Definite Integrals

Alina Gleska

Institute of Mathematics, Faculty of Electrical Engineering,
Poznań University of Technology, Poland

Definite integrals

We consider the following five steps for the function $y = f(x)$.

1. Let f be defined on a closed interval $[a, b]$.
2. Partition the interval $[a, b]$ into n subintervals $[x_{k-1}, x_k]$ of length $\Delta x_k = x_k - x_{k-1}$. Let P denote the partition:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

3. Let $||P||$ be the length of the longest subinterval. The number $||P||$ is called the norm of the partition P.
4. Choose a number x_k^* in each subinterval.
5. Form the sum: $\sum_{i=1}^{n} f(x_k^*)\Delta x_k$.

Such sums for the various partitions of $[a, b]$ are known as Riemann sums and are named after the famous German mathematician, Georg Friedrich Bernhardt Riemann (1826-1866). For a function f defined on an interval $[a, b]$ there are an infinite number of possible Riemann sums for a given partition P of the interval since the numbers x_k^* can be chosen arbitrarily in each subinterval $[x_{k-1}, x_k]$. But if the Riemann sums are close to a number L for every partition P of $[a, b]$ for which the norm $||P||$ is close to 0, we then write:

$$\lim_{||P|| \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k = L$$

and say that L is the definite integral of f on the interval $[a, b]$. If this limit exists, the function f is said to be integrable on the interval.

Definition

Let f be a function defined on a closed interval $[a,b]$. Then the definite integral of f from a to b, denoted by $\int_a^b f(x)dx$, is defined to be

$$\int_a^b f(x)dx = \lim_{||P||\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

The numbers a and b in the preceding definition are called the lower and upper limits of integration, respectively. The integral symbol $\int$, first used by Leibniz, is an elongated S for the word 'sum'.

Theorem

If f is continuous on $[a,b]$*, then* $\int_a^b f(x)dx$ *exists; that is, f is integrable on the interval.*

Theorem

If $f(a)$ *exists, then* $\int_a^a f(x)dx = 0$.

Theorem

If f is integrable on $[a,b]$*, then* $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

Theorem

Let f and g be integrable functions on $[a,b]$. Then:

1. $\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f(x)dx$, *where k is any constant,*
2. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$, *(this point extends to any finite sum of integrable functions on the interval)*
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, *where c is any number in $[a,b]$.*

The independent variable x in a definite integral is called a dummy variable of integration. The value of the integral does not depend on the symbol used. In other words,

$$\int_a^b f(x)dx = \int_a^b f(r)dr = \int_a^b f(t)dt,$$

and so on.

Geometric interpretation of definite intervals

Theorem

For any constant k, $\int_a^b k \cdot dx = k \cdot \int_a^b dx = k(b-a).$

Theorem

If f is continuous on $[a,b]$*, then there exists a point* $c \in [a,b]$ *such that*

$$\int_a^b f(x)dx = (b-a)f(c).$$

Theorem (The Fundamental Theorem of Calculus)

Let f be continuous on $[a, b]$ and let F be any function for which $F'(x) = f(x)$. Then,

$$\int_a^b f(x)dx = F(b) - F(a).$$

Remark

$$\int_a^b f(x)dx = \int f(x)dx|_a^b = F(x)|_a^b = F(b) - F(a).$$

Integration by substitution we can proceed in two ways:

1. evaluate the indefinite integral by means of the substitution $u = g(x)$.Resubstitute $u = g(x)$ in the antiderivative, and then apply the Fundamental Theorem of Calculus by using the original limits of integration $x = a$ and $x = b$;
2. alternatively, the resubstitution can be avoided by changing the limits of integration to correspond to the value of u at $x = a$ and u at $x = b$ (this way is usually quicker).

Theorem (Integration by Substitution)

Let $u = g(x)$ be a function that has a continuous derivative on the interval $[a, b]$, and let f be a function that is continuous on the range of g. If $F'(u) = f(u)$ and $c = g(a)$, $d = g(b)$, then

$$\int_a^b f(g(x))g'(x)dx = F(d) - F(c).$$

Example

Evaluate $\int_0^2 x\sqrt{4 - x^2}dx$.

Theorem (Integration by Parts)

If the functions f, g, f' and g' are continuous on some interval $[a,b]$, then

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx.$$

Example

Evaluate the integral: $\int_0^{\frac{\pi}{2}} x \cdot \cos(x)dx$.

Example

Find the area bounded by the graph of the given functions:

1. $y = x^2$, $y = \frac{1}{2}x^2$, $y = 3x$
2. $y^2 = 4x + 4$, $y = 2 - x$.

ARC LENGTH

If $y = f(x)$ has a continuous first derivative on an interval $[a, b]$, then its graph is said to be smooth and f is called a smooth function. As the name implies, a smooth graph has no sharp points. Now we will find the length of a smooth graph.

Let f have a smooth graph on $[a,b]$, and let P denote the arbitrary partition

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

As usual, let the length of each subinterval be given by $\Delta x_k = x_k - x_{k-1}$ and let $||P||$ be the length of the longest subinterval. The length of the chord between $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$ is an approximation to the length of the graph between these points. The length of the chord is:

$$\Delta S_k = \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}.$$

By the Mean Value Theorem we know that there exists an x_k^* in each open subinterval (x_{k-1}, x_k) such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*)$$

or

$$f(x_k) - f(x_{k-1}) = f'(x_k^*)(x_k - x_{k-1}).$$

Substituting from this last equation into previous one gives

$$\Delta S_k = \sqrt{(x_k - x_{k-1})^2 + (f'(x_k))^2 (x_k - x_{k-1})^2} =$$

$$= \sqrt{1 + (f'(x_k^*))^2}(x_k - x_{k-1}) = \sqrt{1 + (f'(x_k^*))^2}\Delta x_k.$$

The sum:

$$\sum_{k=1}^{n} \Delta S_k = \sum_{k=1}^{n} \sqrt{1 + (f'(x_k^*))^2} \Delta x_k$$

gives an approximation to the total length of the graph on $[a, b]$.
As $||P|| \to 0$, we obtain:

$$\lim_{||P|| \to 0} \sum_{k=1}^{n} \sqrt{1 + (f'(x_k^*))^2} \Delta x_k = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Definition

Let f be a function for which f' is continuous on an interval $[a,b]$. The length S of the graph on the interval, of arc length, is given by

$$S = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Example

Evaluate the length S of the graph of the function $f(x) = \sqrt{2x - x^2}$ for $x \in [0, 1]$.

The Volume V of the Solids of Revolution: Disk and Washer Method

The disk method – let R be the region bounded by the graph of a nonnegative continuous function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$. If this region is revolved about the x-axis, let us find the volume V of the resulting solid of revolution.

The exact volume is: $V = \pi \int_a^b (f(x))^2 dx$.

The washer method – let the region R bounded by the graphs of the continuous functions $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$, be revolved about the x-axis. Then the rectangular element between the two graphs on $[x_{k-1}, x_k]$ will generate a circular ring or washer.

The exact volume of the solid is given by:

$$V = \pi \int_a^b (f(x))^2 dx - \pi \int_a^b (g(x))^2 dx = \pi \int_a^b (f^2(x) - g^2(x)) dx.$$