lemma 5.57 gives immediately the claim.
(iii) We first discuss the fact that $f$ is non convex. We let
\[
\xi = \frac{1}{\sqrt{2}} \left( \begin{array}{cc}
1 & 0 \\
0 & 1 \\
\end{array} \right) \quad \text{and} \quad \eta = \frac{1}{\sqrt{2}} \left( \begin{array}{cc}
0 & 1 \\
-1 & 0 \\
\end{array} \right)
\]
and for $t \in \mathbb{R}$, we define
\[
t \to \varphi(t) := f(\xi + t\eta) = 5(1 + t^2)^{1/2} + 6(1 + t^2)^{-1/2}.
\]
A direct computation shows that
\[
\varphi''(t) = (17t^2 - 1)(1 + t^2)^{-5/2}
\]
and hence $\varphi''(0) = -1 < 0$, which implies that $f$ is non convex.

It therefore remains only to show that $f$ is rank one convex. We divide the proof of this fact into three steps.

**Step 1.** The rank one convexity of $f$ is equivalent to showing that for every fixed $\xi \in \mathbb{R}^{2 \times 2}$, $a, b \in \mathbb{R}^2$ the function
\[
t \to \varphi_{\xi,a,b}(t) := f(\xi + ta \otimes b)
\]
is convex in $t \in \mathbb{R}$.

Since $f(\xi) \geq 0$, we have that if there exists $\alpha \in \mathbb{R}$ such that
\[
\xi = \alpha a \otimes b,
\]
then
\[
f(\xi + ta \otimes b) = f((\alpha + t) a \otimes b) = |\alpha + t|f(a \otimes b)
\]
and thus $\varphi_{\xi,a,b}$ is convex in $t$. From now on we may therefore assume that $\xi$ is not parallel to $a \otimes b$. The function $\varphi_{\xi,a,b}$ is then twice continuously differentiable and its convexity is therefore equivalent to the Legendre-Hadamard condition, namely
\[
\langle \nabla^2 f(\xi) a \otimes b; a \otimes b \rangle \geq 0 \quad (5.102)
\]
for every $\xi \in \mathbb{R}^{2 \times 2}$, $a, b \in \mathbb{R}^2$ with $\xi$ not parallel to $a \otimes b$.

**Step 2.** We now compute the Hessian of $f$. It will be more convenient, in the present analysis, to identify $\mathbb{R}^{2 \times 2}$ with $\mathbb{R}^4$ and, therefore, a matrix $\xi$ will be written as a vector $(\xi_1, \xi_2, \xi_3, \xi_4)$. We then let
\[
\langle \xi; \eta \rangle = \sum_{i=1}^{4} \xi_i \eta_i, \quad |\xi|^2 = \langle \xi; \xi \rangle, \quad \det \xi = \xi_1\xi_4 - \xi_2\xi_3.
\]
Letting
\[
M = \left( \begin{array}{cccc}
9 & 0 & 0 & 1 \\
0 & 6 & 2 & 0 \\
0 & 2 & 6 & 0 \\
1 & 0 & 0 & 9 \\
\end{array} \right)
\]
we can rewrite $f$, when $\xi \neq 0$, as

$$f(\xi) = |\xi| + \frac{\langle M\xi; \xi \rangle}{|\xi|}.$$ 

Computing the Hessian of $f$, when $\xi \neq 0$, we first find, for $\alpha = 1, 2, 3, 4$, that

$$\frac{\partial f(\xi)}{\partial \xi_\alpha} = \frac{\xi_\alpha}{|\xi|} + \frac{2 |\xi| (M\xi)_\alpha - \langle M\xi; \xi \rangle \xi_\alpha}{|\xi|^2}$$

and thus

$$\frac{\partial^2 f(\xi)}{\partial \xi_\alpha \partial \xi_\beta} = \frac{\delta_{\alpha\beta}}{|\xi|} - \frac{\xi_\alpha \xi_\beta}{|\xi|^3} + \frac{1}{|\xi|^3} \left\{ -3 |\xi| \xi_\beta [2 |\xi|^2 (M\xi)_\alpha - \langle M\xi; \xi \rangle \xi_\alpha] \\
+ [4 (M\xi)_\alpha \xi_\beta + 2 |\xi|^2 M_{\alpha\beta} - \langle M\xi; \xi \rangle \delta_{\alpha\beta} - 2 (M\xi)_\beta \xi_\alpha] |\xi|^3 \right\},$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol.

Since the quadratic form $\langle \nabla^2 f(\xi) \lambda; \lambda \rangle$ is homogeneous of degree $-1$ in $\xi$ and $2$ in $\lambda$, we only need to consider the case where $|\xi| = |\lambda| = 1$. We hence get that

$$\sum_{\alpha, \beta=1}^{4} \frac{\partial^2 f(\xi)}{\partial \xi_\alpha \partial \xi_\beta} \lambda_\alpha \lambda_\beta = 1 - (\langle \xi; \lambda \rangle)^2 - 4 \langle M\xi; \lambda \rangle \langle \xi, \lambda \rangle + 2 \langle M\lambda; \lambda \rangle$$

$$- \langle M\xi; \xi \rangle + 3 \langle M\xi; \xi \rangle (\langle \xi; \lambda \rangle)^2.$$

We can still transform this expression into a more amenable one, by choosing a vector $\eta \in \mathbb{R}^4$ and $\theta \in \mathbb{R}$ so that

$$\lambda = \xi \cos \theta + \eta \sin \theta, \text{ with } |\eta| = 1 \text{ and } \langle \xi; \eta \rangle = 0.$$

We therefore obtain that

$$\langle \xi; \lambda \rangle = \cos \theta, \quad \langle M\xi; \lambda \rangle = \langle M\xi; \xi \rangle \cos \theta + \langle M\xi; \eta \rangle \sin \theta$$

$$\langle M\lambda; \lambda \rangle = \langle M\xi; \xi \rangle \cos^2 \theta + 2 \langle M\xi; \eta \rangle \cos \theta \sin \theta + \langle M\eta; \eta \rangle \sin^2 \theta.$$}

Returning to the quadratic form we therefore find that

$$\langle \nabla^2 f(\xi) \lambda; \lambda \rangle = [1 + 2 \langle M\eta; \eta \rangle - \langle M\xi; \xi \rangle] \sin^2 \theta.$$ 

Hence (5.102) is equivalent to showing that

$$1 + 2 \langle M\eta; \eta \rangle - \langle M\xi; \xi \rangle \geq 0 \quad (5.103)$$
for every $\xi, \eta \in \mathbb{R}^4$ and $\theta \in \mathbb{R}$ satisfying

$$|\xi| = |\eta| = 1, \langle \xi; \eta \rangle = 0 \text{ and } \det (\xi \cos \theta + \eta \sin \theta) = 0.$$  \hfill (5.104)

**Step 3.** It therefore remains to show (5.103) whenever (5.104) holds. We start by observing that the matrix $M$ has eigenvalues

$$\mu_1 = 4 \leq \mu_2 = \mu_3 = 8 \leq \mu_4 = 10$$

and corresponding orthonormal eigenvectors

$$\varphi_1 = \frac{1}{\sqrt{2}} (0, 1, -1, 0) \quad \varphi_2 = \frac{1}{\sqrt{2}} (0, 1, 1, 0)$$

$$\varphi_3 = \frac{1}{\sqrt{2}} (1, 0, 0, -1) \quad \varphi_4 = \frac{1}{\sqrt{2}} (1, 0, 1, 0).$$

Note that

$$\det \varphi_1 = \det \varphi_4 = -\det \varphi_2 = -\det \varphi_3 = \frac{1}{2}.$$ 

Expanding the vectors $\xi, \eta \in \mathbb{R}^4$ in this basis we have

$$\xi = \sum_{i=1}^{4} \xi_i \varphi_i, \quad \eta = \sum_{i=1}^{4} \eta_i \varphi_i,$$

and from now on $\xi_i$ and $\eta_i$ will always denote the components of $\xi$ and $\eta$ in this new basis and in particular we find that

$$\det \xi = \frac{1}{2} (\xi_1^2 + \xi_2^2 - \xi_2^2 - \xi_3^2).$$

Moreover, (5.103) is equivalent to showing that

$$2 \langle M\eta, \eta \rangle - \langle M\xi, \xi \rangle = \sum_{i=1}^{4} \mu_i (2\eta_i^2 - \xi_i^2) \geq -1.$$  \hfill (5.105)

Moreover, (5.104) can then be rewritten as

$$|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = |\eta|^2 = \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = 1,$$

$$\langle \xi; \eta \rangle = 0 \Leftrightarrow \xi_1 \eta_1 + \xi_4 \eta_4 = - (\xi_2 \eta_2 + \xi_3 \eta_3),$$

with

$$\det (\xi \cos \theta + \eta \sin \theta) = 0$$

$$\Leftrightarrow (\xi_1^2 + \xi_2^2 - \xi_3^2) \cos^2 \theta + (\eta_1^2 + \eta_4^2 - \eta_2^2 - \eta_3^2) \sin^2 \theta$$

$$+ 2 (\xi_1 \eta_1 + \xi_4 \eta_4 - \xi_2 \eta_2 - \xi_3 \eta_3) \cos \theta \sin \theta = 0.$$

We now argue by contradiction and assume that (5.105) does not hold, meaning that we can find $\xi, \eta \in \mathbb{R}^4$ and $\theta \in \mathbb{R}$ as above and so that

$$\sum_{i=1}^{4} \mu_i (2\eta_i^2 - \xi_i^2) < -1.$$
Observing that
\[ 2\mu_1 |\eta|^2 - \frac{\mu_3 + \mu_4}{2} |\xi|^2 = -1, \]
we can rewrite the above inequality as
\[ 12\eta_4^2 + 5\xi_1^2 + 8(\eta_2^2 + \eta_3^2) < \xi_4^2 - \xi_2^2 - \xi_3^2. \] (5.106)

Similarly, writing
\[ \sum_{i=1}^{4} \mu_i (2\eta_i^2 - \xi_i^2) < -1 < 2 = (\mu_1 + \mu_2) |\eta|^2 - \mu_4 |\xi|^2 \]
we find that
\[ 8\eta_4^2 + 6\xi_1^2 + 2(\xi_2^2 + \xi_3^2) < 4(\eta_1^2 - \eta_2^2 - \eta_3^2). \] (5.107)

From (5.106) and (5.107), we deduce that
\[ 8(\eta_2^2 + \eta_3^2) < \xi_4^2 - \xi_2^2 - \xi_3^2 \quad \text{and} \quad \frac{1}{2}(\xi_2^2 + \xi_3^2) < \eta_1^2 - \eta_2^2 - \eta_3^2. \]

Inserting these inequalities in the identity \( \det (\xi \cos \theta + \eta \sin \theta) = 0 \) and also using the fact that \( \xi_1 \eta_4 - \xi_4 \eta_1 = - (\xi_2 \eta_2 + \xi_3 \eta_3) \) leads to the desired contradiction, namely
\[
0 = \left( \xi_1^2 + \xi_4^2 - \xi_2^2 - \xi_3^2 \right) \cos^2 \theta + \left( \eta_1^2 + \eta_4^2 - \eta_2^2 - \eta_3^2 \right) \sin^2 \theta \\
+ 2 \left( \xi_1 \eta_4 - \xi_4 \eta_1 - \xi_2 \eta_2 - \xi_3 \eta_3 \right) \cos \theta \sin \theta \\
> \left[ \xi_1^2 + 8(\eta_2^2 + \eta_3^2) \right] \cos^2 \theta + \left[ \eta_1^2 + \frac{1}{2}(\xi_2^2 + \xi_3^2) \right] \sin^2 \theta \\
- 4(\xi_2 \eta_2 + \xi_3 \eta_3) \cos \theta \sin \theta \\
\geq 8(\eta_2^2 + \eta_3^2) \cos^2 \theta + \frac{1}{2}(\xi_2^2 + \xi_3^2) \sin^2 \theta - 4|\xi_2 \eta_2 + \xi_3 \eta_3| \cos \theta \sin \theta \\
\geq \frac{1}{2} \left[ 4|\cos \theta| \sqrt{\eta_2^2 + \eta_3^2} - |\sin \theta| \sqrt{\xi_2^2 + \xi_3^2} \right]^2 \geq 0.
\]

This concludes the proof of the theorem. ■

5.3.11 Some more examples

We now give some more examples.

**Theorem 5.58** Let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) and let \(|.|\) denote the Euclidean norm, namely, for \( \xi \in \mathbb{R}^{N \times n} \), we let
\[
|\xi| := \left( \sum_{\alpha=1}^{n} \sum_{i=1}^{N} (\xi_{\alpha}^i)^2 \right)^{1/2}.
\]
(i) Let \( g : \mathbb{R}^+ \to \mathbb{R} \) be such that
\[
f(\xi) = g(|\xi|).
\]
Then
\[
f \text{ convex } \iff f \text{ polyconvex } \iff f \text{ quasiconvex } \iff f \text{ rank one convex}
\]
\[
\iff g \text{ convex and } g(0) = \inf \{g(x) : x \geq 0\}.
\]
(ii) Let \( N = n, 1 \leq \alpha < 2n, h : \mathbb{R} \to \mathbb{R} \) be such that
\[
f(\xi) = |\xi|^\alpha + h(\det \xi).
\]
Then
\[
f \text{ polyconvex } \iff f \text{ quasiconvex } \iff f \text{ rank one convex}
\]
\[
\iff h \text{ convex}.
\]
(iii) Let \( N = n, p > 0, 1 \leq s \leq n - 1 \) and
\[
f(\xi) = \begin{cases} 
\left( \frac{|\text{adj}_s \xi|^{n/s}}{\det \xi} \right)^p & \text{if } \det \xi > 0 \\
+\infty & \text{otherwise.}
\end{cases}
\]
Then
\[
f \text{ polyconvex } \iff f \text{ rank one convex} \iff p \geq \frac{s}{n-s}.
\]

**Remark 5.59** (i) The result (i) was established by Dacorogna [176].

(ii) Case (ii) was proved by Ball-Murat [65]. Note that the hypothesis \( \alpha < 2n \) cannot be dropped in general. Indeed, if \( n = 2 \) and \( \alpha = 4 \), then
\[
f(\xi) = |\xi|^4 - 2(\det \xi)^2
\]
is even convex.

(iii) Case (iii) is interesting in elasticity for slightly compressible materials and was established by Charrier-Dacorogna-Hanouzet-Laborde [144]. It was then generalized by Dacorogna-Maréchal [206]. ♦

**Proof.** (i) Let \( \xi \in \mathbb{R}^{N \times n} \) and
\[
f(\xi) = g(|\xi|).
\]
In view of Theorem 5.3, it remains to show that
\[
f \text{ rank one convex } \Rightarrow g \text{ convex and } g(0) = \inf \{g(x) : x \geq 0\}
\]
which will be proved in Step 1 and
\[
g \text{ convex and } g(0) = \inf \{g(x) : x \geq 0\} \Rightarrow f \text{ convex}
\]
which we will establish in Step 2.

Step 1. Let \( x > 0 \) and define \( \xi \in \mathbb{R}^{N \times n} \) to be such that
\[
\xi_1^1 = x \quad \text{and} \quad \xi_i^j = 0 \quad \text{if} \quad (i, j) \neq (1, 1).
\]
We then deduce that
\[
g(0) = f\left(\frac{\xi - \xi}{2}\right) \leq \frac{1}{2} f(\xi) + \frac{1}{2} f(-\xi) = g(x)
\]
as wished.

Let us now show that \( g \) is convex. Let \( \lambda \in [0, 1], \alpha, \beta \geq 0 \). Define \( \xi, \eta \in \mathbb{R}^{N \times n} \) by
\[
\xi_1^1 = \alpha, \quad \eta_1^1 = \beta \quad \text{and} \quad \xi_i^j = \eta_i^j = 0 \quad \text{if} \quad (i, j) \neq (1, 1).
\]
Observing that \( \text{rank} \left\{ \xi - \eta \right\} \leq 1 \) and using the rank one convexity of \( f \) we get
\[
g(\lambda \alpha + (1 - \lambda) \beta) = f(\lambda \xi + (1 - \lambda) \eta) \\
\leq \lambda f(\xi) + (1 - \lambda) f(\eta) = \lambda g(|\alpha|) + (1 - \lambda) g(|\beta|)
\]
which is indeed the claimed convexity inequality.

Step 2. Note that since \( g \) is convex and \( g(0) = \inf \{ g(x) : x \geq 0 \} \), then \( g \) is non decreasing on \( \mathbb{R}_+ \).

We now want to show that \( g \text{ convex} \Rightarrow f \text{ convex} \). This is immediate since
\[
f(\lambda \xi + (1 - \lambda) \eta) = g(|\lambda \xi + (1 - \lambda) \eta|) \leq g(\lambda |\xi| + (1 - \lambda) |\eta|) \\
\leq \lambda g(|\xi|) + (1 - \lambda) g(|\eta|) = \lambda f(\xi) + (1 - \lambda) f(\eta)
\]
and this achieves the proof of the third part of the theorem.

(ii) Let \( n = N, \xi \in \mathbb{R}^{n \times n}, 1 \leq \alpha < 2n \) and
\[
f(\xi) = |\xi|^\alpha + h(\det \xi).
\]
It follows from Theorem 5.3 that it only remains to prove that
\[
f \text{ rank one convex} \Rightarrow h \text{ convex}.
\]
Let \( \lambda \in (0, 1), a, b \in \mathbb{R} \), we want to show that
\[
h(\lambda a + (1 - \lambda) b) \leq \lambda h(a) + (1 - \lambda) h(b). \quad (5.108)
\]
We will assume, with no loss of generality, that \( a \neq b \) and \( a \neq 0 \). Let \( \epsilon \neq 0 \) with \( \epsilon (b - a) > 0 \) and
\[
\xi := \text{diag}\left(\frac{a \epsilon}{b - a}, \left(\frac{b - a}{\epsilon}\right)^{\frac{1}{n-1}}, \cdots, \left(\frac{b - a}{\epsilon}\right)^{\frac{1}{n-1}}\right) \in \mathbb{R}^{n \times n}.
\]
It is then easy to see that, letting $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$,
\[
\begin{cases}
\det \xi = a, \quad \det (\xi + \epsilon e_1 \otimes e_1) = b \\
\det (\xi + (1 - \lambda) \epsilon e_1 \otimes e_1) = \lambda a + (1 - \lambda) b.
\end{cases}
\]
Since $f$ is rank one convex, we have
\[
|\xi + (1 - \lambda) \epsilon e_1 \otimes e_1|^\alpha + h(\lambda a + (1 - \lambda) b) = f (\lambda \xi + (1 - \lambda) (\xi + \epsilon e_1 \otimes e_1)) \leq \lambda f (\xi) + (1 - \lambda) f (\xi + \epsilon e_1 \otimes e_1) = \lambda |\xi|^\alpha + (1 - \lambda) |\xi + \epsilon e_1 \otimes e_1|^\alpha + \lambda h(a) + (1 - \lambda) h(b).
\]
Observe that
\[
\lambda |\xi|^\alpha + (1 - \lambda) |\xi + \epsilon (e_1 \otimes e_1)|^\alpha - |\xi + (1 - \lambda) \epsilon (e_1 \otimes e_1)|^\alpha = \lambda[(\frac{ae}{b-a})^2 + (n-1)(\frac{b-a}{\epsilon})\frac{2}{n-1}]^{\alpha/2} \\
+ (1 - \lambda)[(\frac{ae}{b-a} + \epsilon)^2 + (n-1)(\frac{b-a}{\epsilon})\frac{2}{n-1}]^{\alpha/2} \\
-[(\frac{ae}{b-a} + (1 - \lambda) \epsilon)^2 + (n-1)(\frac{b-a}{\epsilon})\frac{2}{n-1}]^{\alpha/2} = O(\epsilon^{\frac{2n-\alpha}{n-1}})
\]
where $O(t)$ stands for a term that goes to 0 as $t \to 0$. It is clear that if $1 \leq \alpha < 2n$, then the right hand side in the above identity tends to zero as $\epsilon \to 0$. Thus combining (5.109) and the above identity, as $\epsilon \to 0$, we have indeed obtained (5.108), i.e. that $h$ is convex.

(iii) We decompose the proof into two steps.

Step 1: $p \geq \frac{s}{n-s} \Rightarrow f$ polyconvex. Define first $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by
\[
h(x, \delta) := \begin{cases} x^{np/s} \delta^{-p} & \text{if } x, \delta > 0 \\
\infty & \text{otherwise.}
\end{cases}
\]
It is then easy to see that $h$ is convex if and only if $p \geq \frac{s}{n-s}$. We then let, for $1 \leq s \leq n-1$, $F : \mathbb{R}^{(n)_s} \times \mathbb{R}^{(n)_s} \times \mathbb{R} \to \mathbb{R}$ be defined by
\[
F(\eta, \delta) := h(|\eta|, \delta).
\]
Then from the convexity of $h$ and from the fact that $x \to h(x, \delta)$ is non decreasing in $\mathbb{R}_+$, we deduce that $F$ is convex. Observing that
\[
f(\xi) = F(\text{adj}_s \xi, \det \xi)
\]
we immediately obtain the polyconvexity of \( f \) from the fact that \( p \geq \frac{s}{n-s} \).

**Step 2:** \( f \) rank one convex \( \Rightarrow p \geq \frac{s}{n-s} \). Let \( \xi \in \mathbb{R}^{n \times n}, \alpha, \beta \in \mathbb{R}^n \) be such that
\[
\det (\xi + t \alpha \otimes \beta) > 0, \text{ for every } t > 0.
\]

Then the rank one convexity of \( f \) implies that
\[
t \to \varphi(t) := f(\xi + t \alpha \otimes \beta) = \left( \frac{|\text{adj}_s (\xi + t \alpha \otimes \beta)|^{n/s}}{\det (\xi + t \alpha \otimes \beta)} \right)^p
\]
is convex. We next simplify the notations by letting \( \lambda_1, \ldots, \lambda_5 \) be such that
\[
\begin{cases}
|\text{adj}_s (\xi + t \alpha \otimes \beta)|^2 = \lambda_1^2 t^2 + \lambda_2 t + \lambda_3^2 \\
\det (\xi + t \alpha \otimes \beta) = \lambda_4 t + \lambda_5.
\end{cases}
\]
Such \( \lambda_1, \ldots, \lambda_5 \) exist since
\[
t \to \text{adj}_s (\xi + t \alpha \otimes \beta) \quad \text{and} \quad t \to \det (\xi + t \alpha \otimes \beta)
\]
are linear functions (cf. Proposition 5.65). Combining the above notation with the definition of \( \varphi \), we find
\[
\varphi(t) = (\lambda_1^2 t^2 + \lambda_2 t + \lambda_3^2)^{\frac{np}{2s}} (\lambda_4 t + \lambda_5)^{-p}.
\]

After an elementary computation we obtain
\[
\varphi''(t) = (\lambda_1^2 t^2 + \lambda_2 t + \lambda_3^2)^{\frac{np}{2s}-2} (\lambda_4 t + \lambda_5)^{-p-2}
\]
\[
\times [\lambda_1^4 \lambda_4^2 t^4 \left( \frac{p}{s^2} (n-s)^2 \right)^2 (p - \frac{s}{n-s}) + O(t^3)].
\]

Since \( \varphi \) is convex for \( t > 0 \) we must have \( p \geq \frac{s}{n-s} \).

### 5.4 Appendix: some basic properties of determinants

In the whole of Chapter 5, we have seen the importance of *determinants* in quasiconvex analysis. We gather in this appendix some well known algebraic properties of determinants. In the first part, we carefully introduce the notation for the minors \( \text{adj}_s \xi \) of a given matrix \( \xi \).

We first introduce some notation. Let \( n \in \mathbb{N} \) (the set of positive integers) and let \( 1 \leq s \leq n \). We define
\[
I^n_s := \{ (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s : 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_s \leq n \}.
\]
We call the elements of \( I^n_s \) increasing \( s \)-tuples. The number of elements of \( I^n_s \) is then
\[
\text{card} \ I^n_s = \binom{n}{s} = \frac{n!}{s!(n-s)!}.
\]
We next endow \( I^n_s \) with the following ordering relation:
\[
\alpha = (\alpha_1, \ldots, \alpha_s) \succ (\beta_1, \ldots, \beta_s) = \beta
\]
if and only if
\[
\alpha_k < \beta_k,
\]
where \( k \) is the largest integer less than or equal to \( s \) such that \( \alpha_k \neq \beta_k \) and \( \alpha_l = \beta_l \) for every \( l > k \). (This is the inverse of the lexicographical order when read backward.)

**Example 5.60** (i) \( n = 4, s = 2 \). Then
\[
(1, 2) \succ (1, 3) \succ (2, 3) \succ (1, 4) \succ (2, 4) \succ (3, 4).
\]

(ii) \( n = 5, s = 3 \). Then
\[
(1, 2, 3) \succ (1, 2, 4) \succ (1, 3, 4) \succ (2, 3, 4) \succ (1, 2, 5) \\
\succ (1, 3, 5) \succ (2, 3, 5) \succ (1, 4, 5) \succ (2, 4, 5) \succ (3, 4, 5).
\]

(iii) \( s = n - 1 \). Then
\[
(1, \ldots, n-1) \succ \cdots \succ (1, \ldots, k-1, k+1, \ldots, n) \succ \cdots \succ (2, \ldots, n) .
\]

We then define the map \( \varphi^n_s \)
\[
\varphi^n_s : \{1, 2, 3, \ldots, \binom{n}{s}\} \to I^n_s
\]
as the only bijection that respects the order defined above.

**Example 5.61** (i) \( n = 4, s = 2 \). Then
\[
\varphi^2_2(1) = (3, 4), \ \varphi^2_2(2) = (2, 4), \ \varphi^2_2(3) = (1, 4), \\
\varphi^2_2(4) = (2, 3), \ \varphi^2_2(5) = (1, 3), \ \varphi^2_2(6) = (1, 2).
\]

(ii) \( s = n - 1 \). Then
\[
\varphi^n_{n-1}(1) = (2, \ldots, n) \\
\varphi^n_{n-1}(k) = (1, \ldots, k-1, k+1, \ldots, n) \\
\varphi^n_{n-1}(n) = (1, \ldots, n-1).
\]

We are now in a position to define, for a given matrix \( \xi \in \mathbb{R}^{N \times n} \), the *adjugate* matrix of order \( s \), \( 1 \leq s \leq n \wedge N = \min \{n, N\} \),
\[
\text{adj}_s \xi \in \mathbb{R}^{\binom{N}{s} \times \binom{n}{s}}.
\]
Let \( \xi \in \mathbb{R}^{N \times n} \) be such that

\[
\xi = \begin{pmatrix}
\xi_1^1 & \cdots & \xi_1^n \\
\vdots & \ddots & \vdots \\
\xi_N^1 & \cdots & \xi_N^n 
\end{pmatrix} = \begin{pmatrix}
\xi_1^1 \\
\vdots \\
\xi_N^1
\end{pmatrix} = (\xi_1, \ldots, \xi_n).
\]

We define \( \text{adj}_s \xi \) to be the following matrix in \( \mathbb{R}^{(N_s) \times (n_s)} \):

\[
\text{adj}_s \xi = \begin{pmatrix}
(\text{adj}_s \xi)^1_1 & \cdots & (\text{adj}_s \xi)^1_{(n_s)} \\
\vdots & \ddots & \vdots \\
(\text{adj}_s \xi)^N_1 & \cdots & (\text{adj}_s \xi)^N_{(n_s)}
\end{pmatrix}
\in \mathbb{R}^{(N_s) \times (n_s)}
\]

\[
= \begin{pmatrix}
(\text{adj}_s \xi)^1_1 \\
\vdots \\
(\text{adj}_s \xi)^N_1
\end{pmatrix} = \left( (\text{adj}_s \xi)_1, \ldots, (\text{adj}_s \xi)_{(n_s)} \right),
\]

where

\[
(\text{adj}_s \xi)^i_\alpha = (-1)^{i+\alpha} \det
\begin{pmatrix}
\xi_{\alpha_1}^{i_1} & \cdots & \xi_{\alpha_s}^{i_1} \\
\vdots & \ddots & \vdots \\
\xi_{\alpha_1}^{i_s} & \cdots & \xi_{\alpha_s}^{i_s}
\end{pmatrix}
\]

and \((i_1, \ldots, i_s), (\alpha_1, \ldots, \alpha_s)\) are the \(s\)-tuples corresponding to \(i\) and \(\alpha\) by the bijections \(\varphi^N_s\) and \(\varphi^n_s\), meaning that

\[
\varphi^N_s (i) = (i_1, \ldots, i_s) \quad \text{and} \quad \varphi^n_s (\alpha) = (\alpha_1, \ldots, \alpha_s).
\]

\textbf{Notation 5.62} We sometimes, as in examples (iv) and (vii) below, denote by \(\widehat{\xi}_{i_1, \ldots, i_k}^{\alpha_1, \ldots, \alpha_l}\) the \((N - k) \times (n - l)\) matrix obtained from \(\xi \in \mathbb{R}^{N \times n}\) by suppressing the \(k\) rows \(i_1, \ldots, i_k\) and the \(l\) columns \(\alpha_1, \ldots, \alpha_l\).

\textbf{Example 5.63} (i) \(N = n = 2, s = 1\). Let

\[
\xi = \begin{pmatrix}
\xi_1^1 & \xi_2^1 \\
\xi_1^2 & \xi_2^2
\end{pmatrix}.
\]

Then

\[
I^n_s = I^N_s = \{1, 2\}.
\]
and the bijection $\varphi_1^2 : \{1, 2\} \to \{2, 1\}$. Hence
\[
\adj_1 \xi = \begin{pmatrix}
(\adj_1 \xi)_1^1 & (\adj_1 \xi)_1^2 \\
(\adj_1 \xi)_2^1 & (\adj_1 \xi)_2^2 
\end{pmatrix} = \begin{pmatrix}
\xi_2^2 & -\xi_1^2 \\
-\xi_2^1 & \xi_1^1 
\end{pmatrix}.
\]
(note that $\adj_1 \xi$ is exactly $\tilde{\xi}$ defined in Theorem 5.51 above).

(ii) $N = n = s = 2$. Then
\[
I^n_s = I^N_s = \{(1, 2)\}
\]
and $\varphi_2^2 (1) = (1, 2)$. Hence
\[
\adj_2 \xi = \det \begin{pmatrix}
\xi_1^1 & \xi_1^2 \\
\xi_2^1 & \xi_2^2 
\end{pmatrix} = \det \xi.
\]

(iii) $N = 3$, $s = n = 2$. Then
\[
I^n_s = I^2_2 = \{(1, 2)\}
\]
and $\varphi_2^2 (1) = (1, 2)$, while
\[
I^N_s = I^3_2 = \{(1, 2) ; (1, 3) ; (2, 3)\}
\]
and $\varphi_2^3 (1) = (2, 3)$, $\varphi_2^3 (2) = (1, 3)$, $\varphi_2^3 (3) = (1, 2)$. Therefore, if
\[
\xi = \begin{pmatrix}
\xi_1^1 & \xi_1^2 \\
\xi_2^1 & \xi_2^2 \\
\xi_3^1 & \xi_3^2 
\end{pmatrix} = \begin{pmatrix}
\xi^1 \\
\xi^2 \\
\xi^3 
\end{pmatrix} = (\xi_1, \xi_2),
\]
then
\[
\adj_2 \xi = \begin{pmatrix}
(\adj_2 \xi_1^1)_1 \\
(\adj_2 \xi_1^2)_1 \\
(\adj_2 \xi_1^3)_1 
\end{pmatrix} = \begin{pmatrix}
\det \begin{pmatrix}
\xi_2^2 & \xi_2^3 \\
\xi_3^2 & \xi_3^3 
\end{pmatrix} \\
-\det \begin{pmatrix}
\xi_1^1 & \xi_1^2 \\
\xi_3^1 & \xi_3^2 
\end{pmatrix} \\
\det \begin{pmatrix}
\xi_1^1 & \xi_1^2 \\
\xi_2^1 & \xi_2^2 
\end{pmatrix} 
\end{pmatrix}.
\]

(iv) $N = n + 1$, $s = n$. We let
\[
\xi = \begin{pmatrix}
\xi_1^1 & \cdots & \xi_1^n \\
\vdots & \ddots & \vdots \\
\xi_1^{n+1} & \cdots & \xi_1^{n+1} 
\end{pmatrix} = \begin{pmatrix}
\xi^1 \\
\vdots \\
\xi^{n+1} 
\end{pmatrix} = (\xi_1, \cdots, \xi_n).
Then

\[
\text{adj}_n \xi = \begin{pmatrix}
(\text{adj}_n \xi)_1^1 \\
\vdots \\
(\text{adj}_n \xi)_{n+1}^1 \\
\end{pmatrix} = \begin{pmatrix}
\det \begin{pmatrix}
\xi_1^2 & \cdots & \xi_n^2 \\
\vdots & \ddots & \vdots \\
\xi_1^{n+1} & \cdots & \xi_n^{n+1}
\end{pmatrix} \\
\vdots \\
(-1)^{n+2} \det \begin{pmatrix}
\xi_1^1 & \cdots & \xi_n^1 \\
\vdots & \ddots & \vdots \\
\xi_1^n & \cdots & \xi_n^n
\end{pmatrix}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\det \hat{\xi}_1^1 \\
\vdots \\
(-1)^{n+2} \det \hat{\xi}_{n+1}^1
\end{pmatrix}
\]

where \(\hat{\xi}^k\) denotes the \(n \times n\) matrix obtained by suppressing the \(k\) th row in the matrix \(\xi\).

(v) \(N = n = s = 3\). Then \(I_3^3 = \{(1, 2, 3)\}\) and therefore

\[
\text{adj}_3 \xi = \det \xi.
\]

(vi) \(N = n = 3, s = 2\). Then

\[
\text{adj}_2 \xi = \begin{pmatrix}
\det \begin{pmatrix}
\xi_2^2 & \xi_3^2 \\
\xi_2^3 & \xi_3^3
\end{pmatrix} - \det \begin{pmatrix}
\xi_1^2 & \xi_3^2 \\
\xi_1^3 & \xi_3^3
\end{pmatrix} & \det \begin{pmatrix}
\xi_1^2 & \xi_2^2 \\
\xi_1^3 & \xi_2^3
\end{pmatrix} \\
- \det \begin{pmatrix}
\xi_1^1 & \xi_3^1 \\
\xi_1^3 & \xi_3^3
\end{pmatrix} & \det \begin{pmatrix}
\xi_1^1 & \xi_2^1 \\
\xi_1^3 & \xi_2^3
\end{pmatrix} - \det \begin{pmatrix}
\xi_1^1 & \xi_2^1 \\
\xi_1^3 & \xi_2^3
\end{pmatrix} \\
\det \begin{pmatrix}
\xi_1^1 & \xi_3^1 \\
\xi_1^2 & \xi_3^2
\end{pmatrix} - \det \begin{pmatrix}
\xi_1^1 & \xi_3^1 \\
\xi_1^2 & \xi_3^2
\end{pmatrix} & \det \begin{pmatrix}
\xi_1^1 & \xi_2^1 \\
\xi_1^2 & \xi_2^2
\end{pmatrix}
\end{pmatrix}.
\]

The above expression is the usual transpose of the matrix of cofactors.

(vii) \(N = n\) and \(s = n - 1\). Then

\[
\text{adj}_{n-1} \xi \in \mathbb{R}^{n \times n}
\]

and

\[
(\text{adj}_{n-1} \xi)^i_\alpha = (-1)^{i+\alpha} \det(\hat{\xi}^i_\alpha)
\]

where \(\hat{\xi}^i_\alpha\) is the \((n - 1) \times (n - 1)\) matrix obtained from \(\xi \in \mathbb{R}^{n \times n}\) by suppressing the \(i\) th row and the \(\alpha\) th column.
Remark 5.64  Note that one can write the rows of \( \text{adj}_s \xi \) as

\[
(\text{adj}_s \xi)^i = (-1)^{i+1} \text{adj}_s \begin{pmatrix} 
\xi^{i_1} \\
\vdots \\
\xi^{i_s}
\end{pmatrix}, \quad 1 \leq i \leq \binom{N}{s},
\]

where \( (i_1, \ldots, i_s) = \varphi_s^N(i) \) is the \( s \)-tuple associated to the integer \( i \). So, in particular,

\[
(\text{adj}_s \xi)^1 = \text{adj}_s \begin{pmatrix} 
\xi^{N-s+1} \\
\xi^{N-s+2} \\
\vdots \\
\xi^{N}
\end{pmatrix}, \ldots, (\text{adj}_s \xi)^{\binom{N}{s}} = (-1)^{\binom{N}{s}+1} \text{adj}_s \begin{pmatrix} 
\xi^1 \\
\xi^2 \\
\vdots \\
\xi^s
\end{pmatrix}.
\]

A similar remark applies to the columns of \( \text{adj}_s \xi \). \( \diamond \)

We now give some elementary properties of determinants.

Proposition 5.65  Let \( \xi \in \mathbb{R}^{N \times n} \).

(i) If \( N = n \), then, for every \( \xi \in \mathbb{R}^{n \times n} \),

\[
\langle \xi^\mu; (\text{adj}_{n-1} \xi)^\nu \rangle = \langle \xi^\mu; (\text{adj}_{n-1} \xi)^\nu \rangle = \delta_{\mu\nu} \det \xi, \quad \mu, \nu = 1, 2, \ldots, n,
\]

where \( \langle \cdot; \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^n \) and \( \delta_{\mu\nu} \) denotes the Kronecker symbol.

(ii) If \( N = n \), then, for every \( \xi \in \mathbb{R}^{n \times n} \),

\[
\xi \left( \text{adj}_{n-1} \xi \right)^t = \det \xi \cdot I
\]

where \( I \) is the identity matrix in \( \mathbb{R}^{n \times n} \) and \( \xi^t \) denotes the transpose of the matrix \( \xi \). In particular if \( \det \xi \neq 0 \), then

\[
\xi^{-1} = \frac{1}{\det \xi} \left( \text{adj}_{n-1} \xi \right)^t.
\]

(iii) If \( N = n + 1 \), then, for every \( \xi \in \mathbb{R}^{(n+1) \times n} \),

\[
\langle \xi^\nu; \text{adj}_n \xi \rangle = 0, \quad \nu = 1, \ldots, n,
\]

where \( \langle \cdot; \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^{n+1} \).

(iv) If \( N = n - 1 \), then, for every \( \xi \in \mathbb{R}^{(n-1) \times n} \),

\[
\langle \xi^\nu; \text{adj}_{n-1} \xi \rangle = 0, \quad \nu = 1, \ldots, n - 1,
\]

where \( \langle \cdot; \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^n \).
(v) If \( N = n \), then, for every \( \xi \in \mathbb{R}^{n \times n} \),

\[
\frac{\partial}{\partial \xi_{\alpha}^i} (\det \xi) = (\text{adj}_{n-1} \xi)^\alpha_i, \quad 1 \leq i, \alpha \leq n = N.
\]

(vi) Denote

\[
T (\xi) = (\xi, \text{adj}_2 \xi, \cdots, \text{adj}_{n \wedge N} \xi) \in \mathbb{R}^{\tau(n, N)}
\]

where \( n \wedge N = \min \{n, N\} \) and

\[
\tau(n, N) = \sum_{s=1}^{n \wedge N} \sigma(s) = \sum_{s=1}^{n \wedge N} \binom{n}{s} \binom{N}{s}.
\]

Let \( a \in \mathbb{R}^n \), \( b \in \mathbb{R}^N \). Define

\[
a \otimes b = (a^i b_\alpha)_{1 \leq i \leq N \wedge n} \in \mathbb{R}^{N \times n}.
\]

Let \( t \in [0, 1] \), then, for every \( \xi \in \mathbb{R}^{N \times n} \),

\[
T (\xi + (1 - t) a \otimes b) = tT (\xi) + (1 - t) T (\xi + a \otimes b).
\]

**Proof.**

(i) The case \( \mu = \nu \) is just the way a determinant is computed, by expanding it along the \( \nu \)th row or the \( \nu \)th column. When \( \mu \neq \nu \), then both \( \langle \xi_\mu; (\text{adj}_{n-1} \xi)^\nu \rangle \) and \( \langle \xi_\mu; (\text{adj}_{n-1} \xi)^\nu \rangle \) are again determinants of \( n \times n \) matrices, but the first matrix has twice the row \( \xi_\mu \) and the second has twice the column \( \xi_\mu \). Thus both determinants are equal to 0, as claimed.

(ii) This follows at once from (i).

(iii) Let \( N = n + 1 \) and \( \nu \in \{1, \cdots, n\} \). We have to show that

\[
\langle \xi_\nu; \text{adj}_n \xi \rangle = 0.
\]

Define the matrix \( \eta = [\xi_\nu; \xi] \in \mathbb{R}^{(n+1) \times (n+1)} \) (recall that \( \xi \in \mathbb{R}^{(n+1) \times n} \)). Then \( \eta_1 = \eta_{\nu+1} \) and therefore \( \det \eta = 0 \). Using (i), we obtain

\[
0 = \det \eta = \langle \eta_1; (\text{adj}_n \eta)_1 \rangle = \langle \xi_\nu; \text{adj}_n \xi \rangle.
\]

(iv) This is established exactly as (iii).

(v) This is a direct consequence of (i).

(vi) We divide the proof into three steps.

**Step 1.** The result is equivalent to

\[
\text{adj}_s (\xi + (1 - t) a \otimes b) = t \text{adj}_s \xi + (1 - t) \text{adj}_s (\xi + a \otimes b)
\]

for every \( 1 \leq s \leq n \wedge N \). In terms of components this is equivalent to

\[
(\text{adj}_s (\xi + (1 - t) a \otimes b))_\alpha^i = t (\text{adj}_s \xi)_\alpha^i + (1 - t) (\text{adj}_s (\xi + a \otimes b))_\alpha^i,
\]

(5.110)
1 \leq i \leq \binom{N}{s}, 1 \leq \alpha \leq \binom{n}{s}. Recall that
\[(\text{adj}_s \xi)^i_\alpha = (-1)^{i+\alpha} \det \begin{pmatrix}
\xi_{i1}^\alpha & \cdots & \xi_{i1}^\alpha \\
\vdots & \ddots & \vdots \\
\xi_{i1}^\alpha & \cdots & \xi_{i1}^\alpha
\end{pmatrix}.
\]
By abuse of notation, let
\[\xi = \begin{pmatrix}
\xi_{\alpha1}^1 & \cdots & \xi_{\alpha1}^1 \\
\vdots & \ddots & \vdots \\
\xi_{\alpha1}^i & \cdots & \xi_{\alpha1}^i
\end{pmatrix}, \quad a \otimes b = \begin{pmatrix}
a_{11}^1 b_{\alpha1} & \cdots & a_{11}^i b_{\alpha1} \\
\vdots & \ddots & \vdots \\
a_{1i}^i b_{\alpha1} & \cdots & a_{1i}^i b_{\alpha1}
\end{pmatrix}.
\]
Therefore (5.110) is equivalent to showing that, for every \(\xi \in \mathbb{R}^{s \times s}, a, b \in \mathbb{R}^s, t \in [0, 1],\)
\[\det (\xi + (1 - t) a \otimes b) = t \det \xi + (1 - t) \det (\xi + a \otimes b).
\] (5.111)
This is a standard property of determinants that we prove in the two steps below.

\textit{Step 2.} We start by proving (5.111) when
\[a = b = e^1 = e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^s.
\]
Note that, for every \(x \in \mathbb{R},\) we have
\[(\xi + xe^1 \otimes e_1)^1 = \xi^1 + xe^1 \quad \text{and} \quad (\text{adj}_{s-1} (\xi + xe^1 \otimes e_1))^1 = (\text{adj}_{s-1} \xi)^1.
\]
The first identity is obvious and the second one follows since the components of \((\text{adj}_{s-1} \xi)^1\) are given by determinants where the first row of \(\xi\) does not appear.
We can therefore apply (i) to find
\[
det (\xi + (1 - t) e^1 \otimes e_1)
\]
\[
= \langle (\xi + (1 - t) e^1 \otimes e_1)^1; (\text{adj}_{s-1} (\xi + (1 - t) e^1 \otimes e_1))^1 \rangle
\]
\[
= \langle \xi^1 + (1 - t) e^1; (\text{adj}_{s-1} \xi)^1 \rangle
\]
\[
= t \langle \xi^1; (\text{adj}_{s-1} \xi)^1 \rangle + (1 - t) \langle \xi^1 + e^1; (\text{adj}_{s-1} \xi)^1 \rangle
\]
\[
= t \langle \xi^1; (\text{adj}_{s-1} \xi)^1 \rangle + (1 - t) \langle \xi^1 + e^1; (\text{adj}_{s-1} (\xi + e^1 \otimes e_1))^1 \rangle
\]
\[
= t \det \xi + (1 - t) \det (\xi + e^1 \otimes e_1)
\]
which is the claim of Step 2.
Step 3. The general statement (5.111) follows at once from Step 2 and Theorem 13.3. Indeed, we can find \( R, Q \in O(s) \) such that

\[
R \left( e^1 \otimes e_1 \right) Q = a \otimes b.
\]

We therefore find, using Step 2,

\[
\begin{align*}
\det (\xi + (1-t) a \otimes b) &= \det \left( R \left( R^t \xi Q^t + (1-t) e^1 \otimes e_1 \right) Q \right) \\
&= \det R \det \left( R^t \xi Q^t + (1-t) e^1 \otimes e_1 \right) \det Q \\
&= t \det R \det \left( R^t \xi Q^t \right) \det Q \\
&\quad + (1-t) \det R \det \left( R^t \xi Q^t + e^1 \otimes e_1 \right) \det Q \\
&= t \det \xi + (1-t) \det \left( \xi + R^t \xi Q^t \right) Q \\
&= t \det \xi + (1-t) \det \left( \xi + a \otimes b \right)
\end{align*}
\]

which is the claim. ■

We also have the following useful result (see Buttazzo-Dacorogna-Gangbo [113] and Dacorogna-Maréchal [205]).

**Proposition 5.66** (i) Let \( \xi \in \mathbb{R}^{N \times n}, \eta \in \mathbb{R}^{n \times m} \) and

\[
1 \leq s \leq N \wedge n \wedge m := \min \{N, n, m\}.
\]

Then

\[
\adj_s (\xi \eta) = \adj_s \xi \adj_s \eta.
\]

(ii) Let \( \xi \in \mathbb{R}^{N \times n} \) and \( 1 \leq s \leq N \wedge n \), then

\[
\adj_s \left( \xi^t \right) = (\adj_s \xi)^t.
\]

(iii) If \( N = n \) and \( R \in O(n) \) (respectively \( R \in SO(n) \)), then

\[
\adj_s R \in O \left( \binom{n}{s} \right) \quad \text{(respectively } \adj_s R \in SO \left( \binom{n}{s} \right) \text{)}.
\]

(iv) If \( N = n \) and \( \xi \in \mathbb{R}^{n \times n} \) is invertible, then \( \adj_s \xi \in \mathbb{R}^{n \times n} \) is invertible and

\[
(\adj_s \xi)^{-1} = \adj_s (\xi^{-1}).
\]

(v) If \( N = n \) and if \( R \in SO(n) \), then

\[
\adj_{n-1} R = R.
\]

**Proof.** (i) We have to prove that

\[
(\adj_s (\xi \eta))^i_j = (\adj_s \xi \adj_s \eta)^i_j
\]
for every $1 \leq i \leq \binom{N}{s}$, $1 \leq j \leq \binom{m}{s}$. To simplify the notation, we will write

$$\alpha := \varphi^N_s, \beta := \varphi^n_s, \gamma := \varphi^m_s.$$  

Let the $s$-tuples corresponding to $i$ and $j$ (and later $k$) be given by

$$\alpha(i) = (i_1, \ldots, i_s), \beta(k) = (k_1, \ldots, k_s), \gamma(j) = (j_1, \ldots, j_s).$$

For a matrix $\theta \in \mathbb{R}^{N \times m}$, we let

$$\theta^{\alpha(i)}_{\gamma(j)} := \begin{pmatrix} \theta^{i_1}_{j_1} & \cdots & \theta^{i_s}_{j_s} \\ \vdots & \ddots & \vdots \\ \theta^{i_s}_{j_1} & \cdots & \theta^{i_s}_{j_s} \end{pmatrix} \in \mathbb{R}^{s \times s}$$

and, for $1 \leq \nu \leq m$,

$$\left( \theta^{\alpha(i)}_{\gamma(j)} \right)_{\nu} := \begin{pmatrix} \theta^{i_1}_{\nu} \\ \vdots \\ \theta^{i_s}_{\nu} \end{pmatrix} \in \mathbb{R}^{s}.$$  

For $1 \leq p, q \leq s$, we have that

$$\left( \left( \xi \eta^{\alpha(i)}_{\gamma(j)} \right)^q_p \right) = \left( \xi \eta^q_p \right) \sum_{\nu=1}^{n} \xi^q_p \eta^{\nu}_{jp}.$$  

In other words, the $p$th column vector of the matrix is given by

$$\left( \left( \xi \eta^{\alpha(i)}_{\gamma(j)} \right)^q_p \right) = \begin{pmatrix} \sum_{\nu=1}^{n} \xi^q_p \eta^{\nu}_{jp} \\ \vdots \\ \sum_{\nu=1}^{n} \xi^q_p \eta^{\nu}_{jp} \end{pmatrix} \begin{pmatrix} \xi^{i_1}_{\nu} \\ \vdots \\ \xi^{i_s}_{\nu} \end{pmatrix} = \sum_{\nu=1}^{n} \eta^{\nu}_{jp} \left( \xi^{\alpha(i)}_{\gamma(j)} \right)_{\nu}.$$  

We therefore have, by definition of $\text{adj}_s$, that

$$\left( \text{adj}_s (\xi \eta) \right)^q_j = (-1)^{i+j} \det \left( (\xi \eta)^{\alpha(i)}_{\gamma(j)} \right) = \left( (-1)^{i+j} \det \left( (\xi \eta)^{\alpha(i)}_{\gamma(j)} \right) \right) \left( (\xi \eta)^{\alpha(i)}_{\gamma(j)} \right)^1_1 \cdots (\xi \eta)^{\alpha(i)}_{\gamma(j)} \left( (\xi \eta)^{\alpha(i)}_{\gamma(j)} \right)^1_1 \cdots (\xi \eta)^{\alpha(i)}_{\gamma(j)} \left( (\xi \eta)^{\alpha(i)}_{\gamma(j)} \right)^1_1 \cdots (\xi \eta)^{\alpha(i)}_{\gamma(j)} \left( (\xi \eta)^{\alpha(i)}_{\gamma(j)} \right)^1_1 \cdots (\xi \eta)^{\alpha(i)}_{\gamma(j)} \left( (\xi \eta)^{\alpha(i)}_{\gamma(j)} \right)^1_1.$$
Appendix: some basic properties of determinants

Now, if \( \nu_p = \nu_q \) for two distinct integers \( p, q \in \{1, \cdots, s\} \), we clearly have

\[
\det\left( (\xi^{\alpha(i)})_{\nu_1}, \cdots, (\xi^{\alpha(i)})_{\nu_s} \right) = 0.
\]

Thus, writing \( F_{n,s} \) for all \( s \)-tuples \( (\nu_1, \cdots, \nu_s) \) in \( \{1, \cdots, n\}^s \) such that the \( \nu_p \) are pairwise distinct, we find

\[
(\text{adj}_s (\xi \eta))^{i}_{j} = (-1)^{i+j} \sum_{(\nu_1, \cdots, \nu_s) \in F_{n,s}} \eta^{\alpha_1}_{j_1} \cdots \eta^{\alpha_s}_{j_s} \det((\xi^{\alpha(i)})_{\nu_1}, \cdots, (\xi^{\alpha(i)})_{\nu_s}).
\]

(5.112)

On the other hand we can write

\[
(\text{adj}_s \xi \text{adj}_s \eta)^{i}_{j} = \sum_{k=1}^{n} (\text{adj}_s \xi)^{i}_{k} (\text{adj}_s \eta)^{j}_{k}
\]

\[
= \sum_{k=1}^{n} (-1)^{i+k} \det(\xi^{\alpha(i)}) (-1)^{k+j} \det(\eta^{\beta(k)})
\]

\[
= (-1)^{i+j} \sum_{k=1}^{n} \det(\xi^{\alpha(i)} \eta^{\beta(k)})
\]

Since, for \( 1 \leq p, q, r \leq s \),

\[
(\xi^{\alpha(i)})^q_p = \xi^{i_q}_{k_p} \text{ and } (\eta^{\beta(k)})^p_r = \eta^{k_p}_{j_r}
\]

we find

\[
(\xi^{\alpha(i)} \eta^{\beta(k)})^q_r = \sum_{p=1}^{s} \xi^{i_q}_{k_p} \eta^{k_p}_{j_r}.
\]

Phrased differently, we have that the \( r \)-th column vector of the matrix is given by

\[
\begin{pmatrix}
(\xi^{\alpha(i)} \eta^{\beta(k)})^1_r \\
(\xi^{\alpha(i)} \eta^{\beta(k)})^2_r \\
\vdots \\
(\xi^{\alpha(i)} \eta^{\beta(k)})^s_r
\end{pmatrix}
= \begin{pmatrix}
(\xi^{\alpha(i)} \eta^{\beta(k)})^1_r \\
\vdots \\
(\xi^{\alpha(i)} \eta^{\beta(k)})^s_r
\end{pmatrix}
= \begin{pmatrix}
\sum_{p=1}^{s} \xi^{i_1}_{k_p} \eta^{k_p}_{j_r} \\
\vdots \\
\sum_{p=1}^{s} \xi^{i_s}_{k_p} \eta^{k_p}_{j_r}
\end{pmatrix}
= \begin{pmatrix}
\sum_{p=1}^{s} \eta^{k_p}_{j_r} \\
\vdots \\
\sum_{p=1}^{s} \eta^{k_p}_{j_r}
\end{pmatrix} \begin{pmatrix}
\xi^{i_1}_{k_p} \\
\vdots \\
\xi^{i_s}_{k_p}
\end{pmatrix}
= \sum_{p=1}^{s} \eta^{k_p}_{j_r} (\xi^{\alpha(i)})^{k_p}.\]
We thus deduce that

\[
(\text{adj}_s \xi \text{adj}_s \eta)^i_j = (-1)^{i+j} \sum_{k=1}^{(n\choose s)} \det((\xi^{\alpha(i)}_{\beta(k)} \eta^{\beta(k)}_{\gamma(j)})_1, \ldots, (\xi^{\alpha(i)}_{\beta(k)} \eta^{\beta(k)}_{\gamma(j)})_s)
\]

\[
= (-1)^{i+j} \sum_{k=1}^{(n\choose s)} \det(\sum_{p=1}^s \eta_{j_{kp}}^{k_p}(\xi^{\alpha(i)})_{k_p}, \ldots, \sum_{p=1}^s \eta_{j_{sp}}^{k_p}(\xi^{\alpha(i)})_{k_p})
\]

\[
= (-1)^{i+j} \sum_{k=1}^{(n\choose s)} \det(\sum_{p=1}^s \eta_{j_{kp}}^{k_p}(\xi^{\alpha(i)})_{k_p}, \ldots, \sum_{s=1}^s \eta_{j_{sp}}^{k_p}(\xi^{\alpha(i)})_{k_p})
\]

\[
= (-1)^{i+j} \sum_{k=1}^{(n\choose s)} \gamma(k_{p1}) \cdots \gamma(k_{ps}) \det((\xi^{\alpha(i)})_{k_{p1}}, \ldots, (\xi^{\alpha(i)})_{k_{ps}}).
\]

If \((p_1, \ldots, p_s) \in \{1, \ldots, s\}^s\) is not a permutation of \((1, \ldots, s)\), then

\[
\det((\xi^{\alpha(i)})_{k_{p1}}, \ldots, (\xi^{\alpha(i)})_{k_{ps}}) = 0.
\]

Letting

\[
\nu_r := k_{pr}, \quad r = 1, \ldots, s,
\]

we note that, when \((p_1, \ldots, p_s) \in \{1, \ldots, s\}^s\) is a permutation of \((1, \ldots, s)\) and \(k \in \{1, \ldots, (n\choose s)\}\), then \((\nu_1, \ldots, \nu_s) \in F_{n,s}\), the set of \(s\)-tuples \((\nu_1, \ldots, \nu_s)\) in \(\{1, \ldots, n\}^s\) such that the \(\nu_p\) are pairwise distinct. We therefore get that

\[
(\text{adj}_s \xi \text{adj}_s \eta)^i_j = (-1)^{i+j} \sum_{(\nu_1, \ldots, \nu_s) \in F_{n,s}} \eta_{j_{\nu_1}}^{\nu_1} \cdots \eta_{j_{\nu_s}}^{\nu_s} \det((\xi^{\alpha(i)})_{\nu_1}, \ldots, (\xi^{\alpha(i)})_{\nu_s}).
\]

The above identity and (5.112) imply the result.

(ii) As above, let

\[
\alpha := \varphi^N_s, \quad \beta := \varphi^n_s.
\]

We clearly have, for \(1 \leq i \leq (N\choose s)\) and \(1 \leq j \leq (n\choose s)\), that

\[
(\xi^t)^{\alpha(i)}_{\beta(j)} = \left(\xi^{\beta(j)}_{\alpha(i)}\right)^t.
\]
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since, for \( \alpha (i) = (i_1, \cdots, i_s) \) and \( \beta (j) = (j_1, \cdots, j_s) \), we can write

\[
(\xi^t)_{\beta(j)}^{\alpha(i)} = \begin{pmatrix}
(\xi^t)^{i_1}_{j_1} & \cdots & (\xi^t)^{i_{i_s}}_{j_{i_s}} \\
\vdots & \ddots & \vdots \\
(\xi^t)^{i_{i_s}}_{j_1} & \cdots & (\xi^t)^{i_{i_s}}_{j_{i_s}}
\end{pmatrix} = \begin{pmatrix}
\xi^{j_1}_{i_1} & \cdots & \xi^{j_{i_s}}_{i_{i_s}} \\
\vdots & \ddots & \vdots \\
\xi^{j_{i_s}}_{i_1} & \cdots & \xi^{j_{i_s}}_{i_{i_s}}
\end{pmatrix}^t = \begin{pmatrix}
\xi^{j_1}_{i_1} & \cdots & \xi^{j_{i_s}}_{i_{i_s}} \\
\vdots & \ddots & \vdots \\
\xi^{j_{i_s}}_{i_1} & \cdots & \xi^{j_{i_s}}_{i_{i_s}}
\end{pmatrix} = (\xi^t)_{\alpha(i)}^{\beta(j)}.
\]

We can therefore deduce that

\[
(adj_s (\xi^t))_{ij}^t = (-1)^{i+j} \det((\xi^t)_{\alpha(i)}^{\beta(j)}) = (-1)^{i+j} \det((\xi_{\alpha(i)})^t) = (-1)^{i+j} \det(\xi_{\alpha(i)})^t = (adj_s \xi)^j_i
\]

which is statement (ii).

(iii) From (i) and (ii) we immediately deduce the claim for \( R \in O(n) \), since

\[
adj_s R (adj_s R)^t = adj_s R adj_s R^t = adj_s (RR^t) = adj_s I = I(n)
\]

where for any integer \( m \) we have let \( I_m \) to be the identity matrix in \( \mathbb{R}^{m \times m} \).

We now discuss the case where \( R \in SO(n) \). We already know that

\[
adj_s R \in O\left(\left(\begin{smallmatrix} n \\ s \end{smallmatrix}\right)\right).
\]

It therefore remains to prove that

\[
\det(adj_s R) = 1.
\]

We observe that \( SO(n) \) is a connected manifold, meaning that, for every \( R \in SO(n) \), there exists a continuous function

\[
\theta : [0, 1] \rightarrow SO(n) , \theta(0) = I_n , \theta(1) = R.
\]

We then define, for \( t \in [0, 1] \), the function

\[
f(t) := \det(adj_s \theta(t)) .
\]

We observe that since any \( Q \in SO(n) \subset O(n) \) has

\[
\det(adj_s Q) \in \{ \pm 1 \},
\]
then the function $f$ takes only values in $\{\pm 1\}$. Since it is a continuous function, as a composition of three continuous functions, and since $f(0) = 1$, we deduce that $f(1) = 1$, which is the assertion.

(iv) This follows from (i) exactly as above. Indeed

$$\text{adj}_s \xi \text{adj}_s (\xi^{-1}) = \text{adj}_s I_n = I_{(n)}.$$ 

(v) From (ii) of Proposition 5.65, we have, since $R \in SO(n)$,

$$R (\text{adj}_{n-1} R)^t = I_n$$

and thus the claim. ■

We now want to write, for every $\xi, \eta \in \mathbb{R}^{n \times n}$, $\det (\xi + \eta)$. To this aim let us introduce the following notations.

- Let $\mathcal{N}_{\{1, \ldots, n\}}$ be the set of couples $(I, J)$, each of them ordered, so that

$$I \cup J = \{1, \ldots, n\}, \ I \cap J = \emptyset.$$ 

- For all $(I, J) \in \mathcal{N}_{\{1, \ldots, n\}}$ and all matrices $\xi, \eta \in \mathbb{R}^{n \times n}$, we denote by

$$(\xi^I, \eta^J) \in \mathbb{R}^{n \times n}$$

the $n \times n$ matrix whose row of index $k$ is $\xi^k$ if $k \in I$ or $\eta^k$ if $k \in J$. So, for example, if $n = 3$, $I = \{1, 3\}$, $J = \{2\}$, then

$$(\xi^I, \eta^J) = \begin{pmatrix} \xi^1 \\ \eta^2 \\ \xi^3 \end{pmatrix}.$$ 

**Proposition 5.67** Let $\xi, \eta \in \mathbb{R}^{n \times n}$, then

$$\det (\xi + \eta) = \sum_{(I, J) \in \mathcal{N}_{\{1, \ldots, n\}}} \det(\xi^I, \eta^J).$$

**Proof.** Let us first examine the case $n = 2$, where we trivially have

$$\det (\xi + \eta) = \det(\xi^1, \xi^2) + \det(\xi^1, \eta^2) + \det(\eta^1, \xi^2) + \det(\eta^1, \eta^2).$$

The general case easily follows if we write the determinant as a multilinear form; namely, for $\xi \in \mathbb{R}^{n \times n}$, we write

$$\det \xi = \xi^1 \wedge \cdots \wedge \xi^n.$$
The claim follows by induction, since
\[
\det (\xi + \eta) = \xi^1 \wedge (\xi^2 + \eta^2) \wedge \cdots \wedge (\xi^n + \eta^n) + \eta^1 \wedge (\xi^2 + \eta^2) \wedge \cdots \wedge (\xi^n + \eta^n)
\]
\[
= \sum_{(I,J) \in \mathcal{N}_{\{2, \cdots, n\}}} \det (\xi^I, \xi^J) + \sum_{(I,J) \in \mathcal{N}_{\{2, \cdots, n\}}} \det (\eta^I, \xi^J)
\]
\[
= \sum_{(I,J) \in \mathcal{N}_{\{1, \cdots, n\}}} \det (\xi^I, \eta^J).
\]
This finishes the proof of the proposition. \(\blacksquare\)
Chapter 6

Polyconvex, quasiconvex and rank one convex envelopes

6.1 Introduction

We now proceed with the characterization of the convex $Cf$, polyconvex $Pf$, quasiconvex $Qf$ and rank one convex envelope $Rf$, which are, respectively, defined as the largest convex, polyconvex, quasiconvex and rank one convex function below $f$. In other words, we have, for every $\xi \in \mathbb{R}^{N \times n}$,

$$
Cf (\xi) = \sup \{ g(\xi) : g \leq f \text{ and } g \text{ convex} \}, \\
Pf (\xi) = \sup \{ g(\xi) : g \leq f \text{ and } g \text{ polyconvex} \}, \\
Qf (\xi) = \sup \{ g(\xi) : g \leq f \text{ and } g \text{ quasiconvex} \}, \\
Rf (\xi) = \sup \{ g(\xi) : g \leq f \text{ and } g \text{ rank one convex} \}.
$$

The first notion has already been encountered in Chapter 2, where we gave two different characterizations of $Cf$. The first one, in Section 2.3.3, via Carathéodory theorem and the second one, in Section 2.3.5, via duality and the separation theorems.

In view of Theorem 5.3, we have

$$
Cf \leq Pf \leq Qf \leq Rf \leq f.
$$

In Section 6.2, we start with the polyconvex envelope $Pf$, which is the most similar to the convex envelope $Cf$. We always recall, without proofs, what has already been said about $Cf$ in Chapter 2 to show the resemblance between the two envelopes.
In Section 6.3, we give a representation formula for the quasiconvex envelope, inspired by Carathéodory theorem.

In Section 6.4, we discuss a representation formula for $Rf$, also in the spirit of Carathéodory theorem.

In Section 6.5, we present a result that in some cases can simplify the computations of the different envelopes.

In Section 6.6, we discuss several examples, relevant for applications, where one can compute these envelopes.

6.2 The polyconvex envelope

6.2.1 Duality for polyconvex functions

We first recall (see Section 2.3.5) some facts about duality in convex analysis.

**Definition 6.1** Let $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ with $f \not\equiv +\infty$ and let $f^* : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ be defined as

$$f^*(\xi^*) := \sup_{\xi \in \mathbb{R}^{N \times n}} \{\langle \xi; \xi^* \rangle - f(\xi)\},$$

where $\langle \cdot; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{N \times n}$, and let

$$f^{**} := (f^*)^*,$$

or in other words

$$f^{**}(\xi) = \sup_{\xi^* \in \mathbb{R}^{N \times n}} \{\langle \xi; \xi^* \rangle - f^*(\xi^*)\}.$$

**Remark 6.2** As seen in Chapter 2, we always have that

(i) $f^{***} = f^*$;

(ii) $f^{**}$ is convex and lower semicontinuous and therefore

$$f^{**} \leq Cf \leq f.$$

Moreover, if $Cf$ is lower semicontinuous and $Cf > -\infty$, then

$$f^{**} = Cf.$$

We now proceed in an analogous way for polyconvex functions and follow here the idea of Kohn and Strang [373], [374] as presented in Dacorogna [179]. We also adopt the notation of Chapter 5.

**Definition 6.3** Let $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ with $f \not\equiv +\infty$.

(i) One defines the polyconvex conjugate function of $f$ as

$$f^p : \mathbb{R}^{(n,N)} \to \mathbb{R} \cup \{+\infty\},$$
The polyconvex envelope

where \( \tau (n, N) = \sum_{s=1}^{n \land N} \binom{n}{s} \binom{N}{s} \), by

\[
  f^p (X^*) := \sup_{\xi \in \mathbb{R}^{\tau(n,N)}} \{ \langle T(\xi) ; X^* \rangle - f(\xi) \},
\]

where \( \langle \cdot ; \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^{\tau(n,N)} \) and \( T \) is as in Definition 5.1.

(ii) Let \( (f^p)^* : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \cup \{ \pm \infty \} \) be defined by

\[
  (f^p)^* (X) := \sup_{X^* \in \mathbb{R}^{\tau(n,N)}} \{ \langle X ; X^* \rangle - f^p (X^*) \}.
\]

Finally, let \( \xi \in \mathbb{R}^{N \times n} \) and define the polyconvex biconjugate function of \( f \) as

\[
  f^{pp} (\xi) := (f^p)^* (T(\xi)).
\]

Remark 6.4 (i) It is clear that if \( N = 1 \) or \( n = 1 \)

\[
  f^p = f^\ast \quad \text{and} \quad f^{pp} = f^{\ast\ast}.
\]

(ii) It is also simple to see that

\[
  f^{ppp} = f^p
\]

and that \( f^{pp} \) is polyconvex, lower semicontinuous and less than \( f \) and therefore

\[
  f^{pp} \leq Pf \leq f.
\]

(iii) If \( N = n = 2 \), then \( \tau (n, N) = 5 \) and we can write

\[
  f^p (\xi^*, \delta^*) = \sup_{\xi \in \mathbb{R}^{2 \times 2}} \{ \langle \xi ; \xi^* \rangle + \delta^* \det \xi - f(\xi) \},
\]

where \( \xi^* \in \mathbb{R}^{2 \times 2}, \delta^* \in \mathbb{R} \) and \( \langle \cdot ; \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^{2 \times 2} \). Similarly

\[
  (f^p)^* (\xi, \delta) = \sup_{\xi^* \in \mathbb{R}^{2 \times 2}} \{ \langle \xi ; \xi^* \rangle + \delta^* \delta - f^p (\xi^*, \delta^*) \}
\]

and therefore

\[
  f^{pp} (\xi) = (f^p)^* (\xi, \det \xi). \quad \diamond
\]

We now give a simple example where one can explicitly compute \( f^\ast, f^{\ast\ast}, f^p, f^{pp} \).

Example 6.5 Let \( N = n = 2 \) and

\[
  f(\xi) = \det \xi.
\]

(i) It is easy to show that

\[
  f^\ast (\xi^*) = \sup_{\xi \in \mathbb{R}^{2 \times 2}} \{ \langle \xi ; \xi^* \rangle - \det \xi \} \equiv +\infty
\]
and therefore
\[ f^{**} (\xi) \equiv -\infty. \]

(ii) Similarly,
\[
(f^p)^* (\xi^*, \delta^*) = \sup_{\xi \in \mathbb{R}^{2 \times 2}} \{ \langle \xi; \xi^* \rangle + (\delta^* - 1) \det \xi \}
\]
\[
= \begin{cases} 
0 & \text{if } \delta^* = 1 \text{ and } \xi^* = 0 \\
+\infty & \text{elsewhere.}
\end{cases}
\]

We therefore obtain
\[
(f^p)^* (\xi, \delta) = \sup_{\xi^* \in \mathbb{R}^{2 \times 2}, \delta^* \in \mathbb{R}} \{ \langle \xi; \xi^* \rangle + \delta^* \delta - f^p (\xi^*, \delta^*) \} = \delta
\]
and hence
\[ f^{pp} (\xi) = \det \xi. \]

We now have the following result, which was already proved in Theorem 2.43 for the convex case.

**Theorem 6.6** Let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) (i.e. \( f \) is finite) and let \( f^{**} \) and \( f^{pp} \) be defined as in the preceding section.

**Part 1.** If there exists \( g : \mathbb{R}^{N \times n} \to \mathbb{R} \) convex such that
\[ f (\xi) \geq g (\xi) \text{ for every } \xi \in \mathbb{R}^{N \times n}, \]
then
\[ f^{**} = Cf. \] (6.1)

**Part 2.** If there exists \( g : \mathbb{R}^{N \times n} \to \mathbb{R} \) polyconvex such that
\[ f (\xi) \geq g (\xi) \text{ for every } \xi \in \mathbb{R}^{N \times n}, \]
then
\[ f^{pp} = Pf. \] (6.2)

**Remark 6.7** (i) One can also rewrite (6.1) in the following way (if \( f \) takes only finite values):
\[ Cf (\xi) = \sup \{ g (\xi) : g \leq f \text{ and } g \text{ affine} \}. \]

Similarly, for (6.2),
\[ Pf (\xi) = \sup \{ g (\xi) : g \leq f \text{ and } g \text{ quasiaffine} \}. \]

(ii) It is also interesting to note that (6.1) and (6.2) do not hold if \( f \) is allowed to take the value \( +\infty \). For example, if \( N = n = 1 \) and
\[ f (\xi) = \chi_{(0,1)} (\xi) = \begin{cases} 0 & \text{if } \xi \in (0,1) \\
+\infty & \text{elsewhere.}
\end{cases} \]
then \( f = Cf \) and \( f^{**} = \chi_{[0,1]} \). \( \diamond \)
Proof. As already pointed out, we always have \( f^{pp} \leq Pf \). We now wish to prove the reverse inequality. We divide the proof into two steps.

Step 1. We first show that if \( f \) is polyconvex and finite then
\[
f^{pp} = f. \tag{6.3}
\]

From Theorem 5.6 of Chapter 5, we have that there exists \( F : \mathbb{R}^\tau \to \mathbb{R}, \tau = \tau(n,N) \), convex and finite such that
\[
f(\xi) = F(T(\xi)).
\]

It is obvious from the definition that
\[
F^*(X^*) = \sup_{X \in \mathbb{R}^\tau} \{(X;X^*) - F(X)\}
\]
\[
\geq \sup_{\xi \in \mathbb{R}^{N\times n}} \{(T(\xi);X^*) - F(T(\xi))\} = f^p(X^*)
\]
and therefore
\[
F^{**}(X) \leq (f^p)^*(X).
\]

However since \( F \) is convex and finite, we have
\[
f(\xi) = F(T(\xi)) = F^{**}(T(\xi)) \leq (f^p)^*(T(\xi)) = f^{pp}(\xi).
\]

Since the reverse inequality is trivial, we have indeed established (6.3).

Step 2. Applying Step 1 to \( Pf \) which is polyconvex and finite (since \( f \geq Pf \geq g = Pg \)) we get
\[
Pf = (Pf)^{pp}.
\]

We thus deduce
\[
Pf = (Pf)^{pp} \leq f^{pp} \leq f
\]
and the result follows.

6.2.2 Another representation formula

We start by recalling the notation, valid for an integer \( s \),
\[
\Lambda_s := \{ \lambda = (\lambda_1, \cdots, \lambda_s) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^s \lambda_i = 1 \}\.
\]

In Theorem 2.35, we proved that for \( f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\} \)
\[
 Cf(\xi) = \inf \left\{ \sum_{i=1}^{nN+1} \lambda_i f(\xi_i) : \lambda \in \Lambda_{nN+1}, \sum_{i=1}^{nN+1} \lambda_i \xi_i = \xi \right\}.
\]

We now discuss an analogous formula for the polyconvex envelope, that was first proved by Dacorogna in [176], see also [177], [179].
Theorem 6.8 Let $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$. Let $g : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ be polyconvex and such that

$$f(\xi) \geq g(\xi) \text{ for every } \xi \in \mathbb{R}^{N \times n}.$$ 

Then the following formula holds, for every $\xi \in \mathbb{R}^{N \times n}$,

$$P f(\xi) = \inf \left\{ \sum_{i=1}^{\tau+1} \lambda_i f(\xi_i) : \lambda \in \Lambda_{\tau+1}, \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = T(\xi) \right\}.$$ 

Proof. When there is no ambiguity, we write $\tau$ for $\tau(n, N)$. We first define

$$P' f(\xi) := \inf \left\{ \sum_{i=1}^{I} \lambda_i f(\xi_i) : \lambda \in \Lambda_{I}, I \geq \tau+1, \sum_{i=1}^{I} \lambda_i T(\xi_i) = T(\xi) \right\}.$$ 

(6.4)

We decompose the proof into three steps.

Step 1. We first show that $P' f$ is polyconvex. In view of Theorem 5.6, to show polyconvexity of $P' f$, it is sufficient to see that

$$\sum_{\nu=1}^{\tau+1} \lambda_\nu P' f(\eta_\nu) \geq P' f(\sum_{\nu=1}^{\tau+1} \lambda_\nu \eta_\nu)$$

whenever $\lambda \in \Lambda_{\tau+1}$ and

$$\sum_{\nu=1}^{\tau+1} \lambda_\nu T(\eta_\nu) = T(\sum_{\nu=1}^{\tau+1} \lambda_\nu \eta_\nu).$$

Fix $\epsilon > 0$. From (6.4), we have that there exist, for every $1 \leq \nu \leq \tau+1$,

$$I_\nu \geq \tau+1, \alpha_\nu' \in \Lambda_{I_\nu}, \xi_i^{\nu'} \in \mathbb{R}^{N \times n}$$

such that

$$\begin{cases} 
\epsilon + P' f(\eta_\nu) \geq \sum_{i=1}^{I_\nu} \alpha_i^{\nu'} f(\xi_i^{\nu'}), & 1 \leq \nu \leq \tau+1 \\
\sum_{i=1}^{I_\nu} \alpha_i^{\nu'} T(\xi_i^{\nu'}) = T(\eta_\nu), & 1 \leq \nu \leq \tau+1. 
\end{cases}$$

Relabeling $\alpha_i^{\nu'}$ and $\xi_i^{\nu'}$ as

$$\begin{cases} 
\beta_i = \lambda_1 \alpha_i^1 & X_i = \xi_i^1, \quad 1 \leq i \leq I_1 \\
\beta_{I_1+i} = \lambda_2 \alpha_i^2 & X_{I_1+i} = \xi_i^2, \quad 1 \leq i \leq I_2 \\
\vdots & \vdots \\
\beta_{I_1+\ldots+I_{r+1}+i} = \lambda_{\tau+1} \alpha_i^{\tau+1} & X_{I_1+\ldots+I_{r+1}+i} = \xi_i^{\tau+1}, \quad 1 \leq i \leq I_{\tau+1}
\end{cases}$$

we get that $\beta \in \Lambda_{I_1+\ldots+I_{r+1}}$.

$$\epsilon + \sum_{\nu=1}^{\tau+1} \lambda_\nu P' f(\eta_\nu) \geq \sum_{i=1}^{I_1+\ldots+I_{\tau+1}} \beta_i f(X_i)$$

(6.6)
The quasiconvex envelope

and

\[
I_1 + \cdots + I_{\tau + 1} \sum_{i=1}^{\tau + 1} \beta_i T(X_i) = \sum_{\nu=1}^{\tau + 1} \lambda_\nu T(\eta_\nu) = T(\sum_{\nu=1}^{\tau + 1} \lambda_\nu \eta_\nu)
\]

\[
= T(\sum_{i=1}^{I_1 + \cdots + I_{\tau + 1}} \beta_i X_i).
\]

Using (6.4) in the right hand side of the inequality in (6.6) and the fact that \(\epsilon\) is arbitrary, we have indeed obtained (6.5) and therefore shown that \(P'f\) is polyconvex.

**Step 2.** We next want to prove that \(P'f = Pf\). We have, using Theorem 5.6 and Step 1, that if \(h \leq f\) is any polyconvex function with \(h \geq g\), then \(P'h = h\) and hence

\[
h = P'h \leq P'f \leq f.
\]

Thus \(P'f \geq Pf\), and since \(P'f\) is polyconvex, we have indeed \(P'f = Pf\).

**Step 3.** It now remains to show that in (6.4) we can choose \(I = \tau + 1\). The proof is almost identical to that of Step 2 of Theorem 5.6 and we will not reproduce it here. \(\blacksquare\)

### 6.3 The quasiconvex envelope

The following formula was established by Dacorogna in [172], see also [176], [177].

**Theorem 6.9 (Dacorogna formula)** Let \(f : \mathbb{R}^{N \times n} \to \mathbb{R}\) be locally bounded and Borel measurable. Let \(g : \mathbb{R}^{N \times n} \to \mathbb{R}\) be quasiconvex and such that

\[
f(\xi) \geq g(\xi) \text{ for every } \xi \in \mathbb{R}^{N \times n}.
\]

Then, for every \(\xi \in \mathbb{R}^{N \times n}\),

\[
Qf(\xi) = \inf \left\{ \frac{1}{\text{meas } D} \int_D f(\xi + \nabla \varphi(x)) \, dx : \varphi \in W^{1,\infty}_0(D; \mathbb{R}^N) \right\}
\]

(6.7)

where \(D \subset \mathbb{R}^n\) is a bounded open set. In particular, the infimum in the formula is independent of the choice of \(D\).

**Proof.** We first set the notation. We have

\[
Qf(\xi) := \sup \{ h(\xi) : h \leq f \text{ and } h \text{ quasiconvex} \}.
\]

We next call for \(D \subset \mathbb{R}^n\) a bounded open set

\[
\tilde{Q}f(\xi) := \inf \left\{ \frac{1}{\text{meas } D} \int_D f(\xi + \nabla \varphi(x)) \, dx : \varphi \in W^{1,\infty}_0(D; \mathbb{R}^N) \right\}
\]
the expression given in the right hand side of (6.7) and we finally let

$$Q' f (\xi) := \inf \left\{ \frac{1}{\text{meas } D} \int_D f(\xi + \nabla \varphi(x)) \, dx : \varphi \in A(D) \right\}$$  \hspace{1cm} (6.8)$$

where (see Chapter 12)

$$A(D) := \left\{ \varphi \in W^{1,\infty}_0(D; \mathbb{R}^N) \cap \text{Aff}_{\text{piece}} (\overline{D}; \mathbb{R}^N) : \text{supp } \varphi \subset D \right\}.$$  

Note that since $f \geq g$, then $Qf, Q'f, \tilde{Q}f > -\infty$.

The aim is to show that the definitions of $\tilde{Q}f$ and $Q'f$ are independent of the choice of the set $D$ and that the three definitions coincide, namely

$$Qf = Q'f = \tilde{Q}f.$$  

To achieve this goal we divide the proof into five steps.

Step 1. We first show that the definition of $Q'f$ is independent of the choice of $D$.

Step 2. We then establish that

$$\int_D Q' f (\xi + \nabla \psi(x)) \, dx \geq Q' f (\xi) \text{ meas } D$$

for every $\xi \in \mathbb{R}^{N \times n}$ and for every $\psi \in A(D)$.

Step 3. The previous step easily shows that $Q'f$ is quasiconvex.

Step 4. We then deduce that $Q'f = Qf$.

Step 5. We finally establish (6.7).

We now discuss the details of the different steps.

Step 1. For $D \subset \mathbb{R}^n$ a bounded open set, let

$$Q' f_D (\xi) := \inf \left\{ \frac{1}{\text{meas } D} \int_D f(\xi + \nabla \varphi(x)) \, dx : \varphi \in A(D) \right\}.  \hspace{1cm} (6.9)$$

We wish to show that given two such sets $D$ and $E$, then

$$Q' f_D = Q' f_E .$$

This will, a posteriori, justify the notation $Q' f$ instead of $Q' f_D$ in (6.8). The proof is in the same spirit as that of Proposition 5.11.

(1) We first prove this result for sets $D$ and $E$ which are dilated and translated one from each other, namely

$$E = x_0 + \lambda D,$$

where $x_0 \in \mathbb{R}^n$ and $\lambda > 0$. This is straightforward, if we set for any $\varphi \in A(D)$

$$\varphi_\lambda (y) := \lambda \varphi\left(\frac{y - x_0}{\lambda}\right), \ y \in E$$
and observing that $\varphi_\lambda \in A(E)$ with
\[
\frac{1}{\operatorname{meas} D} \int_D f(\xi + \nabla \varphi(x)) \, dx = \frac{1}{\operatorname{meas} E} \int_E f(\xi + \nabla \varphi_\lambda(x)) \, dx.
\]

(2) We now discuss the case of open sets $D$ and $E$ with
\[
\operatorname{meas}(\partial D) = 0.
\]

The idea is to approximate the set $E$ by dilation and translation of $D$. More precisely, for every $\epsilon > 0$ we can find (see Lemma 5.3 in Giusti [316]) $I^\epsilon$ an integer and disjoint open sets $D_i \subset E$ that are homothetic to $D$ so that
\[
\operatorname{meas}(E - \bigcup_{i=1}^{I^\epsilon} D_i) \leq \epsilon.
\]

We next use (6.9) and the previous observation that $Q' f_D = Q' f_{D_i}$ to find $\varphi^\epsilon_i \in A(D_i)$ such that
\[
\int_{D_i} f(\xi + \nabla \varphi^\epsilon_i(x)) \, dx \leq (\epsilon + Q' f_D(\xi)) \operatorname{meas} D_i.
\]

Define then $\varphi^\epsilon \in A(E)$ by
\[
\varphi^\epsilon(x) = \begin{cases} 
\varphi^\epsilon_i(x) & \text{if } x \in D_i \\
0 & \text{if } x \in E - \bigcup_{i=1}^{I^\epsilon} D_i.
\end{cases}
\]

By the definition of $Q' f_E$ we have
\[
Q' f_E(\xi) \operatorname{meas} E \leq \int_E f(\xi + \nabla \varphi^\epsilon(x)) \, dx
\leq \sum_{i=1}^{I^\epsilon} \int_{D_i} f(\xi + \nabla \varphi^\epsilon_i(x)) \, dx + f(\xi) \operatorname{meas}(E - \bigcup_{i=1}^{I^\epsilon} D_i)
\leq (\epsilon + Q' f_D(\xi)) \operatorname{meas}(\bigcup_{i=1}^{I^\epsilon} D_i) + \epsilon f(\xi).
\]

Since $\epsilon$ is arbitrary, we have indeed shown that
\[
\operatorname{meas}(\partial D) = 0 \Rightarrow Q' f_E \leq Q' f_D.
\]

A similar argument establishes the reverse inequality, namely
\[
\operatorname{meas}(\partial E) = 0 \Rightarrow Q' f_D \leq Q' f_E,
\]

in particular
\[
\operatorname{meas}(\partial E) = \operatorname{meas}(\partial D) = 0 \Rightarrow Q' f_D = Q' f_E.
\]
We finally consider general open sets $D$ and $E$. By definition, we can find, for every $\epsilon > 0$, $\varphi \in A(D)$ such that
\[
Q' f_D (\xi) \text{ meas } D \geq -\epsilon + \int_D f (\xi + \nabla \varphi (x)) \, dx.
\] (6.13)

Since $\text{supp } \varphi \subset D$, we can find an open set $A$, with $\text{meas } (\partial A) = 0$, such that $\text{supp } \varphi \subset A \subset D$.

Returning to (6.13) and using the definition of $Q' f_A$ and the fact that $\varphi \in A(A)$, we find
\[
Q' f_D (\xi) \text{ meas } A + Q' f_D (\xi) \text{ meas } (D - A)
\geq -\epsilon + \int_A f (\xi + \nabla \varphi (x)) \, dx + f (\xi) \text{ meas } (D - A)
\geq -\epsilon + Q' f_A (\xi) \text{ meas } A + f (\xi) \text{ meas } (D - A).
\]

We have thus obtained, since $f \geq Q' f_D$, that
\[
[Q' f_D (\xi) - Q' f_A (\xi)] \text{ meas } A \geq -\epsilon + [f (\xi) - Q' f_D (\xi)] \text{ meas } (D - A)
\geq -\epsilon.
\]

Using (6.12), we find that the above inequality holds for any set $A$ with $\text{meas } (\partial A) = 0$ and thus, since $\epsilon$ is arbitrary, we have obtained that $Q' f_D \geq Q' f_A$. Since $\text{meas } (\partial A) = 0$, we have, appealing to (6.11) and to the preceding inequality, that
\[
Q' f_A = Q' f_D.
\] (6.14)

The same reasoning on $E$ gives that there exists an open set $B$, with $\text{meas } (\partial B) = 0$, such that
\[
Q' f_E = Q' f_B.
\]

Combining the above equality, (6.14) and (6.12), we have
\[
Q' f_E = Q' f_B = Q' f_A = Q' f_D
\]
as wished.

**Step 2.** We now want to show that
\[
\int_D Q' f (\xi + \nabla \psi (x)) \, dx \geq Q' f (\xi) \text{ meas } D
\] (6.15)
for every $\xi \in \mathbb{R}^{N \times n}$ and $\psi \in A(D)$. Note that the inequality (6.15) ensures, up to a density argument (see Step 3), that $Q' f$ is quasiconvex. Since $\psi \in A(D)$, there exist disjoint open sets $D_i \subset D$ with
\[
\text{meas}(D - \bigcup_{i=1}^{\infty} D_i) = 0
\]
and $\eta_i \in \mathbb{R}^{N \times n}$ such that
\[
\nabla \psi(x) = \eta_i, \quad x \in D_i.
\]

Since $f$ is locally bounded and $g \leq Q'f \leq f$, we can find $\gamma = \gamma(\xi, \psi) > 0$ such that
\[
|Q'f(\xi + \nabla \psi(x))|, |f(\xi + \nabla \psi(x))| \leq \gamma, \text{ a.e. } x \in D.
\]

We therefore have, for every $\epsilon > 0$, that there exists an integer $I^\epsilon = I^\epsilon(\epsilon, \xi, \psi)$ such that
\[
\int_{D - \bigcup_{i=1}^{I^\epsilon} D_i} |Q'f(\xi + \nabla \psi(x))| \, dx, \int_{D - \bigcup_{i=1}^{I^\epsilon} D_i} |f(\xi + \nabla \psi(x))| \, dx \leq \epsilon \quad (6.16)
\]
and thus
\[
\int_{D} Q'f(\xi + \nabla \psi(x)) \, dx \geq \sum_{i=1}^{I^\epsilon} Q'f(\xi + \eta_i) \text{ meas } D_i - \epsilon. \quad (6.17)
\]

Fixing $\epsilon > 0$ and using (6.9) we have that there exists $\varphi_i^\epsilon \in A(D_i), 1 \leq i \leq I^\epsilon$, such that
\[
Q'f(\xi + \eta_i) \geq \frac{1}{\text{meas } D_i} \int_{D_i} f(\xi + \eta_i + \nabla \varphi_i^\epsilon(x)) \, dx - \epsilon. \quad (6.18)
\]

Let $\chi \in A(D)$ be defined by
\[
\chi(x) := \begin{cases} 
\psi(x) + \varphi_i^\epsilon(x) & \text{if } x \in D_i, \; i = 1, \ldots, I^\epsilon \\
\psi(x) & \text{if } x \in D - \bigcup_{i=1}^{I^\epsilon} D_i.
\end{cases}
\]

We therefore have
\[
\int_{D} Q'f(\xi + \nabla \psi(x)) \, dx \geq \sum_{i=1}^{I^\epsilon} \int_{D_i} f(\xi + \nabla \chi(x)) \, dx - \epsilon (1 + \text{meas } D)
\]
\[
\geq \int_{D} f(\xi + \nabla \chi(x)) \, dx - \epsilon (2 + \text{meas } D)
\]
\[
\geq Q'f(\xi) \text{ meas } D - \epsilon (2 + \text{meas } D)
\]
where we have used in the first line (6.16), (6.17) and (6.18), in the second line (6.16) and in the last line (6.9). Letting $\epsilon \to 0$, we have obtained (6.15).

**Step 3.** We next want to prove that $Q'f$ is quasiconvex. It will be sufficient to show that (6.15) implies that $Q'f$ is continuous and therefore combining (6.15), the continuity of $Q'f$, the fact that $A(D)$ is dense in $W^{1,\infty}_0(D; \mathbb{R}^N)$ in any $W^{1,p}$ norm, $1 \leq p < \infty$, (see Theorem 12.15) and Lebesgue dominated convergence theorem, we will have that $Q'f$ is quasiconvex.
In order to show that \(Q'f\) is continuous we prove that \(Q'f\) is rank one convex. The continuity will then follow from standard properties of convex functions (see Theorem 2.31).

Let \(t \in [0,1]\), \(\alpha, \beta \in \mathbb{R}^{N \times n}\) be such that \(\text{rank}\{\alpha - \beta\} = 1\). We wish to show that \(Q'f\) is rank one convex, meaning that

\[
t Q'f (\alpha) + (1 - t) Q'f (\beta) \geq Q'f (t\alpha + (1 - t) \beta). \tag{6.19}
\]

We can then find from Lemma 3.11 that, for every \(\epsilon > 0\), there exist \(k = k(\alpha, \beta) > 0\), \(u \in \mathcal{A}(D)\) and disjoint open sets \(D_\alpha, D_\beta \subset D\), so that

\[
\begin{align*}
|\text{meas } D_\alpha - t \text{ meas } D|, & \quad |\text{meas } D_\beta - (1 - t) \text{ meas } D| \leq \epsilon, \\
\nabla u(x) = \begin{cases}
(1 - t) (\alpha - \beta) & \text{in } D_\alpha, \\
-t (\alpha - \beta) & \text{in } D_\beta
\end{cases}, & \quad \|\nabla u\|_{L^\infty} \leq k.
\end{align*}
\]

We then use (6.15) to obtain

\[
\int_D Q'f (t\alpha + (1 - t) \beta + \nabla u(x)) \, dx \geq Q'f (t\alpha + (1 - t) \beta) \text{ meas } D.
\]

Letting \(\epsilon \to 0\), we have indeed obtained (6.19) and thus the continuity of \(Q'f\) and consequently the quasiconvexity of \(Q'f\).

Step 4. We now show that \(Q'f = Qf\). Observe that if \(h\) is quasiconvex then, trivially, \(Q'h = h\). Therefore let \(h \leq f\) be quasiconvex, we then deduce that

\[
h = Q'h \leq Q'f \leq f.
\]

Hence \(Q'f \geq Qf\) and since \(Q'f\) itself is quasiconvex, according to Step 3, we have indeed established that \(Q'f = Qf\).

Step 5. It remains to establish (6.7), i.e. if

\[
\tilde{Q}f (\xi) = \inf \left\{ \frac{1}{\text{meas } D} \int_D f(\xi + \nabla \varphi(x)) \, dx : \varphi \in W^{1,\infty}_0 (D; \mathbb{R}^N) \right\}
\]

then \(Qf = \tilde{Q}f\). From Step 4 we also have

\[
Qf (\xi) = \inf \left\{ \frac{1}{\text{meas } D} \int_D f(\xi + \nabla \varphi(x)) \, dx : \varphi \in \mathcal{A}(D) \right\}.
\]

Since \(\mathcal{A}(D) \subset W^{1,\infty}_0 (D; \mathbb{R}^N)\), we deduce that

\[
Qf (\xi) \geq \tilde{Q}f (\xi). \tag{6.20}
\]

Since \(Qf\) is quasiconvex, we immediately obtain that

\[
\tilde{Q} (Qf) = Qf.
\]

Therefore, combining (6.20) and the above identity we have

\[
Qf \geq \tilde{Q}f \geq \tilde{Q} (Qf) = Qf.
\]

This indeed establishes the result.
6.4 The rank one convex envelope

Recall first that, for any integer $I$, we let

$$
\Lambda_I := \{ \lambda = (\lambda_1, \cdots, \lambda_I) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^I \lambda_i = 1 \}.
$$

Theorem 6.10 Let $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$. Let $g : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ be rank one convex and such that $f(\xi) \geq g(\xi)$ for every $\xi \in \mathbb{R}^{N \times n}$.

**Part 1.** For every $\xi \in \mathbb{R}^{N \times n}$,

$$
Rf(\xi) = \inf \left\{ \sum_{i=1}^I \lambda_i f(\xi_i) : \lambda \in \Lambda_I, \sum_{i=1}^I \lambda_i \xi_i = \xi, (\lambda_i, \xi_i) \text{ satisfy } (H_I) \right\}
$$

where $(H_I)$ is as in Definition 5.14.

**Part 2.** Let $R_0f := f$ and for $k \in \mathbb{N}$ define inductively

$$
R_{k+1}f(\xi) := \inf \left\{ \lambda R_kf(\xi_1) + (1 - \lambda) R_kf(\xi_2) : \lambda \xi_1 + (1 - \lambda) \xi_2 = \xi \text{ with rank } \{\xi_1 - \xi_2\} \leq 1 \right\}.
$$

Then

$$
Rf = \lim_{k \to \infty} R_kf = \inf_{k \in \mathbb{N}} R_kf.
$$

Remark 6.11 Part 1 was established by Dacorogna in [176], see also [177], [179]. Part 2 was proved in Kohn-Strang [373], [374]. The two approaches are very similar and both formulas present a serious defect in the sense that in the first one we cannot prescribe a priori the value of the integer $I$, while in the second one we cannot prescribe that the limit $Rf$ is attained after a given finite number of steps. Therefore such formulas are useful for computing $Rf$ only when there is a hint on the number of steps required in order to get $Rf$. ⊤

**Proof. Part 1.** We first define

$$
R'f(\xi) := \inf \left\{ \sum_{i=1}^I \lambda_i f(\xi_i) : \lambda \in \Lambda_I, \sum_{i=1}^I \lambda_i \xi_i = \xi, (\lambda_i, \xi_i) \text{ satisfy } (H_I) \right\}.
$$

Note that, since $f \geq g$, then $R'f > -\infty$ (cf. Proposition 5.16).

We decompose the proof into three steps.

**Step 1.** We start with a preliminary step where we want to show that if

$$
\left\{ \begin{array}{l}
\lambda \in \Lambda_I \text{ and } (\lambda_i, \xi_i)_{1 \leq i \leq I} \text{ satisfy } (H_I) \\
\mu \in \Lambda_J \text{ and } (\mu_j, \eta_j)_{1 \leq j \leq J} \text{ satisfy } (H_J)
\end{array} \right.
$$

and if

$$
\text{rank}\{\sum_{i=1}^I \lambda_i \xi_i - \sum_{j=1}^J \mu_j \eta_j\} \leq 1
$$

(6.21)
then, for every $t \in [0,1]$, we have

$$
\left( (t \lambda_i, \xi_i)_{1 \leq i \leq I}, \left( (1-t) \mu_j, \eta_j \right)_{1 \leq j \leq J} \right) \text{ satisfy } (H_{I+J}). \tag{6.22}
$$

To show (6.22) we proceed by induction over $I + J$.

The case $I + J = 2$ is trivial since this implies that $I = J = 1$ and therefore (6.22) is equivalent, by definition, to (6.21).

Assume therefore that (6.22) has been established for $I + J - 1$. We may also assume, without loss of generality, that $I \geq 2$. Since $(\lambda_i, \xi_i)$ satisfy $(H_I)$ we have, up to a permutation, that

$$
\text{rank} \{ \xi_1 - \xi_2 \} \leq 1 \tag{6.23}
$$

and if

$$
\begin{align*}
\tilde{\lambda}_1 &= \lambda_1 + \lambda_2, & \tilde{\xi}_1 &= \frac{\lambda_1 \xi_1 + \lambda_2 \xi_2}{\lambda_1 + \lambda_2} \\
\tilde{\lambda}_i &= \lambda_{i+1}, & \tilde{\xi}_i &= \xi_{i+1}, & i \geq 2
\end{align*}
$$

then

$$
\left( (\tilde{\lambda}_i, \tilde{\xi}_i)_{1 \leq i \leq I-1} \right) \text{ satisfy } (H_{I-1}).
$$

Note that (6.21) implies then that

$$ \text{rank}\{I^{-1} \tilde{\lambda}_i \tilde{\xi}_i - \sum_{j=1}^{J} \mu_j \eta_j\} \leq 1. $$

The hypothesis of induction therefore ensures that

$$
\left( (t \tilde{\lambda}_i, \tilde{\xi}_i)_{1 \leq i \leq I-1}, \left( (1-t) \mu_j, \eta_j \right)_{1 \leq j \leq J} \right) \text{ satisfy } (H_{I+J-1}).
$$

Coupling the last statement and (6.23), we have indeed obtained (6.22).

**Step 2.** We now show that $R'f$ is rank one convex, i.e.

$$
t R'f(\xi) + (1-t) R'f(\eta) \geq R'f(t \xi + (1-t) \eta)
$$

for every $\xi, \eta \in \mathbb{R}^{N \times n}$ such that

$$ \text{rank}\{\xi - \eta\} \leq 1. $$

Fix $\epsilon > 0$ and use the definition of $R'f$ to get

$$
\begin{align*}
\epsilon + R'f(\xi) &\geq \sum_{i=1}^{I} \lambda_i f(\xi_i), \ \sum_{i=1}^{I} \lambda_i \xi_i = \xi, \\
\lambda &\in \Lambda_I, \ (\lambda_i, \xi_i)_{1 \leq i \leq I} \text{ satisfy } (H_I)
\end{align*}
$$

and

$$
\begin{align*}
\epsilon + R'f(\eta) &\geq \sum_{j=1}^{J} \mu_j f(\eta_j), \ \sum_{j=1}^{J} \mu_j \eta_j = \eta, \\
\mu &\in \Lambda_J, \ (\mu_j, \eta_j)_{1 \leq j \leq J} \text{ satisfy } (H_J).
\end{align*}
$$
Combining the above two inequalities with Step 1 we get

$$
\epsilon + t R' f (\xi) + (1 - t) R' f (\eta) \geq \sum_{i=1}^{l} t \lambda_i f (\xi_i) + \sum_{j=1}^{J} (1 - t) \mu_j f (\eta_j)
$$

where

$$
\begin{align*}
&\sum_{i=1}^{l} t \lambda_i \xi_i + \sum_{j=1}^{J} (1 - t) \mu_j \eta_j = t \xi + (1 - t) \eta \\
& (t \lambda_i, \xi_i)_{1 \leq i \leq I}, ((1 - t) \mu_j, \eta_j)_{1 \leq j \leq J}) \text{ satisfy } (H_{I+J}).
\end{align*}
$$

Using the definition of $R' f$ and the fact that $\epsilon$ is arbitrary, we have indeed obtained that $R' f$ is rank one convex.

**Step 3.** We may now conclude. Note first that if $h (g \leq h)$ is rank one convex, then, by Proposition 5.16, we have $R' h = h$. Finally let $h \leq f$ be rank one convex and observe that

$$
h = R' h \leq R' f \leq f
$$

and thus $R' f \geq R f$. Since $R' f$ is rank one convex we have indeed the result, namely $R' f = R f$.

**Part 2.** We start by observing that

$$
g \leq R_{k+1} f \leq R_k f
$$

and therefore the following quantities are well defined

$$
R' f := \lim_{k \to \infty} R_k f = \inf_{k \in \mathbb{N}} R_k f.
$$

To prove the result of Part 2, we first note that if $f$ is rank one convex, then trivially

$$
R' f = f.
$$

This implies that $R' (R f) = R f$ and hence

$$
R f = R' (R f) \leq R' f \leq f.
$$

(6.24)

We next prove that $R' f$ is rank one convex, which combined with the inequality (6.24) leads to the claim, namely $R f = R' f$.

It therefore remains to show that for every $t \in [0, 1]$ and every $\xi, \eta \in \mathbb{R}^{N \times n}$ with rank $\{\xi - \eta\} \leq 1$ we have

$$
t R' f (\xi) + (1 - t) R' f (\eta) \geq R' f (t \xi + (1 - t) \eta).
$$

By definition we have that for every $\epsilon > 0$, we can find $i, j \in \mathbb{N}$, such that (we assume, without loss of generality, that $i \leq j$)

$$
R' f (\xi) \geq -\epsilon + R_i f (\xi) \geq -\epsilon + R_j f (\xi) \quad \text{and} \quad R' f (\eta) \geq -\epsilon + R_j f (\eta).
$$
We thus find that
\[
\begin{align*}
tR' f (\xi) + (1-t) R' f (\eta) & \geq -\epsilon + t R_j f (\xi) + (1-t) R_j f (\eta) \\
& \geq -\epsilon + R_{j+1} f (t \xi + (1-t) \eta) \\
& \geq -\epsilon + R' f (t \xi + (1-t) \eta).
\end{align*}
\]

The claim follows by letting \( \epsilon \to 0. \)

\section*{6.5 Some more properties of the envelopes}

\subsection*{6.5.1 Envelopes and sums of functions}

We now give a result that allows us to separate in some cases the computation of the different envelopes.

**Theorem 6.12** Let \( u = u (x, t) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^N, \)
\[
\nabla u = (\nabla_x u, \nabla_t u) \in \mathbb{R}^{N \times n} \times \mathbb{R}^{N \times m}.
\]

Let \( \xi = (\alpha, \beta) \in \mathbb{R}^{N \times n} \times \mathbb{R}^{N \times m} = \mathbb{R}^{N \times (n+m)} \) and
\[
f (\xi) = g (\alpha) + h (\beta)
\]
where \( g \) and \( h \) satisfy the hypotheses of Theorem 2.35 for the convex case (and for the polyconvex, quasiconvex and rank one convex cases, the hypotheses of Theorems 6.8, 6.9 and 6.10, respectively). Then
\[
\begin{align*}
C f &= C g + C h \\
P f &= P g + P h \\
Q f &= Q g + Q h \\
R f &= R g + R h.
\end{align*}
\]

**Remark 6.13** This result is standard for \( C f \) and for the other cases it has been established by Dacorogna [175], [179].

**Proof.** All three formulas are easily proved and we do so only for \( Q f \) and \( R f \) (the formula for \( P f \) is proved in exactly the same way).

**Formula for \( R f \).** It is clear that if \( \alpha \in \mathbb{R}^{N \times n} \) and \( \beta \in \mathbb{R}^{N \times m}, \) then
\[
R g (\alpha) + R h (\beta) \leq R f (\alpha, \beta). \quad (6.25)
\]

It therefore only remains to prove the reverse inequality. For this we first prove that
\[
R f (\alpha, \beta) \leq R g (\alpha) + h (\beta). \quad (6.26)
\]
Fix $\epsilon \geq 0$, then by Theorem 6.10, there exist $(\lambda_i, \alpha_i)_{1 \leq i \leq I}$ such that

$$\lambda \in \Lambda_I, \quad \sum_{i=1}^{I} \lambda_i \alpha_i = \alpha$$

$$\alpha_i \in \mathbb{R}^{N \times n} \text{ with } (\lambda_i, \alpha_i)_{1 \leq i \leq I} \text{ satisfying } (H_I)$$

$$\epsilon + R g (\alpha) \geq \sum_{i=1}^{I} \lambda_i g (\alpha_i).$$

It is clear that if $\beta \in \mathbb{R}^{N \times m}$, then $(\lambda_i, (\alpha_i, \beta))_{1 \leq i \leq I}$ satisfy $(H_I)$ and therefore

$$\sum_{i=1}^{I} \lambda_i (\alpha_i, \beta) = (\alpha, \beta), \quad (\lambda_i, (\alpha_i, \beta))_{1 \leq i \leq I} \text{ satisfy } (H_I)$$

$$\epsilon + R g (\alpha) + h (\beta) \geq \sum_{i=1}^{I} \lambda_i (g (\alpha_i) + h (\beta)).$$

Again using Theorem 6.10 and the fact that $\epsilon$ is arbitrary, we obtain (6.26). A similar argument shows that

$$Rf (\alpha, \beta) \leq g (\alpha) + R h (\beta).$$

We now combine Theorem 6.10, (6.26) and the above inequality to get

$$Rf = R (Rf) \leq R (Rg + h) \leq Rg + Rh$$

which, combined with (6.25), is the claimed result.

**Formula for $Qf$.** We establish the present formula similarly as the one for $Rf$. We first prove that $Qf \leq Qg + h$, then, in exactly the same way, that $Qf \leq g + Qh$ and conclude as above. Therefore we only show that

$$Qf (\alpha, \beta) \leq Qg (\alpha) + h (\beta). \quad (6.27)$$

From Theorem 6.9, we have that if $D \subset \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^m$ are unit cubes, then

$$Qf (\alpha, \beta) = \inf_{\varphi \in \mathcal{W}^{1,\infty}_0 (D \times \Omega; \mathbb{R}^N)} \left\{ \int_D \int_{\Omega} [g (\alpha + \nabla_x \varphi (x, t)) + h (\beta + \nabla_t \varphi (x, t))] \, dx \, dt \right\}.$$ 

Let $\epsilon > 0$ be fixed, then Theorem 6.9 implies that there exists $\sigma \in \mathcal{W}^{1,\infty}_0 (D; \mathbb{R}^N)$ such that

$$\int_D g (\alpha + \nabla_x \sigma (x)) \, dx \leq \epsilon + Qg (\alpha).$$

On extending $\sigma$ by periodicity from $D$ to $\mathbb{R}^n$, we trivially have that, for $\nu \in \mathbb{N}$,

$$\int_D g (\alpha + \nabla_x \sigma (\nu x)) \, dx \leq \epsilon + Qg (\alpha). \quad (6.28)$$

Let $\Omega_\nu \subset \Omega$ be a cube with the same centre as $\Omega$ and such that

$$\text{dist} (\partial \Omega; \Omega_\nu) = \frac{1}{\nu}.$$
We then define \( \psi \in W^{1,\infty}_0(\Omega) \), \( 0 \leq \psi(t) \leq 1 \), \( \|\text{grad } \psi\|_{L^n} \leq L\nu \), for a certain constant \( L \), and such that

\[
\psi(t) = \begin{cases} 
1 & \text{if } t \in \Omega_\nu \subset \Omega \subset \mathbb{R}^m \\
0 & \text{if } t \in \partial \Omega.
\end{cases}
\]

and choose

\[
\varphi(x,t) := \frac{1}{\nu} \sigma(\nu x) \psi(t).
\]

Observe that \( \varphi \in W^{1,\infty}_0(D \times \Omega; \mathbb{R}^N) \). Using the formula for \( Qf(\alpha,\beta) \) we get

\[
Qf(\alpha,\beta) \leq \int_D \int_\Omega \left[ g(\alpha + \psi(t) \nabla_x \sigma(\nu x)) + h(\beta + \frac{1}{\nu} \sigma(\nu x) \otimes \text{grad } \psi(t)) \right] dx dt
\]

where \( \sigma(\nu x) \otimes \text{grad } \psi(t) \) denotes the tensorial product in \( \mathbb{R}^{N \times (n+m)} \). We next use (6.28) to get, recalling that \( \text{meas } \Omega = \text{meas } D = 1 \),

\[
\int_D \int_\Omega g(\alpha + \psi(t) \nabla_x \sigma(\nu x)) dx dt
\]

\[
= \int_D \int_{\Omega_\nu} g(\alpha + \nabla_x \sigma(\nu x)) dx dt + \int_D \int_{\Omega - \Omega_\nu} g(\alpha + \psi(t) \nabla_x \sigma(\nu x)) dx dt
\]

\[
\leq [\epsilon + Qg(\alpha)] \text{meas } \Omega_\nu + \text{meas } (\Omega - \Omega_\nu) \sup \{ |g(\alpha + \psi(t) \nabla_x \sigma(\nu x))| \}.
\]

Similarly we have

\[
\int_D \int_\Omega h(\beta + \frac{1}{\nu} \sigma(\nu x) \otimes \text{grad } \psi(t)) dx dt
\]

\[
\leq h(\beta) \text{meas } \Omega_\nu + \int_D \int_{\Omega - \Omega_\nu} h(\beta + \frac{1}{\nu} \sigma(\nu x) \otimes \text{grad } \psi(t)) dx dt
\]

\[
\leq h(\beta) \text{meas } \Omega_\nu + \text{meas } (\Omega - \Omega_\nu) \sup \{ |h(\beta + \frac{1}{\nu} \sigma(\nu x) \otimes \text{grad } \psi(t))| \}.
\]

Combining (6.29) and the above two inequalities, letting \( \nu \to +\infty \) and using the fact that \( \epsilon \) is arbitrary, we have indeed established (6.27) and thus the result.

\section{Envelopes and invariances}

We now see that, in some cases, the different envelopes inherit the invariance of the function.

\textbf{Theorem 6.14} Let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{ +\infty \} \) satisfy the hypotheses of Theorem 2.35 for the convex case (and for the polyconvex, quasiconvex and rank one convex cases, the hypotheses of Theorems 6.8, 6.9 and 6.10 respectively). Let \( \Gamma_1 \subset \mathbb{R}^{N \times N} \) be a subgroup of \( GL(N) \) and \( \Gamma_2 \subset \mathbb{R}^{n \times n} \) be a subgroup of \( GL(n) \). Assume that \( f \) is \( \Gamma_1 \times \Gamma_2 \)-invariant, meaning that

\[
f(U \xi V) = f(\xi), \ \forall U \in \Gamma_1, \ \forall V \in \Gamma_2.
\]

Then \( Cf, Pf, Qf, Rf \) are \( \Gamma_1 \times \Gamma_2 \)-invariant.
Some more properties of the envelopes

Proof. We first recall that for any integer \( s \), we denote by

\[
\Lambda_s := \{ \lambda = (\lambda_1, \cdots , \lambda_s) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^{s} \lambda_i = 1 \}.
\]

(i) We first prove the result for \( Cf \). We use Theorem 2.35 to write

\[
Cf (\xi) = \inf \left\{ \sum_{i=1}^{nN+1} \lambda_i f (\xi_i) : \lambda \in \Lambda_{nN+1}, \sum_{i=1}^{nN+1} \lambda_i \xi_i = \xi \right\}.
\]

Since matrices in \( \Gamma_1 \) and \( \Gamma_2 \) are invertible, we have, for every \( U \in \Gamma_1, V \in \Gamma_2 \),

\[
\sum_{i=1}^{nN+1} \lambda_i \xi_i = \xi \iff \sum_{i=1}^{nN+1} \lambda_i U \xi_i V = U \xi V
\]

and since, moreover,

\[
\sum_{i=1}^{nN+1} \lambda_i f (\xi_i) = \sum_{i=1}^{nN+1} \lambda_i f (U \xi_i V)
\]

we find immediately

\[
Cf (U \xi V) = Cf (\xi).
\]

(ii) We now discuss the case of \( Pf \). We first invoke Theorem 6.8 that gives

\[
Pf (\xi) = \inf \left\{ \sum_{i=1}^{\tau(n,N)+1} \lambda_i f(\xi_i) : \lambda \in \Lambda_{\tau(n,N)+1}, \sum_{i=1}^{\tau(n,N)+1} \lambda_i T(\xi_i) = T(\xi) \right\}.
\]

Since matrices in \( \Gamma_1 \) and \( \Gamma_2 \) are invertible and appealing to Proposition 5.66 (i), we find that

\[
\sum_{i=1}^{\tau(n,N)+1} \lambda_i T(\xi_i) = T(\xi) \iff \sum_{i=1}^{\tau(n,N)+1} \lambda_i T(U \xi_i V) = T(U \xi V).
\]

Furthermore, the invariance of \( f \), leads immediately to

\[
Pf (U \xi V) = Pf (\xi).
\]

(iii) We next turn our attention to \( Qf \). We first let \( D \subset \mathbb{R}^n \) be a bounded open set, \( U \in \Gamma_1 \) and \( V \in \Gamma_2 \) and we note that

\[
\varphi \in W^{1,\infty}_0 (D; \mathbb{R}^N) \iff \psi \in W^{1,\infty}_0 (VD; \mathbb{R}^N)
\]

where

\[
\varphi (x) = U \psi (V x)
\]

which implies that

\[
\nabla \varphi (x) = U \nabla \psi (V x) V.
\]
The result
\[ Qf (U\xi V) = Qf (\xi) \]
then follows from Theorem 6.9, which states that
\[ Qf (\xi) = \inf \left\{ \frac{1}{\text{meas } D} \int_D f (\xi + \nabla \varphi (x)) \, dx : \varphi \in W^{1,\infty}_0 (D; \mathbb{R}^N) \right\}. \]

(iv) We finally prove the result for \( Rf \). We first note that the invertibility of matrices in \( \Gamma_1 \) and \( \Gamma_2 \) leads to
\[ \lambda \xi_1 + (1 - \lambda) \xi_2 = \xi \iff \lambda U\xi_1 V + (1 - \lambda) U\xi_2 V = U\xi V \]
and
\[ \text{rank} \{ \xi_1 - \xi_2 \} \leq 1 \iff \text{rank} \{ U\xi_1 V - U\xi_2 V \} \leq 1. \]

Theorem 6.10 Part 2 then implies that
\[ R_i f (U\xi V) = R_i f (\xi), \forall i \in \mathbb{N} \]
and thus
\[ Rf (U\xi V) = Rf (\xi). \]

This concludes the proof of the theorem. ■

Note that if \( \Gamma_1 \) is a subgroup of \( GL (n) \), then the set
\[ \Gamma_1^t := \{ M^t : M \in \Gamma_1 \} \]
is also a subgroup of \( GL (n) \). We now have the following elementary proposition.

**Proposition 6.15** Let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{ +\infty \} \), \( f \neq +\infty \), let \( \Gamma_1 \) be a subgroup of \( GL (N) \), and let \( \Gamma_2 \) be a subgroup of \( GL (n) \). Consider the following statements:

(i) \( f \) is \( \Gamma_1 \times \Gamma_2^t \)-invariant;
(ii) \( f^* \) is \( \Gamma_1^t \times \Gamma_2 \)-invariant.

Then (i) implies (ii), and the converse is true if \( f \) is lower semicontinuous and convex.

**Proof.** Suppose that \( f \) is \( \Gamma_1 \times \Gamma_2^t \)-invariant, and let \( U \in \Gamma_1 \) and \( V \in \Gamma_2 \). For every \( \xi, X \in \mathbb{R}^{N \times n} \), we have
\[ \langle U^t \xi V; X \rangle = \text{trace} \left( U^t \xi V X^t \right) = \text{trace} \left( \xi V X^t U^t \right) = \langle \xi; U XV^t \rangle. \]

Thus
\[ f^*(U^t \xi V) = \sup \{ \langle U^t \xi V; X \rangle - f (X) : X \in \mathbb{R}^{N \times n} \} = \sup \{ \langle \xi; U XV^t \rangle - f (U XV^t) : X \in \mathbb{R}^{N \times n} \} = \sup \{ \langle \xi; Y \rangle - f (Y) : Y \in \mathbb{R}^{N \times n} \} \]
Examples

since $X \to UXV'$ is bijective. Therefore, $f^*(U^t\xi V) = f^*(\xi)$, so that $f^*$ is $\Gamma_1^t \times \Gamma_2$-invariant. If $f$ is lower semicontinuous and convex, the converse follows dually, since $f^{**} = f$ in this case. ■

6.6 Examples

We now turn our attention to some examples where one can explicitly compute the different envelopes. Usually one is interested (see Chapter 9) in computing $Qf$ ($Cf$ in the scalar case), which is in general a difficult problem. One way of doing so is to compute $Pf$ and $Rf$ and then show that they are equal, which is, of course, not always true. The examples should be compared with those of Section 5.3.

6.6.1 Duality for $SO(n) \times SO(n)$ and $O(N) \times O(n)$ invariant functions

The results of the present section and Section 6.6.2 are closely related. In this section, we discuss the duality and polyconvex duality aspects of the problem.

We adopt the notation of Section 5.3.3 and Chapter 13. In particular we always assume that $N \geq n$ (a completely analogous treatment holds when $N < n$). For $\xi \in \mathbb{R}^{N \times n}$, we denote its singular values by

$$0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$$

and we let

$$\lambda(\xi) = (\lambda_1(\xi), \cdots, \lambda_n(\xi)).$$

When $N = n$, we denote by

$$0 \leq \mu_1(\xi) \leq \cdots \leq \mu_n(\xi)$$

the signed singular values of $\xi \in \mathbb{R}^{n \times n}$ and we let

$$\mu(\xi) = (\mu_1(\xi), \cdots, \mu_n(\xi)).$$

We now see how the duality in convex analysis is carried for the $SO(n) \times SO(n)$-invariant (respectively $O(N) \times O(n)$-invariant) functions and we follow the presentation of Dacorogna-Maréchal [204].

**Theorem 6.16** (i) Let $f : \mathbb{R}^{n \times n} \to \mathbb{R} \cup \{+\infty\}$ be $SO(n) \times SO(n)$-invariant, $f \neq +\infty$, and let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the unique $\Pi_e(n)$-invariant function such that $f = g \circ \mu$. Then

$$f^* = g^* \circ \mu.$$
(ii) Let $N \geq n$, let $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ be $O(N) \times O(n)$-invariant, $f \not\equiv +\infty$, and let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the unique $\Pi(n)$-invariant function such that $f = g \circ \lambda$. Then

$$f^* = g^* \circ \lambda.$$ 

**Proof.** (i) We start by noticing that

$$f^*(\xi) = \sup_{X \in \mathbb{R}^{n \times n}} \{\langle \xi; X \rangle - f(X)\} = \sup_{X \in \mathbb{R}^{n \times n}} \{\langle \xi; X \rangle - g(\mu(X))\} = \sup_{X \in \mathbb{R}^{n \times n}} \left\{ \sup_{Q,R \in SO(n)} \{\langle \xi; QXR^t \rangle - g(\mu(QXR^t))\} \right\}. $$

We obtain that, for every $Q, R \in SO(n)$,

$$\langle \xi; QXR^t \rangle = \text{trace} (\xi^tQXR^t) = \text{trace} (QXR^t\xi^t) \quad \text{and} \quad \mu(QXR^t) = \mu(X)$$

and therefore the inner supremum is, from Theorem 13.10 (i), equal to

$$\sum_{k=1}^{n} \mu_k(\xi) \mu_k(X) - g(\mu_1(X), \cdots, \mu_n(X)).$$

Furthermore, $\mu(X)$ runs over $K^n := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : |x_1| \leq x_2 \leq \cdots \leq x_n\}$, as $X$ runs over $\mathbb{R}^{n \times n}$. Therefore,

$$f^*(\xi) = \sup_{x \in K^n} \{\langle \mu(\xi); x \rangle - g(x)\}. \quad (6.30)$$

On the other hand, let $x' \in \mathbb{R}^n$ and first find $M \in \Pi_e(n)$ such that

$$x := Mx' \in K^n.$$ 

Next apply Proposition 13.9 and the invariance of $g$ under the action of $\Pi_e(n)$ to get, for every $y \in K^n$,

$$g(x') = g(x) \quad \text{and} \quad \langle y; x' \rangle \leq \langle y; x \rangle.$$ 

We therefore have that, for every $y \in K^n$,

$$g^*(y) := \sup_{x \in \mathbb{R}^n} \{\langle y; x \rangle - g(x)\} = \sup_{x \in K^n} \{\langle y; x \rangle - g(x)\}. \quad (6.31)$$

The result follows from (6.30) and (6.31).
(ii) We first observe that

\[ f^* (\xi) = \sup_{X \in \mathbb{R}^{N \times n}} \{ \langle \xi; X \rangle - f(X) \} \]

\[ = \sup_{X \in \mathbb{R}^{N \times n}} \{ \sup_{Q \in O(N)} \{ \langle \xi; QX R^t \rangle - f(QX R^t) \} \} \]

\[ = \sup_{X \in \mathbb{R}^{N \times n}} \{ \sup_{Q \in O(N)} \{ \langle \xi; QX R^t \rangle - f(X) \} \} \]

By Theorem 13.10 (ii), we get

\[ \sup_{Q \in O(N)} \{ \langle \xi; QX R^t \rangle \} = \sup_{Q \in O(N)} \{ \text{trace}(QX R^t \xi^t) \} = n \sum_{k=1}^{n} \lambda_k (\xi) \lambda_k (X). \]

Furthermore, \( \lambda(X) \) runs over

\[ K^+_n := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \} , \]

as \( X \) runs over \( \mathbb{R}^{N \times n} \). We therefore deduce the following identity

\[ f^* (\xi) = \sup_{x \in K^+_n} \{ \langle \lambda (\xi); x \rangle - g(x) \} . \quad (6.32) \]

On the other hand, let \( x' \in \mathbb{R}^n \) and first find \( M \in \Pi(n) \) such that

\[ x := Mx' \in K^+_n . \]

Next apply Proposition 13.9 and the invariance of \( g \) under the action of \( \Pi(n) \) to get, for every \( y \in K^n_+ \),

\[ g(x') = g(x) \text{ and } \langle y; x' \rangle \leq \langle y; x \rangle . \]

We thus deduce that, for every \( y \in K^n_+ \),

\[ g^*(y) := \sup_{x \in \mathbb{R}^n} \{ \langle y; x \rangle - g(x) \} = \sup_{x \in K^+_n} \{ \langle y; x \rangle - g(x) \} . \quad (6.33) \]

The result follows from (6.32) and (6.33). □

We get the following as an immediate corollary.

**Theorem 6.17** (i) Let \( f : \mathbb{R}^{n \times n} \to \mathbb{R} \cup \{ +\infty \} \) be \( SO(n) \times SO(n) \)-invariant, \( f \neq +\infty \), and let \( g := f \circ \text{diag} \). Then

\[ f^{**} = g^{**} \circ \mu . \]

Furthermore, let \( Cf \) and \( Cg \) denote the convex envelopes of \( f \) and \( g \), respectively. Assume that the relationships \( Cf = f^{**} \) and \( Cg = g^{**} \) hold, which happens notably when \( f \) and \( g \) are finite and bounded below by a convex function. Then

\[ Cf = Cg \circ \mu . \]
(ii) Let \( N \geq n \) and let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\} \) be \( O(N) \times O(n) \)-invariant and \( f \neq +\infty \). Let \( g := f \circ \text{diag}_{N \times n} \). Then

\[
f^{**} = g^{**} \circ \lambda.
\]

Assume that the relationships \( Cf = f^{**} \) and \( Cg = g^{**} \) hold, which happens notably when \( f \) and \( g \) are finite and bounded below by a convex function. Then

\[
Cf = Cg \circ \lambda.
\]

Proof. It is an immediate consequence of Theorem 6.16.

The corollary allows an important simplification in the computation of the convex envelopes of \( SO(n) \times SO(n) \)-invariant (or \( O(N) \times O(n) \)-invariant) functions \( f \). This is true if either one proceeds by duality or by Carathéodory theorem. In this last case, for example, while to compute \( Cf \) we normally need \( Nn + 1 \) matrices, for \( Cg \) it is sufficient to take \( n + 1 \) diagonal matrices.

We now turn to similar results for the polyconvex envelope and have the following result due to Dacorogna-Maréchal [205] (see also Section 6.6.2). But we first need the following definition (compare with Definition 6.3).

**Definition 6.18** Let \( g : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\} \) with \( g \neq +\infty \). We define

\[
g^p : \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}
\]

as

\[
g^p (x_1^*, x_2^*, \delta^*) := \sup_{(x_1, x_2) \in \mathbb{R}^2} \{x_1 x_1^* + x_2 x_2^* + x_1 x_2 \delta^* - g(x_1, x_2)\}.
\]

Similarly, we let \( g^{pp} : \mathbb{R}^2 \to \mathbb{R} \cup \{\pm \infty\} \) be defined as

\[
g^{pp} (x_1, x_2) := (g^p)^* (x_1, x_2, x_1 x_2)
\]

where

\[
(g^p)^* (x_1, x_2, \delta) := \sup_{(x_1^*, x_2^*, \delta^*) \in \mathbb{R}^3} \{x_1 x_1^* + x_2 x_2^* + \delta \delta^* - g^p (x_1^*, x_2^*, \delta^*)\}.
\]

**Theorem 6.19** Let \( f : \mathbb{R}^{2 \times 2} \to \mathbb{R} \cup \{+\infty\} \) be \( SO(2) \times SO(2) \)-invariant, \( f \neq +\infty \) and \( g : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\} \) be the unique \( \Pi_e(2) \)-invariant function such that \( f = g \circ \mu \). The following two properties then hold.

(i) For every \( X^* = (\xi^*, \delta^*) \in \mathbb{R}^{2 \times 2} \times \mathbb{R} \),

\[
f^p (X^*) = g^p (\mu (\xi^*), \delta^*).
\]

(ii) For every \( \xi \in \mathbb{R}^{2 \times 2} \),

\[
f^{pp} (\xi) = g^{pp} (\mu (\xi)),
\]
or differently expressed

\[ f^{pp} = g^{pp} \circ \mu. \]

Furthermore, if \( f : \mathbb{R}^{2 \times 2} \to \mathbb{R} \) and is bounded below by a polyconvex function, then \( Pf = f^{pp} \).

**Proof.** (i) By definition of \( f^p \), we have

\[
f^p(X^*) = f^p (\xi^*, \delta^*) = \sup_{\xi \in \mathbb{R}^{2 \times 2}} \{ \langle \xi; \xi^* \rangle + \delta^* \det \xi - f(\xi) \}
= \sup_{\xi \in \mathbb{R}^{2 \times 2}} \{ \sup_{Q,R \in SO(2)} \{ \langle Q\xi R^t; \xi^* \rangle + \delta^* \det(Q\xi R^t) - g^p(Q\xi R^t) \} \}.
\]

We then invoke Theorem 13.10 and the \( SO(2) \times SO(2) \) invariance of \( \det \) and \( \mu \), to get

\[
f^p(X^*) = \sup_{\xi \in \mathbb{R}^{2 \times 2}} \{ \sum_{j=1}^{2} \mu_j(\xi) \mu_j(\xi^*) + \delta^* \det \xi - g(\mu(\xi)) \}
= \sup_{\xi \in \mathbb{R}^{2 \times 2}} \{ \sum_{j=1}^{2} \mu_j(\xi) \mu_j(\xi^*) + \delta^* \mu_1(\xi) \mu_2(\xi) - g(\mu(\xi)) \}
= \sup_{0 \leq |x_1| \leq x_2} \{ x_1 \mu_1(\xi^*) + x_2 \mu_2(\xi^*) + x_1 x_2 \delta^* - g(x_1, x_2) \}.
\]

Using the definition of \( g^p \), we have

\[
g^p(\mu(\xi^*), \delta^*) := \sup_{(x_1, x_2) \in \mathbb{R}^2} \{ x_1 \mu_1(\xi^*) + x_2 \mu_2(\xi^*) + x_1 x_2 \delta^* - g(x_1, x_2) \}.
\]

Since \( 0 \leq |\mu_1(\xi^*)| \leq \mu_2(\xi^*) \), we deduce that

\[
\sup_{(x_1, x_2) \in \mathbb{R}^2} \{ x_1 \mu_1(\xi^*) + x_2 \mu_2(\xi^*) \} = \sup_{0 \leq |x_1| \leq x_2} \{ x_1 \mu_1(\xi^*) + x_2 \mu_2(\xi^*) \}
\]

and therefore, since the functions \((x_1, x_2) \to x_1 x_2 \) and \( g \) are \( \Pi_e(2) \)-invariant,

\[
g^p(\mu(\xi^*), \delta^*) = \sup_{0 \leq |x_1| \leq x_2} \{ x_1 \mu_1(\xi^*) + x_2 \mu_2(\xi^*) + x_1 x_2 \delta^* - g(x_1, x_2) \}.
\]

The claim has therefore been validated.

Let us show, for further reference, that the function

\[
x^* = (x_1^*, x_2^*) \to g^p(x^*, \delta^*)
\]

is \( \Pi_e(2) \)-invariant, for every \( \delta^* \in \mathbb{R} \). Recall that

\[
\Pi_e(2) := \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.
\]
We therefore have, using the $\Pi_e (2)$-invariance of $g$, that, for every $M \in \Pi_e (2)$,
\[
g^p (Mx^*, \delta^*) = \sup_{x \in \mathbb{R}^2} \left\{ \langle x; Mx^* \rangle + x_1 x_2 \delta - g(x) \right\}
\]
\[
= \sup_{x \in \mathbb{R}^2} \left\{ \langle M^t x; x^* \rangle + (M^t x)_1 (M^t x)_2 \delta^* - g(M^t x) \right\}
\]
\[
= \sup_{y \in \mathbb{R}^2} \left\{ \langle y; x^* \rangle + y_1 y_2 \delta^* - g(y) \right\}
\]
\[
= g^p (x^*, \delta^*).
\]

(ii) The proof is almost identical to that of (i). We have, using the definition, (i) and Theorem 13.10,
\[
f_{pp} (\xi) = (f^p)^* (\xi, \det \xi) = \sup_{\xi^* \in \mathbb{R}^2} \sup_{\delta^* \in \mathbb{R}} \left\{ \langle \xi; \xi^* \rangle + \delta^* \det \xi - f^p (\xi^*, \delta^*) \right\}
\]
\[
= \sup_{\xi^* \in \mathbb{R}^2} \sup_{\delta^* \in \mathbb{R}} \left\{ \langle \xi; \xi^* \rangle + \delta^* \det \xi - g^p (\mu (\xi^*), \delta^*) \right\}
\]
\[
= \sup_{\xi^* \in \mathbb{R}^2} \sup_{\delta^* \in \mathbb{R}} \left\{ \langle \xi; Q \xi^* R^t \rangle + \delta^* \det \xi - g^p (\mu (Q \xi^* R^t), \delta^*) \right\}
\]
\[
= \sup_{\xi^* \in \mathbb{R}^2} \left\{ \sum_{j=1}^2 \mu_j (\xi) \mu_j (\xi^*) + \delta^* \mu_1 (\xi) \mu_2 (\xi) - g^p (\mu (\xi^*), \delta^*) \right\}
\]
\[
= \sup_{0 \leq \|x_1^*\| \leq \|x_2^*\|} \left\{ x_1^* \mu_1 (\xi) + x_2^* \mu_2 (\xi) + \delta^* \mu_1 (\xi) \mu_2 (\xi) - g^p (x_1^*, x_2^*, \delta^*) \right\}.
\]

On the other hand
\[
g_{pp} (x_1, x_2) = (f^p)^* (x_1, x_2, x_1 x_2)
\]
\[
= \sup_{(x_1^*, x_2^*, \delta^*) \in \mathbb{R}^3} \left\{ x_1 x_1^* + x_2 x_2^* + x_1 x_2 \delta - g^p (x_1^*, x_2^*, \delta^*) \right\}
\]
and thus
\[
g_{pp} (\mu (\xi)) = \sup_{(x_1, x_2, \delta^*) \in \mathbb{R}^3} \left\{ x_1^* \mu_1 (\xi) + x_2^* \mu_2 (\xi) + \delta^* \mu_1 (\xi) \mu_2 (\xi) - g^p (x_1^*, x_2^*, \delta^*) \right\}.
\]

Since (cf. (i)) the function
\[
(x_1^*, x_2^*) \rightarrow g^p (x_1^*, x_2^*, \delta^*)
\]
is $\Pi_e (2)$-invariant, for every $\delta^* \in \mathbb{R}$ and $0 \leq |\mu_1 (\xi)| \leq \mu_2 (\xi)$, we obtain
\[
\sup_{(x_1^*, x_2^*) \in \mathbb{R}^2} \left\{ x_1^* \mu_1 (\xi) + x_2^* \mu_2 (\xi) \right\} = \sup_{0 \leq \|x_1^*\| \leq \|x_2^*\|} \left\{ x_1^* \mu_1 (\xi) + x_2^* \mu_2 (\xi) \right\}
\]
and therefore
\[
g_{pp} (\mu (\xi)) = \sup_{0 \leq \|x_1^*\| \leq \|x_2^*\|} \left\{ x_1^* \mu_1 (\xi) + x_2^* \mu_2 (\xi) + \delta^* \mu_1 (\xi) \mu_2 (\xi) - g^p (x_1^*, x_2^*, \delta^*) \right\}.
\]

The statement (ii) is thus established.

(iii) The fact that $ Pf = f_{pp}$ was already discussed in Theorem 6.6.
6.6.2 The case of singular values

As already said, the results of the present section are closely related to those of the previous one, but the emphasis is now on Carathéodory type representation formulas.

We first start with an immediate corollary of Theorem 6.14. We will then see how to improve the theorem in some special cases.

**Theorem 6.20** Let \( f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \cup \{+\infty\} \). Let \( g : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \cup \{+\infty\} \) be respectively convex, polyconvex, quasiconvex (in this case, \( f, g : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \)), rank one convex and such that

\[ f(\xi) \geq g(\xi) \text{ for every } \xi \in \mathbb{R}^{N \times n}. \]

Assume that \( f \) is \( O(N) \times O(n) \)-invariant, meaning that

\[ f(R\xi Q) = f(\xi), \forall R \in O(N), \forall Q \in O(n). \]

Then \( C_f, P_f, Q_f, R_f \) are respectively \( O(N) \times O(n) \)-invariant.

**Remark 6.21** When \( N = n \), a completely analogous result is valid for \( SO(n) \times SO(n) \)-invariant functions.

We next continue our study of the envelopes of \( SO(n) \times SO(n) \)-invariant functions. We recall that we denote the singular values of \( \xi \in \mathbb{R}^{n \times n} \) by \( 0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi) \). We consider functions of the form

\[ f(\xi) = g(\lambda_1(\xi), \cdots, \lambda_{n-1}(\xi), \det \xi). \quad (6.34) \]

The first immediate consequence of Theorem 6.20 is that \( C_f, P_f, Q_f, R_f \) are of the same form. In general, it is difficult to determine these envelopes in function of some convex envelopes of \( g \). Furthermore, in such a general context, the envelopes might all be different. We now examine some examples and counter examples and recall the notation that

\[ K_+^n := \{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \}. \]

**Theorem 6.22** Let \( f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \).

(i) Let \( g : K_+^{n-1} \rightarrow \mathbb{R} \) be upper semicontinuous at \((0, x_2, \cdots, x_{n-1})\) and \( \inf g > -\infty \). If

\[ f(\xi) = g(\lambda_1(\xi), \cdots, \lambda_{n-1}(\xi)), \]

then

\[ Cf(\xi) = Pf(\xi) = Qf(\xi) = Rf(\xi) = \inf g. \]
(ii) There exists a function $g : \mathbb{R} \to \mathbb{R}$, namely
\[
g(t) = \begin{cases} 
1 + t^2 & \text{if } t \neq 0 \\
0 & \text{if } t = 0 
\end{cases}
\]
such that if
\[
f(\xi) = g(\lambda_n(\xi)),
\]
then
\[
Cf < Pf.
\]

(iii) Let $g : K^{n-2} \to \mathbb{R}$ be upper semicontinuous at $(0, x_3, \cdots , x_{n-1})$ and $\inf g > -\infty$. Let $h : \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$ be such that
\[
h(\delta) \geq a\delta + b, \quad \forall \delta \in \mathbb{R}.
\]
If
\[
f(\xi) = g(\lambda_2(\xi), \cdots , \lambda_{n-1}(\xi)) + h(\det \xi)
\]
then
\[
Pf(\xi) = Qf(\xi) = Rf(\xi) = \inf \{g + Ch(\det \xi) \}
\]

(iv) Let $f : \mathbb{R}^{2 \times 2} \to \mathbb{R}$ be defined by
\[
f(\xi) = |\lambda_1(\xi) - 1| + |\det \xi|.
\]

Then there exists $\xi \in \mathbb{R}^{2 \times 2}$ such that
\[
Pf(\xi) \neq |\det \xi|.
\]

Remark 6.23 (i) The first two statements were established in Buttazzo-Dacorogna-Gangbo [113], while the last two were proved in Dacorogna-Pisante-Ribeiro [211].

(ii) Assertion (iii) of the theorem is valid for more general functions of the form
\[
f(\xi) = g(\lambda_2(\xi), \cdots , \lambda_{n-1}(\xi), \det \xi).
\]

(iii) It is interesting to note that (ii) and (iv) of the present theorem show that results (i) and (iii) are not true if some dependence on $\lambda_1$ or $\lambda_n$ is allowed.

Proof. (i) The inequalities
\[
\inf g \leq Cf(\xi) \leq Pf(\xi) \leq Qf(\xi) \leq Rf(\xi),
\]
being obvious, we only need to prove that
\[
Rf(\xi) \leq \inf g, \quad \forall \xi \in \mathbb{R}^{n \times n}.
\]

(6.35)
To show this inequality, we proceed in two steps.

**Step 1.** Let \( x \in K_{+}^{n-1} \) with \( x_1 > 0 \) and denote
\[
E_x := \{ \eta \in \mathbb{R}^{n \times n} : \lambda_i(\eta) = x_i, \; i = 1, \ldots, n-1 \}.
\]

Using the results and the notations of Chapter 7, we find
\[
\text{Rco } E_x = \mathbb{R}^{n \times n}, \tag{6.36}
\]
where \( \text{Rco } E_x \) denotes the rank one convex hull of \( E_x \). Indeed, let \( \xi \in \mathbb{R}^{n \times n} \) and choose \( x_n \geq x_{n-1} \) so large that
\[
\prod_{i=\nu}^{n} \lambda_i(\xi) \leq \prod_{i=\nu}^{n} x_i, \; \nu = 1, \ldots, n
\]
and apply Theorem 7.43 to get that
\[
\xi \in \text{Rco} \left[ E_x \cap \{ \eta \in \mathbb{R}^{n \times n} : \lambda_n(\eta) = x_n \} \right] \subset \text{Rco } E_x
\]
and hence (6.36) is proved.

**Step 2.** Let \( x \in K_{+}^{n-1} \) with \( x_1 > 0 \) and let us show that
\[
Rf(\xi) \leq g(x), \; \forall \xi \in \mathbb{R}^{n \times n}. \tag{6.37}
\]
Define a function \( F : \mathbb{R}^{n \times n} \to \mathbb{R} \) by
\[
F(\eta) := Rf(\eta) - g(x).
\]
Observe that \( F \) is rank one convex and that
\[
F(\eta) = Rf(\eta) - f(\eta), \; \forall \eta \in E_x.
\]
Therefore \( F|_{E_x} \leq 0 \) and hence, by Definition 7.25 and Theorem 7.28, we deduce that
\[
F|_{\text{Rco } E_x} \leq 0
\]
which combined with (6.36) leads to the claimed inequality (6.37).

To prove the desired inequality (6.35), it remains to approximate any \( x \in K_{+}^{n-1} \) by \( x' \in K_{+}^{n-1} \) with \( x'_1 > 0 \) and apply (6.37) with \( x' \) and use the upper semicontinuity of \( g \) at 0 to get (6.35).

(ii) It is sufficient to restrict our attention to the case \( n = 2 \). Since \( \xi \to \lambda_2(\xi) \) is a norm, we have (see Theorem 6.30) that
\[
Cf(\xi) = Cg(\lambda_2(\xi)),
\]
where
\[
Cg(t) = \begin{cases} 
1 + t^2 & \text{if } |t| \geq 1 \\
2 |t| & \text{if } |t| < 1.
\end{cases}
\]
In view of Theorem 6.19, it is sufficient, for computing \( Pf \), to only consider diagonal matrices and therefore, by abuse of notation, we write a diagonal matrix \( \xi \in \mathbb{R}^{2 \times 2} \) whose entries are \( x \) and \( y \) as \( \xi = (x, y) \).

In order to prove the desired inequality, namely \( Cf < Pf \), we show that if

\[
0 < a < 2(\sqrt{2} - 1) < 1
\]

then

\[
Cf (a, a) = 2a < Pf (a, a). \tag{6.38}
\]

Applying Theorem 6.19, we get

\[
Pf (a, a) = \sup_{|x| \leq y, \delta \in \mathbb{R}} \left\{ a(x+y) + \delta a^2 - f^p (x, y, \delta) \right\}, \tag{6.39}
\]

where

\[
f^p (x, y, \delta) = \sup_{|\alpha| \leq \beta} \{ \alpha x + \beta y + \delta \alpha \beta - g(\beta) \}.
\]

We next compute, for \( y \geq 0 \),

\[
f^p (y, y, -1) = \sup_{|\alpha| \leq \beta} \{ (\alpha + \beta) y - \alpha \beta - g(\beta) \} = \max\{0, \sup_{|\alpha| \leq \beta} \{ (\alpha + \beta) y - \alpha \beta - 1 - \beta^2 \} \}.
\]

It is easy to see that the last supremum is attained for \( \alpha = \beta = y/2 \) and hence if we denote

\[
[x]_+ = \begin{cases} 
  x & \text{if } x \geq 0 \\
  0 & \text{otherwise}
\end{cases}
\]

we find

\[
f^p (y, y, -1) = \left[ \frac{y^2}{2} - 1 \right]_+.
\]

Returning to (6.39), we find that

\[
Pf (a, a) \geq \sup_{y \geq 0} \left\{ 2ay - a^2 - f^p (y, y, -1) \right\} \geq 2a\sqrt{2} - a^2.
\]

The last inequality and the fact that \( 0 < a < 2(\sqrt{2} - 1) \) immediately give (6.38).

\textbf{(iii)} We proceed in a way very similar to (i). The inequalities

\[
\inf g + Ch(\det \xi) \leq Pf (\xi) \leq Qf (\xi) \leq Rf (\xi)
\]

being obvious, we only show

\[
Rf (\xi) \leq \inf g + Ch(\det \xi), \forall \xi \in \mathbb{R}^{n \times n}. \tag{6.40}
\]
To establish this inequality, we proceed in two steps. Recall that
\[ K_{+}^{n-2} := \{ x = (x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-2} : 0 \leq x_2 \leq \cdots \leq x_{n-1} \}. \]

**Step 1.** Let \( c \in \mathbb{R} \) and \( x \in K_{+}^{n-2} \) with \( x_2 > 0 \). Denote
\[ E_{x,c} := \{ \xi \in \mathbb{R}^{n \times n} : \lambda_i(\xi) = x_i, \ i = 2, \cdots, n-1, \ \det \xi = c \}. \]

Using the results and the notations of Chapter 7, we find
\[ \text{Rco } E_{x,c} = \{ \xi \in \mathbb{R}^{n \times n} : \det \xi = c \}. \quad (6.41) \]

Indeed, let \( \xi \in \mathbb{R}^{n \times n} \) with \( \det \xi = c \), choose \( x_n \geq x_{n-1} \) so large that
\[ \prod_{i=2}^{n} x_i \leq \prod_{i=2}^{n} x_i, \ \nu = 2, \cdots, n \]
and apply Theorem 7.43 to get that
\[ \xi \in \text{Rco } [E_{x,c} \cap \{ \eta \in \mathbb{R}^{n \times n} : \lambda_n(\eta) = x_n \}] \subset \text{Rco } E_{x,c} \]
and hence (6.41) is proved.

**Step 2.** Let \( x \in K_{+}^{n-2} \) with \( x_2 > 0 \) and let us show that
\[ Rf(\xi) \leq g(x) + Ch(\det \xi), \ \forall \xi \in \mathbb{R}^{n \times n}, \quad (6.42) \]
which will follow if we can prove that, for every \( \xi \in \mathbb{R}^{n \times n} \),
\[ Rf(\xi) \leq g(x) + h(\det \xi). \quad (6.43) \]

In fact, if we get (6.43), then the rank one convex envelope of each member preserves the inequality and since the rank one convex envelope of \( h(\det \xi) \) is \( Ch(\det \xi) \), we get (6.42).

So let \( \xi \) be any matrix in \( \mathbb{R}^{n \times n} \) with \( c := \det \xi \) and let us show (6.43). To this aim, we define a function \( F_\xi : \mathbb{R}^{n \times n} \to \mathbb{R} \) such that
\[ F_\xi(\eta) := Rf(\eta) - g(x) - h(\det \xi). \]

Observe that \( F_\xi \) is rank one convex and that
\[ F_\xi(\eta) = Rf(\eta) - f(\eta), \ \forall \eta \in E_{x,c}. \]

Therefore \( F_\xi|_{E_{x,c}} \leq 0 \) and hence, by Definition 7.25 and Theorem 7.28, we deduce that
\[ F_\xi|_{\text{Rco } E_{x,c}} \leq 0, \]
which means (see (6.41)) that, for every \( \eta \in \mathbb{R}^{n \times n} \) with \( \det \eta = \det \xi = c \),
\[ F_\xi(\eta) = Rf(\eta) - g(x) - h(\det \xi) \leq 0. \]
In particular the above inequality holds for $\xi$, which is exactly (6.43).

To prove the final inequality (6.40) it remains to approximate any $x \in K_{+}^{n-2}$ by $x' \in K_{+}^{n-2}$ with $x'_{2} > 0$ and apply (6.42) with $x'$ and use the upper semi-continuity of $g$ at 0 to get (6.40).

(iv) Let us suppose for the sake of contradiction that $Pf(\xi) = |\det \xi|$ for every $\xi \in \mathbb{R}^{2 \times 2}$. Then, for $\xi$ such that $\lambda_{1}(\xi) = 0$, we get

$$Pf(\xi) = |\det \xi| = \lambda_{1}(\xi)\lambda_{2}(\xi) = 0.$$  

From the representation formula for the polyconvex envelope (see Theorem 6.8), we therefore get that there exist $\xi'_{i} \in \mathbb{R}^{2 \times 2}$, $t_{i} \in [0, 1]$ and $\sum_{i=1}^{6} t_{i} = 1$ such that

$$\lim_{\nu \to \infty} \sum_{i=1}^{6} t_{i} f(\xi'_{i}) = 0 \quad \text{with} \quad \sum_{i=1}^{6} t_{i}(\xi'_{i}, \det \xi'_{i}) = (\xi, \det \xi).$$

In particular, $t_{i} |\lambda_{1}(\xi'_{i}) - 1| \to 0$ and $t_{i} |\det \xi'_{i}| \to 0$, $i = 1, \cdots, 6$. Up to a subsequence, $t_{i} \to t_{i} \in [0, 1]$ with $\sum_{i=1}^{6} t_{i} = 1$. So, there is some $j$ such that $t_{j} \neq 0$ and thus for this $j$ we have

$$\begin{cases} 
|\lambda_{1}(\xi'_{j}) - 1| = \frac{1}{t_{j}} t'_{j} |\lambda_{1}(\xi'_{j}) - 1| \to 0 \\
|\det \xi'_{j}| = \frac{1}{t_{j}} t'_{j} |\det \xi'_{j}| \to 0.
\end{cases}$$

The first condition implies that $\lambda_{1}(\xi'_{j}) \to 1$, which contradicts the second one, since then we would have $|\det \xi'_{j}| \geq (\lambda_{1}(\xi'_{j}))^{2} \to 1$.

6.6.3 Functions depending on a quasiaffine function

The following theorem, established by Dacorogna [176], [179], should be related to Theorem 5.46.

**Theorem 6.24** Let $g : \mathbb{R} \to \mathbb{R}$ be such that there exist $a_{1}, a_{2} \in \mathbb{R}$ with

$$g(\delta) \geq a_{1}\delta + a_{2}, \ \forall \delta \in \mathbb{R}.$$ 

Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$, $\Phi : \mathbb{R}^{N \times n} \to \mathbb{R}$ be quasiaffine not identically constant and

$$f(\xi) = g(\Phi(\xi)).$$

Then

$$Pf = Qf = Rf = Cg \circ \Phi$$

and in general

$$Pf > Cf.$$ 

Before proceeding with the proof, we establish a preliminary lemma.
Lemma 6.25 Let $\Phi : \mathbb{R}^{N \times n} \to \mathbb{R}$ be quasiaffine and not identically constant. Let $\xi \in \mathbb{R}^{N \times n}$ be such that

\[
\nabla \Phi (\xi) = \left( \frac{\partial \Phi}{\partial \xi_j} (\xi) \right)_{1 \leq j \leq n}^{1 \leq i \leq N} \neq 0.
\]

Let $\beta, \gamma \in \mathbb{R}$ and $\lambda \in [0, 1]$ be such that

\[
\Phi (\xi) = \lambda \beta + (1 - \lambda) \gamma.
\]

Then there exist $B, C \in \mathbb{R}^{N \times n}$ such that

\[
\begin{cases}
\xi = \lambda B + (1 - \lambda) C \\
\Phi (B) = \beta, \Phi (C) = \gamma \\
\text{rank} \{B - C\} \leq 1.
\end{cases}
\]

Proof. Since $\nabla \Phi (\xi) \neq 0$, we can find $a \in \mathbb{R}^N, b \in \mathbb{R}^n$ such that

\[
\langle \nabla \Phi (\xi) ; a \otimes b \rangle = \gamma - \beta
\]

where $\langle \cdot ; \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{N \times n}$ and $a \otimes b = (a^i b_j)_{1 \leq i \leq N, 1 \leq j \leq n}$. Define then

\[
\begin{cases}
B := \xi - (1 - \lambda) a \otimes b \\
C := \xi + \lambda a \otimes b.
\end{cases}
\]

In order to obtain the lemma it is therefore sufficient to show that $\Phi (B) = \beta$ and $\Phi (C) = \gamma$. Since $\Phi$ is quasiaffine we have (see Theorem 5.20)

\[
\begin{cases}
\Phi (B) = \Phi (\xi - (1 - \lambda) a \otimes b) = \Phi (\xi) - (1 - \lambda) \langle \nabla \Phi (\xi) ; a \otimes b \rangle = \beta \\
\Phi (C) = \Phi (\xi + \lambda a \otimes b) = \Phi (\xi) + \lambda \langle \nabla \Phi (\xi) ; a \otimes b \rangle = \gamma
\end{cases}
\]

which is indeed the claim.

We may now proceed with the proof of Theorem 6.24.

Proof. It is easy to see that

\[
Rf \geq Qf \geq Pf \geq Cg \circ \Phi.
\]

It therefore remains to show that for every $\xi \in \mathbb{R}^{N \times n}$

\[
Rf (\xi) \leq Cg (\Phi (\xi)). \quad (6.44)
\]

Case 1: $\nabla \Phi (\xi) \neq 0$. Fix $\epsilon > 0$; from Theorem 2.35 we have that there exist $\beta, \gamma \in \mathbb{R}, \lambda \in [0, 1]$ such that

\[
\lambda g (\beta) + (1 - \lambda) g (\gamma) \leq Cg (\Phi (\xi)) + \epsilon
\]

\[
\lambda \beta + (1 - \lambda) \gamma = \Phi (\xi).
\]
Using Lemma 6.25, we have that there exist \( B, C \in \mathbb{R}^{N \times n} \) satisfying the conclusions of the lemma. Using Theorem 6.10, we obtain that

\[
R_f (\xi) \leq \lambda f (B) + (1 - \lambda) f (C) = \lambda g (\beta) + (1 - \lambda) g (\gamma) \leq C g (\Phi (\xi)) + \epsilon.
\]

Since \( \epsilon \) is arbitrary we have indeed obtained (6.44).

**Case 2:** \( \nabla \Phi (\xi) = 0 \). Since \( Rf \) and \( Cg \) are continuous and \( \Phi \) is not identically constant we have (see Theorem 5.20 and Corollary 5.23) that, for every \( \epsilon > 0 \), there exists \( \eta \in \mathbb{R}^{N \times n} \) sufficiently close to \( \xi \) such that \( \nabla \Phi (\eta) \neq 0 \),

\[
C g (\Phi (\eta)) \leq C g (\Phi (\xi)) + \epsilon \quad \text{and} \quad R_f (\xi) \leq R_f (\eta) + \epsilon.
\]

Applying Case 1 to \( \eta \) we find that

\[
R_f (\xi) \leq R_f (\eta) + \epsilon = C g (\Phi (\eta)) + \epsilon \leq C g (\Phi (\xi)) + 2\epsilon.
\]

Since \( \epsilon \) is arbitrary, we have indeed obtained (6.44).

It therefore remains to show that, in general, \( P f > C f \). Choosing, for example, \( N = n \) and

\[
f (\xi) = (\det \xi)^2
\]

we have immediately

\[
R_f (\xi) = Q f (\xi) = P f (\xi) = C g (\det \xi) = f (\xi) > C f (\xi) \equiv 0.
\]

The identity \( C f (\xi) \equiv 0 \) is a consequence of the fact that

\[
0 \leq C f (\xi) \leq \inf \{ \lambda (\det \alpha)^2 + (1 - \lambda) (\det \beta)^2 : \lambda \alpha + (1 - \lambda) \beta = \xi \}
\]

and that the infimum on the right hand side is exactly zero. ■

### 6.6.4 The area type case

The next theorem, established in Dacorogna [176], [179], should be compared with Theorem 5.47.

Recall first the notation (see Example 5.63) that for \( \xi \in \mathbb{R}^{(n+1) \times n} \) we let

\[
\operatorname{adj}_n \xi = \left( \det \hat{\xi}^1, -\det \hat{\xi}^2, \ldots, (-1)^{k+1} \det \hat{\xi}^k, \ldots, (-1)^{n+2} \det \hat{\xi}^{n+1} \right)
\]

where \( \hat{\xi}^k \) is the \( n \times n \) matrix obtained from \( \xi \) by suppressing the \( k \)th row.

**Theorem 6.26** Let \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) be such that there exist \( a \in \mathbb{R}^{n+1} \) and \( b \in \mathbb{R} \) with

\[
g (\delta) \geq \langle a ; \delta \rangle + b, \ \forall \delta \in \mathbb{R}^{n+1}.
\]
Let \( f : \mathbb{R}^{(n+1)\times n} \to \mathbb{R} \) be such that
\[
f(\xi) = g(\text{adj}_n \xi).
\]

Then
\[
Pf = Qf = Rf = Cg \circ \text{adj}_n
\]
and in general
\[
Pf > Cf.
\]

Before proceeding with the proof, we establish a preliminary lemma that is an extension of Lemma 5.49.

**Lemma 6.27** Let \( N = n + 1 \), \( \xi \in \mathbb{R}^{(n+1)\times n} \) and \( \text{adj}_n \xi \neq 0 \). Let \( I \in \mathbb{N} \), \( \lambda_i \geq 0 \) with \( \sum_{i=1}^I \lambda_i = 1 \), \( \beta_i \in \mathbb{R}^{n+1} \) such that
\[
\text{adj}_n \xi = \sum_{i=1}^I \lambda_i \beta_i.
\]

Then there exist \( \xi_i \in \mathbb{R}^{(n+1)\times n} \) such that (see Definition 5.14)
\[
\begin{align*}
\xi &= \sum_{i=1}^I \lambda_i \xi_i, \quad (\lambda_i, \xi_i)_{1 \leq i \leq I} \text{ satisfy } (H_I), \\
\text{adj}_n \xi_i &= \beta_i, \quad 1 \leq i \leq I.
\end{align*}
\]

**Proof.** We proceed by induction on \( I \). The case \( I = 2 \) is precisely Lemma 5.49. Assume therefore that the lemma has been proved up to the order \((I - 1)\) and we wish to prove that it holds for \( I \). We let
\[
\gamma = \frac{1}{1 - \lambda_1} \sum_{i=2}^I \lambda_i \beta_i.
\]

We may assume, upon a possible relabelling, that \( \gamma \neq 0 \). Observe also that we have
\[
\text{adj}_n \xi = \lambda_1 \beta_1 + (1 - \lambda_1) \gamma.
\]

We now apply Lemma 5.49 to \( \beta_1 \) and \( \gamma \) to get \( \xi_1, \eta \in \mathbb{R}^{(n+1)\times n} \) such that
\[
\begin{align*}
\xi &= \lambda_1 \xi_1 + (1 - \lambda_1) \eta, \quad \text{rank} \{ \xi_1 - \eta \} \leq 1 \\
\text{adj}_n \xi_1 &= \beta_1, \quad \text{adj}_n \eta = \gamma.
\end{align*}
\]

(6.45)

We may then use the hypothesis of induction to get that there exist \( \xi_i \in \mathbb{R}^{(n+1)\times n} \) such that
\[
\begin{align*}
\eta &= \sum_{i=2}^I \frac{\lambda_i}{1 - \lambda_1} \xi_i, \quad \left( \frac{\lambda_i}{1 - \lambda_1}, \xi_i \right)_{2 \leq i \leq I} \text{ satisfy } (H_{I-1}) \\
\text{adj}_n \xi_i &= \beta_i, \quad 2 \leq i \leq I.
\end{align*}
\]

(6.46)
Collecting (6.45) and (6.46) we have indeed obtained the lemma. ■

We may now proceed with the proof of Theorem 6.26.

**Proof.** We trivially have that $Rf \geq Qf \geq Pf \geq Cg \circ \text{adj}_n$ and we therefore only need to show that, for every $\xi \in \mathbb{R}^{(n+1) \times n}$,

$$Rf(\xi) \leq Cg(\text{adj}_n \xi).$$  \hfill (6.47)

**Case 1:** $\text{adj}_n \xi \neq 0$. Fix $\epsilon > 0$, from Theorem 2.35 we have that there exist $\beta_i \in \mathbb{R}^{n+1}$, $\lambda_i \geq 0$ with $\sum_{i=1}^{n+2} \lambda_i = 1$ such that

$$\begin{cases}
\sum_{i=1}^{n+2} \lambda_i g(\beta_i) \leq Cg(\text{adj}_n \xi) + \epsilon \\
\sum_{i=1}^{n+2} \lambda_i \beta_i = \text{adj}_n \xi.
\end{cases}$$

We then use Lemma 6.27 to get $\xi_i$ satisfying the conclusions of the lemma. Appealing to Theorem 6.10, we have that

$$Rf(\xi) \leq \sum_{i=1}^{n+2} \lambda_i f(\xi_i) = \sum_{i=1}^{n+2} \lambda_i g(\beta_i) \leq Cg(\text{adj}_n \xi) + \epsilon.$$

The inequality (6.47) follows by letting $\epsilon \to 0$.

**Case 2:** $\text{adj}_n \xi = 0$. Using the continuity of $Rf$ and $Cg$ and an argument similar to that in the proof of Theorem 6.24, we immediately get (6.47).

In order to show that, in general, $Pf > Cf$, we choose

$$f(\xi) = |\text{adj}_n \xi|^2$$

and we get

$$Rf(\xi) = Qf(\xi) = Pf(\xi) = Cg(\text{adj}_n \xi) = f(\xi) > Cf(\xi) \equiv 0.$$

This achieves the proof of the theorem. ■

### 6.6.5 The Kohn-Strang example

We now turn our attention to an important example in optimal design. The following is a result of Kohn and Strang [373], [374] and we only slightly modify their proof.

**Theorem 6.28** Let $N \geq n = 2$ (or $n \geq N = 2$) and, for $\xi \in \mathbb{R}^{N \times 2}$,

$$f(\xi) = \begin{cases}
1 + |\xi|^2 & \text{if } \xi \neq 0 \\
0 & \text{if } \xi = 0,
\end{cases}$$

We now turn our attention to an important example in optimal design. The following is a result of Kohn and Strang [373], [374] and we only slightly modify their proof.

**Theorem 6.28** Let $N \geq n = 2$ (or $n \geq N = 2$) and, for $\xi \in \mathbb{R}^{N \times 2}$,
where $|\cdot|$ denotes the Euclidean norm. If $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is such that

$$
\theta(t) := \begin{cases} 
1 + t^2 & \text{if } t \geq 1 \\
2t & \text{if } 0 \leq t < 1
\end{cases}
$$

then

$$
Cf(\xi) = \theta(|\xi|)
$$

and

$$
Pf(\xi) = Qf(\xi) = Rf(\xi) = \theta((|\xi|^2 + 2|\text{adj}_2 \xi|)^{1/2}) - 2|\text{adj}_2 \xi|.
$$

**Proof.** We first observe that the representation formula for $Cf$ follows at once from Theorem 6.30 below. We therefore only prove the formula for $Pf$, $Qf$ and $Rf$ and to do so we divide the proof into two steps.

**Step 1.** Let $h : \mathbb{R}^{N \times 2} \to \mathbb{R}$ be defined by

$$
h(\xi) := \theta((|\xi|^2 + 2|\text{adj}_2 \xi|)^{1/2}) - 2|\text{adj}_2 \xi|
$$

and let us first show that it is polyconvex. This will be achieved if we can find a convex function

$$
H : \mathbb{R}^{N \times 2} \times \mathbb{R}^{(N)} \to \mathbb{R}, \ H = H(\xi, A),
$$

such that

$$
h(\xi) = H(\xi, \text{adj}_2 \xi).
$$

(6.48)

Let us start with some observations.

(1) Note first that

$$
|\xi|^2 - 2|\text{adj}_2 \xi| \geq 0.
$$

Indeed, since both

$$
\xi \to |\xi|^2 \quad \text{and} \quad \xi \to |\text{adj}_2 \xi|
$$

are $O(N) \times O(2)$-invariant, it is enough to check the inequality on diagonal matrices of the form

$$
\xi = \begin{pmatrix} x & 0 \\
0 & y \\
\vdots & \vdots \\
0 & 0
\end{pmatrix} \in \mathbb{R}^{N \times 2},
$$

in which case it is a trivial inequality, since then

$$
|\xi|^2 = x^2 + y^2 \quad \text{and} \quad |\text{adj}_2 \xi| = |xy|.
$$
(2) Observe next that, if $|\alpha| = 1$ and appealing to the above inequality, we have

$$|\xi|^2 + 2 \langle \alpha; \text{adj}_2 \xi \rangle \geq |\xi|^2 - 2 |\text{adj}_2 \xi| \geq 0,$$

where $\langle ., . \rangle$ denotes the scalar product in $\mathbb{R}^{N/2}$. The function

$$\xi \rightarrow (|\xi|^2 + 2 \langle \alpha; \text{adj}_2 \xi \rangle)^{1/2}$$

is therefore convex (see Corollary 2.53) and thus, since $\theta$ is convex and increasing, we get that

$$\xi \rightarrow \theta((|\xi|^2 + 2 \langle \alpha; \text{adj}_2 \xi \rangle)^{1/2})$$

is also convex.

(3) We then define, for $\alpha \in \mathbb{R}^{N/2}$ with $|\alpha| = 1$, a family of convex (because of the above considerations) functions

$$H_\alpha : \mathbb{R}^{N \times 2} \times \mathbb{R}^{N/2} \to \mathbb{R}, \quad H_\alpha = H_\alpha(\xi, A),$$

by

$$H_\alpha(\xi, A) := \theta((|\xi|^2 + 2 \langle \alpha; \text{adj}_2 \xi \rangle)^{1/2}) - 2 \langle \alpha; A \rangle.$$

(4) Finally, we let

$$H(\xi, A) := \sup \{H_\alpha(\xi, A) : |\alpha| = 1\}.$$

It is clearly a convex function and it therefore only remains to show (6.48). We have to consider two cases. But before that, we should observe that the case $\text{adj}_2 \xi = 0$ is straightforward, since then

$$H_\alpha(\xi, 0) = H(\xi, 0) = \theta(|\xi|) = h(\xi), \quad \forall \alpha \in \mathbb{R}^{N/2}.$$

So, from now on, we will assume that $\text{adj}_2 \xi \neq 0$ (and hence $\xi \neq 0$).

Case 1: $|\xi|^2 + 2 |\text{adj}_2 \xi| \geq 1$. Observe that (recalling that $\theta(t) \leq 1 + t^2$ for every $t \in \mathbb{R}_+$)

$$H_\alpha(\xi, \text{adj}_2 \xi) \leq 1 + |\xi|^2 = h(\xi), \quad \forall \alpha \in \mathbb{R}^{N/2} \text{ with } |\alpha| = 1.$$

Moreover, by choosing

$$\alpha := \frac{\text{adj}_2 \xi}{|\text{adj}_2 \xi|},$$

we have for such $\alpha$ that

$$H_\alpha(\xi, \text{adj}_2 \xi) = 1 + |\xi|^2 = h(\xi).$$

Thus (6.48) holds.
Case 2: $|\xi|^2 + 2|\text{adj}_2 \xi| < 1$. We then have (recall that $\text{adj}_2 \xi \neq 0$)

$$H_\alpha (\xi, \text{adj}_2 \xi) = 2(|\xi|^2 + 2 \langle \alpha; \text{adj}_2 \xi \rangle)^{1/2} - 2 \langle \alpha; \text{adj}_2 \xi \rangle$$

and we have to prove that, for every $\alpha \in \mathbb{R}^{(N_2)}$ with $|\alpha| = 1$, the supremum in $\alpha$ is exactly

$$h (\xi) = 2(|\xi|^2 + 2|\text{adj}_2 \xi|)^{1/2} - 2|\text{adj}_2 \xi|.$$ 

Denoting the Lagrange multiplier by $\lambda$, we find that the stationary points, on $|\alpha| = 1$, of

$$\alpha \rightarrow H_\alpha (\xi, \text{adj}_2 \xi) - \lambda(|\alpha|^2 - 1)$$

satisfy

$$[(|\xi|^2 + 2\langle \alpha; \text{adj}_2 \xi \rangle)^{-1/2} - 1] \text{adj}_2 \xi = \lambda \alpha.$$ 

Multiplying this equation first by $\alpha$, bearing in mind that $|\alpha| = 1$, then by $\text{adj}_2 \xi$, we find

$$\lambda = [(|\xi|^2 + 2\langle \alpha; \text{adj}_2 \xi \rangle)^{-1/2} - 1] \langle \alpha; \text{adj}_2 \xi \rangle \quad \text{and} \quad \alpha = \pm \frac{\text{adj}_2 \xi}{|\text{adj}_2 \xi|}.$$ 

It is easy to see (see below) that the plus sign corresponds to the maximum and the minus sign to the minimum. If this is the case, we have indeed established that

$$H (\xi, \text{adj}_2 \xi) = 2(|\xi|^2 + 2|\text{adj}_2 \xi|)^{1/2} - 2|\text{adj}_2 \xi| = h (\xi)$$ 

as wished. So it only remains to show that

$$(|\xi|^2 + 2|\text{adj}_2 \xi|)^{1/2} - |\text{adj}_2 \xi| \geq (|\xi|^2 - 2|\text{adj}_2 \xi|)^{1/2} + |\text{adj}_2 \xi|$$

whenever $|\xi|^2 + 2|\text{adj}_2 \xi| < 1$. This is equivalent, to showing that

$$(r + 2s)^{1/2} - s \geq (r - 2s)^{1/2} + s \Leftrightarrow s^2 - r + 1 \geq 0$$

whenever $0 \leq 2s \leq r \leq r + 2s < 1$, and this is straightforward.

Step 2. We now prove that

$$Pf = Qf = Pf = h.$$ 

In view of the general results and those of Step 1, we have

$$h \leq Pf \leq Qf \leq Pf \leq f.$$ 

We therefore only have to prove that

$$Rf \leq h.$$

(6.49)
Appealing to Theorem 6.10, we find

\[ R_f (\xi) = \inf \left\{ \sum_{i=1}^{I} \lambda_i f (\xi_i) : \lambda \in \Lambda_I, \sum_{i=1}^{I} \lambda_i \xi_i = \xi, \ (\lambda_i, \xi_i) \text{ satisfy } (H_I) \right\} \]  

(6.50)

where

\[ \Lambda_I = \left\{ \lambda = (\lambda_1, \ldots, \lambda_I) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^{I} \lambda_i = 1 \right\}. \]

Since \( h, f \) and hence \( P_f, Q_f, R_f \) (see Theorem 6.20) are \( O(N) \times O(2) \)-invariant, we can restrict, as in Step 1, our attention to matrices of the form, \( 0 \leq x \leq y, \)

\[
\xi = \begin{pmatrix}
x & 0 \\ 0 & y \\ \vdots & \vdots \\ 0 & 0
\end{pmatrix}.
\]

We also let

\[ g(x, y) := \begin{cases} 
1 + x^2 + y^2 & \text{if } (x, y) \neq 0 \\
0 & \text{if } (x, y) = 0
\end{cases} \]

so that for such a \( \xi \) we have

\[ f(\xi) = g(x, y). \]

Because of the special form of \( \xi \) above, it is clear that for proving (6.49) we can restrict our attention to the case \( n = N = 2 \) and then infer in a straightforward way the general case \( N \geq n = 2 \) (or \( n \geq N = 2 \)).

Observe that if we let, for \( \alpha, \beta \in (0, 1] \) and \( \alpha + \beta \leq 1 \),

\[
\xi_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} (1 - \beta)x/\alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} x & 0 \\ 0 & y/\beta \end{pmatrix}
\]

then, writing \( \lambda_1 = 1 - \alpha - \beta, \ \lambda_2 = \alpha, \ \lambda_3 = \beta, \) we find

\[ \lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_3 = \xi = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \]

and

\[ \det [\xi_1 - \xi_2] = \det [\frac{\lambda_1 \xi_1 + \lambda_2 \xi_2}{\lambda_1 + \lambda_2} - \xi_3] = 0. \]

We therefore obtain

\[ \lambda_1 f(\xi_1) + \lambda_2 f(\xi_2) + \lambda_3 f(\xi_3) = \alpha g((1 - \beta)x/\alpha, 0) + \beta g(x, y/\beta). \]

Returning to (6.50) with the above \( \xi_i \) we find that

\[ R_f (\xi) \leq G(x, y) := \inf_{\alpha, \beta \in [0, 1]} \left\{ \alpha g((1 - \beta)x/\alpha, 0) + \beta g(x, y/\beta) \right\}. \]
It whence remains to show that
\[
G(x, y) \leq \tilde{h}(x, y) := \begin{cases} 
1 + x^2 + y^2 & \text{if } |x| + |y| \geq 1 \\
2(|x| + |y| - |xy|) & \text{if } |x| + |y| < 1.
\end{cases}
\tag{6.51}
\]
and the proof of (6.49) will be complete.

By choosing \(\alpha + \beta = 1\) and then letting \(\beta \to 1\) in the definition of \(G\), it is clear that we always have
\[
G(x, y) \leq 1 + x^2 + y^2.
\]
Since
\[
2(|x| + |y| - |xy|) \leq 1 + x^2 + y^2,
\]
we only need to show (6.51), when \(|x| + |y| < 1\).

The case \((x, y) = (0, 0)\) being trivial, we also assume that \((x, y) \neq (0, 0)\).

Since the function
\[
\alpha \in (0, 1] \to \alpha g((1 - \beta)x/\alpha, 0) + \beta g(x, y/\beta)
\]
is convex (this is obvious if \(\beta = 1\) or \(x = 0\) and is also clear otherwise since then the function equals \(\alpha(1 + (1 - \beta)^2x^2/\alpha^2) + \beta g(x, y/\beta)\)), we see that it attains its minimum (noticing that \(|x| \leq |x| + |y| < 1\)) at
\[
\alpha = |x|(1 - \beta).
\]

We thus deduce that
\[
G(x, y) = \inf_{\beta \in (0, 1]} \{2|x|(1 - \beta) + \beta[1 + x^2 + y^2/\beta^2]\}
= 2|x| + \inf_{\beta \in (0, 1]} \{\beta(1 - |x|)^2 + y^2/\beta\}
= 2(|x| + |y| - |xy|)
\]
as wished. ■

6.6.6 The Saint Venant-Kirchhoff energy function

We now discuss an important function for nonlinear elasticity, namely the Saint Venant-Kirchhoff energy function. Upon rescaling, the function under consideration is, \(\nu \in (0, 1/2)\) being a parameter,
\[
f(\xi) = |\xi^t - I|^2 + \frac{\nu}{1 - 2\nu}(|\xi|^2 - n)^2,
\]
or, in terms of the singular values \(0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)\) of \(\xi \in \mathbb{R}^{n \times n}\),
\[
f(\xi) = \sum_{i=1}^n (\lambda_i^2(\xi) - 1)^2 + \frac{\nu}{1 - 2\nu} \left(\sum_{i=1}^n \lambda_i^2(\xi) - n\right)^2.
\]
It is clearly not a quasiconvex function and we therefore compute its quasiconvex envelope. This was achieved by Le Dret-Raoult [399], [400] when \( n = 3 \) and we now prove their result when \( n = 2 \).

**Theorem 6.29** Let \( f : \mathbb{R}^{2 \times 2} \to \mathbb{R} \) be defined by

\[
f(\xi) = |\xi^t - I|^2 + \frac{\nu}{1 - 2\nu}(|\xi|^2 - 2)^2
= (\lambda_1^2(\xi) - 1)^2 + (\lambda_2^2(\xi) - 1)^2 + \frac{\nu}{1 - 2\nu} (\lambda_1^2(\xi) + \lambda_2^2(\xi) - 2)^2.
\]

Let

\[
g(\xi) := \begin{cases} f(\xi) & \text{if } \xi \notin D_1 \cup D_2 \\ \frac{1}{1 - \nu} (\lambda_2^2(\xi) - 1)^2 & \text{if } \xi \in D_2 \\ 0 & \text{if } \xi \in D_1 \end{cases}
\]

where

\[
D_1 = \left\{ \xi \in \mathbb{R}^{2 \times 2} : (1 - \nu) [\lambda_1(\xi)]^2 + \nu [\lambda_2(\xi)]^2 < 1 \text{ and } \lambda_2(\xi) < 1 \right\}
= \left\{ \xi \in \mathbb{R}^{2 \times 2} : \lambda_1(\xi) \leq \lambda_2(\xi) < 1 \right\}
D_2 = \left\{ \xi \in \mathbb{R}^{2 \times 2} : (1 - \nu) [\lambda_1(\xi)]^2 + \nu [\lambda_2(\xi)]^2 < 1 \text{ and } \lambda_2(\xi) \geq 1 \right\}.
\]

Then

\[
Cf(\xi) = Pf(\xi) = Qf(\xi) = Rf(\xi) = g(\xi).
\]

**Proof.** In view of the general results, we have

\[
0 \leq Cf \leq Pf \leq Qf \leq Rf \leq f.
\]

We therefore only need to show that

\[
Rf(\xi) = Cf(\xi) = g(\xi), \quad \forall \xi \in \mathbb{R}^{2 \times 2}.
\]

Since \( f \) and hence \( Cf, Pf, Qf, Rf \) (see Theorem 6.20) are \( SO(2) \times SO(2) \)-invariant, we can restrict our attention to matrices of the form

\[
\xi = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},
\]

where \( |x| \leq y \).

Recall that, from Theorem 6.10, we have

\[
Rf(\xi) = \inf \left\{ \sum_{i=1}^I \lambda_i f(\xi_i) : \lambda \in \Lambda_I, \sum_{i=1}^I \lambda_i \xi_i = \xi, \ (\lambda_i, \xi_i) \text{ satisfy } (H_I) \right\}.
\]
Before proceeding further, it is convenient to introduce two new functions defined on \( \mathbb{R}^2 \) by

\[
\psi(x, y) := \frac{1}{1-\nu} \left( y^2 - 1 \right) \frac{1}{1-\nu(1-2\nu)} [(1-\nu) (x^2 - 1) + \nu (y^2 - 1)]^2
\]

\[
\varphi(x, y) := \frac{1}{1-\nu} \left( y^2 - 1 \right) \frac{1}{1-\nu(1-2\nu)} [(1-\nu) (x^2 - 1) + \nu (y^2 - 1)]^2
\]

where for \( z \in \mathbb{R} \) we let

\[
[z]^2 = \begin{cases} z^2 & \text{if } z \geq 0 \\ 0 & \text{otherwise.} \end{cases}
\]

For every \( |x| \leq y \), a simple calculation leads to

\[
f \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \psi(x, y) \quad \text{and} \quad g \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \varphi(x, y).
\]

Moreover, the function \( \varphi \) is clearly convex and thus, appealing to Theorem 5.33 (A), we find that \( g \) is convex.

We may now proceed with the proof of the identity (6.52) and we divide it into the study of three cases.

**Case 1:** \( \xi \in \overline{D}_1 \). We first prove that, for \( |x| \leq 1 \), we have

\[
Rf \begin{pmatrix} x & 0 \\ 0 & \pm 1 \end{pmatrix} = 0.
\]

(6.54)

Indeed let \( \lambda = (1+x)/2 \) and

\[
\xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} -1 & 0 \\ 0 & \pm 1 \end{pmatrix}
\]

which implies that

\[
\xi = \lambda \xi_1 + (1-\lambda) \xi_2 \quad \text{and} \quad \det(\xi_1 - \xi_2) = 0.
\]

We therefore have from (6.53) that

\[
0 \leq Cf \begin{pmatrix} x & 0 \\ 0 & \pm 1 \end{pmatrix} \leq Rf \begin{pmatrix} x & 0 \\ 0 & \pm 1 \end{pmatrix}
\]

\[
\leq \lambda f \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} + (1-\lambda) f \begin{pmatrix} -1 & 0 \\ 0 & \pm 1 \end{pmatrix} = 0
\]

as wished.

We now consider the general case \( \xi \in \overline{D}_1 \) and, recalling that \( |x| \leq y \leq 1 \), we let \( \lambda = (1+y)/2 \) and

\[
\xi_1 = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} x & 0 \\ 0 & -1 \end{pmatrix},
\]
which implies that
\[ \xi = \lambda \xi_1 + (1 - \lambda) \xi_2 \quad \text{and} \quad \det (\xi_1 - \xi_2) = 0. \]

We therefore have from (6.53) and (6.54) that
\[
\varphi(x, y) = g(\xi) = 0 \leq Cf \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \leq Rf \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \\
\leq \lambda Rf \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} + (1 - \lambda) Rf \begin{pmatrix} x & 0 \\ 0 & -1 \end{pmatrix} = 0
\]
and hence (6.52) holds and the discussion of Case 1 is complete.

Case 2: \( \xi \in D_2 \). We can then infer that for \( \xi \in D_2 \) we have
\[
\varphi(x, y) = g(\xi) \leq C f(\xi) \leq R f(\xi) \leq f(\xi). \tag{6.55}
\]

Since for \( \xi \in D_2 \) we have
\[ (1 - \nu) x^2 + \nu y^2 < 1 \leq y \]
we can find \(|x| < x_1 < 1\) so that
\[ (1 - \nu) x_1^2 + \nu y^2 = 1. \]

We hence get that
\[
f \left( \pm x_1, 0 \ 0 y \right) = \psi(x_1, y) = \frac{1}{1 - \nu} (y^2 - 1)^2 \\
+ \frac{1}{(1 - \nu)(1 - 2\nu)} [(1 - \nu) (x_1^2 - 1) + \nu (y^2 - 1)]^2 \\
= \frac{1}{1 - \nu} [y^2 - 1]^2 = \varphi(x, y).
\]

Next define \( \lambda = (x + x_1) / 2x_1 \) and
\[ \xi_1 = \begin{pmatrix} x_1 & 0 \\ 0 & y \end{pmatrix}, \xi_2 = \begin{pmatrix} -x_1 & 0 \\ 0 & y \end{pmatrix} \]
which implies that
\[ \xi = \lambda \xi_1 + (1 - \lambda) \xi_2 \quad \text{and} \quad \det (\xi_1 - \xi_2) = 0. \]

We therefore have from (6.53) and (6.55) that
\[
\varphi(x, y) = g(\xi) \leq C f(\xi) \leq R f(\xi) \\
\leq \lambda f \left( \begin{pmatrix} x_1 & 0 \\ 0 & y \end{pmatrix} \right) + (1 - \lambda) f \left( \begin{pmatrix} -x_1 & 0 \\ 0 & y \end{pmatrix} \right) = \varphi(x, y).
\]

Therefore (6.52) holds also in Case 2.
Case 3: $\xi \notin D_1 \cup D_2$. This case is trivial, since we then have
\[ f(\xi) = \varphi(x,y) = g(\xi) \leq Cf(\xi) \leq Rf(\xi) \leq f(\xi) \]
and thus (6.52) is satisfied. ■

6.6.7 The case of a norm

Theorem 6.30 Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}$ with
\[ g(0) = \inf \{ g(x) : x \geq 0 \} \]
be such that
\[ f(\xi) = g(|\xi|), \]
where $|\cdot|$ denotes any norm on $\mathbb{R}^{N \times n}$. Then, for every $\xi \in \mathbb{R}^{N \times n}$,
\[ Cf(\xi) = C g(|\xi|). \]

If $|\cdot|_2$ denotes the Euclidean norm, namely
\[ |\xi|_2 := \left( \sum_{i=1}^N \sum_{j=1}^n (\xi_{ij})^2 \right)^{1/2}, \]
then, in general,
\[ Pf(\xi) > Cf(\xi). \]

If, however, there exists $a \geq 0$ such that
\[ g(a) = g(0) \quad \text{and} \quad C g(x) = g(x) \quad \text{for every} \ x \geq a \]
then
\[ Rf(\xi) = Qf(\xi) = Pf(\xi) = Cf(\xi) = C g(|\xi|_2). \quad (6.56) \]

Remark 6.31 The result of this theorem is surprising when compared with Theorem 5.58, which would have suggested that
\[ Rf = Qf = Pf = Cf. \]

We have already seen in Theorem 6.28 that, in general, $Pf > Cf$. A simpler example is given below. What is even more striking is that there are examples of functions as in the theorem where
\[ Qf > Pf, \]
as shown by Gangbo [300]. ◦

We now give two examples that illustrate the theorem.
Example 6.32  (i) The first one shows that
\[ Rf = Qf = Pf = Cf = Cg. \]

Let
\[ f (\xi) = g (|\xi|_2) = (|\xi|_2^2 - 1)^2. \]

It follows from the theorem that
\[ Rf (\xi) = Qf (\xi) = Pf (\xi) = Cf (\xi) = Cg (|\xi|_2) \]

where
\[ Cg (x) = \begin{cases} 
(x^2 - 1)^2 & \text{if } |x| \geq 1 \\
0 & \text{if } |x| < 1.
\end{cases} \]

(ii) The second one is an example where strict inequality holds between \( Pf \) and \( Cf \). Let \( N = n = 2 \) and \( g : \mathbb{R}_+ \to \mathbb{R} \) be continuous and such that

\[
\begin{aligned}
g (0) &= \inf \{ g (x) : x \geq 0 \} \\
g (x) &\geq a |x|^\alpha, \ a > 0 \text{ and } \alpha > 2 \\
Cg &\text{ strictly increasing and } Cg \neq g
\end{aligned}
\]

and, for \( \xi \in \mathbb{R}^{2\times 2} \), we let
\[ f (\xi) = g (|\xi|_2). \]

One can choose for example
\[ g (x) = \begin{cases}
x & \text{if } x \in [0, 2] \\
-x + 4 & \text{if } x \in [2, 3] \\
x^3 - 26 & \text{if } x \geq 3.
\end{cases} \]

Then
\[ Cg (x) = \begin{cases}
x/3 & \text{if } x \in [0, 3] \\
x^3 - 26 & \text{if } x \geq 3.
\end{cases} \]

It can easily be seen that, for such \( g \), we have \( Pf > Cf = Cg \) and we refer to Dacorogna [176], [179] for a complete discussion of this example.

We may now proceed with the proof of the theorem.

**Proof.** We decompose the proof into three steps.

**Step 1.** We first show that \( Cf = Cg \). Observe first that one always has \( Cg \leq Cf \). We wish to show the reverse inequality. Let \( \epsilon > 0 \) be fixed. Then, from Theorem 2.35, we get that there exist \( \lambda \in [0, 1], b, c \in \mathbb{R}_+ \), such that

\[ \begin{cases}
\epsilon + Cg (|\xi|) \geq \lambda g (b) + (1 - \lambda) g (c) \\
|\xi| = \lambda b + (1 - \lambda) c.
\end{cases} \]
Examples

Then choose (the case $\xi = 0$ is trivial)

$$\beta = \frac{b\xi}{|\xi|}, \quad \gamma = \frac{c\xi}{|\xi|}.$$  

We therefore get

$$\begin{cases} 
\epsilon + Cg(|\xi|) \geq \lambda f(\beta) + (1 - \lambda) f(\gamma) \geq Cf(\xi) \\
\xi = \lambda\beta + (1 - \lambda)\gamma.
\end{cases}$$

Since $\epsilon$ is arbitrary, we have indeed obtained the claimed result.

**Step 2.** As already noted the inequality $Pf > Cf$ has been seen either in Example 6.32 (ii) or in Theorem 6.28.

**Step 3.** It now remains to show (6.56), i.e. that if there exists $a \geq 0$ such that

$$Cg(x) = \begin{cases} 
g(x) & \text{if } x \geq a \\
g(0) = g(a) & \text{if } x < a,
\end{cases}$$

then $Rf = Qf = Pf = Cf = Cg$ (note that the functions $g$ considered in Example 6.32 (ii) or in Theorem 6.28 do not satisfy (6.57)). It is obvious that $Cg(|\xi|_2) \leq Rf(\xi)$, therefore we only need to show that for every $\xi \in \mathbb{R}^{N \times n}$

$$Rf(\xi) \leq Cg(|\xi|_2).$$

(6.58)

Note also that if $|\xi|_2 \geq a$, then (6.58) is trivially satisfied. Therefore we only need to consider the case where $0 < |\xi|_2 < a$. From (6.57), we then obtain

$$Cg(|\xi|_2) = g(a).$$

Let $\xi = (\xi_{ij})_{1 \leq i \leq N, 1 \leq j \leq n}$ (we may assume for notational convenience that $\xi^1 \neq 0$, the general case being handled similarly) and

$$2\lambda := 1 + \frac{|\xi^1|}{(a^2 - |\xi|_2^2 + (\xi^1_1)^2)^{1/2}}.$$  

Then $\frac{1}{2} < \lambda < 1$. Let

$$\eta = (\eta^i_{ij})_{1 \leq i \leq N, 1 \leq j \leq n} \quad \text{with} \quad \eta^i_1 = \frac{2\xi^1_i}{1 - 2\lambda}, \quad \eta^i_j = 0 \quad \text{otherwise}.$$  

Finally let

$$\beta := \xi - (1 - \lambda)\eta \quad \text{and} \quad \gamma := \xi + \lambda\eta.$$  

It is then easy to see that

$$\begin{cases} 
\xi = \lambda\beta + (1 - \lambda)\gamma \\
|\beta|_2 = |\gamma|_2 = a, \quad \text{rank} \{\beta - \gamma\} \leq 1.
\end{cases}$$
From Theorem 6.10 we find that
\[ Rf(\xi) \leq \lambda f(\beta) + (1 - \lambda) f(\gamma) = \lambda g(a) + (1 - \lambda) g(a) = Cg(|\xi|_2), \]
which is precisely (6.58).
Chapter 7

Polyconvex, quasiconvex and rank one convex sets

7.1 Introduction

We now discuss the notions of polyconvex, quasiconvex and rank one convex sets. Contrary to the usual presentation of classical convex analysis, where the notion of a convex set is defined prior to that of a convex function; this is not the case for the generalized notions of convexity. This is of course due to historical reasons. The notions of polyconvex, quasiconvex and rank one convex functions were introduced, as already said, by Morrey in 1952, although the terminology is that of Ball [53]. It was not until the systematic studies of partial differential equations and inclusions by Dacorogna-Marcellini and Müller-Sverak, initiated in 1996 and discussed in Chapter 10, that the equivalent definitions for sets became an important issue. Moreover these notions were essentially seen through the different generalized convex hulls, leading somehow to terminologies that do not exactly cover the same concepts.

We here try to imitate as much as possible the classical approach of convex analysis in the present context. Throughout the two first sections, we follow the presentation of Dacorogna-Ribeiro [213], following earlier results of Dacorogna-Marcellini [202].

In Section 7.2, we define the notions of polyconvex, quasiconvex and rank one convex sets. The first and third ones are straightforward and equivalent, as they should be, to the polyconvexity and rank one convexity of the indicator function. The second one is more delicate. Indeed one would have liked to define it as equivalent to the quasiconvexity of the indicator function, but quasiconvex functions allowed to take the value $+\infty$ are, as we have already seen, poorly understood. We give a definition of quasiconvex set that is compatible with many of the desired properties that should have such a definition. Notably we
have that, for a set $E \subset \mathbb{R}^{N \times n}$,

$$E \text{ convex} \Rightarrow E \text{ polyconvex} \Rightarrow E \text{ quasiconvex} \Rightarrow E \text{ rank one convex}$$

and all counter implications turn out to be false whenever $N, n \geq 2$. This last result is better than the corresponding one for functions, since we have examples of rank one convex functions that are not quasiconvex (see Section 5.3.7) only when $n \geq 2$ and $N \geq 3$.

We then prove separation and Carathéodory type theorems for polyconvex sets.

In Section 7.3, we give the definitions of polyconvex, quasiconvex and rank one convex hulls of a given set $E$ denoted respectively $\text{Pco } E$, $\text{Qco } E$ and $\text{Rco } E$. They are, as they should be, the smallest polyconvex, quasiconvex and rank one convex sets, respectively, that contain $E$. As we already alluded to, these definitions are not exactly the same for all authors working in the field; however, ours is clearly the closest to classical convex analysis.

We also show that if we let

$$\mathcal{F}_\infty^E := \{ f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{ +\infty \} : f|_E \leq 0 \},$$

$$\mathcal{F}^E := \{ f : \mathbb{R}^{N \times n} \to \mathbb{R} : f|_E \leq 0 \},$$

then

$$\text{Pco } E = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every polyconvex } f \in \mathcal{F}_\infty^E \},$$

$$\text{Rco } E = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every rank one convex } f \in \mathcal{F}_\infty^E \},$$

as for the convex case where

$$\text{co } E = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every convex } f \in \mathcal{F}_\infty^E \}. $$

In the convex case, we also have the representation formula for the closure of the convex hull as

$$\overline{\text{co } E} = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every convex } f \in \mathcal{F}^E \}. $$

However, the representation of the closure of the hulls analogous to the above is not true for general sets. We discuss this question in details introducing three more types of hulls, namely

$$\text{Pco}_f E := \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every polyconvex } f \in \mathcal{F}^E \}$$

$$\text{Qco}_f E := \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every quasiconvex } f \in \mathcal{F}^E \}$$

$$\text{Rco}_f E := \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every rank one convex } f \in \mathcal{F}^E \}. $$
It turns out that, in general,
\[ \text{Pco} E \subset \text{Pco}_f E, \quad \text{Qco} E \subset \text{Qco}_f E \quad \text{and} \quad \text{Rco} E \subset \text{Rco}_f E. \]

However, if \( E \) is compact, then, as for the convex case,
\[ \text{Pco} E = \text{Pco}_f E. \]

We end the section by introducing the notion of extreme points in these generalized senses and establish Minkowski type theorems. We moreover define, as for the convex case, the gauge of a polyconvex set and the Choquet function of a polyconvex or a rank one convex set.

In Section 7.4, we consider several sets that are defined in terms of singular values, for example sets of the form
\[ E = \{ \xi \in \mathbb{R}^{n \times n} : \lambda_i(\xi) = \gamma_i, \quad i = 1, \ldots, n \}, \]
where \( 0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi) \) are the singular values of the matrix \( \xi \) and \( 0 < \gamma_1 \leq \cdots \leq \gamma_n \) are given real numbers.

We also study sets such as
\[ E = SO(2)A \cup SO(2)B, \]
where \( A, B \in \mathbb{R}^{2 \times 2} \) are given matrices.

In all these cases, we characterize their convex, polyconvex, quasiconvex and rank one convex hulls.

### 7.2 Polyconvex, quasiconvex and rank one convex sets

#### 7.2.1 Definitions and main properties

We first introduce some notation that is used throughout this chapter.

**Notation 7.1** - For \( s \in \mathbb{N} \), let
\[ \Lambda_s := \{ \lambda = (\lambda_1, \cdots, \lambda_s) : \lambda_i \geq 0, \quad \sum_{i=1}^{s} \lambda_i = 1 \}. \]

- Recall that \( T : \mathbb{R}^{N \times n} \to \mathbb{R}^{\tau(n,N)} \) is such that
\[ T(\xi) := (\xi, \text{adj}_2 \xi, \cdots, \text{adj}_{n\wedge N} \xi), \]
where \( \text{adj}_s \xi \) stands for the matrix of all \( s \times s \) minors of the matrix \( \xi \in \mathbb{R}^{N \times n} \), \( 2 \leq s \leq n \wedge N = \min\{n, N\} \), and
\[ \tau(n, N) := \sum_{s=1}^{n \wedge N} \sigma(s), \quad \text{where} \quad \sigma(s) := \binom{n}{s}, \binom{n}{s}. \]
- $D$ stands for the unit cube $(0,1)^n$ of $\mathbb{R}^n$.
- $\{e_1, \cdots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$.
- $W_{\text{per}}^{1,\infty}(D;\mathbb{R}^N)$ denotes, as usual, the space of periodic functions in $W^{1,\infty}(D;\mathbb{R}^N)$, meaning that $u(x) = u(x + e_i)$ for every $x \in D$ and $i = 1, \cdots, n$.
- $W_{\text{per}}$ denotes the subspace of functions in $W^{1,\infty}_{\text{per}}(D;\mathbb{R}^N)$ and whose gradients take only a finite number of values.

We start by giving the generalized definitions of convexity for sets.

**Definition 7.2** (i) We say that $E \subset \mathbb{R}^m$ is convex if for every $\lambda \in [0,1]$ and $\xi, \eta \in E$, then
\[ \lambda \xi + (1 - \lambda) \eta \in E. \]

(ii) We say that $E \subset \mathbb{R}^{N \times n}$ is polyconvex if there exists a convex set $K \subset \mathbb{R}^{(n,N)}$ such that
\[ \pi(K \cap T(\mathbb{R}^{N \times n})) = E, \]
where $\pi$ (see below) denotes the orthogonal projection of (the first $N \times n$ components of) $\mathbb{R}^{(n,N)}$ in $\mathbb{R}^{N \times n}$. Equivalently, $E$ is polyconvex if there exists a convex set $K \subset \mathbb{R}^{(n,N)}$ such that
\[ \{ \xi \in \mathbb{R}^{N \times n} : T(\xi) \in K \} = E. \]

(iii) We say that $E \subset \mathbb{R}^{N \times n}$ is quasiconvex if we have
\[ \xi + \nabla \varphi(x) R \in E, \text{ a.e. } x \in D, \]
for some $R \in O(n)$ and some $\varphi \in W_{\text{per}}$ \Rightarrow \xi \in E.

(iv) We say that $E \subset \mathbb{R}^{N \times n}$ is rank one convex if for every $\lambda \in [0,1]$ and $\xi, \eta \in E$ such that $\text{rank} \{\xi - \eta\} = 1$, then
\[ \lambda \xi + (1 - \lambda) \eta \in E. \]

(v) We say that $E \subset \mathbb{R}^m$ is separately convex if for every $\lambda \in [0,1]$ and $\xi, \eta \in E$ such that $\xi - \eta = se_i$, for some $s \in \mathbb{R}$ and $i \in \{1, \cdots, m\}$, then
\[ \lambda \xi + (1 - \lambda) \eta \in E. \]

**Remark 7.3** (i) The operator $\pi$ introduced in the above definition is more precisely defined as follows. If
\[ X = (X_1, \cdots, X_{\tau(n,N)}) \Rightarrow \pi(X) = (X_1, \cdots, X_{N \times n}), \]
this implies that

\[ \pi(T(\xi)) = \xi, \quad \forall \xi \in \mathbb{R}^{N \times n}. \]

In particular, if \( N = n = 2 \) and \( X = (\xi, \delta) \in \mathbb{R}^{2 \times 2} \times \mathbb{R} \), then \( \pi(X) = \xi \).

(ii) The definitions of convex, rank one convex and separately convex sets are standard.

(iii) The usual way to define polyconvexity, is with the condition (ii) in Theorem 7.4 below. However, the two conditions turn out to be equivalent. With our definition, given in Dacorogna-Ribeiro [213], we get some coherence with the analogous notion for functions.

We note that one could think that if a set \( E \) is polyconvex, then \( T(E) \) is convex. This, however, is not true. Consider, for example, the polyconvex set \( E = \{I, \xi\} \), where \( I \) is the identity matrix and \( \xi = \text{diag}(2, 0) \). Then

\[ T(E) = \{(I, 1), (\xi, 0)\}, \]

which is not convex.

(iv) The best definition for quasiconvex sets is unclear. Several definitions have already been considered (see Dacorogna-Marcellini [202], Müller [462], Zhang [616]). The one we propose here, following Dacorogna-Ribeiro [213], is consistent with known properties for functions and have most properties which are desirable (see Theorem 7.7 below).

(v) One can replace \( \mathcal{W}_{\text{per}} \) by \( W^{1, \infty}_{\text{per}}(D; \mathbb{R}^N) \) in the definition of quasiconvex sets and keep the main result (Theorem 7.7) still valid. However the definition given above is more convenient for Theorem 7.16.

We first give an equivalent condition for polyconvexity.

**Theorem 7.4** Let \( E \subset \mathbb{R}^{N \times n} \). The following conditions are equivalent.

(i) \( E \) is polyconvex.

(ii) \[
\sum_{i=1}^{I} \lambda_i T(\xi_i) = T(\sum_{i=1}^{I} \lambda_i \xi_i) \quad \Rightarrow \quad \sum_{i=1}^{I} \lambda_i \xi_i \in E.
\]

Moreover one can take \( I = \tau(n, N) + 1 \).

(iii) Denoting by \( \text{co} T(E) \) the convex hull of \( T(E) \), then

\[ E = \pi(\text{co} T(E) \cap T(\mathbb{R}^{N \times n})) \]

or equivalently

\[ E = \{\xi \in \mathbb{R}^{N \times n} : T(\xi) \in \text{co} T(E)\}. \]

**Proof.** (i) \( \Rightarrow \) (ii). Suppose

\[ \sum_{i=1}^{I} \lambda_i T(\xi_i) = T(\sum_{i=1}^{I} \lambda_i \xi_i), \quad (7.1) \]
for some $\xi_i \in E$ and $(\lambda_1, \cdots, \lambda_I) \in \Lambda_I$. By hypothesis, $\xi_i \in \pi(K \cap T(\mathbb{R}^{N \times n}))$ for some convex set $K \subset \mathbb{R}^{\tau(n,N)}$ and so $T(\xi_i) \in K$. Therefore

$$\sum_{i=1}^{I} \lambda_i T(\xi_i) \in \text{co} K = K$$

and, by (7.1), we conclude that $\sum_{i=1}^{I} \lambda_i \xi_i \in E$.

The fact that we can take $I = \tau(n,N) + 1$ in (ii) is a consequence of Carathéodory theorem (see Theorem 5.6).

(ii) $\Rightarrow$ (iii). We have to see that $E = \pi(\text{co} T(E) \cap T(\mathbb{R}^{N \times n}))$. Evidently $E$ is contained in the set on the right hand side. For the reverse inclusion, consider $\xi \in \pi(\text{co} T(E) \cap T(\mathbb{R}^{N \times n}))$. So, $T(\xi) \in \text{co} T(E)$ and we can write

$$T(\xi) = \sum_{i=1}^{I} \lambda_i T(\xi_i)$$

for some $\xi_i \in E$ and $(\lambda_1, \cdots, \lambda_I) \in \Lambda_I$. We then use (ii) to get that $\xi \in E$, as wished.

(iii) $\Rightarrow$ (i). This is immediate. $\blacksquare$

The next result, whose proof is straightforward, shows the relation between the notions of convexity for sets and the corresponding notions for functions.

**Proposition 7.5** Let $E \subset \mathbb{R}^{N \times n}$ and $\chi_E$ denote the indicator function of $E$:

$$\chi_E(\xi) = \begin{cases} 0 & \text{if } \xi \in E \\ +\infty & \text{if } \xi \notin E. \end{cases}$$

Then $E$ is, respectively, convex, polyconvex, rank one convex or separately convex if and only if $\chi_E$ is, respectively, convex, polyconvex, rank one convex or separately convex.

**Remark 7.6** One would have liked to have the same result for quasiconvex sets but, as already discussed, quasiconvex functions taking the value $+\infty$ are not considered here. $\diamond$

The convexity conditions are related in the following way (see [213]).

**Theorem 7.7** Let $E \subset \mathbb{R}^{N \times n}$. The following implications then hold

$$E \text{ convex } \Rightarrow E \text{ polyconvex } \Rightarrow E \text{ quasiconvex}$$

$$\Rightarrow E \text{ rank one convex } \Rightarrow E \text{ separately convex.}$$

All counter implications are false as soon as $N, n \geq 2$.

**Remark 7.8** We will see (see Proposition 7.24) that, as for the convex case: $E$, respectively, polyconvex, quasiconvex, rank one convex or separately convex implies that int $\overline{E}$ is also, respectively, polyconvex, quasiconvex, rank one convex or separately convex. However, this is not true anymore for $\overline{E}$. Indeed we will give (see Proposition 7.24) an example of a bounded polyconvex set $E \subset \mathbb{R}^{2 \times 2}$ with $\overline{E}$ not even separately convex. $\diamond$
Proof. Part 1. We only prove the implications related to the notion of quasiconvexity since the others are trivial.

(i) We prove that if $E$ is polyconvex then $E$ is quasiconvex. Assume that

$$\xi + \nabla \varphi(x)R \in E, \text{ a.e. } x \in D$$

for some $R \in O(n)$ and $\varphi \in \mathcal{W}_{per}$. We can write

$$\nabla \varphi(x)R \in \{\eta_1, \cdots, \eta_k\}, \text{ a.e. } x \in D$$

for some $\eta_i$ such that $\xi + \eta_i \in E$, $i = 1, \cdots, k$. Defining

$$\lambda_i := \text{meas}\{x \in D : \nabla \varphi(x)R = \eta_i\},$$

we have $\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1$. Since $\varphi$ is periodic and the functions $\text{adj}_s$ are quasiaffine ($s = 1, \cdots, N \land n$) we have

$$T(\xi) = \int_D T(\xi + \nabla \varphi(x)R) \, dx = \sum_{i=1}^k \lambda_i T(\xi + \eta_i).$$

Using the polyconvexity of the set $E$ and Theorem 7.4, we obtain that $\xi \in E$.

(ii) We now prove that if a set $E$ is quasiconvex then it is rank one convex. Let $\xi, \eta \in E$ be such that $\text{rank} \{\xi - \eta\} = 1$ and $\lambda \in (0, 1)$. We have to show that $\lambda \xi + (1 - \lambda) \eta \in E$. To achieve this, it is enough to find $R \in O(n)$ and $\varphi \in \mathcal{W}_{per}$ such that

$$\lambda \xi + (1 - \lambda) \eta + \nabla \varphi(x)R \in \{\xi, \eta\}, \text{ a.e. } x \in D$$

or equivalently

$$\nabla \varphi(x)R \in \{(1 - \lambda)(\xi - \eta), -\lambda(\xi - \eta)\}, \text{ a.e. } x \in D.$$

The result then follows from the quasiconvexity of $E$. The construction of such $\varphi$ is standard for relaxation theorems (see, for a more sophisticated version, Lemma 3.11) and we now give the proof. Since $\text{rank} \{\xi - \eta\} = 1$, we can write

$$\xi - \eta = a \otimes \nu$$

with $a \in \mathbb{R}^N$ and $\nu$ a unit vector in $\mathbb{R}^n$. Choose $R \in O(n)$ an orthogonal transformation such that $e_1 R = \nu$. Let $h : \mathbb{R} \to \mathbb{R}$ be periodic, of period 1, and such that

$$h(s) := \begin{cases} 
(1 - \lambda)s & \text{if } 0 \leq s \leq \lambda \\
-\lambda(s - 1) & \text{if } \lambda < s \leq 1 
\end{cases}$$

and define $\varphi \in \mathcal{W}_{per}$ as

$$\varphi(x) := h(x_1) a \Rightarrow \nabla \varphi(x) = h'(x_1) a \otimes e_1 \Rightarrow \nabla \varphi(x) R = h'(x_1) a \otimes \nu.$$
It clearly satisfies the claim and this finishes the proof.

Part 2. We next see that the reverse implications are, in general, not true.

(i) Polyconvexity $\Rightarrow$ convexity. Consider, for example, the set $E = \{\xi, \eta\} \subset \mathbb{R}^{2 \times 2}$, where $\xi = \text{diag}(1, 0)$ and $\eta = \text{diag}(0, 1)$. The set $E$ is polyconvex but not convex.

(ii) Quasiconvexity $\Rightarrow$ polyconvexity. Consider the matrices (cf. Proposition 5.10)

$$
\xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}
$$

and

$$
\eta = \begin{pmatrix} 0 & 0 \\ 2/3 & 1/3 \end{pmatrix} = \frac{1}{3} \xi_1 + \frac{1}{3} \xi_2 + \frac{1}{3} \xi_3.
$$

We have

$$
T(\eta) = \frac{1}{3} T(\xi_1) + \frac{1}{3} T(\xi_2) + \frac{1}{3} T(\xi_3).
$$

The set $E = \{\xi_1, \xi_2, \xi_3\}$ is not a polyconvex set since $\eta \notin E$. However, it is quasiconvex. Suppose $\xi + \nabla \varphi R \in E$ for some $\varphi \in W_{per}$ where $R \in O(2)$. Since $
\text{rank } \{\xi_i - \xi_j\} = 2$ for $i \neq j$, we have from Theorem 7.11 (with $m = 3$) that there exists $\xi_i \in E$ such that

$$
\xi + \nabla \varphi(x) R = \xi_i, \text{ a.e. } x \in D.
$$

Using then the periodicity of $\varphi$, we find

$$
\xi = \int_D (\xi + \nabla \varphi(x) R) \, dx = \xi_i
$$

and thus $\xi = \xi_i \in E$. We then conclude that $E$ is quasiconvex.

(iii) Rank one convexity $\Rightarrow$ quasiconvexity. We should again draw the attention to the fact that our result is better for sets than for functions. We prove this assertion in two steps.

Step 1. From Theorem 7.12, we can find $\{\xi_1, \ldots, \xi_m\} \subset \mathbb{R}^{2 \times 2}$ with

$$
\text{rank } \{\xi_i - \xi_j\} = 2, \forall i \neq j,
$$

$\xi_0 \notin \{\xi_1, \ldots, \xi_m\}$ and $u \in u_{\xi_0} + W^{1, \infty}_0(D; \mathbb{R}^2)$ such that

$$
\nabla u(x) \in \{\xi_1, \ldots, \xi_m\}, \text{ a.e. } x \in D
$$

where $u_{\xi_0}(x) = \xi_0 x$.

Step 2. Let $E = \{\xi_1, \ldots, \xi_m\}$. Since there are no rank one connections between the matrices $\xi_i$, the set $E$ is rank one convex. We now see that $E$ is not quasiconvex. Let $u$ be as in Step 1 and write

$$
\varphi := u - u_{\xi_0}.
$$
We therefore have $\nabla u(x) = \xi_0 + \nabla \varphi(x) \in E$, a.e. in $D$, with
$$\varphi \in W^{1,\infty}_0(D; \mathbb{R}^2) \cap \mathcal{W}_{per}$$
but, by construction, $\xi_0 \notin E$, and thus $E$ is not quasiconvex.

(iv) Separate convexity $\nRightarrow$ rank one convexity. Indeed, the set $E = \{\xi, \eta\} \subset \mathbb{R}^{2 \times 2}$, where
$$\xi = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
is separately convex but not rank one convex. ■

### 7.2.2 Separation theorems for polyconvex sets

We next deal with the problem of separating polyconvex sets generalizing in this way known results in the convex context (see Theorem 2.10 and Corollary 2.11).

**Theorem 7.9** Let $E$ be a polyconvex set of $\mathbb{R}^{N \times n}$.

(i) If $\eta \notin E$ or $\eta \in \partial E$, then there exists $\beta \in \mathbb{R}^{\tau(n,N)}$, $\beta \neq 0$, such that
$$\langle \beta; T(\eta) - T(\xi) \rangle \leq 0, \quad \forall \xi \in E.\]$$

(ii) If $E$ is compact and $\eta \notin E$, then there exists $\beta \in \mathbb{R}^{\tau(n,N)}$, $\beta \neq 0$, such that
$$\langle \beta; T(\eta) \rangle < \inf_{\xi \in E} \{\langle \beta; T(\xi) \rangle\}.\]$$

**Proof.** (i) Since $E$ is polyconvex, if $\eta \notin E$ then (see Theorem 7.4 (iii)) $T(\eta) \notin \text{co} T(E)$; in the case $\eta \in \partial E$ then we get $T(\eta) \in \partial \text{co} T(E)$. In both cases, using the separation theorem for convex sets (see Theorem 2.10 and Corollary 2.11) we obtain the existence of $\beta \neq 0$ satisfying
$$\langle \beta; T(\eta) - X \rangle \leq 0, \quad \forall X \in \text{co} T(E),$$
and, in particular, for $X \in T(E)$ as desired.

(ii) Since $E$ is compact, then $T(E)$ is compact and so is $\text{co} T(E)$ (see Theorem 2.14 (i)). We may then apply Theorem 2.10 (iii) to get the result. ■

As a consequence of the previous separation theorem we have the characterization of a polyconvex set given in the following result. This is an extension of Theorem 2.10 (iv) for convex sets that ensures that a closed convex set is the intersection of the closed half spaces containing the set.

**Theorem 7.10** A compact set $E \subset \mathbb{R}^{N \times n}$ is polyconvex if and only if
$$E = \{\xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every quasiaffine } f \in \mathcal{F}^E\},$$
where

\[ \mathcal{F}^E := \{ f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} : f|_E \leq 0 \} . \]

**Proof.** Call

\[ X := \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0, \text{ for every quasiaffine } f \in \mathcal{F}^E \} . \]

(⇒) The fact that \( E \subset X \) is obvious, so let us show the reverse inclusion. Let \( \xi_0 \in X \) and let us prove that \( \xi_0 \in E \). If this was not the case, then, from Theorem 7.9 (ii), there exist \( \beta \in \mathbb{R}^{\tau(n,N)} - \{0\} \) and \( \epsilon > 0 \) such that

\[ \langle \beta; T(\xi_0) \rangle < -\epsilon + \inf_{\xi \in E} \{ \langle \beta; T(\xi) \rangle \} . \]

Define then

\[ f(\xi) := -\langle \beta; T(\xi) \rangle + \epsilon + \langle \beta; T(\xi_0) \rangle . \]

Observe that \( f \) is quasiaffine and \( f \in \mathcal{F}^E \), although \( f(\xi_0) = \epsilon > 0 \). Therefore \( \xi_0 \notin X \), which is a contradiction.

(⇐) Since \( X \) is clearly polyconvex and \( E = X \), we have the claim. ■

### 7.2.3 Appendix: functions with finitely many gradients

We here gather some results, without proofs, that are used throughout the chapter.

**Theorem 7.11** Let \( n, N \geq 2, D = (0,1)^n \subset \mathbb{R}^n \) and

\[ E = \{ \xi_1, \cdots, \xi_m \} \subset \mathbb{R}^{N \times n} \text{ with } \text{rank} \{ \xi_i - \xi_j \} \geq 2, \forall i \neq j. \]

Let \( u \in W^{1,\infty}(D;\mathbb{R}^N) \) be such that

\[ \nabla u(x) \in E, \text{ a.e. } x \in D. \]

If \( m \leq 4 \), then there exists \( \xi_i \in E \) such that

\[ \nabla u(x) = \xi_i, \text{ a.e. } x \in D. \]

The theorem was established by Ball-James [60] when \( m = 2 \), by Sverak [550], [552] and Zhang [619] when \( m = 3 \) and by Chlebik-Kirchheim [150] when \( m = 4 \).

The result is false as soon as \( m \geq 5 \) (and \( n = N = 2 \)), as was shown by Kirchheim-Preiss [365]. In the same spirit, we now quote a theorem of Kirchheim [364].
Theorem 7.12 Let \( n, N \geq 2 \) and \( D = (0,1)^n \subset \mathbb{R}^n \). Then there exist
\[
\{\xi_1, \cdots, \xi_m\} \subset \mathbb{R}^{N\times n}_{\xi} \text{ with } \operatorname{rank}\{\xi_i - \xi_j\} = n \wedge N, \forall i \neq j,
\]
\( \xi_0 \notin \{\xi_1, \cdots, \xi_m\} \) and \( u \in u_{\xi_0} + W^{1,\infty}_0(D;\mathbb{R}^N) \) such that
\[
\nabla u(x) \in \{\xi_1, \cdots, \xi_m\}, \text{ a.e. } x \in D
\]
where \( u_{\xi_0}(x) = \xi_0 x \).

7.3 The different types of convex hulls

7.3.1 The different convex hulls

Having defined the generalized notions of convexity, we are now in a position to introduce the concept of generalized convex hulls. We follow the same procedure as in the classical convex case.

Definition 7.13 The polyconvex, quasiconvex, rank one convex and separately convex hulls of a set \( E \subset \mathbb{R}^{N\times n} \) are, respectively, the smallest polyconvex, quasiconvex, rank one convex and separately convex sets containing \( E \) and are respectively denoted by \( \text{Pco } E \), \( \text{Qco } E \), \( \text{Rco } E \) and \( \text{Sco } E \).

From the discussion made in Section 7.2.1, the following inclusions hold:
\[
E \subset \text{Sco } E \subset \text{Rco } E \subset \text{Qco } E \subset \text{Pco } E \subset \text{co } E.
\]

As we note below (see Remark 7.26), there are some authors who have adopted other definitions for the rank one convex hull, but this one is more consistent with the convex case.

In the following (see Dacorogna-Marcellini [202] and Dacorogna-Ribeiro [213]) we give some representations of the hulls defined above. We start by giving two characterizations of the polyconvex hull of a set. The second one is a consequence of Carathéodory theorem and is equivalent to that obtained in the convex case.

Theorem 7.14 Let \( E \subset \mathbb{R}^{N\times n} \). Then the following two representation formulas hold (recall that \( \tau = \tau(n,N) \); see Notation 7.1):
\[
\text{Pco } E = \pi(\text{co } T(E) \cap T(\mathbb{R}^{N\times n})),
\]
\[
\text{Pco } E = \left\{ \xi \in \mathbb{R}^{N\times n} : T(\xi) = \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i), \xi_i \in E, \lambda \in \Lambda_{\tau+1} \right\}.
\]
In particular, if \( E \) is compact, then \( \text{Pco } E \) is also compact and if \( E \) is open, then \( \text{Pco } E \) is also open.
Proof. (i) We prove the first representation of $\text{Pco } E$. It is clear, from Theorem 7.4 (iii), that $\text{Pco } E \subset \pi(\text{co } T(E) \cap T(\mathbb{R}^{N \times n}))$. For the other inclusion, we start by noting that, since $\text{Pco } E$ is polyconvex, by definition,

$$\text{Pco } E = \pi(K \cap T(\mathbb{R}^{N \times n}))$$

for some convex set $K \subset \mathbb{R}^{{\tau(n,N)}}$. Since $E \subset \text{Pco } E$, $K$ must contain $T(E)$ and, consequently, must contain $\text{co } T(E)$, thus the desired inclusion follows.

(ii) Let

$$Y := \{\xi \in \mathbb{R}^{N \times n} : T(\xi) = \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i), \xi_i \in E, \lambda \in \Lambda_{\tau+1}\}.$$ 

Let $\xi \in Y$, then there exist $\xi_i \in E \subset \text{Pco } E$ and $\lambda \in \Lambda_{\tau+1}$ such that

$$T(\xi) = \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i).$$

We therefore deduce, from Theorem 7.4 (ii), that $\xi \in \text{Pco } E$.

The reverse inclusion follows from the fact that $E \subset Y$ and that $Y$ is easily seen to be polyconvex (as in Theorems 5.6 and 6.8) and thus $\text{Pco } E \subset Y$.

(iii) Let $E$ be compact, then $\text{Pco } E$ is trivially bounded, so let us show that it is also closed. Then let $\xi_\nu \in \text{Pco } E$ with $\xi_\nu \to \xi$. By the first representation of $\text{Pco } E$, $T(\xi_\nu) \in \text{co } T(E)$, which is a compact set since $T(E)$ is compact and by Theorem 2.14 (i). Then $T(\xi) = \lim T(\xi_\nu) \in \text{co } T(E)$ and thus $\xi \in \text{Pco } E$ as wished.

(iv) Assume (see below) that we have shown that for every $\xi, \xi_i \in \mathbb{R}^{N \times n}$ and $\lambda \in \Lambda_{\tau+1}$, such that

$$T(\xi) = \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i),$$

then

$$T(\xi + \eta) = \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i + \eta), \forall \eta \in \mathbb{R}^{N \times n}. \quad (7.2)$$

From this and (ii), it easily follows that $\text{Pco } E$ is open if $E$ is open.

So it remains to show (7.2). Since $T(\xi) = (\xi, \text{adj}_2 \xi, \cdots, \text{adj}_{n \wedge N} \xi)$, (7.2) is proved if we can show that, for every $1 \leq s \leq n \wedge N$,

$$\text{adj}_r \xi = \sum_{i=1}^{\tau+1} \lambda_i \text{adj}_r \xi_i, \forall 1 \leq r \leq s \Rightarrow \text{adj}_s (\xi + \eta) = \sum_{i=1}^{\tau+1} \lambda_i \text{adj}_s (\xi_i + \eta). \quad (7.3)$$

We prove the claim by induction on $s$. The result is trivially true when $s = 1$, so assume that it has been proved up to $s - 1$ and let us show it for $s$. Since $\text{adj}_s$ consists of $s \times s$ determinants of the matrix $\xi \in \mathbb{R}^{N \times n}$, we find that (7.3)
is proved if we can show that, for \( \alpha, \alpha_i, \beta \in \mathbb{R}^{s \times s} \) and \( \lambda \in \Lambda_{\tau+1} \), then
\[
\text{adj}_r \alpha = \sum_{i=1}^{\tau+1} \lambda_i \text{adj}_r \alpha_i, \quad \forall 1 \leq r \leq s \Rightarrow \det (\alpha + \beta) = \sum_{i=1}^{\tau+1} \lambda_i \det (\alpha_i + \beta).
\]
(7.4)

We use Proposition 5.67 and the hypothesis in (7.4) to get the claim, namely
\[
\sum_{i=1}^{\tau+1} \lambda_i \det (\alpha_i + \beta) = \sum_{i=1}^{\tau+1} \lambda_i \sum_{(I,J) \in \mathcal{N}_{1, \ldots, s}} \det (\alpha^I_i, \beta^J_i)
\]
\[
= \sum_{i=1}^{\tau+1} \lambda_i \det \alpha_i + \sum_{i=1}^{\tau+1} \lambda_i \sum_{(I,J) \in \mathcal{N}_{1, \ldots, s} : J \neq \emptyset} \det (\alpha^I_i, \beta^J_i)
\]
\[
= \det \alpha + \sum_{(I,J) \in \mathcal{N}_{1, \ldots, s} : J \neq \emptyset} \det (\alpha^I_i, \beta^J_i) = \det (\alpha + \beta).
\]

This proves the theorem. ■

We now give a different representation of the polyconvex hull using the separation results of Section 7.2.2.

**Theorem 7.15** Let \( E \subset \mathbb{R}^{N \times n} \) be such that \( \text{Pco} \, E \) is compact. Then
\[
\text{Pco} \, E = \{ \xi \in \mathbb{R}^{N \times n} : f (\xi) \leq 0 \text{ for every quasiaffine } f \in \mathcal{F}^E \},
\]
where
\[
\mathcal{F}^E := \{ f : \mathbb{R}^{N \times n} \to \mathbb{R} : f|_E \leq 0 \}.
\]

**Proof.** Let
\[
X := \{ \xi \in \mathbb{R}^{N \times n} : f (\xi) \leq 0, \text{ for every quasiaffine } f \in \mathcal{F}^E \}
\]
The set \( X \) is clearly polyconvex and contains \( E \), thus \( \text{Pco} \, E \subset X \).

On the other hand, since \( \text{Pco} \, E \) is polyconvex and compact then, by Theorem 7.10, we have
\[
\text{Pco} \, E = \{ \xi \in \mathbb{R}^{N \times n} : f (\xi) \leq 0, \text{ for every quasiaffine } f \in \mathcal{F}^{\text{Pco} \, E} \}.
\]
Since \( \mathcal{F}^{\text{Pco} \, E} \subset \mathcal{F}^E \), we get \( X \subset \text{Pco} \, E \), as claimed. ■

We next give a representation for the quasiconvex hull similar to the second representation formula of Theorem 7.14. This representation is, however, weaker than the one obtained in the polyconvex case since we cannot obtain the representation formula in a prescribed finite number of steps.
Theorem 7.16 Let $E \subset \mathbb{R}^{N \times n}$. Let $Q_0 \co E = E$ and define by induction the sets

$$Q_{i+1} \co E = \left\{ \xi \in \mathbb{R}^{N \times n} : \exists \ R \in O(n), \ \varphi \in W_{\text{per}} \text{ such that } \xi + \nabla \varphi(x)R \in Q_i \co E, \ a.e. \ x \in D \right\}, \ i \geq 0.$$ 

Then $Q \co E = \bigcup_{i \in \mathbb{N}} Q_i \co E$. In particular, if $E$ is open, then $Q \co E$ is also open.

Proof. (i) We first show that

$$\bigcup_{i \in \mathbb{N}} Q_i \co E \subset Q \co E.$$ 

It is sufficient to show that $Q_i \co E \subset Q \co E$, for every $i$. We proceed by induction; the result is, by definition, true for $i = 0$. Since $Q \co E$ is quasiconvex, we have, by definition of quasiconvex sets and by induction, that if $Q_i \co E \subset Q \co E$, then $Q_{i+1} \co E \subset Q \co E$. This proves the claim.

The reverse inclusion follows at once from the fact that $\bigcup_{i \in \mathbb{N}} Q_i \co E$ is, as we now see, a quasiconvex set containing $E$. Let $R \in O(n), \ \varphi \in W_{\text{per}}$ and $\xi + \nabla \varphi(x)R \in \bigcup_{i \in \mathbb{N}} Q_i \co E$, a.e. $x \in D$. One has

$$\nabla \varphi(x)R \in \{ \eta_1, \cdots, \eta_k \} \ a.e. \ x \in D,$$

with

$$\text{meas}\{x \in D : \ \nabla \varphi(x)R = \eta_j\} > 0, \ j = 1, \cdots, k.$$ 

Moreover, $\xi + \eta_j \in Q_{\alpha(j)} \co E$ for some $\alpha(j) \in \mathbb{N}$. Let $s = \max\{\alpha(1), \cdots, \alpha(k)\}$. Since $Q_i \co E \subset Q_{i+1} \co E$, we have, for all $j = 1, \cdots, k$, $\xi + \eta_j \in Q_s \co E$. Thus $\xi + \nabla \varphi(x)R \in Q_s \co E$ and, by definition, we get

$$\xi \in Q_{s+1} \co E \subset \bigcup_{i \in \mathbb{N}} Q_i \co E;$$

thus the quasiconvexity of $\bigcup_{i \in \mathbb{N}} Q_i \co E$ has been proved.

(ii) Since $E$ is open, one easily gets, using an induction argument, that each $Q_i \co E$ is open. By the preceding representation of $Q \co E$ it follows that this set is also open. ■

The analogous representation for the rank one convex hull of a set is given in the result below (see also Dacorogna-Marcellini [202]).

Theorem 7.17 Let $E \subset \mathbb{R}^{N \times n}$. Let $R_0 \co E = E$ and define by induction the sets

$$R_{i+1} \co E = \left\{ \xi \in \mathbb{R}^{N \times n} : \ \xi = \lambda \xi_1 + (1 - \lambda) \xi_2, \ \lambda \in [0, 1], \ \xi_1, \xi_2 \in R_i \co E, \ \text{rank} \{\xi_1 - \xi_2\} \leq 1 \right\}, \ i \geq 0.$$ 

Then $R \co E = \bigcup_{i \in \mathbb{N}} R_i \co E$. In particular, if $E$ is open, then $R \co E$ is also open.
Remark 7.18 (i) A similar construction and results can be obtained for \( \text{Sco} E \).

(ii) In general it is not true that rank one convex hulls or separately convex hulls of compact sets are compact (see Aumann-Hart \[50\] and Kolar \[376\]). ♦

Proof. (i) A straightforward induction leads to \( R_i \text{co} E \subset \text{co} E \) and thus \( \bigcup R_i \text{co} E \subset \text{co} E \). We now show the reverse inclusion. Observe that, by definition,

\[
E \subset \bigcup R_i \text{co} E.
\]

If we can show that \( \bigcup R_i \text{co} E \) is rank one convex, we will have the claim, namely

\[
\text{co} E \subset \bigcup R_i \text{co} E.
\]

So let us show that \( \bigcup R_i \text{co} E \) is rank one convex. Let \( \lambda \in [0, 1] \) and

\[
\xi, \eta \in \bigcup R_i \text{co} E \quad \text{with} \quad \text{rank} \{\xi - \eta\} = 1
\]

and let us prove that

\[
\lambda \xi + (1 - \lambda) \eta \in \bigcup R_i \text{co} E.
\]

By definition there exist \( i, j \in \mathbb{N} \), for notational convenience assume that \( i \geq j \), such that

\[
\xi \in R_i \text{co} E, \quad \eta \in R_j \text{co} E \subset R_i \text{co} E.
\]

We therefore deduce the result, namely

\[
\lambda \xi + (1 - \lambda) \eta \in R_{i+1} \text{co} E \subset \bigcup R_i \text{co} E.
\]

(ii) It is easy to see, by an induction argument that every \( R_i \text{co} E \) is open, provided \( E \) is open; thus \( \text{co} E \) is open. ■

We now consider representations of the convex hulls through functions as we can get in the convex case.

Notation 7.19 Given a set \( E \subset \mathbb{R}^{N \times n} \), we consider the sets of functions

\[
\mathcal{F}_\infty^E := \left\{ f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\} : f|_E \leq 0 \right\},
\]

\[
\mathcal{F}^E := \left\{ f : \mathbb{R}^{N \times n} \to \mathbb{R} : f|_E \leq 0 \right\}.
\]

With the above notation, one has (see Proposition 2.36), for \( E \subset \mathbb{R}^{N \times n} \),

\[
\text{co} E = \left\{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every convex } f \in \mathcal{F}_\infty^E \right\},
\]

\[
\overline{\text{co} E} = \left\{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every convex } f \in \mathcal{F}^E \right\},
\]

where \( \overline{\text{co} E} \) denotes the closure of the convex hull of \( E \).
Representations analogous to (7.5) are obtained in the theorem below for the polyconvex, rank one convex and separately convex cases. However, (7.6) can only be generalized to the polyconvex case if the sets are compact (see Theorem 7.28). When dealing with the other notions of convexity, (7.6) is not true, even if compact sets are considered.

**Theorem 7.20** Let $E \subset \mathbb{R}^{N \times n}$, then

\[
Pco E = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every polyconvex } f \in \mathcal{F}_E^\infty \}
\]
\[
Rco E = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every rank one convex } f \in \mathcal{F}_E^\infty \}
\]

**Remark 7.21** A similar result for separately convex hulls can be proved, namely

\[
Sco E = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every separately convex } f \in \mathcal{F}_E^\infty \}
\]

**Proof.** We prove the first identity, the other one being handled analogously. Let us call $X$ the set on the right hand side. Evidently $X$ is a polyconvex set containing $E$ and thus $Pco E \subset X$.

Consider now $\xi \in X$. Since $\chi_{Pco E}$ is a polyconvex function belonging to $\mathcal{F}_E^\infty$, one has $\chi_{Pco E}(\xi) \leq 0$ and consequently $\xi \in Pco E$, thus obtaining the other inclusion.

We now come to a simple but important result (see Dacorogna-Marcellini [202]). It shows that our definitions of polyconvex and rank one convex hulls are consistent with the notions of polyconvex and rank one convex envelopes defined in Chapter 6.

**Proposition 7.22** Let $E \subset \mathbb{R}^{N \times n}$ and $\chi_E$ be its indicator function. Then

\[
P\chi_E = \chi_{Pco E} \text{ and } R\chi_E = \chi_{Rco E}
\]

where $P\chi_E$ and $R\chi_E$ are, respectively, the polyconvex and rank one convex envelopes of $\chi_E$.

**Remark 7.23** A similar result holds for separately convex hulls.

**Proof.** (i) Since $\chi_{Pco E} \leq \chi_E$ and $\chi_{Pco E}$ is polyconvex, we get that $\chi_{Pco E} \leq P\chi_E$; so it remains to show the reverse inequality. From Theorem 6.8, we have

\[
P\chi_E(\xi) = \inf \left\{ \sum_{i=1}^{\tau+1} \lambda_i \chi_E(\xi_i) : \lambda \in \Lambda_{\tau+1}, \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = T(\xi) \right\}
\]

Note that for every $\xi \in Pco E$ (or equivalently $\chi_{Pco E}(\xi) = 0$), we have from Theorem 7.14 that there exist $\xi_i \in E$, $\lambda \in \Lambda_{\tau+1}$, such that

\[
\sum_{i=1}^{\tau+1} \lambda_i T(\xi_i) = T(\xi)
\]
Therefore $P_{\chi E}(\xi) = 0$ and thus $P_{\chi E} \leq \chi_{Pco E}$.

(ii) We first recall the construction of the rank one convex envelope of a given function $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ (cf. Theorem 6.10). Define by induction $R_0 f = f$ and

$$R_{i+1} f (\xi) := \inf \left\{ \lambda R_i f (\xi_1) + (1 - \lambda) R_i f (\xi_2) : \lambda \xi_1 + (1 - \lambda) \xi_2 = \xi \text{ with } \text{rank} \{\xi_1 - \xi_2\} \leq 1 \right\}.$$  

We then get that the rank one convex envelope of $f$ is given by

$$Rf(\xi) = \inf_{i \in \mathbb{N}} R_i f(\xi).$$

We apply this result to $\chi_E$, the indicator function of $E$. We observe that by induction

$$R_i \chi_E = \chi_{R_i \text{co } E}$$

and thus, invoking Theorem 7.17,

$$R\chi_E = \chi_{\bigcup R_i \text{co } E} = \chi_{Rco E}.$$  (7.7)

The proposition has therefore been proved. ■

We next show, as already mentioned in Remark 7.8, that the interior of a generalized convex set keeps the convexity (in the generalized sense), but that, contrary to the classical convex case (see Proposition 2.4), this is not true for its closure.

**Proposition 7.24** (i) Let $E \subset \mathbb{R}^{N \times n}$ be, respectively, a polyconvex, quasiconvex, rank one convex or separately convex set. Then $\text{int } E$ is also, respectively, polyconvex, quasiconvex, rank one convex or separately convex.

(ii) There is $E \subset \mathbb{R}^{2 \times 2}$ a polyconvex and bounded set such that $\overline{E}$ is not separately convex.

**Proof.** (i) We present the proof in the context of polyconvexity. For the other convexities, the proof is analogous. It is sufficient to prove that $Pco(\text{int } E) = \text{int } E$. The non-trivial inclusion is $Pco(\text{int } E) \subset \text{int } E$. Since $E$ is polyconvex, evidently

$$Pco(\text{int } E) \subset Pco E = E.$$  (7.8)

On the other hand, $\text{int } E$ is open and thus (see Theorem 7.14) $Pco(\text{int } E)$ is also open. From (7.8), the desired inclusion then follows.

(ii) We define (see Figure 7.1)

$$E = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} : 0 < x < 1 \right\}.$$  

It is a bounded set and $\overline{E}$ is not separately convex. In fact, let

$$\xi_1 = \text{diag}(1,0) \text{ and } \xi_2 = \text{diag}(-1,0).$$
One has $\xi_1, \xi_2 \in \overline{E}$, but $\lambda \xi_1 + (1 - \lambda) \xi_2 \notin \overline{E}$ for any $0 < \lambda < 1$.

We now show that $E$ is polyconvex. Let $\xi_1, \cdots, \xi_6 \in E$ and suppose

$$T(\xi) = \sum_{i=1}^{6} \lambda_i T(\xi_i) \text{ for some } \lambda = (\lambda_1, \cdots, \lambda_6) \in \Lambda_6.$$  \hfill (7.9)

We have to see that $\xi \in E$. We can write $\{1, \cdots, 6\} = I_+ \cup I_-$ for some $I_+$ and $I_-$ such that

$$\xi_i = \begin{pmatrix} 1 & 0 \\ 0 & x_i \end{pmatrix} \text{ if } i \in I_+ \quad \text{and} \quad \xi_i = \begin{pmatrix} -1 & 0 \\ 0 & -x_i \end{pmatrix} \text{ if } i \in I_-,$$

where $0 < x_i < 1$, $i = 1, \cdots, 6$. In any case, $\det \xi_i = x_i$.

If $I_+ = \emptyset$ or $I_- = \emptyset$, then it is clear that $\xi \in E$. We will see that the other case, namely $I_+ \neq \emptyset$ and $I_- \neq \emptyset$, is not an admissible one. In fact, from (7.9), we can write

$$\xi = \begin{pmatrix} \sum_{i \in I_+} \lambda_i - \sum_{i \in I_-} \lambda_i \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sum_{i \in I_+} \lambda_i x_i - \sum_{i \in I_-} \lambda_i x_i \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

and

$$\det \xi = \alpha \beta = \sum_{i=1}^{6} \lambda_i x_i.$$  

Then $|\alpha| < \sum_{i=1}^{6} \lambda_i = 1$, $|\beta| < \sum_{i=1}^{6} \lambda_i x_i$ and thus $|\alpha \beta| < \sum_{i=1}^{6} \lambda_i x_i$, which is a contradiction. ■
7.3.2 The different convex finite hulls

We next introduce some new sets that will allow a better understanding of the closure of the different hulls. Recall first that

\[ \mathcal{F}^E := \{ f : \mathbb{R}^{N \times n} \to \mathbb{R} : f|_E \leq 0 \} . \]

**Definition 7.25** For a set \( E \subset \mathbb{R}^{N \times n} \), let

\[ \text{co}_f E := \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every convex } f \in \mathcal{F}^E \} , \]
\[ \text{Pco}_f E := \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every polyconvex } f \in \mathcal{F}^E \} , \]
\[ \text{Qco}_f E := \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every quasiconvex } f \in \mathcal{F}^E \} , \]
\[ \text{Rco}_f E := \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every rank one convex } f \in \mathcal{F}^E \} , \]
\[ \text{Sco}_f E := \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every separately convex } f \in \mathcal{F}^E \} . \]

We call them, respectively, the convex finite, polyconvex finite, rank one convex finite, quasiconvex finite and separately convex finite hulls of \( E \).

**Remark 7.26** (i) We recall that (see Proposition 2.36)

\[ \text{co}_f E = \overline{\text{co} E} . \]

(ii) The above sets are all closed because any separately convex function taking only finite values is continuous. Besides, they are, respectively, (according to our definitions) convex, polyconvex, quasiconvex, rank one convex and separately convex.

(iii) Some authors (see, for example, Müller-Sverak [465], Sverak [554], Zhang [616]), when dealing with quasiconvexity and rank one convexity, have adopted the above definitions for the hull of a set (in the generalized sense). They call a laminate convex hull what we have called \( \text{Rco}_f E \).

As in Theorem 7.14, the following proposition can easily be shown.

**Proposition 7.27** Let \( E \subset \mathbb{R}^{N \times n} \), then

\[ \text{Pco}_f E = \pi(\text{co}_f T(E) \cap T(\mathbb{R}^{N \times n})) . \]

**Proof.** Start by observing that if \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) and \( F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \) are such that

\[ f(\eta) = F(T(\eta)) , \forall \eta \in \mathbb{R}^{N \times n} \]

then

\[ f \in \mathcal{F}^E \iff F \in \mathcal{F}^{T(E)} . \]
Call then
\[ X := \pi(\text{co}_{T}(E) \cap T(\mathbb{R}^{N \times n})) = \{\xi \in \mathbb{R}^{N \times n} : T(\xi) \in \text{co}_{f} T(E)\}. \]

(i) Let us first show that \( X \subset \text{Pco}_{f} E \). So let \( f \in \mathcal{F}^{E} \) be any polyconvex function and \( \xi \in X \). By definition of polyconvexity, we can find a convex function \( F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \) such that
\[
 f(\eta) = F(T(\eta)), \ \forall \eta \in \mathbb{R}^{N \times n}.
\]
Moreover \( F \in \mathcal{F}^{T(E)} \) and therefore we find that
\[
 f(\xi) = F(T(\xi)) \leq 0
\]
which is the claim, namely \( \xi \in \text{Pco}_{f} E \).

(ii) Let \( F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \) be convex and such that \( F \in \mathcal{F}^{T(E)} \) and let \( \xi \in \text{Pco}_{f} E \). Define
\[
 f(\eta) := F(T(\eta)), \ \forall \eta \in \mathbb{R}^{N \times n}
\]
and observe that \( f \) is polyconvex and \( f \in \mathcal{F}^{E} \). We therefore find
\[
 f(\xi) = F(T(\xi)) \leq 0
\]
which means that \( \xi \in X \), as wished.

We next see the relations between the closures of the convex hulls and the sets introduced in the above definition. Recall that we let \( \overline{\text{Pco}} E, \overline{\text{Qco}} E, \overline{\text{Rco}} E \) and \( \overline{\text{Sco}} E \) denote the closure of, respectively, the polyconvex, quasiconvex, rank one convex and separately convex hulls of \( E \).

**Theorem 7.28** Let \( E \subset \mathbb{R}^{N \times n} \), then
\[
 E \subset \text{Sco}_{f} E \subset \text{Rco}_{f} E \subset \text{Qco}_{f} E \subset \text{Pco}_{f} E \subset \text{co}_{f} E
\]
and moreover
\[
 \overline{\text{Pco}} E \subset \text{Pco}_{f} E, \ \overline{\text{Qco}} E \subset \text{Qco}_{f} E, \ \overline{\text{Rco}} E \subset \text{Rco}_{f} E, \ \overline{\text{Sco}} E \subset \text{Sco}_{f} E.
\]
In general, the four inclusions are strict. However, if \( E \) is compact, then
\[
 \text{Pco} E = \overline{\text{Pco}} E = \text{Pco}_{f} E.
\]

**Remark 7.29** (i) We should also draw attention to the fact (see Proposition 7.24) that in general the sets
\[
 \overline{\text{Pco}} E, \ \overline{\text{Qco}} E, \ \overline{\text{Rco}} E, \ \overline{\text{Sco}} E
\]
are not even separately convex. Hence, in particular, \( \overline{\text{Pco}} E \neq \text{Pco}_{f} E \) unless \( E \) is compact.
(ii) Let us emphasize, once more, that all the above analysis shows that there are examples of compact quasiconvex (respectively, rank one convex and separately convex) sets $E$ such that
\[ E \subset \mathcal{Qco} E, \ E \subset \mathcal{Rco} E, \ E \subset \mathcal{Sco} E \]
contrary, by definition, to the following:
\[ E = \mathcal{Qco} E, \ E = \mathcal{Rco} E, \ E = \mathcal{Sco} E. \]

\[ \Diamond \]

**Proof.** (i) The inclusions
\[ E \subset \mathcal{Sco} E \subset \mathcal{Rco} E \subset \mathcal{Qco} E \subset \mathcal{Pco} E \subset \mathcal{co} E. \]
are obvious.

(ii) The inclusions
\[ \mathcal{Pco} E \subset \mathcal{Pco} f E, \ \mathcal{Qco} E \subset \mathcal{Qco} f E, \ \mathcal{Rco} E \subset \mathcal{Rco} f E, \ \mathcal{Sco} E \subset \mathcal{Sco} f E. \]
are also easy, since all the sets in the right hand side of the inclusions are closed, contain $E$ and are, respectively, polyconvex, quasiconvex, rank one convex and separately convex.

(iii) Let us show that the first inclusion ($\mathcal{Pco} E \subset \mathcal{Pco} f E$) is strict. This follows (cf. Proposition 7.24) from the fact that there are polyconvex sets whose closure is not polyconvex though $\mathcal{Pco} f E$ is always a polyconvex set.

(iv) We now deal with the fact that the last three inclusions are strict. We use Example 5.18 which will give at once
\[ \mathcal{Qco} E \subset \mathcal{Qco} f E, \ \mathcal{Rco} E \subset \mathcal{Rco} f E \text{ and } \mathcal{Sco} E \subset \mathcal{Sco} f E. \]
Consider the set
\[ E := \{ \xi_1, \xi_2, \xi_3, \xi_4 \} \subset \mathbb{R}^{2 \times 2} \]
where
\[ \xi_1 = \text{diag}(-1,0), \ \xi_2 = \text{diag}(1,-1), \ \xi_3 = \text{diag}(2,1), \ \xi_4 = \text{diag}(0,2) \]
and let us show that $E$ is quasiconvex (and hence rank one and separately convex); this will imply that
\[ E = \mathcal{Qco} E = \mathcal{Rco} E = \mathcal{Sco} E. \]
Suppose that $\xi + \nabla \varphi R \in E$ for some $\varphi \in \mathcal{W}_{\text{per}}$ and $R \in O(2)$. Since
\[ \text{rank} \{ \xi_i - \xi_j \} = 2 \text{ for } i \neq j, \]
we have from Theorem 7.11 (with \( m = 4 \)) that there exists \( \xi_i \in E \) such that
\[
\xi + \nabla \varphi(x) R = \xi_i, \text{ a.e. } x \in D.
\]
Using then the periodicity of \( \varphi \), we find
\[
\xi = \int_D (\xi + \nabla \varphi(x) R) \, dx = \xi_i
\]
and thus \( \xi = \xi_i \in E \). We then conclude that \( E \) is quasiconvex.

However, any separately convex function \( f \in \mathcal{F}^E \) and consequently any rank one convex or quasiconvex function in \( \mathcal{F}^E \) is such that \( f(0) \leq 0 \), according to (5.31) in Example 5.18 (observing that separately convex functions are rank one convex when restricted to diagonal matrices). Thus \( 0 \in \text{Sco}_f E \), but \( 0 \notin \text{Qco} E \).

(v) If we assume \( E \) to be compact, we then have, as we now see,
\[
\text{Pco} E = \overline{\text{Pco} E} = \text{Pco}_f E.
\]

We have already shown in (ii) that
\[
\overline{\text{Pco} E} \subset \text{Pco}_f E
\]

By Theorem 7.14, in this case, \( \text{Pco} E \) is compact and then \( \text{Pco} E = \overline{\text{Pco} E} \). We therefore have combining this identity and (ii) that
\[
\text{Pco} E = \overline{\text{Pco} E} \subset \text{Pco}_f E.
\]

Let us now show the reverse inclusion: \( \text{Pco}_f E \subset \text{Pco} E \). We start noting that, since \( E \) is compact, \( T(E) \) is compact and thus \( \text{co} T(E) \) is also compact (cf. Theorem 2.14 (i)). Note also that the function
\[
\eta \rightarrow f(\eta) := \text{dist}(T(\eta), \text{co} T(E))
\]
is polyconvex and \( f \in \mathcal{F}^E \). Therefore if \( \xi \in \text{Pco}_f E \) we get
\[
\text{dist}(T(\xi), \text{co} T(E)) = 0.
\]
Since \( \text{co} T(E) \) is closed, we deduce that \( T(\xi) \in \text{co} T(E) \) and thus, \( \xi \in \text{Pco} E \).

Gathering all the results, we can write
\[
\overline{\text{Sco} E} \subset \overline{\text{Rco} E} \subset \overline{\text{Qco} E} \subset \overline{\text{Pco} E} \subset \overline{\text{co} E} = \overline{\text{co} f E}
\]
and also
\[
\text{Sco}_f E \subset \text{Rco}_f E \subset \text{Qco}_f E \subset \text{Pco}_f E \subset \overline{\text{co} E} = \text{co}_f E.
\]
Moreover, the same example and arguments used in the proof of Theorem 7.28 (see also Proposition 7.24) show that, in general,
\[ S_{co}E \not\subseteq R_{co}E, \quad R_{co}fE \not\subseteq Q_{co}E \quad \text{and} \quad Q_{co}fE \not\subseteq P_{co}E. \]
However, if \( E \) is compact one has \( Q_{co}fE \subset P_{co}E \).

We draw attention to the fact that several characterizations of the sets in Definition 7.25 have been used in the literature according to the specific needs of each situation. These sets can be written in terms of measures (see Kirchheim [365], Müller [462]) or using the distance function (see Zhang [617]): if \( E \subset \mathbb{R}^{N \times n} \) is compact, then
\[ Q_{co}fE = \{ \xi \in \mathbb{R}^{N \times n} : Q \text{dist}(\xi, E) = 0 \}, \]
where \( Q \text{dist}(\cdot, E) \) is the quasiconvex envelope of the function \( \text{dist}(\cdot, E) \).

### 7.3.3 Extreme points and Minkowski type theorem for polyconvex, quasiconvex and rank one convex sets

An important tool in convex analysis is the notion of extreme point. In a straightforward manner, we can define it for generalized convex sets as follows (see Dacorogna-Marcellini [202] and Dacorogna-Ribeiro [213]).

**Definition 7.30**

(i) If \( E \subset \mathbb{R}^m \) is convex, \( \xi \in E \) is said to be an extreme point of \( E \) in the convex sense if
\[ \xi = \lambda \xi_1 + (1 - \lambda)\xi_2 \quad \forall \lambda \in (0, 1), \quad \xi_1, \xi_2 \in E \]
\[ \Rightarrow \xi_1 = \xi_2 = \xi. \]
For an arbitrary set \( E \subset \mathbb{R}^m \), the set of extreme points in the convex sense of \( \text{co}E \) is denoted \( E_{ext}^c \) (in Chapter 2, since there was no ambiguity, we have just written for this set \( E_{ext} \)).

(ii) If \( E \subset \mathbb{R}^{N \times n} \) is polyconvex, \( \xi \in E \) is said to be an extreme point of \( E \) in the polyconvex sense if
\[ T(\xi) = \sum_{i=1}^{I} \lambda_i T(\xi_i), \quad I \in \mathbb{N} \]
\[ (\lambda_1, \ldots, \lambda_I) \in \Lambda_I, \quad \lambda_i > 0, \quad \xi_i \in E \]
\[ \Rightarrow \xi_i = \xi, \quad i = 1, \ldots, I. \]
For an arbitrary set \( E \subset \mathbb{R}^{N \times n} \), the set of extreme points in the polyconvex sense of \( \text{Pco}E \) is denoted \( E_{ext}^p \).

(iii) If \( E \subset \mathbb{R}^{N \times n} \) is quasiconvex, \( \xi \in E \) is said to be an extreme point of \( E \) in the quasiconvex sense if
\[ \xi + \nabla \varphi(x)R \in E \text{ a.e. } x \in D, \]
\[ D = (0, 1)^n, \quad R \in O(n), \quad \varphi \in \mathcal{W}_{\text{per}} \]
\[ \Rightarrow \nabla \varphi = 0 \text{ a.e. in } D. \]
For an arbitrary set \( E \subset \mathbb{R}^{N \times n} \), the set of extreme points in the quasiconvex sense of \( \text{Qco}E \) is denoted \( E_{ext}^q \).
Proposition 7.31 Let $E \subset \mathbb{R}^{N \times n}$. Then

$$E_{ext}^c \subset E_{ext}^p \subset E_{ext}^q \subset E_{ext}^r \subset E_{ext}^s \subset E.$$ 

Proof. (i) Let us first show that any of these sets are in $E$. We do it, for example, with $E_{ext}^r$, the others being handled similarly. So let $\xi \in E_{ext}^r \subset Rco E$ and thus, by Theorem 7.17, $\xi \in R_{i+1} co E$ for a certain $i \in \mathbb{N}$. This means that

$$\xi \in \left\{ \begin{array}{l}
\xi \in \mathbb{R}^{N \times n} : \\
\xi = \lambda \xi_1 + (1 - \lambda) \xi_2, \quad \lambda \in [0, 1], \\
\xi_1, \xi_2 \in R_{i} co E \subset Rco E, \quad \text{rank} \{\xi_1 - \xi_2\} \leq 1
\end{array} \right\}$$

and thus, since $\xi \in E_{ext}^r$, we deduce that in fact $\xi \in R_{i} co E$. Iterating the procedure, we find that $\xi \in R_0 co E = E$, as claimed.

(ii) The non-trivial inclusions are those related to $E_{ext}^q$, the set of extreme points of $Qco E$, but it can be obtained with the same arguments used in the proof of Theorem 7.7, Part 1, and we just do it for

$$E_{ext}^q \subset E_{ext}^r;$$

the other one being handled similarly. Let $\xi \in E_{ext}^q \subset E$, we have to show that

$$\xi = \lambda \xi_1 + (1 - \lambda) \xi_2$$

$$\lambda \in (0, 1), \quad \xi_1, \xi_2 \in Rco E, \quad \text{rank} \{\xi_1 - \xi_2\} \leq 1$$

$$\Rightarrow \xi_1 = \xi_2 = \xi.$$ 

So let $\lambda \in (0, 1), \quad \xi_1, \xi_2 \in Rco E, \quad \text{rank} \{\xi_1 - \xi_2\} \leq 1$ and find then, as in the proof of Theorem 7.7, $R \in O(n)$ and $\varphi \in \mathcal{W}_{per}$ such that

$$\nabla \varphi(x) R \in \{(1 - \lambda)(\xi_1 - \xi_2), -\lambda(\xi_1 - \xi_2)\} \text{ a.e. } x \in D.$$
The different types of convex hulls

Since $\xi \in E_{ext}^q$, we then get, from the above construction, that

$$\begin{align*}
\xi + \nabla \varphi(x) R &\in \text{Rco } E \subset \text{Qco } E \text{ a.e. } x \in D, \\
R &\in O(n), \ \varphi \in \mathcal{W}_{\text{per}}
\end{align*}$$

thus $\xi_1 = \xi_2 = \xi$ and hence $\xi \in E_{ext}^r$, as wished.

We now give two examples (see [202]) showing that, in general, the inclusions are strict. The first will show that a point can be extreme in the separately convex sense but not in the usual sense (i.e., the convex sense). The second one will provide an extreme point in the rank one convex sense but not in the polyconvex sense.

**Example 7.32** Let $E = \{ (x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq x + y \leq 1 \}$ (see Figure 7.2),

which is convex (and thus separately convex). Then any element of the line $\{x + y = 1\}$ is an extreme point of $E$ in the separately convex sense but on this line only $(1, 0)$ and $(0, 1)$ are extreme points (in the convex sense) of $E.$

**Example 7.33** (see Proposition 5.10). Let

$$
\xi_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \ \xi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \ \xi_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \ \eta = \begin{pmatrix} 0 \\ 2/3 \\ 0 \\ 1/3 \end{pmatrix}.
$$

Let

$$E = \{\xi_1, \xi_2, \xi_3, \eta\} \text{ and } F = \{\xi_1, \xi_2, \xi_3\}.$$ 

Note that

$$\eta = \frac{1}{3} (\xi_1 + \xi_2 + \xi_3),$$

$$\det \eta = \frac{1}{3} (\det \xi_1 + \det \xi_2 + \det \xi_3).$$
and hence \( \eta \) is not an extreme point in the polyconvex sense of \( \text{Pco} \ E \) (thus \( \text{Pco} \ E = \text{Pco} \ F \)). Moreover, since

\[
\det(\xi_i - \xi_j) \neq 0, \forall i \neq j \quad \text{and} \quad \det(\xi_i - \eta) \neq 0, \forall i,
\]

we deduce that \( \eta \) is an extreme point in the rank one convex sense of \( \text{Rco} \ E = \text{E} \) (thus \( \text{Rco} \ E = \text{E} \supset \text{Rco} \ F = \text{F} \)). What is, however, more interesting is that \( \eta \) is also an extreme point in the rank one convex sense of \( \text{Pco} \ E = \text{Pco} \ F \). Indeed one can easily show that

\[
\text{Pco} \ E = \left\{ \xi \in \mathbb{R}^{2 \times 2} : \xi = \alpha \xi_1 + \beta \xi_2 + (1 - \alpha - \beta) \xi_3 \right\},
\]

with \( 0 \leq \alpha, \beta \leq \alpha + \beta \leq 1 \) and

\[
\beta = \alpha + \frac{1}{2} \pm \frac{1}{2} \sqrt{12 \alpha^2 - 4 \alpha + 1},
\]

\[
\delta = \gamma + \frac{1}{2} \pm \frac{1}{2} \sqrt{12 \gamma^2 - 4 \gamma + 1}.
\]

From \( \eta = t \xi + (1 - t) \xi' \), we deduce that

\[
t \alpha + (1-t) \gamma = t \beta + (1-t) \delta = \frac{1}{3}. \tag{7.10}
\]

Furthermore, from \( \det(\xi - \xi') = 0 \) we find

\[
3(\alpha - \gamma)^2 = [(\alpha - \gamma) - (\beta - \delta)]^2 = \frac{1}{4} \left[ \sqrt{12 \alpha^2 - 4 \alpha + 1} \pm \sqrt{12 \gamma^2 - 4 \gamma + 1} \right]^2.
\]

This identity then implies that \( \alpha = \gamma \), which coupled with (7.10) leads to \( \alpha = \gamma = \beta = \delta = 1/3 \). Thus \( \eta = \xi = \xi' \) as claimed.

\[\Diamond\]

Minkowski theorem (see Theorem 2.20) ensures that the convex hull of a compact set coincides with the convex hull of its extreme points. We next deal with the generalization of this result to the other convexities. We start with the polyconvex case (see also Dacorogna-Tanteri [215]).
Theorem 7.34 Let $E \subset \mathbb{R}^{N \times n}$ be a compact set. Then

$$Pco E = Pco E_{ext}^p.$$ 

Proof. The inclusion $Pco E_{ext}^p \subset Pco E$ is trivial, since $E_{ext}^p \subset Pco E$. We thus show the reverse inclusion. We start noting that

$$Pco E = \pi(\text{co} T(E) \cap T(\mathbb{R}^{N \times n}))$$

$$Pco E_{ext}^p = \pi(\text{co} T(E_{ext}^p) \cap T(\mathbb{R}^{N \times n})).$$

Let $\xi \in Pco E$ and let us prove that $\xi \in Pco E_{ext}^p$. By the above characterization of $Pco E$ we have $T(\xi) \in \text{co} T(E)$. Moreover, by the classical Minkowski theorem (cf. Theorem 2.20), and using the fact that $T(E)$ is compact, we have

$$\text{co} T(E) = \text{co} (T(E)_{ext}^c),$$

where $T(E)_{ext}^c$ is the set of extreme points of $\text{co} T(E)$ (in the convex sense).

We next prove that

$$T(E)_{ext}^c \subset T(E_{ext}^p),$$

which will finish the proof.

Let then $X \in T(E)_{ext}^c$. In particular, $X \in T(E)$ and we can write $X = T(\eta)$ with $\eta \in E$. It suffices then to see that $\eta \in E_{ext}^p$. Suppose that

$$T(\eta) = \sum_{i=1}^{I} \lambda_i T(\eta_i)$$

for some $(\lambda_1, \ldots, \lambda_I) \in \Lambda_I$, $\lambda_i > 0$, $\eta_i \in Pco E$. Observing that, since $\eta_i \in Pco E$ then $T(\eta_i) \in \text{co} T(E)$, it immediately follows, from the fact that $T(\eta)$ is an extreme point of $\text{co} T(E)$, that $\eta_i = \eta$ for every $i$, that is to say $\eta$ is an extreme point of $Pco E$. The proof is finished.

As remarked in Kirchheim [365], the above result is not true for quasiconvex, rank one convex or separately convex hulls. In fact, Example 5.18 considered in the proof of Theorem 7.28 shows that, in general,

$$Qco E_{ext}^q \neq Qco E, \ Rco E_{ext}^r \neq Rco E \text{ and } Sco E_{ext}^s \neq Sco E$$

as we now prove it.

Example 7.35 We consider a set of diagonal matrices that we identify with elements of $\mathbb{R}^2$. In particular, rank one convexity and separate convexity coincide. Let

$$E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5,$$
where
\[ E_1 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}, \]
\[ E_2 = \{(x, 1) \in \mathbb{R}^2 : 1 \leq x \leq 2\}, \]
\[ E_3 = \{(0, y) \in \mathbb{R}^2 : 0 \leq y \leq 2\}, \]
\[ E_4 = \{(y, 0) \in \mathbb{R}^2 : -1 \leq y \leq 0\}. \]
Note that \( E \) is a compact rank one convex set and
\[ E_{ext}^q \subset E_{ext}^r = \{\xi_1, \xi_2, \xi_3, \xi_4\}, \]
where
\[ \xi_1 = (-1, 0), \xi_2 = (1, -1), \xi_3 = (2, 1), \xi_4 = (0, 2). \]
Thus, since there are no rank one connections between the elements \( \xi_i \),
\[ Qco E_{ext}^q = E_{ext}^q \text{ and } Rco E_{ext}^r = E_{ext}^r. \]
However, \( E_{ext}^q \subset E_{ext}^r \subset E = Rco E \subset Qco E \).

We now prove a weaker result than Theorem 7.34 but that is valid in the quasiconvex, rank one convex and separately convex cases. We follow the proof of Matousek-Plechac [439], which can also be adapted to the quasiconvex case (see also Zhang [617] for a different proof in the quasiconvex case).

**Theorem 7.36** Let \( E \subset \mathbb{R}^{N \times n} \) be a bounded set and \( E_{ext}^{qf}, E_{ext}^{rf}, E_{ext}^{sf} \) denote, respectively, the set of extreme points of \( Qco_f E \) (in the quasiconvex sense), the set of extreme points of \( Rco_f E \) (in the rank one convex sense) and the set of extreme points of \( Sco_f E \) (in the separately convex sense). Then
\[ Qco_f E = Qco_f E_{ext}^{qf}, \quad Rco_f E = Rco_f E_{ext}^{rf} \quad \text{and} \quad Sco_f E = Sco_f E_{ext}^{sf}. \]

**Proof.** We divide the proof into two steps. The first is common to the three convexities and we present it in the context of quasiconvexity. In the second step, we consider separately the quasiconvex and the rank one convex cases (the latter being analogous to the separately convex case, which we will not consider explicitly). In all that follows we will denote by \( \overline{E}_{ext}^{qf} \) the closure of \( E_{ext}^{qf} \).

**Step 1.** We remark that, for any set \( K \subset \mathbb{R}^{N \times n} \), since \( Qco_f K \) is automatically closed, \( Qco_f K = Qco_f \overline{K} \). Thus, it is enough to prove that \( Qco_f E = Qco_f \overline{E}_{ext}^{qf} \). The inclusion \( Qco_f \overline{E}_{ext}^{qf} \subset Qco_f E \) is trivial. It remains to verify the reverse inclusion. We argue by contradiction.

Suppose there is some \( \eta \in Qco_f E - Qco_f \overline{E}_{ext}^{qf} \). Then, by definition, there exists a quasiconvex function \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) with \( f \in F \overline{E}_{ext}^{qf} \) such that \( f(\eta) > 0 \).

Now let
\[ M := \max_{Qco_f E} f \quad \text{and} \quad A := \{\xi \in Qco_f E : f(\xi) = M\}. \]
The set $\mathcal{A}$ is non-empty and compact (since $\text{Qco}_f E$ is compact and $f$ is a continuous function). Thus, considering $\mathbb{R}^{N \times n}$ with the lexicographic order (the elements of $\mathbb{R}^{N \times n}$ being seen as vectors), one can consider the maximum element of $\mathcal{A}$, say $\xi_0$. We have $\xi_0 \notin E_{\text{ext}}^f$, which follows from

$$0 < f(\eta) \leq \max_{\text{Qco}_f E} f = M = f(\xi_0).$$

As we will see in Step 2, this leads to the existence of an element in $\mathcal{A}$ greater than $\xi_0$ for the lexicographic order, which is the desired contradiction.

**Step 2. Quasiconvex case.** Since $\xi_0 \in \text{Qco}_f E - E_{\text{ext}}^f$, there are $R \in O(n)$ and $\varphi \in \mathcal{W}_{\text{per}}$ such that

$$\xi_0 + \nabla \varphi(x)R \in \text{Qco}_f E, \text{ a.e. } x \in D, \text{ with } \text{meas}\{x \in D : \nabla \varphi(x) \neq 0\} > 0.$$

We can write

$$\nabla \varphi(x)R \in \{\xi_1, \cdots, \xi_k\} \text{ and } \lambda_i = \text{meas}\{x \in D : \nabla \varphi(x)R = \xi_i\} > 0.$$  

Since $\xi_0 + \xi_i \in \text{Qco}_f E$, we have $f(\xi_0 + \xi_i) \leq M$. Consequently, by the quasiconvexity of $f$ we get

$$M = f(\xi_0) \leq \int_D f(\xi_0 + \nabla \varphi(x)R) \, dx = \sum_{i=1}^{k} \lambda_i f(\xi_0 + \xi_i) \leq M$$

implying $f(\xi_0 + \xi_i) = M$ for every $i = 1, \cdots, k$, that is $\xi_0 + \xi_i \in \mathcal{A}$. Finally, from the fact that $\nabla \varphi \neq 0$ and

$$0 = \int_D \nabla \varphi(x)R \, dx = \sum_{i=1}^{k} \lambda_i \xi_i$$

we conclude that among the elements $\xi_0 + \xi_i$ there must be at least one that is greater than $\xi_0$ (in the lexicographic order), which contradicts the fact that $\xi_0$ is the maximum element of $\mathcal{A}$.

**Rank one convex case.** We recall that in this case the function $f \in \mathcal{F} E_{\text{ext}}^f$ is a rank one convex function. Since $\xi_0 \in \text{Rco}_f E - E_{\text{ext}}^f$, there are $\eta_1, \eta_2 \in \text{Rco}_f E$, with rank $\{\eta_1 - \eta_2\} \leq 1$ such that

$$\xi_0 = \lambda \eta_1 + (1 - \lambda) \eta_2 \text{ and } \xi_0 \neq \eta_1, \xi_0 \neq \eta_2.$$

As in the quasiconvex case we get

$$f(\eta_1) = f(\eta_2) = M$$

and from $\xi_0 = \lambda \eta_1 + (1 - \lambda) \eta_2$ it follows that $\eta_1$ or $\eta_2$ must be greater than $\xi_0$, which is a contradiction. □
7.3.4 Gauges for polyconvex sets

We define, as in convex analysis (see Section 2.3.7), the gauge of a polyconvex set. We follow here the presentation of Dacorogna-Tanteri [215].

**Theorem 7.37** Let $E \subset \mathbb{R}^{N \times n}$ be a non-empty polyconvex set and let

$$
\chi_E(\xi) := \begin{cases} 
0 & \text{if } \xi \in E \\
+\infty & \text{if } \xi \notin E 
\end{cases}
$$

be its indicator function. Let $H^p : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \cup \{+\infty\}$ be defined as

$$
H^p(X^*) := \sup_{\xi \in E} \{ \langle T(\xi) ; X^* \rangle \}.
$$

The following statements then hold.

(i) $H^p$ is lower semicontinuous, convex and positively homogeneous of degree one.

(ii) If $E$ is compact and if

$$(H^p)^* : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \cup \{+\infty\}$$

is the conjugate function of $H^p$ meaning that

$$(H^p)^*(X) = \sup_{X^* \in \mathbb{R}^{\tau(n,N)}} \{ \langle X ; X^* \rangle - H^p(X^*) \},$$

then

$$
\chi_E(\xi) = (H^p)^*(T(\xi)),
$$

$$
E = \{ \xi \in \mathbb{R}^{N \times n} : (H^p)^*(T(\xi)) \leq 0 \}.
$$

(iii) If $0 \in E$, then

$$
H^p(X^*) \geq H^p(0) = 0 \text{ for every } X^* \in \mathbb{R}^{\tau(n,N)};
$$

and if $E$ is compact, then $H^p$ takes only finite values.

(iv) If $0 \in \text{int } E$ and if $E$ is compact, then

$$
H^p(X^*) = 0 \iff X^* = 0;
$$

and in this case

$$
E = \left\{ \xi \in \mathbb{R}^{N \times n} : (H^p)^0(T(\xi)) \leq 1 \right\},
$$

where $(H^p)^0$ (called the gauge in the polyconvex sense of $E$) is the polar of $H^p$, namely

$$
(H^p)^0(X) := \sup_{X^* \neq 0} \left\{ \frac{\langle X ; X^* \rangle}{H^p(X^*)} \right\}.
$$
Remark 7.38  (i) When $N = n = 2$, we have that $H^p : \mathbb{R}^{2 \times 2} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is given by

$$H^p (\xi^*, \delta^*) = \sup_{\xi \in E} \{ \langle \xi ; \xi^* \rangle + \delta^* \det \xi \}$$

and

$$E = \{ \xi \in \mathbb{R}^{2 \times 2} : (H^p)^* (\xi, \det \xi) \leq 0 \} .$$

(ii) Note that $(H^p)^0$ is positively homogeneous of degree one but, of course, this is not the case for the function $\xi \to (H^p)^0 (T(\xi))$.

(iii) In the notation of Section 6.2.1, we have

$$H^p = (\chi_E)^p .$$

Example 7.39 For $\xi \in \mathbb{R}^{2 \times 2}$, let $0 \leq \lambda_1 (\xi) \leq \lambda_2 (\xi)$ denote its singular values, $0 < a_1 \leq a_2$ and

$$E = \{ \xi \in \mathbb{R}^{2 \times 2} : \lambda_2 (\xi) \leq a_2, \lambda_1 (\xi) \lambda_2 (\xi) \leq a_1 a_2 \} ,$$

which is a polyconvex set (see Theorem 7.43). Then

$$(H^p)^0 (\xi^*, \delta^*) = \max \left\{ \frac{\lambda_2 (\xi^*)}{a_2}, \frac{|\delta^*|}{a_1 a_2} \right\}$$

is a gauge for $E$.

Proof.  (i) Since $E$ is non-empty then $H^p > -\infty$. $H^p$ being the supremum of affine functions, is convex and lower semicontinuous. The fact that $H^p$ is positively homogeneous of degree one is easy.

(ii) Since $E$ is compact, the function

$$\chi_{\text{co} T(E)} : \mathbb{R}^{\tau (n, N)} \to \mathbb{R} \cup \{+\infty\}$$

is convex and lower semicontinuous. Moreover since $E$ is polyconvex, we have, according to Theorem 7.4, that

$$E = \{ \xi \in \mathbb{R}^{N \times n} : T(\xi) \in \text{co} T(E) \}$$

and thus

$$\chi_E (\xi) = \chi_{\text{co} T(E)} (T(\xi)) .$$

We then proceed as in the proof of Theorem 6.6 to deduce that

$$H^p (X^*) = \chi^p_E (X^*)$$

$$\chi_E (\xi) = \chi^{pp}_E (\xi) := (H^p)^* (T(\xi))$$

hence the result.
(iii) This is obvious.

(iv) We now show that if \( 0 \in \text{int} \, E \) and if \( E \) is compact then

\[
H^p (X^*) = 0 \iff X^* = 0.
\]

The implication \((\Leftarrow)\) follows from (iii) and we therefore discuss only the reverse one. Let \( \xi \in \mathbb{R}^{N \times n} \) be an arbitrary point, \( \xi \neq 0 \). Since \( 0 \in \text{int} \, E \), we deduce that for every \( \epsilon \) sufficiently small then \( \epsilon \xi / |\xi| \in E \) and therefore

\[
0 = H^p (X^*) \geq \langle T (\epsilon \xi / |\xi|) ; X^* \rangle. \tag{7.11}
\]

Since \( \xi \in \mathbb{R}^{N \times n} \) is arbitrary, the above inequality implies that \( X^* = 0 \), as claimed. We prove this last fact only when \( N = n = 2 \), the general case being proved similarly. The inequality (7.11) reads then (writing \( X^* = (\xi^*, \delta^*) \))

\[
0 = H^p (X^*) \geq \frac{\epsilon}{|\xi|} \langle \xi ; \xi^* \rangle + \epsilon^2 \frac{\det \xi}{|\xi|^2} \delta^*, \forall \xi \in \mathbb{R}^{2 \times 2}, \xi \neq 0.
\]

We therefore get, using the fact that \( \epsilon \) is arbitrary,

\[
\begin{cases}
\langle \xi ; \xi^* \rangle = 0, \forall \xi \in \mathbb{R}^{2 \times 2} \\
\delta^* \det \xi \leq 0, \forall \xi \in \mathbb{R}^{2 \times 2}
\end{cases}
\]

hence \((\xi^*, \delta^*) = (0, 0)\).

The last identity

\[
E = \{ \xi \in \mathbb{R}^{N \times n} : (H^p)^0 (T (\xi)) \leq 1 \}
\]

immediately follows from (ii).

### 7.3.5 Choquet functions for polyconvex and rank one convex sets

We finally define some functions that characterize the extreme points first in the polyconvex and then in the rank one convex sense. In the convex case this is known as the Choquet function (see Section 2.3.8). The first theorem was established by Dacorogna-Tanteri [215].

**Theorem 7.40** Let \( E \subset \mathbb{R}^{N \times n} \) be a non-empty compact polyconvex set and \( E^p_{\text{ext}} \) be its extreme points in the polyconvex sense. Then there exists a polyconvex function

\[
\varphi^E_p : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}
\]

such that

\[
E^p_{\text{ext}} = \{ \xi \in E : \varphi^E_p (\xi) = 0 \},
\]

\[
\varphi^E_p (\xi) \leq 0 \iff \xi \in E.
\]
Proof. We first define

$$f(\xi) := \begin{cases} -|\xi|^2 & \text{if } \xi \in E \\ +\infty & \text{otherwise.} \end{cases}$$

Note that

$$f(\xi) \geq -\sup\{|\xi|^2 : \xi \in E\} > -\infty.$$ 

The Choquet function for polyconvex sets is then defined as

$$\varphi^E_p(\xi) := \begin{cases} Pf(\xi) - f(\xi) & \text{if } \xi \in E \\ +\infty & \text{otherwise.} \end{cases}$$

Note that, letting $\chi_E$ be the indicator function of the set $E$,

$$\varphi^E_p(\xi) = Pf(\xi) + |\xi|^2 + \chi_E(\xi), \forall \xi \in \mathbb{R}^{N \times n}.$$ 

The function $\varphi^E_p : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ is therefore polyconvex and

$$\varphi^E_p(\xi) \leq 0, \text{ if } \xi \in E \text{ and } \varphi^E_p(\xi) = 0, \text{ if and only if } \xi \in E^p_{ext}.$$ 

Indeed the inequality is clear since in $E$ the function $f$ is finite and, by definition, $Pf$ is always not larger than $f$. We now show that

$$\varphi^E_p(\xi) = 0 \iff \xi \in E^p_{ext}.$$ 

Using Theorem 6.8, we find that if $\xi \in E$, then, letting $\tau := \tau(n, N)$,

$$\varphi^E_p(\xi) = |\xi|^2 + \inf_{\xi_i \in E} \{- \sum_{i=1}^{\tau+1} t_i |\xi_i|^2 : T(\xi) = \sum_{i=1}^{\tau+1} t_i T(\xi_i), t \in \Lambda_{\tau+1}\}$$

where for $s \in \mathbb{N}$, we have

$$\Lambda_s := \{\lambda = (\lambda_1, \ldots, \lambda_s) : \lambda_i \geq 0, \sum_{i=1}^{s} \lambda_i = 1\}.$$ 

Therefore if $\xi \in E^p_{ext}$, we deduce, by definition, that in the infimum the only admissible $\xi_i$ are $\xi_i = \xi_i$; and hence we have $\varphi^E_p(\xi) = 0$.

We now show the reverse implication, namely

$$\varphi^E_p(\xi) = 0 \Rightarrow \xi \in E^p_{ext}.$$ 

From the above representation formula we obtain, since $\varphi^E_p(\xi) = 0$ and $\xi \in E$, that

$$|\xi|^2 = \sup_{\xi_i \in E} \{\sum_{i=1}^{\tau+1} t_i |\xi_i|^2 : T(\xi) = \sum_{i=1}^{\tau+1} t_i T(\xi_i), t \in \Lambda_{\tau+1}\}.$$ 

Combining the above with the convexity of the function $\xi \to |\xi|^2$ we get that

$$|\xi|^2 \geq \sum_{i=1}^{\tau+1} t_i |\xi_i|^2 \geq |\sum_{i=1}^{\tau+1} t_i \xi_i|^2 = |\xi|^2;$$
the strict convexity of $\xi \to |\xi|^2$ implies then that $\xi_i = \xi$. Thus $\xi \in E^p_{ext}$. ■

A similar construction can be done for rank one convexity, as was achieved by Ribeiro [512].

**Theorem 7.41** Let $E \subset \mathbb{R}^{N \times n}$ be a non-empty compact rank one convex set and $E^r_{ext}$ be its extreme points in the rank one convex sense. Then there exists $\varphi^E_r : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ a rank one convex function such that

$$E^r_{ext} = \{ \xi \in E : \varphi^E_r (\xi) = 0 \},$$

$$\varphi^E_r (\xi) \leq 0 \iff \xi \in E.$$

**Proof.** We first define

$$f (\xi) := \begin{cases} -|\xi|^2 & \text{if } \xi \in E \\ +\infty & \text{otherwise} \end{cases}$$

and the Choquet function for rank one convex sets is defined as

$$\varphi^E_r (\xi) := \begin{cases} Rf (\xi) - f (\xi) & \text{if } \xi \in E \\ +\infty & \text{otherwise}. \end{cases}$$

Observe that $\varphi^E_r : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\}$ is rank one convex, since

$$\varphi^E_r (\xi) = Rf (\xi) + |\xi|^2 + \chi_E (\xi), \forall \xi \in \mathbb{R}^{N \times n}.$$

We now prove that

$$\varphi^E_r (\xi) \leq 0, \text{ if } \xi \in E \text{ and } \varphi^E_r (\xi) = 0, \text{ if and only if } \xi \in E^r_{ext}.$$

Indeed the inequality is clear since in $E$ the function $f$ is finite and, by definition, $Rf$ is always not larger than $f$. We now show that

$$\varphi^E_r (\xi) = 0 \iff \xi \in E^r_{ext}.$$  

Recall first that (cf. Theorem 6.10) we have

$$Rf = \lim_{k \to \infty} R_k f = \inf_{k \in \mathbb{N}} R_k f$$

where $R_0 f := f$ and $R_k f$ is inductively given by

$$R_{k+1} f (\xi) := \inf \left\{ \lambda R_k f (\xi_1) + (1 - \lambda) R_k f (\xi_2) : \lambda \xi_1 + (1 - \lambda) \xi_2 = \xi \text{ with rank} \{|\xi_1 - \xi_2|\} \leq 1 \right\}.$$

Let us show that for $\xi \in E^r_{ext}$ we have

$$R_k f (\xi) = f (\xi), \text{ for every } k \in \mathbb{N} \quad (7.12)$$
The induction procedure therefore leads to (7.12). We have therefore shown that
\[ \xi \in E^r_{\text{ext}} \Rightarrow \varphi_r^E(\xi) = 0. \]
Let us now prove the reverse implication
\[ \varphi_r^E(\xi) = 0 \Rightarrow \xi \in E^r_{\text{ext}}. \]
From the above representation formula we obtain, since \( \varphi_r^E(\xi) = 0 \) and \( \xi \in E \), that
\[ Rf(\xi) = R_k f(\xi) = f(\xi), \text{ for every } k \in \mathbb{N}. \]
Let \( \xi \in E, \lambda \in (0,1), \xi_1, \xi_2 \in E \) with rank \( \{\xi_1 - \xi_2\} \leq 1 \) and such that
\[ \xi = \lambda \xi_1 + (1 - \lambda) \xi_2 \]
we have to show that \( \xi_1 = \xi_2 = \xi \).
From the fact that \( R_1 f(\xi) = f(\xi) \) we get
\[ |\xi|^2 = \sup_{\eta \in E} \left\{ \frac{\lambda |\eta_1|^2 + (1 - \lambda) |\eta_2|^2}{\lambda \eta_1 + (1 - \lambda) \eta_2 = \xi \text{ with rank } \{\eta_1 - \eta_2\} \leq 1} \right\}. \]
Combining the above with the convexity of the function \( \xi \to |\xi|^2 \) we get that
\[ |\xi|^2 \geq \lambda |\xi_1|^2 + (1 - \lambda) |\xi_2|^2 \geq | \lambda \xi_1 + (1 - \lambda) \xi_2 |^2 = |\xi|^2; \]
the strict convexity of \( \xi \to |\xi|^2 \) implies then that \( \xi_i = \xi \). Thus \( \xi \in E^r_{\text{ext}} \). \( \blacksquare \)

### 7.4 Examples

We now discuss several examples that should be related to those of Sections 6.6, 10.3 and 11.5.

In many instances, we will use below the following elementary lemma or a similar argument to that in the proof.

**Lemma 7.42** Let \( X \subset \mathbb{R}^{N \times n} \) be compact and \( E \) be rank one convex. Then
\[ \partial X \subset E \Rightarrow X \subset E. \]
Proof. Let $\xi \in X$. If $\xi \in \partial X$, then nothing is to be proved; so we assume that $\xi \in \text{int} X$. Let $\eta \in \mathbb{R}^{N \times n}$ be any matrix of rank one and set for $t \in \mathbb{R}$,

$$
\xi_t := \xi + t\eta.
$$

Since $X$ is compact, we can find $t_1 < 0 < t_2$ so that

$$
\xi_{t_1}, \xi_{t_2} \in \partial X \subset E \quad \text{and} \quad \xi = \frac{t_2}{t_2 - t_1} \xi_{t_1} + \frac{-t_1}{t_2 - t_1} \xi_{t_2}.
$$

Since $E$ is rank one convex,

$$
\xi_{t_1}, \xi_{t_2} \in E \quad \text{and} \quad \text{rank} \{\xi_{t_1} - \xi_{t_2}\} = 1
$$

we have that $\xi \in E$. ■

7.4.1 The case of singular values

One of the most general examples of such hulls concerns sets that involve singular values. Let us first recall (see Chapter 13) that the singular values of a given matrix $\xi \in \mathbb{R}^{n \times n}$, denoted by $0 \leq \lambda_1 (\xi) \leq \cdots \leq \lambda_n (\xi)$, are the eigenvalues of $(\xi \xi^T)^{1/2}$.

Our result (see Dacorogna-Tanteri [214], [215] and also [202] for the first two cases and Dacorogna-Ribeiro [212] for the third case) is the following.

Theorem 7.43 Let $0 < \gamma_1 \leq \cdots \leq \gamma_n$.

Part 1. If

$$
E = \{ \xi \in \mathbb{R}^{n \times n} : \lambda_i (\xi) = \gamma_i, \ i = 1, \cdots, n \},
$$

then

$$
\text{co} E = \{ \xi \in \mathbb{R}^{n \times n} : \sum_{i=\nu}^{n} \lambda_i (\xi) \leq \sum_{i=\nu}^{n} \gamma_i, \ \nu = 1, \cdots, n \}
$$

$$
Pco \ E = \text{Qco} \ E = \text{Rco} \ E = \{ \xi \in \mathbb{R}^{n \times n} : \prod_{i=\nu}^{n} \lambda_i (\xi) \leq \prod_{i=\nu}^{n} \gamma_i, \ \nu = 1, \cdots, n \}.
$$

Part 2. If $\alpha \neq 0$,

$$
\prod_{i=1}^{n} \gamma_i = |\alpha|
$$

and

$$
E_\alpha = \{ \xi \in \mathbb{R}^{n \times n} : \lambda_i (\xi) = \gamma_i, \ i = 1, \cdots, n, \ \det \xi = \alpha \}
$$

then

$$
Pco \ E_\alpha = \text{Qco} \ E_\alpha = \text{Rco} \ E_\alpha = \{ \xi \in \mathbb{R}^{n \times n} : \prod_{i=\nu}^{n} \lambda_i (\xi) \leq \prod_{i=\nu}^{n} \gamma_i, \ \nu = 2, \cdots, n, \ \det \xi = \alpha \}.$$
Furthermore,

\[ \text{int} \, Rco \, E_\alpha = \{ \xi \in \mathbb{R}^{n \times n} : \prod_{i=2}^{n} \lambda_i(\xi) < \prod_{i=2}^{n} \gamma_i, \, \nu = 2, \ldots, n, \, \det \xi = \alpha \} \]

where the interior is to be understood relative to the manifold \( \{ \det \xi = \alpha \} \).

Part 3. Let \( \alpha \leq \beta \). If either \( \alpha \neq 0 \) or \( \beta \neq 0 \),

\[ \gamma_2 \prod_{i=2}^{n} \gamma_i \geq \max \{ |\alpha|, |\beta| \} \]

and

\[ E_{\alpha,\beta} = \{ \xi \in \mathbb{R}^{n \times n} : \lambda_i(\xi) = \gamma_i, \, i = 2, \ldots, n, \, \det \xi \in (\alpha, \beta) \} , \]

then

\[ Pco \, E_{\alpha,\beta} = Qco \, E_{\alpha,\beta} = Rco \, E_{\alpha,\beta} = \{ \xi \in \mathbb{R}^{n \times n} : \prod_{i=2}^{n} \lambda_i(\xi) \leq \prod_{i=2}^{n} \gamma_i, \, \nu = 2, \ldots, n, \, \det \xi \in [\alpha, \beta] \} . \]

Moreover, if \( \alpha < \beta \), then

\[ \text{int} \, Rco \, E_{\alpha,\beta} = \{ \xi \in \mathbb{R}^{n \times n} : \prod_{i=2}^{n} \lambda_i(\xi) < \prod_{i=2}^{n} \gamma_i, \, \nu = 2, \ldots, n, \, \det \xi \in (\alpha, \beta) \} . \]

Before proceeding with the proof, let us make some comments. Since \( |\det \xi| = \prod_{i=1}^{n} \lambda_i(\xi) \) and the singular values are ordered as \( 0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi) \), we should, and did, respectively impose in the second and third cases that

\[ \prod_{i=1}^{n} \gamma_i = |\alpha| \]

\[ \gamma_2 \prod_{i=2}^{n} \gamma_i \geq \max \{ |\alpha|, |\beta| \} . \]

**Proof.** First note that the third case contains the other ones as particular cases. Indeed the first one is deduced from the last one by setting \( \beta = -\alpha \) and

\[ \gamma_1 = \beta \left[ \prod_{i=2}^{n} \gamma_i \right]^{-1} , \]

while the second one is obtained by setting \( \beta = \alpha \) and

\[ \gamma_1 = |\alpha| \left[ \prod_{i=2}^{n} \gamma_i \right]^{-1} \]

in the third case.

We therefore divide the proof into four parts: the first dealing with the representation of \( co \, E \), the second with the formulas of the polyconvex, quasiconvex and rank one convex hulls of \( E_{\alpha,\beta} \) (and thus of \( E \) and \( E_{\alpha} \)) and the third and fourth with the representations of \( \text{int} \, Rco \, E_{\alpha,\beta} \) and \( \text{int} \, Rco \, E_{\alpha} \), respectively.
(i) Representation of co $E$. Let
\[ X := \{ \xi \in \mathbb{R}^{n \times n} : \sum_{i=\nu}^{n} \lambda_i(\xi) \leq \sum_{i=\nu}^{n} \gamma_i, \ \nu = 1, \ldots, n \} \]

**Step 1:** co $E \subset X$. The inclusion is easy, since $E \subset X$ and the functions $\xi \to \sum_{i=\nu}^{n} \lambda_i(\xi)$ are convex (see Corollary 5.37), we find that $X$ is convex and thus the inclusion co $E \subset X$.

**Step 2:** $X \subset$ co $E$. Let $\xi \in X$. Since the functions $\xi \to \lambda_i(\xi)$ are invariant by orthogonal transformations, we can assume (see Theorem 13.3), without loss of generality, that
\[
\xi = \text{diag}(x_1, \ldots, x_n) = \begin{pmatrix}
x_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & x_n
\end{pmatrix}
\]
with $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ and
\[ \sum_{i=\nu}^{n} x_i \leq \sum_{i=\nu}^{n} \gamma_i, \ \nu = 1, \ldots, n. \]

We proceed by induction. The case $n = 1$ is easy. Indeed let $\xi = x_1 \in X$ meaning that $0 \leq x_1 \leq \gamma_1$. We then write
\[ x_1 = t\gamma_1 + (1-t)(-\gamma_1) \text{ with } t = \frac{x_1 + \gamma_1}{2\gamma_1} \]
and we can thus deduce that $\xi \in$ co $E$, as claimed.

We now assume that the result has been proved up to the order $n - 1$ and prove the claim for $n$. We divide the study into two cases.

Case 1: $\sum_{i=\nu}^{n} x_i = \sum_{i=\nu}^{n} \gamma_i$ for a certain $\nu \in \{2, \ldots, n\}$. Observe that we can apply the hypothesis of induction to
\[ \{x_1, \ldots, x_{\nu-1}\} \text{ and } \{\gamma_1, \ldots, \gamma_{\nu-1}\} \]
and to
\[ \{x_\nu, \ldots, x_n\} \text{ and } \{\gamma_\nu, \ldots, \gamma_n\}. \]

Indeed for the second one this follows from the hypotheses
\[ \sum_{i=\nu}^{n} x_i \leq \sum_{i=\nu}^{n} \gamma_i, \ \nu = \nu, \ldots, n \]
while for the first one we have, for $\nu = 1, \ldots, \nu - 1$,
\[ \sum_{i=\nu}^{\nu-1} x_i = \sum_{i=\nu}^{n} x_i - \sum_{i=\nu}^{\nu} x_i = \sum_{i=\nu}^{n} x_i - \sum_{i=\nu}^{n} \gamma_i \]
\[ \leq \sum_{i=\nu}^{\nu} \gamma_i - \sum_{i=\nu}^{\nu} \gamma_i = \sum_{i=\nu}^{\nu-1} \gamma_i. \]
We can therefore deduce, by hypothesis of induction, that $\xi \in \text{co } E$.

Case 2: $\sum_{i=\nu}^{n} x_i < \sum_{i=\nu}^{n} \gamma_i$ for every $\nu \in \{2, \cdots, n\}$. We then let

$$L := \left\{ \eta \in \mathbb{R}^{n \times n} : \sum_{i=\nu}^{n} \lambda_i(\eta) \leq \sum_{i=\nu}^{n} \gamma_i, \ \nu = 2, \cdots, n \right\}$$

Observe that $L \subset X$ is compact and that $\xi \in \text{relint } (L)$ (where relint $(L)$ stands for the relative interior of $L$), since

$$\sum_{i=\nu}^{n} \lambda_i(\xi) = \sum_{i=\nu}^{n} x_i < \sum_{i=1}^{n} \gamma_i$$

Note also that by Case 1 we have $\partial L \subset \text{co } E$. We therefore let, for $t \in \mathbb{R}$,

$$\xi_t := \text{diag } (x_1, \cdots, x_{n-2}, x_{n-1} - t, x_n + t).$$

Observe that by compactness of $L$ and since $\xi = \xi_0 \in \text{relint } (L)$ we can find (as in Lemma 7.42) $t_1 < 0 < t_2$ such that

$$\xi_{t_1}, \xi_{t_2} \in \partial L \subset \text{co } E \quad \text{and} \quad \xi = \frac{t_2}{t_2 - t_1} \xi_{t_1} + \frac{-t_1}{t_2 - t_1} \xi_{t_2}.$$

We have therefore obtained that $\xi \in \text{co } E$ and hence the claimed result $X \subset \text{co } E$.

(ii) Formula for $P_{\text{co } E_{\alpha, \beta}}$, $Q_{\text{co } E_{\alpha, \beta}}$ and $R_{\text{co } E_{\alpha, \beta}}$. We let

$$Y := \{ \xi \in \mathbb{R}^{n \times n} : \prod_{i=\nu}^{n} \lambda_i(\xi) \leq \prod_{i=\nu}^{n} \gamma_i, \ \nu = 2, \cdots, n, \ \det \xi \in [\alpha, \beta] \}$$

and we wish to show that

$$Y = P_{\text{co } E_{\alpha, \beta}} = Q_{\text{co } E_{\alpha, \beta}} = R_{\text{co } E_{\alpha, \beta}}.$$

Since we always have $R_{\text{co } E_{\alpha, \beta}} \subset Q_{\text{co } E_{\alpha, \beta}} \subset P_{\text{co } E_{\alpha, \beta}}$, it is sufficient to show that $P_{\text{co } E_{\alpha, \beta}} \subset Y$ and then $Y \subset R_{\text{co } E_{\alpha, \beta}}$.

**Step 1:** $P_{\text{co } E_{\alpha, \beta}} \subset Y$. This is the easy inclusion. Indeed observe that $E_{\alpha, \beta} \subset Y$ and that the functions

$$\xi \rightarrow \pm \det \xi, \quad \xi \rightarrow \prod_{i=\nu}^{n} \lambda_i(\xi), \ \nu = 2, \cdots, n,$$

are polyconvex (see Example 5.41). We therefore have that the set $Y$ is polyconvex and thus the desired inclusion.

**Step 2:** $Y \subset R_{\text{co } E_{\alpha, \beta}}$. Since the set $Y$ is compact (the function $\xi \rightarrow \lambda_n(\xi)$ being a norm), it is enough (see Lemma 7.42) to show that $\partial Y \subset R_{\text{co } E_{\alpha, \beta}}$. So we let $\xi \in \partial Y$ and we wish to prove that $\xi \in R_{\text{co } E_{\alpha, \beta}}$. Note that

$$\partial Y = Y_{\alpha} \cup Y_{\beta} \cup Y_2 \cup \cdots \cup Y_n.$$
where
\[ Y_\alpha := \{ \xi \in Y : \det \xi = \alpha \}, \quad Y_\beta := \{ \xi \in Y : \det \xi = \beta \}, \]
\[ Y_\nu := \{ \xi \in Y : \prod_{i=\nu}^n \lambda_i(\xi) = \prod_{i=\nu}^n \gamma_i \}, \quad \nu = 2, \ldots, n. \]
Since all the functions involved in the definition of \( Y \) are right and left \( \text{SO}(n) \) invariant, there is no loss of generality (see Theorem 13.3) in assuming that \( \xi \) is diagonal
\[ \xi = \text{diag}(x_1, x_2, \ldots, x_n), \]
with \( 0 \leq |x_1| \leq x_2 \leq \cdots \leq x_n \). We therefore have
\[ \lambda_1(\xi) = |x_1|, \quad \lambda_i(\xi) = x_i, \quad i = 2, \ldots, n. \]

We now proceed by induction on the dimension \( n \); when \( n = 1 \) the result is trivial. Several possibilities can then happen, bearing in mind that \( \xi \in \partial Y \).

Case 1: \( \xi \in Y_\nu \) for a certain \( \nu = 2, \ldots, n \), meaning that
\[ n \prod_i x_i = \prod_i \gamma_i. \]

We write \( \xi \in \mathbb{R}^{n \times n} \) as two blocks, one in \( \mathbb{R}^{(\nu-1) \times (\nu-1)} \) and one in \( \mathbb{R}^{(n-\nu+1) \times (n-\nu+1)} \) in the following way:
\[ \xi = \text{diag}(\xi_{\nu-1}, \xi_{n-\nu+1}) \]
where
\[ \xi_{\nu-1} = \text{diag}(x_1, \ldots, x_{\nu-1}) \quad \text{and} \quad \xi_{n-\nu+1} = \text{diag}(x_{\nu}, \ldots, x_n). \]

We then apply the hypothesis of induction on \( \xi_{\nu-1} \) and \( \xi_{n-\nu+1} \) (we will check that we can do so below) and we deduce that \( \xi \in \text{Rco } E_{\alpha, \beta} \). Let us now see that we can apply the hypothesis of induction first for \( \xi_{\nu-1} \).

\[ \gamma_2 \prod_{i=2}^{\nu-1} \lambda_i = \gamma_2 \prod_{i=2}^{\nu-1} \gamma_i (\prod_{i=\nu}^n \gamma_i)^{-1} \geq \max\left\{ \frac{|\alpha|}{\gamma_{\nu-1} \cdots \gamma_n}, \frac{|\beta|}{\gamma_{\nu-1} \cdots \gamma_n} \right\}, \]
\[ \det \xi_{\nu-1} = \prod_{i=1}^{\nu-1} x_i = \prod_{i=1}^{\nu-1} x_i (\prod_{i=\nu}^n x_i)^{-1} = \prod_{i=1}^{\nu-1} x_i (\prod_{i=\nu}^n \gamma_i)^{-1} \]
\[ = \det \xi (\prod_{i=\nu}^n \gamma_i)^{-1} \in \left[ \frac{\alpha}{\gamma_{\nu-1} \cdots \gamma_n}, \frac{\beta}{\gamma_{\nu-1} \cdots \gamma_n} \right], \]
\[ \prod_{i=\nu}^{\nu-1} \lambda_i (\xi_{\nu-1}) = \prod_{i=\nu}^{\nu-1} x_i (\prod_{i=\nu}^n x_i)^{-1} = \prod_{i=\nu}^{\nu-1} x_i (\prod_{i=\nu}^n \gamma_i)^{-1} \]
\[ \leq \prod_{i=\nu}^{\nu-1} \gamma_i, \quad \nu = 2, \ldots, \nu - 1 \]
and thus the result.
Similarly for $\xi_{n-\nu+1}$ since (here the roles for both $\alpha$ and $\beta$ are played by $\prod_{i=\nu}^{n} \gamma_{i}$)

$$
\gamma_{\nu+1} \prod_{i=\nu+1}^{n} \gamma_{i} \geq \prod_{i=\nu}^{n} \gamma_{i},
$$

$$
\det \xi_{n-\nu+1} = \prod_{i=\nu}^{n} x_{i} = \prod_{i=\nu}^{n} \gamma_{i},
$$

$$
\prod_{i=\nu-\nu+1}^{n-\nu+1} \lambda_{i}(\xi_{n-\nu+1}) = \prod_{i=\nu}^{n} x_{i} \leq \prod_{i=\nu}^{n} \gamma_{i}, \nu = \nu, \ldots, n
$$

we have the claim.

**Case 2:** $\xi \in Y_{\alpha}$ (and similarly for the case $\xi \in Y_{\beta}$). We can also assume that $\xi \notin Y_{\nu}, \nu = 2, \ldots, n$, otherwise we apply Case 1. We therefore have

$$
\xi \in \text{int} Y_{\alpha} = \left\{ \eta \in \mathbb{R}^{n \times n} : \det \eta = \alpha, \prod_{i=\nu}^{n} \lambda_{i}(\eta) < \prod_{i=\nu}^{n} \gamma_{i}, \nu = 2, \ldots, n \right\}.
$$

This is clearly an open set (relative to the manifold $\{\det \eta = \alpha\}$). Recall that

$$
\xi = \text{diag}(x_{1}, \ldots, x_{n}) = \begin{pmatrix}
x_{1} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & x_{n}
\end{pmatrix}.
$$

We then set for $t \in \mathbb{R}$

$$
\xi_{t} := \begin{pmatrix}
x_{1} & \cdots & 0 & 0 \\
0 & \ddots & \vdots & \vdots \\
0 & \cdots & x_{n-1} & t \\
0 & \cdots & 0 & x_{n}
\end{pmatrix}
$$

and observe that $\det \xi_{t} = \det \xi = \alpha$. Since $\text{int} Y_{\alpha}$ is bounded we can find (as in Lemma 7.42) $t_{1} < 0 < t_{2}$ so that $\xi_{t_{1}}, \xi_{t_{2}} \in \partial Y_{\alpha}$ which means that $\xi_{t_{i}} \in Y_{\nu_{i}}, i = 1, 2$, for a certain $\nu_{i} = 2, \cdots, n$ and therefore, by Case 1, $\xi_{t_{i}} \in \text{Rco} E_{\alpha, \beta}$ and thus, since $\text{rank}\{\xi_{t_{1}} - \xi_{t_{2}}\} = 1$ and

$$
\xi = \frac{t_{2}}{t_{2} - t_{1}} \xi_{t_{1}} + \frac{-t_{1}}{t_{2} - t_{1}} \xi_{t_{2}},
$$

we deduce that $\xi \in \text{Rco} E_{\alpha, \beta}$ as wished.

**(iii)** Representation formula for int $\text{Rco} E_{\alpha, \beta}$. Let

$$
Z := \left\{ \xi \in \mathbb{R}^{n \times n} : \prod_{i=\nu}^{n} \lambda_{i}(\xi) < \prod_{i=\nu}^{n} \gamma_{i}, \nu = 2, \cdots, n, \det \xi \in (\alpha, \beta) \right\}.
$$

We wish to show that int $\text{Rco} E_{\alpha, \beta} = Z$.

The inclusion $Z \subset \text{int} \text{Rco} E_{\alpha, \beta}$ is clear, since by continuity $Z$ is open and by the representation formula for $\text{Rco} E_{\alpha, \beta}$ we have $Z \subset \text{Rco} E_{\alpha, \beta}$.
We now prove the reverse inclusion \( \text{int} \ Rco E_{\alpha,\beta} \subset Z \). So let \( \xi \in \text{int} \ Rco E_{\alpha,\beta} \). We can find \( R, Q \in SO(n) \) (see Theorem 13.3) so that
\[
\xi = R \text{diag}(\pm \lambda_1(\xi), \cdots, \lambda_n(\xi))Q.
\]
Since \( \xi \in \text{int} \ Rco E_{\alpha,\beta} \), we can find \( \epsilon \) sufficiently small so that \( B_{2\epsilon}(\xi) \subset Rco E_{\alpha,\beta} \) (where \( B_{2\epsilon}(\xi) \) denotes the ball centered at \( \xi \) and of radius \( 2\epsilon \)).

We consider two cases.

Case 1: \( \lambda_\nu(\xi) \neq 0 \) for every \( \nu \in \{1, \cdots, n\} \). Define
\[
\eta_+ := R \text{diag}(\pm \lambda_1(\xi), \cdots, \lambda_{n-1}(\xi), \lambda_n(\xi) + \epsilon)Q,
\]
\[
\eta_- := R \text{diag}(\pm \lambda_1(\xi) - \epsilon, \lambda_2(\xi), \cdots, \lambda_n(\xi))Q.
\]
Since \( |\eta_\pm - \xi| = \epsilon < 2\epsilon \), then \( \eta_\pm \in Rco E_{\alpha,\beta} \), meaning that
\[
\det \eta_\pm \in [\alpha, \beta], \prod_{i=\nu}^n \lambda_i(\eta_\pm) \leq \prod_{i=\nu}^n \gamma_i, \nu = 2, \cdots, n.
\]
This clearly implies that
\[
\det \xi \in (\alpha, \beta), \prod_{i=\nu}^n \lambda_i(\xi) < \prod_{i=\nu}^n \gamma_i, \nu = 2, \cdots, n
\]
which just means that \( \xi \in Z \), as wished.

Case 2: \( \lambda_\nu(\xi) = 0 \) and \( \lambda_{\nu+1}(\xi) > 0 \) for a certain \( \nu \in \{1, \cdots, n\} \) (if \( \nu = n \), this means that \( \xi = 0 \)). Letting \( \delta = \epsilon/\sqrt{\nu+1} \), we define
\[
\eta_\pm := R \text{diag}(\pm \delta, \delta, \cdots, \delta, \lambda_{\nu+1}(\xi), \cdots, \lambda_{n-1}(\xi), \lambda_n(\xi) + \delta)Q.
\]
We therefore have \( |\eta_\pm - \xi| = \epsilon < 2\epsilon \) and thus \( \eta_\pm \in Rco E_{\alpha,\beta} \), meaning that
\[
\det \eta_\pm \in [\alpha, \beta], \prod_{i=\nu}^n \lambda_i(\eta_\pm) \leq \prod_{i=\nu}^n \gamma_i, \nu = 2, \cdots, n.
\]
This obviously implies that
\[
\det \xi \in (\alpha, \beta), \prod_{i=\nu}^n \lambda_i(\xi) < \prod_{i=\nu}^n \gamma_i, \nu = 2, \cdots, n
\]
and hence \( \xi \in Z \), as wished.

(iv) **Representation formula for** \( \text{int} \ Rco E_\alpha \). This is proved exactly as above. We assume that \( \alpha > 0 \), the case \( \alpha < 0 \) being handled similarly. We let
\[
K := \{ \xi \in \mathbb{R}^{n \times n} : \prod_{i=\nu}^n \lambda_i(\xi) < \prod_{i=\nu}^n \gamma_i, \nu = 2, \cdots, n, \det \xi = \alpha \}.
\]
As above the inclusion $K \subseteq \text{int } Rco E_\alpha$ is obvious. Let us show the second one and therefore let $\xi \in \text{int } Rco E_\alpha$ and find $R, Q \in SO(n)$ such that

$$\xi = R \text{ diag}(\lambda_1(\xi), \cdots, \lambda_n(\xi))Q.$$ 

Since $\xi \in \text{int } Rco E_\alpha$, we can find $\varepsilon$ sufficiently small so that $B_\varepsilon(\xi) \subset Rco E_\alpha$ (where $B_\varepsilon(\xi)$ denotes the ball, restricted to the manifold $\text{det } \xi = \alpha$, centered at $\xi$ and of radius $\varepsilon$). Define next, for $\delta > 0$ sufficiently small,

$$\eta := R \text{ diag}(\frac{\lambda_1(\xi)}{1 + \delta}, \cdots, \frac{\lambda_{n-1}(\xi)}{1 + \delta}, (1 + \delta)^{n-1} \lambda_n(\xi))Q,$$

so that $\eta \in B_\varepsilon(\xi) \subset Rco E_\alpha$ and thus

$$\text{det } \eta = \alpha, \prod_{i=\nu}^n \lambda_i(\eta) \leq \prod_{i=\nu}^n \gamma_i, \quad \nu = 2, \cdots, n.$$ 

This clearly shows that $\xi \in K$, as wished. ■

### 7.4.2 The case of potential wells

We now give a representation formula for $Rco E$ where

$$E := SO(2)A \cup SO(2)B$$

and $\text{det } A, \text{det } B > 0$.

Up to rotation and dilation, we can assume without loss of generality that

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$$

with $0 < a_1, a_2, b_1, b_2$ and $\frac{b_1}{a_1} \leq \frac{b_2}{a_2}$; and we assume throughout this section that $A$ and $B$ have this particular form.

We denote the elements of $SO(2)$ by $R_\theta$, i.e.,

$$R_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$ 

The following result was established by Sverak [554].

**Theorem 7.44** Let

$$E := SO(2)A \cup SO(2)B,$$

then

$$\text{co } E = \left\{ \xi \in \mathbb{R}^{2 \times 2} : \begin{array}{c} \xi = \alpha R_a A + \beta R_b B, \quad R_a, R_b \in SO(2), \\
0 \leq \alpha, \beta, \alpha + \beta \leq 1 \end{array} \right\}.$$ 

Furthermore, if $\text{det } (R_\theta A - B) = 0$ for a certain $R_\theta \in SO(2)$, the following results hold.
Case 1. If \( \det B = \det A > 0 \), then

\[
P_{\text{co}} E = \text{Qco} E = \text{Rco} E = \{ \xi \in \mathbb{R}^{2 \times 2} : \xi = \alpha R_a A + \beta R_b B, \ R_a, R_b \in SO(2), \ 0 \leq \alpha, \beta, \alpha + \beta \leq 1 \text{ and } \det \xi = \det A = \det B \}.
\]

Case 2. If \( \det B > \det A > 0 \), then

\[
P_{\text{co}} E = \text{Qco} E = \text{Rco} E = \{ \xi \in \mathbb{R}^{2 \times 2} : \xi = \alpha R_a A + \beta R_b B, \ R_a, R_b \in SO(2), \ 0 \leq \alpha \leq \frac{\det B - \det \xi}{\det B - \det A}, \ 0 \leq \beta \leq \frac{\det \xi - \det A}{\det B - \det A} \}.
\]

Moreover, in this last case, the interior of \( \text{Rco} E \) is given by the same formula with strict inequalities on the right hand side.

Remark 7.45

(i) If the wells are not rank one connected meaning that there exists no \( R_\theta \in SO(2) \) such that \( \det (R_\theta A - B) = 0 \), then it will be obvious from the proof that in this case \( E = \text{Rco} E \). This connection of the wells for \( A, B \) as above is equivalent to

\[
\frac{b_1}{a_1} \leq 1 \leq \frac{b_2}{a_2}.
\]

(ii) When \( 0 < b_1 < a_1 \leq a_2 < b_2 \), then (see Corollary 8.3 in Dacorogna-Marcellini [202]) matrices of the form

\[
A^\delta = \begin{pmatrix} a_1 - \delta & 0 \\ 0 & a_2 + T\delta \end{pmatrix}
\]

are in \( \text{int Rco} E \) for every \( \delta > 0 \) sufficiently small and for \( T \) satisfying

\[
a_2 (b_2 - a_2) (a_1 + b_1) \frac{a_2}{a_1 (b_2 + a_2) (a_1 - b_1)} < T < \frac{b_1 (b_2^2 - a_2^2)}{b_2 (a_1^2 - b_1^2)}.\]

In a similar manner, for appropriate \( S > 0 \), matrices of the form

\[
B^\delta = \begin{pmatrix} b_1 + S\delta & 0 \\ 0 & b_2 - \delta \end{pmatrix}
\]

are in \( \text{int Rco} E \) for every \( \delta > 0 \) sufficiently small. \( \diamond \)

Proof. We start with the following obvious observation. For every \( \alpha, \beta \geq 0, \ R_a, R_b \in SO(2) \), there exist \( \gamma \geq 0 \) and \( R_c \in SO(2) \) such that

\[
\alpha R_a + \beta R_b = \gamma R_c \quad \text{with} \quad \gamma \leq \alpha + \beta.
\]

(7.13)

Indeed, just choose

\[
\gamma^2 = (\alpha \cos a + \beta \cos b)^2 + (\alpha \sin a + \beta \sin b)^2.
\]
and
\[
\cos c = \frac{\alpha \cos a + \beta \cos b}{\gamma}, \quad \sin c = \frac{\alpha \sin a + \beta \sin b}{\gamma}.
\]

**Part 1. Formula for \( \text{co } E \).** We let
\[
X := \{ \xi \in \mathbb{R}^{2 \times 2} : \xi = \alpha R_a A + \beta R_b B, \ R_a, R_b \in SO(2), \ 0 \leq \alpha, \beta, \alpha + \beta \leq 1 \}.
\]

We will prove that \( X = \text{co } E \) in two steps.

**Step 1.** We first show that \( \text{co } E \subset X \). Since \( E \subset X \), we will have the claimed inclusion if we can show that \( X \) is convex. So let \( \xi_1, \xi_2 \in X \) and \( t \in [0, 1] \), then
\[
t \xi_1 + (1 - t) \xi_2 = t (\alpha_1 R_a A + \beta_1 R_b B) + (1 - t) (\alpha_2 R_a A + \beta_2 R_b B).
\]
Using (7.13), we obtain that
\[
t \xi_1 + (1 - t) \xi_2 = \alpha R_a A + \beta R_b B,
\]
\[
0 \leq \alpha \leq t \alpha_1 + (1 - t) \alpha_2, \ 0 \leq \beta \leq t \beta_1 + (1 - t) \beta_2.
\]
Hence
\[
\alpha + \beta \leq t (\alpha_1 + \beta_1) + (1 - t) (\alpha_2 + \beta_2) \leq 1
\]
and thus \( X \) is convex.

**Step 2.** We now prove that \( X \subset \text{co } E \). So let \( \xi \in X \) then
\[
\xi = \alpha R_a A + \beta R_b B
\]
\[
= \frac{1 + \alpha + \beta}{2} \left[ \frac{2\alpha}{1 + \alpha + \beta} R_a A + \frac{1 + \beta - \alpha}{1 + \alpha + \beta} R_b B \right] + \frac{1 - \alpha - \beta}{2} R_{b + \pi} B.
\]
Note that
\[
R_a A, R_b B, R_{b + \pi} B \in E
\]
and hence \( \xi \in \text{co } E \). This achieves the proof of this part.

**Part 2. Formula for \( \text{Pco } E, \text{Qco } E \) and \( \text{Rco } E \).** We first observe that up to rotations and dilations (replacing \( E \) by \( EA^{-1} \) and \( B \) by \( BA^{-1} \)) we can further restrict ourselves to considering
\[
A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \text{with } \lambda \geq \mu \geq 0.
\]
The fact that the wells are rank one connected implies that \( \lambda \geq 1 \geq \mu \); however, if we want the problem to be non-trivial, we also assume that \( \lambda > \mu \).

We let
\[
Y := \{ \xi \in \mathbb{R}^{2 \times 2} : \xi = \alpha R_a + \beta R_b B, \ R_a, R_b \in SO(2), \ 0 \leq \alpha, \beta, \alpha + \beta \leq 1 \text{ and } \det \xi = 1 \}
\]
Polyconvex, quasiconvex and rank one convex sets

and

\[ Z := \left\{ \xi \in \mathbb{R}^{2 \times 2} : \begin{array}{c}
\xi = \alpha R_a + \beta R_b B, \ R_a, R_b \in SO(2), \\
0 \leq \alpha \leq \frac{\det B - \det \xi}{\det B - 1}, \ 0 \leq \beta \leq \frac{\det \xi - 1}{\det B - 1}
\end{array} \right\}. \]

We have to prove that when \( \det B = 1 \), then

\[ Y = \text{Pco } E = \text{Qco } E = \text{Rco } E \]

and similarly when \( \det B \neq 1 \), then

\[ Z = \text{Pco } E = \text{Qco } E = \text{Rco } E. \]

Since we always have \( \text{Rco } E \subset \text{Qco } E \subset \text{Pco } E \), we only need to show that \( Y \subset \text{Rco } E \) and \( \text{Pco } E \subset Y \) and similarly for \( Z \). This is achieved through the following steps.

**Step 1:** \( \text{Pco } E \subset Y \) and \( \text{Pco } E \subset Z \). We clearly have respectively \( E \subset Y \) and \( E \subset Z \); so if we can show that both sets are polyconvex, we will have the desired inclusions. Let \( \xi_i \in Y \) (respectively \( Z \)), \( t \in \Lambda_6 \) be such that

\[ \xi := \sum_{i=1}^{6} t_i \xi_i \text{ and } \sum_{i=1}^{6} t_i \det \xi_i = \det(\sum_{i=1}^{6} t_i \xi_i), \]

where

\[ \Lambda_6 := \left\{ \lambda = (\lambda_1, \cdots, \lambda_6) : \lambda_i \geq 0, \ \sum_{i=1}^{6} \lambda_i = 1 \right\}. \]

We have to show that \( \xi \in Y \) (respectively \( Z \)). We therefore have by iterating (7.13) that

\[ \xi = \sum_{i=1}^{6} t_i \xi_i = \sum_{i=1}^{6} t_i \alpha_i R_{a_i} + (\sum_{i=1}^{6} t_i \beta_i R_{b_i})B \]

with

\[ 0 \leq \alpha \leq \sum_{i=1}^{6} t_i \alpha_i \text{ and } 0 \leq \beta \leq \sum_{i=1}^{6} t_i \beta_i. \]

Using the fact that

\[ \det \xi = \sum_{i=1}^{6} t_i \det \xi_i \]

we deduce that \( \xi \in Y \) (respectively \( Z \)).

**Step 2.** We next establish a decomposition of matrices that keeps the determinant fixed and allows movements in rank one directions. Namely let

\[ \xi = \alpha R_{\theta} + \beta B = \alpha \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} + \beta \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \]

and assume that \( \det \xi > 0 \). We can then find \( s \) and \( \varphi \) such that

\[ \xi = \sqrt{\det \xi} \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} + s \begin{pmatrix} 1 + \sin \varphi & -\cos \varphi \\ \cos \varphi & -1 + \sin \varphi \end{pmatrix}. \quad (7.14) \]
Indeed we have to solve
\[
\begin{align*}
\alpha \cos \theta + \beta \lambda &= \sqrt{\det \xi} \cos \varphi + s (1 + \sin \varphi) \\
\alpha \cos \theta + \beta \mu &= \sqrt{\det \xi} \cos \varphi + s (-1 + \sin \varphi) \\
\alpha \sin \theta &= \sqrt{\det \xi} \sin \varphi - s \cos \varphi.
\end{align*}
\]

We thus choose
\[
s = \frac{\beta}{2} (\lambda - \mu)
\]
(note that if $\beta > 0$ then $s > 0$ since $\lambda > \mu$) and then solve
\[
\begin{cases}
s \sin \varphi + \sqrt{\det \xi} \cos \varphi = \alpha \cos \theta + \frac{\beta}{2} (\lambda + \mu) \\
\sqrt{\det \xi} \sin \varphi - s \cos \varphi = \alpha \sin \theta.
\end{cases}
\]  
(7.15)

Observe that this system is indeed solvable since taking the square of each side of each equation, summing them and using the fact that
\[
\det \xi = \alpha^2 + \beta^2 \lambda \mu + \alpha \beta (\lambda + \mu) \cos \theta
\]
we get that they are compatible. This therefore leads to
\[
\begin{align*}
(\det \xi + s^2) \sin \varphi &= \alpha s \cos \theta + \frac{\beta s}{2} (\lambda + \mu) + \alpha \sqrt{\det \xi} \sin \theta \\
(\det \xi + s^2) \cos \varphi &= \alpha \sqrt{\det \xi} \cos \theta + \frac{\beta}{2} \sqrt{\det \xi} (\lambda + \mu) - \alpha s \sin \theta.
\end{align*}
\]

A similar decomposition can be made for
\[
\xi = \alpha I + \beta R_b B = \begin{pmatrix}
\alpha + \beta \lambda \cos \theta & \beta \mu \sin \theta \\
-\beta \lambda \sin \theta & \alpha + \beta \mu \cos \theta
\end{pmatrix}.
\]  
(7.16)

**Step 3:** $Y \subset R_{co E}$. Let us first observe that if
\[
Y_1 := \left\{ \xi \in \mathbb{R}^{2 \times 2} : \begin{array}{l}
\xi = \alpha R_a + (1 - \alpha) R_b B, \ R_a, R_b \in SO(2), \\
0 \leq \alpha \leq 1 \text{ and } \det \xi = 1
\end{array} \right\},
\]
then $Y_1 = R_1 \text{ co } E$, where we recall that
\[
R_{1 \text{ co } E} = \left\{ \xi \in \mathbb{R}^{2 \times 2} : \begin{array}{l}
\xi = t \xi_1 + (1 - t) \xi_2 \\
\xi_1, \xi_2 \in E, \ \det (\xi_1 - \xi_2) = 0
\end{array} \right\}
\] = \left\{ \xi \in \mathbb{R}^{2 \times 2} : \begin{array}{l}
\xi = t R_a + (1 - t) R_b B, \ R_a, R_b \in SO(2), \\
0 \leq t \leq 1 \text{ and } \det (R_a - R_b B) = 0
\end{array} \right\}.
\]

It is clear that the two sets are equal since no nontrivial rank one connection can be achieved in any of the two wells and
\[
\det (R_a I - R_b B) = 0 \Leftrightarrow \det \xi = t \det I + (1 - t) \det B = 1.
\]
We next show that \( Y \subset R_{2 \text{co}}E \) and thus the claim \( Y \subset R_{\text{co}}E \). We first should observe that on the manifold \( \det \xi = 1 \) we have that \( \partial Y = R_1 \text{co} E \). So let \( \xi \in Y \), then, in view of the previous observation, we can assume that \( \xi \in \text{int} Y \) (understood as relative to the manifold \( \det \xi = 1 \)); otherwise the result is already proved. Furthermore up to a rotation we can assume that \( \xi_1^2 + \xi_2^2 = 0 \). Upon multiplication, if necessary, by \(-I\), we can therefore assume, using (7.14), that there exist \( \alpha, \beta > 0 \), \( \varphi \) and \( s > 0 \) such that

\[
\xi = \alpha R_a + \beta B = \alpha \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} + \beta \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}
\]

\[
= \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} + s \begin{pmatrix} 1 + \sin \varphi & -\cos \varphi \\ \cos \varphi & -1 + \sin \varphi \end{pmatrix}.
\]

Set

\[
\xi_t = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} + t \begin{pmatrix} 1 + \sin \varphi & -\cos \varphi \\ \cos \varphi & -1 + \sin \varphi \end{pmatrix}.
\]

Observe that \( \xi_s = \xi \), \( \det \xi_t \equiv 1 \) and \( \xi_0 \in SO(2) \). Note also that since \( \xi \in \text{int} Y \) and \( Y \) is compact we can find \( t > s \) such that \( \xi_t \in \partial Y \). Therefore

\[
\xi_s = \xi = \left(1 - \frac{s}{t}\right) \xi_0 + \frac{s}{t} \xi_t.
\]

Since \( \xi_0 \in E \), \( \xi_t \in R_1 \text{co} E \) and \( \det (\xi_0 - \xi_t) = 0 \), we deduce that \( \xi \in R_2 \text{co} E \), which is the desired result.

**Step 4:** \( Z \subset R_{\text{co}}E \). We show exactly as in Step 3 that

\[
R_1 \text{co} E = \left\{ \xi \in \mathbb{R}^{2 \times 2} : \begin{array}{l} \xi = \alpha R_a + (1 - \alpha) R_b B, \\
R_a, R_b \in SO(2), \quad \alpha = \frac{\det B \det \xi}{\det B - 1} \end{array} \right\}.
\]

Let \( \xi \in Z \), since \( Z \) is compact, we have that any line containing \( \xi \) will intersect \( \partial Z \) and thus we can write

\[
\begin{aligned}
\xi &= t \xi_1 + (1 - t) \xi_2 \\
t &\in [0, 1], \quad \xi_1, \xi_2 \in \partial Z \\
\det (\xi_1 - \xi_2) &= 0.
\end{aligned}
\]

It is therefore sufficient to prove that \( \partial Z \subset R_{\text{co}}E \). We can write

\[
\partial Z = Z_1 \cup Z_2,
\]

where

\[
Z_1 : = \left\{ \xi \in \mathbb{R}^{2 \times 2} : \begin{array}{l} \xi = \alpha R_a + \beta R_b B, \quad R_a, R_b \in SO(2), \\
\alpha = \frac{\det B \det \xi}{\det B - 1}, \quad 0 \leq \beta \leq \frac{\det \xi - 1}{\det B - 1} \end{array} \right\},
\]

\[
Z_2 : = \left\{ \xi \in \mathbb{R}^{2 \times 2} : \begin{array}{l} \xi = \alpha R_a + \beta R_b B, \quad R_a, R_b \in SO(2), \\
0 \leq \alpha \leq \frac{\det B \det \xi}{\det B - 1}, \quad \beta = \frac{\det \xi - 1}{\det B - 1} \end{array} \right\}.
\]
We only prove that \( Z_2 \subset Rco E \) (the other inclusion being handled similarly but using (7.16)). Note that (on the manifold \( \beta = \frac{\det \xi - 1}{\det B - 1} \)) \( \partial Z_2 = R_1 co E \). So let \( \xi \in Z_2 \); in view of the preceding observation we can therefore assume that \( \xi \in int Z_2 \) (relative to the manifold \( \beta = \frac{\det \xi - 1}{\det B - 1} \)). Up to a rotation, we can assume as in Step 3 (using (7.14)) that
\[
\xi = \alpha R_\theta + \beta B = \sqrt{\det \xi} \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} + s \begin{pmatrix} 1 + \sin \varphi & -\cos \varphi \\ \cos \varphi & -1 + \sin \varphi \end{pmatrix},
\]
where
\[
\beta = \frac{\det \xi - 1}{\det B - 1}.
\]
If we then denote
\[
\xi_t = \sqrt{\det \xi} \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} + t \begin{pmatrix} 1 + \sin \varphi & -\cos \varphi \\ \cos \varphi & -1 + \sin \varphi \end{pmatrix},
\]
we have that
\[
\det \xi_t = \det \xi, \quad \xi_s = \xi \quad \text{and} \quad \det (\xi_{t_1} - \xi_{t_2}) = 0, \quad \forall t_1, t_2 \in \mathbb{R}.
\]
Since \( Z_2 \) is compact and \( \xi \in int Z_2 \), we have that there exist (as in Lemma 7.42) \( t_1 < s < t_2 \) such that \( \xi_{t_1}, \xi_{t_2} \in \partial Z_2 = R_1 co E \). We can hence write
\[
\xi = \xi_s = \frac{t_2 - s}{t_2 - t_1} \xi_{t_1} + \frac{s - t_1}{t_2 - t_1} \xi_{t_2},
\]
\[
\det (\xi_{t_1} - \xi_{t_2}) = 0, \quad \xi_{t_1}, \xi_{t_2} \in R_1 co E.
\]
Thus \( \xi \in Rco E \) and the proof is complete.

**Part 3. Formula for** \( \text{int} \ Rco E \). It is clear by continuity that if for \( \xi \in Rco E \) strict inequalities hold then \( \xi \in \text{int} \ Rco E \). We now show the converse. Assume for the sake of contradiction that \( \xi \in \text{int} \ Rco E \) and that one of the inequality is actually an equality. Without loss of generality we assume that it is the second one, more precisely
\[
\xi = \alpha R_a + \beta R_b B, \quad R_a, R_b \in SO(2),
\]
\[
0 \leq \alpha \leq \frac{\det B - \det \xi}{\det B - 1}, \quad \beta = \frac{\det \xi - 1}{\det B - 1}.
\]
Since \( \xi \in \text{int} \ Rco E \), we deduce that for \( t \) small enough
\[
\xi_t = \xi + tR_\theta \in Rco E, \quad \forall R_\theta \in SO(2),
\]
and hence by the representation formula for \( Rco E \) we deduce that
\[
\beta = \frac{\det \xi - 1}{\det B - 1} \leq \frac{\det \xi_t - 1}{\det B - 1} \Rightarrow \det \xi \leq \det \xi_t, \quad \forall t \text{ small enough}.
\]
Therefore we get
\[ \langle \xi; R_\theta \rangle = 0, \quad \forall R_\theta \in SO(2) \Rightarrow \xi = \begin{pmatrix} \sigma & \tau \\ \tau & -\sigma \end{pmatrix} \]
and hence \( \det \xi \leq 0 \), which is in contradiction with the fact that \( \det \xi \geq 1 \).

7.4.3 The case of a quasiaffine function

We need, prior to the main theorem, an elementary lemma, but we postpone its proof to the end of the present subsection. It will be used to assert that condition (7.17) below can be fulfilled by some \( c_j^i > 0 \) and will also be used in Theorem 10.29.

**Lemma 7.46** Let \( \Phi : \mathbb{R}^{N \times n} \to \mathbb{R} \) be a non-constant quasiaffine function and \( M, m > 0 \). Then there exist \( c_j^i > m, i = 1, \cdots, N, j = 1, \cdots, n \) such that
\[
\inf \{|\Phi(\xi)| : |\xi_j^i| = c_j^i \} > M.
\]

We can now state the main theorem, established by Dacorogna-Ribeiro [212].

**Theorem 7.47** Let \( \Phi : \mathbb{R}^{N \times n} \to \mathbb{R} \) be a non-constant quasiaffine function, \( \alpha < \beta, c_j^i > 0 \), satisfying
\[
\inf \{|\Phi(\xi)| : |\xi_j^i| = c_j^i \} > \max\{|\alpha|, |\beta|\}. \tag{7.17}
\]

Let
\[
E := \{ \xi \in \mathbb{R}^{N \times n} : \Phi(\xi) \in [\alpha, \beta], |\xi_j^i| \leq c_j^i, i = 1, \cdots, N, j = 1, \cdots, n \}.
\]
Then
\[
\text{Rco } E = \{ \xi \in \mathbb{R}^{N \times n} : \Phi(\xi) \in [\alpha, \beta], |\xi_j^i| \leq c_j^i, i = 1, \cdots, N, j = 1, \cdots, n \},
\]
\[
\text{int Rco } E = \{ \xi \in \mathbb{R}^{N \times n} : \Phi(\xi) \in (\alpha, \beta), |\xi_j^i| < c_j^i, i = 1, \cdots, N, j = 1, \cdots, n \}.
\]

**Proof.** Part 1. We let
\[
X := \{ \xi \in \mathbb{R}^{N \times n} : \Phi(\xi) \in [\alpha, \beta], |\xi_j^i| \leq c_j^i, i = 1, \cdots, N, j = 1, \cdots, n \}
\]
and we show that \( X = \text{Rco } E \). The inclusion \( \text{Rco } E \subset X \) follows from the combination of the facts that \( E \subset X \) and that the set \( X \) is rank one convex (the functions \( \Phi, -\Phi \) and \( |\cdot| \) being rank one convex).

We therefore have to show only that \( X \subset \text{Rco } E \). So we let \( \xi \in X \) and we can assume that \( \alpha < \Phi(\xi) < \beta \) otherwise the result is trivial. We observe that
(7.17) implies that for every \( \xi \in X \) there exists \((i, j)\) so that \( |\xi^i_j| < c^i_j \). So let for \( t \in \mathbb{R} \)

\[
\xi^t := \xi + te^i_j \otimes e_j
\]

where \( e^i \) (respectively \( e_j \)) is the \( i \)th (respectively \( j \)th) vector of the canonical basis of \( \mathbb{R}^N \) (respectively \( \mathbb{R}^n \)). Observe that by compactness there exist \( t_1 < 0 < t_2 \) so that \( \xi^{t_\nu} \in \partial X, \nu = 1, 2 \) which implies that either

\[
\Phi(\xi^{t_\nu}) \in \{\alpha, \beta\} \quad \text{or} \quad |(\xi^{t_\nu})^i_j| = c^i_j, \quad \nu = 1, 2.
\]

If the first possibility happens then we are done, if however the second case holds then we restart the process with a different \((i, j)\), since it is not possible by (7.17) that \( |(\xi^{t_\nu})^i_j| = c^i_j \) for every \((i, j)\).

**Part 2.** We now define

\[
Y := \{\xi \in \mathbb{R}^{N \times n} : \Phi(\xi) \in (\alpha, \beta), |\xi^i_j| < c^i_j, \ i = 1, \ldots, N, \ j = 1, \ldots, n\}
\]

and observe that since \( Y \subset \text{Rco} \ E \) and \( Y \) is open, then \( Y \subset \text{int} \ \text{Rco} \ E \). So let us show the reverse inclusion and choose \( \xi \in \text{int} \ \text{Rco} \ E \). Clearly such a \( \xi \) must have \( |\xi^i_j| < c^i_j \). Corollary 5.23 shows also that \( \xi \) should be so that \( \alpha < \Phi(\xi) < \beta \). These observations imply the result. \( \blacksquare \)

We now prove Lemma 7.46.

**Proof.** Since \( \Phi \) is quasiaffine, we can write

\[
\Phi(\xi) = \Phi(0) + \sum_{1 \leq q \leq N \land n} \sum_{1 \leq i_1 < \cdots < i_q \leq N} \mu_{j_1 \cdots j_q}^{i_1 \cdots i_q} \det \begin{pmatrix} \xi_{j_1}^{i_1} & \cdots & \xi_{j_q}^{i_1} \\ \vdots & \ddots & \vdots \\ \xi_{j_1}^{i_q} & \cdots & \xi_{j_q}^{i_q} \end{pmatrix}.
\]

Since \( \Phi \) is not constant we can find \( 1 \leq s \leq N \land n, \ 1 \leq i_1 < \cdots < i_s \leq N \) and \( 1 \leq j_1 < \cdots < j_s \leq N \) so that

\[
\mu_{j_1 \cdots j_s}^{i_1 \cdots i_s} \neq 0 \quad \text{and} \quad \mu_{j_1 \cdots j_q}^{i_1 \cdots i_q} = 0, \forall \ q > s.
\]

Assume, for notational convenience (the general case being handled similarly), that

\[
\mu_{1 \cdots s}^{1 \cdots s} \neq 0. \quad (7.18)
\]

Let us define the set

\[
\Theta = \{\theta \in \mathbb{R}^{N \times n} : \theta_{j}^{i} \in \{\pm 1\}\}
\]

and the product \( A \odot B \in \mathbb{R}^{N \times n} \), for two given matrices \( A, B \in \mathbb{R}^{N \times n} \), as

\[
(A \odot B)^{i}_j := A_{j}^{i} \cdot B_{j}^{i}.
\]
We want to find a matrix \( C \in \mathbb{R}^{N \times n} \) such that its entries satisfy \( c^i_j > m \) and
\[
\xi := C \odot \theta, \ \theta \in \Theta \Rightarrow |\Phi(\xi)| > M.
\]
In fact we prove that the matrix can be chosen of the form \( C = \tau A \) where \( \tau > 0 \) and for \( t > 0 \)
\[
A^i_j := \begin{cases} 
  t & \text{if } 1 \leq i = j \leq s, \\
  1 & \text{if } i \neq j \text{ or if } i = j \geq s + 1.
\end{cases}
\]
We observe that
\[
\Phi(\xi) = \Phi(C \odot \theta)
\]
\[
= \Phi(0) + \sum_{1 \leq q \leq s} \tau^q \left| \sum_{1 \leq i_1 < \ldots < i_q \leq N} \mu^{i_1 \ldots i_q} \det \left( A^{i_1 j_1}_{j_1} \theta_{i_1 j_1} \ldots A^{i_q j_q}_{j_q} \theta_{i_q j_q} \right) \right|
\]
and that for \( \tau \) and \( t \) sufficiently large it is possible to find \( \gamma > 0 \) so that
\[
|\Phi(\xi)| \geq \gamma \tau^s t^n.
\]
So choosing \( \tau \) and \( t \) sufficiently large we have indeed found \( c^i_j > m \) and \( |\Phi(\xi)| > M \) as wished.

## 7.4.4 A problem of optimal design

Recall that the set of \( 2 \times 2 \) symmetric matrices is denoted by \( \mathbb{R}^{2 \times 2}_s \) and that to every \( \xi \in \mathbb{R}^{2 \times 2} \) we associate \( \tilde{\xi} \in \mathbb{R}^{2 \times 2} \) in the following way
\[
\xi = \begin{pmatrix} \xi^1_1 & \xi^1_2 \\ \xi^2_1 & \xi^2_2 \end{pmatrix}, \quad \tilde{\xi} = \begin{pmatrix} \xi^2_2 & -\xi^2_1 \\ -\xi^1_2 & \xi^1_1 \end{pmatrix}.
\]
Our algebraic result, due to Dacorogna-Tantieri [215], is as follows.

**Theorem 7.48** Let
\[
E := \left\{ \xi \in \mathbb{R}^{2 \times 2}_s : \text{trace} \xi \in \{0, 1\}, \det \xi \geq 0 \right\},
\]
then
\[
\text{Rco } E = \text{co } E = \left\{ \xi \in \mathbb{R}^{2 \times 2}_s : 0 \leq \text{trace} \xi \leq 1, \det \xi \geq 0 \right\},
\]
\[
\text{int } \text{Rco } E = \left\{ \xi \in \mathbb{R}^{2 \times 2}_s : 0 < \text{trace} \xi < 1, \det \xi > 0 \right\},
\]
where the interior is understood as relative to \( \mathbb{R}^{2 \times 2}_s \).

**Remark 7.49** Note that it is slightly surprising that the rank one convex hull is in fact convex since the function \( \xi \rightarrow \det \xi \) is not convex.
**Proof.** We call

\[ X := \{ \xi \in \mathbb{R}^{2 \times 2} : 0 \leq \text{trace} \xi \leq 1, \det \xi \geq 0 \}, \]

\[ Y := \{ \xi \in \mathbb{R}^{2 \times 2} : 0 < \text{trace} \xi < 1, \det \xi > 0 \}. \]

**Step 1.** We first prove that

\[ Rco E \subset co E \subset X. \]

The first inclusion always holds and the second one follows from the fact that \( E \subset X \) and that \( X \) is convex. Indeed, let \( \xi, \eta \in X, 0 \leq t \leq 1; \) we wish to show that \( t\xi + (1 - t)\eta \in X. \)

- It is clear that the first inequality in the definition of \( X \) holds since 

\[ \xi \rightarrow \text{trace} \xi \]

is linear.

- We now show the second one. Observe first that since 

\[ \det \xi = \xi_1^1 \xi_2^2 - (\xi_2^1)^2, \]

\[ \det \eta = \eta_1^1 \eta_2^2 - (\eta_2^1)^2 \geq 0 \] and trace \( \xi, \) trace \( \eta \geq 0, \) then \( \xi_1^1, \xi_2^2, \eta_1^1, \eta_2^2 \geq 0 \) and we therefore have (assume below that \( \xi_1^1, \eta_1^1 > 0 \) otherwise, under our assumptions, the inequality below is trivial)

\[ \langle \tilde{\xi}; \eta \rangle = \xi_1^1 \eta_2^2 + \eta_1^1 \xi_2^2 - 2\xi_1^1 \eta_2^2 \]

\[ \geq \xi_1^1 \left( \frac{\eta_1^1}{\eta_1^1} \right)^2 + \eta_1^1 \left( \frac{\xi_1^1}{\xi_1^1} \right)^2 - 2\xi_1^1 \eta_2^2 \]

\[ = \frac{(\xi_1^1 \eta_2^2 - \eta_1^1 \xi_2^2)^2}{\xi_1^1 \eta_1^1} \geq 0. \]

We therefore deduce that

\[ \det (t\xi + (1 - t)\eta) = t^2 \det \xi + t(1 - t) \langle \tilde{\xi}; \eta \rangle + (1 - t)^2 \det \eta \geq 0. \]

**Step 2.** We now show that

\[ X \subset Rco E. \]

Since \( X \) is compact, it is enough to prove (see Lemma 7.42) that \( \partial X \subset Rco E. \)

However, it is easy to see that

\[ \partial X = E \cup \{ \xi \in \mathbb{R}^{2 \times 2}_s : 0 < \text{trace} \xi < 1, \det \xi = 0 \} \]

and therefore the proof will be complete once we show that the second set in the right hand side is contained in \( Rco E. \) Assume that \( \xi \) is such that \( 0 < t = \text{trace} \xi < 1 \) and \( \det \xi = 0. \) We can then write

\[ \xi = \begin{pmatrix} x & \pm \sqrt{x(t-x)} \\ \pm \sqrt{x(t-x)} & t-x \end{pmatrix} = t\xi_1 + (1-t)\xi_2 \]

\[ = t \begin{pmatrix} \alpha & \pm \sqrt{\alpha(1-\alpha)} \\ \pm \sqrt{\alpha(1-\alpha)} & 1-\alpha \end{pmatrix} + (1-t) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]
where \( x = t\alpha \). The result follows from the facts that \( \xi_1, \xi_2 \in E \) and \( \det (\xi_1 - \xi_2) = 0 \).

**Step 3.** Let us now show that \( Y = \text{int} \, \text{Rco} \, E \). The inclusion \( Y \subset \text{int} \, \text{Rco} \, E \) follows from the fact that \( Y \) is open and from the obvious inclusion \( Y \subset X = \text{Rco} \, E \). We therefore only need to prove that \( \text{int} \, \text{Rco} \, E \subset Y \). So let \( \xi \in \text{int} \, \text{Rco} \, E \), we consider two cases (the second one will be shown to be impossible).

**Case 1:** \( \xi_2^2 > 0 \). Find \( \epsilon \) sufficiently small so that \( B_\epsilon (\xi) \subset \text{Rco} \, E \) (where \( B_\epsilon (\xi) \) denotes the ball, restricted to \( \mathbb{R}^{2 \times 2} \), centered at \( \xi \) and of radius \( \epsilon \)). Define

\[
\eta_\pm := \begin{pmatrix} \xi_1^1 \pm \delta & \xi_1^2 \\ \xi_2^1 & \xi_2^2 \end{pmatrix},
\]

where \( \delta \in (0, \epsilon) \) is chosen sufficiently small so that \( \eta_\pm \in B_\epsilon (\xi) \subset \text{Rco} \, E = X \). We thus have

\[
0 \leq \text{trace} \, \eta_\pm \leq 1 \quad \text{and} \quad \det \, \eta_\pm \geq 0.
\]

This immediately leads to

\[
0 < \text{trace} \, \xi < 1 \quad \text{and} \quad \det \, \xi > 0
\]

which is the claim, namely \( \xi \in Y \).

**Case 2:** \( \xi_2^2 = 0 \). Since \( \xi \in \text{int} \, \text{Rco} \, E \subset \text{Rco} \, E \), we deduce that \( \xi_2^1 = 0 \) and hence

\[
\xi = \begin{pmatrix} \xi_1^1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

However, such a \( \xi \) cannot be in \( \text{int} \, \text{Rco} \, E \) since

\[
\begin{pmatrix} \xi_1^1 & \delta \\ \delta & 0 \end{pmatrix} \notin \text{Rco} \, E, \quad \forall \delta \neq 0.
\]

This concludes the proof of the theorem. \( \blacksquare \)
Chapter 8

Lower semi continuity and existence theorems in the vectorial case

8.1 Introduction

We now consider the minimization problem

$$(P) \quad \inf \left\{ I(u) = \int_\Omega f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\},$$

where

- $\Omega \subset \mathbb{R}^n$ is a bounded open set,
- $u : \Omega \to \mathbb{R}^N$ and hence $\nabla u \in \mathbb{R}^{N \times n}$,
- $u_0 \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a given map,
- $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$, $f = f(x, u, \xi)$, is a Carathéodory function.

In Section 8.2, we obtain the main result of this chapter, showing that the integral $I$ is (sequentially) weakly lower semicontinuous, namely

$$\liminf_{\nu \to \infty} I(u_\nu) \geq I(u)$$

for every sequence $u_\nu \rightharpoonup u$ in $W^{1,p}$ if and only if

$$\xi \to f(x, u, \xi)$$

is quasiconvex.

For the clarity of exposition, we prove the result several times. First when there is no dependence on lower order terms, meaning that $f = f(\xi)$. Then in the general case, $f = f(x, u, \xi)$, first with $p = \infty$ and then when $1 \leq p < \infty$, which is the hardest case.
In Section 8.3, we characterize completely the functions $f$ that lead to integrals $I$ that are weakly continuous, meaning that $I$ and $-I$ are weakly lower semicontinuous. These turn out to be the quasiaffine functions. In particular, when $N = n = 2$, we have that if

$$u_\nu \rightharpoonup u \quad \text{in} \quad W^{1,p}(\Omega; \mathbb{R}^2), \quad p > 2,$$

then

$$\det \nabla u_\nu \rightharpoonup \det \nabla u \quad \text{in} \quad L^{p/2}(\Omega).$$

In Section 8.4, we see how to apply the above results to the existence of minimizers for the above problem ($P$).

In Section 8.5, we gather some important properties of the Jacobian determinants that we will use throughout the present chapter.

### 8.2 Weak lower semicontinuity

#### 8.2.1 Necessary condition

The main theorem of this paragraph has already been proved (see Lemma 3.18 of Chapter 3) and we now restate it.

**Theorem 8.1** Let $\Omega$ be an open set of $\mathbb{R}^n$, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and 

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \quad f = f(x, u, \xi),$$

be a Carathéodory function satisfying, for almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$,

$$|f(x, u, \xi)| \leq a(x) + b(u, \xi),$$

where $a, b \geq 0$, $a \in L^1(\mathbb{R}^n)$ and $b \in C(\mathbb{R}^N \times \mathbb{R}^{N \times n})$. Finally, let

$$I(u, \Omega) := \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx$$

and assume that there exists $u_0 \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ such that

$$|I(u_0, \Omega)| < \infty. \quad (8.1)$$

If $I$ is weak $\star$ lower semicontinuous in $W^{1,\infty}(\Omega, \mathbb{R}^N)$, meaning that

$$\liminf_{\substack{u_\nu \rightharpoonup u \\ \Omega}} I(u_\nu, \Omega) \geq I(u, \Omega),$$

then $\xi \to f(x, u, \xi)$ is quasiconvex, i.e.

$$\frac{1}{\text{meas}D} \int_D f(x_0, u_0, \xi + \nabla \varphi(x)) \, dx \geq f(x_0, u_0, \xi_0)$$

for almost every $x_0 \in \Omega$, every $(u_0, \xi_0) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$, for every bounded open set $D \subset \mathbb{R}^n$ and for every $\varphi \in W^{1,\infty}_0(D; \mathbb{R}^n)$. 
Remark 8.2 It is clear that if $I$ is weakly lower semicontinuous in $W^{1,p}$, then $I$ is weak $*$ lower semicontinuous in $W^{1,\infty}$ and therefore the quasiconvexity of $f$ is also necessary for the weak lower semicontinuity of $I$ in $W^{1,p}$.

8.2.2 Lower semicontinuity for quasiconvex functions without lower order terms

We now turn our attention to the sufficiency of the quasiconvexity to obtain weak lower semicontinuity in $W^{1,p}$. We consider here only the case where $f = f(\xi)$, the general case $f = f(x,u,\xi)$ will be discussed in the next sections.

We first introduce a growth condition that should satisfy the function $f$.

Definition 8.3 Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ and $1 \leq p \leq \infty$. Then $f$ is said to satisfy growth condition $(C_p)$ if

1. when $p = \infty$ (C$_\infty$) $|f(\xi)| \leq \eta(|\xi|)$ for every $\xi \in \mathbb{R}^{N \times n}$, where $\eta$ is a continuous and increasing function;

2. when $1 < p < \infty$ (C$_p$) $- \alpha (1 + |\xi|^q) \leq f(\xi) \leq \alpha (1 + |\xi|^p)$ for every $\xi \in \mathbb{R}^{N \times n}$, where $\alpha \geq 0$, $1 \leq q < p$;

3. when $p = 1$ (C$_1$) $|f(\xi)| \leq \alpha (1 + |\xi|)$ for every $\xi \in \mathbb{R}^{N \times n}$, where $\alpha \geq 0$.

We may now state the theorem.

Theorem 8.4 Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ be quasiconvex and satisfying growth condition $(C_p)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and

$$I(u, \Omega) := \int_{\Omega} f(\nabla u(x)) \, dx.$$ 

Then $I$ is weakly lower semicontinuous in $W^{1,p}(\Omega; \mathbb{R}^N)$ (weak $*$ lower semicontinuous if $p = \infty$), i.e.

$$\liminf_{u_\nu \rightharpoonup^* u} I(u_\nu, \Omega) \geq I(u, \Omega).$$

Remark 8.5 (i) The above theorem is essentially due to Morrey [453], [455] and has been refined, notably by Meyers [442], Acerbi-Fusco [3] and Marcellini [423] and we will follow this last proof.
(ii) If \( f \) is convex instead of quasiconvex, then there exists \( \xi^* \in \mathbb{R}^{N \times n} \) such that
\[
f(0) + \langle \xi^*; \xi \rangle \leq f(\xi)
\]
for every \( \xi \in \mathbb{R}^{N \times n} \). Therefore, in the convex case, we impose on \( f \) only the above natural growth condition. As seen in Chapter 5 (in particular, see Section 5.3.8), there is no known equivalent to (8.2) for quasiconvex functions and therefore one needs to impose conditions of the type \((C_p)\) below and above.

(iii) One should also note that the condition \((C_p)\) if \( 1 < p < +\infty \) is optimal in the sense that one cannot allow the lower bound in \((C_p)\) to be of the form \(-\alpha (1 + |\xi|^p)\) with the same \( p \) as in the upper bound \( \alpha (1 + |\xi|^p)\), as for the case \( p = 1 \). This is seen in the example below.

(iv) If \( f \geq 0 \), the result remains valid even if \( \Omega \) is unbounded (this is done as in the proof of Theorem 3.23 or Remark 8.12 (iv)).

\[\square\]

**Example 8.6** We give here an example that is essentially due to Tartar (see Ball-Murat [65]). Let \( N = n = p = 2 \) and
\[
f(\nabla u) = \det \nabla u,
\]
then \((C_2)\) is satisfied only if \( q = p = 2 \). We show that if \( 0 < a < 1 \), \( \Omega = (0, a)^2 \)
and
\[
\nu (x, y) = \frac{1}{\sqrt{\nu}} (1 - y)\nu (\sin \nu x, \cos \nu x),
\]
then
\[
u \rightarrow 0 = \nu \text{ in } W^{1,2}(\Omega; \mathbb{R}^2),
\]
while
\[
\limsup_{\nu \to \infty} \int_{\Omega} f(\nabla \nu) \, dxdy < 0 = \int_{\Omega} f(\nabla \nu) \, dxdy.
\]
The fact that \( \nu \to 0 \) in \( L^\infty(\Omega; \mathbb{R}^2) \) is obvious, while the convergence
\[
\nabla \nu \rightarrow 0 = \nabla \nu \text{ in } L^2(\Omega; \mathbb{R}^{2 \times 2})
\]
is easily obtained from the following computations. We have
\[
\nabla \nu = \begin{pmatrix}
\sqrt{\nu} (1 - y)^\nu \cos \nu x & -\sqrt{\nu} (1 - y)^{\nu-1} \sin \nu x \\
-\sqrt{\nu} (1 - y)^\nu \sin \nu x & -\sqrt{\nu} (1 - y)^{\nu-1} \cos \nu x
\end{pmatrix}
\]
and therefore
\[
\| \nabla \nu \|_{L^2}^2 = \int_{0}^{a} \int_{0}^{a} \nu[(1 - y)^{2\nu} + (1 - y)^{2\nu-2}] \, dxdy
\]
\[
= a\nu \left[ \frac{1}{2\nu + 1} + \frac{1}{2\nu - 1} - \frac{(1 - a)^{2\nu+1}}{2\nu + 1} - \frac{(1 - a)^{2\nu-1}}{2\nu - 1} \right] < 2a
\]
if $\nu \geq 1$. We thus deduce that, up to a subsequence (although in the present case, we do not need to restrict to a subsequence), $\nabla u_\nu \rightharpoonup 0$ in $L^2(\Omega; \mathbb{R}^{2 \times 2})$. However,

$$\int_\Omega f(\nabla u_\nu(x,y)) \, dxdy = \int_0^a \int_0^a (-\nu (1-y)^{2\nu-1}) \, dxdy$$

$$= -\nu a \left[ \frac{1}{2\nu} - \frac{(1-a)^{2\nu}}{2\nu} \right]$$

and therefore the lower semicontinuity inequality does not hold since

$$\limsup_{\nu \to \infty} \int_\Omega f(\nabla u_\nu) = -\frac{a}{2} < \int_\Omega f(\nabla \eta) = 0.$$

We now continue with the proof of Theorem 8.4. But since the proof is long, we prefer to prove the main step in a separate lemma.

Lemma 8.7 Let $D \subset \mathbb{R}^n$ be a cube parallel to the axes, $\xi \in \mathbb{R}^{N \times n}$ and $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ be quasiconvex and satisfying growth condition $(C_p)$. Let $1 \leq p \leq \infty$ and $v_\nu \rightharpoonup 0$ in $W^{1,p}(D; \mathbb{R}^N)$ ($v_\nu \rightharpoonup 0$ in $W^{1,\infty}(D; \mathbb{R}^N)$ if $p = \infty$). Then

$$\liminf_{\nu \to \infty} \int_D f(\xi + \nabla v_\nu(x)) \, dx \geq f(\xi) \text{ meas } D. \quad (8.3)$$

Proof. (Lemma 8.7). To infer (8.3) from the quasiconvexity of $f$, the only problem is to change $v_\nu$ slightly in order to have $v_\nu = 0$ on $\partial D$. This is classical in the calculus of variations, see Chapter 5, Acerbi-Fusco [3], Marcellini [423], Meyers [442], Morrey [455], [453] and others. However since the cases $p = \infty$ and $p = 1$ are simpler and more natural, we will start with those. The more sophisticated case $1 < p < \infty$, which could also include $p = \infty$ and $p = 1$, will be dealt with in Step 2.

Step 1: $p = \infty$ or $p = 1$. We start by fixing an arbitrary number $\epsilon > 0$ and we claim that we can find $\delta = \delta(\epsilon) > 0$ (independent of $\nu$) so that, for any measurable set $E \subset \mathbb{R}^n$, we have

$$\text{meas } E \leq \delta \implies \int_E |\nabla v_\nu(x)| \, dx \leq \epsilon. \quad (8.4)$$

This is clear from the equiintegrability of the sequence $\{\nabla v_\nu\}$ when $p = \infty$ and $p = 1$.

We next construct an open set $D_\epsilon \subset D$ and a function $\eta_\epsilon \in C_0^\infty(D)$ such that $0 \leq \eta_\epsilon \leq 1$ in $D$ and

$$\text{meas}[D - D_\epsilon] \leq \delta, \eta_\epsilon \equiv 1 \text{ on } D_\epsilon \text{ and } |\nabla \eta_\epsilon| \leq c_1/\delta,$$
where \( c_1 > 0 \) is a constant.

We finally define a sequence of functions \( u_\nu \in W^{1,p}_0(D;\mathbb{R}^N) \) by
\[
    u_\nu (x) := \eta_\epsilon(x) v_\nu(x).
\]

We then claim that we can find a constant \( c_2 > 0 \) so that
\[
    \limsup \limits_{\nu \to \infty} \left| \int_{D - D_\epsilon} [f(\xi + \nabla v_\nu(x)) - f(\xi + \nabla u_\nu(x))] \, dx \right| \leq c_2 \epsilon. \tag{8.5}
\]

Assume for a moment that we have proved this last inequality and let us conclude the proof of the lemma. Since \( f \) is quasiconvex, \( u_\nu \in W^{1,p}_0(D;\mathbb{R}^N) \), (8.5) holds and
\[
    \int_{D} f(\xi + \nabla v_\nu(x)) \, dx = \int_{D} f(\xi + \nabla u_\nu(x)) \, dx + \int_{D - D_\epsilon} [f(\xi + \nabla v_\nu(x)) - f(\xi + \nabla u_\nu(x))] \, dx
\]
we obtain that
\[
    \liminf \limits_{\nu \to \infty} \int_{D} f(\xi + \nabla v_\nu(x)) \, dx \geq f(\xi) \text{ meas } D - c_2 \epsilon.
\]

Letting \( \epsilon \to 0 \), we have indeed obtained the lemma.

It therefore remains to show (8.5). We separate the discussion in two cases.

Case 1: \( p = \infty \). Since
\[
    \nabla u_\nu = \eta_\epsilon \nabla v_\nu + \nabla \eta_\epsilon \otimes v_\nu,
\]
\( \|\nabla v_\nu\|_{L^\infty} \) is uniformly bounded and \( v_\nu \to 0 \) uniformly, we clearly have (8.5).

Case 2: \( p = 1 \). Use Proposition 2.32 to deduce that there exists a constant \( c_3 > 0 \) such that
\[
    |f(\xi + \nabla v_\nu(x)) - f(\xi + \nabla u_\nu(x))| \leq c_3 |\nabla v_\nu(x) - \nabla u_\nu(x)|
\]
\[
    = c_3 |(1 - \eta_\epsilon) \nabla v_\nu - \nabla \eta_\epsilon \otimes v_\nu|
\]
\[
    \leq c_3 |\nabla v_\nu| + c_3 |\nabla \eta_\epsilon \otimes v_\nu|
\]
\[
    \leq c_3 |\nabla v_\nu| + \frac{c_3 c_1}{\delta} |v_\nu|.
\]

Appealing to (8.4) and to the fact that \( v_\nu \to 0 \) in \( L^1 \), we have indeed obtained (8.5).

Step 2: \( 1 < p < \infty \). The above procedure is too simple in the present context, because, contrary to the cases \( p = \infty \) and \( p = 1 \), we now lack the equiintegrability of the sequence. In order to fix the boundary datum, we proceed as in Marcellini [423]. Let \( D^0 \subset \subset D \) be a cube and let
\[
    R := \frac{1}{2} \text{ dist } (D^0, \partial D). \tag{8.6}
\]
Let $K$ be an integer and let $D^0 \subset D^k \subset D$ (see Figure 8.1), $1 \leq k \leq K$, be such that

$$\text{dist} \left( D^0, \partial D^k \right) = \frac{k}{K} R, \ 1 \leq k \leq K.$$  

We then choose $\varphi^k \in C^\infty(D), \ 1 \leq k \leq K$, such that

$$0 \leq \varphi^k \leq 1, \ |\nabla \varphi^k| \leq \frac{K}{R}, \ \varphi^k(x) = \begin{cases} 1 & \text{if } x \in D^{k-1} \\ 0 & \text{if } x \in D - D^k, \end{cases}$$

where $a > 0$ is a constant. Let

$$v^k_\nu = \varphi^k v_\nu,$$

then $v^k_\nu \in W^{1,p}_0(D; \mathbb{R}^N)$. We may therefore use the quasiconvexity of $f$ to get

\[
\int_D f(\xi) \, dx \leq \int_D f(\xi + \nabla v^k_\nu(x)) \, dx = \int_{D - D^k} f(\xi) \, dx + \int_{D^k - D^{k-1}} f(\xi + \nabla v^k_\nu(x)) \, dx \nonumber + \int_{D^{k-1}} f(\xi + \nabla v_\nu(x)) \, dx.
\]

We then deduce that

$$\int_{D^k} f(\xi) \, dx \leq \int_{D^k - D^{k-1}} f(\xi + \nabla v^k_\nu(x)) \, dx + \int_{D^{k-1}} f(\xi + \nabla v_\nu(x)) \, dx.$$
We may also rewrite the above inequality in the following way:

\[
\int_{D^k} f(\xi) \, dx \leq \int_D f(\xi + \nabla v_\nu(x)) \, dx - \int_{D-D^{k-1}} f(\xi + \nabla v_\nu(x)) \, dx \\
+ \int_{D^{k-1}-D^{k-1}} f(\xi + \nabla v_\nu^k(x)) \, dx \\
= \int_D f(\xi + \nabla v_\nu(x)) \, dx + A_1 + A_2.
\] (8.7)

We now estimate \(A_1\) and \(A_2\).

**Estimation of \(A_1\), where**

\[
A_1 := -\int_{D-D^{k-1}} f(\xi + \nabla v_\nu(x)) \, dx.
\]

We want to show that by choosing \(R\) sufficiently small (see (8.6)) we have

\[
A_1 \leq \epsilon \tag{8.8}
\]

uniformly in \(\nu\). Use \((C_p)\) to get, \(\alpha'\) being a constant,

\[
A_1 \leq \alpha \int_{D-D^{k-1}} (1 + |\xi + \nabla v_\nu|^q) \, dx \\
\leq \alpha' \int_{D-D^0} (1 + |\xi|^q + |\nabla v_\nu|^q) \, dx.
\]

Since \(q < p\), we use Hölder inequality to obtain

\[
\int_{D-D^0} |\nabla v_\nu|^q \, dx \leq \left( \int_{D-D^0} |\nabla v_\nu|^p \, dx \right)^{q/p} \text{meas} \left( D-D^0 \right)^{(p-q)/p}
\]

and therefore, by choosing \(R\) sufficiently small, we have (8.8).

**Estimation of \(A_2\), where**

\[
A_2 := \int_{D^{k-1}-D^{k-1}} f(\xi + \nabla v_\nu^k(x)) \, dx.
\]

We have, using \((C_p)\) and denoting by \(\alpha_i \geq 0\) constants that are independent of \(K\) and \(\nu\),

\[
A_2 \leq \alpha \int_{D^{k-1}-D^{k-1}} (1 + |\xi + \nabla v_\nu^k|^p) \, dx \\
\leq \alpha_1 \int_{D^{k-1}-D^{k-1}} (1 + |\xi|^p + |\varphi^k \nabla v_\nu + \nabla \varphi^k \otimes v_\nu|^p) \, dx \\
\leq \alpha_2 \int_{D^{k-1}-D^{k-1}} (1 + |\xi|^p + |\nabla v_\nu|^p + (aK/R)^p |v_\nu|^p) \, dx
\]

where we have used the definition of \(\varphi^k\) in the last inequality.
Returning to (8.7), using (8.8) and the above estimate of $A_2$, we find, if we sum the left and right hand sides of (8.7) from $k = 1$ to $K$,

$$K \int_D f(\xi + \nabla v_\nu (x)) \, dx - f(\xi) \sum_{k=1}^K \text{meas } D^k \geq -K \epsilon - \alpha_2 \int_{D^K - D^0} (1 + |\xi|^p + |\nabla v_\nu|^p + (aK/R)^p |v_\nu|^p) \, dx.$$

Dividing the above inequality by $K$ and letting $\nu \to +\infty$ we get (recalling that $v_\nu \rightharpoonup 0$ in $W^{1,p}(D; \mathbb{R}^N)$)

$$\liminf_{\nu \to \infty} \int_D f(\xi + \nabla v_\nu (x)) \, dx - f(\xi) \sum_{k=1}^K \text{meas } D^k \geq -\epsilon - \frac{\alpha_3}{K} \quad (8.9)$$

where $\alpha_3$ is a constant. Noting that

$$\text{meas } D^0 \leq \frac{1}{K} \sum_{k=1}^K \text{meas } D^k \leq \text{meas } D,$$

letting $K \to \infty$ and taking into account the fact that $\epsilon$ and $D^0$ are arbitrary (see (8.6)), we have indeed obtained from (8.9) that

$$\liminf_{\nu \to \infty} \int_D f(\xi + \nabla v_\nu (x)) \, dx \geq f(\xi) \text{meas } D$$

which is the desired result. □

We are now in a position to conclude the proof of Theorem 8.4.

**Proof.** We start by approximating $\Omega$ by a union of cubes parallel to the axes in $\mathbb{R}^n$ and take the average of $\nabla u$ over each of these cubes (so that $\nabla u$ is constant on each cube). More precisely, we let $\delta > 0$ and $h$ be an integer. We approximate $\Omega$ by a union of cubes $D_s$ parallel to the axes and whose edge length is $1/h$. We denote this union by $H_h$. We then choose $h$ large enough so that

$$\text{meas } (\Omega - H_h) \leq \delta \quad \text{where } H_h := \bigcup D_s.$$

We then take the average of $\nabla u$ over each of the $D_s$, namely

$$\xi_s := \frac{1}{\text{meas } D_s} \int_{D_s} \nabla u (x) \, dx.$$

Now fix $\epsilon > 0$. Choosing $\delta$ smaller if necessary, we have

$$\left( \sum_{s} \int_{D_s} |\nabla u (x) - \xi_s|^p \, dx \right)^{1/p} < \epsilon. \quad (8.10)$$

We recall that

$$u_\nu \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N)$$
We consider
\[ I(u_\nu; \Omega) - I(u; \Omega) = \int_\Omega [f(\nabla u_\nu(x)) - f(\nabla u(x))] \, dx \]
(8.11)
where
\[ J_1 := \int_{\Omega - H_h} [f(\nabla u_\nu(x)) - f(\nabla u(x))] \, dx, \]
\[ J_2 := \sum_s \int_{D_s} [f(\nabla (u + (\nabla u_\nu - \nabla u)) - f(\xi_s + (\nabla u_\nu - \nabla u))] \, dx, \]
\[ J_3 := \sum_s \int_{D_s} [f(\xi_s + (\nabla u_\nu - \nabla u)) - f(\xi_s)] \, dx, \]
\[ J_4 := \sum_s \int_{D_s} [f(\xi_s) - f(\nabla u)] \, dx. \]

The difficult term to estimate is \( J_3 \) and this has already been done in Lemma 8.7. We now estimate \( J_1, J_2 \) and \( J_4 \). In the sequel, we denote by \( \alpha_i \geq 0 \) constants that are independent of \( \delta \) and \( \nu \).

**Estimation of \( J_1 \).** We want to prove that if \( \delta \) is chosen small enough, then
\[ J_1 \geq -\alpha_1 \epsilon \] (8.12)
uniformly in \( \nu \).

**Case 1.** If \( p = +\infty \), then (8.12) is trivial since \( \|\nabla u_\nu\|_{L^\infty} \) is uniformly bounded and since \( (C_{\infty}) \) holds.

**Case 2.** If \( 1 < p < \infty \), then use \( (C_p) \) to get
\[ J_1 \geq -\int_{\Omega - H_h} [\alpha (1 + |\nabla u_\nu|^q) + f(\nabla u)] \, dx \]
\[ = -\int_{\Omega - H_h} (\alpha + f(\nabla u)) - \alpha \int_{\Omega - H_h} |\nabla u_\nu|^q. \]

Using Hölder inequality and the fact that \( q < p \) we get
\[ J_1 \geq -\int_{\Omega - H_h} (\alpha + f(\nabla u)) - \alpha \left( \int_{\Omega - H_h} |\nabla u_\nu|^p \right)^{q/p} \, \text{meas}(\Omega - H_h)^{(p-q)/p}. \]
Choosing \( \delta \) small enough, we get (8.12).

**Case 3.** If \( p = 1 \), then by \( (C_1) \) we have
\[ J_1 \geq -\int_{\Omega - H_h} [\alpha (1 + |\nabla u_\nu|) + f(\nabla u)] \, dx. \]
Since \( \nabla u_\nu \rightharpoonup \nabla u \) in \( L^1 \), we may use the equiintegrability of \( \nabla u_\nu \) to get (8.12) immediately.
Estimation of $J_2$. Using Proposition 2.32, if $1 \leq p < +\infty$, we find that there exists a constant $\beta > 0$ such that

$$|J_2| \leq \beta \sum_s \int_{D_s} (1 + |\nabla u_\nu|^{p-1} + |\nabla u_\nu + \xi_s - \nabla u|^{p-1}) |\nabla u - \xi_s| \, dx.$$  

Using Hölder inequality, (8.10) and the fact that $\|\nabla u_\nu\|_{L^p}$ is uniformly bounded, we deduce that, for $\delta$ small enough,

$$|J_2| \leq \alpha_2 \epsilon \quad (8.13)$$

uniformly in $\nu$. If $p = +\infty$, (8.13) is obtained in the same way using $(C_\infty)$ and the fact that $\|\nabla u_\nu\|_{L^\infty}$ is uniformly bounded.

Estimation of $J_4$. This estimate is very similar but simpler than that of $J_2$ and we skip the details. We also find that, for $\delta$ small enough,

$$|J_4| \leq \alpha_4 \epsilon \quad (8.14)$$

We now return to (8.11) gathering (8.12), (8.13) and (8.14). We therefore have, for $\delta$ small enough,

$$I(u_\nu; \Omega) - I(u; \Omega) \geq - (\alpha_1 + \alpha_2 + \alpha_4) \epsilon + \sum_s \int_{D_s} [f(\xi_s + \nabla u_\nu - \nabla u) - f(\xi_s)] \, dx.$$  

Taking the limit as $\nu \to \infty$, we get

$$\liminf_{\nu \to \infty} I(u_\nu; \Omega) - I(u; \Omega) \geq - (\alpha_1 + \alpha_2 + \alpha_4) \epsilon + \sum_s \liminf_{\nu \to \infty} \int_{D_s} [f(\xi_s + (\nabla u_\nu - \nabla u)) - f(\xi_s)] \, dx.$$  

(8.15)

We now invoke Lemma 8.7 to get

$$\liminf_{\nu \to \infty} \int_{D_s} f(\xi_s + (\nabla u_\nu - \nabla u)) \, dx \geq \int_{D_s} f(\xi_s) \, dx.$$  

Then combining (8.15), the above inequality and the fact that $\epsilon$ is arbitrary, we have indeed obtained that $I$ is weakly lower semicontinuous.

8.2.3 Lower semicontinuity for general quasiconvex functions for $p = \infty$

We now turn our attention to general integrands of the type

$$I(u, \Omega) := \int_\Omega f(x, u(x), \nabla u(x)) \, dx.$$  

We start with the easiest case, where $p = \infty$. The strategy is to freeze the lower order terms and then apply Theorem 8.4.
**Theorem 8.8** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary and let

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \ f = f (x, u, \xi),$$

be a Carathéodory function such that $\xi \to f (x, u, \xi)$ is quasiconvex, i.e.

$$\int_D f (x_0, u_0, \xi_0 + \nabla \varphi (x)) \, dx \geq f (x_0, u_0, \xi_0) \text{ meas } D$$

for every bounded open set $D \subset \mathbb{R}^n$, for almost every $x_0 \in \Omega$, every $(u_0, \xi_0) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$ and for every $\varphi \in W^{1, \infty}_0 (D; \mathbb{R}^N)$.

Let

$$I (u, \Omega) := \int_{\Omega} f (x, u (x), \nabla u (x)) \, dx$$

Assume that $f$ satisfies

$$(C_\infty) \ |f (x, u, \xi)| \leq \beta (x) + \alpha (|u|, |\xi|)$$

where $\alpha, \beta \geq 0$, $\beta \in L^1 (\Omega)$ and $\alpha$ is a continuous and increasing (in each argument) function; then $I$ is (sequentially) weak * lower semicontinuous in $W^{1, \infty} (\Omega; \mathbb{R}^N)$.

**Remark 8.9** (i) The theorem is due to Morrey [453], [455], under further hypotheses, and has been refined by Meyers [442], Acerbi-Fusco [3] and Marcellini [423]. We follow here the proof in [3].

(ii) The result remains valid for any open set $\Omega$ (not necessarily bounded) if we assume that $f \geq 0$; see Remark 8.12 (iv) for details.

**Proof.** We divide the proof into two steps.

**Step 1.** We first prove that we can restrict our attention to sets $\Omega$ that are finite unions of disjoint open cubes parallel to the axes and to functions $f$ satisfying

$$(C'_{\infty}) \ 0 \leq f (x, u, \xi) \leq \beta (x) + \alpha (|u|, |\xi|).$$

Indeed, since $u_\nu \rightharpoonup u$ in $W^{1, \infty}$, we can find $\gamma \geq 0$ such that

$$\|u_\nu\|_{W^{1, \infty}}, \|u\|_{W^{1, \infty}} \leq \gamma.$$

Then let

$$g (x, u, \xi) := f (x, u, \xi) + \beta (x) + k$$

where $k = \alpha (\gamma, \gamma)$ ($\alpha$ as in $(C_\infty)$). By hypothesis, we deduce that for almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$ with $\max \{|u|, |\xi|\} \leq \gamma$, then

$$g (x, u, \xi) \geq 0.$$
We next choose \( \Omega_{\mu} \subset \subset \Omega \) to be a union of cubes parallel to the axes, and we obtain
\[
\int_{\Omega} f(x, u_\nu, \nabla u_\nu) \, dx = \int_{\Omega} g(x, u_\nu, \nabla u_\nu) \, dx - \int_{\Omega} [\beta(x) + k] \, dx \\
\geq \int_{\Omega_{\mu}} g(x, u_\nu, \nabla u_\nu) \, dx - \int_{\Omega} [\beta(x) + k] \, dx.
\]

We next apply the lower semicontinuity to \( \Omega_{\mu} \) and to \( g \geq 0 \) to get
\[
\liminf_{\nu \to \infty} \int_{\Omega} f(x, u_\nu, \nabla u_\nu) \, dx \geq \liminf_{\nu \to \infty} \int_{\Omega_{\mu}} g(x, u_\nu, \nabla u_\nu) \, dx - \int_{\Omega} [\beta(x) + k] \, dx \\
\geq \int_{\Omega_{\mu}} g(x, u, \nabla u) \, dx - \int_{\Omega} [\beta(x) + k] \, dx.
\]

Then choosing a sequence of increasing \( \Omega_{\mu} \subset \Omega \) so that \( \Omega_{\mu} \nearrow \Omega \) and applying Lebesgue monotone convergence theorem, we get the result.

From now on, we therefore assume, by working on each cube, that \( \Omega \) itself is a cube parallel to the axes and that \( f \) satisfies \( (C'_\infty) \).

Step 2. We now proceed in a very similar way to that of Theorem 8.4. We let
\[
\|u_\nu\|_{W^{1,\infty}} , \|u\|_{W^{1,\infty}} \leq \gamma,
\]
\[
k := \alpha(\gamma, \gamma).
\]

Let \( \epsilon > 0 \), we can then find \( M = M(\epsilon) \) so that if
\[
E_\epsilon := \{ x \in \Omega : \beta(x) \leq M \}
\]
then
\[
\text{meas} (\Omega - E_\epsilon) < \frac{\epsilon}{2k} , \int_{\Omega - E_\epsilon} \beta(x) \, dx < \frac{\epsilon}{2}
\]
and, in particular,
\[
M \text{meas} (\Omega - E_\epsilon) < \frac{\epsilon}{2}.
\]

Appealing to Theorem 3.8, we can find a compact set \( K_\epsilon \subset \Omega \) with
\[
\text{meas} (\Omega - K_\epsilon) \leq \epsilon/(M + k)
\]
and such that \( f : K_\epsilon \times S \to \mathbb{R} \) is continuous, where
\[
S := \{ (u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n} : |u| + |\xi| \leq \gamma \}.
\]

We next decompose \( \Omega \), which is a cube parallel to the axes, in a finite union of cubes \( D_s \) parallel to the axes and whose edge length is \( 1/h \). We denote this
union by \( H_h \). Choosing \( h \) proportional to the edge length of the cube \( \Omega \), we have

\[
\text{meas}(\Omega - H_h) = 0, \quad \text{where } H_h := \bigcup D_s.
\]

We then take the average of \( u \) over each of the \( D_s \), i.e.

\[
u^s := \frac{1}{\text{meas}\ D_s} \int_{D_s} u(x) \, dx
\]

and we fix \( x^s \in D_s \cap K_\epsilon \cap E_\epsilon \), if it is non-empty.

We next make estimates of the two quantities

\[
A := \int_\Omega f(x, u, \nabla u) \, dx, \quad A_\nu := \int_\Omega f(x, u_\nu, \nabla u_\nu) \, dx.
\]

**Estimation of** \( A \). We observe that

\[
A^1 := \int_{\Omega - E_\epsilon} f(x, u, \nabla u) \, dx \leq \int_{\Omega - K_\epsilon} \left[ \alpha(\gamma, \gamma) + \beta(x) \right] \, dx \leq \epsilon
\]

\[
A^2 := \int_{E_\epsilon - (K_\epsilon \cap E_\epsilon)} f(x, u, \nabla u) \, dx \leq \int_{E_\epsilon - (K_\epsilon \cap E_\epsilon)} \left[ \alpha(\gamma, \gamma) + \beta(x) \right] \, dx
\]

\[
\leq [k + M] \text{meas}(E_\epsilon - (K_\epsilon \cap E_\epsilon)) \leq [k + M] \text{meas}(\Omega - K_\epsilon) \leq \epsilon.
\]

Furthermore, since when letting \( h \to \infty \) we find that \( x^s \to x \) and \( u^s \to u \) and since \( f \) is uniformly continuous over \( K_\epsilon \times S \), we deduce that we can find \( h \) large enough so that

\[
A^3 := \sum_s \int_{D_s \cap K_\epsilon \cap E_\epsilon} \left[ f(x, u, \nabla u) - f(x^s, u^s, \nabla u) \right] \, dx \leq \epsilon.
\]

Combining these estimates, we have

\[
A = \int_\Omega f(x, u, \nabla u) \, dx = \int_{E_\epsilon} f(x, u, \nabla u) \, dx + A^1
\]

\[
= \int_{K_\epsilon \cap E_\epsilon} f(x, u, \nabla u) \, dx + A^1 + A^2
\]

\[
= \sum_s \int_{D_s \cap K_\epsilon \cap E_\epsilon} f(x^s, u^s, \nabla u) \, dx + A^1 + A^2 + A^3
\]

which leads, since \( f \geq 0 \), to

\[
A = \int_\Omega f(x, u, \nabla u) \, dx \leq \sum_s \int_{D_s} f(x^s, u^s, \nabla u) \, dx + 3\epsilon. \quad (8.16)
\]

**Estimation of** \( A_\nu \). We observe that by choosing \( \nu \) sufficiently large we have, since \( f \) is uniformly continuous over \( K_\epsilon \times S \) and \( u_\nu \to u \) uniformly, that

\[
A^1_\nu := \int_{K_\epsilon \cap E_\epsilon} \left[ f(x, u_\nu, \nabla u_\nu) - f(x, u, \nabla u_\nu) \right] \, dx \geq -\epsilon
\]
and similarly, by choosing $h$ sufficiently large, we get

$$A^2_\nu := \sum_s \int_{D_s \cap K_\epsilon \cap E_\epsilon} [f(x, u, \nabla u_\nu) - f(x^s, u^s, \nabla u_\nu)] dx \geq -\epsilon.$$  

We next see that

$$A^3_\nu := -\sum_s \int_{D_s - (D_s \cap K_\epsilon \cap E_\epsilon)} f(x^s, u^s, \nabla u_\nu) dx$$

$$\geq -\sum_s \int_{D_s - (D_s \cap K_\epsilon \cap E_\epsilon)} [\alpha(\gamma, \gamma) + \beta(x^s)] dx$$

$$\geq -[k + M] \text{meas}[\Omega - (K_\epsilon \cap E_\epsilon)]$$

$$\geq -[k + M] \text{meas}[\Omega - K_\epsilon] - [k + M] \text{meas}[\Omega - E_\epsilon]$$

$$\geq -2\epsilon.$$  

We now combine all the inequalities and the fact that $f \geq 0$ to have

$$A_\nu = \int_{\Omega} f(x, u_\nu, \nabla u_\nu) dx \geq \int_{K_\epsilon \cap E_\epsilon} f(x, u_\nu, \nabla u_\nu) dx$$

$$= \int_{K_\epsilon \cap E_\epsilon} f(x, u, \nabla u_\nu) dx + A^1_\nu$$

$$= \sum_s \int_{D_s \cap K_\epsilon \cap E_\epsilon} f(x^s, u^s, \nabla u_\nu) dx + A^1_\nu + A^2_\nu$$

$$= \sum_s \int_{D_s} f(x^s, u^s, \nabla u_\nu) dx + A^1_\nu + A^2_\nu + A^3_\nu$$

$$\geq \sum_s \int_{D_s} f(x^s, u^s, \nabla u_\nu) dx - 4\epsilon.$$  

Moreover, appealing to Theorem 8.4, we find that

$$\liminf_{\nu \to \infty} \int_{\Omega} f(x, u_\nu, \nabla u_\nu) dx \geq \sum_s \int_{D_s} f(x^s, u^s, \nabla u) dx - 4\epsilon.$$  

Combining the above estimate and (8.16), we find

$$\liminf_{\nu \to \infty} \int_{\Omega} f(x, u_\nu, \nabla u_\nu) dx \geq \int_{\Omega} f(x, u, \nabla u) dx - 7\epsilon.$$  

Since $\epsilon$ is arbitrary, we have the claim. □

### 8.2.4 Lower semicontinuity for general quasiconvex functions for $1 \leq p < \infty$

We now state the main theorem, first introducing a growth condition that should satisfy the function $f$.

**Definition 8.10** Let $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ be an open set and let

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \ f = f(x, u, \xi),$$
be a Carathéodory function. The function \( f \) is said to satisfy growth condition \((C_p)\) if for almost every \( x \in \Omega \) and for every \((u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}\) the inequalities

\[
(C_p) \quad -\alpha (|u|^r + |\xi|^q) - \beta (x) \leq f (x, u, \xi) \leq g (x, u) (1 + |\xi|^p)
\]

hold, where \( \alpha, \beta, g \geq 0, \beta \in L^1(\Omega), 1 \leq q < p, 1 \leq r < np/(n-p) \) if \( p < n \) and \( 1 \leq r < \infty \) if \( p \geq n \) and

\[
g : \Omega \times \mathbb{R}^N \to \mathbb{R}_+, \quad g = g (x, u),
\]

is a Carathéodory function. In the case \( p = 1 \) we assume that

\[
(C_1) \quad |f (x, u, \xi)| \leq \alpha (1 + |\xi|)
\]

where \( \alpha \geq 0 \).

We may now state the main theorem.

**Theorem 8.11** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set with a Lipschitz boundary and let

\[
f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \quad f = f (x, u, \xi),
\]

be a Carathéodory function such that \( \xi \to f (x, u, \xi) \) is quasiconvex, i.e.

\[
\int_D f (x_0, u_0, \xi_0 + \nabla \varphi (x)) \, dx \geq f (x_0, u_0, \xi_0) \, \text{meas } D
\]

for every bounded open set \( D \subset \mathbb{R}^n \), for almost every \( x_0 \in \Omega \), every \((u_0, \xi_0) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}\) and for every \( \varphi \in W^{1,\infty} (D; \mathbb{R}^N) \).

Let \( 1 \leq p < \infty \) and assume that \( f \) satisfies \((C_p)\). Let

\[
I (u, \Omega) := \int_\Omega f (x, u (x), \nabla u (x)) \, dx,
\]

then \( I \) is (sequentially) weakly lower semicontinuous in \( W^{1,p} (\Omega; \mathbb{R}^N) \).

**Remark 8.12** (i) The above theorem is due to Morrey [453], [455], under further hypotheses, and has been refined by Meyers [442] and several authors since then, notably by Acerbi-Fusco [3] and Marcellini [423]. We will follow here the proof given in Ansini-Dacorogna [29], which is a combination of that of [3], [423] and a lemma on equiintegrability of Fonseca-Müller-Pedregal [288] and Kristensen [379], see also Pedregal [492].

(ii) If one assumes that \( f \) is quasiconvex and satisfies

\[
|f (x, u, \xi)| \leq \alpha (1 + |u|^p + |\xi|^p)
\]

then Theorem 8.11, as well as Theorem 8.4, is proved in a much simpler way if one wants to show that \( I \) is weakly lower semicontinuous in \( W^{1,p+\epsilon} \), where \( \epsilon > 0 \),
instead of $W^{1,p}$. This observation is useful since for minimization problems it is often possible to see that some minimizing sequences are bounded uniformly in $W^{1,p+\varepsilon}$ instead of $W^{1,p}$ (see, for example, Ekeland-Temam [264], Chapters IX and X, or Marcellini-Sbordone [428], [429]).

(iii) The condition $(C_p)$ is optimal, in the sense that we cannot allow either $p = q$ (when $p > 1$) according to Example 8.6 or $r = p^* = np/(n-p)$ (when $1 \leq p < n$) as the simple example given below shows. However when $p = 1$, we can easily replace the hypothesis by

$$(C_1) \quad |f(x,u,\xi)| \leq \alpha(|u|^r + |\xi|)$$

where $\alpha, \beta \geq 0$, $\beta \in L^1(\Omega)$ and $1 \leq r < n/(n-1)$.

(iv) The result remains valid for any open set $\Omega$ (neither necessarily bounded nor with a Lipschitz boundary) if we assume that $f \geq 0$; see the proof below for details.

We first prove Remark 8.12 (iv).

**Proof.** We here prove that we can restrict our attention to bounded $\Omega$ with smooth boundary if $f \geq 0$. Indeed choose $\Omega_\mu \subset \subset \Omega$ with Lipschitz boundary (in particular $\Omega_\mu$ is bounded) and apply the lower semicontinuity to $\Omega_\mu$ to get

$$\liminf_{\nu \to \infty} \int_\Omega f(x,u_\nu,\nabla u_\nu) \, dx \geq \liminf_{\nu \to \infty} \int_{\Omega_\mu} f(x,u_\nu,\nabla u_\nu) \, dx \geq \int_{\Omega_\mu} f(x,u,\nabla u) \, dx.$$ 

Choosing then a sequence of increasing bounded open sets with smooth boundary $\Omega_\mu \subset \subset \Omega$ so that $\Omega_\mu \nearrow \Omega$ and applying Lebesgue monotone convergence theorem, we get the result.

We now give an example showing that we cannot allow $r = p^* = np/(n-p)$ in the theorem.

**Example 8.13** Let $1 \leq p < n$ and find a sequence $\{u_\nu\}$ such that

$$u_\nu \rightharpoonup 0 \quad \text{in} \quad W^{1,p}(\Omega) \quad \text{with} \quad u_\nu \not\rightarrow 0 \quad \text{in} \quad L^{p^*}(\Omega);$$

more precisely, we assume that

$$b := \lim_{\nu \to \infty} \int_\Omega |u_\nu(x)|^{p^*} \, dx > 0.$$ 

We then let

$$a := \liminf_{\nu \to \infty} \int_\Omega |\nabla u_\nu(x)|^p \, dx$$

and we define

$$I(u,\Omega) = \int_\Omega \left[ |\nabla u(x)|^p - \frac{a+1}{b} |u(x)|^{p^*} \right] dx.$$ 

We clearly have

$$\liminf_{\nu \to \infty} I(u_\nu,\Omega) = -1 < I(0,\Omega) = 0.$$
Before proving the main theorem, we start with a particular case.

**Lemma 8.14** Let $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ be an open cube and let

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \quad f = f(x, u, \xi),$$

be a Carathéodory function satisfying for almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$

$$\left(C'_p\right) \quad \alpha |\xi|^q \leq f(x, u, \xi) \leq \alpha (1 + |\xi|^p),$$

where $\alpha > 0$, $1 \leq q < p$ if $p > 1$ and $q = 1$ if $p = 1$. Assume, in addition, that $\xi \to f(x, u, \xi)$ is quasiconvex, meaning that

$$\int_D f(x_0, u_0, \xi_0 + \nabla \varphi(x)) \, dx \geq f(x_0, u_0, \xi_0) \, \text{meas} D$$

for every bounded open set $D \subset \mathbb{R}^n$, almost every $x_0 \in \Omega$, every $(u_0, \xi_0) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$ and every $\varphi \in W^{1, \infty}_0(D; \mathbb{R}^N)$.

Let

$$u_\nu \rightharpoonup u \quad \text{in} \quad W^{1,p}(\Omega; \mathbb{R}^N) \quad \text{with} \quad \{|
abla u_\nu|^p\} \quad \text{equiintegrable}$$

and let

$$I(u, \Omega) := \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx,$$

then

$$\liminf_{\nu \to \infty} I(u_\nu, \Omega) \geq I(u, \Omega).$$

It is clear that when $p = 1$, the equiintegrability hypothesis is not a restriction.

**Proof.** (Lemma 8.14). The strategy is first to freeze the lower order terms (as in Step 3 of Theorem 3.23) and then to apply Theorem 8.4 to get the result.

**Step 1.** We fix $\epsilon > 0$ and we wish to show that there exists a measurable set $\Omega_{\epsilon} \subset \Omega$ and a subsequence $\nu_j$, with $\nu_j \to \infty$, such that

$$\text{meas} (\Omega - \Omega_{\epsilon}) < \epsilon$$

$$\int_{\Omega_{\epsilon}} |f(x, u_{\nu_j}(x), \nabla u_{\nu_j}(x)) - f(x, u(x), \nabla u_{\nu_j}(x))| \, dx < \epsilon \, \text{meas} \Omega. \quad (8.17)$$

Indeed we first construct a set $\Omega_{\epsilon, \nu} \subset \Omega$ in the following way.

Since $u_\nu \to u$ in $L^p(\Omega)$ and $\nabla u_\nu \rightharpoonup \nabla u$ in $L^p(\Omega)$, we have that, for every $\epsilon > 0$, there exists $M_\epsilon \geq 1$, which is independent of $\nu$, such that if

$$K_{1, \nu}^{\epsilon} := \{x \in \Omega : |u(x)|^p \quad \text{or} \quad |u_\nu(x)|^p \geq M_\epsilon\}$$
We have that there exists $\delta(\epsilon) > 0$ such that

$$|x - y| + |u - v| < \delta(\epsilon) \Rightarrow |f(x, u, \xi) - f(y, v, \xi)| < \epsilon \quad (8.20)$$

for every $x, y \in \Omega_{\epsilon, \nu}^2$, every $|u|^p, |v|^p < 2^p M_\epsilon$ and $|\xi|^p < M_\epsilon$.

Having fixed $\delta(\epsilon)$ in this way and using the fact that $u_\nu \to u$ in $L^p(\Omega)$, we can find $\nu_\epsilon = \nu_{\epsilon, \delta(\epsilon)}$ such that if

$$\Omega_{\epsilon, \nu}^3 := \{ x \in \Omega : |u_\nu(x) - u(x)| < \delta(\epsilon) \},$$

then

$$\text{meas} \left( \Omega - \Omega_{\epsilon, \nu}^3 \right) < \frac{\epsilon}{3} \text{ for every } \nu \geq \nu_\epsilon. \quad (8.21)$$

Therefore, letting

$$\Omega_{\epsilon, \nu} := \Omega_{\epsilon, \nu}^2 \cap \Omega_{\epsilon, \nu}^3,$$

we have from (8.18), (8.19) and (8.21) that

$$\text{meas} \left( \Omega - \Omega_{\epsilon, \nu} \right) < \epsilon \quad (8.22)$$

and from (8.20) that

$$\int_{\Omega_{\epsilon, \nu}} |f(x, u(x), \nabla u_\nu(x)) - f(x, u_\nu(x), \nabla u_\nu(x))| \, dx < \epsilon \text{ meas } \Omega \quad (8.23)$$
for every $\nu \geq \nu_\epsilon$. We now choose $\epsilon_j = \epsilon/2^j$, $j \in \mathbb{N}$. We therefore have that (8.23) holds with $\epsilon$ and $\nu_\epsilon$ replaced by $\epsilon_j, \nu_{\epsilon_j}$. We then choose any $\nu_j \geq \nu_{\epsilon_j}$ with $\lim \nu_j = \infty$ and we let

$$\Omega_\epsilon := \bigcap_{j=1}^{\infty} \Omega_{\epsilon_j, \nu_j}.$$  

We therefore immediately have (8.17), as wished. From now on, in order not to burden the notations, we will still denote the subsequence $\{u_{\nu_j}\}$ by $\{u_\nu\}$.

**Step 2.** We next use the equiintegrability of the sequence $\{\|\nabla u_\nu\|^p\}$ and the fact that $u \in W^{1,p}$ to get that there exists a non negative and increasing function $\eta$ such that $\eta(t) \to 0$ as $t \to 0$, so that, for every measurable set $A \subset \Omega$,

$$\int_A [1 + \max\{|\nabla u(x)|^p, |\nabla u_\nu(x)|^p\}] \, dx \leq \eta(\text{meas } A). \quad (8.24)$$

Combining (8.17) and $(C'_p)$, we get, noting that $|\xi|^q \leq 1 + |\xi|^p$,

$$\int_\Omega |f(x, u(x), \nabla u_\nu(x)) - f(x, u_\nu(x), \nabla u_\nu(x))| \, dx \leq \epsilon \text{meas } \Omega + 2\alpha \int_{\Omega - \Omega_\epsilon} [1 + |\nabla u_\nu(x)|^p] \, dx$$

and thus, appealing to (8.24) and to the fact that $\text{meas } (\Omega - \Omega_\epsilon) < \epsilon$, we infer that

$$\int_\Omega |f(x, u(x), \nabla u_\nu(x)) - f(x, u_\nu(x), \nabla u_\nu(x))| \, dx \leq \epsilon \text{meas } \Omega + 2\alpha \eta(\epsilon). \quad (8.25)$$

**Step 3.** (1) We then decompose the cube $\Omega$ into a finite union of cubes $D_s$ of edge length $1/h$. We denote this union by $H_h$. Choosing $h$ proportional to the edge length of the cube $\Omega$, we have

$$\text{meas } (\Omega - H_h) = 0 \quad \text{where } H_h := \bigcup D_s.$$

(2) We then define

$$u^s := \frac{1}{\text{meas } D_s} \int_{D_s} u(x) \, dx \quad \text{and } u_h(x) := \sum_s u^s 1_{D_s}(x),$$

where

$$1_{D_s}(x) := \begin{cases} 1 & \text{if } x \in D_s \\ 0 & \text{if } x \in \Omega - D_s. \end{cases}$$

Since $u_h \to u$ in $L^p(\Omega)$ as $h \to \infty$, by choosing $h$ sufficiently large, we can assume that there exists $\Omega^\delta = \Omega^\delta(\epsilon) \subset \Omega$ such that

$$\text{meas } (\Omega - \Omega^\delta) < \epsilon \quad \text{and} \quad |u(x) - u^s| < \delta(\epsilon)/2, \quad \forall x \in \Omega^\delta \cap D_s.$$
Moreover for a fixed $x^s \in \Omega_\epsilon \cap D_s$, we have

$$|x - x^s| < \delta(\epsilon)/2, \forall x \in \Omega_\epsilon \cap D_s.$$ 

Combining the two estimates, we get

$$|x - x^s| + |u(x) - u^s| < \delta(\epsilon), \ x \in \Omega_\epsilon \cap \Omega^\delta \cap D_s.$$ 

Note that, since $|u(x)|^p < M_\epsilon$ in $\Omega_\epsilon$ and since we can always assume that $\delta^p \leq M_\epsilon$, we get, for every $x \in \Omega_\epsilon \cap \Omega^\delta \cap D_s$,

$$|u^s|^p \leq 2^{p-1}(|u(x)|^p + |u(x) - u^s|^p) \leq 2^{p-1}(M_\epsilon + |\delta|^p) \leq 2^p M_\epsilon.$$ 

We therefore get, from (8.20) and the fact that $|u(x)|^p, |\nabla u_\nu(x)|^p < M_\epsilon$ in $\Omega_\epsilon$,

$$\int_{\Omega_\epsilon \cap \Omega^\delta \cap D_s} |f(x, u(x), \nabla u_\nu(x)) - f(x^s, u^s, \nabla u_\nu(x))| \, dx < \epsilon \text{ meas} (\Omega_\epsilon \cap \Omega^\delta \cap D_s).$$ 

Summing up the last inequality we get

$$\sum_s \int_{\Omega_\epsilon \cap \Omega^\delta \cap D_s} |f(x, u(x), \nabla u_\nu(x)) - f(x^s, u^s, \nabla u_\nu(x))| \, dx < \epsilon \text{ meas} \Omega.$$ 

From $(C_p')$, with the observation that $|\xi|^q \leq 1 + |\xi|^p$, and the above inequality, we deduce that

$$\sum_s \int_{D_s} |f(x, u(x), \nabla u_\nu(x)) - f(x^s, u^s, \nabla u_\nu(x))| \, dx \leq \epsilon \text{ meas} \Omega + 2\alpha \int_{\Omega - (\Omega_\epsilon \cap \Omega^\delta)} [1 + |\nabla u_\nu(x)|^p] \, dx.$$ 

We thus get, from the fact that $\text{meas} (\Omega - \Omega_\epsilon) < \epsilon$, $\text{meas} (\Omega - \Omega^\delta) < \epsilon$ and from (8.24),

$$\sum_s \int_{D_s} |f(x, u(x), \nabla u_\nu(x)) - f(x^s, u^s, \nabla u_\nu(x))| \, dx \leq \epsilon \text{ meas} \Omega + 2\alpha \eta(2\epsilon).$$ 

(3) For a similar reason we also have

$$\sum_s \int_{D_s} |f(x, u(x), \nabla u(x)) - f(x^s, u^s, \nabla u(x))| \, dx < \epsilon \text{ meas} \Omega + 2\alpha \eta(2\epsilon).$$ 

(8.27)

**Step 4.** We now gather all these inequalities. From (8.25), we have

$$I(u_\nu, \Omega) = \int_{\Omega} f(x, u_\nu(x), \nabla u_\nu(x)) \, dx$$

$$\geq \int_{\Omega} f(x, u(x), \nabla u_\nu(x)) \, dx - \epsilon \text{ meas} \Omega - 2\alpha \eta(2\epsilon).$$

(8.26)
We therefore get from (8.26)

\[ I(u_\nu, \Omega) \geq \sum_s \int_{D_s} f(x^s, u^s, \nabla u_\nu(x)) \, dx - 2\epsilon \text{meas } \Omega - 4\alpha \eta (2\epsilon). \]

We may now apply Theorem 8.4 to

\[ \xi \to f(x^s, u^s, \xi) \]

and we hence find

\[ \liminf_{\nu \to \infty} I(u_\nu, \Omega) \geq \sum_s \int_{D_s} f(x^s, u^s, \nabla u(x)) \, dx - 2\epsilon \text{meas } \Omega - 4\alpha \eta (2\epsilon). \]

Thus, using (8.27), we find

\[ \liminf_{\nu \to \infty} I(u_\nu, \Omega) \geq \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx - 3\epsilon \text{meas } \Omega - 6\alpha \eta (2\epsilon). \]

Since \( \epsilon \) is arbitrary, we have the claim. \( \blacksquare \)

**Proof.** We now continue with the proof of Theorem 8.11.

**Step 1.** We first show that we can restrict ourselves to sets \( \Omega \) that are a finite union of disjoint open cubes and to functions \( f \) satisfying

\[ -\alpha |\xi|^q \leq f(x, u, \xi) \leq g(x, u) (1 + |\xi|^p), \quad (8.28) \]

with \( 1 \leq q < p \) if \( p > 1 \) and \( q = 1 \) if \( p = 1 \).

This is already so when \( p = q = 1 \) and we therefore consider only the case \( p > 1 \). We start by defining

\[ h(x, u, \xi) := f(x, u, \xi) + \alpha |u|^r + \beta(x), \]

which satisfies (8.28), with a different \( g \) than the one in \((C_p)\). By Rellich theorem, we have

\[ u_\nu \to u \text{ in } L^r(\Omega; \mathbb{R}^N). \]

We now use the equiintegrability in \( W^{1,q} \cap L^r \) (since \( 1 \leq q < p \) if \( p > 1 \) and \( q = 1 \) if \( p = 1 \)) of the sequence \( \{u_\nu\} \) to get, for every \( \epsilon > 0 \), \( \delta = \delta(\epsilon) \) so that

\[ \text{meas } A \leq \delta \Rightarrow 0 \leq \int_A [\alpha(|u_\nu(x)|^r + |\nabla u_\nu(x)|^q) + \beta(x)] \, dx \leq \epsilon. \]

We next choose \( \Omega_\mu \subset \Omega \) to be a finite union of cubes and \( \mu \) sufficiently large so that

\[ \text{meas } (\Omega - \Omega_\mu) \leq \delta. \]
We thus obtain, from \((C_p)\) and the equiintegrability, that
\[
\int_\Omega f(x, u_\nu(x), \nabla u_\nu(x)) \, dx \\
\geq \int_{\Omega_\mu} f(x, u_\nu(x), \nabla u_\nu(x)) \, dx - \int_{\Omega_\mu - \Omega}\left[\alpha (|u_\nu|^r + |\nabla u_\nu(x)|^q) + \beta(x)\right] \, dx \\
\geq \int_{\Omega_\mu} h(x, u_\nu(x), \nabla u_\nu(x)) \, dx - \int_{\Omega_\mu}\left[\alpha |u_\nu|^r + \beta(x)\right] \, dx - \epsilon.
\]
Applying the lower semicontinuity for functions \(h\) satisfying (8.28), we can write
\[
\liminf_{\nu \to \infty} \int_\Omega f(x, u_\nu, \nabla u_\nu) \, dx \geq \int_{\Omega_\mu} h(x, u, \nabla u) \, dx - \int_{\Omega_\mu}\left[\alpha |u|^r + \beta(x)\right] \, dx - \epsilon \\
= \int_{\Omega_\mu} f(x, u, \nabla u) \, dx - \epsilon.
\]
Letting \(\mu \to \infty\) and recalling that \(\epsilon > 0\) is arbitrary, we indeed have the claim.

\textbf{Step 2.} We next see that we can further assume that \(f\) verifies for almost every \(x \in \Omega\) and for every \((u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}\) the following condition
\[
-\alpha |\xi|^q \leq f(x, u, \xi) \leq \alpha (1 + |\xi|^p)
\]
where \(\alpha > 0\). This is already so when \(p = q = 1\) and we hence discuss only the case \(p > 1\). First note that we can restrict our attention to \(f \geq 0\). Indeed since \(1 \leq q < p < \infty\), we can find, for every \(\epsilon > 0\), \(k = k(\epsilon) > 0\) so that
\[
\epsilon |\xi|^p + k \geq \alpha |\xi|^q, \quad \forall \xi \in \mathbb{R}^{N \times n}
\]
and thus
\[
f_\epsilon(x, u, \xi) := f(x, u, \xi) + \alpha |u|^r + \beta(x) + \epsilon |\xi|^p + k \geq 0.
\]
Note that \(f_\epsilon \geq 0\) and satisfies \((C_p)\), of course with a different function \(g\). Applying the lower semicontinuity for the non-negative function \(f_\epsilon \geq 0\) satisfying \((C_p)\) and letting \(\epsilon \to 0\), we easily get the claim. So we may now assume that when \(p > 1\)
\[
0 \leq f(x, u, \xi) \leq g(x, u) (1 + |\xi|^p).
\]
It therefore remains to show that we can replace \(g(x, u)\) by \(\alpha\) so as to have (8.29). Indeed, define, for every integer \(i\), a sequence of non-increasing continuous functions
\[
h_i : \mathbb{R}_+ \to [0, 1]
\]
such that
\[
h_i(s) := \begin{cases} 
1 & \text{if } 0 \leq s \leq i - 1 \\
0 & \text{if } s \geq i
\end{cases}
\]
and let
\[ \varphi_i(x, u) := \begin{cases} h_i(|u|) & \text{if } g(x, u) \leq i \\ \frac{i h_i(|u|)}{g(x, u)} & \text{if } g(x, u) > i. \end{cases} \]

Then define
\[ f_i(x, u, \xi) := \varphi_i(x, u) f(x, u, \xi) \]
and observe that it is a non negative Carathéodory function, quasiconvex in the variable \( \xi \) and that it satisfies (8.29) with \( \alpha = i \). Moreover,
\[ \lim_{i \to \infty} f_i(x, u, \xi) = \sup_i f_i(x, u, \xi) = f(x, u, \xi). \]

So, assume that we have proved the theorem for integrands \( f_i \) satisfying (8.29) and let us prove the result for \( f \). We in fact have
\[ \liminf_{\nu \to \infty} \int_{\Omega} f(x, u_\nu(x), \nabla u_\nu(x)) \, dx \geq \liminf_{\nu \to \infty} \int_{\Omega} f_i(x, u_\nu, \nabla u_\nu) \, dx \geq \int_{\Omega} f_i(x, u, \nabla u) \, dx. \]
Taking the supremum over \( i \) on the right hand side, we have indeed obtained the claim.

**Step 3.** So, from now on, we will assume that \( \Omega \) is a finite union of cubes and \( f \) satisfies (8.29). By working on each cube separately, we can even assume that \( \Omega \) itself is a cube. By restricting our attention to a subsequence, still denoted \( \{u_\nu\} \), we can assume that
\[ L := \liminf_{\nu \to \infty} I(u_\nu, \Omega) = \lim_{\nu \to \infty} I(u_\nu, \Omega). \]
We now conclude the proof of the theorem. Applying Lemma 8.15, we can find a subsequence \( \{u_\mu\} \) and \( v_\mu \in W^{1,p}(\Omega; \mathbb{R}^N) \) such that \( \{\nabla v_\mu\} \) is equiintegrable,
\[ v_\mu \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N) \]
and
\[ \lim_{\mu \to \infty} \text{meas } \Omega_\mu = 0, \]
where
\[ \Omega_\mu := \{x \in \Omega : u_\mu(x) \neq v_\mu(x)\} \cup \{x \in \Omega : \nabla u_\mu(x) \neq \nabla v_\mu(x)\}. \]
Using the fact that \( f \) satisfies (8.29), we can write
\[ I(u_\mu, \Omega) = I(u_\mu, \Omega - \Omega_\mu) + I(u_\mu, \Omega_\mu) \geq I(u_\mu, \Omega - \Omega_\mu) - \alpha \int_{\Omega_\mu} |\nabla u_\mu(x)|^q \, dx. \]
Weak lower semicontinuity

Since $v_\mu = u_\mu$ in $\Omega - \Omega_\mu$, we find from the above inequality and from (8.29) that

$$I(u_\mu, \Omega) \geq I(v_\mu, \Omega - \Omega_\mu) - \alpha \int_{\Omega_\mu} |\nabla u_\mu(x)|^q \, dx$$

$$\geq I(v_\mu, \Omega) - \alpha \int_{\Omega_\mu} (1 + |\nabla u_\mu(x)|^q + |\nabla v_\mu(x)|^p) \, dx.$$ 

We then apply Lemma 8.14 (note that (8.29) is $C'_p$ of Lemma 8.14) to the sequence $\{v_\mu\}$, using the equiintegrability of the sequences $\{\nabla v_\mu\}$ and $\{\nabla u_\mu\}$ (since $1 \leq q < p$ if $p > 1$ and $q = 1$ if $p = 1$) and the fact that $\lim_{\mu \to \infty} \text{meas} \, \Omega_\mu = 0$, to obtain the result, namely

$$L = \lim_{\mu \to \infty} I(u_\mu, \Omega) \geq \liminf_{\mu \to \infty} I(v_\mu, \Omega) \geq I(u, \Omega).$$

This achieves the proof of the theorem. \[\blacksquare\]

One important step in the proof of Theorem 8.11 was to replace the original sequence by an equiintegrable sequence, using the following result of Fonseca-Müller-Pedregal [288] (based on ideas contained in [3]) and Kristensen [379].

**Lemma 8.15** Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and

$$u_\nu \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N).$$

Then there exists a subsequence $\{u_\mu\}$ and $v_\mu \in W^{1,p}(\Omega; \mathbb{R}^N)$ such that $\{\nabla v_\mu\}$ is equiintegrable,

$$v_\mu \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N)$$

and

$$\lim_{\mu \to \infty} \text{meas} \, \Omega_\mu = 0$$

where

$$\Omega_\mu := \{x \in \Omega : u_\mu(x) \neq v_\mu(x)\} \cup \{x \in \Omega : \nabla u_\mu(x) \neq \nabla v_\mu(x)\}.$$ 

### 8.2.5 Lower semicontinuity for polyconvex functions

We now discuss the case of polyconvex functions. At first glance, it may seem that, since polyconvex functions are quasiconvex, nothing new is to be proved. However we can now consider functions that have no upper bound and in particular functions that may take the value $+\infty$.

**Theorem 8.16** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary, $p > n \land N$,

$$F : \Omega \times \mathbb{R}^N \times \mathbb{R}^r(n,N) \to \mathbb{R} \cup \{+\infty\}, \quad F = F(x, u, X),$$
be a Carathéodory function that is such that for almost every \( x \in \Omega \) and for every \((u, X) \in \mathbb{R}^N \times \mathbb{R}^{\tau(n,N)}\)

\[
X \to F(x,u,X) \text{ is convex,}
\]

\[
F(x,u,X) \geq \langle a(x);X \rangle + b(x) + c|u|^r,
\]

where \( a \in L^{p'}(\Omega;\mathbb{R}^{\tau(n,n)}) \), \( 1/p + 1/p' = 1 \), \( b \in L^1(\Omega) \), \( 1 \leq r < np/(n-p) \) if \( p < n \) and \( 1 \leq r < \infty \) if \( p \geq n \) and \( c \in \mathbb{R} \). Then

\[
I(u,\Omega) := \int_{\Omega} F(x,u(x),\nabla u(x))\,dx
\]

is (sequentially) weakly lower semicontinuous in \( W^{1,p}(\Omega;\mathbb{R}^N) \).

**Remark 8.17** The restriction \( p > n \land N \) can be slightly relaxed if we replace the convergence

\[
u \nu \to u \text{ in } W^{1,p}(\Omega;\mathbb{R}^N)
\]

by

\[
\begin{aligned}
u \nu \to u \text{ in } L^p(\Omega;\mathbb{R}^N) \\
T(\nabla u_\nu) \to T(\nabla u) \text{ in } L^1(\Omega;\mathbb{R}^{\tau(n,N)})
\end{aligned}
\]

We will see, in Theorem 8.20 and in Theorem 8.31, how this can be obtained with some \( p \leq n \land N \).

**Proof.** Anticipating the results of Theorem 8.20, we find that since \( p > n \land N \), then

\[
T(\nabla u_\nu) \to T(\nabla u) \text{ in } L^{p/n}(\Omega;\mathbb{R}^{\tau(n,N)})
\]

We are therefore in a position to apply Theorem 3.23 and hence the result follows at once.

The above proof relied heavily on the weak continuity of determinants. One can however improve the result in the following context. As a simplification, we assume that \( N = n \) and that the function \( F \geq 0 \) and does not depend on lower order terms.

Assume that \( p > n - 1 \) (note that in the theorem we need \( p > n \)) and \( u, u_\nu \in W^{1,n}(\Omega;\mathbb{R}^n) \). Then

\[
I(u,\Omega) := \int_{\Omega} F(T(\nabla u(x)))\,dx
\]

is weakly lower semicontinuous in \( W^{1,p}(\Omega;\mathbb{R}^N) \).

The above result was proved by Dacorogna-Marcellini [194] when \( p > n - 1 \). Maly [415] then gave a counterexample proving that the result does not hold if \( p < n - 1 \). Later, Celada-Dal Maso [128] and Fusco-Hutchinson [296] showed that Dacorogna-Marcellini result holds if \( p = n - 1 \).
8.3 Weak Continuity

We now turn our attention to results on weak continuity of nonlinear functions. Let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) be continuous. We show that

\[
  f (\nabla u_\nu) \rightharpoonup f (\nabla u) \quad \text{in } \mathcal{D}'(\Omega)
\]

for every sequence \( u_\nu \rightharpoonup u \) in \( W^{1,p}(\Omega; \mathbb{R}^N) \) if and only if \( f \) is quasiaffine (i.e. from Theorem 5.20, \( f \) is a linear combination of minors of the matrix \( \nabla u \)).

Plainly, the existence of nonlinear weakly continuous functions is purely due to the vectorial nature of the problem, since if \( N = 1 \) (or \( n = 1 \)), the only minors of the matrix \( \nabla u \) are just the linear terms \( \partial u/\partial x_i, 1 \leq i \leq n \) (or if \( n = 1 \), the linear terms \( du/\partial x, 1 \leq i \leq N \)).

It is also clear that Theorem 8.1 and Theorem 8.4 applied to \( f, I \) and \(-f, -I\), added to the fact that the domain \( \Omega \) is arbitrary, immediately give the weak continuity if \( p \) is large enough. We use Theorem 8.1 for the necessary condition; however, for reasons explained below, we do not use Theorem 8.4 for the sufficiency result and we give a new proof of the weak continuity of the minors.

The results of this section are essentially due to Reshetnyak [509], [510] and Ball [51], [53]. Considerations on weak continuity have been developed in a more general context, called compensated compactness, by Murat and Tartar [469], [470], [471], [568] (for a presentation of this theory, see also Dacorogna [173]).

Before starting our analysis, we recall the following definition.

**Definition 8.18** Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( f_\nu, f \in L^1_{\text{loc}}(\Omega) \). We say that \( f_\nu \) converges to \( f \) in the sense of distributions and we write

\[
  f_\nu \rightharpoonup f \quad \text{in } \mathcal{D}'(\Omega)
\]

if

\[
\int_\Omega f_\nu(x) \varphi(x) \, dx \to \int_\Omega f(x) \varphi(x) \, dx
\]

for every \( \varphi \in \mathcal{D}(\Omega) \) (the set of \( C^\infty \) functions with compact support).

### 8.3.1 Necessary condition

**Theorem 8.19** Let \( 1 \leq p \leq \infty \), let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) be continuous. If, for every sequence \( u_\nu \rightharpoonup u \) in \( W^{1,p}(\Omega; \mathbb{R}^N) \)

\[
  (u_\nu \rightharpoonup u \text{ if } p = \infty),
\]

\[
  f (\nabla u_\nu) \rightharpoonup f (\nabla u) \quad \text{in } \mathcal{D}'(\Omega)
\]

(8.30)

then \( f \) is quasiaffine, i.e. there exist \( \alpha \in \mathbb{R}, \beta \in \mathbb{R}^{(n,N)} \) such that

\[
  f (\xi) = \alpha + \langle \beta; T(\xi) \rangle
\]

(8.31)
for every $\xi \in \mathbb{R}^{N \times n}$, where $n \wedge N = \min\{n, N\}$ and
\[
\begin{aligned}
T(\xi) &= (\xi, \text{adj}_2 \xi, \cdots, \text{adj}_{n \wedge N} \xi) \\
\tau(n, N) &= \sum_{s=1}^{n \wedge N} \sigma(s), \quad \sigma(s) = \binom{N}{s} \binom{n}{s}
\end{aligned}
\]
and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{\tau(n, N)}$.

**Proof.** Let $\varphi \in \mathcal{D}(\Omega)$ and let

\[
I(u, \Omega) := \int_{\Omega} \varphi(x) f(\nabla u(x)) \, dx
\]
then (8.30) is equivalent to

\[
\lim_{\nu \to \infty} I(u_{\nu}, \Omega) = I(u, \Omega).
\]
We may therefore apply Theorem 8.1 to $I$ and $-I$ and get that $f$ and $-f$ are quasiconvex, i.e. $f$ is quasiaffine. Theorem 5.20 implies then (8.31) and the theorem follows.

### 8.3.2 Sufficient condition

For the clarity of the exposition, we always give the results for the cases $N = n = 2$, $N = n = 3$ and then $N = n$, before giving the general result when $N, n \geq 2$.

We also recall some notations and elementary properties of determinants and adjugate matrices (for more details, see Sections 5.4 and 8.5).

- For $N = n = 3$, we denote

\[
\text{adj}_2 \nabla u = \left( (\text{adj}_2 \nabla u)_i^j \right)_{1 \leq i \leq 3},
\]
where

\[
(\text{adj}_2 \nabla u)_i^j = (-1)^{i+j} \frac{\partial (u^j, u^k)}{\partial (x_\beta, x_\gamma)}
\]
\[
= (-1)^{i+j} \left[ \frac{\partial u^j}{\partial x_\beta} \frac{\partial u^k}{\partial x_\gamma} - \frac{\partial u^j}{\partial x_\gamma} \frac{\partial u^k}{\partial x_\beta} \right],
\]
where $j < k$ with $j, k \neq i$ and $\beta < \gamma$ with $\beta, \gamma \neq \alpha$.

We also have

\[
\text{det} \nabla u = \sum_{i=1}^{3} \frac{\partial u_i^1}{\partial x_i} (\text{adj}_2 \nabla u)_i^1.
\]
- When $N = n$, we recall that

$$\det \nabla u = \frac{\partial (u^1, \cdots, u^n)}{\partial (x_1, \cdots, x_n)}$$

$$= \sum_{\alpha=1}^{n} (-1)^{\alpha+1} \frac{\partial u^1}{\partial x_\alpha} \frac{\partial (u^2, \cdots, u^n)}{\partial (x_1, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots, x_n)}.$$ 

- For $N, n \geq 2$ and $2 \leq s \leq n \land N = \min \{n, N\}$, we have

$$\text{adj}_s \nabla u = \left( (\text{adj}_s \nabla u)^{i}_{\alpha} \right)_{1 \leq i \leq (\begin{smallmatrix} n \\ s \end{smallmatrix}), 1 \leq \alpha \leq (\begin{smallmatrix} n \\ s \end{smallmatrix}))},$$

where

$$(\text{adj}_s \nabla u)^{i}_{\alpha} = (-1)^{i+\alpha} \frac{\partial (u^{i_1}, \cdots, u^{i_s})}{\partial (x_{\alpha_1}, \cdots, x_{\alpha_s})}.$$ 

For the precise relation between $i, i_1, \cdots, i_s$ and $\alpha, \alpha_1, \cdots, \alpha_s$ see Section 5.4.

We now give the main theorem, which shows that these functions are actually weakly continuous.

**Theorem 8.20** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $1 < p < \infty$, and let $u_\nu \rightharpoonup u$ in $W^{1,p} (\Omega; \mathbb{R}^N)$.

**Part 1.** Let $N = n = 2$ and $p \geq 2$. Then

$$\det \nabla u_\nu \rightharpoonup \det \nabla u \text{ in } D' (\Omega)$$

and if $p > 2$, then

$$\det \nabla u_\nu \rightharpoonup \det \nabla u \text{ in } L^{p/2} (\Omega).$$

**Part 2.** Let $N = n = 3$. If $p \geq 2$, then

$$\text{adj}_2 \nabla u_\nu \rightharpoonup \text{adj}_2 \nabla u \text{ in } D' (\Omega; \mathbb{R}^9)$$

and if $p > 2$, then

$$\text{adj}_2 \nabla u_\nu \rightharpoonup \text{adj}_2 \nabla u \text{ in } L^{p/2} (\Omega; \mathbb{R}^9).$$

If $p \geq 3$, then

$$\det \nabla u_\nu \rightharpoonup \det \nabla u \text{ in } D' (\Omega)$$

and if $p > 3$, then

$$\det \nabla u_\nu \rightharpoonup \det \nabla u \text{ in } L^{p/3} (\Omega).$$

**Part 3.** Let $N = n$ and $p \geq n$. Then

$$\det \nabla u_\nu \rightharpoonup \det \nabla u \text{ in } D' (\Omega)$$
and if \( p > n \), then
\[
\det \nabla u_\nu \rightharpoonup \det \nabla u \text{ in } L^{p/n}(\Omega).
\]

Part 4. Let \( N, n \geq 2 \), \( 2 \leq s \leq n \wedge N = \min \{n, N\} \) and \( p \geq s \). Then
\[
\text{adj}_s \nabla u_\nu \rightharpoonup \text{adj}_s \nabla u \text{ in } D'(\Omega; \mathbb{R}^{\sigma(s)}),
\]
where
\[
\sigma(s) = \binom{N}{s} \binom{n}{s} = \frac{N!n!}{(s!)^2 (N-s)! (n-s)!}.
\]
Furthermore, if \( p > s \), then
\[
\text{adj}_s \nabla u_\nu \rightharpoonup \text{adj}_s \nabla u \text{ in } L^{p/s}(\Omega; \mathbb{R}^{\sigma(s)}).
\]

Part 5. Let \( N, n \geq 2 \), \( 2 \leq s \leq n \wedge N \) and assume that
\[
\text{adj}_{s-1} \nabla u_\nu \rightharpoonup \text{adj}_{s-1} \nabla u \text{ in } L^r(\Omega; \mathbb{R}^{\sigma(s-1)}),
\]
where \( r > 1 \) with \( \frac{1}{p} + \frac{1}{r} \leq 1 \). Then
\[
\text{adj}_s \nabla u_\nu \rightharpoonup \text{adj}_s \nabla u \text{ in } D'(\Omega; \mathbb{R}^{\sigma(s)}).
\]

**Remark 8.21** (i) Let \( N = n = 2 \). Note that if \( p = 2 \) and if we know, in addition, that
\[
\det \nabla u_\nu \rightharpoonup f \text{ in } L^1(\Omega),
\]
then the uniqueness of the limit in \( D'(\Omega) \) ensures that \( f = \det \nabla u \).

(ii) Let \( N = n = 2 \). If \( p > 2 \), the statement in Part 1 results immediately from Theorem 8.4, since, trivially,
\[
-\left(1 + |\nabla u|^2\right) \leq -|\nabla u|^2 \leq \det \nabla u \leq |\nabla u|^2 \leq 1 + |\nabla u|^p.
\]

(iii) Let \( N = n = 2 \). If \( p = 2 \), Theorem 8.4 cannot be applied, as seen in Remark 8.5 (iii), there are examples (see Example 8.6) of sequences \( u_\nu \rightharpoonup u \) in \( W^{1,2}(\Omega; \mathbb{R}^2) \) such that \( \det \nabla u_\nu \not\rightharpoonup \det \nabla u \) in \( L^1(\Omega) \). Theorem 8.20 ensures, however, that \( \det \nabla u_\nu \rightharpoonup \det \nabla u \) in \( D'(\Omega) \).

(iv) All the above remarks can be made for the general case \( N, n \geq 2 \).

(v) If \( p = +\infty \), replace everywhere weak convergence by weak * convergence in the appropriate space (\( L^\infty \) or \( W^{1,\infty} \)).

(vi) For some extensions of the theorem, we refer to Müller [458], [459].

The main tool in proving Theorem 8.20 is the observation that any minor of \( \nabla u \) can be expressed as a divergence of a vector field and we are led to introduce the following operators.
Definition 8.22 Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C^2(\Omega; \mathbb{R}^N)$.

(i) For $N = n = 2$, define

$$ \text{Det} \nabla u := \frac{\partial}{\partial x_1} (u^1 \frac{\partial u^2}{\partial x_2}) - \frac{\partial}{\partial x_2} (u^1 \frac{\partial u^2}{\partial x_1}). $$

(ii) For $N = n = 3$, define (see the form of adj$_2 \nabla u$ given above)

$$ \text{Adj}_2 \nabla u := \left( (\text{Adj}_2 \nabla u)^i_\alpha \right)_{1 \leq i \leq 3, 1 \leq \alpha \leq 3}, $$

where

$$ (\text{Adj}_2 \nabla u)^i_\alpha := (-1)^{i+\alpha} \left[ \frac{\partial}{\partial x_\beta} (u^j \frac{\partial u^k}{\partial x_\gamma}) - \frac{\partial}{\partial x_\gamma} (u^j \frac{\partial u^k}{\partial x_\beta}) \right] $$

where $j < k$ with $j, k \neq i$ and $\beta < \gamma$ with $\beta, \gamma \neq \alpha$.

Similarly, we let

$$ \text{Det} \nabla u := \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left( u^1 (\text{Adj}_2 \nabla u)^i_1 \right). $$

(iii) When $N = n$, we let

$$ \text{Det} \nabla u := \sum_{\alpha=1}^n (-1)^{\alpha+1} \frac{\partial}{\partial x_\alpha} \left( u^1 \frac{\partial (u^1, \ldots, u^n)}{\partial (x_1, \ldots, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_n)} \right). $$

(iv) Let $N, n \geq 2$ and $2 \leq s \leq n \wedge N = \min \{n, N\}$. We define (see the form of adj$_s \nabla u$ given above)

$$ \text{Adj}_s \nabla u := \left( (\text{Adj}_s \nabla u)^i_\alpha \right)_{1 \leq i \leq (N)_{\wedge s}, 1 \leq \alpha \leq (n)_{\wedge s}}, $$

where

$$ (\text{Adj}_s \nabla u)^i_\alpha := (-1)^{i+\alpha} \sum_{t=1}^s (-1)^{t+1} \frac{\partial}{\partial x_{\alpha t}} \left( u^{i_1} \frac{\partial (u^{i_2}, \ldots, u^{i_t})}{\partial (x_{\alpha_1}, \ldots, x_{\alpha_{t-1}}, x_{\alpha_{t+1}}, \ldots, x_{\alpha_s})} \right). $$

Remark 8.23 (i) In the case $N = n = 2$, we could have written as well

$$ \text{Det} \nabla u := \frac{\partial}{\partial x_2} (u^2 \frac{\partial u^1}{\partial x_1}) - \frac{\partial}{\partial x_1} (u^2 \frac{\partial u^1}{\partial x_2}), $$

but this does not change anything in the analysis below. A similar remark applies to all other cases.

(ii) One should observe that if $N = n = 3$, Ball [53] defines

$$ \text{Det} \nabla u := \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left( u^1 (\text{Adj}_2 \nabla u)^i_1 \right), $$
which does not correspond to our definition (ii) (note the change from adj to Adj above). The two definitions need not be the same if \( u \in W^{1,p}(\Omega;\mathbb{R}^3) \) and \( p < 2 \), as suggested by the following lemma.

We now see how to relate these operators Det and Adj\(_s\) to the algebraic definitions of det and adj\(_s\).

**Lemma 8.24** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and let \( u \in W^{1,p}(\Omega;\mathbb{R}^N) \), \( 1 < p < \infty \).

Part 1: \( N = n = 2 \). If \( p \geq \frac{4}{3} \), then \( \text{Det} \nabla u \in \mathcal{D}'(\Omega) \). Furthermore, if \( p \geq 2 \), then

\[
\text{Det} \nabla u = \det \nabla u \text{ in } \mathcal{D}'(\Omega).
\]

In particular, if \( u \in C^2(\Omega;\mathbb{R}^2) \), then the above identity holds in the usual sense.

Part 2: \( N = n = 3 \).

(i) If \( p \geq \frac{3}{2} \), then \( \text{Adj}_2 \nabla u \in \mathcal{D}'(\Omega;\mathbb{R}^9) \) and if \( p \geq 2 \), then

\[
\text{Adj}_2 \nabla u = \text{adj}_2 \nabla u \text{ in } \mathcal{D}'(\Omega;\mathbb{R}^9).
\]

In particular, if \( u \in C^2(\Omega;\mathbb{R}^3) \), then the above identity holds in the usual sense.

(ii) If \( p \geq \frac{9}{4} \), then \( \text{Det} \nabla u \in \mathcal{D}'(\Omega) \). Moreover, if \( p \geq 3 \), then

\[
\text{Det} \nabla u = \det \nabla u \text{ in } \mathcal{D}'(\Omega) \text{ and, in particular, if } u \in C^2(\Omega;\mathbb{R}^3), \text{ then the identity holds in the usual sense.}
\]

Part 3: \( N = n \). If \( p \geq \frac{n^2}{n+1} \), then \( \text{Det} \nabla u \in \mathcal{D}'(\Omega) \). Furthermore, if \( p \geq n \), then

\[
\text{Det} \nabla u = \det \nabla u \text{ in } \mathcal{D}'(\Omega) \text{ and thus the identity holds in the usual sense if } u \in C^2(\Omega;\mathbb{R}^n).
\]

Part 4: \( N, n \geq 2 \) and \( 2 \leq s \leq n \wedge N = \min\{n,N\} \). If \( p \geq \frac{sn}{n+1} \), then \( \text{Adj}_s \nabla u \in \mathcal{D}'(\Omega;\mathbb{R}^{\sigma(s)}) \), where \( \sigma(s) = \left( \begin{array}{c} N \\ s \end{array} \right) \) \( \left( \begin{array}{c} n \\ s \end{array} \right) \). Moreover, if \( p \geq s \), then

\[
\text{Adj}_s \nabla u = \text{adj}_s \nabla u \text{ in } \mathcal{D}'(\Omega;\mathbb{R}^{\sigma(s)}) \text{ and thus the identity holds in the usual sense if } u \in C^2(\Omega;\mathbb{R}^N).
\]

Part 5: \( N, n \geq 2 \) and \( 2 \leq s \leq n \wedge N \). Assume that, for \( r > 1 \),

\[
\text{adj}_{s-1} \nabla u \in L^r(\Omega;\mathbb{R}^{\sigma(s-1)}).
\]

(i) If \( \frac{1}{p} + \frac{1}{r} \leq 1 + \frac{1}{s} \), then \( \text{Adj}_s \nabla u \in \mathcal{D}'(\Omega;\mathbb{R}^{\sigma(s)}) \).

(ii) If \( \frac{1}{p} + \frac{1}{r} \leq 1 \), then

\[
\text{Adj}_s \nabla u = \text{adj}_s \nabla u \text{ in } \mathcal{D}'(\Omega;\mathbb{R}^{\sigma(s)}).
\]
Remark 8.25 (i) Let \( N = n = 2 \) and \( \frac{4}{3} \leq p < 2 \). If \( \det \nabla u \) is defined in the usual way, namely
\[
\det \nabla u = \frac{\partial u^1}{\partial x_1} \frac{\partial u^2}{\partial x_2} - \frac{\partial u^2}{\partial x_1} \frac{\partial u^1}{\partial x_2},
\]
then \( \det \nabla u \) is not necessarily a distribution, while \( \text{Det} \nabla u \) is. In fact, \( \text{Det} \nabla u \) is the (unique) extension, as a distribution, by continuity of \( \det \nabla u \), when \( \frac{4}{3} \leq p < 2 \). Note also that if \( p < \frac{4}{3} \), then even \( \text{Det} \nabla u \) need not be a distribution (see Example 8.28).

(ii) Similar remarks apply to the general case \( N, n \geq 1 \).

Proof. (Lemma 8.24). We prove Part 1 for illustration and then prove Parts 4 and 5.

Part 1: \( N = n = 2 \). If \( p \geq \frac{4}{3} \) and since \( u \in W^{1,p} (\Omega; \mathbb{R}^2) \), we have by the Sobolev imbedding theorem that \( u \in L^{1}_{\text{loc}} (\Omega; \mathbb{R}^2) \). Using Hölder inequality, we deduce that
\[
\frac{u^1 \partial u^2}{\partial x_2}, \frac{u^1 \partial u^2}{\partial x_1} \in L^1_{\text{loc}} (\Omega)
\]
and thus \( \text{Det} \nabla u \in D' (\Omega) \).

Moreover, if \( p \geq 2 \), then \( \det \nabla u \in L^1 (\Omega) \). Observe that if \( u \in C^2 (\Omega; \mathbb{R}^2) \), then
\[
\det \nabla u = \frac{\partial u^1}{\partial x_1} \frac{\partial u^2}{\partial x_2} - \frac{\partial u^2}{\partial x_1} \frac{\partial u^1}{\partial x_2} = \text{Det} \nabla u.
\]

Therefore, multiplying the above identity by \( \varphi \in D (\Omega) \) and integrating by parts, we find
\[
\int_{\Omega} \det \nabla u \cdot \varphi \, dx = - \int_{\Omega} \left( u^1 \frac{\partial u^2}{\partial x_2} \frac{\partial \varphi}{\partial x_1} - u^1 \frac{\partial u^2}{\partial x_1} \frac{\partial \varphi}{\partial x_2} \right) \, dx.
\]
Since \( C^2 (\Omega; \mathbb{R}^2) \) is dense in \( W^{1,2}_{\text{loc}} (\Omega; \mathbb{R}^2) \), then the above equality holds for every \( u \in W^{1,2}_{\text{loc}} \); and this concludes Part 1.

Part 4: \( N, n \geq 2 \). We first consider the case where \( p \geq \frac{sn}{n+1} \). Since \( u \in W^{1,p} (\Omega; \mathbb{R}^N) \), we have by the Sobolev imbedding theorem that \( u \in L^{sn/(n+1-s)}_{\text{loc}} (\Omega; \mathbb{R}^N) \). Using Hölder inequality, we also have that
\[
\frac{\partial (u^{i_2}, \ldots, u^{i_s})}{\partial (x_{\alpha_1}, \ldots, x_{\alpha_{t-1}}, x_{\alpha_{t+1}}, \ldots, x_{\alpha_s})} \in L^{sn/(n+1)(s-1)} (\Omega)
\]
and hence
\[
\frac{u^{i_1}}{u^{i_1}} \frac{\partial (u^{i_2}, \ldots, u^{i_s})}{\partial (x_{\alpha_1}, \ldots, x_{\alpha_{t-1}}, x_{\alpha_{t+1}}, \ldots, x_{\alpha_s})} \in L^1_{\text{loc}} (\Omega).
\]
Therefore, using the definition of \( \text{Adj}_s \), we get that \( \text{Adj}_s \nabla u \in D' (\Omega; \mathbb{R}^{\sigma(s)}) \).
We now discuss the case $p \geq s$ and then $\text{adj}_s \nabla u \in L^1 \left( \Omega; \mathbb{R}^{\sigma(s)} \right)$. Observe that if $u \in C^2 \left( \Omega; \mathbb{R}^N \right)$, then (cf. Theorem 8.33)

$$\text{Adj}_s \nabla u = \text{adj}_s \nabla u.$$ 

Multiplying the above equality by $\varphi \in D(\Omega)$ and integrating by parts the left hand side, we find that

$$\int_{\Omega} \left( (\text{adj}_s \nabla u)^i_\alpha \right) \varphi \, dx = \sum_{t=1}^s (-1)^t \int_{\Omega} u^{i_1} \frac{\partial (u^{i_2}, \ldots, u^{i_s})}{\partial (x_{\alpha_1}, \ldots, x_{\alpha_{t-1}}, x_{\alpha_{t+1}}, \ldots, x_{\alpha_s})} \frac{\partial \varphi}{\partial x_{\alpha_t}} \, dx.$$ 

Since $C^2 \left( \Omega; \mathbb{R}^N \right)$ is dense in $W^{1,s}_{\text{loc}} \left( \Omega; \mathbb{R}^N \right)$, we deduce that the equality holds for every $u \in W^{1,s}$ and this concludes Part 4 of the lemma.

**Part 5:** $N, n \geq 2$. The case $p \geq n$ being easier, we assume that $p < n$. Using Sobolev imbedding theorem, we have, since $u \in W^{1,p}$, that $u \in L^{np/(n-p)} \left( \Omega; \mathbb{R}^N \right)$. We now combine the definition of $\text{Adj}_s$, $\text{adj}_s$ and the fact that $\text{adj}_{s-1} \nabla u \in L^r$ with Hölder inequality (recalling that $\frac{1}{p} + \frac{1}{r} - \frac{1}{n} \leq 1$) to deduce that

$$u^{i_1} \left( \text{adj}_{s-1} \nabla u \right)^i_\alpha \in L^1_{\text{loc}} \left( \Omega \right).$$ 

We therefore have $\text{Adj}_s \nabla u \in \mathcal{D}' \left( \Omega; \mathbb{R}^{\sigma(s)} \right)$.

Furthermore, since $\frac{1}{p} + \frac{1}{r} \leq 1$ and $\text{adj}_{s-1} \nabla u \in L^r$, we have that $\text{adj}_s \nabla u \in L^1$. Since, using Theorem 8.33, the identity

$$\text{Adj}_s \nabla u = \text{adj}_s \nabla u$$ 

holds for every $u \in C^2$, we deduce, by density, that

$$\text{Adj}_s \nabla u = \text{adj}_s \nabla u \quad \text{in} \quad \mathcal{D}' \left( \Omega; \mathbb{R}^{\sigma(s)} \right).$$ 

This concludes the proof of the Lemma. □

We are now in a position to show Theorem 8.20.

**Proof.** We prove Part 1 for the sake of illustration, then Parts 4 and 5.

**Part 1.** Let $N = n = 2$ and $p \geq 2$. Let $\varphi \in D(\Omega)$, then by Lemma 8.24 we have

$$\int_{\Omega} \det \nabla u_\nu \cdot \varphi \, dx = -\int_{\Omega} \left( u_\nu \frac{\partial u_\nu^2}{\partial x_2} \frac{\partial \varphi}{\partial x_1} - u_\nu \frac{\partial u_\nu^2}{\partial x_1} \frac{\partial \varphi}{\partial x_2} \right) \, dx.$$ 

Since $u_\nu \rightharpoonup u$ in $W^{1,p} \left( \Omega; \mathbb{R}^2 \right)$, $u_\nu \rightarrow u$ in $L^q_{\text{loc}} \left( \Omega; \mathbb{R}^2 \right)$ with $q < \infty$, and therefore

$$\left( u_\nu \frac{\partial u_\nu^2}{\partial x_2}, u_\nu \frac{\partial u_\nu^2}{\partial x_1} \right) \rightharpoonup \left( u \frac{\partial u^2}{\partial x_2}, u \frac{\partial u^2}{\partial x_1} \right)$$ 

in $L^1_{\text{loc}} \left( \Omega; \mathbb{R}^2 \right)$. 

We therefore deduce that
\[ \int_{\Omega} \det \nabla u_{\nu} \cdot \varphi \, dx \rightarrow \int_{\Omega} \det \nabla u \cdot \varphi \, dx. \]

**Part 4.** Let \( N, n \geq 2, 2 \leq s \leq n \wedge N \) and \( p \geq s \). In order to show the theorem it is sufficient to establish that for every \( \varphi \in \mathcal{D}(\Omega) \) we have
\[ \int_{\Omega} \frac{\partial (u_{\nu}^{i_1}, \ldots, u_{\nu}^{i_s})}{\partial (x_{\alpha_1}, \ldots, x_{\alpha_s})} \varphi \, dx \rightarrow \int_{\Omega} \frac{\partial (u^{i_1}, \ldots, u^{i_s})}{\partial (x_{\alpha_1}, \ldots, x_{\alpha_s})} \varphi \, dx. \] (8.32)

To show (8.32), we proceed by induction on \( s \). Suppose that the theorem has been established up to the order \( s - 1 \) (the case \( s = 2 \) has been dealt with in Part 1). Since \( u_{\nu} \rightharpoonup u \) in \( W^{1,p}(\Omega; \mathbb{R}^N) \) and \( p \geq s \) we deduce that
\[ u_{\nu} \rightharpoonup u \text{ in } L^q_{\text{loc}}(\Omega; \mathbb{R}^N) \text{ with } 1 \leq q < \frac{ns}{n - s}. \] (8.33)

By hypothesis of induction, denoting
\[ (x_1, \ldots, \widehat{x_i}, \ldots, x_s) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s), \]
we have
\[ \frac{\partial (u_{\nu}^{i_2}, \ldots, u_{\nu}^{i_s})}{\partial (x_{\alpha_1}, \ldots, \widehat{x_{\alpha_i}}, \ldots, x_{\alpha_s})} \rightharpoonup \frac{\partial (u^{i_2}, \ldots, u^{i_s})}{\partial (x_{\alpha_1}, \ldots, \widehat{x_{\alpha_i}}, \ldots, x_{\alpha_s})} \text{ in } \mathcal{D}'(\Omega) \]
for every \( 1 \leq t \leq s \). Since the above \((s - 1)\) minor is in \( L^{p/(s-1)}(\Omega) \) and \( p \geq s \), we get that the above convergence is actually in \( L^{p/(s-1)}(\Omega) \). Combining (8.33) with the above convergence, we obtain that, for \( 1 \leq r < n/(n - 1) \),
\[ u_{\nu}^{i_1} \frac{\partial (u_{\nu}^{i_2}, \ldots, u_{\nu}^{i_s})}{\partial (x_{\alpha_1}, \ldots, \widehat{x_{\alpha_i}}, \ldots, x_{\alpha_s})} \rightharpoonup u^{i_1} \frac{\partial (u^{i_2}, \ldots, u^{i_s})}{\partial (x_{\alpha_1}, \ldots, \widehat{x_{\alpha_i}}, \ldots, x_{\alpha_s})} \text{ in } L^r_{\text{loc}}(\Omega) \]
for every \( 1 \leq t \leq s \). We finally combine this convergence result with Lemma 8.24 to get, for every \( \varphi \in \mathcal{D}(\Omega) \),
\[ \int_{\Omega} \frac{\partial (u_{\nu}^{i_1}, \ldots, u_{\nu}^{i_s})}{\partial (x_{\alpha_1}, \ldots, x_{\alpha_s})} \varphi \, dx \]
\[ = - \sum_{t=1}^{s} (-1)^t \int_{\Omega} u_{\nu}^{i_t} \frac{\partial (u_{\nu}^{i_2}, \ldots, u_{\nu}^{i_s})}{\partial (x_{\alpha_1}, \ldots, \widehat{x_{\alpha_t}}, \ldots, x_{\alpha_s})} \, \frac{\partial \varphi}{\partial x_{\alpha_t}} \, dx \]
\[ \begin{align*}
& \quad \quad \rightarrow - \sum_{t=1}^{s} (-1)^t \int_{\Omega} u^{i_t} \frac{\partial (u^{i_2}, \ldots, u^{i_s})}{\partial (x_{\alpha_1}, \ldots, \widehat{x_{\alpha_t}}, \ldots, x_{\alpha_s})} \, \frac{\partial \varphi}{\partial x_{\alpha_t}} \, dx.
\end{align*}
\]
Using again Lemma 8.24 on the right hand side, we obtain (8.32) and thus Part 4.

**Part 5.** Let \( N, n \geq 2, 2 \leq s \leq n \wedge N \) and \( \frac{1}{p} + \frac{1}{r} \leq 1 \). We then have \( \text{adj}_s \nabla u \in L^1(\Omega; \mathbb{R}^{\sigma(s)}) \). Proceeding exactly as in Part 4, we obtain the result and hence the theorem. ■
It is clear that, from Theorem 8.20 and Lemma 8.24 (with the same notations as in the lemma), we immediately get (see Ball [51], [53]) the following.

**Corollary 8.26** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let

$$u_\nu \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N).$$

Part 1. Let $N = n$ and $p > \frac{n^2}{n+1}$, then

$$\det \nabla u_\nu \rightharpoonup \det \nabla u \text{ in } \mathcal{D}'(\Omega).$$

Part 2. Let $N, n \geq 2$, $2 \leq s \leq n \land N$ and $p > \frac{sn}{n+1}$, then

$$\text{Adj}_s \nabla u_\nu \rightharpoonup \text{Adj}_s \nabla u \text{ in } \mathcal{D}' \left( \Omega; \mathbb{R}^{s(s)} \right).$$

**Remark 8.27** (i) The proof of Corollary 8.26 is almost identical to that of Theorem 8.20 using Lemma 8.24.

(ii) The result of the corollary is false if, for example for Part 1, $p \leq \frac{n^2}{n+1}$, see Example 8.28.

**Example 8.28** (Dacorogna-Murat [208]). The result of Part 1 of the corollary is false if $p \leq \frac{n^2}{n+1}$. When $p = \frac{n^2}{n+1}$, the result remains partially true (see Theorem 1 in [208]) in the sense that, up to a subsequence,

$$\det \nabla u_\nu \rightharpoonup \det \nabla u + \mu \text{ in } \mathcal{D}'(\Omega)$$

for a certain $\mu \in \mathcal{D}'(\Omega)$, with, in general, $\mu \neq 0$. We now give an example where weak continuity does not hold. We construct a sequence $\{u_\nu\} \subset W^{1,\infty}(\Omega; \mathbb{R}^n)$ such that

$$u_\nu \rightharpoonup u \equiv 0 \text{ in } W^{1,p}(\Omega; \mathbb{R}^n) \text{ and } \det \nabla u_\nu \not\rightharpoonup \det \nabla u \equiv 0 \text{ in } \mathcal{D}'(\Omega).$$

Note that since $u_\nu, u \in W^{1,\infty}(\Omega; \mathbb{R}^n)$, then

$$\det \nabla u = \det \nabla u = 0 \text{ and } \det \nabla u_\nu = \det \nabla u_\nu \text{ in } \mathcal{D}'(\Omega).$$

We first let $x = (x_1, \cdots, x_n)$, $r^2 = x_1^2 + \cdots + x_n^2$ and

$$\Omega = \{x \in \mathbb{R}^n : r < 1\}.$$

We then consider the sequence

$$u_\nu (x) := f_\nu (r) \begin{pmatrix} 1 \\ x_2/r \\ \vdots \\ x_n/r \end{pmatrix},$$
where for \( p \geq 1 \) and for \( \nu \geq 4 \), we let

\[
f_{\nu}(r) := \begin{cases} 
\nu^{n/p} & \text{if } r \in [0, \frac{1}{\nu}) \\
\nu^{n/p} \left( \frac{2}{\nu} - r \right) & \text{if } r \in \left[ \frac{1}{\nu}, \frac{2}{\nu} \right) \\
0 & \text{if } r \in \left[ \frac{2}{\nu}, 1 \right).
\end{cases}
\]

A direct computation shows that the sequence \( \{u_{\nu}\} \) has all the above properties and in particular that

\[
\det \nabla u_{\nu}(x) = f'_{\nu}(r) \left( \frac{f_{\nu}(r)}{r} \right)^{n-1} \frac{x_1}{r}.
\]

We now claim that we can find \( \varphi \in D(\Omega) \) and \( a \neq 0 \) such that

\[
\lim_{\nu \to \infty} \int_{\Omega} \det \nabla u_{\nu}(x) \varphi(x) dx \rightarrow \begin{cases} 
a & \text{if } p = \frac{n^2}{n+1} \\
\infty & \text{if } 1 \leq p < \frac{n^2}{n+1}.
\end{cases}
\]

Indeed choose

\[
\varphi(x) = -x_1 \rho(x)
\]

where \( \rho \in D(\Omega) \) and \( \rho(x) \equiv 1 \) if \( |x| < 1/2 \). We therefore find, since \( \det \nabla u_{\nu}(x) = 0 \) if \( |x| > 1/2 \), that

\[
\int_{\Omega} \det \nabla u_{\nu}(x) \varphi(x) dx = -\int_{|x|<1/2} \det \nabla u_{\nu}(x) x_1 dx \\
= -\int_{|x|<1/2} f'_{\nu}(r) \left( \frac{f_{\nu}(r)}{r} \right)^{n-1} \frac{x_1^2}{r} dx \\
= -\frac{\sigma_n}{n} \int_{0}^{1/2} r f'_{\nu}(r) \left( f_{\nu}(r) \right)^{n-1} dr \\
= \frac{2\sigma_n}{n^2(n+1)} \nu^2 \nu^{-(n+1)},
\]

where \( \sigma_n \) is the area of the unit sphere in \( \mathbb{R}^n \). The result follows at once.

\[\diamondsuit\]

### 8.4 Existence theorems

We now collect the results of Sections 8.2 and 8.3 to get existence theorems in the classical way. There will be two results:

1. one involving quasiconvex functions that are finite everywhere;
2. one using polyconvex functions that are allowed to take the value \(+\infty\) in a subset of \( \mathbb{R}^{N \times n} \).

#### 8.4.1 Existence theorem for quasiconvex functions

We first combine the lower semicontinuity result obtained in Theorem 8.11 with a coercivity condition to get the first existence theorem.
**Theorem 8.29** Let $p > 1$, $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary. Let

$$ f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, f = f (x, u, \xi), $$

be a Carathéodory function satisfying for almost every $x \in \Omega$, for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$

$$ \xi \to f (x, u, \xi) \text{ is quasiconvex,}$$

$$\alpha_1 |\xi|^p + \beta_1 |u|^q + \gamma_1 (x) \leq f (x, u, \xi) \leq \alpha_2 |\xi|^p + \beta_2 |u|^r + \gamma_2 (x), \quad (8.34)$$

where $\alpha_2 \geq \alpha_1 > 0$, $\beta_1 \in \mathbb{R}$, $\beta_2 \geq 0$, $\gamma_1, \gamma_2 \in L^1 (\Omega)$, $p > q \geq 1$ and $1 \leq r \leq np/(n - p)$ if $p < n$ and $1 \leq r < \infty$ if $p \geq n$. Let

$$ (P) \inf \left\{ I (u) = \int_{\Omega} f (x, u (x), \nabla u (x)) \, dx : u \in u_0 + W_{0}^{1,p} (\Omega; \mathbb{R}^N) \right\}, $$

then $(P)$ admits at least one solution.

**Remark 8.30** The above theorem is due to Acerbi-Fusco [3] and Marcellini [423], improving earlier results by Morrey [453], [455] and Meyers [442].

**Proof.** Observe first that $\inf (P)$ is finite, since, for example, $I (u_0) < +\infty$, by the growth condition (8.34). So let $u_\nu$ be a minimizing sequence, i.e.

$$ I (u_\nu) \to \inf (P). $$

Proceeding exactly as in Theorem 3.30, we obtain that $\|u_\nu\|_{W^{1,p}}$ is uniformly bounded. Since $p > 1$, we then deduce that, up to the extraction of a subsequence still labeled $u_\nu$,

$$ u_\nu \rightharpoonup \tau \text{ in } W^{1,p} (\Omega; \mathbb{R}^N). $$

Using Theorem 8.11, we immediately get that

$$ I (\tau) = \inf (P), $$

which is the claim. ■

**8.4.2 Existence theorem for polyconvex functions**

We now give a theorem that is applicable to functions in a smaller class than the previous one from the point of view of convexity (since $f$ polyconvex $\Rightarrow$ $f$ quasiconvex) but in a larger class from the point of view of growth and coercivity conditions. More precisely, the previous theorem excludes two important cases:

1. functions $f$ allowed to take the value $+\infty$,
2. functions $f$ of the type (if, for example, $N = n = 2$)

$$ f (\xi) = |\xi|^2 + |\det \xi|^2. $$
These two cases are important for applications. For example, the first one is useful when one deals with minimization problems with constraints, as is the case, for example, in elasticity where a natural constraint is $\det \xi > 0$.

Since polyconvex functions are defined through a convex function $F$, the theorem will be stated in terms of the function $F$.

**Theorem 8.31** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary, $p > n \wedge N$,

$$ F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \cup \{+\infty\}, \quad F = F(x, u, X), $$

be a Carathéodory function which is such that for almost every $x \in \Omega$, for every $(u, X) \in \mathbb{R}^N \times \mathbb{R}^{\tau(n,N)}$

$$ X \to F(x, u, X) \text{ is convex}, $$

$$ F(x, u, X) \geq a(x) + b_1 |X_1|^p, \quad (8.35) $$

where $X = (X_1, X_2, \ldots, X_{n\wedge N}) \in \mathbb{R}^{\tau(n,N)}$, $a \in L^1(\Omega)$ and $b_1 > 0$. Let

$$(P) \quad \inf \left\{ I(u) = \int_{\Omega} F(x, u(x), T(\nabla u(x))) \, dx : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\} = m. $$

Assume that

$$ I(u_0) < +\infty. \quad (8.36) $$

Then $(P)$ admits at least one solution.

**Remark 8.32** (i) The above theorem is due to Ball [51, 53] and has been applied to find minima in nonlinear elasticity.

(ii) The hypothesis (8.36) is important to ensure that $m < +\infty$. A way of satisfying (8.36) would be to impose a growth condition of the same type as the coercivity condition (8.35), as was done in Theorem 8.29 and then $u_0$ would trivially satisfy (8.36).

(iii) The coercivity (8.35) ensures, for some appropriate sequence, that

$$ u_\nu \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N) $$

and thus, since $p > n \wedge N$, we have

$$ T(\nabla u_\nu) \rightharpoonup T(\nabla u) \text{ in } L^1(\Omega; \mathbb{R}^{\tau(n,N)}). $$

However, since in the proof we only need this last convergence, any coercivity condition which ensures it is enough. For example (see Step 2 in the proof of the theorem for details), one could consider

$$ F(x, u, X) \geq a(x) + \sum_{s=1}^{n\wedge N} b_s |X_s|^{p_s}, \quad (8.37) $$
where \( X = (X_1, X_2, \cdots, X_{n \wedge N}) \), \( a \in L^1(\Omega) \), \( b_s > 0 \) and \( p_1 \geq 2 \), \( p_s \geq \frac{p_1}{p_1 - 1} \) if \( 2 \leq s < n \wedge N \) and \( p_{n \wedge N} > 1 \).

For example, if \( N = n = 2 \), then (8.37) is read, writing \( X = (\xi, \delta) \in \mathbb{R}^{2 \times 2} \times \mathbb{R} \),

\[
F(x, u, \xi, \delta) \geq \alpha(x) + \beta_1 |\xi|^{p_1} + \beta_2 |\delta|^{p_2}
\]

with \( p_1 \geq 2 \) and \( p_2 > 1 \). Therefore \( f(\xi) = |\xi|^2 + (\det \xi)^2 \) satisfies (8.37).

\( \Box \)

**Proof.** (Theorem 8.31). We divide the proof into three steps.

**Step 1.** Let \( u_\nu \) be a minimizing sequence for \((P)\). Then by (8.35) and (8.36), we have

\[
\int_\Omega a(x) \, dx + b_1 \int_\Omega |\nabla u_\nu(x)|^p \, dx \leq m + 1 < +\infty.
\]

Using Poincaré inequality we find that there exists a constant \( \gamma \) so that

\[
\|u_\nu\|_{W^{1,p}} \leq \gamma
\]

and therefore, up to the extraction of a subsequence still denoted \( u_\nu \), we have

\[
u \rightarrow u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N).
\]

Since \( p > n \wedge N \), we deduce from Theorem 8.20 that

\[
\begin{aligned}
  u_\nu & \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^N) \\
  T(\nabla u_\nu) & \rightarrow T(\nabla u) \text{ in } L^1(\Omega; \mathbb{R}^{\tau(n,N)}).
\end{aligned}
\]

(8.38)

**Step 2.** We now show that (8.38) holds for \( p = p_1 \) when we replace the coercivity condition (8.35) by (8.37). By the same argument as in Step 1, we find that there exists \( u \in W^{1,p_1}(\Omega; \mathbb{R}^N) \) and \( \xi_s \in L^{p_s}(\Omega; \mathbb{R}^{\sigma(s)}) \) so that, up to the extraction of a subsequence,

\[
u \rightarrow u \text{ in } W^{1,p_1}(\Omega; \mathbb{R}^N)
\]

(8.39)

\[
\text{adj}_s \nabla u_\nu \rightarrow \xi_s \text{ in } L^{p_s}(\Omega; \mathbb{R}^{\sigma(s)}), \quad s = 2, \cdots, n \wedge N.
\]

(8.40)

We now show that these two convergences imply \( \xi_s = \text{adj}_s \nabla u \) and thus

\[
\text{adj}_s \nabla u_\nu \rightarrow \text{adj}_s \nabla u \text{ in } L^1(\Omega; \mathbb{R}^{\sigma(s)}).
\]

(8.41)

This implies (8.38). We prove (8.41) by induction on \( s \).

If \( s = 2 \), then Theorem 8.20 combined with (8.39), (8.40) and the fact that \( p_1 \geq 2 \) give immediately (8.41).

Assume that we have proved (8.41) up to \( s - 1 \leq (N \wedge n) - 2 \). We therefore have

\[
\text{adj}_{s-1} \nabla u_\nu \rightarrow \xi_{s-1} = \text{adj}_{s-1} \nabla u \text{ in } L^{p_{s-1}}.
\]
Appendix: some properties of Jacobians

Since $1/p_1 + 1/p_s \leq 1$, we have immediately (8.41) by Part 5 of Theorem 8.20, combined with (8.40), and Step 2 is therefore complete.

Step 3. In view of Step 1 or Step 2, we are now in a position of applying Theorem 8.16 to $F$. We therefore obtain that

$$\liminf_{\nu \to \infty} I(u_{\nu}) = \liminf_{\nu \to \infty} \int_{\Omega} F(x, u_{\nu}(x), T(\nabla u_{\nu}(x))) \, dx \geq \int_{\Omega} F(x, u(x), T(\nabla u(x))) \, dx = I(u)$$

and thus $u$ is a minimizer for $(P)$. ■

8.5 Appendix: some properties of Jacobians

We now let $u : \mathbb{R}^n \to \mathbb{R}^N$ (hence $\nabla u \in \mathbb{R}^{N \times n}$) be a $C^2$ function and study the analytic properties of $\text{adj}_s \nabla u$. We first introduce the notation for $2 \leq s \leq n \wedge N$, $1 \leq i_1 < \cdots < i_s \leq n$ and $1 \leq \alpha_1 < \cdots < \alpha_s \leq N$.

$$\partial \left( \frac{u^{i_1}, \cdots, u^{i_s}}{x_{\alpha_1}, \cdots, x_{\alpha_s}} \right) := \det \begin{pmatrix} \frac{\partial u^{i_1}}{\partial x_{\alpha_1}} & \cdots & \frac{\partial u^{i_1}}{\partial x_{\alpha_s}} \\ \vdots & \ddots & \vdots \\ \frac{\partial u^{i_s}}{\partial x_{\alpha_1}} & \cdots & \frac{\partial u^{i_s}}{\partial x_{\alpha_s}} \end{pmatrix}.$$  

For $1 \leq t \leq s$, we also let

$$(x_1, \cdots, \hat{x}_t, \cdots, x_s) = (x_1, \cdots, x_{t-1}, x_{t+1}, \cdots, x_s).$$

The main result is that Jacobians may be written in divergence form.

Theorem 8.33 Let $u \in C^2(\mathbb{R}^n; \mathbb{R}^N)$. Let $2 \leq s \leq n \wedge N$, $1 \leq i_1 < \cdots < i_s \leq n$ and $1 \leq \alpha_1 < \cdots < \alpha_s \leq N$. Then

$$\sum_{t=1}^{s} (-1)^{t+1} \frac{\partial}{\partial x_{\alpha_2}} \left[ \frac{\partial (u^{i_1}, \cdots, u^{i_s})}{\partial (x_{\alpha_1}, \cdots, \hat{x}_t, \cdots, x_{\alpha_s})} \right] = 0 \quad (8.42)$$

and

$$\frac{\partial (u^{i_1}, \cdots, u^{i_s})}{\partial (x_{\alpha_1}, \cdots, x_{\alpha_s})} = \sum_{t=1}^{s} (-1)^{t+1} \frac{\partial}{\partial x_{\alpha_t}} \left[ u^{i_1} \frac{\partial (u^{i_2}, \cdots, u^{i_s})}{\partial (x_{\alpha_1}, \cdots, \hat{x}_t, \cdots, x_{\alpha_s})} \right]. \quad (8.43)$$

Remark 8.34 (i) Let $N = n = 2$, $u(x_1, x_2) = (u^1, u^2)$. Then (8.42) just expresses the fact that

$$\text{curl} \left( \nabla u^1 \right) = \text{curl} \left( \nabla u^2 \right) = 0.$$
Equation (8.43) is then
\[
\det \nabla u = \frac{\partial}{\partial x_1} (u^1 \frac{\partial u^2}{\partial x_2}) - \frac{\partial}{\partial x_2} (u^1 \frac{\partial u^2}{\partial x_1}) = \frac{\partial}{\partial x_2} (u^2 \frac{\partial u^1}{\partial x_1}) - \frac{\partial}{\partial x_1} (u^2 \frac{\partial u^1}{\partial x_2}).
\]

(ii) It is obvious that in (8.42) and (8.43) one can interchange the role of \(u^{i_1}\) with any \(u^i\).

**Proof.** Note that (8.43) is a direct consequence of (8.42). To show (8.42) we proceed by induction. To simplify notation, we take \((i_1, \cdots, i_s) = (1, \cdots, s)\) and \((\alpha_1, \cdots, \alpha_s) = (1, \cdots, s)\). Assume that (8.42) and (8.43) have been established up to the order \((s - 1)\), the case \(s = 1\) being trivial. Using the hypothesis of induction we have
\[
\frac{\partial}{\partial (x_1, \cdots, \hat{x}_t, \cdots, x_s)} (u^2, \cdots, u^s) = \sum_{\alpha=1}^{s-1} (-1)^{\alpha+1} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial (x_1, \cdots, \hat{x}_t, \cdots, \hat{x}_\alpha, \cdots, x_s)} (u^2, \cdots, u^s)
\]
\[
+ \sum_{\alpha=t+1}^{s} (-1)^\alpha \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial (x_1, \cdots, \hat{x}_t, \cdots, \hat{x}_\alpha, \cdots, x_s)} (u^2, \cdots, u^s),
\]
where we have denoted, as above,
\[
(x_1, \cdots, \hat{x}_t, \cdots, x_s) = (x_1, \cdots, x_{t-1}, x_{t+1}, \cdots, x_s)
\]
and similarly for \((x_1, \cdots, \hat{x}_t, \cdots, \hat{x}_\alpha, \cdots, x_s)\). Returning to (8.42) and using the above identity we have
\[
\sum_{t=1}^{s} (-1)^{t+1} \frac{\partial}{\partial x_t} \left[ \frac{\partial}{\partial (x_1, \cdots, \hat{x}_t, \cdots, x_s)} (u^2, \cdots, u^s) \right] = \sum_{t=1}^{s} \left[ \sum_{\alpha=1}^{t-1} (-1)^{\alpha+t} \frac{\partial^2}{\partial x_t \partial x_\alpha} \frac{\partial}{\partial (x_1, \cdots, \hat{x}_t, \cdots, \hat{x}_\alpha, \cdots, x_s)} (u^2, \cdots, u^s) \right]
\]
\[
+ \sum_{\alpha=t+1}^{s} (-1)^{\alpha+t+1} \frac{\partial^2}{\partial x_t \partial x_\alpha} \frac{\partial}{\partial (x_1, \cdots, \hat{x}_t, \cdots, \hat{x}_\alpha, \cdots, x_s)} (u^2, \cdots, u^s) \right].
\]
Now observe that for any \(r < \beta\)
\[
\frac{\partial^2}{\partial x_t \partial x_\beta} \frac{\partial}{\partial (x_1, \cdots, \hat{x}_r, \cdots, \hat{x}_\beta, \cdots, x_s)} (u^3, \cdots, u^s) \right] \equiv 0.
\]
We therefore have, by combining the last two identities, established (8.42). [□]

Theorem 8.33 allows us to prove now Lemma 5.5 and at the same time to generalize it.
Theorem 8.35 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let

$$T(\xi) = (\xi, \text{adj}_2 \xi, \cdots, \text{adj}_{n\wedge N} \xi).$$

Then

$$\int_{\Omega} T(\xi + \nabla \varphi(x)) \, dx = T(\xi) \cdot \text{meas} \Omega$$  \hspace{1cm} (8.44)

for every $\xi \in \mathbb{R}^{N\times n}$ and for every $\varphi \in W^{1,\infty}_0(\Omega; \mathbb{R}^N)$.

(ii) Let $u \in v + W^{1,p}_0(\Omega; \mathbb{R}^N)$, with $p \geq n \wedge N$. Then

$$\int_{\Omega} T(\nabla u(x)) \, dx = \int_{\Omega} T(\nabla v(x)) \, dx.$$

Observe that when $N = n = 2$, the theorem reads as

$$T(\xi) = (\xi, \det \xi)$$

and hence (8.44) becomes

$$T(\xi) \cdot \text{meas} \Omega = (\xi, \det \xi) \cdot \text{meas} \Omega$$

$$= \int_{\Omega} (\xi + \nabla \varphi(x), \det (\xi + \nabla \varphi(x))) \, dx.$$

We now proceed with the proof of Theorem 8.35.

**Proof.** It is clear that (ii) is more general than (i); however, for the sake of exposition, we still establish (i).

(i) Step 1. We first prove the result when $\varphi \in C_0^\infty(\Omega; \mathbb{R}^N)$. In order to establish (8.44), we only need to show that

$$\int_{\Omega} \text{adj}_s (\xi + \nabla \varphi(x)) \, dx = \text{adj}_s \xi \cdot \text{meas} \Omega$$  \hspace{1cm} (8.45)

for every $1 \leq s \leq n \wedge N$, for every $\xi \in \mathbb{R}^{N\times n}$ and for every $\varphi \in C_0^\infty(\Omega; \mathbb{R}^N)$.

Recall that for $1 \leq i \leq (N_s), \ 1 \leq \alpha \leq (n_s)$, we have

$$(\text{adj}_s \xi)^i_\alpha = (-1)^{i+\alpha} \det \left( \begin{array}{cccc} \xi^i_{\alpha_1} & \cdots & \xi^i_{\alpha_s} \\ \vdots & \ddots & \vdots \\ \xi^i_{s} & \cdots & \xi^i_{s} \\ \end{array} \right).$$

By abuse of notation, let

$$\xi = \left( \begin{array}{cccc} \xi^i_{\alpha_1} & \cdots & \xi^i_{\alpha_s} \\ \vdots & \ddots & \vdots \\ \xi^i_{s} & \cdots & \xi^i_{s} \\ \end{array} \right), \quad \nabla \varphi = \left( \begin{array}{cccc} \frac{\partial \varphi^i_{\alpha_1}}{\partial x_{\alpha_1}} & \cdots & \frac{\partial \varphi^i_{\alpha_s}}{\partial x_{\alpha_s}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi^i_{s}}{\partial x_{\alpha_1}} & \cdots & \frac{\partial \varphi^i_{s}}{\partial x_{\alpha_s}} \\ \end{array} \right).$$
Therefore (8.44) or (8.45) is equivalent to showing that
\[ \int_{\Omega} \det (\xi + \nabla \varphi (x)) \, dx = \det \xi \cdot \text{meas} \Omega \] (8.46)
for every bounded domain \( \Omega \subset \mathbb{R}^s \), for every \( \xi \in \mathbb{R}^{s \times s} \) and for every \( \varphi \in C^\infty_0 (\Omega; \mathbb{R}^s) \). To show (8.46), we proceed by induction on \( s \). The result is trivial if \( s = 1 \). Assume therefore that (8.46) has been established up to the order \( s - 1 \).

Using Proposition 5.65, we have that
\[ \det (\xi + \nabla \varphi) = \langle (\xi + \nabla \varphi)^1 \rangle_{\alpha} \]
\[ = \langle (\xi^1; (\text{adj}_{s-1} (\xi + \nabla \varphi))^1 \rangle_{\alpha} \]
\[ = \sum_{\alpha=1}^s [\xi^1_{\alpha} \langle (\text{adj}_{s-1} (\xi + \nabla \varphi))^1 \rangle_{\alpha} + \partial \varphi_{\alpha} \partial x^\alpha \langle (\text{adj}_{s-1} (\xi + \nabla \varphi))^1 \rangle_{\alpha}] . \]

Integrating the above identity, using the hypothesis of induction on the first part, an integration by part in the second term and (8.42), we have indeed obtained (8.46) and thus the theorem.

\textbf{Step 2.} From Step 1, we know that (8.44) holds for \( \varphi \in C^\infty_0 (\Omega; \mathbb{R}^N) \). By an elementary density argument we have the claim.

(ii) As in Step 1 of (i), it is clearly enough to prove the result when \( N = n \) and for
\[ \int_{\Omega} \det \nabla u (x) \, dx = \int_{\Omega} \det \nabla v (x) \, dx \] (8.47)
for every \( u \in v + W^{1,n}_0 (\Omega; \mathbb{R}^n) \). By density, it will be sufficient to prove the identity for \( u \) and \( v \) of the form
\[ u = v + w \]
with \( v \in C^\infty (\Omega; \mathbb{R}^n) \cap W^{1,n} (\Omega; \mathbb{R}^n) \) and \( w \in C^\infty_0 (\Omega; \mathbb{R}^n) \).

1) As usual we start with the case \( n = 2 \) to illustrate the purpose. We note that
\[ \det \nabla u = \det \nabla v + \det (\nabla v^1, \nabla w^2) + \det (\nabla w^1, \nabla v^2) + \det \nabla w \]
\[ = \det \nabla v + \frac{\partial}{\partial x^2} (w^2 \partial v^1 \partial x_1) - \frac{\partial}{\partial x_1} (w^2 \partial v^1 \partial x_2) \]
\[ + \frac{\partial}{\partial x_1} (w^1 \partial v^2 \partial x_2) - \frac{\partial}{\partial x_2} (w^1 \partial v^2 \partial x_1) + \det \nabla w . \]

Integrating both sides, we have, since \( w \in C^\infty_0 (\Omega; \mathbb{R}^2) \), the identity (8.47).

2) We now proceed with the general case. We appeal to Proposition 5.67 to write
\[ \det \nabla u = \det \nabla v + \sum_{(I,J) \in \mathcal{N}_{1,\ldots,n} \setminus \emptyset} \det (\nabla v^I, \nabla w^J) . \]
Clearly, if we can show that, for every \((I, J) \in N_{\{1, \ldots, n\}}\) with \(J \neq \emptyset\), we have
\[
\int_{\Omega} \det \left( \nabla v^I(x), \nabla w^J(x) \right) \, dx = 0,
\]
then the result (8.47) will follow.

Since \(J \neq \emptyset\), we can choose \(j \in J\) and use (8.43) in Theorem 8.33 to obtain
\[
\det \left( \nabla v^I, \nabla w^J \right) = \sum_{t=1}^{n} (-1)^{t+1} \frac{\partial}{\partial x_t} \left[ w^j \frac{\partial \left( v^I, w^{J-\{j\}} \right)}{\partial \left( x_1, \ldots, \hat{x}_t, \ldots, x_n \right)} \right].
\]
Integrating this last identity, bearing in mind that \(w \in C_0^\infty(\Omega; \mathbb{R}^n)\), we have indeed obtained (8.48). \(\blacksquare\)
Chapter 9

Relaxation theorems

9.1 Introduction

In Chapters 3 and 8 we have seen that in order to get existence theorems for

\[ (P) \quad \inf \left\{ I(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\}, \]

the convexity (or quasiconvexity in the vectorial case) of \( f \), with respect to the
last variable, plays a central role. In this chapter, we study the case where \( f \)
fails to be convex (quasiconvex in the vectorial case).

It is then natural to replace the problem \((P)\) by the so called relaxed problem

\[ (QP) \quad \inf \left\{ \mathcal{T}(u) = \int_{\Omega} Qf(x, u(x), \nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\}, \]

where \( Qf \) is the quasiconvex envelope of \( f \) (with respect to the last variable
\( \nabla u \)). We show that, even though the original \( f \) is not quasiconvex (not convex
in the scalar case) and therefore in general the infimum of \((P)\) is not attained,
one has

\[ \inf (P) = \inf (QP), \]

and with some extra coercivity condition, the infimum of \((QP)\) is attained. More
precisely, if \( \bar{u} \) is a solution of \((QP)\), then there exists a minimizing sequence \( \{u_\nu\} \)
of \((P)\) such that

\[
\begin{align*}
&u_\nu \rightharpoonup \bar{u} \text{ in } W^{1,p}_0(\Omega; \mathbb{R}^N) \\
&I(u_\nu) \rightarrow \mathcal{T}(\bar{u}) = \inf (P) = \inf (QP).
\end{align*}
\]

In other words, even if \((P)\) has no solution in \( W^{1,p}(\Omega; \mathbb{R}^N) \), one can consider
the solutions of \((QP)\) as generalized solutions of \((P)\), in the sense of weak
convergence.

In the case \( N = n = 1 \), this result was proved by L.C. Young [606], [608]
and then generalized by others to the scalar case, \( N = 1 \) or \( n = 1 \), notably
by Berliocchi-Lasry [81], Ekeland [262], [264], Ioffe-Tihomirov [351], MacShane [412], [413] and Marcellini-Sbordone [427], [428]. Note that in this context

\[ Qf = Cf = f^{**}, \]

where \( Cf \) is the usual convex envelope of \( f \) (with respect to the last variable).

The result for the vectorial case (i.e. \( N, n > 1 \); recall also that, in general, we now have \( Qf > Cf \)) was established by Dacorogna in [172], when there is no lower order terms. Following a different approach, it was later also proved by Acerbi-Fusco [3].

9.2 Relaxation Theorems

9.2.1 The case without lower order terms

We now turn our attention to the relaxation theorem when the integrand depends only on the higher order terms. We recall our minimization problem

\[
(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(\nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\},
\]

where \( 1 \leq p \leq \infty \).

We define the relaxed problem associated to \( (P) \) to be

\[
(QP) \quad \inf \left\{ \mathcal{T}(u) = \int_{\Omega} Qf(\nabla u(x)) \, dx : u \in u_0 + W^{1,p}_0(\Omega; \mathbb{R}^N) \right\}.
\]

**Theorem 9.1 (Relaxation theorem)** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set. Let \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) be Borel measurable satisfying, for \( 1 \leq p < \infty \),

\[
g(\xi) \leq f(\xi) \quad \text{and} \quad |g(\xi)|, |f(\xi)| \leq \alpha_1 (1 + |\xi|^p) \quad \text{for every} \quad \xi \in \mathbb{R}^{N \times n}, \quad (9.1)
\]

where \( g : \mathbb{R}^{N \times n} \to \mathbb{R} \) is quasiconvex and \( \alpha_1 > 0 \) is a constant, while for \( p = \infty \) it is assumed that \( f \) is locally bounded and bounded below by \( g \).

Let

\[
Qf(\xi) := \sup \left\{ g(\xi) : g \leq f \quad \text{and} \quad g \text{ quasiconvex} \right\}
\]

be the quasiconvex envelope of \( f \).

Part 1. Then

\[
\inf (P) = \inf (QP).
\]

More precisely, for every \( p \leq q \leq \infty \) and \( u \in W^{1,q}(\Omega; \mathbb{R}^N) \), there exists a sequence \( \{u_\nu\}_{\nu=1}^{\infty} \subset u + W^{1,q}_0(\Omega; \mathbb{R}^N) \) such that

\[
u \to u \quad \text{in} \quad L^q(\Omega; \mathbb{R}^N) \quad \text{as} \quad \nu \to \infty,
\]
\[
\int_{\Omega} f(\nabla u_\nu (x)) \, dx \to \int_{\Omega} Qf(\nabla u (x)) \, dx \quad \text{as } \nu \to \infty.
\]

Part 2. Let \( \alpha_2 > 0 \), \( I \geq 1 \) be an integer, \( \infty > p \geq p_i > 1 \), \( i = 1, \cdots, I \) and \( \Phi_i : \mathbb{R}^{N \times n} \to \mathbb{R} \), \( i = 1, \cdots, I \), be quasiaffine functions satisfying

\[
\max_{i=1, \cdots, I} \{|\Phi_i(\xi)|^{p_i}\} \leq \alpha_2 (1 + |\xi|^p) \quad \text{for every } \xi \in \mathbb{R}^{N \times n}.
\]

Assume, in addition to the above hypothesis and those of Part 1, that there exist \( a, c \in \mathbb{R}, d_i \geq b_i > 0 \) such that

\[
(C) \quad a + \sum_{i=1}^{I} b_i |\Phi_i(\xi)|^{p_i} \leq f(\xi) \leq c + \sum_{i=1}^{I} d_i |\Phi_i(\xi)|^{p_i}
\]

for every \( \xi \in \mathbb{R}^{N \times n} \). Then, in addition to the conclusions of Part 1, the following holds

\[
\Phi_i(\nabla u_\nu) \rightharpoonup \Phi_i(\nabla u) \quad \text{in } L^{p_i}(\Omega), \quad i = 1, \cdots, I, \quad \text{as } \nu \to \infty.
\]

Before making some remarks, we give two significant examples of functions satisfying \((C)\).

Example 9.2 (i) The case where \( f \) satisfies a condition of the type

\[
a + b|\xi|^p \leq f(\xi) \leq c + d|\xi|^p
\]

is a particular case of \((C)\). It suffices to choose \( I = N \times n \), \( p_i = p > 1 \), \( d_i = d \geq b_i = b > 0 \), \( i = 1, \cdots, I \) and for \( \xi = (\xi_i)_{1 \leq i \leq N} \),

\[
\left\{
\begin{array}{l}
\Phi_1(\xi) = \xi_1^1, \cdots, \Phi_n(\xi) = \xi_n^1 \\
\Phi_{n+1}(\xi) = \xi_1^2, \cdots, \Phi_{2n}(\xi) = \xi_n^2 \\
\quad \vdots \\
\Phi_{(N-1)n+1}(\xi) = \xi_1^N, \cdots, \Phi_{Nn}(\xi) = \xi_n^N 
\end{array}
\right.
\]

which are all quasiaffine. In this case, the theorem implies that

\[
u_\nu \rightharpoonup u \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^N) \quad \text{as } \nu \to \infty.
\]

(ii) If \( N = n, \ p > n \) and

\[
a + b|\det \xi|^{p/n} \leq f(\xi) \leq c + d|\det \xi|^{p/n},
\]

then choose in \((C)\) \( I = 1 \), \( p_1 = p/n \) and \( \Phi_1(\xi) = \det \xi \) which is quasiaffine. We therefore also have

\[
\det \nabla u_\nu \rightharpoonup \det \nabla u \quad \text{in } L^{p/n}(\Omega) \quad \text{as } \nu \to \infty,
\]
and if \( f(\xi) = F(\det \xi) \), we then find
\[
\int_{\Omega} F(\det \nabla u_\nu(x)) \, dx \to \int_{\Omega} CF(\det \nabla u(x)) \, dx,
\]
since, by Theorem 6.24, \( Qf = CF = F^{**} \).

We should also note that, in general, the sequence \( \{u_\nu\} \), which converges in \( L^q \), does not converge in any Sobolev space, as the following simple example shows.

**Example 9.3** Consider \( n = N = 1 \), \( f(\xi) = e^{-|\xi|} \) and
\[
(P) \quad \inf \left\{ I(u) = \int_0^1 f(u'(x)) \, dx : u \in W^{1,1}_0(0,1) \right\} = m.
\]
Since \( Cf = Qf \equiv 0 \), we have from the relaxation theorem that
\[
\inf (P) = \inf (QP) = 0.
\]

However, any sequence \( \{u_\nu\}_{\nu=1}^\infty \subset W^{1,1}_0(0,1) \) such that
\[
\int_0^1 f(u'_\nu(x)) \, dx \to \int_0^1 Qf(u'(x)) \, dx = 0 \quad \text{as } \nu \to \infty
\]
also satisfies by Jensen inequality
\[
0 \leq e^{-\int_0^1 |u'_\nu| \, dx} \leq \int_0^1 e^{-|u'_\nu|} \, dx = \int_0^1 f(u'_\nu(x)) \, dx \to 0
\]
and therefore must satisfy
\[
\int_0^1 |u'_\nu| \, dx \to \infty.
\]
Thus it cannot converge weakly to any \( u \) in \( W^{1,1} \), though there exists a sequence satisfying
\[
u \to u \quad \text{in } L^\infty(0,1) \quad \text{as } \nu \to \infty.
\]

We now make some further remarks.

**Remark 9.4** (i) The history of the theorem has already been discussed. This approach of relaxing non-convex (non-quasiconvex in the vectorial case) problems is not the only one. There is a closely related idea due to L.C. Young [606], [608] (and in fact prior to the one presented here); see also MacShane [412], [413] that, instead of replacing \( f \) by \( Qf \), enlarges the space of admissible functions from Sobolev spaces to spaces of parametrized measures (called generalized curve by L.C. Young and nowadays called Young measure). This idea of L.C. Young has been very fruitful in the calculus of variations as well as in optimal control theory and in partial differential equations following the work of Tartar [568]; see also Dacorogna [173] and Pedregal [492].
(ii) Theorem 9.1 implies in particular that if (QP) has a solution \( \overline{u} \), then there exists a minimizing sequence \( \{ u_\nu \} \) for (P) satisfying the conclusions of the theorem. Conversely, since every minimizing sequence \( \{ u_\nu \} \) for (P) is also a minimizing sequence of (QP), then, up to the extraction of a subsequence, \( u_\nu \) converges weakly to a solution \( \overline{u} \) of (QP) (provided it exists). One should understand, in this sense, that solutions of (QP) are generalized solutions of (P).

(iii) Note also that in the scalar case (QP) and (P**) are the same problems where

\[
(P**) \quad \inf \left\{ I** (u) = \int_\Omega f** (\nabla u (x)) \, dx : u \in u_0 + W^{1,p}_0 (\Omega; \mathbb{R}^N) \right\}.
\]

However, this is not true in the vectorial case, one has in general

\[
\inf (P) = \inf (QP) > \inf (P**).
\]

For example, if \( N = n \geq 2 \) and \( f (\xi) = (\det \xi)^2 \), then

\[
f (\xi) = Q f (\xi) = (\det \xi)^2 > C f (\xi) = f** (\xi) \equiv 0
\]

(see Theorem 6.24) and therefore, using Jensen inequality and Theorem 8.35, we have, if \( \det \nabla u_0 > 0 \),

\[
\inf (P) = \inf (QP) = \frac{1}{\text{meas} \Omega} \int_\Omega \det \nabla u_0 (x) \, dx \geq 0 = (\inf P**).
\]

One can prove, for some \( u_0 \) and \( \Omega \), (see Corollary 14.9) that

\[
\inf (P) = \inf (QP) = \left( \frac{1}{\text{meas} \Omega} \int_\Omega \det \nabla u_0 (x) \, dx \right)^2 \text{meas} \Omega
\]

and the infimum of (QP) is attained.

(iv) The above Part 2 of the theorem does not apply to area type problems (see Section 5.3.6), since the growth condition (C) holds in this case with \( p_i = 1 \). However, the minimal surface problem in parametric form can be handled in a similar way; see Dacorogna [171].

(v) With the notation

\[
I (u) = \int_\Omega f (\nabla u (x)) \, dx, \quad T (u) = \int_\Omega Q f (\nabla u (x)) \, dx,
\]

we find that the theorem, under the hypotheses of Example 9.2 (i), implies that

\[
T (u) = \inf \left\{ \liminf [ I (u_\nu) : u_\nu - u \rightharpoonup 0 \text{ in } W^{1,p}_0 (\Omega; \mathbb{R}^N) ] \right\}.
\]

(vi) Recently Anza Hafsa-Mandallena [32], [33], [34], [35] have extended the above theorem so as to take into account some constraints of nonlinear elasticity.

\( \diamond \)
We finally proceed with the proof of the theorem.

**Proof.** We divide the proof into three steps.

**Step 1.** We first show Part 1 when $\Omega = D = (0, 1)^n$ and $u$ is affine on $D$, meaning that there exists $\xi \in \mathbb{R}^{N \times n}$ such that

$$\nabla u(x) = \xi \text{ for every } x \in D.$$ 

Now use Theorem 6.9 to find $\varphi_\nu \in W^{1, \infty}_0(D; \mathbb{R}^N)$ with the property that

$$\int_D f(\xi + \nabla \varphi_\nu(x)) \, dx \geq Q_f(\xi) \geq -\frac{1}{\nu} + \int_D f(\xi + \nabla \varphi_\nu(x)) \, dx. \quad (9.2)$$

Let $s(\nu)$ be an integer such that $s(\nu) \to \infty$ as $\nu \to \infty$ and satisfying

$$\nu \|\varphi_\nu\|_{L^\infty} \leq s(\nu).$$

Extend $\varphi_\nu$ by periodicity, in each variable, from $D$ to $\mathbb{R}^n$ and let

$$\psi_\nu(x) := \frac{1}{s(\nu)} \varphi_\nu(s(\nu)x).$$

Note that $\psi_\nu \in W^{1, \infty}_0(D; \mathbb{R}^N)$. Finally, define

$$u_\nu := u + \psi_\nu.$$ 

Note that, using the periodicity of $\varphi_\nu$, we have

$$\int_D f(\xi + \nabla \psi_\nu(x)) \, dx = \frac{1}{s^N} \int_{sD} f(\xi + \nabla \varphi_\nu(y)) \, dy = \int_D f(\xi + \nabla \varphi_\nu(y)) \, dy. \quad (9.3)$$

Combining the above identity with (9.2), we have indeed shown that $u_\nu$ has all the properties stated in Part 1, more precisely, $u_\nu \in u + W^{1, \infty}_0(\Omega; \mathbb{R}^N)$ and

$$u_\nu \to u \text{ in } L^\infty(\Omega; \mathbb{R}^N) \text{ as } \nu \to \infty,$$

and

$$0 \leq \int_{\Omega} [f(\nabla u_\nu(x)) - Q_f(\nabla u_\nu(x))] w(x) \, dx \to 0 \text{ as } \nu \to \infty.$$

**Step 2.** We now show Part 2, still under the same restrictions as in Step 1. Since the following reasoning applies to each $i = 1, \cdots, I$, we may assume, for notational convenience, that $I = 1$ and we therefore write $\Phi_i = \Phi$ and $p_i = p$. We thus have to show that

$$\Phi(\xi + \nabla \psi_\nu) \to \Phi(\xi) \text{ in } L^p(D) \text{ as } \nu \to \infty;$$

in other words, we have to prove that for every $w \in L^{p'}(D)$

$$\lim_{\nu \to \infty} \int_D [\Phi(\xi + \nabla \psi_\nu(x)) - \Phi(\xi)]w(x) \, dx = 0. \quad (9.4)$$
Relaxation Theorems

From the hypothesis \((C)\), we have that there exists a constant \(\gamma\), independent of \(\nu\), such that
\[
\int_D |\Phi (\xi + \nabla \psi_\nu (x))|^p \, dx \leq \gamma. \tag{9.5}
\]
Since \(p > 1\) and (9.5) holds, it is sufficient in order to show (9.4) to prove that if
\[
E = x_0 + \delta D \subset D,
\]
where \(x_0 \in D\), \(\delta > 0\), then
\[
\int_E \left[ \Phi (\xi + \nabla \psi_\nu (x)) - \Phi (\xi) \right] \, dx \to 0 \text{ as } \nu \to \infty. \tag{9.6}
\]
We will assume that \(x_0 = 0\), since by periodicity we can always get back to this case. We therefore have, letting \(s = s(\nu)\),
\[
\begin{align*}
\int_E \Phi (\xi + \nabla \psi_\nu (x)) \, dx &= \int_E \Phi (\xi + \nabla \varphi_\nu (sx)) \, dx = \frac{1}{s^n} \int_{sE} \Phi (\xi + \nabla \varphi_\nu (y)) \, dy \\
&= \frac{1}{s^n} \int_{[s\delta]D} \Phi (\xi + \nabla \varphi_\nu (y)) \, dy + \frac{1}{s^n} \int_{sE-\([s\delta]\)D} \Phi (\xi + \nabla \varphi_\nu (y)) \, dy \\
&= \left( [s\delta] / s \right)^n \int_D \Phi (\xi + \nabla \varphi_\nu (y)) \, dy + \frac{1}{s^n} \int_{sE-\([s\delta]\)D} \Phi (\xi + \nabla \varphi_\nu (y)) \, dy
\end{align*}
\]
where \([s\delta]\) denotes the integer part of \(s\delta\) and where we have used in the above identity the periodicity of \(\varphi_\nu\) (in a similar way as in (9.3)). Since \(\varphi_\nu = 0\) on \(\partial D\) and \(\Phi\) is quasiaffine, we have
\[
\int_E \Phi (\xi + \nabla \psi_\nu (x)) \, dx = ([s\delta] / s)^n \Phi (\xi) + \frac{1}{s^n} \int_{sE-\([s\delta]\)D} \Phi (\xi + \nabla \varphi_\nu (y)) \, dy. \tag{9.7}
\]
Let us now estimate the last term in the above identity. Observe first that if \(y \in sE - \([s\delta]\) D\), then there exists \(i \in \{1, \ldots, n\}\) such that
\[
[s\delta] \leq y_i \leq s\delta \quad \text{and} \quad 0 \leq y_i \leq s\delta, \text{ for every } i \in \{1, \ldots, n\}.
\]
We then find, using once more the periodicity of \(\varphi_\nu\),
\[
\begin{align*}
\int_{sE-\([s\delta]\)D} |\Phi (\xi + \nabla \varphi_\nu (y))| \, dy &\leq n \int_{[s\delta]}^{[s\delta]+1} \int_{[s\delta]+1}^{[s\delta]+1} \cdots \int_{[s\delta]+1}^{[s\delta]+1} |\Phi (\xi + \nabla \varphi_\nu (y))| \, dy_1 \cdots dy_n \\
&\leq n \int_{[s\delta]}^{[s\delta]+1} \int_{[s\delta]+1}^{[s\delta]+1} \cdots \int_{[s\delta]+1}^{[s\delta]+1} |\Phi (\xi + \nabla \varphi_\nu (y))| \\
&\leq n([s\delta] + 1)^{n-1} \int_D |\Phi (\xi + \nabla \varphi_\nu (y))| \, dy.
\end{align*}
\]
Hence, in view of (9.5), we have
\[ \left| \frac{1}{s^n} \int_{sE-\{s\delta\}D} \Phi(\xi + \nabla \varphi \nu (y)) \, dy \right| \leq \frac{\gamma_1}{s} \]
where \( \gamma_1 \) denotes a constant. Therefore returning to (9.7), we deduce
\[ \left| \int_E \left[ \Phi(\xi + \nabla \psi \nu (x)) - \Phi(\xi) \right] \, dx \right| \leq (\delta^n - [s\delta]^n / s^n) \left| \Phi(\xi) \right| + \frac{\gamma_1}{s}. \]
Letting \( \nu \to \infty \) and thus \( s \to \infty \), we have indeed obtained (9.6) and hence Step 2.

**Step 3.** We finally remove the assumptions of Steps 1 and 2, namely that \( \Omega = D = (0,1)^n \) and \( u \) is affine on \( D \). First observe that Steps 1 and 2 remain unchanged if the unit cube is replaced by any cube with faces parallel to the axes.

We start with an approximation of the given function \( u \). Let \( \epsilon > 0 \) be arbitrary, we can then find disjoint open cubes with faces parallel to the axes \( \Omega_1, \ldots, \Omega_k \subseteq \Omega, \xi_1, \ldots, \xi_k \in \mathbb{R}^{N \times n} \), \( \gamma \) independent of \( \epsilon \) and \( v \in u + W^{1,q}_{0} (\Omega; \mathbb{R}^N) \) such that
\[
\begin{align*}
\text{meas} \left[ \Omega - \bigcup_{j=1}^k \Omega_j \right] &\leq \epsilon \\
\|u\|_{W^{1,q}}, \|v\|_{W^{1,q}} &\leq \gamma, \|u - v\|_{W^{1,q}} \leq \epsilon \\
\nabla v (x) = \xi_j &\text{ if } x \in \Omega_j
\end{align*}
\]
(if \( q = \infty \), we only have \( \|u - v\|_{W^{1,r}} \leq \epsilon \) for every \( 1 \leq r < \infty \)). We can then find a non-negative increasing function \( \eta \) satisfying \( \eta (t) \to 0 \) as \( t \to 0 \) and such that, using the continuity of \( Qf \) and the growth condition on \( f \),
\[
\int_{\Omega} \left| Qf(\nabla u (x)) - Qf(\nabla v (x)) \right| \, dx \leq \eta (\epsilon), \quad (9.9)
\]
\[
0 \leq \int_{\Omega - \bigcup_{j=1}^k \Omega_j} \left[ f(\nabla v (x)) - Qf(\nabla v (x)) \right] \, dx \leq \eta (\epsilon). \quad (9.10)
\]
Indeed, let us discuss the case \( 1 \leq p < \infty \), the case \( p = \infty \) being easier. Recall that (see Theorem 5.3) any quasiconvex function is locally Lipschitz continuous. Since it also satisfies (9.1), we can find \( \beta > 0 \) (see Proposition 2.32) such that
\[
\left| Qf(\nabla u) - Qf(\nabla v) \right| \leq \beta \left( 1 + |\nabla u|^{p-1} + |\nabla v|^{p-1} \right) |\nabla u - \nabla v|.
\]
Using Hölder inequality, we obtain
\[
\int_{\Omega} \left| Qf(\nabla u) - Qf(\nabla v) \right| \, dx 
\leq \beta \left[ \int_{\Omega} \left[ (1 + |\nabla u|^{p-1} + |\nabla v|^{p-1}) \right]^{p-1} \, dx \right]^{\frac{1}{p}} \left[ \int_{\Omega} |\nabla u - \nabla v|^p \right]^{\frac{1}{p}}
\]
and (9.9) therefore follows from (9.8). The inequality (9.10) follows from (9.8), since \( f(\nabla v), Qf(\nabla v) \in L^1 \).
Make the construction of Step 1 on every \( \Omega_j \) and find
\[
u_j, \nu \in v + W_0^{1, \infty} (\Omega_j; \mathbb{R}^N).
\]
Then define
\[
u(x) = \begin{cases} u_{j, \nu}(x) & \text{if } x \in \Omega_j, j = 1, \cdots, k \\ v(x) & \text{if } x \in \Omega - \bigcup_{j=1}^k \Omega_j. \end{cases}
\]
We get that \( \nu \in u + W_0^{1,q}(\Omega; \mathbb{R}^N) \) and
\[
u \to u \text{ in } L^q(\Omega; \mathbb{R}^N), \text{ as } \nu \to \infty.
\]
Let \( \epsilon > 0 \) be fixed, then, for \( \nu \) sufficiently large, we have from Step 1 that
\[
0 \leq \int_{\bigcup_{j=1}^k \Omega_j} [f(\nabla u_\nu(x)) - Qf(\nabla v(x))] dx \leq \epsilon \text{ meas } \bigcup_{j=1}^k \Omega_j.
\]
Using (9.10), we get
\[
0 \leq \int_{\Omega - \bigcup_{j=1}^k \Omega_j} [f(\nabla u_\nu(x)) - Qf(\nabla v(x))] dx
\]
\[
= \int_{\Omega - \bigcup_{j=1}^k \Omega_j} [f(\nabla v(x)) - Qf(\nabla v(x))] dx \leq \eta(\epsilon).
\]
In other words, combining these inequalities, we have proved that
\[
0 \leq \int_{\Omega} [f(\nabla u_\nu(x)) - Qf(\nabla v(x))] dx \leq \eta(\epsilon) + \epsilon \text{ meas } \Omega.
\]
Invoking (9.9), we find
\[
| \int_{\Omega} [f(\nabla u_\nu(x)) - Qf(\nabla u(x))] dx | \leq 2\eta(\epsilon) + \epsilon \text{ meas } \Omega.
\]
Letting \( \epsilon \to 0 \) (and thus \( \nu \to \infty \)), we have indeed obtained Part 1 of the theorem.

Let us now show Part 2. Let \( w \in L^{p'}(\Omega) \), then we have
\[
\int_{\Omega} [\Phi_i(\nabla u_\nu) - \Phi_i(\nabla u)] w = \int_{\Omega} [\Phi_i(\nabla u_\nu) - \Phi_i(\nabla v)] w
\]
\[
+ \int_{\Omega} [\Phi_i(\nabla v) - \Phi_i(\nabla u)] w
\]
\[
= \sum_{j=1}^k \int_{\Omega_j} [\Phi_i(\nabla u_{j, \nu}) - \Phi_i(\nabla v)] w
\]
\[
+ \int_{\Omega} [\Phi_i(\nabla v) - \Phi_i(\nabla u)] w.
\]
Passing to the limit, appealing to Step 2, as \( \nu \to \infty \) on every \( \Omega_j \), we get
\[
\lim_{\nu \to \infty} \int_{\Omega} [\Phi_i(\nabla u_\nu) - \Phi_i(\nabla u)] w = \int_{\Omega} [\Phi_i(\nabla v) - \Phi_i(\nabla u)] w.
\]
Next, using (9.8) and the fact that $\epsilon$ is arbitrary, we have indeed obtained Part 2 of the theorem.

### 9.2.2 The general case

We now generalize the result of Theorem 9.1 to integrands that depend not only on $\nabla u$ but also on $x$ and $u$.

We start with some general considerations on quasiconvex envelopes. The following proposition was established, in the scalar case $N = 1$, by Marcellini-Sbordone [428], following earlier work of Ekeland-Temam [264]. The vectorial version is, essentially, in Acerbi-Fusco [3] and Marcellini [423].

**Proposition 9.5** Let $D \subset \mathbb{R}^n$ be a bounded open set and

$$f : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \quad f = f(x, u, \xi),$$

be a Carathéodory function. Assume that there exist a Carathéodory function

$$g : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \quad g = g(x, u, \xi),$$

quasiconvex in the last variable (i.e. $\xi \to g(x, u, \xi)$ is quasiconvex for almost every $x \in \mathbb{R}^n$ and every $u \in \mathbb{R}^N$), $\beta \geq 0$, $\beta \in L^1(\mathbb{R}^n)$ and $\alpha$, a continuous and increasing (in each argument) function, satisfying

$$g(x, u, \xi) \leq f(x, u, \xi), \quad |f(x, u, \xi)| \leq \beta(x) + \alpha(|u|, |\xi|)$$

for almost every $x \in \mathbb{R}^n$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$.

For almost every $x \in \mathbb{R}^n$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$, let

$$Q_f(x, u, \xi) := \inf \left\{ \frac{1}{\text{meas } D} \int_D f(x, u, \xi + \nabla \phi(y)) \, dy : \phi \in W^{1,\infty}_0(D; \mathbb{R}^N) \right\}$$

and, for $r > 0$, set

$$Q_{f_r}(x, u, \xi) := \inf \left\{ \frac{1}{\text{meas } D} \int_D f(x, u, \xi + \nabla \phi(y)) \, dy : \phi \in W^{1,\infty}_0(D; \mathbb{R}^N), \quad \|\xi + \nabla \phi\|_{L^\infty} \leq r \right\}.$$

- **Part 1.** The function $Q_f : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ is measurable in $x$, upper semicontinuous in $u$ and quasiconvex in $\xi$ (and hence continuous). Moreover, the definition of $Q_f$ is independent of the choice of the set $D$.

- **Part 2.** The function $Q_{f_r} : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ is a Carathéodory function on $|\xi| < r$, the definition of $Q_{f_r}$ is independent of the choice of $D$ and, furthermore,

$$Q_f = \lim_{r \to \infty} Q_{f_r}.$$  

- **Part 3.** Moreover, $Q_f$ is a Carathéodory function provided $f$ satisfies any of the following conditions:
(a) \( f(x, u, \xi) = f_1(x, u) f_2(x, \xi) + f_3(x, u) \) with \( f_1(x, u) \geq 0 \);

(b) \( f \) is continuous in \( u \), uniformly with respect to \( \xi \);

(c) there exist \( p > 1, \gamma_1 > 0, \gamma_2 \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( \theta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) a non-negative Carathéodory function, increasing in the last argument, with \( \theta(x, 0) = 0 \) such that

\[
\gamma_2(x) + \gamma_1 |\xi|^p \leq f(x, u, \xi),
\]

\[
|f(x, u, \xi) - f(x, v, \xi)| \leq \theta(x, |u - v|) (1 + |\xi|^p),
\]

for almost every \( x \in \mathbb{R}^n \), every \( u, v \in \mathbb{R}^N \) and every \( \xi \in \mathbb{R}^{N \times n} \).

It should be emphasized that, in general, the function \( Qf \) is not a Carathéodory function, even in the scalar case \( N = 1 \) or \( n = 1 \), where \( Qf \) coincides with the usual convex envelope \( Cf = f^{**} \), as the following example, given by Marcellini-Sbordone [428], shows.

**Example 9.6** Let \( N = n = 1 \) and consider the function

\[
f(u, \xi) = (|\xi| + 1)^{|u|}.
\]

An easy computation (recall that here \( Qf = Cf = f^{**} \) and that the convex envelope is understood as the envelope only with respect to the variable \( \xi \)) gives that

\[
f^{**}(u, \xi) = \begin{cases}f(u, \xi) & \text{if } |u| \geq 1 \\ 1 & \text{if } |u| < 1.\end{cases}
\]

Clearly the function \( u \to f^{**}(u, \xi) \) is not continuous, but only upper semicontinuous.

**Proof.** Part 1. The measurability in \( x \) follows from Part 2 (ii) and (iii). The quasiconvexity in \( \xi \), as well as the independence of the definition on the set \( D \), follows from Theorem 6.9. The only thing that remains to be proved is the upper semicontinuity in \( u \).

Without loss of generality, we may assume that \( \beta \) and \( f \) are defined for every \( (x, u, \xi) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \) and that the inequalities assumed in the theorem hold everywhere. Moreover, choose \( x \in \mathbb{R}^n \) so that the function

\[
(u, \xi) \to f(x, u, \xi)
\]

is continuous.

So let \( u_\nu, u \in \mathbb{R}^N \) be such that

\[
u \to u \text{ as } \nu \to \infty.
\]

(assume without loss of generality that \( |u_\nu| \leq |u| + 1 \)) and let \( \epsilon > 0 \) be arbitrary. By definition, we have (letting \( D \) be the unit cube of \( \mathbb{R}^n \)) that there exists
\( \varphi \in W^{1,\infty}_0 (D; \mathbb{R}^N) \) such that

\[
Qf (x, u, \xi) \geq -\epsilon + \int_D f (x, u, \xi + \nabla \varphi (y)) \, dy
\]

\[
\geq -\epsilon - \int_D \lambda_\nu (y) \, dy + \int_D f (x, u_\nu, \xi + \nabla \varphi (y)) \, dy
\]

\[
\geq -\epsilon - \int_D \lambda_\nu (y) \, dy + Qf (x, u_\nu, \xi)
\]

where

\[
\lambda_\nu (y) : = |f (x, u, \xi + \nabla \varphi (y)) - f (x, u_\nu, \xi + \nabla \varphi (y)) |.
\]

Observe that, by hypothesis, \( \lambda_\nu (y) \to 0 \) a.e. \( y \in D \) and

\[
0 \leq \lambda_\nu (y) \leq 2 \left[ \beta (x) + \alpha (|u| + 1, |\xi| + \| \nabla \varphi \|_{L^\infty}) \right].
\]

We may therefore conclude that

\[
Qf (x, u, \xi) \geq -\epsilon - \lim_{\nu \to \infty} \int_D \lambda_\nu (y) \, dy + \limsup_{\nu \to \infty} Qf (x, u_\nu, \xi)
\]

\[
\geq -\epsilon + \limsup_{\nu \to \infty} Qf (x, u_\nu, \xi).
\]

Since \( \epsilon > 0 \) is arbitrary, we have the claim.

**Part 2.** (i) The fact that the definition of \( Qf_r \) is independent of the choice of the set \( D \) is shown exactly as in Theorem 6.9.

(ii) We next prove that

\[
Qf = \lim_{r \to \infty} Qf_r.
\]

Observe first that trivially

\[
r \geq s > 0 \Rightarrow Qf \leq Qf_r \leq Qf_s.
\]

From the definition of \( Qf \) (choosing \( D \) to be the unit cube of \( \mathbb{R}^n \)) we can find, for every \( \nu \in \mathbb{N} \), \( \varphi_\nu \in W^{1,\infty}_0 (D; \mathbb{R}^N) \) such that

\[
Qf (x, u, \xi) \geq -\frac{1}{\nu} + \int_D f (x, u, \xi + \nabla \varphi_\nu (y)) \, dy.
\]

Denote

\[
r (\nu) : = \| \xi + \nabla \varphi_\nu \|_{L^\infty}.
\]

From the definition of \( Qf_r \), we find

\[
\int_D f (x, u, \xi + \nabla \varphi_\nu (y)) \, dy \geq Qf_r (\nu) (x, u, \xi).
\]
Combining the three inequalities, we get

\[ Qf_r(\nu)(x, u, \xi) \geq Qf(x, u, \xi) \geq -\frac{1}{\nu} + Qf_r(\nu)(x, u, \xi). \]

Letting \( \nu \to \infty \), we have the claim.

(iii) We now establish that, for every \((u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}\), the function

\[ x \to h_r(x) := Qf_r(x, u, \xi) \]

is measurable. Choose \( r > 0 \) sufficiently large so that \(|u|, |\xi| < r\) and let

\[ B_r := \{ (u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n} : |u|, |\xi| \leq r \}. \]

It follows from Scorza-Dragoni theorem (Theorem 3.8) that, for every bounded open set \( \Omega \subset \mathbb{R}^n \) and every \( \epsilon > 0 \), there exists a compact set \( K_\epsilon \subset \Omega \) such that

\[ \text{meas} (\Omega - K_\epsilon) \leq \epsilon \quad \text{and} \quad f|_{K_\epsilon \times B_r} \text{ is continuous.} \]

Therefore \( h_r|_{K_\epsilon} \) is upper semicontinuous. Since \( \epsilon > 0 \) is arbitrary we deduce that \( h_r \) is measurable in \( \Omega \) and thus, since \( \Omega \) is arbitrary, we have the claim.

(iv) Let us now show that \( Qf_r \) is a Carathéodory function on \(|\xi| < r\), where \( r > 0 \) is given. We already discussed the measurability in \( x \), so we now consider the continuity in \((u, \xi)\). Let \( \epsilon > 0 \) and \((u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}\), with \(|\xi| < r\). Let

\[ \theta := \frac{r - |\xi|}{2} > 0. \]

We have to prove that, for almost every \( x \in \mathbb{R}^n \), we can find \( \lambda = \lambda(\epsilon, x, u, \xi, r) > 0 \) such that

\[ |u - v| + |\xi - \eta| \leq \lambda \Rightarrow |Qf_r(x, u, \xi) - Qf_r(x, v, \eta)| \leq \epsilon. \quad (9.11) \]

(1) Since \( f \) is a Carathéodory function, we have, for almost every \( x \in \mathbb{R}^n \), that there exists \( \delta = \delta(\epsilon, x, u, r) \) such that if

\[ |A|, |B| \leq r \quad \text{and} \quad |v|, |w| \leq |u| + 1 \]

then

\[ |v - w| + |A - B| \leq \delta \Rightarrow |f(x, v, A) - f(x, w, B)| \leq \epsilon/2. \]

(2) From the definition of \( Qf_r \) (choosing \( D \) to be the unit cube of \( \mathbb{R}^n \)), we can find, for every \( \nu \in \mathbb{N} \), \( \varphi_\nu \in W^{1,\infty}_0(D; \mathbb{R}^N) \) such that

\[ \|B + \nabla \varphi_\nu\|_{L^\infty} \leq r \]

\[ Qf_r(x, w, B) \geq -\frac{1}{\nu} + \int_D f(x, w, B + \nabla \varphi_\nu(y)) dy. \]
(3) We now let \( t := \delta/2r \), \( \lambda := t\theta \).

Note that \( \lambda < \delta \) (since \( \theta \leq r/2 \)). Choosing \( \delta \) smaller if necessary, we can assume that \( \delta < 2r \), and hence \( t \in (0, 1) \) and \( 0 < \lambda < \theta \).

(4) From now on we assume that 
\[ |A - B| \leq \lambda \quad \text{and} \quad |A|, |B| \leq r - \theta. \]

(5) Defining, for \( t \) as above,
\[ \psi_\nu (y) := (1 - t) \varphi_\nu (y) \]
we have \( \psi_\nu \in W^{1,\infty}_0 (D; \mathbb{R}^N) \) and, using (2) and (4),
\[ \|B + \nabla \psi_\nu\|_{L^\infty} \leq t |B| + (1 - t) \|B + \nabla \varphi_\nu\|_{L^\infty} \leq t (r - \theta) + (1 - t) r = r - t\theta \leq r. \]

Furthermore, using the above inequality and the definition of \( t \) and \( \lambda \), we find, recalling that \( |A - B| \leq \lambda \) and \( |A|, |B| \leq r - \theta \),
\[ \|A + \nabla \psi_\nu\|_{L^\infty} \leq |A - B| + \|B + \nabla \psi_\nu\|_{L^\infty} \leq \lambda + r - t\theta = r. \]

Finally, noting that
\[ \|\nabla \varphi_\nu\|_{L^\infty} \leq |B| + \|B + \nabla \varphi_\nu\|_{L^\infty} \leq 2r \]
we have
\[ \|(B + \nabla \psi_\nu) - (B + \nabla \varphi_\nu)\|_{L^\infty} = t \|\nabla \varphi_\nu\|_{L^\infty} \leq 2rt = \delta. \]

We now combine (1), (2) and (5) to get, for \( |w| \leq |u| + 1 \) and \( |B| \leq r - \theta \),
\[ Q_{f_r} (x, w, B) \geq \frac{-1}{\nu} + \int_D f (x, w, B + \nabla \varphi_\nu (y)) \, dy \]
\[ \geq \frac{-1}{\nu} + \int_D f (x, w, B + \nabla \psi_\nu (y)) \, dy \]
\[ - \int_D |f (x, w, B + \nabla \varphi_\nu (y) - f (x, w, B + \nabla \psi_\nu (y))| \, dy \]
\[ \geq \frac{-1}{\nu} - \frac{\epsilon}{2} + \int_D f (x, w, B + \nabla \psi_\nu (y)) \, dy. \]

Again using (5) we find, from the definition of \( Q_{f_r} \),
\[ Q_{f_r} (x, v, A) \leq \int_D f (x, v, A + \nabla \psi_\nu (y)) \, dy. \]
Therefore, if $|A - B| \leq \lambda$, $|A|, |B| \leq r - \theta$ and $|v|, |w| \leq |u| + 1$, the two inequalities lead to

$$Qf_r (x, v, A) - Qf_r (x, w, B) \leq \frac{1}{\nu} + \epsilon + \frac{\epsilon}{2} + \int_D [f(x, v, A + \nabla \psi_v(y)) - f(x, w, B + \nabla \psi_v(y))] dy.$$ 

We have therefore obtained, from (1) and the above inequality, that if $|A|, |B| \leq r - \theta$, $|v|, |w| \leq |u| + 1$ and

$$|v - w| + |A - B| \leq \lambda$$

then

$$Qf_r (x, v, A) - Qf_r (x, w, B) \leq \frac{1}{\nu} + \epsilon.$$ 

Letting $\nu \to \infty$, we have found that

$$Qf_r (x, v, A) - Qf_r (x, w, B) \leq \epsilon.$$ 

Since the inequality $Qf_r (x, w, B) - Qf_r (x, v, A) \leq \epsilon$ can be obtained in the same way, we have indeed obtained (9.11).

Part 3. (a) The first statement results from the observation that

$$Qf (x, u, \xi) = f_1 (x, u) Qf_2 (x, \xi) + f_3 (x, u)$$

and from the fact that $Qf_2 : \mathbb{R}^n \times \mathbb{R}^{N \times n} \to \mathbb{R}$ is, according to Part 1, a Carathéodory function.

(b) Let $\epsilon > 0$ and $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$. We have to find, for almost every $x \in \mathbb{R}^n$, $\delta = \delta (\epsilon, x, u, \xi) > 0$ such that

$$|u - v| + |\xi - \eta| \leq \delta \Rightarrow |Qf (x, u, \xi) - Qf (x, v, \eta)| \leq \epsilon. \quad (9.12)$$

(1) Since $f$ is continuous in $u$ uniformly with respect to the last variable, we can find, for almost every $x \in \mathbb{R}^n$ and for every $A \in \mathbb{R}^{N \times n}$, $\delta_1 = \delta_1 (\epsilon, x, u) > 0$ such that

$$|u - v| \leq \delta_1 \Rightarrow |f(x, u, A) - f(x, v, A)| \leq \epsilon/4.$$ 

Using the definition of $Qf$ (choosing $D$ to be the unit cube of $\mathbb{R}^n$), we have $\varphi_\epsilon \in W^{1,\infty}_0 (D; \mathbb{R}^N)$ such that

$$Qf (x, v, \eta) \geq -\frac{\epsilon}{4} + \int_D f(x, v, \eta + \nabla \varphi_\epsilon(y)) dy$$

$$\geq -\frac{\epsilon}{4} - \int_D \lambda(y) dy + \int_D f(x, u, \eta + \nabla \varphi_\epsilon(y)) dy,$$

where

$$\lambda(y) := |f(x, u, \eta + \nabla \varphi_\epsilon(y)) - f(x, v, \eta + \nabla \varphi_\epsilon(y))|.$$
We therefore get from the definition of $Qf(x, u, \eta)$ that, for $|u - v| \leq \delta_1$,

$$Qf(x, u, \eta) - Qf(x, v, \eta) \leq \epsilon/2.$$ 

Since the opposite inequality is obtained in a similar manner, we get

$$|u - v| \leq \delta_1 \Rightarrow |Qf(x, u, \eta) - Qf(x, v, \eta)| \leq \epsilon/2.$$

(2) The function $Qf$ being continuous in $\xi$, by Part 1, we can find, for almost every $x \in \mathbb{R}^n$, $\delta_2 = \delta_2(\epsilon, x, u, \xi) > 0$ such that

$$|\xi - \eta| \leq \delta_2 \Rightarrow |Qf(x, u, \eta) - Qf(x, u, \xi)| \leq \epsilon/2.$$

Letting $\delta = \min \{\delta_1, \delta_2\}$ and combining the two inequalities, we have indeed obtained (9.12).

(c) The argument is very similar to the one in (b).

(1) We start by observing that, because of the coercivity condition on $f$, we have, for almost every $x \in \mathbb{R}^n$ and every $(v, \eta) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$ with $|v|, |\eta| \leq R$, that, if $\varphi_\epsilon \in W_{0}^{1,\infty}(D; \mathbb{R}^N)$ is such that

$$Qf(x, v, \eta) \geq -\epsilon + \int_D f(x, v, \eta + \nabla \varphi_\epsilon(y)) \, dy,$$

then there exists a constant $\gamma = \gamma(x, R)$, independent of $\epsilon$, such that

$$\|\eta + \nabla \varphi_\epsilon\|_{L^p} \leq \gamma.$$

(2) By hypothesis we have, for almost every $x \in \mathbb{R}^n$ and for every $A \in \mathbb{R}^{N \times n}$, that

$$|f(x, u, A) - f(x, v, A)| \leq \theta(x, |u - v|) (1 + |A|^p).$$

Choose $\delta_1 = \delta_1(\epsilon, x)$ such that

$$|u - v| \leq \delta_1 \Rightarrow \theta(x, |u - v|) \leq \epsilon.$$

Using the definition of $Qf$ (choosing $D$ to be the unit cube of $\mathbb{R}^n$), we can find $\varphi_\epsilon \in W_{0}^{1,\infty}(D; \mathbb{R}^N)$ such that

$$Qf(x, v, \eta) \geq -\epsilon + \int_D f(x, v, \eta + \nabla \varphi_\epsilon(y)) \, dy \geq -\epsilon - \int_D \lambda(y) \, dy + \int_D f(x, u, \eta + \nabla \varphi_\epsilon(y)) \, dy,$$

where

$$\lambda(y) := |f(x, u, \eta + \nabla \varphi_\epsilon(y)) - f(x, v, \eta + \nabla \varphi_\epsilon(y))|.$$
We therefore get from the definition of $Qf(x, u, \eta)$ that, for $|u - v| \leq \delta_1$,

$$Qf(x, u, \eta) - Qf(x, v, \eta) \leq (2 + \|\eta + \nabla \varphi_\epsilon\|_{L^p}) \epsilon.$$  

Invoking the constant $\gamma$ in (1), we get that

$$Qf(x, u, \eta) - Qf(x, v, \eta) \leq (2 + \gamma) \epsilon.$$  

Since the opposite inequality is obtained in a similar manner, we get

$$|u - v| \leq \delta_1 \Rightarrow |Qf(x, u, \eta) - Qf(x, v, \eta)| \leq (2 + \gamma) \epsilon.$$  

(3) The function $Qf$ being continuous in $\xi$ by Part 1, we can find, for almost every $x \in \mathbb{R}^n$, $\delta_2 = \delta_2(\epsilon, x, u, \xi) > 0$ such that

$$|\xi - \eta| \leq \delta_2 \Rightarrow |Qf(x, u, \eta) - Qf(x, u, \xi)| \leq \epsilon.$$  

Letting $\delta = \min\{\delta_1, \delta_2\}$ and combining the two inequalities, we have indeed obtained

$$|u - v| + |\xi - \eta| \leq \delta \Rightarrow |Qf(x, u, \xi) - Qf(x, v, \eta)| \leq (3 + \gamma) \epsilon.$$  

This concludes the proof of the proposition. ■

We are now in a position to state the main theorem, but prior to that we express the growth condition that should satisfy the function.

**Definition 9.7** Let $1 \leq p \leq \infty$ and

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \ f = f(x, u, \xi),$$

be a Carathéodory function. We say that $f$ satisfies growth condition $(G_p)$ if there exists a Carathéodory function

$$g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \ g = g(x, u, \xi),$$

quasiconvex in the last variable (i.e. $\xi \to g(x, u, \xi)$ is quasiconvex for almost every $x \in \Omega$ and every $u \in \mathbb{R}^N$) and

$$g(x, u, \xi) \leq f(x, u, \xi)$$

for almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$.

Moreover the following inequalities hold for almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$.

(i) When $1 \leq p < \infty$

$$|g(x, u, \xi)|, |f(x, u, \xi)| \leq \alpha(1 + |u|^p + |\xi|^p),$$
where $\alpha \geq 0$ is a constant.

(ii) If $p = \infty$, it verifies

$$(G_\infty) \quad |g(x,u,\xi)|, |f(x,u,\xi)| \leq \beta(x) + \alpha(|u|,|\xi|),$$

where $\alpha, \beta \geq 0$, $\beta \in L^1(\Omega)$ and $\alpha$ is a continuous and increasing (in each argument) function.

**Theorem 9.8** Let $1 \leq p \leq \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded open set and

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}, \quad f = f(x,u,\xi),$$

a Carathéodory function satisfying growth condition $(G_p)$ (see Definition 9.7). For almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$, let

$$Qf(x,u,\xi) := \inf \left\{ \frac{1}{\text{meas}D} \int_D f(x,u,\xi + \nabla \varphi(y)) \, dy : \varphi \in W^{1,\infty}_0(D; \mathbb{R}^N) \right\}$$

($D \subset \mathbb{R}^n$ being a bounded open set), which is the quasiconvex envelope (with respect to the last variable) of $f$. Assume that

$$Qf : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$$

is a Carathéodory function.

**Part 1.** Let $p \leq q \leq \infty$ and $u \in W^{1,q}(\Omega; \mathbb{R}^N)$, then there exists a sequence $\{u_\nu\}_{\nu=1}^\infty \subset u + W^{1,q}_0(\Omega; \mathbb{R}^N)$ such that

$$u_\nu \rightarrow u \quad \text{in} \quad L^q(\Omega; \mathbb{R}^N) \quad \text{as} \quad \nu \rightarrow \infty,$$

$$\int_\Omega f(x,u_\nu(x),\nabla u_\nu(x)) \, dx \rightarrow \int_\Omega Qf(x,u(x),\nabla u(x)) \, dx \quad \text{as} \quad \nu \rightarrow \infty.$$

**Part 2.** Assume, in addition to the hypotheses of Part 1, that $1 \leq p < \infty$ and there exist $\alpha_2 > 0$ and $\alpha_3 \in \mathbb{R}$ such that, for almost every $x \in \Omega$ and for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$,

$$f(x,u,\xi) \geq \alpha_2 |\xi|^p + \alpha_3. \quad (9.13)$$

Then, in addition to the conclusions of Part 1, the following holds:

$$u_\nu \rightharpoonup u \quad \text{in} \quad W^{1,p}(\Omega; \mathbb{R}^N) \quad \text{as} \quad \nu \rightarrow \infty.$$

**Remark 9.9** (i) In the scalar case $N = 1$ or $n = 1$, we recall that $Qf$ is nothing else than $Cf = f^{**}$, which is the convex envelope of $f$ with respect to the last variable.

(ii) Note that both hypotheses of Parts 1 and 2 of the present theorem are stronger than the corresponding ones in Theorem 9.1.
- Indeed, in Part 1 we have required continuity of \( f \) in the variable \( \xi \). As it will be seen in the proof, this is not necessary (upon some modification of the proof), since the main requirement of continuity (with respect to \( \xi \)) is on \( Q_f \), which is automatically continuous.

- In Part 2, we could also impose, instead of (9.13), a coercivity condition of the type (C) of Theorem 9.1, getting the corresponding conclusion.

We now proceed with the proof of the theorem.

**Proof.** We divide the proof into three steps (the first two corresponding to Part 1 and the other to Part 2).

**Step 1.** We start by proving the theorem when \( p = \infty \) and \( u \in C^\infty (\Omega; \mathbb{R}^N) \) (this last restriction will be removed at the end of Step 1).

1. Let \( r > 0 \) so that \( \| u \|_{W^{1,\infty}} < r/2. \)

Let \( Q_f_r \) be as in Proposition 9.5, namely

\[
Q_f_r(x, u, \xi) := \inf \left\{ \frac{1}{\text{meas } D} \int_D f(x, u, \xi + \nabla \varphi(y))dy : \varphi \in W^{1,\infty}_0(D; \mathbb{R}^N), \| \xi + \nabla \varphi \|_{L^\infty} \leq r \right\}.
\]

Observe that \( Q_f_r \) also satisfies \((G_\infty)\).

2. Let \( \eta \) be a non-negative increasing function satisfying \( \eta(t) \to 0 \) as \( t \to 0 \) and such that, for every measurable set \( A \subset \Omega \),

\[
0 \leq \int_A \beta(x) \, dx \leq \eta \left( \text{meas } A \right),
\]

\[
\int_A |g(x, u(x), \nabla u(x))| \, dx, \int_A |Q_f_r(x, u(x), \nabla u(x))| \, dx \leq \eta \left( \text{meas } A \right).
\]

3. Let \( \epsilon > 0 \), we can then find \( M = M(\epsilon) \) such that if

\[
E_\epsilon := \{ x \in \Omega : \beta(x) \leq M \},
\]

then, letting \( k = \alpha(2r, 2r) \),

\[
\text{meas } (\Omega - E_\epsilon) \leq \min \{ \epsilon, \epsilon/2k \}
\]

and, in particular,

\[
M \text{meas } (\Omega - E_\epsilon) \leq \eta(\epsilon).
\]

4. Appealing to Theorem 3.8, we can find a compact set \( K_\epsilon \subset \Omega \) with

\[
\text{meas } (\Omega - K_\epsilon) \leq \min \{ \epsilon, \epsilon/(M + 2k) \}
\]

and such that \( f : K_\epsilon \times S_{2r} \to \mathbb{R} \) and \( Q_f_r : K_\epsilon \times S_{r/2} \to \mathbb{R} \) are continuous, where

\[
S_r := \{(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{N \times n} : |u|, |\xi| \leq r \}.
\]
We can therefore find $\delta = \delta(\epsilon)$ such that, if $x, y \in K_\epsilon$, $(u, \xi), (v, \eta) \in S_r$, then
\[
|x - y| + |u - v| + |\xi - \eta| \leq \delta \Rightarrow |f(x, u, \xi) - f(y, v, \eta)| \leq \epsilon/\text{meas } \Omega
\]
and, for $x, y \in K_\epsilon$, $(u, \xi), (v, \eta) \in S_{r/2}$, we also have
\[
|x - y| + |u - v| + |\xi - \eta| \leq \delta \Rightarrow |Qf_r(x, u, \xi) - Qf_r(y, v, \eta)| \leq \epsilon/\text{meas } \Omega.
\]

(5) We then let $h > 0$ be small and decompose $\Omega$ in a finite union of disjoint open sets $\Omega^s$ so that
\[
\text{meas}(\Omega - \bigcup_{s=1}^S \Omega^s) = 0 \quad \text{and} \quad \text{meas } \Omega^s \leq h.
\]
We then fix $x^s \in \Omega^s \cap K_\epsilon \cap E_\epsilon$, whenever this set is non-empty, and define
\[
(u^s, \xi^s) := \frac{1}{\text{meas } \Omega^s} \int_{\Omega^s} (u(x), \nabla u(x)) \, dx.
\]
Note that
\[
|u^s|, |\xi^s| < r/2
\]
and, by choosing $h$ sufficiently small, we can assume (recall that $u \in C^\infty$) that, for every $x \in \Omega^s$,
\[
|x - x^s| + |u(x) - u^s| + |\nabla u(x) - \xi^s| \leq \delta/2.
\]

(6) This has a direct consequence that
\[
|\int_{\Omega} Qf_r(x, u(x), \nabla u(x)) \, dx - \sum_{s=1}^S Qf_r(x^s, u^s, \xi^s) \text{meas } \Omega^s| \leq 3\epsilon + 3\eta(\epsilon).
\]
Indeed, letting
\[
\lambda^s(x) := |Qf_r(x, u(x), \nabla u(x)) - Qf_r(x^s, u^s, \xi^s)|,
\]
we have, using (4) and (5), that
\[
\sum_{s=1}^S \int_{\Omega^s \cap K_\epsilon \cap E_\epsilon} \lambda^s(x) \, dx \leq \epsilon.
\]
Furthermore, since $Qf_r$ also satisfies $(G_\infty)$ and $x^s \in E_\epsilon$, we have
\[
\sum_{s=1}^S \int_{\Omega^s - (K_\epsilon \cap E_\epsilon)} \lambda^s(x) \, dx
\leq \sum_{s=1}^S [\beta(x^s) + 2\alpha(r/2, r/2)] \text{meas } [\Omega^s - (K_\epsilon \cap E_\epsilon)] + \int_{\Omega - (K_\epsilon \cap E_\epsilon)} \beta(x) \, dx
\leq [M + 2k] [\text{meas } (\Omega - E_\epsilon) + \text{meas } (\Omega - K_\epsilon)] + 2\eta(\epsilon) \leq 2\epsilon + 3\eta(\epsilon).
Combining the two estimates, we have, as wished,

$$\sum_{s=1}^{S} \int_{\Omega^s} \lambda^s(x) \, dx \leq 3\epsilon + 3\eta(\epsilon).$$

(7) Using the definition of $Qf_r$ and the same argument as in Theorem 9.1, we can find for every $s$ a function $\varphi^s_\nu \in W^{1,\infty}_0(\Omega^s; \mathbb{R}^N)$ with the properties that

$$\varphi^s_\nu \rightarrow 0 \quad \text{in} \quad L^\infty(\Omega^s; \mathbb{R}^N) \quad \text{as} \quad \nu \rightarrow \infty, \quad \|\xi^s + \nabla \varphi^s_\nu\|_{L^\infty} \leq r,$$

$$0 \leq \int_{\Omega^s} [f(x^s, u^s, \xi^s + \nabla \varphi^s_\nu(x)) - Qf_r(x^s, u^s, \xi^s)] \, dx \leq \epsilon \frac{\text{meas} \Omega^s}{\text{meas} \Omega}.$$

Letting

$$\varphi_\nu(x) := \varphi^s_\nu(x), \quad \text{for} \quad x \in \Omega^s$$

we have constructed a function $\varphi_\nu \in W^{1,\infty}_0(\Omega; \mathbb{R}^N)$ satisfying

$$\varphi_\nu \rightarrow 0 \quad \text{in} \quad L^\infty(\Omega; \mathbb{R}^N) \quad \text{as} \quad \nu \rightarrow \infty, \quad \sup_{1 \leq s \leq S} \|\xi^s + \nabla \varphi_\nu\|_{L^\infty} \leq r,$$

$$0 \leq \sum_{s=1}^{S} \int_{\Omega^s} [f(x^s, u^s, \xi^s + \nabla \varphi_\nu(x)) - Qf_r(x^s, u^s, \xi^s)] \, dx \leq \epsilon.$$  

(8) We now estimate, as in (6) above,

$$\sum_{s=1}^{S} \int_{\Omega^s} |f(x^s, u^s, \xi^s + \nabla \varphi_\nu(x)) - f(x^s, u^s, \xi^s + \nabla \varphi_\nu(x))| \, dx.$$  

We denote

$$\lambda^s_\nu(x) := |f(x^s, u^s, \xi^s + \nabla \varphi_\nu(x)) - f(x^s, u^s, \xi^s + \nabla \varphi_\nu(x))|.$$  

We then choose $\nu$ sufficiently large so that $\|\varphi_\nu\|_{L^\infty} \leq \delta/2$ and hence, using (4) and (5),

$$\sum_{s=1}^{S} \int_{\Omega^s \cap K_\epsilon \cap E_\epsilon} \lambda^s_\nu(x) \, dx \leq \epsilon.$$  

Furthermore, since $f$ satisfies $(G_\infty)$ and $x^s \in E_\epsilon$, we have

$$\sum_{s=1}^{S} \int_{\Omega^s \setminus (K_\epsilon \cap E_\epsilon)} \lambda^s_\nu(x) \, dx$$

$$\leq \sum_{s=1}^{S} [\beta(x^s) + 2\alpha(2r, 2r)] \text{meas} [\Omega^s \setminus (K_\epsilon \cap E_\epsilon)] + \int_{\Omega^s \setminus (K_\epsilon \cap E_\epsilon)} \beta(x) \, dx$$

$$\leq [M + 2k] [\text{meas} (\Omega \setminus E_\epsilon) + \text{meas} (\Omega \setminus K_\epsilon)] + 2\eta(\epsilon) \leq 2\epsilon + 3\eta(\epsilon).$$
Combining the two estimates, we have
\[ \sum_{s=1}^{S} \int_{\Omega^s} \lambda_{\nu}^s (x) \, dx \leq 3\epsilon + 3\eta (\epsilon). \]

We finally collect the estimates (6), (7) and (8) to get that, writing \( u_{\nu} := u + \varphi_{\nu} \), for every \( \| u \|_{W^{1,\infty}} < r/2 \), there exists \( \nu_{\epsilon, r} = \nu (\epsilon, r) \) such that for \( \nu \geq \nu_{\epsilon, r} \)
\[ \left| \int_{\Omega} Q f_{r} (x, u (x), \nabla u (x)) \, dx - \int_{\Omega} f (x, u_{\nu} (x), \nabla u_{\nu} (x)) \, dx \right| \leq 7\epsilon + 6\eta (\epsilon). \]

Using Proposition 9.5 and Lebesgue dominated convergence theorem, we can find \( r = r (\epsilon) \) so that
\[ \int_{\Omega} \left| Q f_{r} (x, u (x), \nabla u (x)) - Q f (x, u (x), \nabla u (x)) \right| \, dx \leq \epsilon. \]
Combining (9.14) and the above inequality, we find that for every \( \epsilon > 0 \) we can find \( \nu_{\epsilon} = \nu (\epsilon, r (\epsilon)) \) such that for \( \nu \geq \nu_{\epsilon} \)
\[ \left| \int_{\Omega} Qf (x, u (x), \nabla u (x)) \, dx - \int_{\Omega} f (x, u_{\nu} (x), \nabla u_{\nu} (x)) \, dx \right| \leq 8\epsilon + 6\eta (\epsilon). \]

Letting \( \epsilon \to 0 \), we have indeed obtained the result for any function \( u \in C^{\infty} (\overline{\Omega}; \mathbb{R}^N) \).

In order to have the claim for \( u \in W^{1,\infty} (\Omega; \mathbb{R}^N) \) we proceed as follows. Let \( \epsilon > 0 \) be arbitrary, we can find an open set \( O \subset \Omega \) with smooth boundary, \( \gamma > 0 \) independent of \( \epsilon \) and \( v \in u + W_{0}^{1,\infty} (\Omega; \mathbb{R}^N) \) such that
\[ \begin{align*}
\text{meas} [\Omega - \overline{O}] &\leq \epsilon,
\| u \|_{W^{1,\infty} (\Omega)} , \| v \|_{W^{1,\infty} (\Omega)} &\leq \gamma,
\| u - v \|_{W^{1,1} (\Omega)} &\leq \epsilon.
\end{align*} \]

We can therefore find a non-negative increasing function \( \eta \) satisfying \( \eta (t) \to 0 \) as \( t \to 0 \) and so that
\[ \begin{align*}
\int_{\Omega} \left| Q f (x, u (x), \nabla u (x)) - Q f (x, v (x), \nabla v (x)) \right| \, dx &\leq \eta (\epsilon) \\
\int_{\Omega - O} \left| Q f (x, v (x), \nabla v (x)) - f (x, v (x), \nabla v (x)) \right| \, dx &\leq \eta (\epsilon).
\end{align*} \]

We then apply the above construction to \( v \) and \( O \) to get \( u_{\nu} \in v + W_{0}^{1,\infty} (\Omega; \mathbb{R}^N) \) such that
\[ \left| \int_{O} Q f (x, v (x), \nabla v (x)) \, dx - \int_{O} f (x, u_{\nu} (x), \nabla u_{\nu} (x)) \, dx \right| \leq \epsilon. \]

Letting \( u_{\nu} = v \) in \( \Omega - O \) and combining the three estimates, we have indeed obtained the claim for \( p = \infty \).

**Step 2.** We now discuss the case \( 1 \leq p < \infty \). We first approximate \( u \in W^{1,q} (\Omega; \mathbb{R}^N) \). Indeed, let \( \epsilon > 0 \) be arbitrary, we can then find an open set \( O \subset \Omega \)
with a smooth boundary, $\gamma > 0$ independent of $\epsilon$ and $v \in u + W^{1,q}_0(\Omega; \mathbb{R}^N)$ such that
\[
\begin{cases}
\text{meas } [\Omega - \overline{O}] \leq \epsilon, \ v \in W^{1,\infty}(O; \mathbb{R}^N) \\
\|u\|_{W^{1,q}(\Omega)}, \|v\|_{W^{1,q}(\Omega)} \leq \gamma, \ \|u - v\|_{W^{1,p}(\Omega)} \leq \epsilon.
\end{cases}
\]
Since $Qf$ satisfies $(G_p)$, we can find a non-negative increasing function $\eta$ satisfying $\eta(t) \to 0$ as $t \to 0$ and so that
\[
\int_{\Omega} |Qf(x, u(x), \nabla u(x)) - Qf(x, v(x), \nabla v(x))| \, dx \leq \eta(\epsilon)
\]
\[
\int_{\Omega - O} |Qf(x, v(x), \nabla v(x)) - f(x, v(x), \nabla v(x))| \, dx \leq \eta(\epsilon).
\]
Apply Step 1 to find \(\{u_\nu\}_{\nu=1}^\infty \subset v + W^{1,\infty}_0(O; \mathbb{R}^N)\) such that
\[
u \to v \text{ in } L^\infty(O; \mathbb{R}^N) \text{ as } \nu \to \infty,
\]
\[
\int_{O} f(x, u_\nu(x), \nabla u_\nu(x)) \, dx \to \int_{O} Qf(x, v(x), \nabla v(x)) \, dx \text{ as } \nu \to \infty.
\]
Letting $u_\nu \equiv v$ in $\Omega - O$, we have indeed, combining the two above estimates, established Part 1 of the theorem.

**Step 3.** We now conclude with Part 2. The coercivity condition (9.13) and Part 1 imply that, up to the extraction of a subsequence (still denoted \(\{u_\nu\}\)), we have
\[
\nu \to u \text{ in } W^{1,p}(\Omega; \mathbb{R}^N) \text{ as } \nu \to \infty,
\]
as wished. ■
Chapter 10

Implicit partial differential equations

10.1 Introduction

In this chapter, we discuss the existence of solutions \( u \in W^{1,\infty}(\Omega; \mathbb{R}^N) \) for the Dirichlet problem involving differential inclusions of the form

\[
\begin{align*}
\nabla u(x) &\in E \quad \text{a.e. } x \in \Omega \\
u(x) &= \varphi(x) \quad x \in \partial \Omega,
\end{align*}
\]

where \( \varphi \in W^{1,\infty}(\Omega; \mathbb{R}^N) \) is a given map and \( E \subset \mathbb{R}^{N \times n} \) is a given set. Closely related is the implicit partial differential equation

\[
\begin{align*}
F(\nabla u(x)) &= 0 \quad \text{a.e. } x \in \Omega \\
u(x) &= \varphi(x) \quad x \in \partial \Omega,
\end{align*}
\]

where \( F : \mathbb{R}^{N \times n} \to \mathbb{R} \) is a given function. It suffices to set

\[ E := \{ \xi \in \mathbb{R}^{N \times n} : F(\xi) = 0 \}. \]

The results obtained here, combined with the relaxation theorems of Chapter 9, will lead in Chapter 11 to proving the existence of minimizers in problems of the calculus of variations without appealing to lower semicontinuity theorems.

In the scalar case \( (n = 1 \text{ or } N = 1) \), a sufficient condition, for finding a solution of our problem, is

\[ \nabla \varphi(x) \in E \cup \text{int co } E \text{ a.e. in } \Omega, \]

where \( \text{int co } E \) stands for the interior of the convex hull of \( E \). This fact was observed by several authors with different proofs and different levels of generality, notably by Bressan-Flores [103], Cellina [134], Dacorogna-Marcellini [196],
[198], [202], De Blasi-Pianigiani [230] or Friesecke [291]. It should be noted that this sufficient condition is, in some sense, a necessary one (see Theorem 10.24).

When turning to the vectorial case \((n, N \geq 2)\), the problem becomes considerably harder and no result with such a degree of elegance is available. The first general results were obtained by Dacorogna-Marcellini (see the bibliography, in particular [202]) and by Müller-Sverak [464] with the help of the method of convex integration of Gromov.

The presentation follows Dacorogna-Marcellini [202], to which we constantly refer. In particular, we do not discuss the case with lower order terms, meaning equations of the form

\[ F(x, u(x), \nabla u(x)) = 0 \quad \text{a.e.} \quad x \in \Omega, \]

which is considered in [202]. However, we mention several new results, which we discuss in detail.

The chapter is organized as follows. In Section 10.2, we present an abstract existence theorem based on Baire category theorem. In Section 10.3, we give several examples, notably one involving singular values and one concerning potential wells, where the abstract theorem applies.

10.2 Existence theorems

10.2.1 An abstract theorem

We start by recalling the notation for various convex hulls of sets (see Section 7.3).

**Notation 10.1** For \(E \subset \mathbb{R}^{N \times n}\), we let

\[
\mathcal{F}_\infty^E := \{ f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\} : f|_E \leq 0 \},
\]

\[
\mathcal{F}^E := \{ f : \mathbb{R}^{N \times n} \to \mathbb{R} : f|_E \leq 0 \}.
\]

We then have, respectively, that the convex, polyconvex and rank one convex hulls of \(E\) satisfy (see Theorem 7.20)

\[
\text{co} E = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every convex } f \in \mathcal{F}_\infty^E \},
\]

\[
\text{Pco} E = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every polyconvex } f \in \mathcal{F}^E \},
\]

\[
\text{Rco} E = \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every rank one convex } f \in \mathcal{F}_\infty^E \},
\]

while the finite quasiconvex hull of \(E\) is defined as

\[
\text{Qco}_f E := \{ \xi \in \mathbb{R}^{N \times n} : f(\xi) \leq 0 \text{ for every quasiconvex } f \in \mathcal{F}^E \}.
\]

The following definition was introduced by Dacorogna-Marcellini in [201] (see also [202]) and it is the key condition to get the existence of solutions.
Existence theorems

Definition 10.2 (Relaxation property) Let $E, K \subset \mathbb{R}^{N \times n}$. We say that $K$ has the relaxation property with respect to $E$ if for every bounded open set $\Omega \subset \mathbb{R}^n$, for every affine map $u_\xi$ satisfying

$$\nabla u_\xi(x) = \xi \in K,$$

there exists a sequence $\{u_\nu\} \subset \text{Aff}_{\text{piec}}(\Omega; \mathbb{R}^N)$ with the following properties

$$u_\nu \in u_\xi + W^{1,\infty}_0(\Omega; \mathbb{R}^N), \nabla u_\nu(x) \in E \cup K, \text{ a.e. in } \Omega,$$

$$u_\nu \rightharpoonup^{\star} u_\xi \text{ in } W^{1,\infty}, \int_{\Omega} \text{dist}(\nabla u_\nu(x); E) \, dx \to 0 \text{ as } \nu \to \infty.$$

Remark 10.3 (i) With the exception of the condition $\nabla u_\nu(x) \in E \cup K$, whose status as a necessary condition is unclear, the relaxation property is obviously a natural condition for solving differential inclusions of the form $u \in u_\xi + W^{1,\infty}_0(\Omega; \mathbb{R}^N)$ and

$$\nabla u(x) \in E$$

for every $\xi \in K$. Indeed it then just states that there is an approximate solution of the problem under consideration.

(ii) It is interesting to note that in the scalar case ($n = 1$ or $N = 1$), $K = \text{int co } E$ has the relaxation property with respect to $E$.

(iii) In the vectorial case, we have that, if $K$ is bounded and has the relaxation property with respect to $E$, then necessarily

$$K \subset \text{Qco}_f E.$$ 

Indeed first recall that the definition of quasiconvexity implies that, for every quasiconvex $f \in \mathcal{F}^E$,

$$f(\xi) \text{ meas } \Omega \leq \int_{\Omega} f(\nabla u_\nu(x)) \, dx.$$ 

Combining this last result with the fact that $\{\nabla u_\nu\}$ is uniformly bounded, the fact that any quasiconvex function is continuous and the last property in the definition of the relaxation property, we get the inclusion $K \subset \text{Qco}_f E$. $\diamond$

We now give the main abstract theorem.

Theorem 10.4 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $E, K \subset \mathbb{R}^{N \times n}$ be such that $E$ is compact and $K$ is bounded. Assume that $K$ has the relaxation property with respect to $E$. Let $\varphi \in \text{Aff}_{\text{piec}}(\Omega; \mathbb{R}^N)$ be such that

$$\nabla \varphi(x) \in E \cup K \text{ a.e. in } \Omega.$$ 

Then there exists (a dense set of) $u \in \varphi + W^{1,\infty}_0(\Omega; \mathbb{R}^N)$ such that

$$\nabla u(x) \in E \text{ a.e. in } \Omega.$$
**Remark 10.5** (i) We will see in Section 10.3.1 that, in the scalar case, the largest such $K$ is

$$K = E \cup \text{int co } E.$$ 

(ii) Although we will not discuss the details, we can consider (using the results in Chapter 10 in [202]) the more general boundary datum $\varphi$, if we make the following extra hypotheses:

- in the scalar case (see Corollary 10.11 in [202]), if $K$ is open, $\varphi$ can even be taken in $W^{1,\infty}(\Omega; \mathbb{R}^N)$ and considering the previous remark, we should have (see Theorem 10.18 below) that

$$\nabla \varphi(x) \in E \cup \text{int co } E;$$

- in the vectorial case, if the set $K$ is open, $\varphi$ can be taken in $C^1_{\text{piec}}(\Omega; \mathbb{R}^N)$ (see Corollary 10.15 or Theorem 10.16 in [202]) with $\nabla \varphi(x) \in E \cup K$; while if $K$ is open and convex, $\varphi$ can be taken in $W^{1,\infty}(\Omega; \mathbb{R}^N)$ provided

$$\nabla \varphi(x) \in C \text{ a.e. in } \Omega,$$

where $C \subset K$ is compact (see Corollary 10.21 in [202]).

(iii) The present theorem was first proved by Dacorogna-Marcellini in [201] (see also Theorem 6.3 in [202]) under the further hypothesis that

$$E = \{ \xi \in \mathbb{R}^{N \times n} : F_i(\xi) = 0, i = 1, 2, \cdots, I \},$$

where $F_i : \mathbb{R}^{N \times n} \to \mathbb{R}, i = 1, 2, \cdots, I$, are quasiconvex. This hypothesis was later removed by Sychev in [559] (see also Müller-Sychev [468]). Kirchheim in [364] pointed out that using a classical result (see Theorem 10.15 below), the proof of Dacorogna-Marcellini was still valid without the extra hypothesis on $E$. Kirchheim’s idea, combined with the proof of [202], is used below, following Dacorogna-Pisante [210].

(iv) The theorem can be extended to the case with higher derivatives or lower order terms, see Dacorogna-Marcellini [202], Dacorogna-Pisante [210], Müller-Sychev [468] and also Theorem 10.9 below.

**Proof.** By working on each piece where $\varphi$ is affine, we can assume, without loss of generality, that $\varphi$ itself is affine. We next let $\mathbf{V}$ be the closure in $L^\infty(\Omega; \mathbb{R}^N)$ of

$$V := \bigl\{ u \in \text{Aff}_{\text{piec}}(\Omega; \mathbb{R}^N) : u \in \varphi + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \text{ and } \nabla u(x) \in E \cup K \bigr\}.$$

$V$ is non empty since $\varphi \in V$. Let, for $k \in \mathbb{N}$,

$$V^k := \text{int} \left\{ u \in \mathbf{V} : \int_\Omega \text{dist}(\nabla u(x); E) \, dx \leq \frac{1}{k} \right\}.$$
where "int" stands for the interior of the set.

We claim that $V^k$ is, in addition to be open, dense in the complete metric space $\nabla$. Postponing the proof of this fact for the end of the proof, we conclude by Baire category theorem that

$$\bigcap_{k=1}^{\infty} V^k \subset \{ u \in \nabla : \text{dist}(\nabla u(x); E) = 0, \text{ a.e. in } \Omega \} \subset \nabla$$

is dense, and hence non empty, in $\nabla$. The result then follows, since $E$ is compact.

We finally show that $V^k$ is dense in $\nabla$. So let $u \in \nabla$ and $\epsilon > 0$ be arbitrary. We wish to find $v \in V^k$ so that

$$\|u - v\|_{L^\infty} \leq \epsilon.$$ 

We recall (cf. Section 10.2.3) that

$$\omega_\nabla(u) := \lim_{\delta \to 0} \sup_{\varphi, \psi \in B_\delta(u)} \|\nabla \varphi - \nabla \psi\|_{L^1}$$

where

$$B_\delta(u) := \{ v \in \nabla : \|u - v\|_{L^\infty} < \delta \}.$$

- We start by finding $\alpha \in \nabla$ a point of continuity of the operator $\nabla$ (in particular $\omega_\nabla(\alpha) = 0$) so that

$$\|u - \alpha\|_{L^\infty} \leq \epsilon/3.$$ 

This is always possible by virtue of Theorem 10.15 and Proposition 10.17.

- We next approximate $\alpha \in \nabla$ by $\beta \in V$ so that, using Proposition 10.14,

$$\|\beta - \alpha\|_{L^\infty} \leq \epsilon/3 \quad \text{and} \quad \omega_\nabla(\beta) < 1/4k.$$ 

- Finally we use the relaxation property on every piece where $\nabla \beta$ is constant and we then construct $v \in V$, by patching all the pieces together, such that

$$\|\beta - v\|_{L^\infty} \leq \epsilon/3, \quad \omega_\nabla(v) < 1/2k \quad \text{and} \quad \int_\Omega \text{dist}(\nabla v(x); E) dx < 1/2k.$$ 

Moreover, since $\omega_\nabla(v) < 1/2k$, we can find $\delta = \delta(k, v) > 0$ so that

$$\|v - \psi\|_{L^\infty} \leq \delta \Rightarrow \|\nabla v - \nabla \psi\|_{L^1} \leq 1/2k$$

and hence

$$\int_\Omega \text{dist}(\nabla \psi(x); E) dx \leq \int_\Omega \text{dist}(\nabla v(x); E) dx + \|\nabla \psi - \nabla v\|_{L^1} < 1/k$$

for every $\psi \in B_\delta(v)$; which implies that $v \in V^k$.

Combining these three facts we have indeed obtained the desired density result. \qed
10.2.2 A sufficient condition for the relaxation property

We now give a sufficient condition that ensures the relaxation property. In concrete examples, this condition is usually much easier to check than the relaxation property. It resembles the so-called approximation in the convex integration method of Gromov as revisited by Müller and Sverak. We start with a definition.

**Definition 10.6 (Approximation property)** Let \( d \) be an integer and \( E \subset K \subset M \subset \mathbb{R}^d \).

We say that \((E, K, M)\) (when \( M = \mathbb{R}^d \) we simply write \((E, K)\)) has the approximation property if there exists a family of sets \( E_\delta \) and \( K(E_\delta) \), \( \delta > 0 \), such that

1. \( E_\delta \subset K(E_\delta) \subset \text{int} M \) for every \( \delta > 0 \) (where \( \text{int} M \) stands for the interior relative to \( M \) and \( A \subset \subset B \) means that \( A \subset B \) and is compact);

2. for every \( \epsilon > 0 \) there exists \( \delta_0 = \delta_0(\epsilon) > 0 \) such that \( \text{dist}(\eta; E) \leq \epsilon \) for every \( \eta \in E_\delta \) and \( \delta \in (0, \delta_0] \);

3. if \( \eta \in \text{int} M \subset \text{int} K \), then \( \eta \in K(E_\delta) \) for every \( \delta > 0 \) sufficiently small.

Before proceeding further, we first recall the notation (see Chapter 12) for the higher derivatives. The aim is to write in a simple way the matrix \( \nabla^m u \) of all partial derivatives of order \( m \) of a map \( u : \mathbb{R}^n \to \mathbb{R}^N \).

**Notation 10.7** (i) Let \( N, n, m \geq 1 \) be integers. We denote by \( \mathbb{R}^{N \times n^m}_s \) the set of matrices

\[
A = (A_{i_{j_1\cdots j_m}}^{j_1\cdots j_m})_{1\leq i \leq N, \ 1\leq j_1\cdots j_m \leq n} \in \mathbb{R}^{N \times n^m}_s
\]

such that for every permutation \( \sigma \) of \( \{j_1, \cdots, j_m\} \) we have

\[
A_{i_{\sigma(j_1\cdots j_m)}}^{j_1\cdots j_m} = A_{i_{j_1\cdots j_m}}^{j_1\cdots j_m}.
\]

The number of different entries (because of the different symmetries) is

\[
N \times \binom{n + m - 1}{m}.
\]

- When \( m = 1 \), we have \( \mathbb{R}^{N \times n}_s = \mathbb{R}^{N \times n} \).

- When \( N = 1 \) and \( m = 2 \), we get \( \mathbb{R}^{n^2}_s = \mathbb{R}^{n \times n}_s \), i.e. the set of symmetric matrices.

(ii) Let \( u : \mathbb{R}^n \to \mathbb{R}^N \). We therefore have

\[
\nabla^m u = \left( \frac{\partial^m u^i}{\partial x_{j_1} \cdots \partial x_{j_m}} \right)_{1 \leq i \leq N, \ 1 \leq j_1, \cdots, j_m \leq n} \in \mathbb{R}^{N \times n^m}_s.
\]

- When \( m = 1 \), this is the usual gradient map.
- When $N = 1$ and $n = m = 2$, we have
\[
\nabla^2 u = \begin{pmatrix}
\frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} \\
\frac{\partial^2 u}{\partial x_1 \partial x_2} & \frac{\partial^2 u}{\partial x_2^2}
\end{pmatrix} \in \mathbb{R}_{s}^{2 \times 2}.
\]

(iii) Given $\alpha \in \mathbb{R}^n$ denote by $\alpha^{\otimes m} = \alpha \otimes \alpha \cdots \otimes \alpha$ ($m$ times), it is a matrix in $\mathbb{R}^{n^m}$. Therefore a generic matrix of rank one in $\mathbb{R}^{N \times n}$ is of the form
\[
\beta \otimes \alpha^{\otimes m} = \left( \beta^i \alpha_{j_1} \cdots \alpha_{j_m} \right)_{1 \leq i \leq N, 1 \leq j_1, \cdots, j_m \leq n},
\]
where $\beta \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}^n$.

- When $m = 1$, this is the usual tensorial product; i.e.
\[
\beta \otimes \alpha = \left( \beta^i \alpha_j \right)_{1 \leq i \leq N, 1 \leq j \leq n}.
\]

- When $N = 1$ and $n = m = 2$, we have
\[
\alpha^{\otimes 2} = \alpha \otimes \alpha = \begin{pmatrix}
\alpha_1^2 & \alpha_1 \alpha_2 \\
\alpha_1 \alpha_2 & (\alpha_2)^2
\end{pmatrix} \in \mathbb{R}_{s}^{2 \times 2}. \diamond
\]

The next result gives the appropriate generalization to higher derivatives of Lemma 3.11. Its proof is very similar and we will not reproduce it here; we refer to Lemma 6.8 in Dacorogna-Marcellini [202] for details.

**Lemma 10.8** Let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure. Let $t \in [0, 1]$ and $A, B \in \mathbb{R}^{N \times n}$ with rank $\{A - B\} = 1$. Let $\varphi$ be such that
\[
\nabla^m \varphi(x) = tA + (1 - t)B, \quad \forall x \in \Omega.
\]

Then, for every $\epsilon > 0$, there exist $u \in \text{Aff}_m^\text{piec} (\overline{\Omega}; \mathbb{R}^N)$ and disjoint open sets $\Omega_A, \Omega_B \subset \Omega$, such that
\[
\begin{align*}
|\text{meas } \Omega_A - t \text{ meas } \Omega|, & \quad |\text{meas } \Omega_B - (1 - t) \text{ meas } \Omega| \leq \epsilon \\
& \quad u \equiv \varphi \text{ near } \partial \Omega, \quad ||u - \varphi||_{W^{m-1, \infty}} \leq \epsilon \\
& \quad \nabla^m u(x) = \begin{cases} 
A & \text{in } \Omega_A \\
B & \text{in } \Omega_B
\end{cases} \\
& \quad \text{dist } (\nabla^m u(x), \text{co } \{A, B\}) \leq \epsilon \quad \text{a.e. in } \Omega.
\end{align*}
\]

We now give the following theorem (see Theorem 6.14 in [202] and for a slightly more flexible one, see Theorem 6.15 in [202]), which will be used in an important way in the examples.

**Theorem 10.9** Let $m, N, n \in \mathbb{N}$ and
\[
E \subset \mathcal{M} = \mathbb{R}^{N \times n} \subset \mathbb{R}^{N \times n}
\]
with $E$ compact. Assume that there exist $E_\delta$ such that $E_\delta, K(E_\delta) = \text{Rco } E_\delta$ satisfy (1), (2) and (3) in the definition of the approximation property with $K = \text{int } \mathcal{M} \text{Rco } E$. Then $K$ has the relaxation property with respect to $E$. 


Remark 10.10 (i) We recall that $\text{Rco} \ E$ stands for the rank one convex hull of $E$ (see Section 7.3.1). Note also that $\mathcal{M}$ is convex and thus rank one convex. Consequently, if $E \subset \mathcal{M}$, then $\text{Rco} \ E \subset \mathcal{M}$.

(ii) The theorem contains, in particular, the case where $m = 1$ and hence

$$\mathcal{M} = \mathbb{R}^{N \times n}.$$ 

The theorem will always be applied when $m = 1$ in the examples, but in Section 10.3.5 where $N = 1$ and $n = m = 2$. ♦

**Proof.** We wish to show that for every bounded open set $\Omega \subset \mathbb{R}^n$, for every affine map $u_\xi \in \text{Aff}_{\text{piec}}(\overline{\Omega}; \mathbb{R}^{N \times n^{m-1}})$ satisfying

$$\nabla u_\xi (x) = \xi \in K,$$

there exists a sequence $\{u_\nu\} \subset \text{Aff}_{\text{piec}}(\overline{\Omega}; \mathbb{R}^{N \times n^{m-1}})$ such that

$$u_\nu \in u_\xi + W_0^{1,\infty}(\Omega; \mathbb{R}^{N \times n^{m-1}}), \ \nabla u_\nu (x) \in E \cup K, \ \text{a.e. in} \ \Omega \ \nu \rightarrow \infty.$$

Since $\xi \in \text{int}_M \text{Rco} \ E$, we have, by (3) in the definition of the approximation property, that, for any $\delta > 0$ sufficiently small,

$$\xi \in \text{Rco} \ E_\delta.$$

We can therefore find (cf. Theorem 7.17) an integer $I$ so that

$$\xi \in \text{Rco} \ E_\delta.$$

We then proceed by induction on $I$.

**Case:** $I = 1$. We can thus find $t \in [0, 1]$ and

$$A, B \in E_\delta \ \text{with} \ \text{rank} \{A - B\} = 1$$

so that

$$\xi = tA + (1 - t)B.$$ 

We then use Lemma 10.8 (with $\varphi$ such that $\nabla^m \varphi = \xi$) to get that there exists a sequence

$$\{v_\nu\} \subset \text{Aff}_{\text{piec}}^m(\overline{\Omega}; \mathbb{R}^N)$$

such that, if we write

$$u_\nu = \nabla^{m-1}v_\nu \in \text{Aff}_{\text{piec}}(\overline{\Omega}; \mathbb{R}^{N \times n^{m-1}})$$
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then (letting \( \tilde{\Omega} := \Omega_A \cup \Omega_B \) and \( \epsilon = 1/\nu \) in the conclusion of the lemma)

\[
\begin{align*}
0 \leq \text{meas } \Omega - \text{meas } \tilde{\Omega} & \leq 1/\nu \\
u\nu \in u_\xi + W^{1,\infty}_0(\Omega; \mathbb{R}^{N \times n^{m-1}}) \\
\|u_\nu - u_\xi\|_{L^\infty} & \leq 1/\nu \\
\nabla u_\nu(x) & \in E_\delta, \text{ a.e. in } \tilde{\Omega} \\
\text{dist } (\nabla u_\nu(x), R_1 \text{ co } E_\delta) & \leq 1/\nu, \text{ a.e. in } \Omega. 
\end{align*}
\]

Since \( \nabla u_\nu \in \mathcal{M}, \text{Rco } E_\delta \subset K \) (by (1) in the definition of the approximation property), \( K \) is a bounded set and (2) in the definition of the approximation property holds, we have the claim.

**Case:** \( I > 1 \). We now assume that the result has been proved for \( I - 1 \) and let us show the claim for \( I \). We can thus find \( t \in [0, 1] \) and

\[ A, B \in R_{I-1} \text{ co } E_\delta \text{ with } \text{rank } \{ A - B \} = 1 \]

so that

\[ \xi = tA + (1 - t)B. \]

Appealing to Lemma 10.8 (with \( \varphi \) such that \( \nabla^m \varphi = \xi \)), we get that there exists a sequence

\[ \{ \psi_\nu \} \subset \text{Aff}_{\text{piec}}^{m}(\overline{\Omega}; \mathbb{R}^N) \]

such that, if we write

\[ \varphi_\nu = \nabla^{m-1} \psi_\nu \in \text{Aff}_{\text{piec}}^{m}(\overline{\Omega}; \mathbb{R}^{N \times n^{m-1}}) \]

then

\[
\begin{align*}
|\text{meas } (\Omega_A \cup \Omega_B) - \text{meas } \Omega| & \leq 1/\nu \\
\varphi_\nu & \in u_\xi + W^{1,\infty}_0(\Omega; \mathbb{R}^{N \times n^{m-1}}) \\
\|\varphi_\nu - u_\xi\|_{L^\infty} & \leq 1/\nu \\
\nabla \varphi_\nu(x) & = \begin{cases} 
A & \text{ in } \Omega_A \\
B & \text{ in } \Omega_B 
\end{cases} \\
\text{dist } (\nabla \varphi_\nu(x), R_I \text{ co } E_\delta) & \leq 1/\nu, \text{ a.e. in } \Omega. 
\end{align*}
\]

We now use the hypothesis of induction on \( \Omega_A, \Omega_B \) and \( A, B \in R_{I-1} \text{ co } E_\delta \). We therefore find sequences

\[ \{ \alpha_\mu \} \subset \text{Aff}_{\text{piec}}(\overline{\Omega_A}; \mathbb{R}^{N \times n^{m-1}}), \{ \beta_\mu \} \subset \text{Aff}_{\text{piec}}(\overline{\Omega_B}; \mathbb{R}^{N \times n^{m-1}}) \]

such that

\[ \alpha_\mu \in \varphi_\nu + W^{1,\infty}_0(\Omega_A; \mathbb{R}^{N \times n^{m-1}}), \nabla \alpha_\mu(x) \in E \cup K, \text{ a.e. in } \Omega \]

\[ \alpha_\mu \overset{*}{\rightharpoonup} \varphi_\nu \text{ in } W^{1,\infty}, \int_{\Omega_A} \text{dist } (\nabla \alpha_\mu(x); E) \, dx \to 0 \text{ as } \mu \to \infty \]
and similarly for the sequence $\{\beta_\mu\}$.

We next write, taking a diagonal sequence,

$$u_\nu(x) = \begin{cases} 
\varphi_\nu(x) & \text{in } \Omega - (\Omega_A \cup \Omega_B) \\
\alpha_\nu(x) & \text{in } \Omega_A \\
\beta_\nu(x) & \text{in } \Omega_B 
\end{cases}$$

and use the facts that $\nabla u_\nu \in \mathcal{M}$, $\text{Rco } E_\delta \subset K$, $K$ is a bounded set and (2) in the definition of the approximation property holds, to get the claim. 

Finally we discuss an example showing that the approximation property is not always fulfilled. We consider the case $N = n = 2$ and denote by $\mathbb{R}^{2 \times 2}_d$ the set of $2 \times 2$ diagonal matrices, we write any such matrix as a vector of $\mathbb{R}^2$.

**Example 10.11** Let $E = \{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6\} \subset \mathcal{M} = \mathbb{R}^{2 \times 2}_d$ (see Figure 10.1) be defined by

$\xi_1 = (1, 0), \xi_2 = (1, -1), \xi_3 = (0, -1), \xi_4 = (-1, 0), \xi_5 = (-1, 1), \xi_6 = (0, 1)$.

It is easy to find that

$$\text{Rco } E = \{\xi : \xi = (x, y) \in [0, 1] \times [-1, 0]\} \cup \{\xi : \xi = (x, y) \in [-1, 0] \times [0, 1]\}$$

and its interior (relative to $\mathbb{R}^{2 \times 2}_d$) is given by

$$\text{int}_{\mathcal{M}} \text{Rco } E = \{\xi : \xi = (x, y) \in (0, 1) \times (-1, 0)\} \cup \{\xi : \xi = (x, y) \in (-1, 0) \times (0, 1)\}.$$
However, there is no way of finding a set $E_\delta$ satisfying the approximation property with $K(E_\delta) = \text{Rco } E_\delta$. In fact, condition (3) in the definition of the approximation property will be violated.

\[\diamondsuit\]

### 10.2.3 Appendix: Baire one functions

In this appendix, we recall some well-known facts about so-called Baire one functions (see, for example, Oxtoby [488] or Yosida [605]). We start with the following definitions.

#### Definition 10.12

Let $X, Y$ be metric spaces and $f : X \to Y$. We define the oscillation of $f$ at $x_0 \in X$ as

$$\omega_f(x_0) := \lim_{\delta \to 0} \sup_{x, y \in B_X(x_0, \delta)} d_Y(f(y), f(x)),$$

where $B_X(x_0, \delta) := \{x \in X : d_X(x, x_0) < \delta\}$ and $d_X, d_Y$ are the metrics on the spaces $X$ and $Y$, respectively.

#### Definition 10.13

A function $f$ is said to be a Baire one function (or a function of first class) if it can be represented as the pointwise limit of an everywhere convergent sequence of continuous functions.

In the next proposition, we recall some elementary properties of the oscillation function $\omega_f$.

#### Proposition 10.14

Let $X, Y$ be metric spaces and $f : X \to Y$.

(i) $f$ is continuous at $x_0 \in X$ if and only if $\omega_f(x_0) = 0$.

(ii) The set $\Omega_f := \{x \in X : \omega_f(x) < \epsilon\}$ is an open set in $X$.

Using the notion of oscillation and Proposition 10.14, we can write the set $D_f$ of all points at which a given function $f$ is discontinuous as an $F_\sigma$ set as follows

$$D_f = \bigcup_{\nu=1}^{\infty} \{x \in X : \omega_f(x) \geq 1/\nu\}. \quad (10.1)$$

We therefore have the following theorem, due to Baire, for Baire one functions. For the convenience of the reader, we give a proof of this theorem (see also Theorem 7.3 in Oxtoby [488] or Yosida [605] page 12).

#### Theorem 10.15

Let $X, Y$ be metric spaces, with $X$ complete, and $f : X \to Y$. If $f$ is a Baire one function, then $D_f$ is a set of first category.

**Proof.** Using the representation (10.1) of $D_f$, it suffices to show that, for every $\epsilon > 0$, the set $F := \{x \in X : \omega_f(x) \geq 5\epsilon\}$ is nowhere dense.

Let $f(x) := \lim_{\nu \to \infty} f_\nu(x)$, with $f_\nu$ continuous and define the sets

$$E_\nu := \bigcap_{i, j \geq \nu} \{x \in X : d_Y(f_i(x), f_j(x)) \leq \epsilon\}, \quad \forall \nu \in \mathbb{N}.$$
Then \( E_\nu \) is closed in \( X \), by continuity of \( f_\nu \), and \( E_\nu \subset E_{\nu+1} \). Moreover, 
\( \bigcup_{\nu\in\mathbb{N}} E_\nu = X \), since for every \( x \in X \) the sequence \( \{f_\nu(x)\} \) is convergent and 
thus a Cauchy sequence in \( Y \).

Consider any closed set with non-empty interior \( I \subset X \). Since \( I = \bigcup(E_\nu \cap I) \), 
the sets \( E_\nu \cap I \) cannot all be nowhere dense. Indeed (see Yosida [605] page 12) 
in this case the complement of \( I \) in \( X \), \( I^c \), should be a dense set as a complement 
of a set of first category by Baire category theorem and this is a contradiction 
with the fact that \( I \) has a non-empty interior. Hence for some positive integer \( \nu \), 
\( E_\nu \cap I \) contains an open subset \( J \), by definition (see Yosida [605] page 11) of 
a nowhere dense set.

We have \( d_Y(f_j(x), f_i(x)) \leq \epsilon \) for all \( x \in J \) and for all \( i, j \geq \nu \). Putting \( j = \nu \) 
and letting \( i \) tend to \( \infty \), we find that \( d_Y(f_\nu(x), f(x)) \leq \epsilon \) for all \( x \in J \). By the 
continuity of \( f_\nu \), for any \( x_0 \in J \), there exists a neighborhood \( I(x_0) \subset J \) such 
that \( d_Y(f_\nu(x), f_\nu(x_0)) \leq \epsilon \) for all \( x \in I(x_0) \) and hence 
\[
   d_Y(f(x), f_\nu(x_0)) \leq 2\epsilon, \quad \forall x \in I(x_0).
\]

Therefore
\[
   d_Y(f(x), f(y)) \leq d_Y(f(x), f_\nu(x_0)) + d_Y(f(y), f_\nu(x_0)) \leq 4\epsilon, \quad \forall x, y \in I(x_0),
\]
then \( \omega_f(x_0) \leq 4\epsilon \), and so no point of \( J \) belongs to \( F \). Thus, for every closed set 
\( I \) with non-empty interior there is an open set \( J \subset I \cap F^c \). This shows that \( F \) 
is nowhere dense and therefore \( D_f \) is of first category. \( \blacksquare \)

**Remark 10.16** From Theorem 10.15 and the Baire category theorem, it follows 
in particular that the set of points of continuity of a Baire one function from a 
complete metric space \( X \) to any metric space \( Y \) (i.e. the set \( D_f \) complement of \( D_f \) ) is a dense \( G_\delta \) set. Indeed, for any \( \epsilon > 0 \), the set 
\[
   \Omega_f^\epsilon := \{x \in X : \omega_f(x) < \epsilon\}
\]
is open and dense in \( X \). \( \diamondsuit \)

In the proof of our main theorem, we have used Theorem 10.15 applied to the 
following, quite surprising, special case of a Baire one function. This result was 
observed by Kirchheim in [364], [365] for complete sets of Lipschitz functions 
and the same argument in fact gives the result for general complete subsets of 
\( W^{1,\infty}(\Omega) \) functions.

**Proposition 10.17** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and let \( V \subset W^{1,\infty}(\Omega) \cap 
W^{1,\infty}_{\text{loc}}(\mathbb{R}^n) \) be a non-empty complete space with respect to the \( L^\infty \) metric. Then 
the gradient operator \( \nabla : V \to L^p(\Omega; \mathbb{R}^n) \) is a Baire one function for any \( 1 \leq p < \infty \).

**Proof.** For \( h \neq 0 \), we let 
\[
   \nabla^h = (\nabla_{1}^h, \ldots, \nabla_{n}^h) : V \to L^p(\Omega; \mathbb{R}^n)
\]
be defined, for every $u \in V$ and $x \in \Omega$, by

$$
\nabla_i^h u(x) := \begin{cases} 
\frac{u(x + he_i) - u(x)}{h} & \text{if } \text{dist}(x, \Omega^c) > |h| \\
0 & \text{elsewhere}
\end{cases}
$$

for $i = 1, \cdots, n$, where $e_1, \cdots, e_n$ stand for the vectors from the Euclidean basis.

The claim follows once we have proved that for any fixed $h$ the operator $\nabla^h$ is continuous and that, for any sequence $h \to 0$,

$$
\lim_{h \to 0} \| \nabla_i^h u - \nabla_i u \|_{L^p(\Omega)} = 0
$$

for any $i = 1, \cdots, n$, $u \in V$.

The continuity of $\nabla^h$ follows easily by observing that for every $i = 1, \cdots, n$, $\epsilon > 0$ and $u, v \in V$ we have that

$$
\| \nabla_i^h u - \nabla_i^h v \|_{L^p(\Omega)} \leq \frac{1}{|h|} \left( \int_{\Omega_h} |u(x) - v(x) + v(x + he_i) - u(x + he_i)|^p \, dx \right)^{\frac{1}{p}} 
\leq \frac{2(\text{meas } \Omega)^{\frac{1}{p}}}{|h|} \| u - v \|_{L^\infty(\Omega)},
$$

where $\Omega_h := \{ x \in \Omega : \text{dist}(x, \Omega^c) > |h| \}$.

For the second claim we start observing that for any $h$ and for any $u \in V$ we have

$$
\| \nabla_i^h u \|_{L^\infty(\Omega)} \leq \frac{1}{h} \| u(x + he_i) - u(x) \|_{L^\infty(\Omega_h)} \leq \| \nabla_i u \|_{L^\infty(\Omega)} < +\infty.
$$

Moreover by Rademacher theorem (see Theorems 6.2.1 and 6.2.2 in Evans-Gariepy [273]), for any sequence $h \to 0$,

$$
\lim_{h \to 0} \nabla_i^h u(x) = \nabla_i u(x), \text{ a.e. } x \in \Omega.
$$

The result follows by Lebesgue dominated convergence theorem. ■

### 10.3 Examples

We now give several examples of existence theorems that follow from the abstract ones.

#### 10.3.1 The scalar case

The first one concerns the scalar case, where we can even get sharper results (see Bressan-Flores [103], Cellina [134], Dacorogna-Marchese [196], [198], [202], De Blasi-Pianigiani [230] or Friesecke [291]).

**Theorem 10.18** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $E \subset \mathbb{R}^n$. Let $\varphi \in \text{Aff}_{\text{piecey}}(\Omega)$ satisfy

$$
\nabla \varphi(x) \in E \cup \text{int co } E \text{ a.e. } x \in \Omega
$$

(10.2)
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(Where \text{int co} \ E \ stands \ for \ the \ interior \ of \ the \ convex \ hull \ of \ E). \ Then \ there \ exists \ \ u \in \varphi + W^{1,\infty}_0 (\Omega) \ such \ that

\[ \nabla u(x) \in E \ \text{a.e.} \ x \in \Omega. \quad (10.3) \]

Remark 10.19 \ (i) \ The \ theorem \ easily \ follows \ from \ the \ abstract \ Theorems \ 10.9 \ and \ 10.4, \ but \ we \ prefer \ to \ give \ here \ a \ different \ proof, \ based \ on \ the \ method \ of \ pyramids \ introduced \ by \ Cellina \ [134] \ and \ Friesecke \ [291]. \ Below \ we \ follow \ the \ proof \ of \ Lemma \ 2.11 \ in \ Dacorogna-Marcellini \ [202], \ showing, \ in \ particular, \ that \ u \in \text{Aff}_{\text{piec}} (\Omega).

(ii) \ The \ theorem \ can \ also \ be \ proved (see \ Theorem \ 2.10 \ in \ Dacorogna-Marcellini \ [202]) \ if \ \ \varphi \in W^{1,\infty}(\Omega) \ satisfy

\[ \nabla \varphi(x) \in E \cup \text{int co} \ E \ \text{a.e.} \ x \in \Omega. \quad \diamond \]

Proof. \ Step 1. By working on each piece, where \varphi \ is \ affine, \ we \ can \ assume \ that \ in \ fact \ \varphi \ is \ affine \ and \ therefore

\[ \nabla \varphi = \xi_0 \in \text{int co} \ E, \ \forall x \in \Omega, \]

the case \ \xi_0 \in E \ being \ trivial. \ Invoking \ Corollary \ 2.16, \ we \ can \ find

\[ \xi_1, \xi_2, \ldots, \xi_m \in E, \ m \geq n + 1, \]

such that \ \{\xi_1 - \xi_0, \xi_2 - \xi_0, \ldots, \xi_m - \xi_0\} \ spans \ the \ whole \ of \ \mathbb{R}^n,

\[ \xi_0 \in \text{int co} \{\xi_1, \xi_2, \ldots, \xi_m\} \]

and \ there \ exist \ \ s_i > 0, \ i = 1, 2, \ldots, m, \ with \ \sum_{i=1}^{m} s_i = 1 \ such \ that

\[ \sum_{i=1}^{m} s_i (\xi_i - \xi_0) = 0. \quad (10.4) \]

Step 2. \ Let \ \ x_0 \in \Omega \ and \ define \ for \ \ r > 0 \ the \ function

\[ v_{r,x_0}(x) := r + \min_{i=1,\ldots,m} \{\langle \xi_i - \xi_0; x - x_0 \rangle \} \]

which we \ call \ a "pyramid". \ Let

\[ G(r, x_0) := \{x \in \mathbb{R}^n : v_{r,x_0}(x) > 0 \} \]

and \ observe (see \ Step 3) \ that \ this \ set \ is \ bounded. \ Finally, \ let

\[ u(x) := \varphi(x) + v_{r,x_0}(x). \]

Then \ \ u \in \varphi + W^{1,\infty}_0 (G(r, x_0)), \ with \ \ u \in \text{Aff}_{\text{piec}}(G(r, x_0)), \ and

\[ \nabla u(x) \in \{\xi_1, \ldots, \xi_m\} \subset E \ \text{a.e.} \ x \in G(r, x_0). \]
Covering Ω by dilation and translation of sets of the form \( G(r, x_0) \) and appealing to Vitali covering theorem (see Corollary 10.6 in [202]), we have the result.

**Step 3.** It remains to prove that \( G(r, x_0) \) is bounded. Let us assume, for the sake of contradiction, that for some \( x_0 \in \mathbb{R}^n \) and \( r > 0 \) the set \( G(r, x_0) \) is not bounded. Then there exists a sequence \( x_k \in \mathbb{R}^n, k \in \mathbb{N}, \) such that

\[
\lim_{k \to \infty} |x_k| = +\infty \quad \text{and} \quad r + \langle \xi_i - \xi_0; x_k - x_0 \rangle \geq 0, \forall i = 1, 2, \ldots, m, \forall k \in \mathbb{N}.
\]

Let

\[
y_k := \frac{x_k}{|x_k|}.
\]

Then, up to the extraction of a subsequence that we still denote by \( y_k \), we have that \( y_k \to y_0 \) for some \( y_0 \in \mathbb{R}^n \) with \( |y_0| = 1 \). Passing to the limit as \( k \to +\infty \) on both sides of the inequality, we obtain

\[
0 = \langle 0; y_0 \rangle = \sum_{i=1}^{m} s_i \langle \xi_i - \xi_0; y_0 \rangle.
\]

Since

\[
s_i > 0, \langle \xi_i - \xi_0; y_0 \rangle \geq 0, \forall i = 1, 2, \ldots, m,
\]

we deduce that

\[
\langle \xi_i - \xi_0; y_0 \rangle = 0, \forall i = 1, 2, \ldots, m.
\]

Recall that the set \( \{ \xi_1 - \xi_0, \xi_2 - \xi_0, \ldots, \xi_m - \xi_0 \} \) spans the whole of \( \mathbb{R}^n \). Therefore there exist \( c_i \in \mathbb{R}, i = 1, 2, \ldots, m, \) such that \( y_0 = \sum_{i=1}^{m} c_i (\xi_i - \xi_0) \). Combining all the above identities, we obtain the desired contradiction, namely

\[
1 = |y_0|^2 = \langle y_0; y_0 \rangle = \sum_{i=1}^{m} c_i \langle \xi_i - \xi_0; y_0 \rangle = 0.
\]

This finishes the proof. \( \blacksquare \)

We have as immediate corollaries the following.

**Corollary 10.20** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and \( F : \mathbb{R}^n \to \mathbb{R} \) be continuous and such that \( \lim_{|\xi| \to \infty} F(\xi) = +\infty \). Let \( \varphi \in \text{Aff}_{\text{piece}}(\overline{\Omega}) \) be such that

\[
F(\nabla \varphi(x)) \leq 0 \text{ a.e. } x \in \Omega.
\]

Then there exists \( u \in \varphi + W^{1,\infty}_0(\Omega) \) such that

\[
F(\nabla u(x)) = 0 \text{ a.e. } x \in \Omega.
\]
Remark 10.21 As in Remark 10.19 (ii), the corollary still holds for \( \varphi \in W^{1,\infty}(\Omega) \).

Proof. Let

\[ E = \{ \xi \in \mathbb{R}^n : F(\xi) = 0 \}, \quad K = \{ \xi \in \mathbb{R}^n : F(\xi) \leq 0 \} \]

and observe that, under our hypotheses,

\[ K \subset E \cup \text{int co } E. \]

Indeed let \( \xi \in K \) and observe that if \( F(\xi) = 0 \), then the inclusion is trivially true; we therefore assume that \( F(\xi) < 0 \).

We then let \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n \), \( t \in \mathbb{R} \) and

\[ \xi_t := \xi + te_1. \]

Since \( F \) is continuous and \( \lim_{|\xi| \to \infty} F(\xi) = +\infty \), we can find \( t_- < 0 < t_+ \) such that

\[ F(\xi_t) < 0, \quad \forall t \in (t_-, t_+) \quad \text{and} \quad F(\xi_{t_\pm}) = 0, \text{ i.e. } \xi_{t_\pm} \in E. \]

We can therefore write

\[ \xi = \frac{t_+}{t_+ - t_-} \xi_- + \frac{-t_-}{t_+ - t_-} \xi_+ \in \text{co } E. \]

Since \( F(\xi) < 0 \), it is easy to see that, in fact, \( \xi \in \text{int co } E \). Thus \( K \subset E \cup \text{int co } E \).

We are therefore in a position to apply Theorem 10.18 to get the result. \( \blacksquare \)

The above corollary generalizes in a straightforward way to the vectorial case.

Corollary 10.22 Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and \( F : \mathbb{R}^{N \times n} \to \mathbb{R} \) be continuous and such that \( \lim_{|\xi| \to \infty} F(\xi) = +\infty \). Let \( \varphi \in \text{Aff}_{\text{piece}}(\overline{\Omega}; \mathbb{R}^N) \) be such that

\[ F(\nabla \varphi(x)) \leq 0 \text{ a.e. } x \in \Omega. \]

Then there exists \( u \in \varphi + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \) such that

\[ F(\nabla u(x)) = 0 \text{ a.e. } x \in \Omega. \]

Proof. By working on each piece where \( \varphi \) is affine, we can assume that it is affine and thus \( \nabla \varphi = \xi_0 \in \mathbb{R}^{N \times n} \). We can also assume that \( \xi_0 = 0 \), otherwise make a translation. Set for \( \xi \in \mathbb{R}^{N \times n} \)

\[ \xi = (\xi^1, \ldots, \xi^N), \text{ where } \xi^i \in \mathbb{R}^n \]
and
\[ G(\xi^1) := F(\xi^1, 0, \cdots, 0). \]

Observe that \( G \) satisfies all the hypotheses of Corollary 10.20 and \( G(0) = F(0) \leq 0; \)
therefore there exists \( u^1 \in W^{1,\infty}_0(\Omega) \) such that
\[ G(\nabla u^1(x)) = 0, \text{ a.e. } x \in \Omega. \]

Setting \( u(x) := (u^1(x), 0, \cdots, 0) \),
we have indeed obtained the claim. \( \square \)

We now conclude with an approximation result of Müller-Sychev [468] (compare it with Lemma 3.11).

**Corollary 10.23** Let \( n \geq 2, \ N \geq 1, \ \Omega \subset \mathbb{R}^n \) be a bounded open set. Let \( t \in [0, 1] \) and \( A, B \in \mathbb{R}^{N \times n} \) such that
\[ A - B = a \otimes b \]
with \( a \in \mathbb{R}^N \) and \( b \in \mathbb{R}^n \). Let \( b_3, \cdots, b_k \in \mathbb{R}^n, \ k \geq n + 2, \) such that
\[ 0 \in \text{int } \text{co}\{b, -b, b_3, \cdots, b_k\}. \quad (10.5) \]

Let \( \varphi \) be an affine map such that
\[ \nabla \varphi(x) = \xi_0 = tA + (1 - t)B, \quad x \in \overline{\Omega} \]
(i.e. \( A = \xi_0 + (1 - t)a \otimes b \) and \( B = \xi_0 - ta \otimes b \)). Then, for every \( \epsilon > 0 \), there exist \( u \in \text{Aff}_{\text{piece}}(\overline{\Omega}; \mathbb{R}^N) \) and disjoint open sets \( \Omega_A, \ \Omega_B \subset \Omega \), such that
\[
\begin{cases}
|\text{meas } \Omega_A - t \text{meas } \Omega|, \ |\text{meas } \Omega_B - (1 - t) \text{meas } \Omega| \leq \epsilon \\
\quad u(x) = \varphi(x), \ x \in \partial \Omega \text{ and } |u(x) - \varphi(x)| \leq \epsilon, \ x \in \Omega \\
\quad \nabla u(x) = \begin{cases} A & \text{in } \Omega_A \\ B & \text{in } \Omega_B \end{cases} \\
\quad \nabla u(x) \in \xi_0 + \{(1 - t)a \otimes b, -ta \otimes b, a \otimes b_3, \cdots, a \otimes b_k\} \text{ a.e. in } \Omega.
\end{cases}
\]

**Proof.** We follow here the proof of Kirchheim [365]. We divide the proof into three steps.

**Step 1.** There is no loss of generality if we assume the two next hypotheses.
1) \( \xi_0 = 0 \), by setting \( u := \tilde{u} + \varphi \).
and solving the problem for $u$ replaced by $\tilde{u}$ and $\varphi$ by 0.

2) $N = 1$, by letting

$$u(x) := v(x)a$$

where we have to find $v \in \text{Aff}_{\text{piece}}(\Omega)$ and disjoint open sets $\Omega_A$, $\Omega_B \subset \Omega$, such that

$$|\text{meas } \Omega_A - t \text{ meas } \Omega|, |\text{meas } \Omega_B - (1 - t) \text{ meas } \Omega| \leq \epsilon$$

$$v(x) = 0, \ x \in \partial \Omega \text{ and } |v(x)| \leq \epsilon, \ x \in \Omega$$

$$\nabla v(x) = \begin{cases} 
(1 - t)b & \text{in } \Omega_A \\
-tb & \text{in } \Omega_B 
\end{cases}$$

(10.6)

$$\nabla v(x) \in \{(1 - t)b, -tb, b_3, \ldots, b_k\}, \ a.e. \ in \ \Omega.$$ 

**Step 2.** We now prove (10.6), without the conclusion $|v(x)| \leq \epsilon$, for $\Omega$ of the form

$$G := \left\{ x \in \mathbb{R}^n : 1 + \min_{i=1,\ldots,k} \{\langle b_i; x \rangle\} > 0 \right\}$$

where $b_1 = b$ and $b_2 = -b$. It is easily proved (cf. Step 3 in the proof of Theorem 10.18) that under our hypothesis (10.5) the set $G$ is bounded.

Define next a non negative periodic function, of period 1, $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$ and

$$h'(\tau) = \begin{cases} 
(1 - t) & \text{if } \tau \in (0, t) \\
-t & \text{if } \tau \in (t, 1) 
\end{cases}.$$ 

Finally let, for $\nu \in \mathbb{N}$,

$$v(x) := \min\{1 + \min_{i=3,\ldots,k} \{\langle b_i; x \rangle\}, \frac{1}{\nu} h(\nu \langle b; x \rangle)\}.$$ 

We claim that, by choosing $\nu$ sufficiently large, the function $v$ has all the desired properties in (10.6), with the exception of $|v(x)| \leq \epsilon$.

Note that, since $h \geq 0$ and by definition of $G$,

$$v(x) \geq 0, \ x \in G.$$ 

Since at a boundary point of $G$, either the first term in the minimum in $v(x)$ vanishes or $|\langle b; x \rangle| = 1$ (recall that $b_1 = b = -b_2$), which implies

$$h(\nu \langle b; x \rangle) = 0;$$

thus

$$v(x) = 0, \ x \in \partial G.$$ 

It is also clear that

$$\nabla v(x) \in \{(1 - t)b, -tb, b_3, \ldots, b_k\}, \ a.e. \ in \ G.$$
So it remains to show that, by choosing \( \nu \) sufficiently large, the sets

\[
\begin{align*}
\Omega_A &:= \{ x \in G : \nabla v(x) = (1 - t) b \} \\
\Omega_B &:= \{ x \in G : \nabla v(x) = -tb \}
\end{align*}
\]
satisfy the desired estimates on their measures. Indeed note that, for every \( s \in [0, 1] \), we have \((1 - s) G \subset G\), and

\[
1 + \min_{i=3, \ldots, k} \{ \langle b_i; x \rangle \} \geq s, \quad \forall x \in (1 - s) G.
\]

Therefore by choosing \( \nu \) sufficiently large, \( \nu > 1/s \), we have that

\[
v(x) = \frac{1}{\nu} \left( \nu \langle b; x \rangle \right), \quad \forall x \in (1 - s) G.
\]

Therefore

\[
\text{meas } \Omega_A \geq (1 - s)^n t \text{ meas } G \text{ and } \text{meas } \Omega_B \geq (1 - s)^n (1 - t) \text{ meas } G.
\]

Thus choosing \( s > 0 \) sufficiently small so that

\[
\epsilon = [1 - (1 - s)^n] \text{ meas } G
\]

we get

\[
t \text{ meas } G - \text{meas } \Omega_A \leq t \epsilon \text{ and } (1 - t) \text{ meas } G - \text{meas } \Omega_B \leq (1 - t) \epsilon \quad (10.7)
\]

which gives one set of inequalities for the measures in (10.6). To prove the second ones, we proceed by contradiction and assume, for example, that

\[
|\text{meas } \Omega_A - t \text{ meas } G| > \epsilon.
\]

Since the first inequality in (10.7) holds, this implies that

\[
\text{meas } \Omega_A - t \text{ meas } G > \epsilon.
\]

Combining this inequality with the second one in (10.7), we get

\[
0 \geq \text{meas } \Omega_A + \text{meas } \Omega_B - \text{meas } G > \epsilon - (1 - t) \epsilon = t \epsilon
\]

which is the desired contradiction. Therefore the inequalities for the measures in (10.6) have been proved.

**Step 3.** The conclusion for general \( \Omega \) follows by using Vitali covering theorem (cf. Corollary 10.6 in [202]), covering \( \Omega \) by dilations and translations of the above set \( G \). More precisely if \( x_0 \in \Omega \) and \( r > 0 \), we consider sets of the form

\[
G(r, x_0) := \left\{ x \in \mathbb{R}^n : r + \min_{i=1, \ldots, k} \{ \langle b_i; x - x_0 \rangle \} > 0 \right\}.
\]
By choosing \( r > 0 \) sufficiently small, we can also ensure that \(|v(x)| \leq \epsilon\). This concludes the proof. □

We finally show that (10.2) is in fact also a necessary condition, at least when \( \varphi \) is affine; for the general case, see Section 2.4 in Dacorogna-Marcellini [202]. For the affine case, the result is in some of the above mentioned articles (notably in Cellina [134] or Friesecke [291]), but we follow here Bandyopadhyay-Barroso-Dacorogna-Matias [68].

**Theorem 10.24** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set, \( E \subset \mathbb{R}^n \), \( \xi_0 \in \mathbb{R}^n \) and \( u \in u_{\xi_0} + W^{1,\infty}_0(\Omega) \) (\( u_{\xi_0} \) being such that \( \nabla u_{\xi_0} = \xi_0 \)) such that

\[
\nabla u(x) \in E \text{ a.e. } x \in \Omega
\]

then

\[
\xi_0 \in E \cup \text{int co } E.
\]

**Proof.** Assume that \( \xi_0 \notin E \), otherwise nothing is to be proved. It is easy to see (see Proposition 2.36) that, by Jensen inequality and since \( \nabla u(x) \in E \),

\[
\xi_0 = \frac{1}{\text{meas } \Omega} \int_{\Omega} \nabla u(x) \, dx \in \overline{\text{co } E}.
\]

Let us show that we cannot have \( \xi_0 \in \partial (\overline{\text{co } E}) \). If we can prove this, we will deduce that \( \xi_0 \in \text{int co } E \). Since \( \text{int co } E = \text{int co } E \) (cf. Theorem 2.6) we will have the result.

If \( \xi_0 \in \partial (\overline{\text{co } E}) \), we find from the separation theorem (see Corollary 2.11) that there exists \( \alpha \in \mathbb{R}^n \), \( \alpha \neq 0 \), such that

\[
\langle \alpha; z - \xi_0 \rangle \geq 0, \quad \forall z \in \overline{\text{co } E}.
\]

We therefore have that

\[
\langle \alpha; \nabla u(x) - \xi_0 \rangle \geq 0, \quad \text{a.e. } x \in \Omega.
\]

Recalling that \( u \in u_{\xi_0} + W^{1,\infty}_0(\Omega) \), we find that

\[
\int_{\Omega} \langle \alpha; \nabla u(x) - \xi_0 \rangle \, dx = 0,
\]

which coupled with the above inequality leads to

\[
\langle \alpha; \nabla u(x) - \xi_0 \rangle = 0, \quad \text{a.e. } x \in \Omega.
\]

Applying Lemma 11.17, we get that \( u \equiv u_{\xi_0} \) and hence \( \xi_0 \in E \), a contradiction with the hypothesis made at the beginning of the proof. Therefore \( \xi_0 \notin \partial (\overline{\text{co } E}) \) as claimed and hence the theorem is proved. □
10.3.2 The case of singular values

The next example, studied by Dacorogna-Ribeiro [212], deals with the singular values case that we encountered in Section 7.4.1.

**Theorem 10.25 (Singular values)** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set, \( \alpha < \beta \) and \( 0 < \gamma_2 \leq \cdots \leq \gamma_n \) be such that

\[
\max \{ |\alpha|, |\beta| \} < \gamma_2 \prod_{i=2}^n \gamma_i.
\]

Let \( \varphi \in \text{Aff}_{\text{piece}}(\overline{\Omega}; \mathbb{R}^n) \) be such that, for almost every \( x \in \Omega \),

\[
\alpha < \det \nabla \varphi(x) < \beta \quad \text{and} \quad \prod_{i=\nu}^n \lambda_i(\nabla \varphi(x)) < \prod_{i=\nu}^n \gamma_i, \quad \nu = 2, \cdots, n.
\]

Then there exists \( u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n) \) such that, for almost every \( x \in \Omega \),

\[
\det \nabla u(x) \in \{ \alpha, \beta \} \quad \text{and} \quad \lambda_\nu(\nabla u(x)) = \gamma_\nu, \quad \nu = 2, \cdots, n.
\]

**Remark 10.26** (i) If \( \alpha = -\beta < 0 \) and if we set \( \gamma_1 = \beta [\prod_{i=2}^n \gamma_i]^{-1} \),

we recover the result of Dacorogna-Marcellini [202], namely that if

\[
\prod_{i=\nu}^n \lambda_i(\nabla \varphi(x)) < \prod_{i=\nu}^n \gamma_i, \quad \nu = 1, \cdots, n,
\]

then there exists \( u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n) \) such that

\[
\lambda_\nu(\nabla u) = \gamma_\nu, \quad \nu = 1, \cdots, n, \quad \text{a.e. in } \Omega.
\]

(ii) If \( \alpha = \beta \neq 0 \), it can also be proved, as in Dacorogna-Tanteri [215], that if

\[
\det \nabla \varphi(x) = \alpha, \quad \prod_{i=\nu}^n \lambda_i(\nabla \varphi(x)) < \prod_{i=\nu}^n \gamma_i, \quad \nu = 2, \cdots, n,
\]

then there exists \( u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n) \) such that

\[
\lambda_\nu(\nabla u) = \gamma_\nu, \quad \nu = 2, \cdots, n \quad \text{and} \quad \det \nabla u = \alpha, \quad \text{a.e. in } \Omega.
\]

**Proof.** We let

\[
E := \{ \xi \in \mathbb{R}^{n \times n} : \det \xi \in \{ \alpha, \beta \}, \lambda_i(\xi) = \gamma_i, \; i = 2, \cdots, n \}
\]

and recall that from Theorem 7.43 we have

\[
\text{Rco } E = \{ \xi \in \mathbb{R}^{n \times n} : \det \xi \in [\alpha, \beta], \prod_{i=\nu}^n \lambda_i(\xi) \leq \prod_{i=\nu}^n \gamma_i, \; \nu = 2, \cdots, n \}
\]
and a similar formula for \( \text{int} \, R^c E \). Since \( \varphi \in \text{Aff}_{\text{piec}}(\Omega; \mathbb{R}^n) \) and \( \nabla \varphi \in \text{int} \, R^c E \), we only need, in order to apply Theorems 10.9 and 10.4 to get the result, to verify that \( E \) and \( R^c E \) have the approximation property (cf. Definition 10.6).

For \( \delta > 0 \) such that \( \gamma_2 - \delta > 0 \) and \( \alpha + \delta < \beta - \delta \), let

\[
E_\delta := \{ \xi \in \mathbb{R}^{n \times n} : \det \xi \in \{\alpha + \delta, \beta - \delta\}, \lambda_i(\xi) = \gamma_i - \delta, \ i = 2, \ldots, n \}.
\]

For a sufficiently small \( \delta \) we have

\[
(\gamma_2 - \delta) \prod_{i=2}^{n} (\gamma_i - \delta) \geq \max\{|\alpha + \delta|, |\beta - \delta|\}
\]

and thus Theorem 7.43 ensures that

\[
R^c E_\delta = \left\{ \xi \in \mathbb{R}^{n \times n} : \det \xi \in [\alpha + \delta, \beta - \delta], \prod_{i=\nu}^{n} \lambda_i(\xi) \leq \prod_{i=\nu}^{n} (\gamma_i - \delta), \ \nu = 2, \ldots, n \right\}.
\]

We have to verify the three conditions of Definition 10.6. The first one is obvious. We next verify the second condition. Since \( \eta \in E_\delta \), we assume that \( \det \eta = \alpha + \delta \), the case \( \det \eta = \beta - \delta \) being handled in an analogous way. The set \( E_\delta \) being left and right \( SO(n) \) invariant, we can assume that

\[
\eta = \text{diag}(\frac{\alpha + \delta}{(\gamma_2 - \delta) \cdots (\gamma_n - \delta)}, \gamma_2 - \delta, \cdots, \gamma_n - \delta).
\]

If we let

\[
\xi = \text{diag}(\frac{\alpha}{\gamma_2 \cdots \gamma_n}, \gamma_2, \cdots, \gamma_n)
\]

we have \( \xi \in E \) and

\[
\text{dist}(\eta; E) \leq \max\{ |\frac{\alpha + \delta}{(\gamma_2 - \delta) \cdots (\gamma_n - \delta)} - \frac{\alpha}{\gamma_2 \cdots \gamma_n} |, \delta \} \to 0, \text{ as } \delta \to 0.
\]

The second condition of Definition 10.6 then follows.

The third condition of the approximation property follows from the continuity of the functions involved in the definition of \( R^c E_\delta \).

We also find the following immediate corollary.

**Corollary 10.27** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and \( \varphi \in \text{Aff}_{\text{piec}}(\Omega; \mathbb{R}^n) \) be such that

\[
|\det \nabla \varphi(x)| < 1 \text{ a.e. } x \in \Omega.
\]

Then there exists \( u \in \varphi + W^{1,\infty}_0(\Omega; \mathbb{R}^n) \) such that, for almost every \( x \in \Omega \),

\[
|\det \nabla u(x)| = 1 \text{ a.e. } x \in \Omega.
\]
10.3.3 The case of potential wells

With the help of Theorem 7.44 and the abstract results of the present chapter, we can prove the following existence theorem. The result was proved by Müller-Sverak [464] using the method of convex integration of Gromov [324] and by Dacorogna-Marcellini in [196] and [202].

**Theorem 10.28** Let $\Omega \subset \mathbb{R}^2$ be a bounded open set,

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix},$$

where $0 < b_1 < a_1 \leq a_2 < b_2$ and $a_1a_2 < b_1b_2$. Let

$$E := SO(2)A \cup SO(2)B$$

and $\xi_0 \in \text{int} \ Rco E$. Denote by $u_{\xi_0}$ an affine map such that $\nabla u_{\xi_0} = \xi_0$. Then there exists $u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^2)$ such that

$$\nabla u(x) \in E \ a.e. \ in \ \Omega.$$

Before proceeding with the proof, let us make some comments on the hypotheses (see Section 7.4.2).

(i) The hypothesis $a_1a_2 < b_1b_2$ guarantees that $\det A \neq \det B$. The case of equality can also be handled but requires a special treatment (see Dacorogna-Tanteri [215] and Müller-Sverak [466]).

(ii) Up to rotations, we can always assume that the matrices $A$ and $B$ are diagonal.

(iii) The hypothesis $0 < b_1 < a_1 \leq a_2 < b_2$ ensures that there exists $R \in SO(2)$ such that

$$\det (RA - B) = 0$$

and guarantees also that $\text{int} \ Rco E \neq \emptyset$.

We now proceed with the proof of the theorem.

**Proof.** We recall that (see Theorem 7.44)

$$Rco E = \left\{ \xi \in \mathbb{R}^{2\times 2} : \xi = \alpha RaA + \beta RbB, \ R_a, R_b \in SO(2), \ 0 \leq \alpha \leq \frac{\det B - \det \xi}{\det B - \det A}, \ 0 \leq \beta \leq \frac{\det \xi - \det A}{\det B - \det A} \right\}$$

and its interior is given by the same formula with strict inequalities on the right hand side.

Moreover,

$$A^\delta = \begin{pmatrix} a_1 - \delta & 0 \\ 0 & a_2 + T\delta \end{pmatrix}, \quad B^\delta = \begin{pmatrix} b_1 + S\delta & 0 \\ 0 & b_2 - \delta \end{pmatrix}$$
both belong to \( \text{int} Rco E \), for appropriate \( S, T > 0 \) and for every \( \delta > 0 \) sufficiently small.

Then write
\[
E_\delta := SO(2)A^\delta \cup SO(2)B^\delta.
\]

Note that
\[
\text{Rco } E_\delta \subset \subset \text{int} Rco E,
\]
which is (1), with \( K(E_\delta) = \text{Rco } E_\delta \), in the definition of the approximation property (see Definition 10.6). Since the properties (2) and (3) in Definition 10.6 are also true, we can apply Theorems 10.9 and 10.4 to get the theorem. \( \blacksquare \)

### 10.3.4 The case of a quasiaffine function

The next theorem will use Theorem 7.47.

**Theorem 10.29** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set, \( \alpha < \beta \), \( \Phi : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \) a non-constant quasiaffine function and \( \varphi \in \text{Aff}_{\text{piece}}(\overline{\Omega}; \mathbb{R}^N) \) such that, for almost every \( x \in \Omega \),
\[
\alpha < \Phi(\nabla \varphi(x)) < \beta.
\]

Then there exists \( u \in \varphi + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \) satisfying
\[
\Phi(\nabla u) \in \{ \alpha, \beta \} \text{ a.e. in } \Omega.
\]

**Proof.** By working on each piece where \( \varphi \) is affine, we can assume that \( \varphi \) is affine. By Lemma 7.46, we can find constants \( c^i_j \) such that \( |\partial \varphi^i(x)/\partial x_j| < c^i_j \) and
\[
\inf_{i=1,\cdots,N, \atop j=1,\cdots,n} \{|\Phi(\xi)| : |\xi^i_j| = c^i_j \} > \max\{|\alpha|, |\beta|\}. \tag{10.8}
\]

We then define
\[
E := \{ \xi \in \mathbb{R}^{N \times n} : \Phi(\xi) \in (\alpha, \beta), \ |\xi^i_j| \leq c^i_j, \ i = 1, \cdots, N, \ j = 1, \cdots, n \}.
\]

As before we only need to verify that the sets \( E \) and \( \text{Rco } E \) have the approximation property. Let, for \( \delta > 0 \) sufficiently small,
\[
E_\delta := \left\{ \xi \in \mathbb{R}^{N \times n} : \begin{array}{c}
\Phi(\xi) \in (\alpha + \delta, \beta - \delta), \\
|\xi^i_j| \leq c^i_j - \delta, \ i = 1, \cdots, N, \ j = 1, \cdots, n
\end{array} \right\}.
\]

We first observe that, by continuity, it follows from (10.8) that
\[
\inf_{i=1,\cdots,N, \atop j=1,\cdots,n} \{|\Phi(\xi)| : |\xi^i_j| = c^i_j - \delta \} > \max\{|\alpha + \delta|, |\beta - \delta|\}.
\]
We can then apply Theorem 7.47 to find
\[ \text{Rco } E_\delta = \left\{ \xi \in \mathbb{R}^{N \times n} : \Phi(\xi) \in [\alpha + \delta, \beta - \delta], \quad |\xi^i_j| \leq c^i_j - \delta, \quad i = 1, \ldots, N, \quad j = 1, \ldots, n \right\}. \]

It immediately follows that the first and third conditions of Definition 10.6 are verified. It therefore remains to check the second one.

We proceed by contradiction and assume that there exist \( \epsilon > 0 \) and a sequence \( \eta_\nu \in E_{1/\nu} \) with dist(\( \eta_\nu, E \)) \( > \epsilon \). As \( |(\eta_\nu)^i_j| \leq c^i_j \) we can extract a convergent subsequence, still denoted \( \eta_\nu \), and \( \eta \in E \) so that \( \eta_\nu \to \eta \), which contradicts the fact that dist(\( \eta_\nu, E \)) \( > \epsilon \).

We can therefore invoke Theorems 10.9 and 10.4 to conclude the proof.

10.3.5 A problem of optimal design

We now turn our attention to the problem already considered in Sections 6.6.5 and 7.4.4. The present result will be fully used in Theorem 11.35.

**Theorem 10.30** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set and
\[ E := \left\{ \xi \in \mathbb{R}^{2 \times 2}_s : \det \xi \geq 0 \quad \text{and} \quad \text{trace} \xi \in \{0, 1\} \right\}, \]
where \( \mathbb{R}^{2 \times 2}_s \) denotes the set of \( 2 \times 2 \) symmetric matrices. Let \( \xi_0 \in \mathbb{R}^{2 \times 2}_s \) be such that
\[ \det \xi_0 > 0 \quad \text{and} \quad 0 < \text{trace} \xi_0 < 1 \]
and denote by \( u_{\xi_0} \) an affine map such that \( \nabla u_{\xi_0} = \xi_0 \). Then there exists \( u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^2) \) such that
\[ \nabla u(x) \in E \text{ a.e. in } \Omega. \]

**Proof.** According to Theorem 7.48, we have that
\[ \xi_0 \in \text{int Rco } E \]
and that \( \text{Rco } E = \text{co } E \). So let \( \delta \in (0, 1) \) and let
\[ E_\delta := \delta \xi_0 + (1 - \delta) E. \]
It is easy to see that
\[ \text{Rco } E_\delta = \delta \xi_0 + (1 - \delta) \text{ co } E \subset \subset \text{ int Rco } E \]
which is (1), with \( K(E_\delta) = \text{Rco } E_\delta \), in the definition of the approximation property (see Definition 10.6). Since the properties (2) and (3) in Definition 10.6 are also true, we can apply Theorems 10.9 and 10.4 to get the result.
Chapter 11

Existence of minima for non-quasiconvex integrands

11.1 Introduction

In this chapter, we discuss the existence of minimizers for the problem

\[ (P) \quad \inf \left\{ \int_{\Omega} f(\nabla u(x)) \, dx : u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \right\}, \]

where

- \( \Omega \subset \mathbb{R}^n \) is a bounded open set,
- \( u : \Omega \to \mathbb{R}^N \) and \( \nabla u = \left( \frac{\partial u^i}{\partial x_j} \right) \in \mathbb{R}^{N \times n} \),
- \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) is lower semicontinuous, locally bounded and non-negative,
- \( u_{\xi_0} \) is a given affine map (i.e. \( \nabla u_{\xi_0} = \xi_0 \), where \( \xi_0 \in \mathbb{R}^{N \times n} \) is a fixed matrix).

If the function \( f \) is quasiconvex (recall that in the scalar case \( n = 1 \) or \( N = 1 \), quasiconvexity and ordinary convexity are equivalent), then the problem \( (P) \), trivially, has \( u_{\xi_0} \) as a minimizer.

The aim of the present chapter is to study the case where \( f \) fails to be quasiconvex. The general rule is that the problem has no solution, as already seen even in the simplest case \( N = n = 1 \) in Chapter 4. However, there are still many instances where solutions do exist, although the direct methods do not apply. We now explain how to deal with such problems. The first step is to apply the relaxation theorem (see Chapter 9). It has as a direct consequence (see Theorem 11.1) that \( (P) \) has a solution \( \overline{u} \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \) if and only if

\[ f(\nabla \overline{u}(x)) = Q f(\nabla \overline{u}(x)) \text{ a.e. } x \in \Omega, \]
\[
\int_{\Omega} Qf(\nabla \overline{u}(x)) \, dx = Qf(\xi_0) \, \text{meas } \Omega,
\]

where \(Qf\) is the \textit{quasiconvex envelope} (see Section 6.3) of \(f\), namely

\[
Qf(\xi) := \sup \{g(\xi) : g \leq f \text{ and } g \text{ quasiconvex}\}.
\]

The problem is then to discuss the existence or non-existence of a \(\overline{u}\) satisfying the two equations. The two equations are not really of the same nature. The first one is an \textit{implicit partial differential equation} of the type studied in Chapter 10. The second one is more geometric in nature and has to do with some "quasiaffinity" of the quasiconvex envelope \(Qf\).

The scalar case \((n = 1 \text{ or } N = 1)\) has been intensively studied by many authors including: Aubert-Tahraoui [42], [43], [46], Bauman-Phillips [72], Buttazzo-Ferone-Kawohl [116], Celada-Perrotta [130], [131], Cellina [133], [134], Cellina-Colombo [135], Cesari [141], [143], Cutri [169], Dacorogna [179], Ekeland [262], Friescke [291], Fusco-Marcellini-Ornelas [297], Giachetti-Schianchi [306], Klötzer [369], Marcellini [419], [420], [426], Mascolo [433], Mascolo-Schianchi [437], [438], Monteiro Marques-Ornelas [451], Ornelas [485], Raymond [502], [503], [504], Sychev [558], Tahraoui [564], [565], Treu [580] and Zagatti [610], [611].

The vectorial case has been investigated for some special examples notably by Allaire-Francfort [15], Cellina-Zagatti [138], Dacorogna-Ribeiro [212], Dacorogna-Tanteri [215], Mascolo-Schianchi [436], Müller-Sverak [464], Raymond [505] and Zagatti [612]. A more systematic study was achieved by Dacorogna-Marcellini in [195], [202], [203] and Dacorogna-Pisante-Ribeiro [211]. We will closely follow the survey article of Dacorogna [184], which is based on [195] and [211].

We have, throughout this chapter, made two important restrictions:

- \(f\) does not depend on lower order terms, i.e. \(f(x, u, \xi) = f(\xi)\);
- the boundary datum \(u_0\) is affine, i.e. there exists \(\xi_0 \in \mathbb{R}^{N \times n}\) such that

\[
\nabla u_0 = \xi_0.
\]

In the above literature, some authors have considered either of the two more general cases. The results are then much less general and essentially apply only to the scalar case.

We now briefly describe the content of the chapter. We start by making some abstract considerations on sufficient (in Section 11.2) and necessary (in Section 11.3) conditions. We then apply these abstract results first to the scalar case (see Section 11.4), getting general existence theorems, particularly in the case of single integrals (i.e. \(n = 1\)). In the vectorial case, we investigate several examples that are relevant for applications and that we have already encountered in the previous chapters.
11.2 Sufficient conditions

With the help of the relaxation theorem (see Theorem 9.1) and Theorem 10.4, we are now in a position to discuss some existence results for the problem \((P)\). The following theorem (see [195]) is elementary and gives a necessary and sufficient condition for the existence of minima. It will be crucial in several of our arguments.

**Theorem 11.1** Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set, \(f : \mathbb{R}^{N \times n} \to \mathbb{R}\) a lower semicontinuous, locally bounded and non-negative function, \(\xi_0 \in \mathbb{R}^{N \times n}\) and \(u_{\xi_0}\) be such that \(\nabla u_{\xi_0} = \xi_0\). The problem \((P)\) has a solution if and only if there exists \(u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N)\) such that

\[
\begin{align*}
    f(\nabla u(x)) &= Qf(\nabla u(x)) \text{ a.e. } x \in \Omega, \quad (11.1) \\
    \int_{\Omega} Qf(\nabla u(x)) \, dx &= Qf(\xi_0) \text{ meas } \Omega. \quad (11.2)
\end{align*}
\]

**Proof.** By the relaxation theorem (see Theorem 9.1) and since \(u_{\xi_0}\) is affine, we have

\[
\inf (P) = \inf (QP) = Qf(\xi_0) \text{ meas } \Omega.
\]

Moreover, since we always have \(f \geq Qf\) and we have a solution of (11.1) satisfying (11.2), we get that \(u\) is a solution of \((P)\). The fact that (11.1) and (11.2) are necessary for the existence of a minimum for \((P)\) follows in the same way. ■

The previous theorem explains why the set

\[
K := \{\xi \in \mathbb{R}^{N \times n} : Qf(\xi) < f(\xi)\}
\]

plays a central role in the existence theorems that follow. In order to ensure (11.1), we will have to consider differential inclusions of the form studied in Chapter 10, namely: find \(\overline{\nabla} \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N)\) such that

\[
\nabla \overline{\nabla}(x) \in \partial K \text{ a.e. } x \in \Omega.
\]

In order to deal with the second condition (11.2), we will have to impose some hypotheses of the type ”\(Qf\) is quasiaffine on \(K\)”.

The main abstract theorem is the following.

**Theorem 11.2** Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set, \(\xi_0 \in \mathbb{R}^{N \times n}\), \(f : \mathbb{R}^{N \times n} \to \mathbb{R}\) a lower semicontinuous, locally bounded and non-negative function and let

\[
K := \{\eta \in \mathbb{R}^{N \times n} : Qf(\eta) < f(\eta)\}.
\]

Assume that there exists \(K_0 \subset K\) such that

- \(\xi_0 \in K_0\),
- \(K_0\) is bounded and has the relaxation property (see Definition 10.2) with respect to \(K_0 \cap \partial K\),
• \( Qf \) is quasiaffine on \( K_0 \).

Let \( u_{\xi_0}(x) = \xi_0 x \). Then the problem

\[
(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(\nabla u(x)) \, dx : \ u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \right\}
\]

has a solution \( \bar{u} \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \).

**Remark 11.3** (i) Although this theorem applies to functions \( f \) that take only finite values, it can sometimes be extended to functions \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{+\infty\} \).

(ii) Of course, if \( \xi_0 \notin K \), then \( u_{\xi_0} \) is a minimizer of \((P)\).

(iii) The last hypothesis in the theorem means that

\[
\int_{\Omega} Qf(\xi + \nabla \varphi(x)) \, dx = Qf(\xi) \text{ meas } \Omega
\]

for every \( \xi \in K_0 \) and every \( \varphi \in W^{1,\infty}_0(\Omega; \mathbb{R}^N) \) with

\[
\xi + \nabla \varphi(x) \in K_0 \text{ a.e. in } \Omega.
\]

**Proof.** Since \( \xi_0 \in K_0 \) and \( K_0 \) is bounded and has the relaxation property with respect to \( K_0 \cap \partial K \), we can find, appealing to Theorem 10.4, a map \( \bar{u} \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \) satisfying

\[
\nabla \bar{u} \in K_0 \cap \partial K, \text{ a.e. in } \Omega,
\]

which means that (11.1) of Theorem 11.1 is satisfied. Moreover, since \( Qf \) is quasiaffine on \( K_0 \), we have that (11.2) of Theorem 11.1 holds and thus the claim. \( \blacksquare \)

The second hypothesis in the theorem is clearly the most difficult to verify; nevertheless, there are some cases when it is automatically satisfied. For example, if \( K \) is bounded, we can take \( K_0 = K \) (see Corollary 11.8).

We will see that, in many applications, the set \( K \) turns out to be unbounded and in order to apply Theorem 11.2 we need to find some weaker conditions on \( K \) that guarantee the existence of a subset \( K_0 \) of \( K \) satisfying the requested properties. With this aim in mind, we give the following notation and definitions.

**Notation 11.4** Let \( K \subset \mathbb{R}^{N \times n} \) be open and \( \lambda \in \mathbb{R}^{N \times n} \).

(i) For \( \xi \in K \), we denote by \( L_K(\xi, \lambda) \) the largest segment of the form \( [\xi + t\lambda, \xi + s\lambda] \), \( t < 0 < s \), such that \( (\xi + t\lambda, \xi + s\lambda) \subset K \).

(ii) If \( L_K(\xi, \lambda) \) is bounded, we denote by \( t_-(\xi) < 0 < t_+ (\xi) \) the elements such that \( L_K(\xi, \lambda) = [\xi + t_-\lambda, \xi + t_+\lambda] \). They therefore satisfy

\[
\xi + t_\pm \lambda \in \partial K \quad \text{and} \quad \xi + t\lambda \in K, \forall t \in (t_-, t_+).
\]

(iii) If \( H \subset K \), we let

\[
L_K(H, \lambda) := \bigcup_{\xi \in H} L_K(\xi, \lambda).
\]

\( \blacksquare \)
Definition 11.5 Let $K \subset \mathbb{R}^{N \times n}$ be open, $\xi_0 \in K$ and $\lambda \in \mathbb{R}^{N \times n}$.

(i) We say that $K$ is bounded at $\xi_0$ in the direction $\lambda$ if $L_K(\xi_0, \lambda)$ is bounded.

(ii) We say that $K$ is stably bounded at $\xi_0$ in the rank one direction $\lambda = \alpha \otimes \beta$ (with $\alpha \in \mathbb{R}^N$ and $\beta \in \mathbb{R}^n$) if there exists $\epsilon > 0$ such that $L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda)$ is bounded, where we have denoted

$$\xi_0 + \alpha \otimes B_\epsilon := \{\xi \in \mathbb{R}^{N \times n} : \xi = \xi_0 + \alpha \otimes b \text{ with } |b| < \epsilon\}.$$

Clearly a bounded open set $K$ is bounded at every point $\xi \in K$ in any direction $\lambda$ and consequently it is also stably bounded.

We now give an example of a globally unbounded set that is bounded in certain directions.

Example 11.6 Let $N = n = 2$, $\xi_0 \in \mathbb{R}^{2 \times 2}$, $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \det \xi_0 < \beta$ and

$$K = \{\xi \in \mathbb{R}^{2 \times 2} : \alpha < \det \xi < \beta\}.$$

The set $K$ is clearly unbounded.

(i) If $\xi_0 = I$, then $K$ is bounded, and even stably bounded, at $\xi_0$, in a direction of rank one, for example with

$$\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \lambda = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) However, if $\xi_0 = 0$, then $K$ is unbounded in any rank one direction but is bounded in any rank two direction.

In the following result, we deal with sets $K$ that are bounded in a rank one direction only. This corollary says, roughly speaking, that if $K$ is bounded at $\xi_0$ in a rank one direction $\lambda$ and this boundedness (in the same direction) is preserved under small perturbations of $\xi_0$ along rank one $\lambda$-compatible directions, then we can ensure the relaxation property required in the main existence theorem.

Corollary 11.7 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ a lower semicontinuous, locally bounded and non-negative function and let $\xi_0 \in K$ where

$$K := \{\xi \in \mathbb{R}^{N \times n} : Qf(\xi) < f(\xi)\}.$$

If there exists a rank one direction $\lambda \in \mathbb{R}^{N \times n}$ (meaning that $\lambda = \alpha \otimes \beta$ with $\alpha \in \mathbb{R}^N$ and $\beta \in \mathbb{R}^n$) such that

(i) $K$ is stably bounded at $\xi_0$ in the direction $\lambda$,

(ii) $Qf$ is quasiaffine on the set $L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda),$

then the problem

$$\inf \left\{ I(u) = \int_{\Omega} f(\nabla u(x)) \, dx : u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \right\}$$

has a solution $\bar{u} \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N).$
The following corollary is strictly contained in the previous one but, since it takes a much simpler form, we state it now.

**Corollary 11.8** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ a lower semicontinuous, locally bounded and non-negative function and let $\xi_0 \in K$ where $$K := \{ \xi \in \mathbb{R}^{N \times n} : Qf(\xi) < f(\xi) \}.$$ If the connected component of $K$ containing $\xi_0$ is bounded and if $Qf$ is quasi-affine on this connected component, then the problem $$\inf \left\{ I(u) = \int_{\Omega} f(\nabla u(x)) \, dx : u \in u_{\xi_0} + W^{1,\infty}(\Omega; \mathbb{R}^N) \right\}$$ has a solution $\bar{u} \in u_{\xi_0} + W^{1,\infty}(\Omega; \mathbb{R}^N)$.

We now proceed with the proof of Corollary 11.7.

**Proof.** We divide the proof into two steps.

**Step 1.** Assume that $|\beta| = 1$, otherwise replace it by $\beta/|\beta|$, and let $\beta_k \in \mathbb{R}^n$, $k \geq n + 2$, with $|\beta_k| = 1$, be such that $$0 \in H := \text{int} \, \text{co} \{ \beta, -\beta, \beta_3, \cdots, \beta_k \} \subset B_1(0) := \{ x \in \mathbb{R}^n : |x| < 1 \}.$$ Since $K$ is stably bounded at $\xi_0$, we can find $\epsilon > 0$ so that $L_K(\xi_0 + \alpha \otimes \mathbf{B}_\epsilon, \lambda)$ is bounded. Define then $$K_0 := (\xi_0 + \alpha \otimes \epsilon H) \cup \left[ \partial K \cap L_K(\xi_0 + \alpha \otimes \epsilon H, \lambda) \right].$$ We therefore have that $\xi_0 \in K_0$ and, by hypothesis, that $K_0$ is bounded, since $$K_0 \subset \overline{K_0} \subset L_K(\xi_0 + \alpha \otimes \mathbf{B}_\epsilon, \lambda).$$ We furthermore have $$\overline{K_0} \cap \partial K = \partial K \cap L_K(\xi_0 + \alpha \otimes \epsilon H, \lambda).$$

In order to deduce the corollary from Theorem 11.2, we only need to show that $K_0$ has the relaxation property (cf. Definition 10.2) with respect to $\overline{K_0} \cap \partial K$. This is achieved in the next step.

**Step 2.** We now prove that $K_0$ has the relaxation property with respect to $\overline{K_0} \cap \partial K$. Let $\xi \in K_0$ and let us find a sequence $u_\nu \in \text{Aff} \, \text{piece} (\overline{\Omega}; \mathbb{R}^N)$ so that $$u_\nu \in u_\xi + W^{1,\infty}(\Omega; \mathbb{R}^N), \quad \nabla u_\nu(x) \in (\overline{K_0} \cap \partial K) \cup K_0, \text{ a.e. in } \Omega,$$ $$u_\nu \rightharpoonup^* u_\xi \text{ in } W^{1,\infty}, \quad \int_{\Omega} \text{dist} (\nabla u_\nu(x); \overline{K_0} \cap \partial K) \, dx \to 0 \text{ as } \nu \to \infty. \quad (11.3)$$ If $\xi \in \partial K \cap L_K(\xi_0 + \alpha \otimes \epsilon H, \lambda)$, nothing is to be proved; so we assume that $\xi \in \xi_0 + \alpha \otimes \epsilon H$. By hypothesis (i), we can find $t_-(\xi) < 0 < t_+(\xi)$ so that $$\xi_\pm := \xi + t_\pm \lambda \in \partial K \quad \text{and} \quad \xi + t\lambda \in K, \quad \forall \, t \in (t_-, t_+)$$
Sufficient conditions

and hence \( \xi_{\pm} \in \overline{K}_0 \cap \partial K \). We moreover have that

\[
\xi = \frac{-t_-}{t_+ - t_-} \xi_+ + \frac{t_+}{t_+ - t_-} \xi_- \quad \text{with} \quad \xi_{\pm} \in \overline{K}_0 \cap \partial K. \tag{11.4}
\]

Furthermore, since \( \xi \in \xi_0 + \alpha \otimes \epsilon H \), we can find \( \gamma \in \epsilon H \) such that

\[
\xi = \xi_0 + \alpha \otimes \gamma. \tag{11.5}
\]

The set \( H \) being open we have that \( \overline{B}_\delta(\gamma) \subset \epsilon H \), for every sufficiently small \( \delta > 0 \). Moreover since for every \( \delta > 0 \), we have

\[
0 \in \delta H = \text{int co}\{\pm \delta \beta, \delta \beta_3, \cdots, \delta \beta_k\}
\]

and since for every sufficiently small \( \delta > 0 \), we have

\[
\pm \delta \beta \in \text{co}\{\pm (t_+ - t_-) \beta\} \subset \text{co}\{\pm (t_+ - t_-) \beta, \delta \beta_3, \cdots, \delta \beta_k\},
\]

we get that

\[
0 \in \delta H = \text{int co}\{\pm \delta \beta, \delta \beta_3, \cdots, \delta \beta_k\} \subset \text{int co}\{\pm (t_+ - t_-) \beta, \delta \beta_3, \cdots, \delta \beta_k\}.
\]

We are therefore in a position to apply Corollary 10.23 to

\[
a = \alpha, \ b = (t_+ - t_-) \beta, \ b_j = \delta \beta_j \quad \text{for} \quad j = 3, \cdots, k, \ t = \frac{-t_-}{t_+ - t_-},
\]

\[
A = \xi_+ = \xi + \frac{t_+}{t_+ - t_-} \alpha \otimes (t_+ - t_-) \beta = \xi + (1 - t) a \otimes b,
\]

\[
B = \xi_- = \xi + \frac{t_-}{t_+ - t_-} \alpha \otimes (t_+ - t_-) \beta = \xi - ta \otimes b
\]

and find \( u_\delta \in \text{Aff}_{\text{piec}}(\overline{\Omega}; \mathbb{R}^N) \), disjoint open sets \( \Omega_+, \Omega_- \subset \Omega \), such that

\[
\begin{cases}
0 \leq \text{meas} \ \Omega - \text{meas} \ (\Omega_+ \cup \Omega_-) \leq \delta \\
u_\delta(x) = u_\xi(x), \ x \in \partial \Omega \quad \text{and} \quad |u_\delta(x) - u_\xi(x)| \leq \delta, \ x \in \Omega \\
\nabla u_\delta(x) = \xi_\pm \quad \text{a.e. in} \ \Omega_\pm \\
\n\nabla u_\delta(x) \in \xi + \{t_+ \alpha \otimes \beta, t_- \alpha \otimes \beta, \alpha \otimes \delta \beta_3, \cdots, \alpha \otimes \delta \beta_k\}, \ \text{a.e. in} \ \Omega.
\end{cases}
\]

Since \( \xi_{\pm} \in \overline{K}_0 \cap \partial K \) and

\[
\xi + \alpha \otimes \delta \beta_j \in \xi + \alpha \otimes \delta H = \xi_0 + \alpha \otimes (\gamma + \delta H) \subset \xi_0 + \alpha \otimes \epsilon H \subset K_0 \quad \text{for} \quad j = 3, \cdots, k,
\]

we deduce, by choosing \( \delta = 1/\nu \) as \( \nu \to \infty \) in (11.5), the relaxation property (11.4). This achieves the proof of Step 2 and thus of the corollary.

We finally want to point out that, as a particular case of Corollary 11.7, we find the existence theorem (Theorem 3.1) proved by Dacorogna-Marcellini in [195].
11.3 Necessary conditions

Recall that we are considering the minimization problem

\[ (P) \inf \left\{ I(u) = \int_{\Omega} f(\nabla u(x)) \, dx : u \in u_{\xi_0} + W_{0}^{1,\infty}(\Omega; \mathbb{R}^N) \right\}, \]

where \( \Omega \) is a bounded open set of \( \mathbb{R}^n \), \( u_{\xi_0} \) is affine (i.e. \( \nabla u_{\xi_0} = \xi_0 \)) and \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) is a lower semicontinuous, locally bounded and non-negative function. In order to avoid the trivial case, we always assume that

\[ Qf(\xi_0) < f(\xi_0). \]

Most non-existence results for problem \((P)\) follow by showing that the relaxed problem \((QP)\) has a unique solution, namely \(u_{\xi_0}\), which by hypothesis is not a solution of \((P)\). This approach was strongly used in Marcellini [420], Dacorogna-Marcellini [195] and Dacorogna-Pisante-Ribeiro [211]. We should point out that we will give an example (see Proposition 11.38 in Section 11.5.5) related to area type integrands, where non-existence occurs, while the relaxed problem has infinitely many solutions, none of them being a solution of \((P)\).

The right notion in order to have uniqueness of the relaxed problem is the following.

**Definition 11.9** A quasiconvex function \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) is said to be strictly quasiconvex at \( \xi_0 \in \mathbb{R}^{N \times n} \) if for some bounded open set \( U \subset \mathbb{R}^n \) the equality

\[ \int_{U} f(\xi_0 + \nabla \varphi(x)) \, dx = f(\xi_0) \, \text{meas} \, U \]

holds for some \( \varphi \in W_{0}^{1,\infty}(U; \mathbb{R}^N) \), then necessarily \( \varphi \equiv 0 \).

We should observe that as in Proposition 5.11 the notion of strict quasiconvexity is independent of the choice of the set \( U \), more precisely we have the following.

**Proposition 11.10** If a function \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) is strictly quasiconvex at \( \xi_0 \in \mathbb{R}^{N \times n} \) for one bounded open set \( U \subset \mathbb{R}^n \), it is so for any such set.

**Proof.** Let \( V \subset \mathbb{R}^n \) be a bounded open set and \( \psi \in W_{0}^{1,\infty}(V; \mathbb{R}^N) \) be such that

\[ \int_{V} f(\xi_0 + \nabla \psi(x)) \, dx = f(\xi_0) \, \text{meas} \, V \]  \hspace{1cm} (11.6)

and let us conclude that we necessarily have \( \psi \equiv 0 \).

Choose first \( a > 0 \) sufficiently large so that

\[ V \subset Q_a = (-a, a)^n \]

and then define

\[ v(x) := \begin{cases} \psi(x) & \text{if } x \in V \\ 0 & \text{if } x \in Q_a - V \end{cases} \]
so that \( v \in W^{1,\infty}_0(Q_a; \mathbb{R}^N) \).

Let then \( x_0 \in U \) and choose \( \nu \) sufficiently large so that
\[
x_0 + \frac{1}{\nu} Q_a = x_0 + \left(-\frac{a}{\nu}, \frac{a}{\nu}\right)^n \subset U.
\]

Define next
\[
\varphi(x) := \begin{cases} \frac{1}{\nu} v(\nu(x-x_0)) & \text{if } x \in x_0 + \frac{1}{\nu} Q_a \\ 0 & \text{if } x \in U - [x_0 + \frac{1}{\nu} Q_a]. \end{cases}
\]

Observe that \( \varphi \in W^{1,\infty}_0(U; \mathbb{R}^N) \) and
\[
\int_U f(\xi_0 + \nabla \varphi(x)) \, dx = f(\xi_0) \, \text{meas}(U - [x_0 + \frac{1}{\nu} Q_a])
\]
\[
+ \int_{x_0 + \frac{1}{\nu} Q_a} f(\xi_0 + \nabla v(\nu(x-x_0))) \, dx
\]
\[
= f(\xi_0) \left[ \text{meas} U - \frac{\text{meas} Q_a}{\nu^n} \right] + \frac{1}{\nu^n} \int_{Q_a} f(\xi_0 + \nabla v(y)) \, dy
\]
\[
= f(\xi_0) \left[ \text{meas} U - \frac{\text{meas} Q_a}{\nu^n} + \frac{\text{meas}(Q_a \setminus V)}{\nu^n} \right]
\]
\[
+ \frac{1}{\nu^n} \int_V f(\xi_0 + \nabla \psi(y)) \, dy.
\]

Appealing to (11.6), we deduce that
\[
\int_U f(\xi_0 + \nabla \varphi(x)) \, dx = f(\xi_0) \, \text{meas} U.
\]

Since \( f \) is strictly quasiconvex at \( \xi_0 \in \mathbb{R}^{N_n} \) for the set \( U \), we deduce that \( \varphi \equiv 0 \), which in turn implies that
\[
v(y) \equiv 0, \text{ for every } y \in Q_a.
\]

This finally implies that \( \psi \equiv 0 \) as claimed. \( \blacksquare \)

We will see below some sufficient conditions that can ensure strict quasiconvexity, but let us start with the elementary following non-existence theorem.

**Theorem 11.11** Let \( f : \mathbb{R}^{N_n} \to \mathbb{R} \) be lower semicontinuous, locally bounded and non-negative, \( \xi_0 \in \mathbb{R}^{N_n} \) with \( Qf(\xi_0) < f(\xi_0) \) and \( Qf \) be strictly quasiconvex at \( \xi_0 \). Then the relaxed problem \((QP)\) has a unique solution, namely \( u_{\xi_0} \), while \((P)\) has no solution.

**Proof.** The fact that \((QP)\) has only one solution follows by definition of the strict quasiconvexity of \( Qf \) and Proposition 11.10. Assume for the sake of contradiction that \((P)\) has a solution \( \overline{u} \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \). We should have from Theorem 11.1 that (writing \( \overline{u}(x) = \xi_0 x + \varphi(x) \))
\[
f(\xi_0 + \nabla \varphi(x)) = Qf(\xi_0 + \nabla \varphi(x)), \text{ a.e. } x \in \Omega
\]
\[ \int_{\Omega} Qf (\xi_0 + \nabla \varphi (x)) \, dx = Qf (\xi_0) \text{ meas } \Omega. \]

Since \( Qf \) is strictly quasiconvex at \( \xi_0 \), we deduce from the last identity that \( \varphi \equiv 0 \). Hence we have, from the first identity, that \( Qf (\xi_0) = f (\xi_0) \), which is in contradiction with the hypothesis. \( \blacksquare \)

We now want to give some criteria that can ensure the strict quasiconvexity of a given function. The first one was introduced by Dacorogna-Marcellini in [195].

**Definition 11.12** A convex function \( f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \) is said to be strictly convex at \( \xi_0 \in \mathbb{R}^{N \times n} \) in at least \( N \) directions if there exist \( \alpha^i \in \mathbb{R}^n \), \( \alpha^i \neq 0 \) for every \( i = 1, \cdots, N \), such that: if for some \( \eta \in \mathbb{R}^{N \times n} \) the identity

\[
\frac{1}{2} f (\xi_0 + \eta) + \frac{1}{2} f (\xi_0) = f (\xi_0 + \frac{1}{2} \eta)
\]

holds, then necessarily

\[ \langle \alpha^i; \eta^i \rangle = 0, \quad i = 1, \cdots, N. \]

In order to better understand the generalization of this notion to polyconvex functions (see Proposition 11.18), it might be enlightening to state the definition in the following way.

**Proposition 11.13** Let \( f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \) be a convex function and, for \( \xi \in \mathbb{R}^{N \times n} \), denote by \( \partial f (\xi) \) the subdifferential of \( f \) at \( \xi \). The following two conditions are then equivalent:

(i) \( f \) is strictly convex at \( \xi_0 \in \mathbb{R}^{N \times n} \) in at least \( N \) directions;

(ii) there exist \( \alpha^i \in \mathbb{R}^n \) with \( \alpha^i \neq 0 \) for every \( i = 1, \cdots, N \), so that whenever

\[ f (\xi_0 + \eta) - f (\xi_0) - \langle \lambda; \eta \rangle = 0 \]

for some \( \eta \in \mathbb{R}^{N \times n} \) and for some \( \lambda \in \partial f (\xi_0) \), then

\[ \langle \alpha^i; \eta^i \rangle = 0, \quad i = 1, \cdots, N. \]

**Proof.** Step 1. We start with a preliminary observation that if

\[ \frac{1}{2} f (\xi_0 + \eta) + \frac{1}{2} f (\xi_0) = f (\xi_0 + \frac{1}{2} \eta), \quad (11.7) \]

then, for every \( t \in [0, 1] \), we have

\[ tf (\xi_0 + \eta) + (1 - t) f (\xi_0) = f (\xi_0 + t \eta). \quad (11.8) \]

Let us show this under the assumption that \( t > 1/2 \) (the case \( t < 1/2 \) is handled similarly). We can therefore find \( \alpha \in (0, 1) \) such that

\[ \frac{1}{2} = \alpha t + (1 - \alpha) 0 = \alpha t. \]
From the convexity of $f$ and by hypothesis, we obtain

$$\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) = f(\xi_0 + \frac{1}{2}\eta) \leq \alpha f(\xi_0 + t\eta) + (1 - \alpha) f(\xi_0).$$

Assume, for the sake of contradiction, that

$$f(\xi_0 + t\eta) < tf(\xi_0 + \eta) + (1 - t) f(\xi_0).$$

Then combine this inequality with the previous one to get

$$\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) < \alpha [tf(\xi_0 + \eta) + (1 - t) f(\xi_0)] + (1 - \alpha) f(\xi_0) = \frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0),$$

which is clearly a contradiction. Therefore the convexity of $f$ and the above contradiction imply (11.8) and also that

$$f'(\xi_0, \eta) := \lim_{t \to 0^+} \frac{f(\xi_0 + t\eta) - f(\xi_0)}{t} = f(\xi_0 + \eta) - f(\xi_0).$$

Applying Theorem 2.50, we get that there exists $\lambda \in \partial f(\xi_0)$ such that

$$f(\xi_0 + \eta) - f(\xi_0) - \langle \lambda; \eta \rangle = 0, \forall t \in [0, 1]. \quad (11.9)$$

We have therefore proved that (11.7) implies (11.9). Since the converse is obviously true, we conclude that they are equivalent.

Step 2. Let us show the equivalence of the two conditions.

(i) $\Rightarrow$ (ii). We first observe that, for any $\mu \in \mathbb{R}^{N \times n}$, we have

$$\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) - f(\xi_0 + \frac{1}{2}\eta)$$

$$= \frac{1}{2} [f(\xi_0 + \eta) - f(\xi_0) - \langle \mu; \eta \rangle] - [f(\xi_0 + \frac{1}{2}\eta) - f(\xi_0) - \frac{1}{2} \langle \mu; \eta \rangle]. \quad (11.10)$$

Assume that, for $\lambda \in \partial f(\xi_0)$, we have

$$f(\xi_0 + \eta) - f(\xi_0) - \langle \lambda; \eta \rangle = 0.$$

From (11.10) applied to $\mu = \lambda$, from the definition of $\partial f(\xi_0)$ and from the convexity of $f$, we get

$$0 \leq \frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) - f(\xi_0 + \frac{1}{2}\eta)$$

$$= -[f(\xi_0 + \frac{1}{2}\eta) - f(\xi_0) - \frac{1}{2} \langle \lambda; \eta \rangle] \leq 0.$$

Using the above identity, we then are in the framework of (i) and we deduce that $\langle \alpha^i; \eta^i \rangle = 0$, $i = 1, \cdots, N$, and thus (ii).
(ii) ⇒ (i). Assume now that we have (11.7), namely
\[
\frac{1}{2}f(\xi_0 + \eta) + \frac{1}{2}f(\xi_0) - f(\xi_0 + \frac{1}{2}\eta) = 0
\]
which, by Step 1, implies that there exists \(\lambda \in \partial f(\xi_0)\) such that
\[
f(\xi_0 + t\eta) - f(\xi_0) - t\langle \lambda; \eta \rangle = 0, \quad \forall t \in [0,1].
\]
We are therefore choosing \(t = 1\) in the framework of (ii) and we get
\[
\langle \alpha^i; \eta^i \rangle = 0, \quad i = 1, \cdots, N,
\]
as wished.

Of course any strictly convex function is strictly convex in at least \(N\) directions, but the above condition is much weaker. For example, in the scalar case, \(N = 1\), it is enough that the function is not affine in a neighborhood of \(\xi_0\) to guarantee the condition (see below).

We now have the following result established by Dacorogna-Marcellini in [195].

**Proposition 11.14** If a convex function \(f : \mathbb{R}^{N \times n} \to \mathbb{R}\) is strictly convex at \(\xi_0 \in \mathbb{R}^{N \times n}\) in at least \(N\) directions, then it is strictly quasiconvex at \(\xi_0\).

Theorem 11.11, combined with the above proposition, immediately gives a sharp result for the scalar case, namely the following corollary.

**Corollary 11.15** Let \(f : \mathbb{R}^n \to \mathbb{R}\) be lower semicontinuous, locally bounded and non-negative, \(\xi_0 \in \mathbb{R}^n\) with \(Cf(\xi_0) < f(\xi_0)\) and \(Cf\) not affine in a neighborhood of \(\xi_0\). Then \((P)\) has no solution.

**Remark 11.16** In the scalar case this result has been obtained by several authors, in particular Cellina [133], Friesecke [291] and Dacorogna-Marcellini [195]. It also gives (see Theorem 11.26), combined with the result of the preceding section, that, provided some appropriate boundedness is assumed, a necessary and sufficient condition for the existence of minima for \((P)\) is that \(f\) be affine on the connected component of \(\{\xi : Cf(\xi) < f(\xi)\}\) that contains \(\xi_0\).

Before proceeding with the proof of Proposition 11.14, we need the following elementary lemma.

**Lemma 11.17** Let \(\Omega\) be a bounded open set of \(\mathbb{R}^n\) and \(\varphi \in W^{1,\infty}_0(\Omega; \mathbb{R}^N)\) be such that
\[
\langle \alpha^i; \nabla \varphi^i(x) \rangle = 0 \text{ a.e. } x \in \Omega, \quad i = 1, \cdots, N,
\]
for some \(\alpha^i \neq 0, i = 1, \cdots, N\), then \(\varphi \equiv 0\).

**Proof.** (Lemma 11.17). Working component by component, we can assume that \(N = 1\) and therefore we will drop the indices. So let \(\varphi \in W^{1,\infty}_0(\Omega)\) satisfy, for some \(\alpha \in \mathbb{R}^n, \alpha \neq 0,\)
\[
\langle \alpha; \nabla \varphi(x) \rangle = 0 \text{ a.e. } x \in \Omega.
\]
We then choose \( \alpha_2, \ldots, \alpha_n \in \mathbb{R}^n \) such that \( \{\alpha, \alpha_2, \ldots, \alpha_n\} \) generate a basis of \( \mathbb{R}^n \). Let \( a > 0 \) and for \( m \) an integer
\[
Q_a^m := (-a, a)^m.
\]

Let \( x \in \Omega \) and let \( a > 0 \) and \( t > 0 \) be sufficiently small so that
\[
x + \tau \alpha + \tau_2 \alpha_2 + \cdots + \tau_n \alpha_n \in \Omega, \text{ for every } \tau \in (0, t) \text{ and } (\tau_2, \ldots, \tau_n) \in Q_a^{n-1}.
\]

Observe that if \( \varphi \in C_0^1(\Omega) \), then
\[
\int_{Q_a^{n-1}} \left[ \varphi(x + t\alpha + \tau_2 \alpha_2 + \cdots + \tau_n \alpha_n) - \varphi(x + \tau_2 \alpha_2 + \cdots + \tau_n \alpha_n) \right] d\tau_2 \cdots d\tau_n
= \int_{Q_a^{n-1}} \int_0^t \frac{d}{d\tau} \left[ \varphi(x + \tau \alpha + \tau_2 \alpha_2 + \cdots + \tau_n \alpha_n) \right] d\tau d\tau_2 \cdots d\tau_n
= \int_{Q_a^{n-1}} \int_0^t \langle \nabla \varphi(x + \tau \alpha + \tau_2 \alpha_2 + \cdots + \tau_n \alpha_n); \alpha \rangle d\tau d\tau_2 \cdots d\tau_n.
\]

By a standard regularization procedure, the above identity also holds for any \( \varphi \in W^{1,\infty}_0(\Omega) \). Since \( \langle \alpha; \nabla \varphi \rangle = 0 \), we deduce that
\[
\int_{Q_a^{n-1}} \left[ \varphi(x + t\alpha + \tau_2 \alpha_2 + \cdots + \tau_n \alpha_n) - \varphi(x + \tau_2 \alpha_2 + \cdots + \tau_n \alpha_n) \right] d\tau_2 \cdots d\tau_n = 0.
\]

Since \( \varphi \) is continuous, we deduce, by dividing by the measure of \( Q_a^{n-1} \) and letting \( a \to 0 \), that, for every \( t \) sufficiently small so that \( x + t\alpha \in \Omega \),
\[
\varphi(x + t\alpha) = \varphi(x).
\]

Choosing \( t \) so that
\[
x + \tau \alpha \in \Omega, \ \forall \tau \in [0, t) \quad \text{and} \quad x + t\alpha \in \partial \Omega,
\]
we obtain the claim, namely
\[
\varphi(x) = 0, \ \forall x \in \Omega.
\]

This achieves the proof of the lemma. \( \blacksquare \)

**Proof.** (Proposition 11.14). Assume that for a certain bounded open set \( U \subset \mathbb{R}^n \) and for some \( \varphi \in W^{1,\infty}_0(U; \mathbb{R}^N) \) we have
\[
\int_U f(\xi_0 + \nabla \varphi(x)) \, dx = f(\xi_0) \, \text{meas} \, U
\]
and let us show that \( \varphi \equiv 0 \).

Since \( f \) is convex and the above identity holds, we find
\[
f(\xi_0) \, \text{meas} \, U = \int_U \left[ \frac{1}{2} f(\xi_0) + \frac{1}{2} f(\xi_0 + \nabla \varphi(x)) \right] dx
\geq \int_U f(\xi_0 + \frac{1}{2} \nabla \varphi(x)) dx \geq f(\xi_0) \, \text{meas} \, U,
\]
which implies that
\[ \int_U \left[ \frac{1}{2} f(\xi_0) + \frac{1}{2} f(\xi_0 + \nabla \varphi(x)) - f(\xi_0 + \frac{1}{2} \nabla \varphi(x)) \right] dx = 0. \]
The convexity of \( f \) implies then that, for almost every \( x \) in \( U \), we have
\[ \frac{1}{2} f(\xi_0) + \frac{1}{2} f(\xi_0 + \nabla \varphi(x)) - f(\xi_0 + \frac{1}{2} \nabla \varphi(x)) = 0. \]
The strict convexity in at least \( N \) directions leads to
\[ \langle \alpha_i; \nabla \varphi^i(x) \rangle = 0, \text{ a.e. } x \in \Omega, \ i = 1, \cdots, N. \]

Lemma 11.17 gives the claim.

We now generalize Proposition 11.14. Since the notation in the next result is involved, we first write the proposition when \( N = n = 2 \). We also adopt the notation of Definition 5.1.

**Proposition 11.18**  
Let \( f : \mathbb{R}^{N \times n} \rightarrow \mathbb{R} \) be polyconvex, \( \xi_0 \in \mathbb{R}^{N \times n} \) and \( \lambda = \lambda(\xi_0) \in \mathbb{R}^{\tau(n,N)} \) such that
\[ f(\xi_0 + \eta) - f(\xi_0) - \langle \lambda; T(\xi_0 + \eta) - T(\xi_0) \rangle \geq 0 \text{ for every } \eta \in \mathbb{R}^{N \times n}. \]

(i) Let \( N = n = 2 \) and assume that there exist \( \alpha_1^{1,1}, \alpha_1^{1,2}, \alpha_2^{2,2} \in \mathbb{R}^2, \ \alpha_1^{1,1} \neq 0, \ \alpha_2^{2,2} \neq 0, \ \beta \in \mathbb{R}, \) so that if for some \( \eta \in \mathbb{R}^{2 \times 2} \) the equality
\[ f(\xi_0 + \eta) - f(\xi_0) - \langle \lambda; T(\xi_0 + \eta) - T(\xi_0) \rangle = 0 \]
holds, then necessarily
\[ \langle \alpha_2^{2,2}; \eta^2 \rangle = 0 \quad \text{and} \quad \langle \alpha_1^{1,1}; \eta^1 \rangle + \langle \alpha_1^{1,2}; \eta^2 \rangle + \beta \det \eta = 0. \]

Then \( f \) is strictly quasiconvex at \( \xi_0 \).

(ii) Let \( N, n \geq 2 \) and assume that there exist, for every \( \nu = 1, \cdots, N, \)
\[ \alpha_\nu^{\nu,\nu}, \alpha_\nu^{\nu,\nu+1}, \cdots, \alpha_\nu^{\nu,N} \in \mathbb{R}^n, \ \alpha_\nu^{\nu,\nu} \neq 0, \]
\[ \beta_\nu^s \in \mathbb{R}^{\binom{N-\nu+1}{s} \times \binom{n}{s}}, \ 2 \leq s \leq n \wedge (N - \nu + 1), \]
such that if for some \( \eta \in \mathbb{R}^{N \times n} \) the equality
\[ f(\xi_0 + \eta) - f(\xi_0) - \langle \lambda; T(\xi_0 + \eta) - T(\xi_0) \rangle = 0 \]
holds, then necessarily
\[ \sum_{s=\nu}^{N} \langle \alpha_\nu^{\nu,s}; \eta^s \rangle + \sum_{s=2}^{n \wedge (N-\nu+1)} \langle \beta_\nu^s; \text{adj}_s(\eta^\nu, \cdots, \eta^N) \rangle = 0, \nu = 1, \cdots, N. \]

Then \( f \) is strictly quasiconvex at \( \xi_0 \).
Remark 11.19 (i) The existence of a $\lambda$ as in the hypotheses of the proposition is automatically guaranteed by the polyconvexity of $f$ (see Theorem 5.6, which corresponds in the case of a convex function to an element of $\partial f (\xi_0)$).

(ii) We have adopted the convention that if $l > k > 0$ are integers, then

$$\sum_{l}^{k} = 0.$$  

Example 11.20 Let $N = n = 2$ and consider the function

$$f (\eta) = (\eta_2^2 + (\eta_1^1 + \det \eta)^2).$$

This function is trivially polyconvex and according to the proposition it is also strictly quasiconvex at $\xi_0 = 0$ (choose $\lambda = 0 \in \mathbb{R}^5$, $\alpha^{2,2} = (0,1)$, $\alpha^{1,2} = (0,0)$, $\alpha^{1,1} = (1,0)$, $\beta = 1$).

Proof. We prove the proposition only in the case $N = n = 2$, the general case being handled similarly.

Assume that for a certain bounded open set $U \subset \mathbb{R}^2$ and for some $\varphi \in W^{1,\infty}_0 (U; \mathbb{R}^2)$ we have

$$\int_{U} f (\xi_0 + \nabla \varphi (x)) \, dx = f (\xi_0) \, \text{meas} \, U$$

and let us prove that $\varphi \equiv 0$. This is equivalent, for every $\mu \in \mathbb{R}^{\tau(2,2)}$, to

$$\int_{U} \left[ f (\xi_0 + \nabla \varphi (x)) - f (\xi_0) - \langle \mu; T (\xi_0 + \nabla \varphi (x)) - T (\xi_0) \rangle \right] \, dx = 0.$$  

Choosing $\mu = \lambda$ (as in the statement of the proposition) in the previous equation and using the polyconvexity of the function $f$, we get

$$f (\xi_0 + \nabla \varphi (x)) - f (\xi_0) - \langle \lambda; T (\xi_0 + \nabla \varphi (x)) - T (\xi_0) \rangle = 0, \, \text{a.e.} \, x \in \Omega.$$  

We hence infer that, for almost every $x \in \Omega$, we have

$$\langle \alpha^{2,2}; \nabla \varphi \rangle = 0 \, \text{ and } \, \langle \alpha^{1,1}; \nabla \varphi \rangle + \langle \alpha^{1,2}; \nabla \varphi \rangle + \beta \det \nabla \varphi = 0.$$  

Lemma 11.17, applied to the first equation, implies that $\varphi^2 \equiv 0$. Using this result in the second equation we get

$$\langle \alpha^{1,1}; \nabla \varphi \rangle = 0$$  

and hence, appealing once more to the lemma, we have the claim, namely $\varphi^1 \equiv 0.$

Summarizing the results of Theorem 11.11, Proposition 11.14 and Proposition 11.18, we get the following corollary.
Corollary 11.21 Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ be lower semicontinuous, locally bounded and non-negative and $\xi_0 \in \mathbb{R}^{N \times n}$ with

$$Qf(\xi_0) < f(\xi_0).$$

If either one of the two conditions

(i) $Qf(\xi_0) = Cf(\xi_0)$ and $Cf$ is strictly convex at $\xi_0$ in at least $N$ directions;

(ii) $Qf(\xi_0) = Pf(\xi_0)$ and $Pf$ is strictly polyconvex at $\xi_0$ (in the sense of Proposition 11.18);

holds, then $(QP)$ has a unique solution, namely $u_{\xi_0}$, while $(P)$ has no solution.

Proof. The proof is almost identical under both hypotheses and so we establish the corollary only in the first case. The result follows from Theorem 11.11 if we can show that $Qf$ is strictly quasiconvex at $\xi_0$. So assume that

$$\int_{\Omega} Qf(\xi_0 + \nabla \varphi(x)) \, dx = Qf(\xi_0) \text{ meas } \Omega$$

for some $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ and let us prove that $\varphi \equiv 0$. Using Jensen inequality combined with the hypothesis $Qf(\xi_0) = Cf(\xi_0)$ and the fact that $Qf \geq Cf$, we find that the above identity implies

$$\int_{\Omega} Cf(\xi_0 + \nabla \varphi(x)) \, dx = Cf(\xi_0) \text{ meas } \Omega.$$ 

The hypotheses on $Cf$ and Proposition 11.14 imply that $\varphi \equiv 0$, as wished. ■

We now conclude this section with a different necessary condition that is based on Carathéodory theorem (see Theorem 2.13).

Recall first that, for any integer $s$, we let

$$\Lambda_s := \{ \lambda = (\lambda_1, \cdots, \lambda_s) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^s \lambda_i = 1 \}. $$

Theorem 11.22 Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ be lower semicontinuous, locally bounded and non-negative. If $(P)$ has a solution $\varphi \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$, then there exist $\mu \in \Lambda_{Nn+1}$ and $\xi_0 \in \mathbb{R}^{N \times n}$, $|\xi_0| \leq \| \varphi \|_{W^{1,\infty}}$, $1 \leq \nu \leq Nn + 1$ such that

$$Qf(\xi_0) \geq \sum_{\nu=1}^{Nn+1} \mu_\nu f(\xi_\nu) \quad \text{and} \quad \xi_0 = \sum_{\nu=1}^{Nn+1} \mu_\nu \xi_\nu.$$ 

Moreover, if either $n = 1$ or $N = 1$, the inequality becomes an equality, namely

$$Cf(\xi_0) = \sum_{\nu=1}^{Nn+1} \mu_\nu f(\xi_\nu) \quad \text{and} \quad \xi_0 = \sum_{\nu=1}^{Nn+1} \mu_\nu \xi_\nu.$$
Remark 11.23 The theorem is just a curiosity in the vectorial case \( n, N > 1 \). However, in the scalar case \( n > N = 1 \), under some extra hypotheses (see Theorem 11.26), one of them being

\[
\xi_0 \in \text{int co} \{\xi_1, \ldots, \xi_{n+1}\},
\]

it turns out that the necessary condition is also sufficient. But it is in the case \( N \geq n = 1 \) that it is particularly interesting since then this condition is also sufficient (see Theorem 11.24). \( \diamond \)

Proof. We decompose the proof into three steps.

Step 1. Let \( \pi \in u_{\xi_0} + W^{1,\infty}_0 (\Omega; \mathbb{R}^N) \) be a solution of \((P)\). It should therefore satisfy

\[
\int_{\Omega} f (\nabla \pi(x)) \, dx = \inf (P) = \inf (QP) = Qf (\xi_0) \text{ meas } \Omega. \tag{11.11}
\]

Let \( r = \|\pi\|_{W^{1,\infty}} \) and use the fact that \( f \) is locally bounded to find \( R = R(r) \) such that

\[
0 \leq f (\nabla \pi(x)) \leq R \text{ a.e. } x \in \Omega.
\]

Denote

\[
K_r := \{ (\xi, y) \in \mathbb{R}^{N \times n} \times \mathbb{R} : |\xi| \leq r \text{ and } |y| \leq R \},
\]

\[
\text{epi } f := \{ (\xi, y) \in \mathbb{R}^{N \times n} \times \mathbb{R} : f (\xi) \leq y \},
\]

\[
E := \text{epi } f \cap K_r.
\]

Note that since \( f \) is lower semicontinuous then \( \text{epi } f \) is closed and hence \( E \) is compact. Therefore its convex hull \( \text{co } E \) is also compact.

Observe that, for almost every \( x \in \Omega \), we have

\[
(\nabla \pi(x), f (\nabla \pi(x))) \in E
\]

and thus by Jensen inequality and (11.11) we deduce that

\[
(\xi_0, Qf (\xi_0)) = \frac{1}{\text{meas } \Omega} \int_{\Omega} (\nabla \pi(x), f (\nabla \pi(x))) \, dx \in \text{co } E.
\]

Appealing to Carathéodory theorem, we can find \( \lambda \in \Lambda_{Nn+2}, (\xi_i, y_i) \in E, 1 \leq i \leq Nn + 2 \) (in particular, \( f (\xi_i) \leq y_i \)) such that

\[
Qf (\xi_0) = \sum_{i=1}^{Nn+2} \lambda_i y_i \geq \sum_{i=1}^{Nn+2} \lambda_i f (\xi_i) \quad \text{and} \quad \xi_0 = \sum_{i=1}^{Nn+2} \lambda_i \xi_i.
\]

(Note, in passing, that if \( f \) is continuous, we can replace \( \text{epi } f \) in the above argument by

\[
\text{graph } f := \{ (\xi, y) \in \mathbb{R}^{N \times n} \times \mathbb{R} : f (\xi) = y \},
\]
Step 2. To obtain Part 1 of the theorem it therefore remains to show that one can take only \((N_n + 1)\) elements. This is a classical procedure in convex analysis and we have encountered it in Theorem 2.35. The result is equivalent to showing that there exist \(\mu_i, 1 \leq i \leq N_n + 2\), such that

\[
\begin{align*}
\mu_i &\geq 0, \quad N_{n+2} \sum_{i=1}^{N_{n+2}} \mu_i = 1, \text{ at least one of the } \mu_i = 0 \\
\sum_{i=1}^{N_{n+2}} \mu_i f(\xi_i) &\leq \sum_{i=1}^{N_{n+2}} \lambda_i f(\xi_i), \quad \xi_0 = \sum_{i=1}^{N_{n+2}} \mu_i \xi_i. \quad (11.12)
\end{align*}
\]

meaning in fact that \(\mu \in \Lambda_{N_{n+1}}\) as wished.

Assume that \(\lambda_i > 0, 1 \leq i \leq N_n + 2\); otherwise nothing is to be proved. Observe first that \(\xi_0 \in \Co\{\xi_1, \ldots, \xi_{N_n+2}\} \subset \mathbb{R}^{N \times n}\). Thus it follows from Carathéodory theorem that there exist \(\nu \in \Lambda_{N_n+2}\) with at least one of the \(\nu_i = 0\) (i.e. \(\nu \in \Lambda_{N_n+1}\)) such that

\[
\xi_0 = \sum_{i=1}^{N_{n+2}} \nu_i \xi_i.
\]

Assume, without loss of generality, that

\[
\sum_{i=1}^{N_{n+2}} \nu_i f(\xi_i) > \sum_{i=1}^{N_{n+2}} \lambda_i f(\xi_i); \quad (11.13)
\]

otherwise, choosing \(\mu_i = \nu_i\) we would immediately have (11.12). Let

\[
J := \{ i \in \{1, \ldots, N_n + 2\} : \lambda_i - \nu_i < 0 \}.
\]

Observe that \(J \neq \emptyset\), since otherwise \(\lambda_i \geq \nu_i \geq 0\) for every \(i\) and since at least one of the \(\nu_i = 0\), we would have a contradiction with \(\sum \nu_i = \sum \lambda_i = 1\) and \(\lambda_i > 0\) for every \(i\). We then define

\[
\gamma := \min_{i \in J} \left\{ \frac{\lambda_i}{\nu_i - \lambda_i} \right\}.
\]

We clearly have that \(\gamma > 0\). Finally, let

\[
\mu_i = \lambda_i + \gamma (\lambda_i - \nu_i), \quad 1 \leq i \leq N_n + 2.
\]

We immediately get that

\[
\mu_i \geq 0, \quad \sum_{i=1}^{N_{n+2}} \mu_i = 1, \text{ at least one of the } \mu_i = 0. \quad (11.14)
\]

From (11.13), we obtain

\[
\sum_{i=1}^{N_{n+2}} \mu_i f(\xi_i) = \sum_{i=1}^{N_{n+2}} \lambda_i f(\xi_i) + \gamma (\sum_{i=1}^{N_{n+2}} \lambda_i f(\xi_i) - \sum_{i=1}^{N_{n+2}} \nu_i f(\xi_i)) \\
\leq \sum_{i=1}^{N_{n+2}} \lambda_i f(\xi_i).
\]
The combination of the above with (11.14) (assuming for the sake of notations that $\mu_{Nn+2} = 0$) immediately gives

$$Qf (\xi_0) \geq \sum_{i=1}^{Nn+1} \mu_i f (\xi_i) \text{ and } \xi_0 = \sum_{i=1}^{Nn+1} \mu_i \xi_i.$$ 

Step 3. The result for the scalar case follows from the fact that $Qf (\xi_0) = Cf (\xi_0)$ and from Theorem 2.35.

11.4 The scalar case

We now see how to apply the above abstract considerations to the case where either $n = 1$ or $N = 1$. We recall that

$$(P) \inf \left\{ I (u) = \int_{\Omega} f (\nabla u (x)) \, dx : u \in u_{\xi_0} + W_0^{1, \infty} (\Omega; \mathbb{R}^N) \right\}.$$ 

We first treat the more elementary case where $n = 1$ and then the case $N = 1$.

11.4.1 The case of single integrals

In this very elementary case, we can get much simpler and sharper results.

**Theorem 11.24** Let $N \geq 1$ and $f : \mathbb{R}^N \to \mathbb{R}$ be non-negative, locally bounded and lower semicontinuous. Let $a < b$, $\alpha, \beta \in \mathbb{R}^N$ and

$$(P) \inf \left\{ I (u) = \int_a^b f (u' (x)) \, dx : u \in X \right\},$$

where

$$X := \left\{ u \in W^{1, \infty} ((a, b); \mathbb{R}^N) : u (a) = \alpha, \, u (b) = \beta \right\}.$$ 

The following two statements are then equivalent:

(i) problem $(P)$ has a minimizer;

(ii) there exist $\lambda_\nu \geq 0$ with $\sum_{\nu=1}^{N+1} \lambda_\nu = 1$, $\gamma_\nu \in \mathbb{R}^N$, $1 \leq \nu \leq N + 1$ such that

$$Cf \left( \frac{\beta - \alpha}{b - a} \right) = \sum_{\nu=1}^{N+1} \lambda_\nu f (\gamma_\nu) \text{ and } \frac{\beta - \alpha}{b - a} = \sum_{\nu=1}^{N+1} \lambda_\nu \gamma_\nu,$$  \hspace{1cm} (11.15)

where $Cf$ is the convex envelope of $f$.

Furthermore, if (11.15) is satisfied and if

$$I_p := \left[ a + (b - a) \sum_{\nu=1}^{p-1} \lambda_\nu, \, a + (b - a) \sum_{\nu=1}^{p} \lambda_\nu \right], \; 1 \leq p \leq N + 1,$$
then
\[ \pi(x) = \gamma_p (x - a) + (b - a) \sum_{\nu=1}^{p} \lambda_{\nu} (\gamma_{\nu} - \gamma_p) + \alpha, \quad x \in I_p , \ 1 \leq p \leq N + 1, \]
is a solution of \((P)\).

**Remark 11.25**  
(i) The sufficiency of (11.15) is implicitly or explicitly proved in the papers mentioned in the introduction of the present chapter. The necessity is less known but is also implicit in the literature. The theorem as stated can be found in Dacorogna [179].

(ii) Recall that by Carathéodory theorem (see Theorem 2.35) we always have
\[ Cf ((\beta - \alpha) / (b - a)) = \inf \left\{ \sum_{\nu=1}^{N+1} \lambda_{\nu} f (\gamma_{\nu}) : \sum_{\nu=1}^{N+1} \lambda_{\nu} \gamma_{\nu} = \frac{\beta - \alpha}{b - a} \right\}. \quad (11.16) \]
Therefore (11.15) states that a necessary and sufficient condition for existence of solutions is that the infimum in (11.16) be attained. Note also that if \( f \) is convex or \( f \) coercive (in the sense that \( f (\xi) \geq a |\xi|^p + b \) with \( p > 1, \ a > 0 \)), then the infimum in (11.16) is always attained.

(iii) Therefore if \( f(x,u,\xi) = f(\xi) \), counterexamples to existence must be non-convex and non-coercive; see Example 4.4, where

\[ (P) \quad \inf \left\{ I(u) = \int_{0}^{1} e^{-u'(x)^2} dx : u \in W^{1,\infty}_1 (0,1) \right\} \]
(i.e. \( f(\xi) = e^{-\xi^2} \)), then \( Cf (\xi) \equiv 0 \) and therefore by the relaxation theorem
\[ \inf (P) = \inf (QP) = 0. \]
However, it is obvious that \( I(u) \neq 0 \) for every \( u \in W^{1,\infty}_1 (0,1) \) and hence the infimum of \((P)\) is not attained.

(iv) A similar proof to that of Theorem 11.24 (see for example Marcellini [419]) shows that a sufficient condition to ensure existence of minima to
\[ (P) \quad \inf \left\{ I(u) = \int_{a}^{b} f(x,u'(x)) dx : u \in X \right\} \]
is (11.15), where \( \lambda_{\nu} \) and \( \gamma_{\nu} \) are then measurable functions. Of course, if \( f \) depends explicitly on \( u \), the example of Bolza (see Example 4.8) shows that the theorem is then false.

\[ \text{Proof. (Theorem 11.24).} \] It is easy to see that we can reduce our study to the case where
\[ a = 0, \ b = 1 \text{ and } \alpha = 0. \]

**Sufficient condition.** The sufficiency part is elementary. Let
\[ (QP) \quad \inf \left\{ \mathcal{T}(u) = \int_{0}^{1} Cf(u'(x)) dx : u \in X \right\} \]
The scalar case

where now
\[ X := \{ u \in W^{1,\infty} ((0, 1); \mathbb{R}^N) : u(0) = 0, u(1) = \beta \} . \]

Then \( \tilde{u}(x) = \beta x \) is trivially a solution of \((QP)\) and therefore
\[ \inf (QP) = Cf(\beta) . \]

Let now \( u \) be as in the statement of the theorem. Observe first that \( u \in W^{1,\infty} ((0, 1); \mathbb{R}^N) \) and \( u(0) = 0, u(1) = \beta \). We now compute
\[
\mathbf{T}(\mathbf{u}) = \int_0^1 f(\nabla (\mathbf{u}(x))) \, dx = \sum_{p=1}^{N+1} \int_{I_p} f(\nabla (\mathbf{u}(x))) \, dx = \sum_{p=1}^{N+1} f(\gamma_p) \, \text{meas} \, I_p \\
= \sum_{p=1}^{N+1} \lambda_p f(\gamma_p) = Cf(\beta) = \inf (QP) \leq \inf (P) .
\]

_Necessary condition._ This has already been proved in Theorem 11.22. ■

11.4.2 The case of multiple integrals

We now discuss the case \( n > N = 1 \). This is of course a more difficult case than the preceding one and no such simple result like Theorem 11.24 is available. However we immediately have from Sections 11.2 and 11.3 (Theorem 10.18 and Corollary 11.15) the theorem stated below. For some historical comments on this theorem, see the remark following Corollary 11.15.

But let us first recall the problem and the notation. We have
\[
(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(\nabla u(x)) \, dx : u \in u_{\xi_0} + W^{1,\infty}_0(\Omega) \right\} ,
\]
where \( \Omega \) is a bounded open set of \( \mathbb{R}^n \), \( u_{\xi_0} \) is affine (i.e. \( \nabla u_{\xi_0} = \xi_0 \)) and \( f : \mathbb{R}^n \to \mathbb{R} \) is a lower semicontinuous, locally bounded and non-negative function. Let
\[ Cf(\xi) := \sup \{ g(\xi) : g \leq f \quad \text{and} \quad g \text{ convex} \} . \]

In order to avoid the trivial situation, we assume that
\[ Cf(\xi_0) < f(\xi_0) . \]

We next set
\[ K := \{ \xi \in \mathbb{R}^n : Cf(\xi) < f(\xi) \} \]
and we assume that it is connected, otherwise we replace it by its connected component that contains \( \xi_0 \).
Theorem 11.26 Necessary condition. If \((P)\) has a minimizer, then \( Cf \) is affine in a neighborhood of \( \xi_0 \).

Sufficient condition. If there exists \( E \subset \partial K \) such that \( \xi_0 \in \text{int co} \, E \) and \( \text{Cf}|_{E \cup \{\xi_0\}} \) is affine, then \((P)\) has a solution.

Remark 11.27 (i) By \( \text{Cf}|_{E \cup \{\xi_0\}} \) affine we mean that there exist \( \alpha \in \mathbb{R}^n, \beta \in \mathbb{R} \) such that
\[
\text{Cf} (\xi) = \langle \alpha; \xi \rangle + \beta \quad \text{for every} \quad \xi \in E \cup \{\xi_0\}.
\]
Usually one proves that \( \text{Cf} \) is affine on the whole of \( \text{co} \, E \).

(ii) The theorem applies, of course, to the case where \( E = \partial K \) and \( \text{Cf} \) is affine on the whole of \( K \) (since \( K \) is open and \( \xi_0 \in K \subset \text{int co} \, K \)). However, in many simple examples such as the one given below, it is not realistic to assume that \( E = \partial K \).

Proof. The necessary part is just Corollary 11.15. We therefore discuss only the sufficient part. We use Theorem 10.18 to find \( \pi \in u_{\xi_0} + W^{1,\infty}_0 (\Omega) \) such that
\[
\nabla \pi (x) \in E \subset \partial K, \text{ a.e. } x \in \Omega
\]
and hence
\[
f (\nabla \pi (x)) = \text{Cf} (\nabla \pi (x)), \text{ a.e. } x \in \Omega.
\]
Then use the fact that \( \text{Cf}|_{E \cup \{\xi_0\}} \) is affine to deduce that
\[
\int \Omega \, \text{Cf} (\nabla \pi (x)) \, dx = \text{Cf} (\xi_0) \, \text{meas} \, \Omega.
\]
The conclusion then follows from Theorem 11.1.

We would now like to give two simple examples.

Example 11.28 Let \( N = 1, n = 2, \Omega = (0,1)^2, u_0 \equiv 0, a \geq 0, \xi = (\xi_1, \xi_2) \) and
\[
f (\xi) = ((\xi_1)^2 - 1)^2 + ((\xi_2)^2 - a^2)^2.
\]
We find that
\[
\text{Cf} (\xi) = \left[ (\xi_1)^2 - 1 \right]_+^2 + \left[ (\xi_2)^2 - a^2 \right]_+^2,
\]
where
\[
[x]_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}
\]
We therefore have that
\[
K = \{ \xi \in \mathbb{R}^2 : |\xi_1| < 1 \text{ or } |\xi_2| < a \}
\]
and note that it is unbounded and that \( \text{Cf} \) is not affine on the whole of \( K \).
Let us discuss the two different cases.

Case 1: \( a = 0 \). Then clearly \( Cf \) is not affine in the neighborhood of \( \xi_0 = 0 \), since it is strictly convex in the direction \( e_2 = (0, 1) \). Hence \((P)\) has no solution.

Case 2: \( a > 0 \). We let

\[
E := \{ \xi \in \mathbb{R}^2 : |\xi_1| = 1 \text{ and } |\xi_2| = a \} \subset \partial K.
\]

Note that \( \xi_0 = 0 \in \text{int} \, \text{co} \, E \) and \( Cf|_{\text{co} \, E} \equiv 0 \) is affine. Therefore the theorem applies and we obtain that \((P)\) has a solution.

\[\diamond\]

11.5 The vectorial case

We now consider several examples of the form studied in the previous sections, namely

\[
(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(\nabla u(x)) \, dx : \ u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \right\},
\]

\textbf{Example 11.29} (see Marcellini [420] and Dacorogna-Marcellini [195]). Let \( n \geq 2 \) and

\[
f(\nabla u) = g(|\nabla u|),
\]

where \( g : \mathbb{R} \to \mathbb{R} \) is lower semicontinuous, locally bounded and non-negative with

\[
g(0) = \inf \{ g(t) : t \geq 0 \}.
\]

Theorem 6.30 implies that \( Cf = Cg \). Let

\[
S := \{ t \geq 0 : Cg(t) < g(t) \}
\]

\[
K := \{ \xi \in \mathbb{R}^n : Cf(\xi) < f(\xi) \} = \{ \xi \in \mathbb{R}^n : |\xi| \in S \}.
\]

Assume that \( \xi_0 \in K \) and that \( S \) is connected, otherwise replace it by its connected component containing \( |\xi_0| \).

We then have to consider two cases.

Case 1: \( Cg \) is strictly increasing at \( |\xi_0| \). Then clearly \( Cf \) is not affine in any neighborhood of \( \xi_0 \) and hence \((P)\) has no solution.

Case 2: \( Cg \) is constant on \( S \). Assume that \( S \) is bounded, this can be guaranteed if, for example,

\[
\lim_{t \to +\infty} \frac{g(t)}{t} = +\infty.
\]

So let \( |\xi_0| \in S = (\alpha, \beta) \) and choose in the sufficient part of the theorem

\[
E := \{ \xi \in \mathbb{R}^n : |\xi| = \beta \}
\]

and apply the theorem to find a minimizer for \((P)\).

\[\diamond\]
where $\Omega$ is a bounded open set of $\mathbb{R}^n$, $u_{\xi_0}$ is affine (i.e. $\nabla u_{\xi_0} = \xi_0$) and $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ is a lower semicontinuous, locally bounded and non-negative function.

All the cases have already been encountered on several occasions.

(1) We consider in Section 11.5.1 (see also Sections 6.6.2 and 10.3.2) the case where $N = n$ and

$$f(\xi) = g(\lambda_2(\xi), \cdots, \lambda_{n-1}(\xi), \det \xi),$$

where $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ are the singular values of $\xi \in \mathbb{R}^{n \times n}$.

(2) In Section 11.5.2 (see also Sections 6.6.3 and 10.3.4), we deal with the case

$$f(\xi) = g(\Phi(\xi))$$

where $\Phi : \mathbb{R}^{N \times n} \to \mathbb{R}$ is quasiaffine (so in particular we can have, when $N = n$, $\Phi(\xi) = \det \xi$, as in the previous case).

(3) We next discuss in Section 11.5.3 (see also Section 6.6.6) the Saint Venant-Kirchhoff energy functional. Up to rescaling, the function under consideration is (here $N = n$ and $\nu \in (0, 1/2)$ is a parameter)

$$f(\xi) = |\xi\xi^t - I|^2 + \frac{\nu}{1 - 2\nu}(|\xi|^2 - n)^2$$

or in terms of the singular values $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ of $\xi \in \mathbb{R}^{n \times n}$

$$f(\xi) = \sum_{i=1}^{n} (\lambda_i^2 - 1)^2 + \frac{\nu}{1 - 2\nu} \left( \sum_{i=1}^{n} \lambda_i^2 - n \right)^2.$$  

(4) In Section 11.5.4 (see also Sections 6.6.5 and 10.3.5), we consider a problem of optimal design where $N = n = 2$ and

$$f(\xi) = \begin{cases} 
1 + |\xi|^2 & \text{if } \xi \neq 0 \\
0 & \text{if } \xi = 0.
\end{cases}$$

(5) In Section 11.5.5 (see also Section 6.6.4), we deal with the area type case, namely when $N = n + 1$ and $f(\xi) = g(\text{adj}_n \xi)$.

(6) Finally, in Section 11.5.6 (see also Sections 7.4.2 and 10.3.3), we discuss the case of potential wells.

\subsection*{11.5.1 The case of singular values}

In this section, we let $N = n$ and we denote by $\lambda_1(\xi), \cdots, \lambda_n(\xi)$ the singular values of $\xi \in \mathbb{R}^{n \times n}$ with $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ and by $K_{n-2}^+$ the set

$$K_{n-2}^+ := \{x = (x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-2} : 0 \leq x_2 \leq \cdots \leq x_{n-1}\},$$
which is the natural set where to consider \((\lambda_2(\xi), \cdots, \lambda_{n-1}(\xi))\) for \(\xi \in \mathbb{R}^{n \times n}\).

The following theorem has been established by Dacorogna-Pisante-Ribeiro [211].

**Theorem 11.30** Let

\[
f(\xi) = g(\lambda_2(\xi), \cdots, \lambda_{n-1}(\xi)) + h(\det \xi),
\]

where \(g : K_{\xi_0}^{n-2} \to \mathbb{R}\) is upper semi continuous and verifies

\[
\inf g = g(m_2, \cdots, m_{n-1}), \text{ with } 0 < m_2 \leq \cdots \leq m_{n-1}
\]

and \(h : \mathbb{R} \to \mathbb{R}\) is a lower semicontinuous, locally bounded and non-negative function such that

\[
\lim_{|t| \to +\infty} \frac{h(t)}{|t|} = +\infty.
\]

(11.17)

Let \(\xi_0 \in \mathbb{R}^{n \times n}\) be such that

\[
Ch(\det \xi_0) < h(\det \xi_0).
\]

(11.18)

Then

\[
(P) \inf \left\{ I(u) = \int_{\Omega} f(\nabla u(x)) \, dx : u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^n) \right\}
\]

has a solution.

**Remark 11.31** It can be shown (see the proof for details) that the condition (11.18) is not needed and that the conclusion is valid for every \(\xi_0 \in \mathbb{R}^{n \times n}\).

**Proof.** We note that, by Theorem 6.22, \(Qf(\xi) = \inf g + Ch(\det \xi)\). Letting

\[
K := \{\xi \in \mathbb{R}^{N \times n} : Qf(\xi) < f(\xi)\}
\]

we see that

\[
K = L_1 \cup L_2
\]

where

\[
L_1 := \{\xi \in \mathbb{R}^{n \times n} : Ch(\det \xi) < h(\det \xi)\}
\]

\[
L_2 := \{\xi \in \mathbb{R}^{n \times n} : Ch(\det \xi) = h(\det \xi), \text{ inf } g < g(\lambda_2(\xi), \cdots, \lambda_{n-1}(\xi))\}\.
\]

Clearly, if \(\xi_0 \notin K\) then \(u_{\xi_0}\) is a solution of \((P)\). Let us suppose that \(\xi_0 \in K\). Our hypothesis (11.18) ensures that \(\xi_0 \in L_1\). The case \(\xi_0 \in L_2\) is more delicate and can be handled as in Dacorogna-Pisante-Ribeiro [211] (cf. also [184]), but we do not discuss here the details.
We first observe that hypothesis (11.17) allows us to write $S$ as an, at most denumerable, union of intervals, namely

$$S := \{ t \in \mathbb{R} : Ch(t) < h(t) \} = \bigcup_{j \in \mathbb{N}} (\alpha_j, \beta_j),$$

$Ch$ being affine in each interval $(\alpha_j, \beta_j)$; thus $Qf$ is quasiaffine on each connected component of $L_1$ and

$$L_1 = \{ \xi \in \mathbb{R}^{n \times n} : \det \xi \in \bigcup_{j \in \mathbb{N}} (\alpha_j, \beta_j) \}.$$

Let $(\alpha_j, \beta_j)$ be an interval as above such that

$$\det \xi_0 \in (\alpha_j, \beta_j).$$

We then choose $m_n \geq m_{n-1}$ sufficiently large so that

$$\prod_{i=\nu}^{n} \lambda_i(\xi_0) < \prod_{i=\nu}^{n} m_i, \quad \nu = 2, \cdots, n \quad \text{and} \quad \max \{|\alpha_j|, |\beta_j|\} < m_2 \prod_{i=2}^{n} m_i.$$

We are then in a position to apply Theorem 10.25 to find $u \in u_{\xi_0} + W^{1, \infty}_0(\Omega; \mathbb{R}^n)$ so that, for almost every $x \in \Omega$,

$$\det \nabla u(x) \in \{ \alpha_j, \beta_j \}, \quad \lambda_{\nu}(\nabla u(x)) = m_\nu, \quad \nu = 2, \cdots, n.$$

Since $Qf$ is quasiaffine on the connected component of $L_1$ containing $\xi_0$, we can apply Theorem 11.1 to get the result.

### 11.5.2 The case of quasiaffine functions

We next study the minimization problem

$$(P) \quad \inf \left\{ \int_{\Omega} g(\Phi(\nabla u(x))) \, dx : \ u \in u_{\xi_0} + W^{1, \infty}_0(\Omega; \mathbb{R}^N) \right\},$$

where $\Omega$ is a bounded open set of $\mathbb{R}^n$, $\nabla u_{\xi_0} = \xi_0$ and

- $g : \mathbb{R} \to \mathbb{R}$ is a lower semicontinuous, locally bounded and non-negative function,
- $\Phi : \mathbb{R}^{N \times n} \to \mathbb{R}$ is quasiaffine and non-constant.

We recall that in particular we can have, when $N = n$, $\Phi(\xi) = \det \xi$.

The relaxed problem is then

$$(QP) \quad \inf \left\{ \int_{\Omega} Cg(\Phi(\nabla u(x))) \, dx : \ u \in u_{\xi_0} + W^{1, \infty}_0(\Omega; \mathbb{R}^N) \right\},$$

where $Cg$ is the convex envelope of $g$ (see Theorem 6.24).

The existence result is the following.
Theorem 11.32 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $g : \mathbb{R} \to \mathbb{R}$ a lower semicontinuous, locally bounded and non-negative function such that

$$\lim_{|t| \to +\infty} \frac{g(t)}{|t|} = +\infty$$

(11.19)

and $u_{\xi_0}(x) = \xi_0 x$ with $\xi_0 \in \mathbb{R}^{N \times n}$. Then there exists $\bar{u} \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N)$ solution of

$$\begin{aligned}
(P) \quad & \inf \left\{ \int_{\Omega} g(\Phi(\nabla u(x))) \, dx : u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N) \right\}.
\end{aligned}$$

Remark 11.33 This result was first established by Mascolo-Schianchi [436] and later by Dacorogna-Marcellini [195] for the case of the determinant. The general case is due to Cellina-Zagatti [138] and later to Dacorogna-Ribeiro [212]. Here we see that it can be obtained as a particular case of Theorem 11.1. ♦

Proof. We first let

$$S := \{ t \in \mathbb{R} : Cg(t) < g(t) \}.$$  

From the hypothesis on $g$ we can write

$$S = \bigcup_{j \in \mathbb{N}} (\alpha_j, \beta_j)$$

with $Cg$ affine in each interval $(\alpha_j, \beta_j)$.

Case 1: $\Phi(\xi_0) \notin S$. Then $u_{\xi_0}$ is a solution of $(P)$.

Case 2: $\Phi(\xi_0) \in (\alpha_j, \beta_j) \subset S$ for some $\alpha_j$ and $\beta_j$. We apply Theorem 10.29 to find $\bar{u} \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^N)$ satisfying

$$\Phi(\nabla \bar{u}) \in \{ \alpha_j, \beta_j \}, \ a.e. \ in \ \Omega.$$  

Note also that $Qf = Cg \circ \Phi$ is quasiaffine on the connected component of

$$K := \{ \xi \in \mathbb{R}^{N \times n} : Qf(\xi) < f(\xi) \}$$

containing $\xi_0$. Invoking then Theorem 11.1, we have the claim.

The problem under consideration is sufficiently flexible that we could also proceed as in Dacorogna-Marcellini [195], using Corollary 11.7. Indeed if $\nabla \Phi(\xi_0) \neq 0$ (in the case $\Phi(\xi) = \det \xi$ this means that rank $\xi_0 \geq n - 1$), we can apply the corollary, since the connected component of $K$ containing $\xi_0$ is bounded, in the neighborhood of $\xi_0$, in a direction of rank one. We do not discuss the details of this different approach. ■
11.5.3 The Saint Venant-Kirchhoff energy

We recall that the Saint Venant-Kirchhoff function is given by

$$f(\xi) = |\xi^t - I|^2 + \frac{\nu}{1 - 2\nu} (|\xi|^2 - n)^2$$

where $\nu \in (0, 1/2)$ is a parameter. We here discuss only the case $n = 2$ and we recall (see Theorem 6.29) that

$$Qf(\xi) = Cf(\xi),$$

where

$$Qf(\xi) := \begin{cases} 
 f(\xi) & \text{if } \xi \notin D_1 \cup D_2 \\
 \frac{1}{1 - \nu}((\lambda_2)^2 - 1)^2 & \text{if } \xi \in D_2 \\
 0 & \text{if } \xi \in D_1
\end{cases}$$

where

$$D_1 = \left\{ \xi \in \mathbb{R}^{2 \times 2} : (1 - \nu)[\lambda_1(\xi)]^2 + \nu[\lambda_2(\xi)]^2 < 1 \text{ and } \lambda_2(\xi) < 1 \right\}$$

$$= \left\{ \xi \in \mathbb{R}^{2 \times 2} : \lambda_1(\xi) \leq \lambda_2(\xi) < 1 \right\},$$

$$D_2 = \left\{ \xi \in \mathbb{R}^{2 \times 2} : (1 - \nu)[\lambda_1(\xi)]^2 + \nu[\lambda_2(\xi)]^2 < 1 \text{ and } \lambda_2(\xi) \geq 1 \right\}.$$

The existence theorem, which was first studied in Dacorogna-Marcellini [195], is then the following.

**Theorem 11.34** Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, $f : \mathbb{R}^{2 \times 2} \to \mathbb{R}$ be as above, $\xi_0 \in \mathbb{R}^{2 \times 2}$ and

$$(P) \quad \inf \left\{ \int_\Omega f(\nabla u(x)) \, dx : u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^2) \right\}.$$

The following statements then hold.

(i) If $\xi_0 \in D_1$ or $\xi_0 \notin D_1 \cup D_2$ then $(P)$ has a solution.

(ii) If $\xi_0 \in \text{int } D_2$ then $(P)$ has no solution.

**Proof.** (i) The case where $\xi_0 \notin D_1 \cup D_2$ corresponds to the trivial case, where $Qf = f$.

The case $\xi_0 \in D_1$ was not settled in [195] and can be treated as follows. From Theorem 10.25, we find $\overline{\tau} \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^2)$ such that

$$\lambda_1(\nabla \overline{\tau}) = \lambda_2(\nabla \overline{\tau}) = 1, \text{ a.e. in } \Omega.$$ 

Note that $Qf$ is quasiaffine on $D_1$ (in fact $Qf(\xi) \equiv 0$) and therefore we can apply Theorem 11.1, to find that $\overline{\tau}$ is indeed a minimizer of $(P)$.

(ii) It was shown in [195] that if $\xi_0 \in \text{int } D_2$ then the function $Qf$ is strictly quasiconvex at $\xi_0$ and therefore $(P)$ has no solution. We refer for details to [195].
11.5.4 A problem of optimal design

We now consider the case, studied by many authors following the pioneering work of Kohn-Strang [374], where

\[ (P) \inf \left\{ \int_{\Omega} f(\nabla u(x)) \, dx : u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^2) \right\}, \]

\(\Omega\) is a bounded open set of \(\mathbb{R}^2\), \(\nabla u_{\xi_0} = \xi_0\) and \(f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0. \end{cases}\)

We have seen in Theorem 6.28 that the quasiconvex envelope is then

\[ Qf(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } |\xi|^2 + 2|\det \xi| \geq 1 \\ 2(|\xi|^2 + 2|\det \xi|)^{1/2} - 2|\det \xi| & \text{if } |\xi|^2 + 2|\det \xi| < 1. \end{cases} \]

The existence of minimizers for problem \((P)\) was then established by Dacorogna-Marcellini in [195] and [202], namely the following.

**Theorem 11.35** Let \(\Omega \subset \mathbb{R}^2\), \(f : \mathbb{R}^{2\times 2} \to \mathbb{R}\) be as above and \(\xi_0 \in \mathbb{R}^{2\times 2}\). Then a necessary and sufficient condition for \((P)\) to have a solution is that one of the following conditions hold:

(i) \(\xi_0 = 0\) or \(|\xi_0|^2 + 2|\det \xi_0| \geq 1\), (i.e. \(f(\xi_0) = Qf(\xi_0)\))

(ii) \(\det \xi_0 \neq 0\).

**Proof.** We refer for the necessary part to [195]. Observe that if \(\xi_0\) satisfy (i), we are in the trivial situation; so we assume from now on that

\(|\xi_0|^2 + 2|\det \xi_0| < 1\) and \(\det \xi_0 \neq 0\).

Since \(f\) is \(O(2) \times O(2)\)-invariant and \(\det \xi_0 \neq 0\), we can assume, without loss of generality, that \(\xi_0 \in K_0\) where (denoting by \(\mathbb{R}^{2\times 2}_s\) the set of \(2 \times 2\) symmetric matrices)

\[ K_0 := \{ \xi \in \mathbb{R}^{2\times 2}_s : \det \xi > 0 \text{ and } \text{trace} \xi \in (0, 1) \}. \]

Using Theorem 10.30, we can find, letting

\[ E = \{ \xi \in \mathbb{R}^{2\times 2}_s : \det \xi \geq 0 \text{ and } \text{trace} \xi \in \{0, 1\} \} = \{0\} \cup \{ \xi \in \mathbb{R}^{2\times 2}_s : \det \xi \geq 0 \text{ and } \text{trace} \xi = 1 \} \]

\(u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^2)\) such that

\(\nabla u(x) \in E\), a.e. in \(\Omega\).
This last condition means that 
\[ f(\nabla u(x)) = Qf(\nabla u(x)), \text{ a.e. } x \in \Omega. \]

Since \( Qf \) is quasiaffine on \( K_0 \) (\( Qf(\xi) = 2 \text{trace } \xi - 2 \det \xi \)), we have that 
\[ \int_{\Omega} Qf(\nabla u(x)) \, dx = Qf(\xi_0) \, \text{meas } \Omega. \]

Theorem 11.1 implies that \( \overline{u} \) is a minimizer of \((P)\). \( \blacksquare \)

### 11.5.5 The area type case

Following Dacorogna-Pisante-Ribeiro [211], we now deal with the case where \( N = n + 1 \) and 
\[ f(\xi) = g(\text{adj}_n \xi). \]

The minimization problem is then 
\[ (P) \inf \left\{ \int_{\Omega} g(\text{adj}_n(\nabla u(x))) \, dx : u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^{n+1}) \right\}, \]
where \( \Omega \) is a bounded open set of \( \mathbb{R}^n \), \( \nabla u_{\xi_0} = \xi_0 \) and \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) is a non-negative, lower semicontinuous and locally bounded non-convex function.

From Theorem 6.26, we have 
\[ Qf(\xi) = Cg(\text{adj}_n \xi). \]

We next set 
\[ S := \{ y \in \mathbb{R}^{n+1} : Cg(y) < g(y) \} \]
and assume, in order to avoid the trivial situation, that \( \text{adj}_n \xi_0 \in S \). We also assume that \( S \) is connected, otherwise we replace it by its connected component that contains \( \text{adj}_n \xi_0 \).

Observe that 
\[ K := \{ \xi \in \mathbb{R}^{(n+1)\times n} : Qf(\xi) < f(\xi) \} = \{ \xi \in \mathbb{R}^{(n+1)\times n} : \text{adj}_n \xi \in S \}. \]

**Theorem 11.36** If \( S \) is bounded, \( Cg \) is affine in \( S \) and rank \( \xi_0 \geq n - 1 \), then 
\((P)\) has a solution.

**Remark 11.37** The fact that \( Cg \) be affine in \( S \) is not a necessary condition for existence of minima, as seen in Proposition 11.38. \( \diamond \)

**Proof.** The result follows if we choose a convenient rank one direction \( \lambda = \alpha \otimes \beta \in \mathbb{R}^{(n+1)\times n} \) satisfying the hypothesis of Corollary 11.7. We remark that, since we suppose \( Cg \) affine in \( S \), \( Qf \) is quasiaffine in \( L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda) \) (cf. ...
Notation 11.4 and Definition 11.5) independently of the choice of \( \lambda \). So we only have to prove that \( K \) is stably bounded at \( \xi_0 \) in a direction \( \lambda = \alpha \otimes \beta \).

Firstly we observe that we can find (cf. Theorem 13.3) \( P \in O(n+1) \), \( Q \in SO(n) \) and \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_n \), so that

\[
\xi_0 = PAQ, \quad \text{where } \Lambda = \text{diag}_{(n+1) \times n}(\lambda_1, \ldots, \lambda_n);
\]
in particular when \( n = 2 \) we have

\[
\Lambda = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 \\
0 & 0
\end{pmatrix}.
\]

Since rank \( \xi_0 \geq n - 1 \) we have that \( \lambda_2 > 0 \). We also note that

\[
\text{adj}_n \xi_0 = \text{adj}_n P \cdot \text{adj}_n \Lambda \quad \text{and} \quad \text{adj}_n \Lambda = \begin{pmatrix}
0 \\
\vdots \\
(\Lambda)^n \lambda_1 \cdots \lambda_n
\end{pmatrix}.
\]

Without loss of generality we assume \( \xi_0 = \Lambda \). We then choose \( \lambda = \alpha \otimes \beta \) where \( \alpha = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1} \) and \( \beta = (1, 0, \ldots, 0) \in \mathbb{R}^n \). We will see that \( L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda) \) is bounded for every small \( \epsilon > 0 \). Let \( \eta \in L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda) \) then we can write \( \eta = \xi_0 + \alpha \otimes \gamma_\epsilon + t\lambda \) for some \( \gamma_\epsilon \in B_\epsilon \) and \( t \in \mathbb{R} \). By definition of \( L_K(\xi_0 + \alpha \otimes B_\epsilon, \lambda) \) we have \( \text{adj}_n \eta \in S \). Since \( S \) is bounded and

\[
|\text{adj}_n \eta| = |\lambda_1 + \gamma_\epsilon^1 + t| \lambda_2 \cdots \lambda_n
\]

it follows, using the fact that rank \( \xi_0 \geq n - 1 \), that \( |t| \) is bounded by a constant depending on \( S, \xi_0 \) and \( \epsilon \). Consequently \( |\eta| \leq |\xi_0| + |\alpha \otimes \gamma_\epsilon| + |t| |\lambda| \) is bounded for any fixed positive \( \epsilon \) and we get the result.

As already alluded in Section 11.3, we now obtain a result of non-existence although the integrand of the relaxed problem is not strictly quasiconvex. We consider the case where \( N = 3, n = 2 \) and \( f : \mathbb{R}^{3 \times 2} \to \mathbb{R} \) is given by

\[
f(\xi) = g(\text{adj}_2 \xi)
\]

where \( g : \mathbb{R}^3 \to \mathbb{R} \) is defined by, letting \( \nu = (\nu_1, \nu_2, \nu_3) \),

\[
g(\nu) = ((\nu_1)^2 - 4)^2 + (\nu_2)^2 + (\nu_3)^2.
\]

We therefore get \( Qf(\xi) = Cg(\text{adj}_2 \xi) \) and

\[
Cg(\nu) = [(\nu_1)^2 - 4]_+^2 + (\nu_2)^2 + (\nu_3)^2,
\]
where
\[ [x]_+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \]

We choose the boundary datum
\[
\begin{pmatrix}
    u_{\xi_0}^1 (x) = \alpha_1 x_1 + \alpha_2 x_2 \\
    u_{\xi_0}^2 (x) = 0 \\
    u_{\xi_0}^3 (x) = 0
\end{pmatrix}
\]

and hence
\[
\nabla u_{\xi_0} (x) = \xi_0 = \begin{pmatrix}
    \alpha_1 \\
    0 \\
    0
\end{pmatrix}, \quad \text{adj}_2 \nabla u_{\xi_0} (x) = \text{adj}_2 \xi_0 = \begin{pmatrix}
    0 \\
    0 \\
    0
\end{pmatrix}.
\]

The problem is then
\[
(P) \quad \inf \left\{ I(u) = \int_{\Omega} f(\nabla u(x)) \, dx : u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^3) \right\}.
\]

Note also that \( Qf (\xi_0) = 0 < f (\xi_0) = 16 \).

In terms of the previous notation, we have
\[
S = \{ y \in \mathbb{R}^3 : Cg(y) < g(y) \} = \{ y = (y_1, y_2, y_3) \in \mathbb{R}^3 : |y_1| < 2 \},
\]
\[
K = \{ \xi \in \mathbb{R}^{3 \times 2} : Qf(\xi) < f(\xi) \} = \{ \xi \in \mathbb{R}^{3 \times 2} : \text{adj}_2 \xi \in S \}
\]

and we observe that \( Cg \) is not affine on \( S \), which in turn implies that \( Qf \) is not quasiaffine on \( K \).

The following result shows that the hypothesis of strict quasiconvexity of \( Qf \) is not necessary for non-existence.

**Proposition 11.38** \( (P) \) has a solution if and only if \( u_{\xi_0} \equiv 0 \). Moreover, \( Qf \) is not strictly quasiconvex at any \( \xi_0 \in \mathbb{R}^{3 \times 2} \) of the form

\[
\xi_0 = \begin{pmatrix}
    \alpha_1 \\
    0 \\
    0
\end{pmatrix}.
\]

**Proof.** Step 1. We first show that if \( (P) \) has a solution then \( u_{\xi_0} \equiv 0 \). If \( u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^3) \) is a solution of \( (P) \) we necessarily have, denoting by \( \nu (\xi) = \text{adj}_2 \xi \),
\[
|\nu_1 (\nabla u)| = 2, \quad \nu_2 (\nabla u) = \nu_3 (\nabla u) = 0,
\]

since
\[
Qf (\nabla u_{\xi_0}) = Cg(\text{adj}_2 \nabla u_{\xi_0}) = Cg(0) = 0.
\]
The three equations read as

\[
\begin{aligned}
&\left| u_{x_1}^2 u_{x_2}^3 - u_{x_2}^2 u_{x_1}^3 \right| = 2 \\
u_{x_1}^1 u_{x_2}^3 - u_{x_2}^1 u_{x_1}^3 = 0 \\
u_{x_1}^1 u_{x_2}^2 - u_{x_2}^1 u_{x_1}^2 = 0.
\end{aligned}
\] (11.20)

Multiplying the second equation of (11.20) first by \(u_{x_1}^2\), then by \(u_{x_2}^2\), using the third equation of (11.20), we get

\[
0 = u_{x_1}^2 u_{x_1}^3 u_{x_2}^2 - u_{x_1}^2 u_{x_2}^3 u_{x_1}^2 = u_{x_1}^2 u_{x_1}^3 u_{x_2}^2 - u_{x_1}^2 u_{x_2}^3 u_{x_1}^2 = u_{x_1}^2 u_{x_1}^3 u_{x_2}^2 - u_{x_1}^2 u_{x_2}^3 u_{x_1}^2 = u_{x_1}^2 u_{x_1}^3 u_{x_2}^2 - u_{x_1}^2 u_{x_2}^3 u_{x_1}^2.
\]

Combining these last equations with the first one of (11.20), we find

\[
u_{x_1}^1 = u_{x_2}^1 = 0, \text{ a.e.}
\]

We therefore find that any solution of \((P)\) should have \(\nabla u^1 = 0\) a.e. and hence \(u^1 \equiv \text{constant on each connected component of } \Omega\). Since \(u^1\) agrees with \(u_{\xi_0}^1\) on the boundary of \(\Omega\), we deduce that \(u_{\xi_0}^1 \equiv 0\) and thus \(u_{\xi_0}^1 \equiv 0\), as claimed.

**Step 2.** We next show that if \(u_{\xi_0}^1 \equiv 0\), then \((P)\) has a solution. It suffices to choose \(u^1 \equiv 0\) and to solve

\[
\begin{aligned}
&\left| u_{x_1}^2 u_{x_2}^3 - u_{x_2}^2 u_{x_1}^3 \right| = 2 \text{ a.e. in } \Omega \\
u^2 = u^3 = 0 \text{ on } \partial \Omega.
\end{aligned}
\]

This is possible by virtue of Corollary 10.27.

**Step 3.** We finally prove that \(Qf\) is not strictly quasiconvex at any \(\xi_0 \in \mathbb{R}^{3 \times 2}\) of the form given in the statement of the proposition. Indeed let \(0 < R_1 < R_2 < R\) and denote by \(B_R\) the ball centered at 0 and of radius \(R\). Choose \(\lambda, \mu \in C^{\infty}(BR)\) such that

1) \(\lambda = 0\) on \(\partial BR\) and \(\lambda \equiv 1\) on \(BR_2\).

2) \(\mu \equiv 0\) on \(BR - \overline{BR}_2\), \(\mu \equiv 1\) on \(BR_1\), and

\[
\left| \mu^2 + \mu (x_1 \mu x_1 + x_2 \mu x_2) \right| < 2, \text{ for every } x \in BR.
\]

This last condition (which is a restriction only in \(BR_2 - \overline{BR}_1\)) is easily ensured by choosing appropriately \(R_1\), \(R_2\) and \(R\).

We then choose \(u(x) = u_{\xi_0}(x) + \varphi(x)\) where

\[
\varphi^1(x) = -\lambda(x) u_{\xi_0}^1(x), \quad \varphi^2(x) = \mu(x) x_1 \quad \text{and} \quad \varphi^3(x) = \mu(x) x_2.
\]

We therefore have that \(\varphi \in W^{1,\infty}(BR; \mathbb{R}^3), \text{adj}_2 \nabla u \equiv 0\) on \(BR - \overline{BR}_2\), while on \(BR_2\) we have

\[
\text{adj}_2 \nabla u = \left( \mu^2 + \mu (x_1 \mu x_1 + x_2 \mu x_2), 0, 0 \right).
\]
We have thus obtained that $Cg(\operatorname{adj}_2 \nabla u) \equiv 0$ and hence
$$Qf(\xi_0 + \nabla \varphi) \equiv Qf(\xi_0) = 0.$$ This implies that $(QP)$ has infinitely many solutions. However since $\varphi$ does not vanish identically, we deduce that $Qf$ is not strictly quasiconvex at any $\xi_0$ of the given form.

### 11.5.6 The case of potential wells

The general problem of potential wells has been intensively studied by many authors in conjunction with crystallographic models involving fine microstructures. The reference paper on the subject is Ball and James [60]. It has since then been studied by many authors including Bhattacharya-Firoozye-James-Kohn, Dacorogna-Marcellini, De Simone-Dolzmann, Dolzmann-Müller, Ericksen, Firoozye-Kohn, Fonseca-Tartar, Kinderlehrer-Pedregal, Kohn, Luskin, Müller-Sverak, Pipkin and Sverak and we refer to [202] for full bibliographic references.

In mathematical terms the problem of potential wells can be described as follows. Find a minimizer of the problem

$$(P) \quad \inf \left\{ \int_\Omega f(\nabla u(x)) \, dx : u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^n) \right\},$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $u_{\xi_0}$ is an affine map with $\nabla u_{\xi_0} = \xi_0$ and $f : \mathbb{R}^{n \times n} \to \mathbb{R}^+$ is such that

$$f(\xi) = 0 \iff \xi \in E := \bigcup_{i=1}^m SO(n) A_i.$$ The $m$ wells are $SO(n) A_i$, $1 \leq i \leq m$ (and $SO(n)$ denotes the set of matrices $U$ such that $U^t U = U U^t = I$ and $\det U = 1$).

The interesting case is when

$$\xi_0 \in \operatorname{int} \operatorname{Rco} E$$

and we have, since $\operatorname{Rco} E \subset \operatorname{Qco} E \subset \operatorname{Qco}_f E$ (see Theorem 7.28), that

$$Qf(\xi_0) = 0.$$ Therefore, by the relaxation theorem, we have

$$\inf (P) = \inf (QP) = 0.$$ The existence of minimizers, since $Qf$ is affine on $\operatorname{Rco} E$ (indeed $Qf \equiv 0$), for $(P)$ is then reduced to finding a function $u \in u_{\xi_0} + W^{1,\infty}_0(\Omega; \mathbb{R}^n)$ such that

$$\nabla u(x) \in E = \bigcup_{i=1}^m SO(n) A_i.$$
The vectorial case

The problem is relatively well understood only in the cases of two wells, i.e. $m = 2$, and in dimension $n = 2$. It is this case that we briefly discuss now. We therefore now have $A, B \in \mathbb{R}^{2 \times 2}$ and we assume that

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix},$$

where $0 < b_1 < a_1 \leq a_2 < b_2$ and $a_1 a_2 < b_1 b_2$. We want to find $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^2)$, where $\Omega \subset \mathbb{R}^2$ is a bounded open set, satisfying

$$\nabla u(x) \in SO(2)A \cup SO(2)B \ a.e. \ in \ \Omega.$$ 

The first important result is to identify the set where the gradient of the boundary datum, $\xi_0$, should lie. We have seen in Theorem 7.44 that

$$Rco \ E = \left\{ \xi \in \mathbb{R}^{2 \times 2} : \begin{array}{l} \xi = \alpha R_a A + \beta R_b B, \ R_a, R_b \in SO(2), \\ 0 \leq \alpha \leq \frac{\det B - \det \xi}{\det B - \det A}, \ 0 \leq \beta \leq \frac{\det \xi - \det A}{\det B - \det A} \end{array} \right\},$$

while the interior is given by the same formulas with strict inequalities on the right hand side.

We therefore have the following (see Theorem 10.28).

**Theorem 11.39** Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and $\xi_0 \in \text{int} \ Rco \ E$.

Then there exists $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^2)$ such that

$$\nabla u(x) \in E = SO(2)A \cup SO(2)B \ a.e. \ in \ \Omega$$

and therefore $(P)$ has a solution.

As already discussed in Section 10.3.3, the case where $\det A = \det B > 0$ can also be handled (see Müller-Sverak [466] and also Dacorogna-Tantieri [215]), using the representation formula of Sverak [554] (see Theorem 7.44), namely

$$Rco \ E = \left\{ \xi \in \mathbb{R}^{2 \times 2} : \begin{array}{l} \xi = \alpha R_a A + \beta R_b B, \ R_a, R_b \in SO(2), \\ 0 \leq \alpha, \beta, \alpha + \beta \leq 1 \ \text{and} \ \det \xi = \det A = \det B \end{array} \right\}.$$
Chapter 12

Function spaces

12.1 Introduction

We have gathered in this chapter the notation and the most important results on different function spaces that we have used or will use throughout the book. We precisely fix the notations and state the theorems. But we provide almost no proof, since the results are standard.

12.2 Main notation

We first recall the usual notation for derivatives.

(i) If \( u : \mathbb{R}^n \to \mathbb{R}, u = u(x_1, \cdots, x_n) \), we denote partial derivatives by either of the following ways

\[
D_j u = u_{x_j} = \frac{\partial u}{\partial x_j}
\]

and

\[
\nabla u = \text{grad } u = (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}) = (u_{x_1}, \cdots, u_{x_n}) \in \mathbb{R}^n.
\]

(ii) For maps \( u : \mathbb{R}^n \to \mathbb{R}^N \), we write \( u = (u^1, \cdots, u^N) \) and

\[
\nabla u = \left( \frac{\partial u^i}{\partial x_j} \right)_{1 \leq i \leq N, 1 \leq j \leq n} \in \mathbb{R}^{N \times n}.
\]

(iii) For higher derivatives, we proceed as follows. Let \( m \geq 1 \) be an integer; an element of

\[
A_m := \{ a = (a_1, \cdots, a_n) \in \mathbb{N}^n : \sum_{j=1}^n a_j = m \}
\]
is called a multi-index of order $m$. We also write for such elements
\[ |a| = \sum_{j=1}^{n} a_j = m. \]

For $a \in A_m$, we write
\[ D^a u = D_1^{a_1} \cdots D_n^{a_n} u = \frac{\partial |a|}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}. \]

- Let $N, n, m \geq 1$ be integers. For $u : \mathbb{R}^n \to \mathbb{R}^N$ we write
\[ \nabla^m u = \left( \frac{\partial^m u}{\partial x_{j_1} \cdots \partial x_{j_m}} \right)_{1 \leq i \leq N}^{1 \leq j_1, \cdots, j_m \leq n} \in \mathbb{R}^{N \times n^m}. \]
(The index $s$ here stands for all the natural symmetries implied by the interchange of the order of differentiation.) When $m = 1$, we have
\[ \mathbb{R}^{N \times n} = \mathbb{R}^{N \times n}, \]
while if $N = 1$ and $m = 2$, we obtain
\[ \mathbb{R}^{n^2} = \mathbb{R}^{n \times n} \]
(i.e., the usual set of symmetric matrices).

- We also let
\[ \nabla^{[m]} u = (u, \nabla u, \cdots, \nabla^m u) \]
stand for the matrix of all partial derivatives of $u$ up to the order $m$. Note that
\[ \nabla^{[m-1]} u \in \mathbb{R}^{N \times M} = \mathbb{R}^{N \times N \times n} \times \mathbb{R}^{N \times n^2} \times \cdots \times \mathbb{R}^{N \times n^{(m-1)}}, \]
where
\[ M := 1 + n + \cdots + n^{(m-1)} = \frac{n^m - 1}{n - 1}. \]
Hence
\[ \nabla^{[m]} u = (\nabla^{[m-1]} u, \nabla^m u) \in \mathbb{R}^{N \times M} \times \mathbb{R}^{N \times n^m}. \]

We next define some function spaces.

**Definition 12.1** Let $\Omega \subset \mathbb{R}^n$ be an open set.

(i) $C^0(\Omega) = C(\Omega)$ is the set of continuous functions $u : \Omega \to \mathbb{R}$.

(ii) $C^0(\overline{\Omega}) = C(\overline{\Omega})$ is the set of continuous functions $u : \Omega \to \mathbb{R}$, which can be continuously extended to $\overline{\Omega}$. The norm over $C(\overline{\Omega})$ is given by
\[ \|u\|_{C^0} = \sup_{x \in \overline{\Omega}} |u(x)|. \]
The support of a function $u : \Omega \to \mathbb{R}$ is defined as
$$\text{supp } u := \{x \in \Omega : u(x) \neq 0\}.$$

$(iv)$ $C_0(\Omega) := \{u \in C(\Omega) : \text{supp } u \subset \Omega \text{ is compact}\}$.

We now proceed similarly for the spaces involving derivatives.

**Definition 12.2** Let $n, m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be an open set.

(i) The set of functions $u : \Omega \to \mathbb{R}$ that have all partial derivatives, $D^a u$, $a \in \mathcal{A}_k$, $0 \leq k \leq m$, continuous is denoted by $C^m(\Omega)$.

(ii) $C^m(\Omega)$ is the set of $C^m(\Omega)$ functions whose derivatives up to the order $m$ can be extended continuously to $\overline{\Omega}$. It is equipped with the following norm
$$\|u\|_{C^m} = \max_{0 \leq |a| \leq m} \sup_{x \in \Omega} |D^a u(x)|.$$

(iii) $\text{Aff}^m(\Omega)$ stands for the set of polynomials of degree $m$; in particular, if $u \in \text{Aff}^m(\Omega)$, there exists $\xi \in \mathbb{R}^n$ such that $\nabla^m u(x) = \xi$ for every $x \in \Omega$.

Most of the time when $m = 1$, we let $\text{Aff}(\Omega)$ instead of $\text{Aff}^1(\Omega)$.

(iv) $C^m_0(\Omega) := C^m(\Omega) \cap C_0(\Omega)$.

(v) $C^\infty(\Omega) := \bigcap_{m=0}^{\infty} C^m(\Omega)$, $C^\infty(\overline{\Omega}) := \bigcap_{m=0}^{\infty} C^m(\overline{\Omega})$.

(vi) $C^\infty_0(\Omega) = \mathcal{D}(\Omega) := C^\infty(\Omega) \cap C_0(\Omega)$.

(vii) When dealing with maps, $u : \Omega \to \mathbb{R}^N$, we accordingly write $C^m(\Omega; \mathbb{R}^N)$, $C^m(\overline{\Omega}; \mathbb{R}^N)$ or $\text{Aff}^m(\overline{\Omega}; \mathbb{R}^N)$ and similarly for the other notations.

We often have to consider the above spaces as split in several pieces and we therefore have the following definitions.

**Definition 12.3** Let $n, m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be an open set.

(i) A function $u \in C^m_{\text{piec}}(\Omega)$ if $u \in C^{m-1}_{\text{piec}}(\Omega)$ and $\nabla^m u$ is piecewise continuous, meaning that there exists a partition of $\Omega$ into a countable union of disjoint open sets $\Omega_k \subset \Omega$ for every $k \in \mathbb{N}$, more precisely
$$\Omega_h \cap \Omega_k = \emptyset, \forall h, k \in \mathbb{N}, h \neq k, \text{ and } \text{meas } (\Omega \setminus \bigcup_{k \in \mathbb{N}} \Omega_k) = 0$$
and so that $\nabla^m u \in C(\overline{\Omega_k}; \mathbb{R}^n)$ for every $k \in \mathbb{N}$.

(ii) $\text{Aff}_{\text{piec}}^m(\Omega)$ stands for the subset of $C^m_{\text{piec}}(\Omega)$ so that $\nabla^m u$ is piecewise (in the above sense) constant. Most of the time when $m = 1$, we let $\text{Aff}_{\text{piec}}(\Omega)$ instead of $\text{Aff}_{\text{piec}}^1(\Omega)$.

(iii) Similarly for maps, $u : \Omega \to \mathbb{R}^N$, we accordingly write $C^m_{\text{piec}}(\Omega; \mathbb{R}^N)$ or $\text{Aff}_{\text{piec}}^m(\Omega; \mathbb{R}^N)$.
On several occasions we used the following definition.

**Definition 12.4** Let $\Omega \subset \mathbb{R}^n$ be an open set and $u_\nu : \Omega \to \mathbb{R}$ be a sequence of measurable functions. We say that $\{u_\nu\}$ is equiintegrable, if there exists an increasing function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\eta(t) \to 0$ as $t \to 0$, so that

$$\int_A |u_\nu(x)| \, dx \leq \eta(\text{meas } A),$$

for every measurable set $A \subset \Omega$.

We recall (see Dunford-Pettis theorem) that if the sequence $\{u_\nu\}$ converges weakly in $L^1$, then it is equiintegrable.

### 12.3 Some properties of Hölder spaces

We recall here some basic properties of Hölder spaces. We use as references on this part: Adams [5], Dacorogna [180], Gilbarg and Trudinger [313] or Hörmander [343].

**Definition 12.5** Let $D \subset \mathbb{R}^n$, $u : D \to \mathbb{R}$ and $0 < \alpha \leq 1$. We let

$$[u]_{\alpha,D} := \sup_{x,y \in D, x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} \right\}.$$

Let $\Omega \subset \mathbb{R}^n$ be open and $m \geq 0$ be an integer. We define the different spaces of Hölder continuous functions in the following way.

(i) $C^{0,\alpha}(\Omega)$ is the set of $u \in C(\Omega)$ such that

$$[u]_{\alpha,K} = \sup_{x,y \in K, x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} \right\} < \infty$$

for every compact set $K \subset \Omega$.

(ii) $C^{0,\alpha}(\overline{\Omega})$ is the set of functions $u \in C(\overline{\Omega})$ such that

$$[u]_{\alpha,\overline{\Omega}} < \infty.$$

It is equipped with the norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} := \|u\|_{C^0(\overline{\Omega})} + [u]_{\alpha,\overline{\Omega}}.$$

If there is no ambiguity, we drop the dependence on the set $\overline{\Omega}$ and write simply

$$\|u\|_{C^{0,\alpha}} := \|u\|_{C^0} + [u]_{\alpha}.$$
(iii) \( C^{m,\alpha}(\Omega) \) is the set of \( u \in C^m(\Omega) \) such that
\[
[D^a u]_{\alpha, K} < \infty
\]
for every compact set \( K \subset \Omega \) and every \( a \in A_m \).
(iv) \( C^{m,\alpha}(\overline{\Omega}) \) is the set of functions \( u \in C^m(\overline{\Omega}) \) such that
\[
[D^a u]_{\alpha, \overline{\Omega}} < \infty
\]
for every multi-index \( a \in A_m \). We equip \( C^{m,\alpha}(\overline{\Omega}) \) with the following norm:
\[
\|u\|_{C^{m,\alpha}} := \|u\|_{C^m} + \max_{a \in A_m} [D^a u]_{\alpha}.
\]
(v) For maps, \( u : \Omega \to \mathbb{R}^N \), we write \( C^{m,\alpha}(\Omega; \mathbb{R}^N) \).

Remark 12.6 (i) \( C^{m,\alpha}(\overline{\Omega}) \) with its norm \( \|\cdot\|_{C^{m,\alpha}} \) is a Banach space.
(ii) By abuse of notation we write \( C^m(\Omega) = C^{m,0}(\Omega) \), or, in other words, the set of continuous functions is identified with the set of Hölder continuous functions with exponent 0.
(iii) Similarly, when \( \alpha = 1 \), we see that \( C^{0,1}(\overline{\Omega}) \) is in fact the set of Lipschitz continuous functions, namely the set of functions \( u \) such that there exists a constant \( \gamma > 0 \) such that
\[
|u(x) - u(y)| \leq \gamma |x - y|, \quad \forall x, y \in \overline{\Omega}.
\]
The best such constant is \( \gamma = [u]_{C^{0,1}} \).

We now list some important properties of Hölder spaces.

Proposition 12.7 Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set with a Lipschitz boundary, \( m \geq 0 \) an integer and \( 0 \leq \alpha \leq 1 \). The following properties then hold.
(i) If \( u, v \in C^{m,\alpha}(\overline{\Omega}) \), then \( uv \in C^{m,\alpha}(\overline{\Omega}) \). More precisely, if \( u, v \in C^{m,\alpha} \), then there exists a constant \( \gamma > 0 \) such that
\[
\|uv\|_{C^{m,\alpha}} \leq \gamma (\|u\|_{C^m} \|v\|_{C^m} + \|u\|_{C^m} \|v\|_{C^{m,\alpha}}) \leq 2\gamma \|u\|_{C^{m,\alpha}} \|v\|_{C^{m,\alpha}}.
\]
Moreover, if \( \Omega \) is convex or its boundary is \( C^{m,\alpha} \), then
\[
\|uv\|_{C^{m,\alpha}} \leq \gamma (\|u\|_{C^{m,\alpha}} \|v\|_{C^m} + \|u\|_{C^{m,\alpha}} \|v\|_{C^{m,\alpha}}).
\]
(ii) Let \( O \subset \mathbb{R}^N \) be open bounded and with a Lipschitz boundary, \( m \geq 1 \), \( v \in C^{m,\alpha}(\overline{\Omega}; \overline{O}) \) and \( u \in C^{m,\alpha}(\overline{O}) \). Then
\[
u \circ v \in C^{m,\alpha}(\overline{\Omega}) \)
(iii) If \( 0 \leq \alpha \leq \beta \leq 1 \), then
\[
C^m(\overline{\Omega}) \supset C^{m,\alpha}(\overline{\Omega}) \supset C^{m,\beta}(\overline{\Omega}) \supset C^{m,1}(\overline{\Omega}) \supset C^{m+1}(\overline{\Omega}) \).
(iv) Let \( m \geq l \) be non-negative integers and \( 0 < \beta \leq \alpha < 1 \) be such that
\[
l + \beta < m + \alpha.
\]
Then, for every \( \epsilon > 0 \) and every \( u \in C^{m,\alpha}(\overline{\Omega}) \), there exists \( v \in C^{\infty}(\overline{\Omega}) \) such that
\[
\|u - v\|_{C^{l,\beta}} \leq \epsilon.
\]

We conclude with the following lower semicontinuity result.

**Proposition 12.8** Let \( m \geq 1 \) be an integer and \( 0 < \alpha < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set with a Lipschitz boundary. Let \( r > 0 \) and
\[
C_r := \{ u \in C^{m,\alpha}(\overline{\Omega}) : \|u\|_{C^{m,\alpha}} \leq r \}.
\]
Let \( \{u_\nu\} \subset C_r \) be a sequence such that
\[
u \rightarrow u \text{ in } L^\infty(\Omega) \text{ as } \nu \rightarrow \infty,
\]
then \( u \in C_r \) and
\[
\|u\|_{C^{m,\alpha}} \leq \liminf_{\nu \rightarrow \infty} \|u_\nu\|_{C^{m,\alpha}}.
\]

**Proof.** We divide the proof into two steps.

**Step 1.** We recall that
\[
\|u\|_{C^{m,\alpha}} = \|u\|_{C^m} + \max_{a \in A_m} [D^a u]_\alpha
\]
and observe that since (see Section 12.4 for the definition and properties of Sobolev spaces)
\[
\|u\|_{C^m} = \|u\|_{W^{m,\infty}}
\]
we can deduce that, up to the extraction of a subsequence still labeled \( \{u_\nu\} \), there exists \( v \in W^{m,\infty}(\Omega) \) so that
\[
u \rightarrow \ast \text{ } v \text{ in } W^{m,\infty}(\Omega) \text{ as } \nu \rightarrow \infty.
\]
By uniqueness of the limit we can identify \( u \) and \( v \). We therefore have that
\[
u \rightarrow u \text{ in } W^{m-1,\infty}(\Omega) \text{ as } \nu \rightarrow \infty,
\]
\[
u \in W^{m,\infty}(\Omega) \text{ and }
\|u\|_{W^{m,\infty}} \leq \liminf_{\nu \rightarrow \infty} \|u_\nu\|_{W^{m,\infty}}.
\]

**Step 2.** We prove the claim only for the case \( m = 1 \). The general case follows in a similar manner, since we have (12.2). We already know that \( u \in W^{1,\infty}(\Omega) \) and
\[
\|u\|_{W^{1,\infty}} \leq \liminf_{\nu \rightarrow \infty} \|u_\nu\|_{W^{1,\infty}} \text{ (12.3)}
\]
so it remains to show that \( u \in C^{1,\alpha}(\Omega) \) and that
\[
[D_i u]_\alpha \leq \liminf_{\nu \to \infty} [D_i u_\nu]_\alpha, \quad i = 1, \ldots, n. \tag{12.4}
\]
The combination of (12.3) and (12.4) gives the proposition.

Since \( u \in W^{1,\infty}(\Omega) \) and \( \partial \Omega \) is regular, we have that \( u \) is Lipschitz and therefore by Rademacher theorem we have that \( u \) is almost everywhere differentiable (see for example Evans [272]). So let \( x, y \in \Omega \) be points of differentiability of \( u \), i.e.
\[
D_i u(x) = \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h}, \quad D_i u(y) = \lim_{h \to 0} \frac{u(y + he_i) - u(y)}{h}. \tag{12.5}
\]
Observe next that
\[
| \frac{u(x + he_i) - u(x)}{h} - \frac{u(y + he_i) - u(y)}{h} | \leq \frac{4}{h} \| u_\nu - u \|_{L^\infty} + \left| \frac{u_\nu(x + he_i) - u_\nu(x)}{h} - \frac{u_\nu(y + he_i) - u_\nu(y)}{h} \right|. \tag{12.6}
\]
Since \( u_\nu \in C^{1,\alpha}(\Omega) \), we can find \( \theta_\nu^x \) and \( \theta_\nu^y \) so that
\[
| \theta_\nu^x - x |, \quad | \theta_\nu^y - y | \leq h
\]
and
\[
D_i u_\nu(\theta_\nu^x) = \frac{u_\nu(x + he_i) - u_\nu(x)}{h}, \quad D_i u_\nu(\theta_\nu^y) = \frac{u_\nu(y + he_i) - u_\nu(y)}{h}.
\]
This leads to
\[
| \frac{u(x + he_i) - u(x)}{h} - \frac{u(y + he_i) - u(y)}{h} | \leq [D_i u_\nu]_\alpha | \theta_\nu^x - \theta_\nu^y |^\alpha
\]
\[
\leq [D_i u_\nu]_\alpha (2h + |x - y|)^\alpha.
\]
Combining this last inequality with (12.6), we get
\[
| \frac{u(x + he_i) - u(x)}{h} - \frac{u(y + he_i) - u(y)}{h} | \leq \frac{4}{h} \| u_\nu - u \|_{L^\infty} + [D_i u_\nu]_\alpha (2h + |x - y|)^\alpha.
\]

Letting first \( \nu \to \infty \) and then \( h \to 0 \) and appealing to (12.5), we have obtained that, for almost every \( x, y \in \Omega \),
\[
|D_i u(x) - D_i u(y)| \leq \liminf_{\nu \to \infty} [D_i u_\nu]_\alpha |x - y|^\alpha.
\]
This easily leads to the conclusion of Step 2 and thus the proposition is proved.

## 12.4 Some properties of Sobolev spaces

For more details concerning Sobolev spaces, we refer to Adams [5], Brézis [105], Dacorogna [180], Dacorogna-Marcellini [202], Ekeland-Temam [264], Evans [272], Gilbarg and Trudinger [313], Giusti [316], Kufner-John-Fučik [384], Ladyzhenskaya-Ural'tseva [388] or Morrey [455].
12.4.1 Definitions and notations

We first recall the definition of Sobolev spaces.

**Definition 12.9** Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( 1 \leq p \leq \infty \).

(i) We let \( W^{1,p}(\Omega) \) be the set of functions \( u : \Omega \to \mathbb{R}, u \in L^p(\Omega) \), whose weak partial derivatives \( u_{x_i} \in L^p(\Omega) \) for every \( i = 1, \ldots, n \). We endow this space with the following norm

\[
\|u\|_{W^{1,p}} := (\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p)^{1/p} \quad \text{if} \ 1 \leq p < \infty,
\]

\[
\|u\|_{W^{1,\infty}} := \max \{\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}\} \quad \text{if} \ p = \infty.
\]

(ii) If \( 1 \leq p < \infty \), the set \( W_0^{1,p}(\Omega) \) is defined as the closure of \( C^\infty_0(\Omega) \) functions in \( W^{1,p}(\Omega) \).

(iii) We also write \( u \in u_0 + W_0^{1,p}(\Omega) \), meaning that \( u, u_0 \in W^{1,p}(\Omega) \) and \( u - u_0 \in W_0^{1,p}(\Omega) \).

(iv) We let \( W_0^{0,\infty}(\Omega) := W^{1,\infty}(\Omega) \cap W_0^{1,1}(\Omega) \).

(v) Analogously, we define the Sobolev spaces with higher derivatives as follows. If \( m > 0 \) is an integer, we let (by abuse of notation, we will write \( W^{0,p} := L^p \)) \( W^{m,p}(\Omega) \) be the set of functions \( u : \Omega \to \mathbb{R} \) whose weak partial derivatives \( D^a u \in L^p(\Omega) \) for every multi-index \( a \in A_k, 0 \leq k \leq m \). The norm is then

\[
\|u\|_{W^{m,p}} := \begin{cases} 
\left( \sum_{0 \leq |a| \leq m} \|D^a u\|_{L^p}^p \right)^{1/p} & \text{if} \ 1 \leq p < \infty, \\
\max_{0 \leq |a| \leq m} (\|D^a u\|_{L^\infty}) & \text{if} \ p = \infty.
\end{cases}
\]

(vi) If \( 1 \leq p < \infty \), \( W^{m,p}_0(\Omega) \) denotes the closure of \( C^\infty_0(\Omega) \) in \( W^{m,p}(\Omega) \) and \( W^{m,\infty}_0(\Omega) := W^{m,\infty}(\Omega) \cap W^{m,1}_0(\Omega) \).

(vii) For maps \( u : \Omega \to \mathbb{R}^N, u = (u^1, \ldots, u^N) \), we say that \( u \in W^{1,p}(\Omega; \mathbb{R}^N) \) if \( u^i \in W^{1,p}(\Omega) \) for every \( i = 1, \ldots, N \). Similar definitions apply to \( W^{1,p}_0(\Omega; \mathbb{R}^N), W^{m,p}(\Omega; \mathbb{R}^N) \) or \( W^{m,p}_0(\Omega; \mathbb{R}^N) \).

12.4.2 Imbeddings and compact imbeddings

We recall here the Sobolev and the Rellich-Kondrachov theorems. We start with the definition of Lipschitz boundary of a given set.

**Definition 12.10** Let \( \Omega \subset \mathbb{R}^n \) be open and bounded. We say that \( \Omega \) is a bounded open set with a Lipschitz boundary if for every \( x \in \partial \Omega \), there exist a neighborhood \( U \subset \mathbb{R}^n \) of \( x \) and a one-to-one and onto map \( H : Q \to U \), where (see Figure 12.1)

\[
Q := \{x \in \mathbb{R}^n : |x_j| < 1, \ j = 1, 2, \ldots, n\},
\]
Some properties of Sobolev spaces

$H \in C^{0,1} (\overline{Q})$, $H^{-1} \in C^{0,1} (\overline{U})$, $H (Q_+)= U \cap \Omega$, $H (Q_0)= U \cap \partial \Omega$,

with $Q_+ := \{x \in Q : x_n > 0\}$ and $Q_0 := \{x \in Q : x_n = 0\}$.

Similarly, we say that $\Omega$ has a $C^{m,\alpha}$ boundary, if the above $H \in C^{m,\alpha}$ as well as $H^{-1} \in C^{m,\alpha}$.

![Figure 12.1: Regularity of the boundary](image)

**Theorem 12.11 (Sobolev imbedding theorem)** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary.

Case 1. If $1 \leq p < n$, then

$$W^{1,p} (\Omega) \subset L^q (\Omega)$$

for every $q \in [1, p^*]$, where $p^*$ is the Sobolev exponent defined by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \text{ i.e. } p^* = \frac{np}{n-p}.$$

More precisely, for every $q \in [1, p^*]$ there exists $c = c (\Omega, p, q)$ such that

$$\|u\|_{L^q} \leq c \|u\|_{W^{1,p}}.$$

Case 2. If $p = n$, then

$$W^{1,n} (\Omega) \subset L^q (\Omega) \text{ for every } q \in [1, \infty).$$

More precisely, for every $q \in [1, \infty)$ there exists $c = c (\Omega, p, q)$ such that

$$\|u\|_{L^q} \leq c \|u\|_{W^{1,n}}.$$


Case 3. If $p > n$, then
\[ W^{1,p} (\Omega) \subset C^{0,\alpha} (\overline{\Omega}) \text{ for every } \alpha \in [0, 1 - n/p]. \]
In particular, there exists a constant $c = c(\Omega, p)$ such that
\[ \|u\|_{L^\infty} \leq c \|u\|_{W^{1,p}}. \]

We continue with compact imbeddings.

**Theorem 12.12 (Rellich-Kondrachov theorem)** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary.

Case 1. If $1 \leq p < n$, then the imbedding of $W^{1,p}$ in $L^q$ is compact for every $q \in [1, p^*)$. This means that any bounded set of $W^{1,p}$ is precompact (i.e., its closure is compact) in $L^q$ for every $1 \leq q < p^*$ (the result is false if $q = p^*$).

Case 2. If $p = n$, then the imbedding of $W^{1,n}$ in $L^q$ is compact for every $q \in [1, \infty)$.

Case 3. If $p > n$, then the imbedding of $W^{1,p}$ in $C^{0,\alpha} (\overline{\Omega})$ is compact for every $0 \leq \alpha < 1 - n/p$.

In particular, in all cases (i.e., $1 \leq p \leq \infty$), the imbedding of $W^{1,p} (\Omega)$ in $L^p (\Omega)$ is compact.

Remark 12.13 (i) Similar results for $W^{k,p}$ spaces exist, but we will not need them.

(ii) If in both theorems we replace the spaces $W^{1,p}$ by $W^{1,p}_0$, then no regularity on the boundary of the domain is required.

### 12.4.3 Approximation by smooth and piecewise affine functions

We now turn our attention to density results for Sobolev functions. The first result concerns the approximation by smooth functions.

**Theorem 12.14** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $1 \leq p < \infty$. Then the space $C^\infty (\Omega) \cap W^{1,p} (\Omega)$ is dense in $W^{1,p} (\Omega)$. Moreover, if $\Omega$ is a bounded domain with Lipschitz boundary, then $C^\infty (\overline{\Omega})$ is also dense in $W^{1,p} (\Omega)$.

The second result deals with approximation by piecewise affine functions.

**Theorem 12.15** Let $\Omega$ be a bounded open set of $\mathbb{R}^n$, $1 \leq p \leq \infty$ and $u \in W^{1,p}_0 (\Omega)$. Then, for every $\epsilon > 0$, there exists $u_\epsilon \in \text{Aff}_{\text{piec}} (\overline{\Omega})$ such that
\[
\begin{cases}
    u_\epsilon \in W^{1,p}_0 (\Omega) \\
    \|u_\epsilon\|_{W^{1,p} (\Omega)} \leq \|u\|_{W^{1,p} (\Omega)} + \epsilon \\
    \|u_\epsilon - u\|_{W^{1,p} (\Omega)} \leq \epsilon.
\end{cases}
\]

(If $p = \infty$, we only have $\|u_\epsilon - u\|_{W^{1,q} (\Omega)} \leq \epsilon$ for any $q < \infty$.)
More sophisticated approximation results exist (see Corollaries 10.11, 10.13, 10.14, 10.15 or 10.21 in Dacorogna-Marcellini [202]), such as the next one.

**Theorem 12.16** Let $\Omega$ be an open set of $\mathbb{R}^n$. Let $A, B$ be disjoint sets of $\mathbb{R}^n$, with $A$ open and $B$ possibly empty. Let $u \in W^{1, \infty}(\Omega)$ such that

$$\nabla u(x) \in A \cup B \text{ a.e. } x \in \Omega.$$ 

Then, for every $\epsilon > 0$, there exists a function $v \in W^{1, \infty}(\Omega)$ and an open set $\Omega' \subset \Omega$ ($\Omega' = \Omega$ if $B = \emptyset$) such that

$$\begin{cases}
  v \in \text{Aff}_{\text{piec}}(\Omega'),
  v = u \text{ on } \partial \Omega,
  \|v - u\|_{L^\infty(\Omega)} < \epsilon,
  \nabla v(x) \in A \text{ a.e. } x \in \Omega',
  \nabla v(x) = \nabla u(x) \in B \text{ a.e. } x \in \Omega - \Omega'.
\end{cases}$$
Chapter 13

Singular values

13.1 Introduction
In this chapter, we refer to the following books: Bellman [74], Ciarlet [153],
Dacorogna-Marcellini [202], Horn-Johnson [345] and [346], Marshall-Olkin [432]
and Serre [531].

13.2 Definition and basic properties
We collect here the definition and some properties of singular values of matrices
(see Ciarlet [153] Theorems 1.2.1 and 1.2.2; Section 7.3 in Horn-Johnson [345]
and Section 3.1 in [346]). First we recall our notation for matrices \( \xi \in \mathbb{R}^{N \times n} \),
which we write as
\[
\begin{pmatrix}
\xi_1^1 & \cdots & \xi_n^1 \\
\vdots & \ddots & \vdots \\
\xi_1^N & \cdots & \xi_n^N
\end{pmatrix}
= \left( \begin{pmatrix} \xi_1^1 \\ \vdots \\ \xi_n^N \end{pmatrix} \right) = (\xi_1, \cdots, \xi_n).
\]
We start with the following well known notation (in the sequel \( n \) and \( N \) denote
positive integers).

**Definition 13.1**

(i) We denote by \( GL(n) \) the set of invertible matrices in \( \mathbb{R}^{n \times n} \).

(ii) The set of orthogonal matrices is denoted by \( O(n) \). It is the set of
matrices \( R \in \mathbb{R}^{n \times n} \) such that
\[
RR^t = I,
\]
where \( I \) denotes the identity matrix in \( \mathbb{R}^{n \times n} \).

(iii) The set of special orthogonal matrices, denoted by \( SO(n) \), is the subset
of \( O(n) \) where the matrices satisfy
\[
det R = 1.
\]
(iv) We denote by $\mathbb{R}^{N \times n}_d$ the set of diagonal matrices, meaning that $\xi \in \mathbb{R}^{N \times n}_d$ if and only if $\xi_{ji} = 0$ whenever $i \neq j$.

In particular, if $N \geq n$ (and similarly if $N < n$), such a matrix is written as

$$\xi = \text{diag}_{N \times n}(\xi_1^1, \ldots, \xi_n^n)$$

and when $N = n$, we simply let

$$\xi = \text{diag}(\xi_1^1, \ldots, \xi_n^n).$$

We now give the definition of the singular values.

**Definition 13.2** (i) Let $N \leq n$ and $\xi \in \mathbb{R}^{N \times n}_d$. The singular values of $\xi$, denoted by $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_N(\xi)$, are defined to be the square root of the eigenvalues of the symmetric and positive semidefinite matrix $\xi\xi^t \in \mathbb{R}^{N \times N}$.

(ii) Let $N \geq n$ and $\xi \in \mathbb{R}^{N \times n}_d$. The singular values of $\xi$, denoted by $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$, are defined to be the square root of the eigenvalues of the symmetric and positive semidefinite matrix $\xi^t\xi \in \mathbb{R}^{n \times n}$.

The following theorem is the standard decomposition theorem (see Theorem 7.3.5 in [345] or Theorem 3.1.1 in [346], for example).

**Theorem 13.3 (Singular values decomposition theorem)** (i) Let $N \leq n$, $\xi \in \mathbb{R}^{N \times n}_d$ and $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_N(\xi)$ be its singular values. Then there exists $R \in O(N)$ such that, $\delta_{ij}$ denoting the Kronecker symbol,

$$R\xi = \tilde{\xi} = \begin{pmatrix} \tilde{\xi}_1^1 \\ \vdots \\ \tilde{\xi}_N^N \end{pmatrix}, \text{ with } \langle \tilde{\xi}_i^1; \tilde{\xi}_j^j \rangle = |\tilde{\xi}_i^1| |\tilde{\xi}_j^j| \delta_{ij}, \lambda_i(\xi) = |\tilde{\xi}_i^i|.$$

Furthermore, there exists $Q \in O(n)$ such that

$$(R\xi Q)^i_j = \begin{cases} 0 & \text{if } i \neq j \\ \lambda_i(\xi) & \text{if } i = j \end{cases}$$

or equivalently $R\xi Q \in \mathbb{R}^{N \times n}_d$ and

$$R\xi Q = \text{diag}_{N \times n}(\lambda_1(\xi), \ldots, \lambda_N(\xi)).$$
(ii) Let $N \geq n$, $\xi \in \mathbb{R}^{N \times n}$ and $0 \leq \lambda_1 (\xi) \leq \cdots \leq \lambda_n (\xi)$ be its singular values. Then there exists $Q \in O (n)$ such that, $\delta_{ij}$ denoting the Kronecker symbol,

$$\xi Q = \tilde{\xi} = \left( \tilde{\xi}_1, \cdots, \tilde{\xi}_n \right), \text{ with } \langle \tilde{\xi}_i; \tilde{\xi}_j \rangle = | \tilde{\xi}_i | | \tilde{\xi}_j | \delta_{ij}, \lambda_i (\xi) = | \tilde{\xi}_i |.$$

Moreover, there exists $R \in O (N)$ such that

$$\begin{aligned}
(R\xi Q)^i_j &= \begin{cases}
0 & \text{if } i \neq j \\
\lambda_i (\xi) & \text{if } i = j
\end{cases}
\end{aligned}$$

or equivalently $R\xi Q \in \mathbb{R}^{N \times n}$ and

$$R\xi Q = \text{diag}_N (\lambda_1 (\xi), \cdots, \lambda_n (\xi)).$$

We gather below some elementary facts about the singular values, that can be deduced easily from the above theorem (see Dacorogna-Marcellini [202] page 171).

**Proposition 13.4** Let $0 \leq \lambda_1 (\xi) \leq \cdots \leq \lambda_n (\xi)$ be the singular values of the matrix $\xi \in \mathbb{R}^{n \times n}$. Then

$$|\xi|^2 = \sum_{i,j=1}^n (\xi^i_j)^2 = \sum_{i=1}^n (\lambda_i (\xi))^2,$$

$$|\text{adj}_s \xi|^2 = \sum_{i_1 < \cdots < i_s} (\lambda_{i_1} (\xi))^2 \cdots (\lambda_{i_s} (\xi))^2,$$

$$|\det \xi| = \prod_{i=1}^n \lambda_i (\xi),$$

where $\text{adj}_s \xi \in \mathbb{R}^{(s^s) \times (s)}$ denotes the matrix obtained by forming all the $s \times s$ minors, $2 \leq s \leq n - 1$, of the matrix $\xi$. (See Section 5.4 for the notation.) In particular, if $n = 3$, then

$$|\xi|^2 = \sum_{i,j=1}^3 (\xi^i_j)^2 = (\lambda_1 (\xi))^2 + (\lambda_2 (\xi))^2 + (\lambda_3 (\xi))^2,$$

$$|\text{adj}_2 \xi|^2 = (\lambda_1 (\xi) \lambda_2 (\xi))^2 + (\lambda_1 (\xi) \lambda_3 (\xi))^2 + (\lambda_2 (\xi) \lambda_3 (\xi))^2,$$

$$|\det \xi| = \lambda_1 (\xi) \lambda_2 (\xi) \lambda_3 (\xi).$$

**Proof.** We decompose the proof into two steps.

*Step 1.* We recall some elementary facts about $\text{adj}_s \xi$ (see Proposition 5.66). Let $\xi, \eta \in \mathbb{R}^{n \times n}$ and $1 \leq s \leq n$ (by abuse of notation, we write $\xi = \text{adj}_1 \xi$ and $\det \xi = \text{adj}_n \xi$). Then

$$\text{adj}_s (\xi \eta) = \text{adj}_s \xi \text{ adj}_s \eta, \quad \text{adj}_s (\xi^t) = (\text{adj}_s \xi)^t.$$
This implies that if \( U \in O(n) \) then
\[
\text{adj}_s U \in O\left(\binom{n}{s}\right).
\]
Indeed, if for \( N \) an integer we denote by \( I_N \) the identity matrix in \( \mathbb{R}^{N \times N} \), we have
\[
(\text{adj}_s U) (\text{adj}_s U)^t = \text{adj}_s (UU^t) = \text{adj}_s (I_n) = I_n.
\]

**Step 2.** Observe first that by Step 1 and Theorem 13.3, we can find, for every \( \xi \in \mathbb{R}^{n \times n} \), \( Q, R \in O(n) \) such that
\[
R \xi Q = \Lambda := \text{diag}\left(\lambda_1(\xi), \cdots, \lambda_n(\xi)\right)
\]
\[
\text{adj}_s R \text{adj}_s \xi \text{adj}_s Q = \text{adj}_s \Lambda.
\]
Note next that if \( \zeta \in \mathbb{R}^{N \times N} \) and if we denote by
\[
|\zeta|^2 = \sum_{i,j=1}^N (\zeta_{ij})^2
\]
then it is well known (see, for example, Theorem 1.4.4 in [153]) that
\[
|\zeta U| = |U \zeta| = |\zeta| \quad \text{for every} \quad U \in O(N).
\]
Combining the two above observations, the second one applied to \( \zeta = \text{adj}_s \xi \), we immediately deduce the proposition. ■

**Remark 13.5** From the above proof, it is clear that if \( \xi \in \mathbb{R}^{n \times n} \) has singular values \( 0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi) \), and if \( 1 \leq s \leq n - 1 \), then \( \text{adj}_s \xi \) has singular values \( \lambda_{i_1}(\xi) \cdots \lambda_{i_s}(\xi) \), where \( 1 \leq i_1 < \cdots < i_s \leq n \). ♦

When \( n = 2 \), a simple formula for the singular values can be obtained from Proposition 13.4.

**Proposition 13.6** Let \( \xi \in \mathbb{R}^{2 \times 2} \) and \( 0 \leq \lambda_1(\xi) \leq \lambda_2(\xi) \) be its singular values. Then
\[
\lambda_1(\xi) = \frac{1}{2} \left[ \sqrt{|\xi|^2 + 2|\det \xi|} - \sqrt{|\xi|^2 - 2|\det \xi|} \right],
\]
\[
\lambda_2(\xi) = \frac{1}{2} \left[ \sqrt{|\xi|^2 + 2|\det \xi|} + \sqrt{|\xi|^2 - 2|\det \xi|} \right].
\]

In dimension two there is also a standard way of decomposing matrices that is useful for computing singular values (see Alibert-Dacorogna [14] and Section 5.3.8), namely the following.
Signed singular values and von Neumann type inequalities

Remark 13.7 For \( \xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \end{pmatrix} \) define

\[ \tilde{\xi} = \begin{pmatrix} \xi_2^2 & -\xi_1^2 \\ -\xi_2^1 & \xi_1^1 \end{pmatrix} \]

and

\[ \xi^+ = \frac{1}{2} (\xi + \tilde{\xi}), \quad \xi^- = \frac{1}{2} (\xi - \tilde{\xi}). \]

Then the following properties hold (where we denote the scalar product in \( \mathbb{R}^{2 \times 2} \) by \( \langle ; , \rangle \))

\[ \xi = \xi^+ + \xi^-, \quad \tilde{\xi} = \xi^+ - \xi^- \]

\[ 2 \det \xi^+ = |\xi^+|^2, \quad 2 \det \xi^- = -|\xi^-|^2 \]

\[ |\xi|^2 = |\xi^+|^2 + |\xi^-|^2 \]

\[ 2 \det \xi = |\xi^+|^2 - |\xi^-|^2 = \langle \xi; \tilde{\xi} \rangle = 2 \det \xi^+ + 2 \det \xi^- \]

\[ |\xi^+|^2 = \frac{1}{2} [ |\xi|^2 + 2 \det \xi], \quad |\xi^-|^2 = \frac{1}{2} [ |\xi|^2 - 2 \det \xi]. \]

In particular, from the last identities and the above proposition, we deduce that

\[ \lambda_1 (\xi) = \frac{1}{\sqrt{2}} \left| |\xi^+| - |\xi^-| \right|, \]

\[ \lambda_2 (\xi) = \frac{1}{\sqrt{2}} \left( |\xi^+| + |\xi^-| \right). \]

\[ \diamond \]

13.3 Signed singular values and von Neumann type inequalities

We now define the concept of signed singular values, valid only when \( N = n \). Given \( \xi \in \mathbb{R}^{n \times n} \), we denote by

\[ 0 \leq |\mu_1 (\xi)| \leq \cdots \leq \mu_n (\xi) \]

the signed singular values; they are defined by

\[ \mu_1 (\xi) = \lambda_1 (\xi) \text{sign} (\det \xi) \quad \text{and} \quad \mu_j (\xi) = \lambda_j (\xi), \quad j = 2, \cdots, n, \]

where \( 0 \leq \lambda_1 (\xi) \leq \cdots \leq \lambda_n (\xi) \) are the singular values of the matrix \( \xi \in \mathbb{R}^{n \times n} \).

From Theorem 13.3, we immediately deduce that, for every \( \xi \in \mathbb{R}^{n \times n} \), there exist \( Q, R \in SO (n) \) so that

\[ R\xi Q = \text{diag} (\mu_1 (\xi), \cdots, \mu_n (\xi)). \]
In the sequel we write 
\[ \lambda(\xi) = (\lambda_1(\xi), \cdots, \lambda_n(\xi)) \quad \text{and} \quad \mu(\xi) = (\mu_1(\xi), \cdots, \mu_n(\xi)). \]

When we consider matrices \( \xi \in \mathbb{R}^{N \times n} \), we always assume that \( N \geq n \), the case \( N < n \) being immediately inferred from the previous one by transposition.

This section is devoted to von Neumann type inequalities (see Theorem 13.10 below). We follow here the approach of Dacorogna-Maréchal [204], inspired by Rosakis [516]. It combines a variational argument and the resolution of some discrete optimization problem. The main advantage of our proof is that we get the classical von Neumann inequality as a by-product, while Rosakis uses it in his proof. We need the following technical result.

**Lemma 13.8**  
(i) Let \( D \in \mathbb{R}^{n \times n} \) be diagonal, with diagonal entries whose absolute values are pairwise distinct. If \( M \in \mathbb{R}^{n \times n} \) is such that both \( MD \) and \( DM \) are symmetric, then \( M \) is diagonal.

(ii) Let \( D \in \mathbb{R}^{N \times n} \) be diagonal (\( N > n \)), with nonzero diagonal entries whose absolute values are pairwise distinct. If \( M \in \mathbb{R}^{n \times N} \) is such that both \( MD \) and \( DM \) are symmetric, then \( M \) is diagonal.

**Proof.**  
(i) Let \( D = \text{diag}(d_1, \cdots, d_n) \). Assuming that \( MD \) and \( DM \) are symmetric, we have 
\[ MD^2 = DM^tD = D^2M \]
where \( D^2 \) is diagonal and has pairwise distinct diagonal entries. We therefore have for every \( i, j \in \{1, \cdots, n\} \)
\[ (MD^2)_i^j = m_i^j d_j^2 \quad \text{and} \quad (D^2M)_i^j = m_i^j d_i^2. \]
If \( i \neq j \), then \( d_i^2 \neq d_j^2 \) and thus \( m_i^j = 0 \).

(ii) Let us write 
\[ D = \begin{bmatrix} \Delta \\ Z \end{bmatrix} \quad \text{with} \quad \Delta = \text{diag}(d_1, \cdots, d_n) \quad \text{and} \quad Z = 0 \in \mathbb{R}^{(N-n) \times n}. \]
Assuming that \( MD \) and \( DM \) are symmetric, we get 
\[ MDD^t = D^tM^tD^t = D^tDM. \]
Writing 
\[ M = [M_1, M_2] \quad \text{with} \quad M_1 \in \mathbb{R}^{n \times n} \quad \text{and} \quad M_2 \in \mathbb{R}^{n \times (N-n)} \]
we find that the above equation implies that 
\[ M_1\Delta^2 = \Delta^2M_1 \quad \text{and} \quad \Delta^2M_2 = 0. \]
Part (i) then shows that $M_1$ is diagonal and, since $\Delta^2$ is diagonal with nonzero entries, we get $M_2 = 0$. The proof is therefore complete. ■

The following proposition may be regarded as a primary version of our generalized von Neumann inequality for diagonal matrices.

**Proposition 13.9** Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ and $\tau$ be a permutation of \{1, $\ldots$, $n$\} such that

$|b_1| \leq b_2 \leq \cdots \leq b_n$ and $|a_{\tau(1)}| \leq \cdots \leq |a_{\tau(n)}|.$

(i) If $\prod_{j=1}^n a_j \geq 0$, then

$$\sum_{j=1}^n a_j b_j \leq |a_{\tau(1)}| b_1 + \sum_{j=2}^n |a_{\tau(j)}| b_j.$$  

(ii) If $\prod_{j=1}^n a_j < 0$, then

$$\sum_{j=1}^n a_j b_j \leq -|a_{\tau(1)}| b_1 + \sum_{j=2}^n |a_{\tau(j)}| b_j.$$  

In other words, if $b$ belongs to the set

$$K^n := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_1| \leq x_2 \leq \cdots \leq x_n\},$$

then, letting $\Pi_e(n)$ be as in Notation 5.30,

$$\max_{M \in \Pi_e(n)} \langle Ma; b \rangle = \langle \mu (\text{diag } a); b \rangle.$$  

**Proof.** Step 1: $n = 2$. It says that, if $|b_1| \leq b_2$ and if $\tau$ is a permutation of \{1, 2\} such that $|a_{\tau(1)}| \leq |a_{\tau(2)}|$, then

(i') $a_1 a_2 \geq 0$ implies

$$a_1 b_1 + a_2 b_2 \leq |a_{\tau(1)}| b_1 + |a_{\tau(2)}| b_2.$$  

(ii') $a_1 a_2 < 0$ implies

$$a_1 b_1 + a_2 b_2 \leq -|a_{\tau(1)}| b_1 + |a_{\tau(2)}| b_2.$$  

Let us prove the claim.

**Case 1:** $a_1 a_2 \geq 0$. Note that

$$(|a_{\tau(2)}| - a_2) b_2 + (|a_{\tau(1)}| - a_1) b_1 = \begin{cases} 
(|a_2| - a_2) b_2 + (|a_1| - a_1) b_1 & \text{if } |a_2| \geq |a_1| \\
(|a_1| - a_2) b_2 + (|a_2| - a_1) b_1 & \text{if } |a_2| \leq |a_1|.
\end{cases}$$  

Since $|b_1| \leq b_2$, a direct inspection shows that both quantities are non-negative and the claim (i') follows.
Case 2: $a_1a_2 < 0$. Observe that
\[
|a_{\tau(2)}| - a_2b_2 - (|a_{\tau(1)}| + a_1)b_1 = \begin{cases} 
(a_2 - a_2)b_2 - (a_1 + a_1)b_1 & \text{if } |a_2| \geq |a_1| \\
(a_1 - a_2)b_2 - (a_2 + a_1)b_1 & \text{if } |a_2| \leq |a_1|.
\end{cases}
\]
The fact that $|b_1| \leq b_2$ easily leads to the claim (ii').

Step 2: $n > 2$. We will use the rules of Step 1 to prove the result in the general case. The given permutation $\tau$ will be decomposed as a well chosen product of transpositions, each of them giving rise to an inequality via (i') or (ii'). For example, assuming that $|a_k| \geq |a_{k+1}|$ for some $k$, we can write, if $a_k a_{k+1} \geq 0$,
\[
a_1b_1 + \cdots + a_kb_k + a_{k+1}b_{k+1} + \cdots + a_nb_n \\
\leq a_1b_1 + \cdots + |a_{k+1}|b_k + |a_k|b_{k+1} + \cdots + a_nb_n \tag{13.1}
\]
or, if $a_k a_{k+1} < 0$,
\[
a_1b_1 + \cdots + a_kb_k + a_{k+1}b_{k+1} + \cdots + a_nb_n \\
\leq a_1b_1 + \cdots - |a_{k+1}|b_k + |a_k|b_{k+1} + \cdots + a_nb_n. \tag{13.2}
\]
Since the $b_k$ will keep the same place throughout, we will symbolize inequalities such as (13.1) and (13.2) by
\[
(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n) \to (a_1, \ldots, |a_{k+1}|, |a_k|, \ldots, a_n), \tag{13.3}
\]
\[
(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n) \to (a_1, \ldots, -|a_{k+1}|, |a_k|, \ldots, a_n), \tag{13.4}
\]
respectively.

Case 1: $b_1 \geq 0$. We have to consider two subcases.

Case 1.1: $\prod_{j=1}^n a_j \geq 0$. Clearly,
\[
(a_1, \ldots, a_n) \to (|a_1|, \ldots, |a_n|).
\]
Now, $|a_{\tau(n)}|$ can migrate rightward by means of a transposition of type (13.3). Thus
\[
(|a_1|, \ldots, |a_n|) \to (|a_1|, \ldots, |a_{\tau(n)-1}|, |a_{\tau(n)+1}|, \ldots, |a_n-1|, |a_{\tau(n)}|).
\]
Repeating this process, with $|a_{\tau(n-1)}|$, $|a_{\tau(n-2)}|$ and so on, gives rise to the desired inequality.

Case 1.2: $\prod_{j=1}^n a_j < 0$. In this case, we decide to replace all but one of the negative $a_j$ by their absolute values: for example, if $a_k$ is negative,
\[
(a_1, \ldots, a_n) \to (|a_1|, \ldots, |a_{k-1}|, -|a_k|, |a_{k+1}|, \ldots, |a_n|).
\]
Now we let $|a_{\tau(n)}|$ migrate rightward using either a transposition of type (13.3) or a transposition of type (13.4) according to the signs of the elements under
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consideration. Each transposition leaves one negative element. Repeating this
process, with $|a_{\tau(n-1)}|, |a_{\tau(n-2)}|$ and so on, eventually sorts the $|a_j|$ according
to $\tau$ and gives rise to

$$
(|a_1|, \ldots, |a_{k-1}|, -|a_k|, |a_{k+1}|, \ldots, |a_n|)
$$

$$
\rightarrow
(|a_{\tau(1)}|, |a_{\tau(2)}|, \ldots, -|a_{\tau(l)}|, \ldots, |a_{\tau(n-1)}|, |a_{\tau(n)}|).
$$

Finally, it is clear that the minus sign is allowed to migrate leftward, since all
elements are now sorted in increasing order. Therefore,

$$
(|a_{\tau(1)}|, |a_{\tau(2)}|, \ldots, -|a_{\tau(l)}|, \ldots, |a_{\tau(n-1)}|, |a_{\tau(n)}|)
$$

$$
\rightarrow
(-|a_{\tau(1)}|, |a_{\tau(2)}|, \ldots, |a_{\tau(n)}|)
$$

and we are done.

Case 2: $b_1 < 0$. This is easily obtained from the above strategy by observing
that

$$
a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = (-a_1)(-b_1) + a_2 b_2 + \cdots + a_n b_n.
$$

This achieves the proof of the proposition. \[ \square \]

We are now ready to prove the main theorem of this section.

**Theorem 13.10** (i) Let $\xi, \eta \in \mathbb{R}^{n \times n}$. Then

$$
\max_{Q, R \in SO(n)} \{ \text{trace}(Q \xi R^t \eta^t) \} = \sum_{j=1}^{n} \mu_j(\xi) \mu_j(\eta)
$$

and consequently

$$
\text{trace}(\xi \eta^t) \leq \sum_{j=1}^{n} \mu_j(\xi) \mu_j(\eta).
$$

(ii) Let $\xi, \eta \in \mathbb{R}^{N \times n}$, where $N \geq n$. Then

$$
\max_{Q \in O(N)} \{ \text{trace}(Q \xi R^t \eta^t) \} = \sum_{j=1}^{n} \lambda_j(\xi) \lambda_j(\eta)
$$

and consequently

$$
\text{trace}(\xi \eta^t) \leq \sum_{j=1}^{n} \lambda_j(\xi) \lambda_j(\eta).
$$

**Remark 13.11** The set of all transformations $\xi \rightarrow U \xi V^t$ with $U, V \in SO(n)$, 
edowed with the composition, is obviously a group that is isomorphic to the 
product group $SO(n) \times SO(n)$. By abuse of notation, we may denote this group 
by $SO(n) \times SO(n)$. It results from Theorems 13.3 and 13.10 that the system

$$(\mathbb{R}^{n \times n}, SO(n) \times SO(n), \text{diag} \circ \mu)$$
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\( \text{(i) diag} \circ \mu \) is \( SO(n) \times SO(n) \)-invariant;

\( \text{(ii) for all } \xi \in \mathbb{R}^{n \times n}, \text{ there exists } (U, V) \in SO(n) \times SO(n) \text{ such that } \xi = U \text{diag} (\mu (\xi)) V^t; \)

\( \text{(iii) for all } \xi, \eta \in \mathbb{R}^{n \times n}, \) trace\( (\xi \eta^t) \leq \text{trace} (\text{diag} (\mu (\xi)) \text{diag} (\mu (\eta))). \)

According to the terminology of Lewis [401],

\( (\mathbb{R}^{n \times n}, SO(n) \times SO(n), \text{diag} \circ \mu) \)

is a normal decomposition system. Our preceding results also show that, similarly,

\( (\mathbb{R}^{N \times n}, O(N) \times O(n), \text{diag}_{N \times N} \circ \lambda) \)

is a normal decomposition system.

\( \text{♦} \)

**Proof.** (i) As already said, the beginning of our proof follows the one of Rosakis [516]. Observe first that we can assume that \( \eta \) satisfies

\[ \eta = \text{diag} (\mu_1 (\eta), \ldots, \mu_n (\eta)). \]  

(13.5)

As a matter of fact, suppose that the result is proved in this case. Let \( \zeta \) be any element of \( \mathbb{R}^{n \times n} \), and let \( U, V \in SO(n) \) be such that (cf. Theorem 13.3) \( \zeta = U MV^t \), with \( M := \text{diag} (\mu_1 (\zeta), \ldots, \mu_n (\zeta)) \). For all \( Q, R \in SO(n) \),

\[ \text{trace}(Q \xi R^t \zeta^t) = \text{trace}(Q \xi R^t V M U^t) = \text{trace}((U^t Q) (R^t V) M). \]

Since \( U^t SO(n) = SO(n) V = SO(n) \), we see that

\[
\max_{Q, R \in SO(n)} \{\text{trace}(Q \xi R^t \zeta^t)\} = \max_{Q_1, R_1 \in SO(n)} \{\text{trace}(Q_1 \xi R_1^t M)\}
\]

\[
= \sum_{j=1}^n \mu_j (\xi) \mu_j (M)
\]

\[
= \sum_{j=1}^n \mu_j (\xi) \mu_j (\zeta),
\]

where the second equality results from the fact that \( M \) satisfies (13.5).

Notice that we can also assume, in addition to (13.5), that \( \eta \) satisfies \( |\mu_1 (\eta)| < \mu_2 (\eta) < \cdots < \mu_n (\eta) \), since a continuity argument will then allow to extend the result to the case of wide inequalities.

Since \( SO(n) \times SO(n) \) is compact and the function \( (Q, R) \rightarrow \text{trace}(Q \xi R^t \eta^t) \) is continuous, there exist \( Q_0, R_0 \in SO(n) \) such that

\[ \text{trace}(Q_0 \xi R_0^t \eta^t) = \max_{Q, R \in SO(n)} \{\text{trace}(Q \xi R^t \eta^t)\}. \]  

(13.6)
We now prove that $Q_0$ and $R_0$ must be such that $Q_0 \xi R_0^t$ is diagonal. Let $A$ and $B$ be skew-symmetric matrices, that is, $A^t = -A$ and $B^t = -B$. For all $s \in \mathbb{R}$, let

$$Q(s) := e^{sA}Q_0 \quad \text{and} \quad R(s) := e^{sB}R_0.$$ 

Clearly, $Q(s)$ and $R(s)$ are in $SO(n)$, and the function

$$\varphi(s) := \text{trace}(Q(s) \xi R(s)^t \eta^t)$$

is differentiable. The optimality condition (13.6) implies that $s = 0$ maximizes $\varphi$. Consequently,

$$0 = \varphi'(0) = \text{trace}(AQ_0 \xi R_0^t \eta^t) + \text{trace}(Q_0 \xi R_0^t B^t \eta^t).$$

We have therefore shown that, for all skew-symmetric matrices $A$ and $B$,

$$\text{trace}(AQ_0 \xi R_0^t \eta^t) = \langle A; (Q_0 \xi R_0^t \eta^t)^t \rangle = 0,$$

$$\text{trace}(\eta^t Q_0 \xi R_0^t B^t) = \langle \eta^t Q_0 \xi R_0^t; B \rangle = 0.$$

Recall that $\mathbb{R}^{n \times n}$ is the orthogonal direct sum of $\mathbb{R}_s^{n \times n}$ and $\mathbb{R}_{as}^{n \times n}$, the subspaces of symmetric and skew-symmetric matrices, respectively. Therefore, the above conditions tell us that $Q_0 \xi R_0^t \eta^t$ and $\eta^t Q_0 \xi R_0^t$ must be symmetric. Lemma 13.8 (i) then implies that $Q_0 \xi R_0^t$ is diagonal. We have thus shown so far that

$$\max_{Q,R \in SO(n)} \{\text{trace}(Q \xi R^t \eta^t)\} = \text{trace}(Q_0 \xi R_0^t \eta^t),$$

where $Q_0, R_0 \in SO(n)$ are such that $Q_0 \xi R_0^t$ is diagonal. It remains to see that $Q_0$ and $R_0$ are such that

$$Q_0 \xi R_0^t = \text{diag}(\mu_1(\xi), \cdots, \mu_n(\xi)).$$

But this is an immediate consequence of Proposition 13.9.

(ii) The case where $N = n$, which results immediately from Part (i), corresponds to Von Neumann inequality itself. Thus, let us assume that $N > n$. The argument is analogous to that of Part (i), so we merely outline the main steps. We can assume that $\eta$ satisfies

$$\eta = \text{diag}_{n \times n}(\lambda_1(\eta), \cdots, \lambda_n(\eta)), \quad (13.7)$$

with $0 < \lambda_1(\eta) < \cdots < \lambda_n(\eta)$, the case of non strict inequalities being deduced by a passage to the limit. The compactness of $O(N) \times O(n)$ and the continuity of the function $(Q, R) \rightarrow \text{trace}(Q \xi R^t \eta^t)$ imply the existence of $Q_0 \in O(N)$ and $R_0 \in O(n)$ such that

$$\text{trace}(Q_0 \xi R_0^t \eta^t) = \max_{Q \in O(N)} \{\text{trace}(Q \xi R^t \eta^t)\}. \quad (13.8)$$
The same variational argument as in Part (i), together with Lemma 13.8 (ii), shows that $Q_0$ and $R_0$ must be such that $Q_0 \xi R_0^t$ is diagonal. Finally, it is clear (similarly to Proposition 13.9) that, among all diagonal $(N \times n)$-matrices $\xi'$ with prescribed singular values $\lambda_1(\xi), \ldots, \lambda_n(\xi)$, the matrix

$$\text{diag}_{N \times n}(\lambda_1(\xi), \ldots, \lambda_n(\xi))$$

maximizes $\text{trace}(\xi \eta^t)$. Thus we must have

$$Q_0 \xi R_0^t = \text{diag}_{N \times n}(\lambda_1(\xi), \ldots, \lambda_n(\xi)),$$

and the result follows. ■

Observe that, in the square case,

$$-\text{trace}(\xi \eta^t) = \text{trace}(-\xi \eta^t) \leq \sum_{j=1}^n \lambda_j (-\xi) \lambda_j (\eta) = \sum_{j=1}^n \lambda_j (\xi) \lambda_j (\eta),$$

so that

$$|\text{trace}(\xi \eta^t)| \leq \sum_{j=1}^n \lambda_j (\xi) \lambda_j (\eta)$$

for all $\xi, \eta \in \mathbb{R}^{n \times n}$. This is the classical von Neumann inequality (see von Neumann [591], Mirsky [447] or Section 7.4 in Horn-Johnson [345]).

It is worth noticing that the analogous inequality for signed singular values holds as well if $n$ is even.

**Corollary 13.12** Let $\xi, \eta \in \mathbb{R}^{n \times n}$. If $n$ is even, then

$$|\text{trace}(\xi \eta^t)| \leq \sum_{j=1}^n \mu_j (\xi) \mu_j (\eta). \quad (13.9)$$

If $n$ is odd, (13.9) is false in general.

**Proof.** If $n$ is even, then $\det(-\xi) = \det \xi$ and $\mu_j (-\xi) = \mu_j (\xi)$ for all $j = 1, \ldots, n$. Since

$$\text{trace}(-\xi \eta^t) = -\text{trace}(\xi \eta^t),$$

we conclude that both $\text{trace}(\xi \eta^t)$ and $-\text{trace}(\xi \eta^t)$ are not larger than $\sum_{j=1}^n \mu_j (\xi) \mu_j (\eta)$.

If $n$ is odd, counterexamples are easy to construct. For example, if $n = 3$, let $\xi := \text{diag} (-1, 1, 1)$ and $\eta := \text{diag} (1, -1, -1)$. Then

$$\text{trace}(\xi \eta^t) = -3 \quad \text{and} \quad \sum_{j=1}^3 \mu_j (\xi) \mu_j (\eta) = 1.$$

This finishes the proof. ■

The next result, which we do not use, relates the eigenvalues and the singular values of a given matrix (see Theorem 3.3.2 in Horn-Johnson [346]). It should be compared, at least formally, to Theorem 7.43 in Chapter 7 (see also Buliga [108]).
Theorem 13.13 (Weyl theorem) Let $\xi \in \mathbb{R}^{n \times n}$ and denote by $0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)$ its singular values and by $\sigma_1(\xi), \cdots, \sigma_n(\xi)$ its eigenvalues, which are complex in general, ordered by their modulus ($0 \leq |\sigma_1(\xi)| \leq \cdots \leq |\sigma_n(\xi)|$). Then the following result holds

$$\prod_{i=1}^{n} |\sigma_i(\xi)| = \prod_{i=1}^{n} \lambda_i(\xi) = |\det \xi|.$$
Chapter 14

Some underdetermined partial differential equations

14.1 Introduction

In this chapter, we consider Dirichlet problems associated with some underdetermined equations, that are important in geometry as well as in applications, notably in fluid mechanics and elasticity.

We start in Section 14.2.2 by studying the equation $\text{div } u = f$, while in Section 14.2.3 we investigate the system $\text{curl } u = f$ in dimension 3 (a similar result holds in any dimension, see below). We solve both problems in Hölder spaces; of course similar results exist (see the bibliography below) in $L^p$ spaces.

Both problems are part of a more general program, that of solving the Dirichlet problem associated with $du = f$, where $u$ is a $k$ form and $d$ stands for the exterior derivative. This problem is fundamental in differential geometry and algebraic topology. The literature on the subject is vast and we refer for further references to Dacorogna [183], where the problem is solved in Hölder spaces $C^{m,\alpha}$ and to Schwarz [528], where the problem is studied in $L^p$ spaces.

In Section 14.3 we discuss a Dirichlet problem associated with the non-linear equation $\text{det } \nabla u = f$. Note that, in terms of fluid mechanics, this last equation is the Lagrangian version of the equation $\text{div } u = f$. The nonlinear problem is solved in Hölder spaces.

14.2 The equations $\text{div } u = f$ and $\text{curl } u = f$

14.2.1 A preliminary lemma

We start with this elementary lemma, whose proof can be found in Dacorogna-Moser [207]. This lemma is used to fix the boundary data. We denote by $\| \cdot \|_{C^{m,\alpha}}$ the $C^{m,\alpha}$ norm (see Section 12.3 for details).
Lemma 14.1 Let \( m \geq 1 \) be an integer and \( 0 < \alpha < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded connected open set with a \( C^{m+2,\alpha} \) boundary consisting of finitely many connected components (\( \nu \) denotes the outward unit normal). Let \( c \in C^{m,\alpha} (\overline{\Omega}) \), then there exists \( b \in C^{m+1,\alpha} (\overline{\Omega}) \) satisfying
\[
\text{grad} \, b = c \nu \text{ on } \partial \Omega.
\]
Furthermore, there exists \( K = K (\alpha, m, \Omega) > 0 \) such that
\[
\|b\|_{C^{m+1,\alpha}} \leq K \|c\|_{C^{m,\alpha}}.
\]

Proof. If one is not interested in the sharp regularity result, a solution of the problem is given by
\[
b(x) := -c(x) \zeta (\text{dist} (x, \partial \Omega)),
\]
where \( \text{dist} (x, \partial \Omega) \) stands for the distance from \( x \) to the boundary and \( \zeta \) is a smooth function such that \( \zeta (0) = 0, \zeta' (0) = 1 \) and \( \zeta \equiv 0 \) outside a small neighborhood of \( 0 \).

To construct a smoother solution we proceed as follows. First find a \( C^{m+1,\alpha} (\overline{\Omega}) \) solution of (see Gilbarg-Trudinger [313] or Ladyzhenskaya-Uraltseva [388])
\[
\begin{align*}
\Delta d &= \int_{\partial \Omega} c \, d\sigma / \text{meas } \Omega \quad \text{in } \Omega \\
\partial d / \partial \nu &= c \quad \text{on } \partial \Omega.
\end{align*}
\]
Moreover there exists \( K = K (\alpha, m, \Omega) > 0 \) such that
\[
\|d\|_{C^{m+1,\alpha}} \leq K \|c\|_{C^{m,\alpha}}. \tag{14.1}
\]
We then let \( \chi, \zeta \in C^\infty (\mathbb{R}) \) be such that \( \chi, \zeta \equiv 0 \) outside a small neighborhood of \( 0 \) and
\[
\chi (0) = 1, \; \zeta (0) = 0, \; \chi' (0) = 0, \; \zeta' (0) = 1.
\]
Define
\[
b(x) := d(x) - \chi (\text{dist} (x, \partial \Omega)) d(\psi (x)) \tag{14.2}
\]
and
\[
\psi (x) := x - \zeta (\text{dist} (x, \partial \Omega)) \text{grad} (\text{dist} (x, \partial \Omega)).
\]
It remains to check that \( b \) has the claimed property. Indeed, if \( x \in \partial \Omega \) (note that \( \psi (x) = x \) on \( \partial \Omega \)), then
\[
\begin{align*}
\text{grad} \, b (x) &= \text{grad} \, d (x) - \text{grad} \, d (\psi (x)) \nabla \psi (x) \\
&= \text{grad} \, d (x) - \text{grad} \, d (x) [I - \text{grad} (\text{dist} (x, \partial \Omega)) \otimes \text{grad} (\text{dist} (x, \partial \Omega))] \\
&= \text{grad} \, d (x) \left[ \nu \otimes \nu \right] = \frac{\partial d}{\partial \nu} \nu \\
&= c \nu.
\end{align*}
\]
The equations \( \text{div } u = f \) and \( \text{curl } u = f \)

From (14.1), (14.2) and the fact that the distance function is (near the boundary) as smooth as the boundary itself provided the boundary is at least \( C^2 \) (see, for example, Gilbarg-Trudinger [313]), we deduce that

\[
\|b\|_{C^{m+1,\alpha}} \leq K \|c\|_{C^{m,\alpha}}
\]

as wished. \( \blacksquare \)

Once the functions \( \chi \) and \( \zeta \) are fixed, the above construction has defined a bounded linear operator

\[
A : C^{m,\alpha}(\overline{\Omega}) \to C^{m+1,\alpha}(\overline{\Omega})
\]

that to every \( c \in C^{m,\alpha}(\overline{\Omega}) \) associates a unique \( b \in C^{m+1,\alpha}(\overline{\Omega}) \) such that

\[
\text{grad } b = c\nu \text{ on } \partial\Omega.
\]

14.2.2 The case \( \text{div } u = f \)

**Theorem 14.2** Let \( m \geq 0 \) be an integer and \( 0 < \alpha < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded connected open set with a \( C^{m+3,\alpha} \) boundary consisting of finitely many connected components (\( \nu \) denotes the outward unit normal). The following conditions are then equivalent.

(i) \( f \in C^{m,\alpha}(\overline{\Omega}) \) satisfies

\[
\int_{\Omega} f(x) \, dx = 0.
\]

(ii) There exists \( u \in C^{m+1,\alpha}(\overline{\Omega}; \mathbb{R}^n) \) verifying

\[
\begin{align*}
\text{div } u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega,
\end{align*}
\]

where \( \text{div } u = \sum_{i=1}^n \frac{\partial u^i}{\partial x_i} \). Furthermore, there exists \( K = K(\alpha, m, \Omega) > 0 \) such that

\[
\|u\|_{C^{m+1,\alpha}} \leq K \|f\|_{C^{m,\alpha}}.
\]

**Remark 14.3** (i) If the set \( \Omega \) is disconnected, then the result holds true if the compatibility condition is understood on each connected component.

(ii) This problem has been investigated by several authors, in particular Bogovski [89], Borchers-Sohr [92], Dacorogna [183], Dacorogna-Moser [207], Dautray-Lions [221], Galdi [298], Girault-Raviart [314], Kapitanski-Pileckas [359], Ladyzhenskaya [386], Ladyzhenskaya-Solonnikov [387], Necas [473], Tartar [567] and Von Wahl [592], [593]. We follow here the presentation of Dacorogna [183], which is, however, similar to many of the above mentioned articles.
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(iii) Similar type of results hold in $L^p, 1 < p < \infty$, see the above bibliography. However, the result is false if $p = 1$ or $p = \infty$ and it is also false in $C^{0,\alpha}$ when $\alpha = 0$ or $\alpha = 1$, see Bourgain-Brézis [97], Dacorogna-Fusco-Tartar [187], McMullen [408] and Preiss [500].

(iv) In fact, the proof of the theorem shows that if

$$X := \{ f \in C^{m,\alpha}(\overline{\Omega}) : \int_{\Omega} f(x) \, dx = 0 \},$$

$$Y := \{ u \in C^{m+1,\alpha}(\overline{\Omega} ; \mathbb{R}^n) : u = 0 \text{ on } \partial \Omega \},$$

then we can construct a bounded linear operator $L : X \to Y$ which associates to every $f \in X$, a unique $u = Lf \in Y$ satisfying (14.3). 

\[ \Box \]

**Proof.** (ii) $\Rightarrow$ (i). This implication is just the divergence theorem.

(i) $\Rightarrow$ (ii). We split the proof into two steps.

*Step 1.* We first find $a \in C^{m+2,\alpha}$ (see Gilbarg-Trudinger [313] or Ladyzhenskaya-Uraltseva [388]) satisfying

$$\begin{align*}
\Delta a &= f \quad \text{in } \Omega \\
\partial a / \partial \nu &= 0 \quad \text{on } \partial \Omega.
\end{align*}$$

Moreover there exists $K = K(\alpha, m, \Omega) > 0$ such that

$$\|a\|_{C^{m+2,\alpha}} \leq K \|f\|_{C^{m,\alpha}}. \quad (14.4)$$

*Step 2.* We then write

$$u = \text{curl}^* b + \text{grad} a \quad (14.5)$$

where $b = (b_{ij})_{1 \leq i < j \leq n} \in \mathbb{R}^{n(n-1)/2}$,

$$\text{curl}^* b := ((\text{curl}^* b)_1, \ldots, (\text{curl}^* b)_n)$$

and

$$(\text{curl}^* b)_i := \sum_{j=1}^{i-1} \frac{\partial b_{ji}}{\partial x_j} - \sum_{j=i+1}^{n} \frac{\partial b_{ij}}{\partial x_j}.$$

Since $\text{div} \text{curl}^* b = 0$ it remains to find $b \in C^{m+2,\alpha}$ such that

$$\text{curl}^* b = - \text{grad} a \text{ on } \partial \Omega.$$

An easy computation (using the fact that $\partial a / \partial \nu = 0$) shows that a solution of this problem is given by

$$\text{grad} b_{ij} = \left( \frac{\partial a}{\partial x_i} \nu_j - \frac{\partial a}{\partial x_j} \nu_i \right) \nu \text{ on } \partial \Omega.$$
The equations $\text{div } u = f$ and $\text{curl } u = f$

whose solvability is ensured by Lemma 14.1 and moreover there exists $K = K(\alpha, m, \Omega) > 0$ such that

$$\|b\|_{C^{m+2,\alpha}} \leq K \|a\|_{C^{m+2,\alpha}}.$$  \hspace{1cm} (14.6)

The combination of (14.4), (14.5) and (14.6) leads to the proof of the theorem. As in the proof of Lemma 14.1, we have also proved Remark 14.3 (iv).

In order to clarify the link with differential forms, we rewrite the proof in this terminology (see Dacorogna [183] for details). We consider $u$ as a 1 form and therefore the problem we want to solve is

$$\begin{cases}
\delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

where $\delta$ is the codifferential. We therefore write

$$u = da + \delta b$$

(where $a$ is a 0 form and $b$ is a 2 form). This leads to

$$f = \delta u = \delta da = \Delta a$$

since $\delta \delta b = 0$, $\Delta a = \delta da + d \delta a$ and $\delta a = 0$, $a$ being a 0 form. (The fact that $\Delta a = \delta da$ makes easier the case of 1 forms $u$ in comparison with $k$ forms $k \geq 2$).

We also observe that (for the exact definition of $d_\nu$ and $\delta_\nu$, see [183])

$$\delta_\nu (da) := \langle \text{grad } a; \nu \rangle = \frac{\partial a}{\partial \nu}$$

which leads to our choice in Step 1.

Now in order to have $u = 0$ on the boundary it remains to solve (cf. Step 2)

$$\delta b = -da \text{ on } \partial \Omega.$$  $$\Box$$

The idea is then to find a solution, via Lemma 14.1, of

$$\text{grad } b_{ij} = -[d_\nu da]_{ij} \nu = (\frac{\partial a}{\partial x_i} \nu_j - \frac{\partial a}{\partial x_j} \nu_i) \nu \text{ on } \partial \Omega$$

and then to check that such $b$ satisfies $\delta b = -da \text{ on } \partial \Omega$.

14.2.3 The case $\text{curl } u = f$

The problem under investigation is important in fluid mechanics and has been considered by Borchers-Sohr [92], Dacorogna [183], Dautray-Lions [221], Griesinger [322] and Von Wahl [592], [593]. In this section, we follow the approach of Dacorogna [183], which is inspired by that of Von Wahl [592], [593].
Theorem 14.4 Let $m \geq 1$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^3$ be a bounded convex set with a $C^{m+3,\alpha}$ boundary and $\nu$ denote the outward unit normal. The following conditions are then equivalent.

(i) $f \in C^{m,\alpha}(\overline{\Omega}; \mathbb{R}^3)$ verifies
\[
\text{div } f = 0 \text{ in } \Omega \quad \text{and} \quad \langle f; \nu \rangle = 0 \text{ on } \partial \Omega.
\]

(ii) There exists $u \in C^{m+1,\alpha}(\overline{\Omega}; \mathbb{R}^3)$ satisfying
\[
\begin{cases}
\text{curl } u = f & \text{in } \Omega \\
\text{div } u = 0 & \text{in } \Omega \\
u \wedge u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where if $u = (u^1, u^2, u^3)$, then
\[
\text{curl } u = \left(\frac{\partial u^3}{\partial x_2} - \frac{\partial u^2}{\partial x_3}, \frac{\partial u^1}{\partial x_3} - \frac{\partial u^3}{\partial x_1}, \frac{\partial u^2}{\partial x_1} - \frac{\partial u^1}{\partial x_2}\right).
\]

Proof. (ii) $\Rightarrow$ (i) The fact that $\text{div } f = 0$ is obvious. We now show that $\langle f; \nu \rangle = 0$ on $\partial \Omega$. For this purpose, we let $\psi \in C^2(\overline{\Omega})$ be an arbitrary function. The integration by parts formula and the facts that $u = 0$ on $\partial \Omega$, $\text{curl } u = f$ and $\text{div } f = 0$ lead to
\[
\int_{\Omega} \langle \text{grad } \psi; f \rangle \, dx = \int_{\Omega} \langle \text{grad } \psi; \text{curl } u \rangle \, dx = 0,
\]
\[
\int_{\Omega} \langle \text{grad } \psi; f \rangle \, dx = \int_{\partial \Omega} \psi \langle f; \nu \rangle \, d\sigma.
\]
Combining these two equations and the fact that $\psi$ is arbitrary, we have indeed obtained that $\langle f; \nu \rangle = 0$ on $\partial \Omega$.

(i) $\Rightarrow$ (ii) We divide the proof into two steps.

Step 1. We first find $w \in C^{m+1,\alpha}$ that solves the system (denoting the vectorial product by $w \wedge \nu$)
\[
\begin{cases}
\text{curl } w = f & \text{in } \Omega \\
\text{div } w = 0 & \text{in } \Omega \\
w \wedge \nu = 0 & \text{on } \partial \Omega.
\end{cases}
\]
This is possible, using a result for the existence part due to Kress [378] (see also Duff-Spencer [253] and Morrey [455] Sections 7.7 and 7.8). The regularity then follows from standard arguments (see Morrey [455]).

In terms of the notations of differential forms, we are in fact solving (considering $w$ as a 1 form and $f$ as a 2 form)
\[
\begin{cases}
d w = \tilde{f} & \text{in } \Omega \\
\delta w = 0 & \text{in } \Omega \\
d_\nu w = 0 & \text{on } \partial \Omega,
\end{cases}
\]
The equation \( \det \nabla u = f \)

where for \( f = (f_{12}, f_{13}, f_{23}) \) we let

\[ \tilde{f} = (f_{23}, -f_{13}, f_{12}). \]

The compatibility conditions for solving this problem are exactly

\[ d \tilde{f} = \text{div } f = 0 \text{ in } \Omega \quad \text{and} \quad d_{\nu} \tilde{f} = 0 \Leftrightarrow \langle f; \nu \rangle = 0 \text{ on } \partial \Omega. \]

**Step 2.** A solution of our problem is then given by

\[ u = w + \text{grad } v, \]

where \( v \in C^{m+2,\alpha} \) solves on \( \partial \Omega \)

\[ \text{grad } v = -w. \]

Indeed, this is possible by Lemma 14.1 and by the fact that \( w \wedge \nu = 0 \).

We conclude this section by discussing the case where \( \Omega \) is not necessarily convex. We assume that \( \Omega \subset \mathbb{R}^3 \) is a bounded connected set with smooth boundary (\( \nu \) then denotes the outward unit normal) consisting of finitely many connected components.

Denote

\[ D_2(\Omega) := \left\{ \psi \in C^0(\overline{\Omega};\mathbb{R}^3) \cap C^1(\Omega;\mathbb{R}^3) : \right. \]

\[ \left. \text{curl } \psi = 0, \text{ div } \psi = 0 \text{ in } \Omega \quad \text{and} \quad \langle \psi; \nu \rangle = 0 \text{ on } \partial \Omega \right\}, \]

which is the set of 2 harmonic fields with a Dirichlet boundary condition. If \( \Omega \) is convex or more generally contractible, we have \( D_2(\Omega) = \{0\} \). In general, however, this is not the case and the dimension of \( D_2(\Omega) \) is then related to the Betti numbers of \( \Omega \) (see Duff-Spencer [253] and Kress [378]).

Theorem 14.4 remains valid for such general sets if we add the following necessary condition

\[ \int_{\Omega} \langle f; \psi \rangle \, dx = 0, \quad \forall \psi \in D_2(\Omega). \]

### 14.3 The equation \( \det \nabla u = f \)

#### 14.3.1 The main theorem and some corollaries

We now state the main result of this section and we follow the presentation of Dacorogna-Moser [207]. We start first with the following notation.

**Notation 14.5** Let \( \Omega, O \subset \mathbb{R}^n \) be bounded open sets, \( m \geq 1 \) (including \( m = \infty \)) be an integer and \( 0 < \alpha \leq 1 \). We denote by \( \text{Diff}^m(\overline{\Omega};\overline{O}) \) (respectively \( \text{Diff}^{m,\alpha}(\overline{\Omega};\overline{O}) \)) the set of diffeomorphisms \( u : \overline{\Omega} \to \overline{O} \) such that \( u \in C^m(\overline{\Omega};\mathbb{R}^n) \) (respectively \( C^{m,\alpha}(\overline{\Omega};\mathbb{R}^n) \)) and \( u^{-1} \in C^m(\overline{O};\mathbb{R}^n) \) (respectively \( C^{m,\alpha}(\overline{O};\mathbb{R}^n) \)). When \( \Omega = O \), we simply write \( \text{Diff}^m(\overline{\Omega}) \) (respectively \( \text{Diff}^{m,\alpha}(\overline{\Omega}) \)). \( \diamond \)
Theorem 14.6 (Dacorogna-Moser theorem) Let $m \geq 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with a $C^{m+3,\alpha}$ boundary consisting of finitely many connected components. Let $f \in C^{m,\alpha}(\overline{\Omega})$, $f > 0$ in $\overline{\Omega}$ and
\[
\int_{\Omega} f(x) \, dx = \text{meas} \, \Omega. \tag{14.7}
\]
Then there exists $u \in \text{Diff}^{m+1,\alpha}(\overline{\Omega})$ satisfying
\[
\begin{cases}
\det \nabla u(x) = f(x) & x \in \Omega \\
u(x) = x & x \in \partial \Omega.
\end{cases} \tag{14.8}
\]

Conversely, if there exists $u \in \text{Diff}^{m+1,\alpha}(\overline{\Omega})$ satisfying (14.8), then $f \in C^{m,\alpha}(\overline{\Omega})$ and (14.7) holds.

Remark 14.7 (i) The theorem is due to Dacorogna-Moser [207]. It finds its origins in Moser [457], who proved the result for manifolds without boundary and in the $C^\infty$ case. His result was improved notably by Banyaga [69], Reimann [507] and Zehnder [614]. Independently, Tartar [569] and Dacorogna [170] proved a similar result for the case where $\Omega$ is the unit ball of $\mathbb{R}^2$ and $\mathbb{R}^3$. Some counter examples or extensions of the above theorem can be found in Burago-Kleiner [109], MacMullen [408], Rivièreme-Ye [513] and Ye [604].

(ii) Of course no uniqueness is to be expected. For example, if $n = 2$, $\Omega$ is the unit disk and $f \equiv 1$. Indeed, writing $u$ in polar coordinates,
\[
u(x) = u(x_1, x_2) = \left(r \cos \left(\theta + 2k\pi r^2\right), r \sin \left(\theta + 2k\pi r^2\right)\right),
\]
with $r \in [0, 1]$, $k \in \mathbb{Z}$ (the set of integers); we find that $u$ satisfies (14.8) for every $k$. ♦

We first mention the following immediate corollary.

Corollary 14.8 Let $m \geq 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with a $C^{m+3,\alpha}$ boundary consisting of finitely many connected components. Let $f, g \in C^{m,\alpha}(\overline{\Omega})$, $f, g > 0$ in $\overline{\Omega}$ and
\[
\int_{\Omega} f(x) \, dx = \int_{\Omega} g(x) \, dx.
\]
Then there exists $u \in \text{Diff}^{m+1,\alpha}(\overline{\Omega})$ satisfying
\[
\begin{cases}
g(u(x)) \det \nabla u(x) = f(x) & x \in \Omega \\
u(x) = x & x \in \partial \Omega.
\end{cases}
\]
The equation $\det \nabla u = f$

Proof. (Corollary 14.8). It suffices to set $u = v^{-1} \circ w$ where (see Theorem 14.6) $v$ and $w$ satisfy

$$
\begin{align*}
\det \nabla w(x) &= \frac{f(x) \text{meas } \Omega}{\int_{\Omega} f(x) \, dx} \quad x \in \Omega \\
\det \nabla v(x) &= \frac{g(x) \text{meas } \Omega}{\int_{\Omega} g(x) \, dx} \quad x \in \Omega \\
w(x) &= v(x) = x \quad x \in \partial \Omega.
\end{align*}
$$

This concludes the proof of the corollary. \[\Box\]

The theorem has also as a direct consequence the following result.

Corollary 14.9 Let $m \geq 1$ be an integer and $0 < \alpha < 1$. Let $\Omega, O \subset \mathbb{R}^n$ be bounded connected open sets with a $C^{m+3,\alpha}$ boundary consisting of finitely many connected components. Let $u_0 \in \text{Diff}^{m,\alpha}(\overline{\Omega}; O)$ such that

$$\det \nabla u_0 > 0 \text{ in } \overline{\Omega}.$$ 

Let $g : \mathbb{R} \to \mathbb{R}$ be convex and

$$(P) \quad \inf \left\{ I(u) = \int_{\Omega} g(\det \nabla u(x)) \, dx : u \in u_0 + W^{1,\infty}_0(\Omega; \mathbb{R}^n) \right\}.$$ 

Then there exists a minimizer $\overline{u} \in X$ of $(P)$ and moreover $\overline{u} \in \text{Diff}^{m,\alpha}(\overline{\Omega}; \overline{O})$.

Remark 14.10 (i) This problem was first considered in Dacorogna [170]. It should be pointed out that, although the function

$$f(\xi) := g(\det \xi)$$

is quasiconvex (and even polyconvex), since $g$ is convex, the direct methods of Chapter 8 do not apply, because we lack the appropriate coercivity hypothesis.

(ii) The non-convex case has been considered by Mascolo-Schianchi [437], see also Theorem 11.32. \[\Diamond\]

Proof. We let $f : \overline{O} \to \mathbb{R}$ be defined by

$$f(y) := \frac{\det \nabla u_0^{-1}(y)}{\text{meas } \Omega} \int_{\Omega} \det \nabla u_0(z) \, dz.$$ 

Note that $f > 0$ in $\overline{O}$ and

$$\int_{\partial O} f(y) \, dy = \text{meas } O.$$

We may therefore apply Theorem 14.6 to find $v \in \text{Diff}^{m,\alpha}(\overline{O})$ satisfying

$$
\begin{align*}
\det \nabla v(y) &= f(y) \quad y \in O \\
v(y) &= y \quad y \in \partial O.
\end{align*}
$$
Some underdetermined partial differential equations

Setting

\[ \overline{u} = v \circ u_0 \]

it is easy to see that

\[
\begin{align*}
\det \nabla \overline{u}(x) &= \int_{\Omega} \det \nabla u_0(z) \, dz / \text{meas } \Omega \quad x \in \Omega \\
\overline{u}(x) &= u_0(x) \quad x \in \partial \Omega.
\end{align*}
\]

(14.9)

We now claim that \( \overline{u} \) is indeed a minimizer of \((P)\). Take any \( u \in u_0 + W^{1,\infty}_0(\Omega; \mathbb{R}^n) \) and apply first Jensen inequality, then the fact that \( u = u_0 \) on \( \partial \Omega \) (combined with Theorem 8.35) and finally (14.9) to get

\[
I(u) = \int_{\Omega} g(\det \nabla u(x)) \, dx \geq \text{meas } \Omega \cdot g\left(\frac{1}{\text{meas } \Omega} \int_{\Omega} \det \nabla u(x) \, dx\right) \\
= \text{meas } \Omega \cdot g\left(\frac{1}{\text{meas } \Omega} \int_{\Omega} \det \nabla u_0(x) \, dx\right) \\
= \text{meas } \Omega \cdot g(\det \nabla u_0(x)) = \int_{\Omega} g(\det \nabla u_0(x)) \, dx = I(\overline{u})
\]

as wished. \( \blacksquare \)

We now describe roughly the idea of the proof of the theorem. We give several ways of constructing solutions of (14.8). All of them require as a first step to solve the linearized problem (setting \( u(x) = x + v(x) \))

\[
\begin{align*}
\text{div } v &= f - 1 \quad \text{in } \Omega \\
v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(14.10)

This was already achieved in Theorem 14.2. Although the solution of this problem is clearly not unique, our construction provides a well defined solution (see Remark 14.3 (iv)).

We now present the different ways of solving the nonlinear problem (14.8); for a still different approach, see Dacorogna [170] or Dacorogna-Moser [207].

- In Section 14.3.2 (see Lemma 14.11), we find a \( C^{m,\alpha} \) solution by a deformation argument, i.e. by solving the ordinary differential equations

\[
\begin{align*}
\frac{d}{dt} \Phi_t(x) &= \frac{v(\Phi_t(x))}{t + (1 - t) f(\Phi_t(x))} \\
\Phi_0(x) &= x,
\end{align*}
\]

where \( v \) is as in (14.10). Standard properties of ordinary differential equations give that \( u(x) = \Phi_1(x) \) is a solution of (14.8), but it is only in \( C^{m,\alpha} \) and not in \( C^{m+1,\alpha} \) as wished.

- In Section 14.3.3 (see Lemma 14.12), using (14.10) and a smallness assumption on the \( C^{0,\beta} \) norm, \( 0 < \beta \leq \alpha < 1 \), of \( f - 1 \), we obtain a \( C^{m+1,\alpha} \) solution by linearizing the equation around the identity.
The equation \( \det \nabla u = f \)

- Finally, in Section 14.3.4, we give two proofs of the theorem, obtaining the claimed regularity conclusion and removing the smallness assumption on \( f - 1 \). This is achieved in two different ways, one as a combination of Lemmas 14.11 and 14.12 and the other by several iterations of Lemma 14.12.

### 14.3.2 A deformation argument

We turn our attention to proving Theorem 14.6 with a weaker regularity than stated in the conclusion of the theorem. We follow here the original proof of Moser [457] and Dacorogna-Moser [207].

**Lemma 14.11** Let \( m \geq 1 \) be an integer and \( 0 < \alpha < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded connected open set with a \( C^{m+3,\alpha} \) boundary consisting of finitely many connected components. Let \( f \in C^{m,\alpha}(\overline{\Omega}) \), \( f > 0 \) in \( \Omega \) and

\[
\int_\Omega f(x) \, dx = \text{meas} \, \Omega.
\]

Then there exists \( u \in \text{Diff}^{m,\alpha}(\overline{\Omega}) \) satisfying

\[
\begin{align*}
\det \nabla u(x) &= f(x) \quad x \in \Omega \\
u(x) &= x \quad x \in \partial \Omega.
\end{align*}
\]  

(14.11)

**Proof.** We decompose the proof into two steps. 

*Step 1.* For \( t \in [0, 1] \) and \( z \in \Omega \), let

\[
v_t(z) := \frac{v(z)}{t + (1-t) f(z)}.
\]  

(14.12)

where \( v \in C^{m+1,\alpha}(\overline{\Omega}; \mathbb{R}^n) \) (but \( v_t \in C^{m,\alpha}(\overline{\Omega}; \mathbb{R}^n) \)) satisfies

\[
\begin{align*}
\text{div} \, v &= f - 1 \quad \text{in} \, \Omega \\
v &= 0 \quad \text{on} \, \partial \Omega.
\end{align*}
\]  

(14.13)

(Such a \( v \) exists by Theorem 14.2.)

We then define \( \Phi_t(x) : [0, 1] \times \Omega \to \mathbb{R}^n \) as the solution of

\[
\begin{align*}
\frac{d}{dt}[\Phi_t(x)] &= v_t(\Phi_t(x)) \quad t > 0 \\
\Phi_0(x) &= x.
\end{align*}
\]  

(14.14)

First note that \( \Phi_t \in C^{m,\alpha}(\overline{\Omega}; \mathbb{R}^n) \) for every \( t \) and that \( \Phi_t \) is uniquely defined on \( [0, 1] \); moreover, \( \Phi_t \) is, by construction, a diffeomorphism. Observe also that, for every \( t \in [0, 1] \), we have

\[
\Phi_t(x) \equiv x \quad \text{if} \quad x \in \partial \Omega.
\]
This follows from the observation that if \( x \in \partial \Omega \), then \( x \) is a solution of (14.14), since \( v = 0 \) on \( \partial \Omega \); the uniqueness then implies that \( \Phi_t (x) \equiv x \) for every \( x \in \partial \Omega \).

We now show that \( u (x) := \Phi_1 (x) \) is a solution of (14.11). The boundary condition has already been verified, so we only need to check that \( \det \nabla \Phi_1 (x) = f (x) \). To prove this, we let

\[
h (t, x) := [\det \nabla \Phi_t (x)] \cdot [t + (1 - t) f (\Phi_t (x))] .
\]  
(14.15)

If we show (see Step 2) that

\[
\frac{\partial}{\partial t} h (t, x) \equiv 0
\]  
(14.16)

we will have the result from the fact that \( h (1, x) = h (0, x) \).

**Step 2.** We therefore only need to show (14.16). Let \( A \) be an \( n \times n \) matrix, then it is a well known fact (see Coddington-Levinson [160], page 28) that if \( \psi \) satisfies \( \psi' (t) = A (t) \psi (t) \), then

\[
(\det \psi)' = \text{trace} (A) \cdot \det \psi,
\]

where \( \text{trace} (A) \) stands for the trace of \( A \). We therefore get that

\[
\frac{\partial}{\partial t} [\det \nabla \Phi_t (x)] = \det \nabla \Phi_t (x) \cdot \text{div} v_t (\Phi_t (x)).
\]  
(14.17)

We now differentiate (14.15) to get

\[
\frac{\partial}{\partial t} h (t, x) = \frac{\partial}{\partial t} [\det \nabla \Phi_t] \cdot [t + (1 - t) f (\Phi_t)]
\]

\[
+ [\det \nabla \Phi_t] [1 - f (\Phi_t) + (1 - t) \langle \nabla f (\Phi_t); \frac{d}{dt} \Phi_t \rangle].
\]

Using (14.14) and (14.17), we obtain

\[
\frac{\partial}{\partial t} h (t, x) = [\det \nabla \Phi_t] [(t + (1 - t) f (\Phi_t)) \text{div} v_t (\Phi_t)
\]

\[
+ (1 - t) \langle \nabla f (\Phi_t); v_t (\Phi_t) \rangle + (1 - f (\Phi_t))].
\]

Using the definition of \( v_t \) (see (14.12)), we deduce that

\[
\text{div} v (y) = (t + (1 - t) f (y)) \text{div} v_t (y) + (1 - t) \langle \nabla f (y); v_t (y) \rangle .
\]

Combining the two identities, we have

\[
\frac{\partial}{\partial t} h (t, x) = [\det \nabla \Phi_t] \cdot [\text{div} v (\Phi_t) + (1 - f (\Phi_t))].
\]

The definition of \( v \) (see (14.13)) immediately gives (14.16) and thus the lemma.
14.3.3 A proof under a smallness assumption

We now prove Theorem 14.6 under a smallness assumption on the $C^{0,\beta}$ norm of $f - 1$.

**Lemma 14.12** Let $\Omega, m, \alpha$ and $f \in C^{m,\alpha}(\Omega)$ be as in Theorem 14.6. Let $0 < \beta \leq \alpha < 1$. Then there exists $\epsilon = \epsilon(\alpha, \beta, m, \Omega) > 0$ such that if $\|f - 1\|_{C^{0,\beta}} \leq \epsilon$, then there exists $u \in \text{Diff}^{m+1,\alpha}(\Omega)$ such that

$$\begin{cases} 
\det \nabla u(x) = f(x) & x \in \Omega \\
u(x) = x & x \in \partial \Omega.
\end{cases}$$

(14.18)

**Remark 14.13** (i) A similar result can be found in Zehnder [614].

(ii) We use below some elementary properties of Hölder continuous functions that are gathered in Section 12.3.

**Proof.** We divide the proof into two steps.

**Step 1.** We start by defining two constants $K_1, K_2$ as follows.

(i) Let

$$X := \left\{ b \in C^{m,\alpha}(\Omega) : \int_{\Omega} b(x) \, dx = 0 \right\},$$

$$Y := \left\{ a \in C^{m+1,\alpha}(\Omega; \mathbb{R}^n) : a = 0 \text{ on } \partial \Omega \right\}.$$

As seen in Theorem 14.2, we can then define a bounded linear operator $L : X \to Y$ that associates to every $b \in X$ a unique $a \in Y$ such that

$$\begin{cases} 
\text{div } a = b & \text{in } \Omega \\
a = 0 & \text{on } \partial \Omega.
\end{cases}$$

Furthermore, there exists $K_1 > 0$ such that

$$\|Lb\|_{C^{1,\beta}} \leq K_1 \|b\|_{C^{0,\beta}}$$

(14.19)

$$\|Lb\|_{C^{m+1,\alpha}} \leq K_1 \|b\|_{C^{m,\alpha}}.$$  

(14.20)

(ii) For $\xi$ any $n \times n$ matrix, let

$$Q(\xi) := \det (I + \xi) - 1 - \text{trace}(\xi),$$

(14.21)

where $I$ stands for the identity matrix and $\text{trace}(\xi)$ for the trace of $\xi$. Note that $Q$ is a sum of monomials of degree $t$, $2 \leq t \leq n$. We can therefore find (see Proposition 12.7) $K_2 > 0$ such that if $v, w \in C^{m+1,\alpha}$ with $\|v\|_{C^{1,\beta}}, \|w\|_{C^{1,\beta}} \leq 1$, then

$$\|Q(\nabla v) - Q(\nabla w)\|_{C^{0,\beta}} \leq K_2 (\|v\|_{C^{1,\beta}} + \|w\|_{C^{1,\beta}}) \|v - w\|_{C^{1,\beta}},$$

$$\|Q(\nabla v)\|_{C^{m,\alpha}} \leq K_2 \|v\|_{C^{1}} \|v\|_{C^{m+1,\alpha}}.$$  

(14.22)
Some underdetermined partial differential equations

Step 2. In order to solve (14.18), we set \( v(x) = u(x) - x \) and we rewrite it as

\[
\begin{cases}
\text{div } v = f - 1 - Q(\nabla v) & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega.
\end{cases}
\] (14.23)

If we set

\[ N(v) := f - 1 - Q(\nabla v), \]

then (14.23) is satisfied for any \( v \in Y \) with

\[ v = LN(v). \] (14.24)

Note first that the equation is well defined (i.e. \( N : Y \to X \)), since if \( v = 0 \) on \( \partial \Omega \) then \( \int_{\Omega} N(v(x)) \, dx = 0 \). Indeed, from (14.21) we have that

\[
\int_{\Omega} N(v(x)) \, dx = \int_{\Omega} [f(x) - 1 - Q(\nabla v(x))] \, dx
\]

\[
= \int_{\Omega} [f(x) + \text{div } v(x) - \det (I + \nabla v(x))] \, dx;
\]

since \( v = 0 \) on \( \partial \Omega \) and \( \int_{\Omega} f = \text{meas } \Omega \), it follows immediately (see Theorem 8.35)) that the right hand side of the above identity is 0.

We now solve (14.24) by the contraction principle. We first let for \( r > 0 \)

\[ B_r := \left\{ u \in C^{m+1,\alpha}(\overline{\Omega}, \mathbb{R}^n) : u = 0 \text{ on } \partial \Omega, \, \|u\|_{C^{1,\beta}} \leq r, \, \|u\|_{C^{m+1,\alpha}} \leq 2K_1 \|f - 1\|_{C^{m,\alpha}} \right\}. \]

We endow \( B_r \) with the \( C^{1,\beta} \) norm. We observe that \( B_r \) is complete (see Proposition 12.8) and we will show that by choosing \( \|f - 1\|_{C^{0,\beta}} \) and \( r \) small enough, then \( LN : B_r \to B_r \) is a contraction mapping. The contraction principle will then immediately lead to a solution \( v \in B_r \) and hence in \( C^{m+1,\alpha} \) of (14.24).

Indeed, let

\[ r := 2K_1 \|f - 1\|_{C^{0,\beta}}. \] (14.26)

If \( v, w \in B_r \) (note that by construction \( r \leq 1 \)), we then have

\[ \|LN(v) - LN(w)\|_{C^{1,\beta}} \leq \frac{1}{2} \|v - w\|_{C^{1,\beta}} \] (14.27)

\[ \|LN(v)\|_{C^{1,\beta}} \leq r, \, \|LN(v)\|_{C^{m+1,\alpha}} \leq 2K_1 \|f - 1\|_{C^{m,\alpha}}. \] (14.28)
The inequality (14.27) follows from (14.19), (14.22), (14.25) and (14.26) through
\[ \| LN(v) - LN(w) \|_{C^{1,\beta}} \leq K_1 \| N(v) - N(w) \|_{C^{0,\beta}} \]
\[ = K_1 \| Q(\nabla v) - Q(\nabla w) \|_{C^{0,\beta}} \]
\[ \leq K_1 K_2 \left( \| v \|_{C^{1,\beta}} + \| w \|_{C^{1,\beta}} \right) \| v - w \|_{C^{1,\beta}} \]
\[ \leq 2r K_1 K_2 \| v - w \|_{C^{1,\beta}} \]
\[ = 4K_1^2 K_2 \| f - 1 \|_{C^{0,\beta}} \| v - w \|_{C^{1,\beta}} \]
\[ \leq \frac{1}{2} \| v - w \|_{C^{1,\beta}}. \]

To obtain the first inequality in (14.28) we observe that
\[ \| LN(0) \|_{C^{1,\beta}} \leq K_1 \| N(0) \|_{C^{0,\beta}} = K_1 \| f - 1 \|_{C^{0,\beta}} = \frac{r}{2} \]
and hence combining (14.27) with the above inequality, we have immediately the first inequality in (14.28). To obtain the second one, we just have to observe that
\[ \| LN(v) \|_{C^{m+1,\alpha}} \leq K_1 \| N(v) \|_{C^{m,\alpha}} \leq K_1 \| f - 1 \|_{C^{m,\alpha}} + K_1 \| Q(\nabla v) \|_{C^{m,\alpha}} \]
(14.29)
and use the second inequality in (14.22) to get
\[ \| Q(\nabla v) \|_{C^{m,\alpha}} \leq K_2 \| v \|_{C^{1,\beta}} \| v \|_{C^{m+1,\alpha}} \leq K_2 \| v \|_{C^{1,\beta}} \| v \|_{C^{m+1,\alpha}} \]
where we have used in the third inequality the fact that
\[ \| v \|_{C^{1,\beta}} \leq r = 2K_1 \| f - 1 \|_{C^{0,\beta}}. \]
The above inequality combined with (14.25) gives
\[ \| Q(\nabla v) \|_{C^{m,\alpha}} \leq \frac{1}{4K_1} \| v \|_{C^{m+1,\alpha}}. \]
Combining this last inequality, (14.29) and the fact that \( v \in B_r \), we deduce that
\[ \| LN(v) \|_{C^{m+1,\alpha}} \leq 2K_1 \| f - 1 \|_{C^{m,\alpha}}. \]
Thus the contraction principle gives immediately the existence of a \( C^{m+1,\alpha} \) solution.

It now remains to show that \( u(x) = v(x) + x \) is a diffeomorphism. This is a consequence of the fact that \( \det \nabla u = f > 0 \) and \( u(x) = x \) on \( \partial \Omega \) (see, for example, Corollary 2 on page 79 in Meisters-Olech [441]).

### 14.3.4 Two proofs of the main theorem
We may now turn to the first proof of Theorem 14.6.
Some underdetermined partial differential equations

Proof. The fact that if there exists \( u \in \text{Diff}^{m+1,\alpha} (\Omega) \) satisfying (14.8), then \( f \in C^{m,\alpha} (\overline{\Omega}) \) and (14.7) holds, is straightforward (cf. Theorem 8.35), using the fact that if \( u(x) = x \) on \( \partial \Omega \), then

\[
\int_{\Omega} \det \nabla u(x) \, dx = \text{meas} \Omega
\]

and hence the claim.

We now prove the converse and we divide the proof into two steps.

Step 1. Let us first show that, if \( f \in C^{m,\alpha} (\Omega) \), \( f > 0 \) in \( \Omega \) and if \( 0 < \beta < \alpha < 1 \), we can then find, for every \( \epsilon > 0 \), a function \( g \in C^\infty (\Omega) \), \( g > 0 \) in \( \Omega \) such that

\[
\left\| \frac{f}{g} - 1 \right\|_{C^{0,\beta}} \leq \epsilon \quad \text{and} \quad \int_{\Omega} \frac{f(x)}{g(x)} \, dx = \text{meas} \Omega. \tag{14.30}
\]

1) We first start by observing that if \( h \in C^{0,\beta} (\overline{\Omega}) \) and

\[
h(x) \geq \overline{h} > 0, \quad \text{for every } x \in \overline{\Omega}
\]

then

\[
\left\| \frac{1}{h} \right\|_{C^{0,\beta}} \leq \frac{1}{\overline{h}} + \frac{1}{\overline{h}^2} \left[ h \right]_{\beta} \leq \frac{1}{\overline{h}^2} \| h \|_{C^{0,\beta}}. \tag{14.31}
\]

2) Let

\[
\overline{f} := \frac{1}{2} \inf \{ f(x) : x \in \overline{\Omega} \}.
\]

From Proposition 12.7 (v), we can find, for every \( \delta > 0 \) sufficiently small, a function \( f_\delta \in C^\infty (\overline{\Omega}) \) such that

\[
\| f - f_\delta \|_{C^{0,\beta}} \leq \delta \quad \text{and} \quad f_\delta (x) \geq \overline{f} > 0, \quad \text{for every } x \in \overline{\Omega}. \tag{14.32}
\]

Note that from Proposition 12.7 (i), (14.31) and (14.32), we have

\[
\left\| \frac{f}{f_\delta} - 1 \right\|_{C^{0,\beta}} \leq 2C \| f - f_\delta \|_{C^{0,\beta}} \left\| \frac{1}{f_\delta} \right\|_{C^{0,\beta}} \leq \frac{2C}{\overline{f}} \delta \| f_\delta \|_{C^{0,\beta}}
\]

\[
\leq \delta' := \frac{2C}{\overline{f}} \delta \left[ \| f \|_{C^{0,\beta}} + \delta \right].
\]

3) We next set

\[
t := \frac{1}{\text{meas} \Omega} \int_{\Omega} f_\delta (x) \, dx
\]

and observe that \( |t - 1| \leq \delta' \). Defining

\[
g := tf_\delta
\]

and choosing \( \delta \) and thus \( \delta' \) small enough, we have indeed shown that \( g \) satisfies (14.30).
The equation \( \det \nabla u = f \)

Step 2. Choose now \( \epsilon \) as in Lemma 14.12 and apply Step 1 to find \( g \in C^\infty (\overline{\Omega}) \), \( g > 0 \) in \( \overline{\Omega} \), satisfying (14.30). We then define \( b \in \text{Diff}^{m+1,\alpha} (\overline{\Omega}) \) to be a solution, which exists by (14.30) and Lemma 14.12, of

\[
\begin{cases}
\det \nabla b (x) = f (x) & x \in \Omega \\
b (x) = x & x \in \partial \Omega.
\end{cases}
\]

We further let \( a \in \text{Diff}^{m+1,\alpha} (\overline{\Omega}) \) to be a solution of

\[
\begin{cases}
\det \nabla a (y) = g (b^{-1} (y)) & y \in \Omega \\
a (y) = y & y \in \partial \Omega.
\end{cases}
\]

Such a solution exists by Lemma 14.11 since \( g \circ b^{-1} \in C^{m+1,\alpha} (\overline{\Omega}) \) (cf. Proposition 12.7 (ii)) and

\[
\int_{\Omega} g (b^{-1} (y)) \, dy = \int_{\Omega} g (x) \, \det \nabla b (x) \, dx = \int_{\Omega} f (x) \, dx = \text{meas} \, \Omega.
\]

Finally observe that the function \( u = a \circ b \) has all the claimed properties. \( \square \)

We conclude with a second proof of Theorem 14.6 that does not use the flow, as in Lemma 14.11, but only appeals to Lemma 14.12, with \( \beta = \alpha \) (see Ye [604] for a similar procedure in Sobolev spaces).

Proof. We proceed in two steps.

Step 1. We start by defining for \( s = 0, \cdots, N+1, N \) an integer,

\[
f_s := (1 - \frac{s}{N+1}) f + \frac{s}{N+1} \in C^{m,\alpha} (\overline{\Omega}).
\]

(1) Note that \( f_0 = f \) and \( f_{N+1} \equiv 1 \). Recall that

\[
\int_{\Omega} f (x) \, dx = \text{meas} \, \Omega,
\]

which implies in particular that \( \min \{ f \} \leq 1 \leq \max \{ f \} \). We moreover have, for every \( x \in \overline{\Omega}, \)

\[
0 < \overline{f} := \min_{x \in \overline{\Omega}} \{ f (x) \} \leq \min \{ f (x), 1 \} \leq f_s (x) \leq \max \{ f (x), 1 \} \leq \| f \|_{C^0}.
\]

(14.34)

(2) We also have the following estimates for every \( s = 0, \cdots, N+1 : \)

\[
\| f_s \|_{C^0} \leq \| f \|_{C^0} \quad \text{and} \quad [f_s]_{\alpha} = (1 - \frac{s}{N+1}) [f]_{\alpha} \leq [f]_{\alpha}
\]

thus

\[
\| f_s \|_{C^{0,\alpha}} \leq \| f \|_{C^{0,\alpha}}
\]

and

\[
\left\| \frac{1}{f_s} \right\|_{C^{0,\alpha}} \leq \frac{1}{\overline{f}} + \frac{1}{\overline{f}^2} [f_s]_{\alpha} \leq \frac{1}{\overline{f}^2} (\| f \|_{C^0} + [f_s]_{\alpha}) \leq \frac{1}{\overline{f}^2} \| f \|_{C^{0,\alpha}}.
\]
We also clearly have
\[ \| f_{s+1} - f_s \|_{C^{0,\alpha}} \leq \frac{\| f - 1 \|_{C^{0,\alpha}}}{N+1} , \quad s = 0, \ldots, N. \]

(3) We next set
\[ t_s := \frac{1}{\text{meas } \Omega} \int_{\Omega} f_s(x) \, dx , \quad s = 0, \ldots, N + 1 \]
and observe that \( t_0 = t_{N+1} = 1 \). Moreover, from (14.33) and (14.34), we obtain
\[ \frac{1}{\| f \|_{C^{0}}} \leq t_s \leq \frac{1}{\| f \|_{C^{0}}} , \quad s = 0, \ldots, N + 1 \]
as well as, for \( s = 0, \ldots, N + 1 \),
\[ |t_s - t_{s+1}| \leq \frac{1}{\text{meas } \Omega} \int_{\Omega} f(x) \left| \frac{1}{f_s(x)} - \frac{1}{f_{s+1}(x)} \right| \, dx \]
\[ \leq \left\| \frac{1}{f_s} - \frac{1}{f_{s+1}} \right\|_{C^{0}} \leq \| f - 1 \|_{C^{0,\alpha}}. \]

Defining
\[ g_s := t_s f_s , \quad s = 0, \ldots, N + 1 \]
we find \( g_s \in C^{m,\alpha}(\overline{\Omega}) \),
\[ \frac{f}{\| f \|_{C^{0}}} \leq g_s \leq \frac{1}{\| f \|_{C^{0}}} , \quad s = 0, \ldots, N + 1 \]
and
\[ \int_{\Omega} \frac{f(x)}{g_s(x)} \, dx = \text{meas } \Omega , \quad s = 0, \ldots, N + 1. \]

Finally note that, from Proposition 12.7 (i) and the above estimates, we get
\[ \left\| \frac{g_s}{g_{s+1}} - 1 \right\|_{C^{0,\alpha}} = \left\| \frac{t_s f_s}{t_{s+1}f_{s+1}} - 1 \right\|_{C^{0,\alpha}} = \left\| \frac{t_s(f_s - f_{s+1}) + (t_s - t_{s+1})f_{s+1}}{t_{s+1}f_{s+1}} \right\|_{C^{0,\alpha}} \]
\[ \leq \frac{2C}{t_{s+1}} \left\| \frac{1}{f_{s+1}} \right\|_{C^{0,\alpha}} \left[ t_s \| f_{s+1} - f_s \|_{C^{0,\alpha}} + |t_s - t_{s+1}| \| f_{s+1} \|_{C^{0,\alpha}} \right] \]
\[ \leq \frac{2C}{f} \| f \|_{C^{0,\alpha}} \| f \|_{C^{0,\alpha}} \left[ \frac{1}{f} \left\| f - 1 \right\|_{C^{0,\alpha}} \| f \|_{C^{0,\alpha}} \| f \|_{C^{0,\alpha}} + \frac{f - 1}{f^2} \left[ f + \| f \|_{C^{0,\alpha}} \right] \right]. \]

Setting
\[ \gamma := \frac{2C}{f} \| f \|_{C^{0,\alpha}} \| f - 1 \|_{C^{0,\alpha}} \left[ f + \| f \|_{C^{0,\alpha}} \right] \]
we get that
\[ \left\| \frac{g_s}{g_{s+1}} - 1 \right\|_{C^{0,\alpha}} \leq \frac{\gamma}{N+1}. \]
(4) We therefore choose $N$ sufficiently large so that, for $\epsilon$ as in Lemma 14.12, we have $g_0 = f$, $g_{N+1} \equiv 1$ and, for every $s = 0, \cdots, N+1$, 
\[
\left\| \frac{g_s}{g_{s+1}} - 1 \right\|_{C^{0,\alpha}} \leq \epsilon \quad \text{and} \quad \int_{\Omega} g_s(x) \, dx = \text{meas} \, \Omega.
\]

**Step 2.** We then set 
\[
u_0(x) := x \quad \text{and} \quad g_0 = f
\]
and define inductively, with the help of Lemma 14.12, $u_{s+1} \in \text{Diff}^{m+1,\alpha}(\overline{\Omega})$, $s = 0, \cdots, N$, satisfying
\[
\begin{cases}
\det \nabla u_{s+1}(x) = \frac{g_s(\hat{u}_s^{-1}(x))}{g_{s+1}(\hat{u}_s^{-1}(x))} & x \in \Omega \\
u_{s+1}(x) = x & x \in \partial \Omega,
\end{cases}
\tag{14.35}
\]
where
\[
\hat{u}_s := u_s \circ \cdots \circ u_1 \circ u_0.
\]
If such a $u_s$ exists, then a straightforward induction procedure shows that 
\[
\det \nabla \hat{u}_{s+1}(x) = \frac{f(x)}{g_{s+1}(x)}, \quad s = 0, \cdots, N
\tag{14.36}
\]
and hence $\hat{u}_{N+1}$ is the claimed solution of the theorem.

It therefore remains to show that there exists $u_s \in \text{Diff}^{m+1,\alpha}(\overline{\Omega})$ satisfying (14.35) and we proceed by induction. Indeed if $s = 0$, we have
\[
\frac{g_0}{g_1} \in C^{m,\alpha}(\overline{\Omega}), \quad \frac{g_0}{g_1} > 0, \quad \left\| \frac{g_0}{g_1} - 1 \right\|_{C^{0,\alpha}} \leq \epsilon
\]
and
\[
\int_{\Omega} g_0(x) \, dx = \text{meas} \, \Omega.
\]
So we may apply Lemma 14.12 to get $u_1 \in \text{Diff}^{m+1,\alpha}(\overline{\Omega})$.

Assume now that we have proved the result up to $s$ and let us prove it for $s+1$. We clearly have
\[
\frac{g_s(\hat{u}_s^{-1})}{g_{s+1}(\hat{u}_s^{-1})} \in C^{m,\alpha}(\overline{\Omega}), \quad \frac{g_s}{g_{s+1}} > 0, \quad \left\| \frac{g_s}{g_{s+1}} - 1 \right\|_{C^{0,\alpha}} \leq \epsilon.
\]
Moreover we see, from (14.36), that
\[
\int_{\Omega} \frac{g_s(\hat{u}_s^{-1}(x))}{g_{s+1}(\hat{u}_s^{-1}(x))} \, dx = \int_{\Omega} \frac{g_s(y)}{g_{s+1}(y)} \, \det \nabla \hat{u}_s(y) \, dy = \int_{\Omega} \frac{f(y)}{g_{s+1}(y)} \, dy = \text{meas} \, \Omega.
\]
So we may again apply Lemma 14.12 to find, as wished, $u_{s+1} \in \text{Diff}^{m+1,\alpha}(\overline{\Omega})$ satisfying (14.35). This concludes the induction procedure and thus the proof of the theorem.
Chapter 15

Extension of Lipschitz functions on Banach spaces

15.1 Introduction

In this chapter, we deal with the extension of Lipschitz maps and we follow the presentation of Dacorogna-Gangbo [188].

We consider two Banach spaces \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\). We ask when any map \(u : D \subset E \to F\) satisfying

\[
\|u(x) - u(y)\|_F \leq \|x - y\|_E, \quad x, y \in D
\]

(15.1)
can be extended to the whole of \(E\) so as to preserve the inequality.

This is by now a classical problem and we revisit this question in the following sections. In applications to the calculus of variations, we often use this type of extension for \(E = \mathbb{R}^n\) and \(F = \mathbb{R}^N\).

15.2 Preliminaries and notation

Throughout this chapter \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) are normed spaces and even in most cases Banach spaces. We denote by \(S^E\) the unit sphere in \(E\), namely the set of \(x \in E\) such that \(\|x\|_E = 1\). The convex hull of \(S^E\) is the closed ball \(\overline{B}^E\) of interior \(B^E\).

**Definition 15.1** (i) We say that \(u : E \to F\) is a contraction on \(D \subset E\) or \(u\) is 1-Lipschitz on \(D\) if

\[
\|u(x) - u(y)\|_F \leq \|x - y\|_E \quad \text{for every } x, y \in D.
\]

In this case, we write that \(u \in \text{Lip}_1(D, F)\).

(ii) When \(u \in \text{Lip}_1(E, F)\), we simply say that \(u\) is a contraction.
Definition 15.2 We say that \([E; F]\) has the extension property for contractions on \(D\) if every \(u \in \text{Lip}_1(D, F)\) has an extension \(\tilde{u} \in \text{Lip}_1(E, F)\). If \([E; F]\) has the extension property for contractions for every \(D \subset E\), we simply say that \([E; F]\) has the extension property for contractions.

Many extension theorems of Lipschitz maps can be derived from a principle due to Minty [445] that we state in Theorem 15.3. It gives a sufficient condition for extending Lipschitz maps from sets of cardinality \(k\) into sets of cardinality \(k + 1\).

We recall the following notation. When \(k\) is an integer, we let

\[
\Lambda_k := \{ \lambda = (\lambda_1, \ldots, \lambda_k) \in [0, 1]^k : \sum_{i=1}^k \lambda_i = 1 \}.
\]

We next need the function \(F : \Lambda_k \times \Lambda_k \to \mathbb{R}\) (for more general types of functions \(F\), see [188]) defined by

\[
F(\lambda, \mu) := \sum_{i=1}^k \lambda_i \left[ \| y_i - \sum_{j=1}^k \mu_j y_j \|_F^p - \| x_i - x \|_E^p \right],
\]

where \(x, x_1, \ldots, x_k \in E, y_1, \ldots, y_k \in F\) are kept fixed and \(p \geq 1\).

Theorem 15.3 (Minty theorem) Assume that \(k + 1\) points \(x, x_1, \ldots, x_k \in E\) and \(k\) points \(y_1, \ldots, y_k \in F\) are given and are such that

\[
F(\lambda, \lambda) \leq 0 \quad (15.2)
\]

for every \(\lambda \in \Lambda_k\). Then there exists \(y \in \text{co}\{y_1, \cdots, y_k\}\) such that

\[
\| y_i - y \|_F \leq \| x_i - x \|_E
\]

for every \(i = 1, \cdots, k\).

Proof. Clearly, \(\lambda \to F(\lambda, \mu)\) is concave (in fact affine) for every \(\mu\) and \(\mu \to F(\lambda, \mu)\) is convex for every \(\lambda\). Since \(\Lambda_k\) is a convex compact set, the minimax theorem holds (see Zeidler [615] III page 458) and there exists \((\bar{\lambda}, \bar{\mu}) \in \Lambda_k \times \Lambda_k\) such that

\[
\min_{\mu \in \Lambda_k} \max_{\lambda \in \Lambda_k} F(\lambda, \mu) = F(\bar{\lambda}, \bar{\mu}) = \max_{\lambda \in \Lambda_k} \min_{\mu \in \Lambda_k} F(\lambda, \mu). \quad (15.3)
\]

One can readily conclude from (15.3) that \((\bar{\lambda}, \bar{\mu})\) is a saddle point in the sense that

\[
F(\lambda, \bar{\mu}) \leq F(\bar{\lambda}, \bar{\mu}) \leq F(\lambda, \mu) \quad (15.4)
\]

for every \(\lambda, \mu \in \Lambda_k\). Setting \(\mu = \bar{\lambda}\) in (15.4) and using (15.2), we obtain that

\[
F(\lambda, \bar{\mu}) \leq F(\bar{\lambda}, \bar{\mu}) \leq F(\lambda, \lambda) \leq 0
\]

for every \(\lambda \in \Lambda_k\). We set \(y = \sum_{j=1}^k \bar{\mu}_j y_j\) and choose \(\lambda^i \in \Lambda_k\) such that \(\lambda^i_j = 0\) for \(j \neq i\) and \(\lambda^i_i = 1\). Note that \(F(\lambda^i, \bar{\mu}) \leq 0\) is equivalent to

\[
\| y_i - y \|_F \leq \| x_i - x \|_E
\]

which is the claim. ■
15.3 Norms induced by an inner product

We start by collecting some well known facts about inner product spaces. One can consult, as a general reference, Amir [26]; Lemmas 15.7 and 15.9 will be explicitly used in the proofs of the next section.

**Definition 15.4** An ellipse centered at 0 in $\mathbb{R}^n$ is a set

$$\Sigma^\alpha := \{ x \in \mathbb{R}^n : \sum_{i=1}^n \alpha_i^2 x_i^2 = 1 \},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in (0, +\infty)^n$. We refer to the convex hull of $\Sigma^\alpha$ as the region enclosed by $\Sigma^\alpha$ and we denote it by $B^\alpha$.

The next lemma is due to Löwner in an apparently unpublished work.

**Lemma 15.5 (Löwner theorem)** Assume that $n \geq 2$ and that $E = \mathbb{R}^n$. Then there exist a unique ellipse $\Sigma_{\text{max}}$ of maximal volume inscribed in $S^E$ and a unique ellipse of minimal volume $\Sigma_{\text{min}}$ circumscribed about $S^E$. Furthermore, both $\Sigma_{\text{max}} \cap S^E$ and $\Sigma_{\text{min}} \cap S^E$ contain at least $2n$ distinct points.

**Proof.** Existence of ellipses of maximal volume. If $\Sigma^\alpha$ is inscribed in $B^E$, then

$$\sum_{i=1}^n \alpha_i^2 x_i^2 \geq \| x \|_E^2$$

(15.5)

for every $x \in \mathbb{R}^n$. Assume that, for some $\epsilon > 0$, we have

$$\epsilon \leq \text{vol}(B^\alpha) = \frac{\omega_n}{\prod \alpha_i},$$

(15.6)

where $\omega_n$ is the volume of the unit Euclidean ball. The set of $\alpha$ such that $\alpha_i > 0$

![Figure 15.1: Ellipses of maximal and minimal volume](image)

and (15.5)-(15.6) hold is a compact subset $K_\epsilon \subset \mathbb{R}^n$. Every maximizing sequence of the set of ellipses inscribed in $B^E$, of maximal volume, has its accumulation points in $K_\epsilon$ for some small $\epsilon > 0$. This shows that there exists an ellipse $\Sigma_{\text{max}}$ inscribed in $S^E$ and of maximal volume (see Figure 15.1). Similarly, one obtains an ellipse $\Sigma_{\text{min}}$ circumscribed about $S^E$ and of minimal volume.
Uniqueness of ellipses of maximal volume. Assume that $\Sigma^a, \Sigma^c$ are two ellipses inscribed in $S^E$ and of maximal volume. By an affine transformation, we may assume that $c = (1, \cdots, 1)$ so that the volumes of these two regions are

$$\omega_n = \text{vol}(B^c) = \text{vol}(B^a) = \text{vol}(B^c) / \prod_{i=1}^n a_i.$$ 

We therefore deduce that $\prod_{i=1}^n a_i = 1$.

Let $\|\cdot\|^0_E$ be the polar conjugate of $\|\cdot\|_E$ defined by

$$\|z\|^0_E := \sup_x \{ \langle x; z \rangle : \|x\|_E \leq 1 \}.$$ 

Denote (see Section 2.3.7) by $\rho_{\Sigma^a}$ (respectively $\rho_{\Sigma^c}$) the gauge associated to $B^a$ (respectively $B^c$) and let $\rho_{\Sigma^a}^0$ (respectively $\rho_{\Sigma^c}^0$) be its polar; we more precisely have

$$\rho_{\Sigma^a}(z) = \left( \sum_{i=1}^n a_i^2 z_i^2 \right)^{1/2} \quad \text{and} \quad \rho_{\Sigma^a}^0(z) = \left( \sum_{i=1}^n z_i^2 / a_i^2 \right)^{1/2}$$

$$\rho_{\Sigma^c}(z) = \rho_{\Sigma^c}^0(z) = \left( \sum_{i=1}^n z_i^2 \right)^{1/2}.$$ 

Since $\Sigma^a, \Sigma^c$ are inscribed in $S^E$, we have that $\|\cdot\|_E \leq \rho_{\Sigma^a}, \rho_{\Sigma^c}$ and so $\rho_{\Sigma^a}^0, \rho_{\Sigma^c}^0 \leq \|\cdot\|^0_E$. Hence,

$$[\rho^0(z)]^2 := \frac{1}{2} \sum_{i=1}^n (1 + 1/a_i^2) z_i^2 = \frac{1}{2} \left( \sum_{i=1}^n z_i^2 / a_i^2 + \sum_{i=1}^n z_i^2 \right)$$

$$= \frac{1}{2} (\rho_{\Sigma^a}(z)^2 + \rho_{\Sigma^c}(z)^2) \leq (\|z\|^0_E)^2$$

holds for every $z \in \mathbb{R}^n$. The previous inequality yields that $\rho^2 \geq \|\cdot\|^2_E$, which means that

$$\sum_{i=1}^n \frac{2}{1 + 1/a_i^2} x_i^2 \geq \|x\|_E^2 \quad \text{for every } x \in \mathbb{R}^n. \quad (15.7)$$

Letting $b_i^2 = \frac{2}{1 + 1/a_i^2}, b = (b_1, \cdots, b_n)$, we find from (15.7) that $\Sigma^b$ is inscribed in $S^E$.

We now show that $\Sigma^a$ and $\Sigma^c$ coincide and we proceed by contradiction assuming that they are distinct. Then, $a_i \neq 1$ for at least one $i = 1, \cdots, n$. The volume of the region enclosed by $\Sigma^b$ is (recalling that $\prod_{i=1}^n a_i = 1$)

$$\text{vol}(B^b) = \omega_n(\prod_{i=1}^n (1 + 1/a_i^2) / 2)^{1/2}$$

$$= \omega_n(\prod_{i=1}^n [(1 - 1/a_i^2) / 2 + 1/a_i])^{1/2} > \omega_n(\prod_{i=1}^n 1/a_i)^{1/2} = \omega_n.$$ 

This contradicts the maximality of the volume of $\Sigma^c$. Thus, $\Sigma^c = \Sigma^a$ and so we have a unique ellipse of maximal volume in $S^E$. Replacing $\rho_{\Sigma^a}$ and $\|\cdot\|_E$ by their polar conjugates, we conclude that $\Sigma_{\text{min}}$ is also unique.

Intersection of the maximal ellipse with $S^E$. As before, we assume that $\Sigma_{\text{max}} = \Sigma^\alpha$ where $\alpha = (1, \cdots, 1)$. Since $\Sigma_{\text{max}}$ and $S^E$ are compact sets, they have
a non-empty intersection; otherwise the maximality of $\Sigma_{\text{max}}$ would be contradicted. By symmetry there are therefore at least two points in $S^E \cap \Sigma_{\text{max}}$. Let us show that if we have $2s$ points in $S^E \cap \Sigma_{\text{max}}$, $1 \leq s < n$, then in fact we have at least $2(s+1)$ points in the intersection, therefore showing the claim. Up to a rotation, we may assume that the points $\pm p_1, \cdots, \pm p_s \in S^E \cap \Sigma_{\text{max}}$ lay in the subspace generated by the first $s$ elements $\{e_1, \cdots, e_s\}$ of the standard Euclidean basis, which means that for every $j = 1, \cdots, s$, we have

$$p_i^j = 0 \text{ for every } i \geq s+1.$$  

For $\epsilon \in (0,1)$, define

$$\alpha_\epsilon := ((1-\epsilon)^{-1}, \cdots, (1-\epsilon)^{-1}, (1-\epsilon)^s, 1, \cdots, 1).$$

Since $\Sigma_{\text{max}}$ is unique and

$$\text{vol}(\Sigma^{\alpha_\epsilon}) = \omega_n = \text{vol}(\Sigma_{\text{max}}),$$

we conclude that $\Sigma^{\alpha_\epsilon}$ is not inscribed in $S^E$. Consequently, there exists $p^\epsilon = (p_1^\epsilon, \cdots, p_n^\epsilon) \notin \overline{B}^E$, which is in $B^{\alpha_\epsilon}$, the region enclosed by $\Sigma^{\alpha_\epsilon}$, and hence we have

$$\|p^\epsilon\|_E > 1$$  \hspace{1cm} (15.8)

and

$$1 \geq \rho_{\Sigma_{\text{max}}}^2 (p^\epsilon) = \rho_{\Sigma_{\text{max}}}^2 (p^\epsilon) + ((1-\epsilon)^{-2} - 1) \sum_{i=1}^s (p_i^\epsilon)^2 - (1 - (1-\epsilon)^{2s}) (p_{s+1}^\epsilon)^2.$$  \hspace{1cm} (15.9)

Because $p^\epsilon \notin \Sigma_{\text{max}} \subset \overline{B}^E$, (15.9) implies that

$$((1-\epsilon)^{-2} - 1) \sum_{i=1}^s (p_i^\epsilon)^2 \leq (1 - (1-\epsilon)^{2s}) (p_{s+1}^\epsilon)^2.$$  

Dividing both sides of the previous inequality by $\epsilon$, we get

$$\frac{2 - \epsilon}{(1-\epsilon)^2} \sum_{i=1}^s (p_i^\epsilon)^2 \leq \frac{1 - (1-\epsilon)^{2s}}{\epsilon} (p_{s+1}^\epsilon)^2.$$  \hspace{1cm} (15.10)

Let $\{p_{\epsilon}^{\nu}\}_{\nu=1}^\infty$ be a subsequence of $\{p^\epsilon\}_{0<\epsilon<1}$ converging, as $\epsilon_\nu \to 0$, to some $p \in E$. We use (15.8)-(15.10) to obtain that

$$\rho_{\Sigma_{\text{max}}} (p) \leq 1 \leq \|p\|_E \text{ and } \sum_{i=1}^s p_i^2 \leq s p_{s+1}^2.$$  \hspace{1cm} (15.11)

The first two inequalities in (15.11) and the fact that $\rho_{\Sigma_{\text{max}}} \geq \|p\|_E$ yield that $p \in S^E \cap \Sigma_{\text{max}}$. The last inequality in (15.11) gives that $p \notin \text{span} \{e_1, \cdots, e_s\}$ (in particular, $p \neq \pm p_1, \cdots, \pm p_s$) and thus by symmetry $\pm p \in S^E \cap \Sigma_{\text{max}}$. This
proves that \( S^E \cap \Sigma_{\text{max}} \) has at least \( 2(s + 1) \) distinct points, if \( s < n \). Iterating
the process, we have indeed shown that \( S^E \cap \Sigma_{\text{max}} \) has at least \( 2n \) distinct points. Existence of at least \( 2n \) distinct points in \( S^E \cap \Sigma_{\text{min}} \) is obtained in a
similar manner. ■

In [357], Jordan and von Neumann gave a condition that characterizes a
norm induced by an inner product.

**Lemma 15.6 (Jordan-von Neumann theorem)** Assume that \( \dim E \geq 2 \). Then, the norm \( \| \cdot \|_E \) is induced by an inner product if and only if the parallelogram rule holds for every \( x, y \in E \), namely
\[
2(\| x \|_E^2 + \| y \|_E^2) = \| x + y \|_E^2 + \| x - y \|_E^2 .
\]

**Proof.** The fact that every norm induced by an inner product satisfies (15.12)
can be checked by direct computation. Conversely, if (15.12) holds, one defines
\[
\langle x; y \rangle := \frac{\| x + y \|_E^2 - \| x - y \|_E^2}{4}
\]
and check that, for every \( x, y \in E \), we have
\[
\langle x; y \rangle = \langle y; x \rangle , \quad \langle x; x \rangle = \| x \|_E^2 , \quad \langle x; 0 \rangle = 0 , \quad \langle -x; y \rangle = - \langle x; y \rangle
\]
Direct computations give that if \( x, y, z \in E \) then
\[
\langle x + y; z \rangle + \langle x - y; z \rangle = 2 \langle x; z \rangle .
\]
In particular, if we set first \( x = y \), then \( \overline{x} = x + y \) and \( \overline{y} = x - y \) in (15.13), we obtain that
\[
\langle 2x; z \rangle = 2 \langle x; z \rangle , \quad \langle \overline{x} + \overline{y}; z \rangle = \langle \overline{x}; z \rangle + \langle \overline{y}; z \rangle .
\]
By induction, if \( m \) is an integer, we get
\[
\langle mx; z \rangle = m \langle x; z \rangle \quad \text{and} \quad \langle \frac{x}{m}; z \rangle = \frac{1}{m} \langle x; z \rangle .
\]
We conclude that
\[
\langle \frac{m}{n} x; z \rangle = \frac{m}{n} \langle x; z \rangle
\]
for every \( m, n \) integers. By continuity of \( \| \cdot \|_E \) we conclude that \( \langle tx; z \rangle = t \langle x; z \rangle \) for every \( t \in \mathbb{R} \). Thus, \( \langle \cdot; \cdot \rangle \) is an inner product that induces \( \| \cdot \|_E \). ■

The following lemma, which is a corollary of Lemma 15.5, is directly used
in the proof of Theorem 15.12.

**Lemma 15.7** Assume that \( \dim E \geq 2 \). If \( \| \cdot \|_E \) is not induced by an inner prod-
uct, then there exist \( x, y, X, Y \in S^E \) such that
\[
\| x + y \|_E^2 + \| x - y \|_E^2 < 4 < \| X + Y \|_E^2 + \| X - Y \|_E^2 .
\]
Proof. As usual, it is enough to establish the result for \( E = \mathbb{R}^2 \). Let us show the first inequality, the second one being obtained dually by replacing \( \Sigma_{\text{max}} \) by \( \Sigma_{\text{min}} \). Since \( \Sigma_{\text{max}} \) is inscribed in \( S^E \), we have
\[
\|z\|_E \leq \rho_{\Sigma_{\text{max}}} (z) \quad \text{for every } z \in E,
\]
where \( \rho_{\Sigma_{\text{max}}} \) is the gauge associated to \( \Sigma_{\text{max}} \). It is also clear that we cannot have (see below)
\[
\|x + y\|_E = \rho_{\Sigma_{\text{max}}} (x + y)
\]
for every \( x, y \in \Sigma_{\text{max}} \cap S^E \). Therefore choose \( x, y \in \Sigma_{\text{max}} \cap S^E \) such that
\[
\|x + y\|_E < \rho_{\Sigma_{\text{max}}} (x + y).
\]
Since we always have \( \|x - y\|_E \leq \rho_{\Sigma_{\text{max}}} (x - y) \) and \( \rho_{\Sigma_{\text{max}}} \) satisfies the parallelogram rule, we have indeed established the claimed inequality.

We now show, by contradiction, that (15.14) does not hold. Up to an affine transformation, we may assume that \( \Sigma_{\text{max}} \) is the Euclidean disk:
\[
\Sigma_{\text{max}} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}.
\]
By Lemma 15.5, \( \Sigma_{\text{max}} \cap S^E \) contains at least four distinct points \( p_1^1, p_2^1, p_3^1, p_4^1 \) (ordered in the clockwise direction, in particular \( p_3^1 = -p_1^1 \) and \( p_4^1 = -p_2^1 \)) and we denote by \( F_1 := \{p_1^1, p_2^1, p_3^1, p_4^1\} \). Note that
\[
\rho_{\Sigma_{\text{max}}} (p_{i+1}^1 - p_i^1) \leq \pi
\]
(with the convention that \( p_5^1 = p_1^1 \)) for every point in \( F_1 \).

We next use (15.14) for every \( x, y \in F_1 \) to obtain a family \( F_2 \subset \Sigma_{\text{max}} \cap S^E \) of eight distinct points that contains \( F_1 \) (see Figure 15.2). More precisely, we set
\[
p_2^1 = p_1^1, \quad p_3^1 = p_2^1, \quad p_5^1 = p_3^1, \quad p_7^1 = p_4^1,
\]
\[
p_2^2 = \frac{p_1^1 + p_2^1}{\rho_{\Sigma_{\text{max}}} (p_1^1 + p_2^1)}, \quad p_4^2 = \frac{p_3^1 + p_4^1}{\rho_{\Sigma_{\text{max}}} (p_3^1 + p_4^1)},
\]
\[
p_5^2 = \frac{p_3^1 + p_4^1}{\rho_{\Sigma_{\text{max}}} (p_3^1 + p_4^1)}, \quad p_8^2 = \frac{p_1^1 + p_2^1}{\rho_{\Sigma_{\text{max}}} (p_1^1 + p_2^1)}.
\]
We clearly have that \( \rho_{\Sigma_{\text{max}}} (p_{i+1}^2 - p_i^2) \leq \pi/2 \) (with the convention that \( p_5^2 = p_1^2 \)). We iterate this process to inductively obtain families
\[
F_n \subset F_{n+1} \subset \Sigma_{\text{max}} \cap S^E
\]
such that \( F_n = \{p_i^n\}_{i=1}^{2^{n+1}} \) has \( 2^{n+1} \) distinct points and
\[
\rho_{\Sigma_{\text{max}}} (p_{i+1}^n - p_i^n) \leq \pi/2^{n-1}
\]
Figure 15.2: The points $p^i_j$

(with the convention that $p^j_{2n+1} = p^i_n$). This gives that $\bigcup_{n=1}^{\infty} F_n$ is dense in $\Sigma_{\text{max}}$ and in $S^E$. Consequently, $\Sigma_{\text{max}} = S^E$ and thus $\|\cdot\|_E$ is induced by an inner product, which is the desired contradiction.

We immediately obtain as a corollary the following result established by Day [224], which is a refinement of the theorem of Jordan-von Neumann.

**Corollary 15.8** Assume that $\dim E \geq 2$. The norm $\|\cdot\|_E$ is induced by an inner product if and only if for every $x, y \in S^E$ the following identity holds:

$$\|x + y\|_E^2 + \|x - y\|_E^2 = 4.$$  \hfill (15.15)

**Proof.** The fact that (15.15) is a necessary condition for $\|\cdot\|_E$ to be induced by an inner product is a by-product of the parallelogram rule (15.12) proved in Lemma 15.6. Conversely, we proceed by contradiction and assume that the norm $\|\cdot\|_E$ is not induced by an inner product. By Lemma 15.7, we have that (15.15) does not hold and thus the claim.

We conclude with Nordlander inequality [477].

**Lemma 15.9 (Nordlander inequality)** Assume that $\dim E \geq 2$ and that $0 < t < 1$. Then

$$\inf\{\|x + y\|_E : (x, y) \in S_t\} \leq 2\sqrt{1 - t^2} \leq \sup\{\|x + y\|_E : (x, y) \in S_t\}, \hfill (15.16)$$

where

$$S_t := \{(x, y) \in S^E \times S^E : \|x - y\|_E = 2t\}.$$

**Proof.** Observe that it suffices to prove that (15.16) holds on a subspace of $E$ of dimension 2. For that we may assume without loss of generality that $\dim E = 2$. 
Since $S^E$ is the boundary of a convex set, we can find
\[ s \to u(s) = (u_1(s), u_2(s)) \]
a Lipschitz parametrization of $S^E$ (in the counterclockwise direction).

We next consider, for every $s$, the set $u(s) + 2tS^E$. It intersects $S^E$ at two distinct points and we let $v(s)$ denote the one on the "left" of $u(s)$. Thus (see Figure 15.3)
\[ s \to v(s) = (v_1(s), v_2(s)) \]
is another Lipschitz parametrization in the counterclockwise direction of $S^E$.

By Green formula, we get
\[
\text{area}(B^E) = \int_{S^E} u_1 \, dv_2 = \int_{S^E} v_1 \, dv_2. \tag{15.17}
\]
Let $E_t$ be the region enclosed by the curve $s \to (u(s) + v(s))/2$. The curve
\[ C_t : s \to (u(s) - v(s))/2 \]
is a closed curve contained in $tS^E$. Hence, it coincides with $tS^E$ and so the region enclosed by $C_t$ is $tB^E$. We use again Green formula and (15.17) to obtain that
\[
\text{area}(tB^E) = \int_{S^E} \frac{(u_1 - v_1)}{2} \, d\left(\frac{u_2 - v_2}{2}\right) = \frac{1}{2} \text{area}(B^E) - \frac{1}{4} \int_{S^E} (v_1 \, du_2 + u_1 \, dv_2)
\]
and
\[
\text{area}(E_t) = \int_{S^E} \frac{(u_1 + v_1)}{2} \, d\left(\frac{u_2 + v_2}{2}\right) = \frac{1}{2} \text{area}(B^E) + \frac{1}{4} \int_{S^E} (v_1 \, du_2 + u_1 \, dv_2).
\]
We add up both sides of the above identities to conclude that
\[ \text{area}(E_t) = (1 - t^2) \text{area}(B^E). \]
This last identity implies that $E_t$ neither strictly contains nor is strictly contained in the ball of radius $\sqrt{1 - t^2}$ as asserted either on the left-hand side or the right-hand side of (15.16).

15.4 Extension from a general subset of $E$ to $E$

We now present the main results of this chapter. We discuss some necessary and sufficient conditions on the spaces $E$ and $F$, which in most of our analysis are Banach spaces, ensuring that $[E;F]$ has the extension property for contractions.

The earliest result in this direction is the celebrated MacShane lemma [410] (see also Whitney [600]) asserting that if dim $F = 1$, then $[E;F]$ has the extension property for contractions for any $E$. It turns out that this is also true for any $F$ if dim $E = 1$.

At the same time, Kirszbraun [368] proved that if $E$ and $F$ are both finite dimensional spaces whose norms are induced by a scalar product, then $[E;F]$ has the extension property for contractions. This result, known as Kirszbraun theorem, has been proved, and at the same time extended to Hilbert spaces, in several different ways, notably by Valentine [583], [584], Grünbaum [325], Minty [445] and others; one could also consult books such as those of Federer [275] or Schwartz [527].

When turning to necessary conditions, it was established by Schönbeck [524] that if dim $E$, dim $F \geq 2$ and if the unit sphere $S^F$ of $F$ is strictly convex (see below for a precise definition), then $[E;F]$ has the extension property for contractions if and only if both $E$ and $F$ are Hilbert spaces. It can also be shown that $[E;F]$ has the extension property for contractions if and only if for every set $D \subset D'$ of respective cardinality $3,4$, every map $u \in \text{Lip}_1(D,F)$ admits an extension $\tilde{u} \in \text{Lip}_1(D',F)$. When $E = F$, one can prove some stronger results, see De Figueiredo-Karlovitz [235], [236], Edelstein-Thompson [256] and Schönbeck [525].

It is one of our goals to give a still different, and somehow more elementary and more self contained, proof of the result of Schönbeck (see Theorem 15.12). The approach used to obtain this result involves the smallest norm above $\| \cdot \|_E$ that is induced by an inner product. This norm is precisely the gauge $\rho_{\Sigma_{\max}^E}$ of the ellipse $\Sigma_{\max}^E$ of maximal volume, inscribed in $S^E$. Similarly, one also considers the largest norm below $\| \cdot \|_E$ that is induced by an inner product. This norm turns out to be the gauge $\rho_{\Sigma_{\min}^E}$ of the ellipse of minimal volume, circumscribed about $S^E$. One seeks conditions under which

$$\rho_{\Sigma_{\max}^E} = \| \cdot \|_E = \rho_{\Sigma_{\min}^E} \quad \text{and} \quad \rho_{\Sigma_{\max}^F} = \| \cdot \|_F = \rho_{\Sigma_{\min}^F}.$$  

We start with a definition that is used in the main theorem.
Definition 15.10 The unit sphere $S^F$ is said to be strictly convex if it has no flat part, meaning that
$$
\|(1 - t)x + ty\|_F < (1 - t)\|x\|_F + t\|y\|_F = 1
$$
for every $t \in (0, 1)$ and every $x, y \in S^F$ such that $x \neq y$.

Let us recall that, for $1 \leq p \leq \infty$, the Hölder norms $|x|_p$ over $\mathbb{R}^n$ are defined as
$$
|x|_p := \begin{cases} 
\left(\sum^n_{i=1} |x_i|^p\right)^{1/p} & \text{if } 1 \leq p < \infty \\
\max_{1 \leq i \leq n} \{|x_i|\} & \text{if } p = \infty
\end{cases}
$$
When $n \geq 2$, the unit sphere for $|\cdot|_p$ is strictly convex if and only if $1 < p < \infty$.

We can now state our main theorems.

Theorem 15.11 (i) Let $(E, \|\cdot\|_E)$ be a normed space. Then $[E; \mathbb{R}]$ has the extension property for contractions.

(ii) Let $(F, \|\cdot\|_F)$ be a Banach space. Then $[\mathbb{R}; F]$ has the extension property for contractions.

We next turn our attention to the case where both $E$ and $F$ have dimension at least 2 and we give a theorem that characterizes the Banach spaces for which $[E, F]$ has the extension property for contractions.

Theorem 15.12 Assume that $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are Banach spaces such that $\dim E, \dim F \geq 2$ and that the unit sphere in $F$ is strictly convex. Assume also that every closed set $D \subset E$ contains a countable set $D_c \subset D$ whose closure is $D$. Then, the following three properties are equivalent:

(i) $\|\cdot\|_E$ and $\|\cdot\|_F$ are induced by an inner product;

(ii) $[E; F]$ has the extension property for contractions;

(iii) for every $x \in E$ and every $S := \{x_1, x_2, x_3\} \subset E$, every $u \in \text{Lip}_1(S, F)$ has an extension $\tilde{u} \in \text{Lip}_1(S \cup \{x\}, F)$.

Remark 15.13 (i) We should point out that if $S$ consists of only two points $x, y \in E$, $x \neq y$, then the extension to any third point is always possible. Indeed assume that
$$
\|u(x) - u(y)\|_F \leq \|x - y\|_E.
$$
Then let $z \in E$ and define
$$
t := \min\{1, \|z - y\|_E / \|x - y\|_E\} \quad \text{and} \quad u(z) := tu(x) + (1 - t)u(y).
$$
It is immediate to check that
$$
\|u(x) - u(z)\|_F \leq \|x - z\|_E \quad \text{and} \quad \|u(z) - u(y)\|_F \leq \|z - y\|_E
$$
as wished.
(ii) Interestingly enough, if one drops the assumption that $S^F$ is strictly convex, the extension property for contractions may hold for $[E; F]$ even if none of the norm is induced by an inner product. This happens, for example, in the following cases.

- If $F = \mathbb{R}^2$ (or $\mathbb{R}^n, n \geq 2$) and $\|\cdot\|_F = |\cdot|_{\infty}$, MacShane lemma (Theorem 15.11) applied to each component of a vector valued map ensures that $[E; F]$ has the extension property for contractions for every normed space $E$.

- If $F = \mathbb{R}^2$ with $\|\cdot\|_F = |\cdot|_1$, then $[E; F]$ has the extension property for contractions for any normed space $E$. This follows from the simple observation that if $R = 1/2 \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, then $|Ry|_1 = |y|_{\infty}$ for any $y \in \mathbb{R}^2$. This, together with the above argument for the $|\cdot|_{\infty}$ norm, gives that $[E; \mathbb{R}^2]$ has the extension property for contractions for any normed space $E$.

(iii) Proceeding by contradiction in the proof that (iii) $\Rightarrow$ (i), we find $S := \{x_1, x_2, x_3\}$, $\overline{\mathbf{r}} \in (x_1, x_2)$ and $u \in \text{Lip}_1(S, F)$ such that there is no extension $\tilde{u} \in \text{Lip}_1(S \cup \{\overline{\mathbf{r}}\}, F)$. A continuity argument can show that there is also no extension $\tilde{u} \in \text{Lip}_1(S \cup \{\overline{\mathbf{r}}\}, F)$ where for $\delta > 0$ small enough

$$\overline{x}_\delta = \overline{\mathbf{r}} + \delta (x_3 - \overline{\mathbf{r}}).$$

Observe that therefore $\overline{x}_\delta \in \text{int} \text{co}\{x_1, x_2, x_3\}$. ♦

In the proof of Theorem 15.12, we need the following lemma.

**Lemma 15.14** Assume that $\dim E, \dim F \geq 2$ and that at least one of these norms is not induced by an inner product. Then there exist $y_1, y_2 \in F$ and $x_1, x_2 \in E$ so that

$$\|x_1\|_E = \|x_2\|_E = \|y_1\|_F = \|y_2\|_F = 1 \quad \text{and} \quad \|y_1 \pm y_2\|_F < \|x_1 \pm x_2\|_E.$$

**Proof.** It is enough to prove the lemma when $\dim E = \dim F = 2$. We assume that $\|\cdot\|_F$ is not induced by a scalar product; a similar argument holds if $\|\cdot\|_E$ is not induced by a scalar product. By Lemma 15.7, we can therefore find $y_1, y_2 \in \mathbb{R}^2$ so that

$$\|y_1\|_F = \|y_2\|_F = 1 \quad \text{and} \quad \|y_1 - y_2\|_F^2 + \|y_1 + y_2\|_F^2 < 4.$$

Let

$$s = \frac{1}{2} \|y_1 - y_2\|_F$$

and use the triangle inequality to see that $0 < s < 1$. We therefore have

$$\|y_1 + y_2\|_F < 2\sqrt{1 - s^2}.$$
We next choose $t \in (s, 1)$ so that
\[
\|y_1 + y_2\|_F < 2\sqrt{1 - t^2} < 2\sqrt{1 - s^2}.
\]
We then apply Nordlander inequality (15.16) to get that there exist $x_1, x_2 \in \mathbb{R}^2$ so that
\[
\|x_1\|_E = \|x_2\|_E = 1 \quad \text{and} \quad \|x_1 - x_2\|_E = 2t, \quad \|x_1 + x_2\|_E \geq 2\sqrt{1 - t^2}.
\]
Combining all these results we have indeed found $y_1, y_2 \in F$ and $x_1, x_2 \in E$ satisfying
\[
\|y_1\|_F = \|y_2\|_F = \|x_1\|_E = \|x_2\|_E = 1, \quad \|y_1 - y_2\|_F = 2s < 2t = \|x_1 - x_2\|_E \quad \text{and} \quad \|y_1 + y_2\|_F < 2\sqrt{1 - t^2} \leq \|x_1 + x_2\|_E,
\]
as claimed in the lemma.

It is interesting to see how to construct elements satisfying the conclusions of Lemma 15.14 in the case of Hölder norms.

Example 15.15 Assume that $E = F = \mathbb{R}^2$, $\|\cdot\|_F = |\cdot|_q$ and $\|\cdot\|_E = |\cdot|_p$, where $1 < p, q < \infty$. Denote also by $p'$ and $q'$ the conjugate exponents of $p$ and $q$. We then have the following cases.

Case 1. If $q > p$, we set $x_1 = y_1 = (0, 1), \ x_2 = y_2 = (1, 0)$ and observe that
\[
|y_1 - y_2|_q = |y_1 + y_2|_q = 2^{1/q} < |x_1 - x_2|_p = |x_1 + x_2|_p = 2^{1/p}.
\]

Case 2. If $p > q$, we set $x_1 = 2^{-1/p}(1, 1), \ x_2 = 2^{-1/p}(1, -1), \ y_1 = 2^{-1/q}(1, 1), \ y_2 = 2^{-1/q}(1, -1)$ and observe that
\[
|y_1 - y_2|_q = |y_1 + y_2|_q = 2^{1/q'} < |x_1 - x_2|_p = |x_1 + x_2|_p = 2^{1/p'}.
\]

Case 3. We assume here that $p = q$.

(i) If $q > p'$, we set $x_1 = 2^{-1/p}(1, 1), \ x_2 = 2^{-1/p}(1, -1), \ y_1 = (1, 0), \ y_2 = (0, 1)$ and observe that
\[
|y_1 - y_2|_q = |y_1 + y_2|_q = 2^{1/q} < |x_1 - x_2|_p = |x_1 + x_2|_p = 2^{1/p'}.
\]

(ii) If $q < p'$, we let $x_1 = (1, 0), \ x_2 = (0, 1), \ y_1 = 2^{-1/q}(1, 1), \ y_2 = 2^{-1/q}(1, -1)$ to obtain that
\[
|y_1 - y_2|_q = |y_1 + y_2|_q = 2^{1/q'} < |x_1 - x_2|_p = |x_1 + x_2|_p = 2^{1/p}.
\]

We can now proceed with the proofs of the theorems stated above.

Proof. (Theorem 15.11). (i) In fact, the arguments used in the proof of this part of the theorem are still valid in metric spaces. The fact that $[E, \mathbb{R}]$ has the extension property for contractions is, as already discussed, MacShane lemma.
We recall that if $D \subset E$ and $u \in \text{Lip}_1(D, \mathbb{R})$ then both of the functions below are extensions of $u$ that belong to $\text{Lip}_1(E, \mathbb{R})$:

$$u^+(x) := \inf_{y \in D} \{ u(y) + \| x - y \|_E \}, \quad u^-(x) := \sup_{y \in D} \{ u(y) - \| x - y \|_E \}.$$ 

Furthermore, if $\tilde{u} \in \text{Lip}_1(E, \mathbb{R})$ is another extension of $u$, then $u^- \leq \tilde{u} \leq u^+$.

(ii) We now check that $[\mathbb{R}, F]$ has the extension property for contractions. So we assume that we have $D \subset \mathbb{R}$ and $u : D \to F$ satisfying

$$\| u(x) - u(y) \|_F \leq |x - y| \quad \text{for every } x, y \in D.$$ 

We wish to show that we can find $\tilde{u} : \mathbb{R} \to F$, an extension of $u$, satisfying

$$\| \tilde{u}(x) - \tilde{u}(y) \|_F \leq |x - y| \quad \text{for every } x, y \in \mathbb{R}.$$ 

We proceed in two steps.

Step 1. If $D$ is not closed, we extend $\tilde{u}$ to $\overline{D}$ by continuity. More precisely, let $x \in \overline{D}$ and $x_\nu \in D$ converging to $x$. Observe that $\{ u(x_\nu) \}$ is a Cauchy sequence, since

$$\| u(x_\nu) - u(x_\mu) \|_F \leq |x_\nu - x_\mu|.$$ 

It therefore converges to an element of $F$, independent of the choice of the sequence, denoted by $\tilde{u}(x)$. With this definition, we clearly deduce that

$$\| \tilde{u}(x) - \tilde{u}(y) \|_F \leq |x - y| \quad \text{for every } x, y \in \overline{D}.$$ 

Step 2. From now on we assume that $D$ is closed. Let

$$\alpha := \inf \{ x : x \in D \} \quad \text{and} \quad \beta := \sup \{ x : x \in D \}.$$ 

Then

$$\text{int co } D = (\alpha, \beta).$$ 

For $x \in \mathbb{R}$, we define

$$x^+ := \inf \{ y : y \in D \text{ and } y \geq x \} \quad \text{and} \quad x^- := \sup \{ y : y \in D \text{ and } y \leq x \}.$$ 

Since $D$ is closed, if $x \in \text{int co } D$, we deduce that $x^\pm \in D$. Moreover if $x \in D$, we have that $x^\pm = x$, while if $x \in \text{int co } D$ but $x \notin D$, we find $x^- < x < x^+$. If $\alpha < x < \beta$, then $-\infty < x^- \leq x \leq x^+ < +\infty$ and therefore there exists a unique $t = t(x) \in [0, 1]$ such that

$$x = tx^+ + (1 - t)x^-.$$ 

We are now in a position to define $\tilde{u} : \mathbb{R} \to F$ through

$$\tilde{u}(x) := \begin{cases} 
 u(\alpha) & \text{if } x \leq \alpha \\
 tu(x^+) + (1 - t)u(x^-) & \text{if } \alpha < x < \beta \\
 u(\beta) & \text{if } x \geq \beta.
\end{cases}$$
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In the above definition, it is understood that if $\alpha = -\infty$ (respectively $\beta = +\infty$), then the first (respectively the third) possibility does not happen. Furthermore, since when $x \in D$ we have that $x^\pm = x$, we deduce that $\tilde{u}$ is indeed an extension of $u$. The fact that $\tilde{u} \in \text{Lip}_1(\mathbb{R}, F)$ is easily checked once we note that

$$\|\tilde{u}(x) - \tilde{u}(x^+)\|_F \leq x^+ - x \quad \text{and} \quad \|\tilde{u}(x) - \tilde{u}(x^-)\|_F \leq x - x^-$$

if $\alpha < x < \beta$. ■

We continue with the proof of Theorem 15.12.

**Proof.** (i) $\Rightarrow$ (ii). When $E$ and $F$ are finite dimensional spaces, the fact that (i) implies (ii) is Kirszbraun theorem and we present here the proof of Minty [445].

In view of Zorn lemma, it is sufficient to prove that $[E; F]$ has the extension property for contractions for finitely many points (for more details, see Dacorogna-Gangbo [188]).

So assume that we are given $x_1, \ldots, x_k \in E$, $y_1, \ldots, y_k \in F$ such that

$$\|y_i - y_j\|_F \leq \|x_i - x_j\|_E, \ i, j = 1, \ldots, k$$

(15.18)

and let us show that for any $x \in E$, we can find $y \in F$ such that

$$\|y_i - y\|_F \leq \|x_i - x\|_E, \ i = 1, \ldots, k.$$  

(15.19)

In order to obtain this result, we first check that the condition (15.2) with $p = 2$ is satisfied. Theorem 15.3 then implies the claim (15.19).

We therefore have to prove that, for every $\lambda \in \Lambda_k$,

$$F(\lambda, \lambda) := \sum_{i=1}^k \lambda_i \|y_i - \sum_{j=1}^k \lambda_j y_j\|_F^2 - \sum_{i=1}^k \lambda_i \|x_i - x\|_E^2 \leq 0,$$  

(15.20)

where

$$\Lambda_k := \{\lambda = (\lambda_1, \ldots, \lambda_k) \in [0, 1]^k : \sum_{i=1}^k \lambda_i = 1\}.$$  

Note that since the norm is induced by an inner product, the identity

$$\sum_{i,j=1}^k \lambda_i \lambda_j \|y_i - y_j\|_F^2 = 2 \sum_{i=1}^k \lambda_i \|y_i - \sum_{j=1}^k \lambda_j y_j\|_F^2$$  

(15.21)

holds for every $\lambda \in \Lambda_k$. Similarly, the inequality

$$\sum_{i,j=1}^k \lambda_i \lambda_j \|x_i - x_j\|_E^2 \leq 2 \sum_{i=1}^k \lambda_i \|x_i - x\|_E^2$$  

(15.22)

holds for every $x \in E$ and every $\lambda \in \Lambda_k$. In fact, the right-hand side of (15.22) is minimized by the average value

$$\bar{x} := \sum_{i=1}^k \lambda_i x_i.$$  

We combine (15.18), (15.21) and (15.22) to conclude that (15.20) holds.

(ii) $\Rightarrow$ (iii). This implication is obvious.
(iii) ⇒ (i). We proceed by contradiction, assuming that either $\|\cdot\|_E$ or $\|\cdot\|_F$ is not induced by an inner product. We will construct

$$u : S := \{x_1, x_2, x_3\} \subset E \to \{u(x_1) = y_1, u(x_2) = y_2, u(x_3) = y_3\} \subset F$$

so that $u \in \text{Lip}_1(S, F)$, but there is no extension $\tilde{u} \in \text{Lip}_1(S \cup \{\mathbf{x} = 0\}, F)$.

We proceed in two steps.

**Step 1.** From Lemma 15.14, there exist $y_1, \tilde{y}_3 \in F$ and $x_1, x_3 \in E$ such that

$$\|y_1\|_F = \|\tilde{y}_3\|_F = \|x_1\|_E = \|x_3\|_E = 1 \quad \text{and} \quad \|y_1 \pm \tilde{y}_3\|_F < \|x_1 \pm x_3\|_E.$$ 

We can therefore find $\epsilon > 0$ sufficiently small so that if

$$y_3 = (1 + \epsilon)\tilde{y}_3$$

we still have

$$\|y_1 \pm y_3\|_F \leq \|x_1 \pm x_3\|_E.$$ 

Letting $y_2 = -y_1$ and $x_2 = -x_1$ we find that

$$\|y_1\|_F = \|y_2\|_F = 1, \quad \|y_3\|_F = 1 + \epsilon, \quad \|x_1\|_E = \|x_2\|_E = \|x_3\|_E = 1,$$

$$\|y_1 - y_2\|_F = \|2y_1\|_F = 2 \to \|2x_1\|_E = \|x_1 - x_2\|_E,$$

$$\|y_1 - y_3\|_F \leq \|x_1 - x_3\|_E,$$

$$\|y_2 - y_3\|_F = \|y_1 + y_3\|_F \leq \|x_1 + x_3\|_E = \|x_2 - x_3\|_E.$$ 

Hence $u \in \text{Lip}_1(S, F)$, meaning that

$$\|y_i - y_j\|_F \leq \|x_i - x_j\|_E \quad \forall i, j = 1, 2, 3. \quad \text{(15.23)}$$

**Step 2.** The claim that there is no extension $\tilde{u} \in \text{Lip}_1(S \cup \{\mathbf{x} = 0\}, F)$ will follow if we can show that no $y \in F$ can verify

$$\|y - y_j\|_F \leq \|x_j\|_E = 1, \quad \forall j = 1, 2, 3,$$

which is equivalent to showing that

$$\mathcal{A} := \{y \in F : \|y - y_j\|_F \leq 1, \forall j = 1, 2, 3\} = \emptyset.$$ 

To prove this, we only need to show that

$$\mathcal{B} := \{y \in F : \|y - y_1\|_F, \|y - y_2\|_F = \|y + y_1\|_F \leq 1\} = \{0\}$$

and use that $\|y_3\|_F = 1 + \epsilon$ to obtain the claim. If $y \in \mathcal{B}$, we obtain

$$1 = \|y_1\|_F = \|\frac{1}{2}(y_1 - y) + \frac{1}{2}(y_1 + y)\|_F \leq \frac{1}{2}\|y_1 - y\|_F + \frac{1}{2}\|y_1 + y\|_F \leq 1.$$
and consequently
\[ \|y_1\|_F = \frac{1}{2}\|y_1 - y\|_F + \frac{1}{2}\|y_1 + y\|_F = 1. \]
Since \( y \in B \), we get that
\[ \|y_1\|_F = \|y_1 - y\|_F = \|y_1 + y\|_F = 1. \]
Since the unit sphere \( S^F \) is strictly convex, we obtain
\[ y_1 - y = y_1 + y \Rightarrow y = 0 \]
as wished.

### 15.5 Extension from a convex subset of \( E \) to \( E \)

In many applications it is important to know if for every closed convex set \( \Omega \subset E \), every 1–Lipschitz map \( u : \Omega \rightarrow F \) admits a 1–Lipschitz extension over \( E \). These questions have been investigated by De Figueiredo-Karlovitz in [234], [235], [236] in the case where \( E = F \) and \( \|\cdot\|_E = \|\cdot\|_F \). The general case, which still remains open, is apparently closely related to whether or not projections on convex sets are contractions. In this section, we address the extension property for contractions for convex sets in simple cases where \( E \) is a Hilbert space.

Throughout this subsection, we assume that \( E \) is a reflexive Banach space (mostly a Hilbert space) and that \( \Omega \subset E \) is a closed convex set. We will specify it when we need to impose that \( \partial \Omega \), the boundary of \( \Omega \), is strictly convex. This means that \((1 - t)x + ty \in \text{int} \Omega \) whenever \( t \in (0, 1) \) and \( x, y \in \partial \Omega \), \( x \neq y \). Here, \( \text{int} \Omega \) denotes the interior of \( \Omega \). The following result should be related to Theorem 2.9.

**Lemma 15.16** (i) For every \( x \in E \), there exists \( x_\infty \in \Omega \) minimizing
\[ z \rightarrow \|x - z\|_E \]
over \( \Omega \). Moreover, if \( x \notin \text{int} \Omega \), then \( x_\infty \in \partial \Omega \).

(ii) If in addition either \( S^E \) is strictly convex or \( \partial \Omega \) is strictly convex, then \( x_\infty \) is uniquely determined. In that case, the map
\[ x \rightarrow p_{\Omega}(x) := x_\infty \]
is well defined and is referred to as the projection map onto \( \Omega \).

**Proof.** (i) Let \( x \in E \) and let \( \{x_\nu\}_{\nu=1}^\infty \subset \Omega \) be such that
\[ \lim_{\nu \rightarrow +\infty} \|x - x_\nu\|_E = \inf_{z \in \Omega} \|x - z\|_E. \quad (15.24) \]
The set \( \{ x_\nu \}_{\nu=1}^\infty \), being bounded, is weakly precompact and so has a subsequence that we still label \( \{ x_\nu \}_{\nu=1}^\infty \), converging weakly to some \( x_\infty \in \Omega \). Since \( \| \cdot \|_E \) is convex, we conclude that \( \| \cdot \|_E \) is weakly lower semicontinuous and hence

\[
\| x - x_\infty \|_E \leq \lim_{\nu \to +\infty} \| x - x_\nu \|_E.
\]

This, together with (15.24), yields that \( x_\infty \) is a minimizer of \( \| x - z \|_E \) over \( \Omega \).

Let us show that if \( x \in \text{int} \, \Omega \), then \( x_\infty \in \partial \Omega \). By contradiction, if \( x_\infty \in \text{int} \, \Omega \), we would have for \( t \in (0, 1) \) small enough that

\[
x_t = (1 - t) x_\infty + t x \in \Omega
\]

and thus

\[
\| x - x_t \|_E = (1 - t) \| x - x_\infty \|_E < \| x - x_\infty \|_E
\]

contradicting the definition of \( x_\infty \).

(ii) Let \( x \notin \Omega \) and \( x_\infty, \bar{x}_\infty \in \Omega \) be two minimizers of \( \| x - z \|_E \) over \( \Omega \). Since \( x_\infty, \bar{x}_\infty \in \partial \Omega \), we find that

\[
x_0 := \frac{x_\infty + \bar{x}_\infty}{2} \in \Omega
\]

is another minimizer of \( \| x - z \|_E \). Assume for the sake of contradiction that \( x_\infty \neq \bar{x}_\infty \). If \( \partial \Omega \) is strictly convex, then \( x_0 \notin \partial \Omega \), which yields a contradiction. On the other hand, if \( S^E \) is strictly convex, we have from the fact that

\[
r := \| x - x_\infty \|_E = \| x - \bar{x}_\infty \|_E > 0
\]

that \( \| x - x_0 \|_E < r \), which also yields a contradiction. This proves that the minimizer of \( \| x - z \|_E \) over \( \Omega \) is unique.

**Lemma 15.17** If \( E \) is a Hilbert space, then \( p_\Omega : E \to E \) is a contraction.

**Proof.** Every Hilbert space is reflexive. Furthermore, the parallelogram rule (15.12) gives that \( S^E \) is strictly convex. Hence, by Lemma 15.16, \( p_\Omega \) is well defined.

Since for every \( x \in E \), \( t \in [0, 1] \) and \( z \in \Omega \), we have

\[
\| x - p_\Omega(x) \|_E^2 \leq g(t) := \| x - [(1 - t) p_\Omega(x) + tz] \|_E^2
\]

we find, since \( g'(0) \geq 0 \), that \( p_\Omega(x) \) satisfies

\[
\langle x - p_\Omega(x); z - p_\Omega(x) \rangle \leq 0 \text{ for every } z \in \Omega.
\]

(15.25)

If \( x_1, x_2 \in E \), we use (15.25), once with \( z = p_\Omega(x_2) \) and once with \( z = p_\Omega(x_1) \), to obtain that

\[
\langle x_1 - p_\Omega(x_1); p_\Omega(x_2) - p_\Omega(x_1) \rangle \leq 0 \quad \text{and} \quad \langle x_2 - p_\Omega(x_2); p_\Omega(x_1) - p_\Omega(x_2) \rangle \leq 0.
\]
Adding up these two inequalities yields that
\[ \|p_\Omega(x_1) - p_\Omega(x_2)\|_E^2 \leq \langle p_\Omega(x_1) - p_\Omega(x_2); x_1 - x_2 \rangle. \]
This, together with Cauchy-Schwarz inequality, leads to
\[ \|p_\Omega(x_1) - p_\Omega(x_2)\|_E \leq \|x_1 - x_2\|_E, \]
which is the claim.

**Corollary 15.18** Assume that $E$ is a Hilbert space and $F$ is a normed space. Then every contraction $u : \Omega \subset E \rightarrow F$ has an extension $\tilde{u} : E \rightarrow F$ that is still a contraction.

**Proof.** By Lemma 15.17, $p_\Omega$ is a contraction and thus the map
\[
\tilde{u} := u \circ p_\Omega
\]
is a contraction as a composition of two contractions.

**Remark 15.19** Let $E$ be a finite dimensional normed space (not necessarily induced by a scalar product) and consider the radial map
\[
x \rightarrow p_E(x) := \frac{x}{\max\{1, \|x\|_E\}}.
\]
(i) In [234], under the assumption that $\dim E \geq 3$, De Figueiredo-Karlovitz, proved that: $p_E \in \text{Lip}_1(E, E)$ if and only if $\|\cdot\|_E$ is induced by an inner product.
(ii) As is well known, we next verify that $p_E$ satisfies
\[
\|x - p_E(x)\|_E \leq \|x - z\|_E \quad \text{for every} \quad z \in \overline{B}^E. \tag{15.26}
\]
Since the result is trivial if $x \in \overline{B}^E$, we assume that $x \in E \setminus \overline{B}^E$. We then let $\rho = \|\cdot\|_E$ and observe that it trivially is the gauge of $B^E$. Let $\rho^0$ be its polar; then, according to Proposition 2.55,
\[
p \in \partial \rho(x) \Rightarrow \rho^0(p) = 1,
\]
where $\partial \rho(x)$ denotes the subdifferential of $\rho$ at $x$. So let $p \in \partial \rho(x)$ and $z \in \overline{B}^E$; we then have
\[
\|x - z\|_E \geq \|x\|_E - \langle p; z \rangle \geq \|x\|_E - \rho^0(p) \|z\|_E \geq \|x\|_E - 1 = \|x - p_E(x)\|_E
\]
as claimed in (15.26).
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Notation

General notation

- For a given set $E \subset \mathbb{R}^N$, $\overline{E}$, respectively $\partial E$, int $E$ and $E^c$ stand for the closure, respectively the boundary, the interior and the complement of $E$.

- The sum of two sets $E, F \subset \mathbb{R}^N$ is denoted by $E + F$; see Section 2.2.2.

- $B_\epsilon(x) := \left\{ y \in \mathbb{R}^N : |y - x| < \epsilon \right\}$ and $\overline{B}_\epsilon(x) := \left\{ y \in \mathbb{R}^N : |y - x| \leq \epsilon \right\}$.

- $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^N$.

- The following is used throughout:

$$
\Lambda_s := \{ \lambda = (\lambda_1, \cdots, \lambda_s) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^s \lambda_i = 1 \};
$$

see Section 2.2.3.

- For a set $E \subset \mathbb{R}^N$, we denote by $\chi_E$ the indicator function and by $1_E$ the characteristic function of $E$ (see Sections 2.3.1 and 3.2.6) and they are given by

$$
\chi_E(x) := \begin{cases} 
0 & \text{if } x \in E \\
+\infty & \text{if } x \notin E
\end{cases}
\quad \text{and} \quad
1_E(x) := \begin{cases} 
1 & \text{if } x \in E \\
0 & \text{if } x \notin E
\end{cases}.
$$

- The domain of a function $f$ is defined as

$$
\text{dom } f := \left\{ x \in \mathbb{R}^N : f(x) < +\infty \right\}.
$$

- The support of a function $f$ is denoted by supp $f$; see Section 12.2.

- Weak and weak* convergence are denoted by $\rightharpoonup$ and $\rightharpoonup^*$.

- For integers $1 \leq s \leq n$, we let

$$
\binom{n}{s} = \frac{n!}{s!(n-s)!}.
$$

Convex analysis

- For a given convex set $E \subset \mathbb{R}^N$, the relative interior and the relative boundary of $E$ are denoted respectively by $\text{ri } E$ and $\text{rbd } E$; see Section 2.2.1.
- The projection on a convex set $E$ is denoted by $p_E$; see Section 2.2.2.
- The convex hull of a set $E \subset \mathbb{R}^N$ is denoted by $\text{co} E$; see Section 2.2.3.
- The set of extreme points of a convex set $E$ is denoted by $E_{\text{ext}}$; see Section 2.2.4.
- The gauge and the support function of a convex set $E$ are denoted respectively by $\rho_E$ and $\chi_E^*$; see Section 2.3.1.
- The epigraph and the level set of a function $f$ are denoted respectively by $\text{epi} f$ and $\text{level}_\alpha f$; see Section 2.3.1.
- The convex envelope of a function $f$ is denoted by $C f$; see Section 2.3.3.
- The sets
  \[
  \mathcal{F}_E := \{ f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\} : f|_E \leq 0 \}
  \]
  \[
  \mathcal{F}^E := \{ f : \mathbb{R}^N \to \mathbb{R} : f|_E \leq 0 \},
  \]
  are defined in Section 2.3.3.
- The dual and bidual of a function $f$ are respectively denoted by $f^*$ and $f^{**}$; see Section 2.3.5.
- The subdifferential $\partial f (x)$ and the directional derivative $f' (x, y)$ are defined in Section 2.3.6.
- The polar of a gauge $\rho$ is denoted by $\rho^0$; see Section 2.3.7.

**Determinants and singular values**

- $\mathbb{R}^{N \times n}$ stands for the set of $N \times n$ real matrices $\xi$,

  $\xi = \begin{pmatrix}
  \xi_1^1 & \cdots & \xi_1^n \\
  \vdots & \ddots & \vdots \\
  \xi_N^1 & \cdots & \xi_N^n
  \end{pmatrix} = \begin{pmatrix}
  \xi^1 \\
  \vdots \\
  \xi^N
  \end{pmatrix} = (\xi_1, \cdots, \xi_n).

  For such a matrix, $\xi^t \in \mathbb{R}^{n \times N}$ denotes the transpose and, if $N = n$, its trace is denoted by trace ($\xi$).
- $\mathbb{R}_s^{n \times n}$ is the set of $n \times n$ symmetric matrices; see Section 7.4.4.
- $\mathbb{R}_d^{N \times n}$ is the subspace of $\mathbb{R}^{N \times n}$ consisting of diagonal matrices. When $n \leq N$ (and similarly when $n > N$) and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we let

  \[
  \text{diag}_{N \times n} (x_1, \cdots, x_n)
  \]

  denote the diagonal matrix whose entries are the $x_i$ (when $N = n$, we simply write $\text{diag} (x_1, \cdots, x_n)$); see Section 5.3.3.
- For $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$, we let $a \otimes b \in \mathbb{R}^{N \times n}$ be defined (see Section 5.2.1) by

  \[
  a \otimes b = (a^i b_\alpha)_{1 \leq i \leq N}^{1 \leq \alpha \leq n}.
  \]
- We denote
  \[ n \wedge N = \min\{n, N\}, \quad \sigma(s) = \binom{n}{s} \binom{N}{s}, \]
  and
  \[ \tau(n, N) = \sum_{s=1}^{n \wedge N} \sigma(s) = \sum_{s=1}^{n \wedge N} \binom{n}{s} \binom{N}{s}; \]
  see Section 5.4.

- For \( \xi \in \mathbb{R}^{N \times n} \) and \( 1 \leq s \leq n \wedge N \), \( \text{adj}_s \xi \in \mathbb{R}^{(s) \times (s)} \) stands for the adjugate matrix and
  \[ T(\xi) = (\xi, \text{adj}_2 \xi, \cdots, \text{adj}_{n \wedge N} \xi) \in \mathbb{R}^{\tau(n,N)}; \]
  see Section 5.4.

- For matrices \( \xi = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_1 & \xi_2 \end{pmatrix} \in \mathbb{R}^{2 \times 2} \), the notations
  \[ \tilde{\xi} = \begin{pmatrix} \xi_2 & -\xi_1 \\ -\xi_2 & \xi_1 \end{pmatrix}, \quad \xi^+ = \frac{1}{2}(\xi + \tilde{\xi}) \quad \text{and} \quad \xi^- = \frac{1}{2}(\xi - \tilde{\xi}) \]
  are used; see Remark 13.7.

- Let \( N \geq n \) (similarly if \( N < n \)) and \( \xi \in \mathbb{R}^{N \times n} \). The singular values of \( \xi \) are denoted
  \[ 0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi); \]
  see Section 13.2.

- The signed singular values of \( \xi \in \mathbb{R}^{n \times n} \) are denoted by
  \[ 0 \leq |\mu_1(\xi)| \leq \cdots \leq |\mu_n(\xi)|; \]
  see Section 13.3.

- \( GL(n) \), \( \Pi(n) \), \( \Pi_e(n) \), \( S(n) \), stand for some subsets of \( \mathbb{R}^{n \times n} \) matrices; see Section 5.3.3.

- \( O(n) \) and \( SO(n) \) denote respectively the sets of orthogonal and special orthogonal matrices; see Section 13.2.

- For \( u : \mathbb{R}^n \rightarrow \mathbb{R}^N \) (hence \( \nabla u \in \mathbb{R}^{N \times n} \)), we denote (see Section 8.5), for \( 2 \leq s \leq n \wedge N, \ 1 \leq i_1 < \cdots < i_s \leq n \) and \( 1 \leq \alpha_1 < \cdots < \alpha_s \leq N \),
  \[ \frac{\partial (u^{i_1}, \ldots, u^{i_s})}{\partial (x_{\alpha_1}, \ldots, x_{\alpha_s})} := \det \begin{pmatrix} \frac{\partial u^{i_1}}{\partial x_{\alpha_1}} & \cdots & \frac{\partial u^{i_1}}{\partial x_{\alpha_s}} \\ \vdots & \ddots & \vdots \\ \frac{\partial u^{i_s}}{\partial x_{\alpha_1}} & \cdots & \frac{\partial u^{i_s}}{\partial x_{\alpha_s}} \end{pmatrix}. \]
Quasiconvex analysis

- Hypothesis \((H_I)\); see Section 5.2.5.

- The polyconvex, quasiconvex and rank one convex envelopes of a function \(f\) are respectively denoted \(Pf\), \(Qf\) and \(Rf\); see Sections 6.2, 6.3 and 6.4.

- The polyconvex conjugate and biconjugate of a function \(f\) are respectively denoted \(f^p\) and \(f^{pp}\); see Section 6.2.

- The polyconvex, quasiconvex, rank one convex and separately convex hulls of a set \(E \subset \mathbb{R}^{N \times n}\) are respectively denoted by

\[
Pco E, \ Qco E, \ Rco E \quad \text{and} \quad Sco E;
\]

see Section 7.3.1.

- The convex, polyconvex, quasiconvex, rank one convex and separately convex finite hulls of a set \(E \subset \mathbb{R}^{N \times n}\) are respectively denoted by

\[
co_f E, \ Pco_f E, \ Qco_f E, \ Rco_f E \quad \text{and} \quad Sco_f E;
\]

see Section 7.3.2.

- The extreme points in the convex, polyconvex, quasiconvex, rank one convex and separately convex senses are respectively denoted by \(E_{ext}^c\) (also denoted \(E_{ext}\) in Chapter 2), \(E_{ext}^p\), \(E_{ext}^q\), \(E_{ext}^r\) and \(E_{ext}^s\); see Section 7.3.3.

Function spaces

Let \(m \geq 0\) be an integer, \(1 \leq p \leq \infty\) and \(0 < \alpha \leq 1\).

- \(C^m, C_0^m, C^\infty, C_0^\infty, C_{\text{piec}}^m, \text{Aff}^m, \text{Aff}_{\text{piec}}^m, \text{Aff}_{\text{piec}}^m\); see Section 12.2.

- \(C^{m,\alpha}\) denote Hölder spaces; see Section 12.3.

- \(W^{m,p}, W_0^{m,p}\), denote Sobolev spaces; see Section 12.4.
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