Surveys in Noncommutative Geometry

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Surveys in Noncommutative Geometry

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on Noncommutative Geometry

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Mount Holyoke College
South Hadley, MA

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10 9 8 7 6 5 4 3 2 1 11 10 09 08 07 06

Contents

Preface

In June 2000, the Clay Mathematics Institute organized an Instructional Symposium on Noncommutative Geometry in conjunction with the AMS-IMS-SIAM Joint Summer Research Conference. These events were held at Mount Holyoke College in Massachusetts from June 18 to 29, 2000. The Instructional Symposium consisted of several series of expository lectures which were intended to introduce key topics in noncommutative geometry to mathematicians unfamiliar with the subject. Those expository lectures have been edited and are reproduced in this volume. Specifically, the lectures of Rosenberg and Weinberger discussed various applications of noncommutative geometry to problems in "ordinary" topology, and the lectures of Lagarias and Tretkoff discussed the Riemann hypothesis and the possible application of the methods of noncommutative geometry in number theory. This book also contains an account by Higson of the "residue index theorem" of Connes and Moscovici.

At the conference, Higson and Roe gave an overview of noncommutative geometry which was intended to provide a point of entry to the later, more advanced lectures. This preface represents a highly compressed version of that introduction. We hope that it will provide sufficient orientation to allow mathematicians new to the subject to read the later, more technical essays in this volume.

Noncommutative geometry—as we shall use the term—is to an unusual extent the creation of a single mathematician, Alain Connes; his book [**12**] is the central text of the subject. The present volume could perhaps be regarded as a sort of extended introduction to that dense and fascinating book. With that in mind, let us review some of the central notions in Connes' work.

To see what is meant by the phrase "noncommutative geometry," consider ordinary geometry: for example, the geometry of a closed surface S in three-dimensional Euclidean space $\mathbb{R}^3$. Following Descartes, we study the geometry of S using coordinates. These are just the three functions x, y, z on $\mathbb{R}^3$, and their restrictions to S generate, in an appropriate sense, the algebra $C(S)$ of all continuous functions on S. All the geometry of S is encoded in this algebra $C(S)$; in fact, the points of S can be recovered simply as the algebra homomorphisms from $C(S)$ to $\mathbb{C}$. In the language of physics, one might say that the transition from S to $C(S)$ is a transition from a "particle picture" to a "field picture" of the same physical situation.

This idea that features of geometric spaces are reflected within the algebras of their coordinate functions is familiar to everyone. Further examples abound:

- If X is a compact Hausdorff space, then (complex) vector bundles over X correspond to finitely generated and projective modules over the ring of continuous, complex-valued functions on X.

- If M is a smooth manifold, then vector fields on M correspond to derivations of the algebra of smooth, real-valued functions on M.
- If Σ is a measure space, then ergodic transformations of Σ correspond to automorphisms of the algebra of essentially bounded measurable functions on Σ which fix no non-constant function.
- If G is a compact and Hausdorff topological group, then the algebra of complex-valued representative functions on G (those whose left translations by elements of G span a finite-dimensional space) determines, when supplemented by a comultiplication operation, not only the group G as a space, but its group operation as well.

We could cite many others, and so of course can the reader.

The algebra $C(S)$ of continuous functions on the space S is commutative. The basic idea of noncommutative geometry is to view noncommutative algebras as coordinate rings of "noncommutative spaces." Such noncommutative spaces must necessarily be "delocalized," in the sense that there are not enough "points" (homomorphisms to $\mathbb{C}$) to determine the coordinates. This has a natural connection with the Heisenberg uncertainty principle of quantum physics. Indeed, a major strength of noncommutative geometry is that compelling examples of noncommutative spaces arise in a variety of physical and geometric contexts.

Connes' work is particularly concerned with aspects of the space–algebra correspondence which, on the algebra side, involve Hilbert space methods, especially the spectral theory of operators on Hilbert space. Motivation for this comes from a number of sources. As we hinted above, one is physics: quantum mechanics asserts that physical observables such as position and momentum should be modeled by elements of a noncommutative algebra of Hilbert space operators. Another is a long series of results, dating back to Hermann Weyl's asymptotic formula, connecting geometry to the spectral theory of the Laplace operator and other operators. A third comes from the application of operator algebra K-theory to formulations of the Atiyah-Singer index theorem and surgery theory. Here the notion of positivity which is characteristic of operator algebras plays a key role. A fourth motivation, actively discussed at the Mount Holyoke conference, involves the reformulation of the Riemann hypothesis as a Hilbert-space-theoretic positivity statement.

Connes introduces several successively more refined kinds of geometric structure in noncommutative geometry: measure theory, topology (including algebraic topology), differential topology (manifold theory), and differential (Riemannian) geometry. To begin with the coarsest of these, noncommutative measure theory means the theory of *von Neumann algebras.* Let H be a Hilbert space and let $B(H)$ be the set of bounded linear operators on it. A *von Neumann algebra* is an involutive subalgebra M of $B(H)$ which is closed in the weak operator topology on $B(H)$. The commutative example to keep in mind is $H = L^2(X, \mu)$, where (X, μ) is a measure space, and $M = L^\infty(X, \mu)$ acts by multiplication; the weak topology coincides with the topology of pointwise almost everywhere convergence on bounded sets.

Of special importance in von Neumann algebra theory are the *factors*—von Neumann algebras with trivial center (every von Neumann algebra decomposes into factors, in the same way that group representations decompose into isotypical components). For example the von Neumann algebra generated by $L^\infty(X, \mu)$ and an ergodic transformation of (X, μ) is a factor. Already in the early work of Murray and von Neumann [**25**] one finds the classification of factors into three *types*: Type I, which are matrix algebras, possibly infinite-dimensional; Type II, admitting a "dimension function" assigning real-valued (rather than integer-valued) ranks to the ranges of projections in the factor; and Type III, admitting no dimension function. Of these, the factors of type III are the most mysterious. Much of Connes' early work was devoted to elucidating their structure, and an account of these deep results may be found in Chapter V of [**12**].

The fundamental discovery, due to Tomita [**27**], Takesaki [**26**] and Connes [**7**], is that a von Neumann algebra has an *intrinsic* dynamical character—a one-parameter group of automorphisms determined up to inner conjugacy only by the algebra itself —and that by twisting with these automorphisms one can reduce Type III to Type II. This played a major role in Connes' classification of *hyperfinite* factors (those generated by an increasing family of finite-dimensional subalgebras). The final result may be explained as follows. After twisting a factor by its canonical automorphisms a new von Neumann algebra is obtained, which has its own one-parameter group of canonical automorphisms. The center of the twisted von Neumann algebra is necessarily of the form $L^\infty(X,\mu)$. The corresponding one-parameter action on X is ergodic and is Connes' invariant of the original factor, called the *module* of the factor. As Mackey observed much earlier [**24**], ergodic actions of $\mathbb{R}$ can be thought of as *virtual subgroups* of $\mathbb{R}$. There arises then a sort of Galois correspondence between hyperfinite factors and virtual subgroups of $\mathbb{R}$. In a fascinating paper [**13**] Connes has pointed out hopeful parallels with the Brauer theory of central simple algebras, and speculated on connections with class field theory and the Riemann zeta function.

An example of noncommutative space which is significant in many parts of noncommutative geometry comes from the theory of foliations. Let $(V,\mathcal{F})$ be a foliated manifold. We want to study the "space of leaves" $V/\mathcal{F}$—initially only as a measure space, but then successively in the more refined categories mentioned above. If the foliation is ergodic—think of the irrational-slope flow on a torus or geodesic flow on the sphere bundle of a higher genus surface—then the quotient measure space in the usual sense is trivial. But we can form a noncommutative von Neumann algebra whose elements are measurable families of operators on the L^2-spaces of the leaves of the foliations. See Chapter 1 of [**12**]. When the foliation is a fibration, this von Neumann algebra is equivalent to the one arising from the usual measure-theoretic quotient. However, the von Neumann algebra we have constructed is interesting even in the general case; for instance, the von Neumann algebras of the ergodic foliations listed above are hyperfinite of type II. Moreover, in the case of the geodesic flow example, the action of the geodesic flow implements the Galois-type correspondence in the classification of factors: if Γ is a (virtual) subgroup of $\mathbb{R}$ then the associated (virtual) fixed point algebra has module Γ. See [**13**] for some further details.

Just as noncommutative measure theory can be identified with the theory of von Neumann algebras, noncommutative topology depends on the theory of C^*-algebras—involutive subalgebras of $B(H)$ which are closed for the *norm* topology. According to the Gelfand-Naimark theorem the only commutative examples are the algebras of continuous functions vanishing at infinity on locally compact Hausdorff spaces. Hence the study of commutative C^*-algebras is exactly the same as the theory of locally compact Hausdorff spaces. Important noncommutative examples can be obtained from foliations by means of a refinement of the von Neumann algebra construction sketched above. See [**8**]. Another example of particular interest arises by completing the complex algebra of a discrete group G so as to obtain the *(reduced) group* C^**-algebra* $C^*_\lambda(G)$, for which the corresponding noncommutative space is the (reduced) unitary dual of G. Important connections with geometric topology arise when G is the fundamental group of a manifold. For example, noncommutative geometry presents the opportunity to carry over geometric constructions involving the Pontrjagin dual of an abelian group to the non-abelian case.

The development of a significant body of work in noncommutative topology was made possible by the discovery in the 1970s that topological K-theory and its dual theory, K-homology, have natural extensions to the category of C^*-algebras. Moreover, there is a natural relationship with index theory. Atiyah and Singer showed that the index of a family of elliptic operators over a base B should be thought of as an element of the K-theory group $K^*(B)$. Analogously, a family of elliptic operators along the leaves of a foliation, or on the fibers of various other quotients, has an index in the K-theory of the C^*-algebra representing the quotient space. A very general conjecture is then that the whole C^*-algebra K-theory group is generated in a precise way by indices of this kind. This is known as the *Baum-Connes* conjecture [**2, 3, 19**], and it has been verified in a large number of different situations. It also has noteworthy implications: for example, the Baum-Connes conjecture for a discrete group G implies the Novikov conjecture for G, which (via surgery theory) is crucial to the topology of high-dimensional manifolds having G as fundamental group.

Let us briefly review the key definitions of C^*-algebra K-theory (see for example [**6**] or Chapter II of [**12**]). If X is a compact Hausdorff space and V a complex vector bundle over X, then the space of sections of V is a module over the algebra $C(X)$ of functions on X (via pointwise multiplication). According to a theorem of Serre and Swan, this module is finitely generated and projective, and every finitely generated projective module arises in this way from a vector bundle. One therefore defines $K_0(A)$, for a unital C^*-algebra A, to be the abelian group with one generator for each isomorphism class of finitely generated projective A-modules and with relations requiring that the direct sum of modules should correspond to addition in the group. By construction, if X is a compact Hausdorff space, then $K_0(C(X)) = K^0(X)$, the K-group of Atiyah and Hirzebruch.

One can similarly define a group $K_1(A)$, generated by unitaries or invertibles in matrix algebras over A. The key result of elementary K-theory is that if

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

is a short exact sequence of C^*-algebras, then there is a corresponding six-term exact sequence of K-groups

$$\begin{tikzcd}
K_0(J) \arrow[r] & K_0(A) \arrow[r] & K_0(A/J) \arrow[d] \\
K_1(A/J) \arrow[u] & K_1(A) \arrow[l] & K_1(J). \arrow[l]
\end{tikzcd}$$

This incorporates as a special case the *Bott periodicity theorem*, which states that $K_0(A) \equiv K_1(A \otimes C_0(\mathbb{R}))$. The vertical maps are closely connected with index theory: when A is the algebra of bounded operators on a Hilbert space, and J is the ideal of compact operators, the map $K_1(A/J) \to K_0(J)$ is exactly the one which assigns to every Fredholm operator—an element of A invertible modulo J—its Fredholm index.

Kasparov and others have developed extensive bivariant generalizations of operator algebra K-theory (see [**23, 22, 21**], or [**18**] for an introduction to Kasparov's theory). An element of Kasparov's group $KK(A, B)$ determines a morphism from $K(A)$ to $K(B)$. His theory encompasses the construction of the boundary maps in the K-theory long exact sequences, the construction of index maps appearing in the proof of the Atiyah-Singer index theorem, and many other things.

The next stage is the development of noncommutative differential topology, which is organized around Connes' cyclic cohomology theory. If M is a smooth manifold, then the cohomology of M may be obtained from the de Rham complex of differential forms. This is organized around functions on M rather than points of M, although the functions are of course smooth, not merely continuous. Various features of smooth manifold theory (de Rham's theorem, of course, but also Sard's theorem and transversality) make it possible to overcome this issue and effectively apply the techniques of smooth manifold theory to topological questions. The situation in noncommutative geometry is not quite so satisfactory. Cyclic cohomology is a fascinating substitute for de Rham theory in the noncommutative context, but in the absence of generally valid counterparts to de Rham's theorem and other results, the application of cyclic theory to problems in noncommutative topology requires considerable care.

The lectures of Higson provide more details on cyclic theory (and the reader should certainly also refer to the original papers of Connes [**9**]), but one way to motivate the definition is to consider the algebra $A = C^\infty(M)$ of smooth functions on a closed manifold, and the multilinear functionals on A given by

$$\phi\colon (f^0, \dots, f^k) \mapsto \int_N f^0 df^1 \cdots df^k,$$

where N is a closed (oriented) k-submanifold of M. Calculation shows that this functional ϕ has the properties

$$\text{(a)}\quad \phi(f^0 f^1, \dots, f^{k+1}) - \phi(f^0, f^1 f^2, \dots, f^{k+1}) + \cdots \\ + (-1)^k \phi(f^0, \dots, f^k f^{k+1}) + (-1)^{k+1} \phi(f^{k+1} f^0, \dots, f^k) = 0$$

and

$$\text{(b)}\qquad \phi(f^0, \dots, f^k) + (-1)^{k+1} \phi(f^k, f^0, \cdots, f^{k-1}) = 0.$$

These are the properties which define a *cyclic cocycle* as a multilinear functional over A, and the expression (a) above gives the boundary map for *cyclic cohomology.* Interestingly it is property (b) which follows, via Stokes' theorem, from the fact that N is closed, while (a) is more generally valid, despite the fact that formula (a) does in fact define a coboundary operation on multilinear functionals. Moreover it is quite remarkable that the coboundary of a functional with the symmetry property (b)—invariance under the cyclic group C_{k+1}—again has the symmetry property—which is now invariance under the different cyclic group C_k. It therefore requires deep insight to see that (a) and (b) are the crucial properties needed to carry over de Rham theory to the noncommutative context. Given a cyclic k-cocycle ϕ (and taking k even, for simplicity), and an idempotent $e \in A$, Connes shows that the expression

$$\phi(e, e, \cdots, e)$$

depends only on the homotopy class of e and the cohomology class of ϕ, and thus gives rise to a pairing of cyclic cohomology and K-theory which generalizes the classical Chern character (a homomorphism from topological K-theory to ordinary cohomology).

One of Connes' deepest early applications of cyclic theory was to the construction of the *transverse fundamental class* for a foliation [**10**]: this is a map from the K-theory of the leaf space of a foliation (i.e. the K-theory of the C^*-algebra of the foliation) to the real numbers. The construction is elementary when there is an invariant transverse Riemannian metric, but in general no such metric can be found. Connes noticed that this problem was analogous to one that he had already surmounted in his work on factors — namely, the reduction of type III to type II. Following this insight, and combining ideas from cyclic theory (here the choice of an appropriate noncommutative algebra of "smooth functions" on the leaf space is a very delicate matter) and sophisticated versions of the Thom isomorphism in K-theory, Connes was able to construct the desired map. Geometric results about foliations follow, including remarkable links between the von Neumann algebra of a foliation and the more classical invariants. For example Connes showed that if the Godbillon-Vey class of a codimension-one foliation is non-zero, then the module of the associated von Neumann algebra admits a finite invariant measure (recall that the module is an ergodic action of $\mathbb{R}$). This is a beautiful refinement of the result due to Hurder and Katok [**20**] that if the Godbillon-Vey class of a codimension-one foliation is non-zero, then the associated von Neumann algebra has a type III component.

A second motivation for cyclic cohomology, equally significant, is the idea of "quantized calculus". If A is an algebra of operators on some $\mathbb{Z}/2$-graded Hilbert space H, and F an odd operator on H with $F^2 = 1$, then the supercommutator[1] $da = [F, a]_s$ obeys the basic rule $d^2 = 0$ of de Rham theory. If in addition F and A are "almost commuting" in the sense that any product of sufficiently many commutators $[F, a]$ is of trace class, then the trace can be used to define an integral of these quantized forms: if ε is the grading operator on H then

$$\int a^0 da^1 \dots da^k = \mathrm{Tr}(\varepsilon a^0 [F, a^1] \cdots [F, a^k]).$$

This is a cyclic cocycle (see [**9**] and Chapter IV of [**12**]). An important example is $A = C^\infty(M)$, M a closed manifold of dimension n , and F an order zero pseudodifferential involution. The commutators are then pseudodifferential of order -1, and any product of more than n of them is therefore trace-class. In this example we see that the notion of *dimension* in quantized calculus is related to the "degree of traceability" of certain commutators.

Not all cyclic classes arise in this way. In fact, such a cyclic class belongs to the image of a Chern character map from K-homology. It follows, therefore, that if such a cyclic class is paired with an element of K-theory the result must be an *integer*, because the evaluation of pairing between K-theory and K-homology ultimately

[1]The supercommutator of homogeneous elements a and b is defined as $[a, b]_s = ab - (-1)^{\partial(a)\partial(b)} ba$.

reduces to the computation of a Fredholm index. In this way one obtains *integrality* results somewhat analogous to those provided by the Atiyah-Singer index theorem. An interesting application is the work of J. Bellissard [**5, 4**] on the existence of integer plateaux of conductivity in the quantum Hall effect. This concerns the flow of electricity in a crystal subjected to electric and magnetic fields. The observables in this problem generate an algebra isomorphic to that associated to the irrational slope foliation on a torus, and the integrality arises from an associated cyclic 2-cocycle.

The most refined level of structure which can be explored in the noncommutative world is the noncommutative counterpart of *geometry* proper—for instance, of Riemannian geometry. It was an early observation of Connes that knowledge of the *Dirac operator* suffices to reconstruct the Riemannian geometry. This is because one can identify the smooth functions f with gradient ≤ 1 as those for which the commutator $[D, f]$ has operator norm ≤ 1; and from knowledge of this class of smooth functions one can reconstruct the Riemannian distance function

$$d(x, y) = \sup\{|f(x) - f(y)| : |\nabla f| \leq 1\}$$

and therefore the Riemannian metric. This led Connes to his notion of a *spectral triple*—the noncommutative counterpart of a Dirac (or other first order elliptic) operator, and therefore the noncommutative counterpart of a geometric space. A *spectral triple* (called a *K-cycle* in Chapter VI of [**12**]) is made up of

- a Hilbert space H,
- a representation of A as bounded operators on H, and
- an unbounded, selfadjoint operator D on H with compact resolvent,

such that $[a, D]$ extends to a bounded operator on H for every $a \in A$. To connect with cyclic cohomology, one also imposes a *summability condition*, for example that $(1 + D^2)^{-\frac{p}{2}}$ is trace-class, for large enough p. Assuming for simplicity that D is invertible and suitably graded, the phase $F = D|D|^{-1}$ defines a quantized calculus and therefore a cyclic cocycle on A.

A variety of examples of spectral triples are described in [**12**] and elsewhere. Particularly fascinating are Connes' efforts to model space-time itself as a noncommutative space [**11**] and the manner in which he fits the standard model of particle physics into this framework.

The archetypal result in Riemannian geometry is perhaps the Gauss-Bonnet theorem, which expresses the Euler characteristic, a global quantity, as the integral of the Gaussian curvature, a local invariant. Similar results in noncommutative geometry will require adequate notions of "integral" and "local invariant." It might seem as though the search for these would be destined to fail since noncommutative spaces are inevitably somewhat delocalized, almost by definition. Connes addresses this through a consideration of eigenvalue asymptotics which recalls Weyl's theorem for the eigenvalues of the Laplace operator. Weyl's theorem asserts that the rate of growth of eigenvalues of the Laplace operator on a manifold M is governed by the dimension of M and the volume of M. With this in mind Connes defines a new trace of a positive compact operator T not by summing the eigenvalues λ_j but by forming the limit

$$\mathrm{Tr}_\omega(T) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{j=1}^{N} \lambda_j,$$

when it exists. This trace was originally studied by Dixmier; it has the important property that it *vanishes* on operators in the ordinary trace class, thanks to which it is often unexpectedly computable. Moreover, Weyl's formula

$$\mathrm{Tr}_\omega(f \cdot \Delta^{-\frac{n}{2}}) = c_n \int_M f(x)\,dx,$$

where Δ is the Laplacian on M and f is a function on M, hints that the Dixmier trace might serve as a powerful substitute for integration in noncommutative situations. Indeed, Connes has developed an extensive calculus around the Dixmier trace, with the trace serving as the integral, compact operators as infinitesimals (with order governed by rate of decrease of their singular values), and so on.

Once we accept that it is appropriate to think of the Dixmier trace as a generalized integral, and of expressions involving the Dixmier trace as local expressions in noncommutative geometry, we can ask for a local version of the integral of quantized forms we described earlier. Since these integrals determine the Chern character in cyclic theory, a local version might be thought of a local index formula, somewhat in the spirit of the work of Atiyah, Bott, Patodi [**1**] and Gilkey [**17**]. Such a local index theorem was proved by Connes and Moscovici [**15**], and it is discussed in Higson's lectures in this volume. The implications of this formula for the index theory of leaf spaces of foliation have been worked out in detail by Connes and Moscovici [**16**]. The implications in other contexts are only just beginning to be studied. Connes' paper [**14**] on the quantum group $SU_q(2)$ is one very interesting recent example.

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June 2006

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A Minicourse on Applications of Non-Commutative Geometry to Topology

Jonathan Rosenberg

Contents

1. Group Von Neumann Algebras in Topology: L^2-cohomology, Novikov-Shubin invariants

1.1. Motivation. The idea of this part of the book is to explain how non-commutative geometry can be applied to problems in geometry and topology (in the more usual sense of those words). In this chapter, we begin by applying the first major new idea to emerge out of non-commutative geometry, namely, the concept of *continuous dimension* as developed by Murray and von Neumann.

This concept starts to come into play when we compare the spectral decomposition of the Laplacian (or more exactly, the Laplace-Beltrami operator of Riemannian geometry) in the two cases of a compact (Riemannian) manifold and a complete non-compact manifold. The comparison can be seen in the following table:

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Compact manifolds	**Non-compact manifolds**
Discrete spectrum	Continuous spectrum
Finite-dimensional kernel	Infinite-dimensional kernel

TABLE 1. Spectrum of the Laplacian

So, on a non-compact manifold, the dimension of the kernel of the Laplacian is not usually very interesting (it's often ∞) and knowing the eigenvalues of the Laplacian usually does not yield much information about the operator. (For example, there may be no point spectrum at all, yet the spectral decomposition of the operator may be very rich.) In the special case where the non-compact manifold is a normal covering of a compact manifold with covering group π, we will get around these difficulties by using the group von Neumann algebra of π to measure the "size" of the infinite-dimensional kernel and the "thickness" of the continuous spectrum near 0.

1.2. An Algebraic Set-Up. Here we follow the ideas of Michael Farber [**23**], as further elaborated by him (in [**25**], [**24**], and [**26**]) and by Wolfgang Lück ([**56**], [**56**], and [**57**]). Let π be a discrete group. It acts on both the right and left on $L^2(\pi)$. The von Neumann algebras $\lambda(\pi)''$ and $\rho(\pi)''$ generated by the left and by the right regular representations λ and ρ are isomorphic, and $\rho(\pi)'' = \lambda(\pi)'$. These von Neumann algebras are *finite*, with a canonical (faithful finite normal) trace τ defined by

$$\tau(\lambda(g)) = \begin{cases} 1, & g = 1 \\ 0, & g \neq 1, \end{cases}$$

and similarly for ρ. Call a finite direct sum of copies of $L^2(\pi)$, with its left action of π, a *finitely generated free Hilbert π-module*, and the cut-down of such a module by a projection in the commutant $\lambda(\pi)' = \rho(\pi)''$ a *finitely generated projective Hilbert π-module*. (We keep track of the topology but forget the inner product.)

The finitely generated projective Hilbert π-modules form an additive category $\mathcal{H}(\pi)$. The morphisms are continuous linear maps commuting with the π-action. Each object A in this category has a dimension $\dim_\tau(A) \in [0, \infty)$, via

$$\dim_\tau n \cdot L^2(\pi) = n, \quad \dim_\tau eL^2(\pi) = \tau(e),$$

for each projection e in $\rho(\pi)''$ (or more generally in $M_n(\rho(\pi)'')$, to which we extend the trace the usual way, with $\tau(1_n) = n$, 1_n the identity matrix in $M_n(\rho(\pi)'')$). When π is an *ICC* (infinite conjugacy class) group, $\rho(\pi)''$ is a factor, hence a projection $e \in M_n(\rho(\pi)'')$ is determined up to unitary equivalence by its trace, and objects of $\mathcal{H}(\pi)$ are determined by their dimensions.

The category $\mathcal{H}(\pi)$ is *not abelian*, since a morphism need not have closed range, and thus there is no good notion of cokernel. It turns out, however, that there is a natural way to complete it to get an *abelian* category $\mathcal{E}(\pi)$. The finitely generated projective Hilbert π-modules are the projectives in $\mathcal{E}(\pi)$. Each element of the larger category is a direct sum of a projective and a torsion element (representing infinitesimal spectrum near 0). A torsion element is an equivalence class of pairs

(A, α), where A is a projective Hilbert π-module and $\alpha = \alpha^*\colon A \to A$ is a positive π-module endomorphism of A (in other words, a positive element of the commutant of the π-action) with $\ker \alpha = 0$. (Note that this implies α has dense range, but not that it has a bounded inverse.) Two such pairs (A, α), (A', α'), are identified if we can write

$$(A, \alpha) \cong (A_1, \alpha_1) \oplus (A_2, \alpha_2), \quad (A', \alpha') \cong (A'_1, \alpha'_1) \oplus (A'_2, \alpha'_2),$$

with α_2 and α'_2 invertible and with $(A_1, \alpha_1) \cong (A'_1, \alpha'_1)$, in the sense that there is commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\beta", "\cong"'] \arrow[d, "\alpha"] & A'_1 \arrow[d, "\alpha'"] \\
A_1 \arrow[r, "\beta", "\cong"'] & A'_1.
\end{tikzcd}$$

Thus we can always "chop off" the part of α corresponding to the spectral projection for $[\varepsilon, \infty)$ ($\varepsilon > 0$) without changing the equivalence class of the object, and only the "infinitesimal spectrum near 0" really counts. The dimension function $\dim_\tau$ extends to a map, additive on short exact sequences, from objects of $\mathcal{E}(\pi)$ to $[0, \infty)$, under which torsion objects go to 0.

To phrase things another way, the idea behind the construction of $\mathcal{E}(\pi)$ is that we want to be able to study indices of elliptic operators $D\colon A \to B$, where A and B are Hilbert π-modules and D commutes with the π-action. By the usual tricks, we can assume D is a bounded operator. In the case of a compact manifold (with no π around), A and B would then be Hilbert spaces, D would be Fredholm, and the index would be defined as $\operatorname{Ind} D = \dim \ker D - \dim \operatorname{coker} D$. But in the case of a non-compact manifold, we run into the problem that D usually does not have closed range. However, if we fix $\varepsilon > 0$ and let e, f be the spectral projections for $[\varepsilon, \infty)$ for D^*D and DD^*, respectively, then the restriction of D to eA maps this projective π-module isomorphically onto fB, and $D\colon (1-e)A \to (1-f)B$ represents a formal difference of torsion elements $((1-e)A, D^*D)$ and $((1-f)B, DD^*)$ of $\mathcal{E}(\pi)$ when we pass to the limit as $\varepsilon \to 0$.

The most interesting invariant of a torsion element $\mathcal{X}$ represented by $\alpha = \alpha^*\colon A \to A$ is the rate at which

$$F_\alpha(t) = \dim_\tau(E_t A), \quad E_t = \text{spectral projection for } \alpha \text{ for } [0, t),$$

approaches 0 as $t \to 0$. Of course we need to find a way to study this that is invariant under the equivalence relation above, but it turns out (see Exercise 1.7) that F_α is well-defined modulo the equivalence relation

$$F \sim G \Leftrightarrow \exists C,\, \varepsilon > 0,\; G\left(\tfrac{t}{C}\right) \le F(t) \le G(tC),\; t < \varepsilon.$$

The *Novikov-Shubin capacity* of $\mathcal{X}$ is defined to be

$$c(\mathcal{X}) = \limsup_{t \to 0^+} \frac{\log t}{\log F_\alpha(t)}.$$

Note that this is well-defined modulo the equivalence relation above, since if $G\left(\frac{t}{C}\right) \leq F(t) \leq G(tC)$ for t sufficiently small, then

$$\begin{aligned}
\limsup_{t\to 0^+} \frac{\log t}{\log G(t)} &= \limsup_{t\to 0^+} \frac{\log t - \log C}{\log G(t)} = \limsup_{t\to 0^+} \frac{\log(t/C)}{\log G(t)} \\
&= \limsup_{t\to 0^+} \frac{\log(tC/C)}{\log G(tC)} \\
&= \limsup_{t\to 0^+} \frac{\log t}{\log G(tC)} \leq \limsup_{t\to 0^+} \frac{\log t}{\log F(t)} \\
&\leq \limsup_{t\to 0^+} \frac{\log t}{\log G\left(\frac{t}{C}\right)} \\
&= \limsup_{t\to 0^+} \frac{\log(tC/C)}{\log G\left(\frac{t}{C}\right)} \\
&= \limsup_{t\to 0^+} \frac{\log t + \log C}{\log G(t)} = \limsup_{t\to 0^+} \frac{\log t}{\log G(t)}.
\end{aligned}$$

The Novikov-Shubin capacity satisfies

$$c(\mathcal{X}_1 \oplus \mathcal{X}_2) = \max(c(\mathcal{X}_1),\, c(\mathcal{X}_2))$$

and for exact sequences

$$0 \to \mathcal{X}_1 \to \mathcal{X} \to \mathcal{X}_2 \to 0,$$

$$\max(c(\mathcal{X}_1),\, c(\mathcal{X}_2)) \leq c(\mathcal{X}) \leq c(\mathcal{X}_1) + c(\mathcal{X}_2).$$

Many people work instead with the inverse invariant

$$\liminf_{t\to 0^+} \frac{\log F_\alpha(t)}{\log t},$$

called the *Novikov-Shubin invariant* or *Novikov-Shubin number*, but the advantage of the capacity is that "larger" torsion modules have larger capacities. If the Novikov-Shubin invariant is $\gamma > 0$, that roughly means that $\dim_\tau(E_t A) \approx t^\gamma$.

Now consider a connected CW complex X with fundamental group π and only finitely many cells in each dimension. The cellular chain complex $C_\bullet(\widetilde{X})$ of the universal cover $\widetilde{X}$ (with complex coefficients) is a chain complex of finitely generated free (left) $\mathbb{C}[\pi]$-modules. We can complete to $L^2(\pi) \otimes_\pi C_\bullet(\widetilde{X})$, a chain complex in $\mathcal{H}(\pi) \subseteq \mathcal{E}(\pi)$, and get homology and cohomology groups

$$\mathcal{H}_i(X, L^2(\pi)) \in \mathcal{E}(\pi), \quad \mathcal{H}^i(X, L^2(\pi)) \in \mathcal{E}(\pi),$$

called (extended) L^2*-homology and cohomology*, which are homotopy invariants of X. The numbers $\beta_i(X, L^2(\pi)) =$

$$\dim_\tau(\mathcal{H}_i(X), L^2(\pi)) = \dim_\tau(\mathcal{H}^i(X), L^2(\pi))$$

are called the (reduced) L^2*-Betti numbers* of X. Similarly one has *Novikov-Shubin invariants* (first introduced in [**65**], but analytically, using the Laplacian) defined from the spectral density of the torsion parts (though by the Universal Coefficient Theorem, the torsion part of $\mathcal{H}^i(X, L^2(\pi))$ corresponds to the torsion part of $\mathcal{H}_{i-1}(X, L^2(\pi))$, so that there is some confusion in the literature about indexing).

1.3. Calculations.

THEOREM 1.1. *Suppose M is a compact connected smooth manifold with fundamental group π. Fix a Riemannian metric on M and lift it to the universal cover $\widetilde{M}$. Then the L^2-Betti numbers of M as defined above agree with the τ-dimensions of*

$$\left(L^2 \text{ closed } i\text{-forms on } \widetilde{M}\right) / \overline{d\left(L^2 \ (i-1)\text{-forms on } \widetilde{M}\right) \cap \left(L^2 \ i\text{-forms}\right)}.$$

Similarly the Novikov-Shubin invariants can be computed from the spectral density of Δ on $\widetilde{M}$ (as measured using τ).

SKETCH OF PROOF. This is a kind of a de Rham theorem. There are two published proofs, one by Farber ([**25**], §7) and one by Shubin [**90**]. Let $\Omega^\bullet(M)$ be the de Rham complex of differential forms on M. Then a fancy form of the usual de Rham theorem says that the complexes $\Omega^\bullet(M)$ and $C^\bullet(M)$ (the latter being the cellular cochains with coefficients in $\mathbb{C}$ for some cellular decomposition) are chain homotopy equivalent. The same is therefore true for the complexes $L^2(\pi)\otimes_\pi \Omega^\bullet(M)$ and $L^2(\pi)\otimes_\pi C^\bullet(M)$. Unfortunately the first of these is a complex of Fréchet spaces, not of Hilbert π-modules, so Farber's theory doesn't directly apply to it. However, there is a trick: we can also consider the complex $\Omega^\bullet_{\text{Sobolev}}(\widetilde{M})$ of Sobolev spaces of forms on $\widetilde{M}$. More precisely, fix $m \geq n = \dim M$ and consider the complex

$$\Omega^\bullet_{\text{Sobolev}}(\widetilde{M}) \colon \Omega^0_{(m)}(\widetilde{M}) \xrightarrow{d} \Omega^1_{(m-1)}(\widetilde{M}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^j_{(m-j)}(\widetilde{M}) \xrightarrow{d} \cdots,$$

where $\Omega^j_{(m-j)}(\widetilde{M})$ consists of j-forms with distributional derivatives up to order $m-j$ in L^2 (with respect to the Riemannian metric on M). Then the spaces in this complex are all Hilbert spaces on which π acts by a(n infinite) multiple of the left regular representation, and the differentials are all bounded operators commuting with the action of π (since we lose one derivative with each application of d). An extension of Farber's original construction shows that the spaces in such a complex can also be viewed as sitting in an "extended" abelian category (in effect one just needs to drop the finite generation condition in the definition of $\mathcal{E}(\pi)$). Then one shows that the dense inclusion $L^2(\pi) \otimes_\pi \Omega^\bullet(M) \hookrightarrow \Omega^\bullet_{\text{Sobolev}}(\widetilde{M})$ is a chain homotopy equivalence. (The proof depends on elliptic regularity; the spectral decomposition of the Laplacian can be used to construct the inverse chain map $\Omega^\bullet_{\text{Sobolev}}(\widetilde{M}) \to L^2(\pi) \otimes_\pi \Omega^\bullet(M)$ that "smooths out" Sobolev-space-valued forms to smooth ones.) Putting everything together, we then have a chain homotopy equivalence $L^2(\pi) \otimes_\pi C^\bullet(M) \to \Omega^\bullet_{\text{Sobolev}}(\widetilde{M})$ in a suitable extended abelian category, and thus the cohomology (with values in this extended category) of the two complexes is the same. Since the L^2-Betti numbers and Novikov-Shubin invariants are obtained by looking at the projective and torsion parts of the extended cohomology, the theorem follows. □

EXAMPLE 1.2. Let $M = S^1$ and $\pi = \mathbb{Z}$, so $\mathbb{C}[\pi] = \mathbb{C}[T, T^{-1}]$. Then $L^2(\pi)$ is identified via Fourier series with $L^2(S^1)$, and the group von Neumann algebra with $L^\infty(S^1)$, which acts on $L^2(S^1)$ by pointwise multiplication. The trace τ is then identified with the linear functional $f \mapsto \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(e^{i\theta})\, d\theta$. Take the usual cell decomposition of S^1 with one 0-cell and one 1-cell. Then the chain complexes of the universal cover become:

$$C_\bullet(\widetilde{M})\colon \mathbb{C}[T,T^{-1}] \xrightarrow{T-1} \mathbb{C}[T,T^{-1}]$$

$$C_\bullet(\widetilde{M}, L^2(\pi))\colon L^2(S^1) \xrightarrow{e^{i\theta}-1} L^2(S^1).$$

So the L^2-Betti numbers are both zero, but the Novikov-Shubin invariants are non-trivial (in fact equal to 1), corresponding to the fact that if

$$\alpha = |e^{i\theta} - 1|\colon L^2(S^1) \to L^2(S^1),$$

then $F_\alpha(t) \approx \text{const} \cdot t$ for t small.

Generalizing one aspect of this is:

THEOREM 1.3 (Cheeger-Gromov [**13**]). *If X is an aspherical CW complex (i.e., $\pi_i(X) = 0$ for $i \neq 1$) with only finitely many cells of each dimension, and if $\pi = \pi_1(X)$ is amenable and infinite, then all L^2-Betti numbers of X vanish.*

There is a nice treatment of this theorem in [**21**], §4.3. The reader not familiar with amenable groups can consult [**34**] or [**67**] for the various forms of the definition, but one should know at least that finite groups and solvable groups are amenable and free groups on two or more generators (or any groups containing such a free group) are not. It is not known then (at least to the author) if, for an aspherical CW complex with infinite amenable fundamental group, one of the Novikov-Shubin capacities is always positive. However, this is true in many cases for which one can do direct calculations, such as nilmanifolds modeled on stratified nilpotent Lie groups [**85**].

As we will see, amenability is definitely relevant here; for non-amenable groups, the L^2-Betti numbers can be non-zero.

EXAMPLE 1.4. Let M be a compact Riemann surface of genus $g \geq 2$, $\widetilde{M}$ the hyperbolic plane, π a discrete torsion-free cocompact subgroup of $G = PSL(2, \mathbb{R})$. In this case, it's easiest to use the analytic picture, since $L^2(\widetilde{M}) \cong L^2(G/K)$, $K = SO(2)/\{\pm 1\}$. As a representation space of G, this is a direct integral of the principal series representations, and Δ corresponds to the Casimir operator, which has spectrum bounded away from 0. So $\beta_0 = 0$, and also $\beta_2 = 0$ by Poincaré duality.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Then the tangent bundle of G/K is the homogeneous vector bundle induced from the representation space $\mathfrak{g}/\mathfrak{k}$ of K, and the cotangent bundle is similarly induced from $(\mathfrak{g}/\mathfrak{k})^*$. Thus the L^2 sections of $\Omega^1(\widetilde{M})$ may be identified with the unitarily induced representation $\operatorname{Ind}_K^G (\mathfrak{g}/\mathfrak{k})^*$, which contains, in addition to the continuous spectrum, two discrete series representations with Casimir eigenvalue 0. Thus $\beta_1 \neq 0$. The Atiyah L^2-index theorem (to be discussed in the next chapter) implies $\beta_1 = 2(g-1)$. There are no additional Novikov-Shubin invariants, since these measure the non-zero spectrum of Δ close to 0, but the continuous spectrum of Δ is bounded away from 0.

One can also do the calculation of β_1 combinatorially. As is well known, one can construct M by making suitable identifications along the boundary of a $2g$-gon. (See Figure 1.) This construction gives a cell decomposition of M with one 0-cell, one 2-cell, and $2g$ 1-cells. Hence we can take $L^2(\pi) \otimes_\pi C^\bullet(M)$ to be a complex of free Hilbert π-modules of dimensions 1, $2g$, and 1. Since the extended L^2-cohomology vanishes in degrees 0 and 2, it then follows (by the Euler-Poincaré principle in the category $\mathcal{E}(\pi)$) that $\mathcal{H}^1(X, L^2(\pi))$ must be free, of dimension $2g-2$.

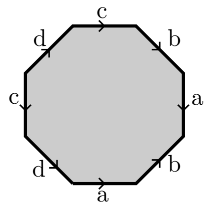

FIGURE 1. Identifications to form a closed surface (the case $g = 2$)

The vanishing of β_0 in Example 1.4 is not an accident. In fact, Brooks ([**6**] and [**7**]) proved the following:

THEOREM 1.5 (Brooks [**6**]). *Let M be a compact Riemannian manifold with fundamental group π. Then 0 lies in the spectrum of the Laplacian on the universal cover $\widetilde{M}$ of M if and only if π is amenable.*

Note that this implies (if π is non-amenable) that $\beta_0(M)$ and the Novikov-Shubin capacity in dimension 0 must be 0, and in fact, via Theorem 1.1 (the de Rham theorem for extended cohomology), that the extended cohomology group $\mathcal{H}^0(M, L^2(\pi))$ must vanish in $\mathcal{E}(\pi)$.

Generalizing one aspect of Example 1.4 is the following result, confirming a conjecture of Singer:

THEOREM 1.6 (Jost-Zuo [**42**] and Cao-Xavier [**12**]). *If M is a compact connected Kähler manifold of non-positive sectional curvature and complex dimension n, then all L^2-Betti numbers of M vanish, except perhaps for β_n.*

As we will see in Exercise 1.10 or via the Atiyah L^2-index theorem of the next chapter, this implies that

$$\beta_n = (-1)^n \chi(M),$$

where χ is the usual Euler characteristic.

Note. One should not be misled by these examples into thinking that the L^2-Betti numbers are always integers, or that most of them usually vanish. However, the *Atiyah Conjecture* asserts that they are always rational numbers. If true, this would have important implications, such as the *Zero Divisor Conjecture* that $\mathbb{Q}[G]$ has no zero divisors when G is a torsion-free group [**21**]. For more details on this and related matters, the reader is referred to the excellent surveys by Lück: [**58**] and [**59**].

1.4. Exercises.

EXERCISE 1.7. Fill in one of the details above by showing that if $\alpha = \alpha^* \colon A \to A$ and $\beta = \beta^* \colon B \to B$ represent the same torsion element $\mathcal{X}$ of the category $\mathcal{E}(\pi)$, and if $F_\alpha(t)$ and $G_\beta(t)$ are the associated spectral growth functions, defined by applying τ to the spectral projections of α (resp., β) for $[0, t)$, then there exist C, $\varepsilon > 0$ such that

$$G_\beta\left(\tfrac{t}{C}\right) \leq F_\alpha(t) \leq G_\beta(tC), \ t < \varepsilon.$$

EXERCISE 1.8. Use Example 1.2 to show that the 0-th Novikov-Shubin invariant of the n-torus $T^n = (S^1)^n$ is equal to n. (In fact this is true for all the other Novikov-Shubin invariants also, since the Laplacian on p-forms simply looks like a direct sum of $\binom{n}{p}$ copies of the Laplacian on functions.)

EXERCISE 1.9. Let X be a wedge of $n \geq 2$ circles, which has fundamental group $\pi = F_n$, a free group on n generators. This space has a cell decomposition with one 0-cell and n 1-cells. Compute the L^2-Betti numbers of X directly from the chain complex $L^2(\pi) \otimes_\pi C^\bullet(\widetilde{X})$.

EXERCISE 1.10. Let X be a finite CW complex with fundamental group π. Use the additivity of $\dim_\tau$ and the Euler-Poincaré principle to show that

$$\sum_{i=0}^{\dim X} (-1)^i \beta_i(X, L^2(\pi)) = \chi(X),$$

the ordinary Euler characteristic of X.

EXERCISE 1.11. Prove the combinatorial analogue of Brooks' Theorem (1.5) as follows. Let X be a finite connected CW complex with fundamental group π. Show that $\mathcal{H}^0(X, L^2(\pi)) = 0$ if and only if π is non-amenable, following this outline. Without loss of generality, one may assume X has exactly one 0-cell, and has 1-cells indexed by a finite generating set $g_1, \ldots, g_n$ for π. First show that $d^*d\colon L^2(\pi) \otimes_\pi C^0(\pi) \to L^2(\pi) \otimes_\pi C^0(\pi)$ can be identified with right multiplication by $\Delta = (g_1 - 1)^*(g_1 - 1) + \cdots + (g_n - 1)^*(g_n - 1)$ on $L^2(\pi)$. So the problem is to determine when 0 is in the spectrum of Δ. This happens if and only if for each $\varepsilon > 0$, there is a unit vector ξ in $L^2(\pi)$ such that $\|\rho(g_i)\xi - \xi\| < \varepsilon$ for $i = 1, \ldots, n$, where ρ denotes the right regular representation. But this means that the trivial representation is weakly contained in ρ, which is equivalent to amenability of π by Hulanicki's Theorem (see [**34**], Theorem 3.5.2, or [**67**], Theorem 4.21).

As pointed out to me by my colleague Jim Schafer, this combinatorial version of Brooks' Theorem is essentially equivalent to a classic theorem of Kesten on random walks on discrete groups [**52**].

EXERCISE 1.12. Deduce from the Cheeger-Gromov Theorem and Exercise 1.10 that if X is a finite aspherical CW complex with nontrivial amenable fundamental group, then $\chi(X) = 0$. See [**81**] and [**86**] for the history of results like this one.

2. Von Neumann Algebra Index Theorems: Atiyah's L^2-Index Theorem and Connes' Index Theorem for Foliations

2.1. Atiyah's L^2-Index Theorem. As we saw in the last chapter, it is not always so easy to compute all of the L^2-Betti numbers of a space directly from the definition, though sometimes we can compute *some* of them. It would be nice to have constraints from which we could then determine the others. Such a constraint, and more, is provided by the following index theorem. The context, as with many index theorems, is that of linear elliptic pseudodifferential operators. The reader who doesn't know what these are exactly can think of the differential operator $d + d^*$ on a Riemannian manifold, or of the operator $\overline{\partial} + \overline{\partial}^*$ on a Kähler manifold. These special cases are fairly typical of the sorts of operators to which the theorem can be applied.

THEOREM 2.1 (Atiyah [**1**]). *Suppose*

$$D\colon C^\infty(M, E_0) \to C^\infty(M, E_1)$$

is an elliptic pseudodifferential operator (we'll abbreviate this phrase hereafter as ψDO*), acting between sections of two vector bundles* E_0 *and* E_1 *over a closed manifold* M*, and* $\widetilde{M}$ *is a normal covering of* M *with covering group* π*. Let*

$$\widetilde{D}\colon C^\infty(\widetilde{M}, \widetilde{E}_0) \to C^\infty(\widetilde{M}, \widetilde{E}_1)$$

be the lift of D *to* $\widetilde{M}$*. Then* $\ker \widetilde{D}$ *and* $\ker \widetilde{D}^*$ *have finite* τ*-dimension, and*

$$\begin{aligned} &\operatorname{Ind} D (= \dim\ker D - \dim\ker D^*) \\ &= L^2\text{-}\operatorname{Ind} \widetilde{D} (= \dim_\tau \ker \widetilde{D} - \dim_\tau \ker \widetilde{D}^*). \end{aligned}$$

SKETCH OF PROOF. For simplicity take D to be a first-order differential operator, and consider the formally self-adjoint operator

$$P = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

acting on sections of $E = E_0 \oplus E_1$. Since D is elliptic, PDE theory shows that the solution of the "heat equation" for P, $H_t = \exp(-tP^2)$, is a *smoothing operator*—an integral operator with smooth kernel—for $t > 0$. And as $t \to \infty$, $H_t \to$ projection on $\ker D \oplus \ker D^*$, so that if

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{cases} 1 & \text{on } E_0 \\ -1 & \text{on } E_1, \end{cases}$$

then γ commutes with H_t and $\operatorname{Ind} D = \lim_{t\to\infty} \operatorname{Tr}(\gamma H_t)$.

Define similarly

$$\widetilde{P} = \begin{pmatrix} 0 & \widetilde{D}^* \\ \widetilde{D} & 0 \end{pmatrix}, \quad \widetilde{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{H} = e^{-t\widetilde{P}^2},$$

acting on sections of $\widetilde{E} = \widetilde{E}_0 \oplus \widetilde{E}_1$. Then $L^2\text{-}\operatorname{Ind}\widetilde{D} = \lim_{t\to\infty} \tau\left(\widetilde{\gamma}\widetilde{H}_t\right)$.

Here we extend τ to matrices over the group von Neumann algebra in the obvious way. So we just need to show that

$$\operatorname{Tr}(\gamma H_t) = \tau\left(\widetilde{\gamma}\widetilde{H}_t\right). \tag{1}$$

Now, in fact both sides of (1) are constant in t, since, for instance,

$$\begin{aligned} \frac{d}{dt}\operatorname{Tr}(\gamma H_t) &= \frac{d}{dt}\operatorname{Tr}\left(\gamma e^{-tP^2}\right) = \operatorname{Tr}\frac{d}{dt}\left(\gamma e^{-tP^2}\right) \\ &= \operatorname{Tr}\left(\frac{d}{dt}e^{-tD^*D} - \frac{d}{dt}e^{-tDD^*}\right) \\ &= \operatorname{Tr}\left(DD^*e^{-tDD^*} - D^*De^{-tD^*D}\right). \end{aligned}$$

But

$$\begin{aligned}
\operatorname{Tr}\left(DD^* e^{-tDD^*}\right) &= \operatorname{Tr}\left(\overbrace{e^{-tDD^*/2}D}\overbrace{D^* e^{-tDD^*/2}}\right) \\
&= \operatorname{Tr}\left(D^* e^{-tDD^*/2} e^{-tDD^*/2} D\right) \\
&= \operatorname{Tr}\left(D^* e^{-tDD^*} D\right) \\
&= \operatorname{Tr}\left(D^* \left(1 - tDD^* + \cdots\right) D\right) \\
&= \operatorname{Tr}\left(D^* D \left(1 - tD^* D + \cdots\right)\right) \\
&= \operatorname{Tr}\left(D^* D e^{-tD^* D}\right).
\end{aligned}$$

So it's enough to show that

$$\lim_{t \to 0^+} \left(\operatorname{Tr}\left(\gamma H_t\right) - \tau\left(\widetilde{\gamma}\widetilde{H}_t\right)\right) = 0.$$

But for small t, the solution of the heat equation is almost local. (See [**74**] for further explanation.) In other words, H_t and $\widetilde{H}_t$ are given by integration against smooth kernels almost concentrated on the diagonal, and the kernel $\widetilde{k}$ for $\widetilde{H}_t$ is practically the lift of the kernel k for H_t, since, locally, M and $\widetilde{M}$ look the same. But for a π-invariant operator $\widetilde{S}$ on $\widetilde{E}$, obtained by lifting the kernel function k for a smoothing operator on M to a kernel function to $\widetilde{k}$, one can check that

$$\begin{aligned}
\tau(\widetilde{S}) &= \int_F \widetilde{k}(\widetilde{x}, \widetilde{x})\, d\operatorname{vol}(\widetilde{x}) \\
&= \int_M k(x, x)\, d\operatorname{vol}(x) \\
&= \operatorname{Tr}(S),
\end{aligned}$$

F a fundamental domain for the action of π on $\widetilde{M}$. So that does it. □

For applications to L^2-Betti numbers, we can fix a Riemannian metric on M and take $E_0 = \bigoplus \Omega^{2i}$, $E_1 = \bigoplus \Omega^{2i+1}$, D the "Euler characteristic operator" $D = d + d^*$, so that $\operatorname{Ind} D = \chi(M)$ by the Hodge Theorem, while L^2-$\operatorname{Ind} \widetilde{D}$ is the alternating sum of the L^2-Betti numbers, $\sum(-1)^i \beta_i$. Thus we obtain an analytic proof of the equality $\sum(-1)^i \beta_i = \chi(M)$, for which a combinatorial proof was given in Exercise 1.10.

Another application comes from taking M closed, connected, and oriented, of dimension $4k$. Then harmonic forms in the middle degree $2k$ can be split into ± 1 eigenspaces for the Hodge $*$-operator, and so the middle Betti number b_{2k} splits as $b_{2k}^+ + b_{2k}^-$. The *signature* of M can be defined to be the difference $b_{2k}^+ - b_{2k}^-$. This can be identified with the signature of the intersection pairing

$$\langle x, y \rangle = \langle x \cup y, [M] \rangle$$

on $H^{2k}(M, \mathbb{R})$, since if we represent cohomology classes x and y by closed forms φ and ψ, then $\langle x, y \rangle = \int_M \varphi \wedge \psi$, while $\int_M \varphi \wedge *\psi$ is the L^2 inner product of φ and ψ, so that the intersection pairing is positive definite on the $+1$ eigenspace of $*$ and negative definite on the -1 eigenspace of $*$.

Now, as observed by Atiyah and Singer, the signature can also be computed as the index of the elliptic differential operator $D = d + d^*$ sending E_0 to E_1, where

$E_0 \oplus E_1$ is a splitting of the complex differential forms defined using the Hodge $*$-operator as well as the grading by degree. More precisely, E_0 and E_1 are the ± 1 eigenspaces of the involution τ sending a complex-valued p-form ω to $i^{p(p-1)+2k} * \omega$. This formula is concocted so that the contributions of p-forms and $(4k-p)$-forms will cancel out as long as $p \neq 2k$, and so that $\tau = *$ on forms of middle degree. If we apply Theorem 2.1 to this D, we see that $\beta_{2k}^+ - \beta_{2k}^- = b_{2k}^+ - b_{2k}^-$, with the splitting of β_{2k} into ± 1 eigenspaces of $*$ defined similarly. (Once again, the contributions from forms of other degrees cancel out.)

Example 2.2. Let M be a compact quotient of the unit ball $\widetilde{M}$ in $\mathbb{C}^2$. Then $\widetilde{M}$ can be identified with the homogeneous space G/K, where $G = SU(2,1)$ and K is its maximal compact subgroup $U(2)$. The signature of M must be nonzero by the "Hirzebruch proportionality principle,"[1] since G/K is the noncompact dual of the compact symmetric space $\mathbb{CP}^2$, which has signature 1. Hence the L^2-Betti number β_2 of M must be non-zero by the identity $\beta_2^+ - \beta_2^- = \operatorname{sign} M$. In this case, we have $\beta_0 = \beta_4 = 0$ by Brooks' Theorem (Theorem 1.5 and Exercise 1.11) and Poincaré duality, since the fundamental group of M is a lattice in G and is thus non-amenable. And in addition, $\beta_1 = \beta_3 = 0$ by Theorem 1.6, so as pointed out before, one has $\beta_2 = \chi(M)$. Together with the identities $\beta_2^+ - \beta_2^- = \operatorname{sign} M$ and $\beta_2^+ + \beta_2^- = \beta_2$, this makes it possible to compute $\beta_2^\pm$ exactly. (Note: for this example, vanishing of the L^2-cohomology in dimensions $\neq 2$ can also be proved using the representation theory of G, as in Example 1.4.)

2.2. Connes' Index Theorem for Foliations. Another important application to topology of finite von Neumann algebras is Connes' index theorem for tangentially elliptic operators on foliations with an invariant transverse measure.

2.3. Prerequisites. We begin by reviewing a few facts about foliations. A *foliation* $\mathcal{F}$ of a compact smooth manifold M^n is a partition of M into (not necessarily closed) connected submanifolds L^p called *leaves*, all of some fixed dimension p and codimension $q = n - p$.[2] The leaves are required to be the integral submanifolds of some integrable subbundle of TM, which we identify with $\mathcal{F}$ itself. *Locally*, M looks like $L^p \times \mathbb{R}^q$, but it can easily happen that every leaf is dense. See Figure 2. When x and y lie on the same leaf, "sliding along the leaf" along a path in the leaf from x to y gives a germ of a homeomorphism from a transversal to the foliation at x to a transversal to the foliation at y, which is called the *holonomy*. This holonomy only depends on the homotopy class of the path chosen from x to y, and so defines a certain connected cover of the leaf, called the *holonomy cover*, which is trivial if the leaf is simply connected.

For purposes of Connes' index theorem we will need to do a sort of integration over the "space of leaves $M/\mathcal{F}$," even though this space may not even be T_0, let alone Hausdorff. So we will assume $(M, \mathcal{F})$ has an *invariant transverse measure* μ.

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[1]This principle asserts that the characteristic numbers of M must be proportional to those of the compact dual symmetric space $\mathbb{CP}^2$, and thus the signature of M is nonzero since the signature of $\mathbb{CP}^2$ is nonzero. The logic behind this is that characteristic numbers are computed from integrals of universal polynomials in the curvature forms, and these forms are determined by the structure of the Lie algebra of G, hence agree for the compact and non-compact symmetric spaces except for a sign.

[2]Admittedly, there is a problem with the notation here; it seems to imply that the leaves are all diffeomorphic to one another, but this is not necessarily the case.

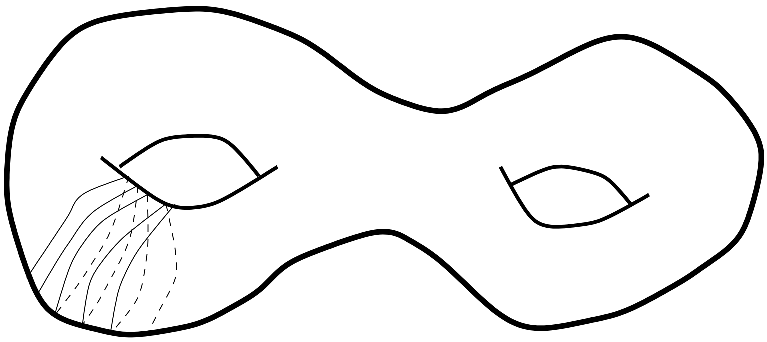

FIGURE 2. Schematic picture of a piece of a typical foliation

This is a map $\mu\colon (T \looparrowright M) \mapsto \mu(T)$ from (immersed) q-dimensional submanifolds of M with compact closure, transverse to the leaves of $\mathcal{F}$, to the reals. It is required to satisfy countable additivity as well as the invariance property, that μ assigns the same value to every pair of transversals $T_1, T_2 \hookrightarrow M$ which are obtained from one another by a holonomy transformation. When the foliation $\mathcal{F}$ is a fibration $L^p \to M^n \to B^q$, where the base B can be identified with the space of leaves, then an invariant transverse measure μ is simply a measure on B. If the leaves are consistently oriented, then given a p-form on M, we can integrate it over the leaves, getting a function on the base B, and then integrate against μ. More generally, without any conditions on $\mathcal{F}$ except that it be orientable, an invariant transverse measure μ defines a closed *Ruelle-Sullivan current* [**84**] C_μ on M of dimension p. To review, a p-current is to a differential form of degree p what a distribution is to a function; it is a linear functional on (compactly supported) p-forms. The current C_μ is defined as follows: on a small open subset of M diffeomorphic to $D^p \times D^q$ (with $\mathcal{F}$ tangent to the subsets $D^p \times \{\text{pt}\}$), given a p-form ω supported in this set, one has

$$\langle C_\mu, \omega\rangle = \int \Big(\int_{D^p \times \{x\}} \omega \Big)\, d\mu(x).$$

There is a differential ∂ on currents dual to the exterior differential d on forms, and since C_μ is basically just a smeared out version of integration along the leaves, one immediately sees that $\partial C_\mu = 0$, so that C_μ defines a de Rham homology class $[C_\mu]$ in $H_p(M, \mathbb{R})$.

From the data M, $\mathcal{F}$, and μ, one can construct (see [**17**], [**16**], and [**18**]) a finite von Neumann algebra $A = W^*(M, \mathcal{F})$ with a trace τ coming from μ. Let's quickly review how this is constructed. When the holonomy covers of the leaves of $\mathcal{F}$ are trivial—for instance, when all leaves are simply connected—consider the graph G of the equivalence relation $\sim$ on M of "being on the same leaf." In other words, $G = \{(x, y) \in M \times M \mid x \text{ and } y \text{ on the same leaf}\}$. Note that G can be identified with a (possibly noncompact) manifold of dimension $n + p = q + 2p$. The algebra A is then the completion of the convolution algebra of functions (or to be more canonical, half-densities) on G of compact support, for the action of this algebra on a suitable Hilbert space defined by μ. The construction in the general case is similar, except that we replace the graph of $\sim$ by the *holonomy groupoid* G, consisting of triples $(x, y, [\gamma])$ with x and y on the same leaf and $[\gamma]$ a class of paths from x to

y all with the same holonomy.[3] (In fact usually this nicety doesn't matter much in the von Neumann algebra context since leaves for which the holonomy cover is trivial are "generic"—see [**11**], Theorem 2.3.12.)

Now suppose there is a differential operator D on M which only involves differentiation in directions tangent to the leaves and is elliptic when restricted to any leaf. (Examples: the Euler characteristic operator or the Dirac operator "along the leaves.") Such an operator is called *tangentially elliptic*. Since the leaves are usually not compact, we can't compute an index for the restriction of D to one leaf. But since M, the union of the leaves, is compact, it turns out one can make sense of a numerical index $\operatorname{Ind}_\tau D$ for D. In the special case where $\mathcal{F}$ has closed leaves, the foliation is a fibration $L^p \to M \xrightarrow{\text{proj}} X^q$, and μ is a probability measure on X, this reduces to $\operatorname{Ind}_\tau D = \int_X \operatorname{Ind}(D|_{L_x})\, d\mu(x)$, where $L_x = \operatorname{proj}^{-1}(x)$. In general, $\operatorname{Ind}_\tau D$ is roughly the "average with respect to μ" of the L^2-index of $D|_{L_x}$, as x runs over the "space of leaves." Here we give each leaf the Riemannian structure defined by a choice of metric on the bundle $\mathcal{F}$.

Example 2.3. Let M_1 and M_2 be compact connected manifolds, and let π be the fundamental group of M_2. If π acts on $M_1 \times \widetilde{M_2}$ with trivial action on the first factor and the usual action on the second factor, then the quotient is $M_1 \times (\widetilde{M_2}/\pi) = M_1 \times M_2$. But suppose we take *any* action of π on M_1 and then take the diagonal action of π on $M_1 \times \widetilde{M_2}$. Then $M = (M_1 \times \widetilde{M_2})/\pi$ is compact, and projection to the second factor gives a fibration onto M_2 with fiber M_1. But M is also foliated by the images of $\{x\} \times \widetilde{M_2}$, usually non-compact. A measure μ on M_1 invariant under the action of π is an invariant transverse measure for this foliation $\mathcal{F}$. If D is the Euler characteristic operator along the leaves and all the leaves are $\cong \widetilde{M_2}$, then $\operatorname{Ind}_\tau D$ just becomes the average L^2-Euler characteristic of $\widetilde{M}$, the alternating sum of the L^2-Betti numbers, and Connes' Theorem will reduce to Atiyah's.

2.4. Connes' Theorem.

Theorem 2.4 (Connes [**14**], [**17**]). *Let $(M, \mathcal{F})$ be a compact foliated manifold with an invariant transverse measure μ, and let $W^*(M, \mathcal{F})$ be the associated von Neumann algebra with trace τ coming from μ. Let*

$$D\colon C^\infty(M, E_0) \to C^\infty(M, E_1)$$

be elliptic along the leaves. Then the L^2 kernels of

$$P = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

on the various leaves assemble to a (graded) Hilbert $W^*(M, \mathcal{F})$-module $K_0 \oplus K_1$, and

$$\operatorname{Ind}_\tau D = \dim_\tau K_0 - \dim_\tau K_1 = \int \operatorname{Ind}_{\text{top}} \sigma(D)\, d\mu,$$

where $\sigma(D)$ denotes the symbol of D and the "topological index" $\operatorname{Ind}_{\text{top}}$ is computed from the characteristic classes of $\sigma(D)$ just as in the usual Atiyah-Singer index theorem.

[3] One has to be a little careful what one means by this when G is non-Hausdorff, but the general idea is still the same even in this case.

We omit the proof, which is rather complicated if one puts in all the details (but see [**14**], [**17**], and [**64**]). However, the basic outline of the proof is similar to that for Theorem 2.1, except that one must replace the group von Neumann algebra by the von Neumann algebra of the measured foliation.

2.5. An Application to Uniformization. If we specialize the Connes index theorem to the Euler characteristic operator along the leaves for foliations with 2-dimensional leaves, it reduces to:

THEOREM 2.5 (Connes). *Let $(M, \mathcal{F})$ be a compact foliated manifold with 2-dimensional leaves and $\mathcal{F}$ oriented. Then for every invariant transverse measure μ, the μ-average of the L^2-Euler characteristic of the leaves is equal to $\langle e(\mathcal{F}), [C_\mu]\rangle$, where $e(\mathcal{F}) \in H^2(M, \mathbb{Z})$ is the Euler class of the oriented 2-plane bundle associated to $\mathcal{F}$, and $[C_\mu] \in H_2(M, \mathbb{R})$ is the Ruelle-Sullivan class attached to μ.*

The result also generalizes to compact *laminations* with 2-dimensional leaves. (That means we replace M by any compact Hausdorff space X locally of the form $\mathbb{R}^2 \times T$, where T is allowed to vary.) The only difference in this case is that we have to use *tangential* de Rham theory. This variant of Connes' Theorem is explained in [**64**].

COROLLARY 2.6. *Suppose $(X, \mathcal{F})$ is a compact laminated space with 2-dimensional oriented leaves and a smooth Riemannian metric g. Let ω be the curvature 2-form of g. If there is an invariant transverse measure μ with $\langle [\omega], [C_\mu]\rangle > 0$, then $\mathcal{F}$ has a set of closed leaves of positive μ-measure. If there is an invariant transverse measure μ with $\langle [\omega], [C_\mu]\rangle < 0$, then $\mathcal{F}$ has a set of (conformally) hyperbolic leaves of positive μ-measure. If all the leaves are (conformally) parabolic, then $\langle [\omega], [C_\mu]\rangle = 0$ for every invariant transverse measure.*

PROOF. By Chern-Weil theory, the de Rham class of $\frac{\omega}{2\pi}$ represents the Euler class of $\mathcal{F}$. So by Theorem 2.5, $\langle [\omega], [C_\mu]\rangle$ is the μ-average of the L^2-Euler characteristic of the leaves. The only oriented 2-manifold with positive L^2-Euler characteristic is S^2. Every hyperbolic Riemann surface has negative L^2-Euler characteristic. And every parabolic Riemann surface (one covered by $\mathbb{C}$ with the flat metric) has vanishing L^2-Euler characteristic. $\square$

This has been used in:

THEOREM 2.7 (Ghys [**33**]). *Under the hypotheses of Corollary 2.6, if every leaf is parabolic, then $(X, \mathcal{F}, g)$ is* approximately uniformizable, *i.e., there are real-valued functions u_n (smooth on the leaves) with the curvature form of $e^{u_n} g$ tending uniformly to 0.*

Note incidentally that the reason for using the curvature *form* here, as opposed to the Gaussian curvature, is that the form, unlike the Gaussian curvature, is invariant under rescaling of the metric by a constant factor.

SKETCH OF PROOF. The proof depends on two facts about 2-dimensional Riemannian geometry. First of all, if g is a metric on a surface, and if K is its curvature, then changing g to the conformal metric $e^u g$ changes the curvature form $K\, d\mathrm{vol}_g$ to $K' d\mathrm{vol}_{e^u g} = (K - \Delta u)\, d\mathrm{vol}_g$, where Δ is the Laplacian (normalized to be a *negative* operator). So if K is the curvature function for the lamination and $\Delta^{\mathcal{F}}$ is the leafwise Laplacian, it's enough to show that K is in the uniform closure of functions of the form $\Delta^{\mathcal{F}}(u)$. (For then if $\Delta^{\mathcal{F}}(u_n) \to K$, the curvature forms of $e^{u_n} g$ tend to 0.)

The second fact we need is that there exist *harmonic measures* ν on X, that is, measures with the property that ν annihilates all functions of the form $\Delta^{\mathcal{F}}(u)$, and that a function lies in the closure of functions of the form $\Delta^{\mathcal{F}}(u)$ if and only if it is annihilated by the harmonic measures. Indeed, by the Hahn-Banach Theorem, the uniform closure of the functions of the form $\Delta^{\mathcal{F}}(u)$ is exactly the set of functions annihilated by measures ν with $\int_X \Delta^{\mathcal{F}}(u)\, d\nu = 0$ for all leafwise smooth functions u. But on a subset of X of the form $U \times T$, where U is an open subset of a leaf, such measures consist exactly of integrals (with respect to some measure on T) of measures of the form $h(x)\, d\mathrm{vol}(x)$ on each leaf, with h a harmonic function. Thus, in the case where all the leaves are parabolic, it turns out (since there are no nonconstant positive harmonic functions on $\mathbb{C}$) that harmonic measures are just obtained by integrating the leafwise area measure with respect to an invariant transverse measure. Since $\langle K\, d\mathrm{vol}_g, [C_\mu]\rangle = 0$ for every invariant transverse measure by Corollary 2.6, the result follows. $\square$

Another known fact is:

THEOREM 2.8 (Candel [**10**]). *Under the hypotheses of Corollary 2.6, if every leaf is hyperbolic, then* $(X, \mathcal{F}, g)$ is uniformizable, *i.e., there is a real-valued function* u (*smooth along the leaves*) *with* $e^u g$ *hyperbolic on each leaf.*

2.6. Exercises.

EXERCISE 2.9. Let M be a compact Kähler manifold of complex dimension n. Then each Betti number $b_r(M)$ splits as $b_r(M) = \sum_{p+q=r} h^{p,q}(M)$, where the Hodge number $h^{p,q}(M)$ is the dimension of the part of the de Rham cohomology in dimension r coming from forms of type (p,q) (*i.e.*, locally of the form $f\, dz_1 \wedge \cdots \wedge dz_p \wedge d\overline{z_{p+1}} \wedge \cdots \wedge d\overline{z_{p+q}}$). And $\sum_q (-1)^q h^{0,q}(M)$, the index of the operator $\overline{\partial} + \overline{\partial}^*$ on forms of type $(0,*)$, graded by parity of the degree, turns out to be given by the *Todd genus* $\mathrm{Td}(M)$ [**41**]. For example, if M is a compact Riemann surface (Kähler manifold of complex dimension 1) of genus g, then $h^{1,0}(M) = h^{0,1}(M) = g$ and $\mathrm{Td}(M) = 1 - g$. Apply the Atiyah L^2-index theorem and see what it says about the L^2 Hodge numbers (associated to the universal cover). For example, compute the L^2 Hodge numbers when M is a compact Riemann surface of genus $g \geq 1$ (see Example 1.4).

EXERCISE 2.10. Part of the idea for this problem comes from [**2**] and [**3**], though we have been able to simplify things considerably by restricting to the easiest special case. Suppose $G = SL(2,\mathbb{R})$ and $K = SO(2)$. Then attached to each character of K, which we can think of as being given by an integer parameter n by $e^{i\theta} \mapsto e^{in\theta}$, $e^{i\theta} \in SO(2) \cong S^1$, is a homogeneous holomorphic line bundle $\widetilde{L}_n$ over $\widetilde{M} = G/K$. Let π be a discrete cocompact subgroup of G, so that $G/K \twoheadrightarrow \pi\backslash G/K$ is the universal cover of a compact Riemann surface M of genus $g > 1$. Note that $\widetilde{L}_n$ descends in a natural way to a holomorphic line bundle L_n over M. Apply the L^2-index theorem, together with the classical Riemann-Roch Theorem for L_n, to compute the L^2-index of the $\overline{\partial}$ operator on the line bundle $\widetilde{L}_n$.

Then, combine this result with a vanishing theorem to show that $\widetilde{L}_n$ has L^2 holomorphic sections (with respect to the G-invariant measure on G/K) if and only if $n \geq 2$. Here is a sketch of the proof of the vanishing theorem. Let $\mathfrak{g}$ and

$\mathfrak{k}$ be the complexified Lie algebras of G and of K, respectively, and suppose the Hilbert space $\mathcal{H}_n$ of L^2 holomorphic sections of L_n is non-zero. We have a splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \oplus \overline{\mathfrak{p}}$, with $\mathfrak{p}$ corresponding to the holomorphic tangent space of G/K, and $\overline{\mathfrak{p}}$ corresponding to the antiholomorphic tangent space. Also, $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{p}$, $\overline{\mathfrak{p}}$ are its root spaces. By definition, we have

$$\mathcal{H}_n = \{f \in L^2(G) : f(gk) = k^{-n} f(g),\ g \in G,\ k \in K \cong S^1,\ d\rho(\overline{\mathfrak{p}})f = 0\}.$$

Here ρ is the right regular representation, and G acts on $\mathcal{H}_n$ by the left regular representation λ. Then $\mathcal{H}_n$ carries a unitary representation of G and (because of the Cauchy integral formula) must be a "reproducing kernel" Hilbert space, that is, there must be a distinguished vector $\xi \in \mathcal{H}_n$ such that $\langle s, \xi \rangle = s(e)$ for all $s \in \mathcal{H}_n$. (Here e denotes the identity element of G, which corresponds to $0 \in \mathbb{C}$ in the unit disk model of G/K.) Since $\mathcal{H}_n \neq 0$, ξ can't vanish. It turns out that $\xi \in \mathcal{H}_n$ is a "lowest weight vector," an eigenvector for $\mathfrak{k}$ (corresponding to the character $e^{i\theta} \mapsto e^{in\theta}$ of K) that is killed by $\overline{\mathfrak{p}}$. This determines the action of $\mathfrak{g}$, hence of G, on ξ, and the vanishing theorem is deduced from the requirement that ξ lie in L^2. (See for example [**53**] for more details.)

Exercise 2.11. Let M be a compact manifold with $H^2(M, \mathbb{R}) = 0$, and suppose M admits a foliation with 2-dimensional leaves and an invariant transverse measure. Deduce from Corollary 2.6 that the "average L^2 Euler characteristic" of the leaves must vanish, and in particular, that the leaves cannot all be hyperbolic (uniformized by the unit disk). (Compare the combination of Theorems 12.3.1 and 12.5.1 in [**11**].)

Exercise 2.12. **(Connes [17], §4)** Let Λ_1 and Λ_2 be lattices in $\mathbb{C}$ (that is, discrete subgroups, each of rank 2) and assume that $\Lambda_1 \cap \Lambda_2 = \emptyset$. Consider the 4-torus $M = (\mathbb{C}/\Lambda_1) \times (\mathbb{C}/\Lambda_2)$ and let p_j, $j = 1, 2$, be the projections of M onto $\mathbb{C}/\Lambda_j$. Fix points $z_1, z_2 \in \mathbb{C}$ and let E_1 and E_2 be the holomorphic line bundles on $\mathbb{C}/\Lambda_j$ attached to the divisors $-[z_1]$ and $[z_2]$, respectively. Then let $E = p_1^*(E_1) \otimes p_2^*(E_2)$. Consider the foliation $\mathcal{F}$ of M obtained by pushing down the foliation of the universal cover $\mathbb{C}^2$ by the complex planes $\{(z, w + z) : z \in \mathbb{C}\}$, $w \in \mathbb{C}$. The leaves of $\mathcal{F}$ may be identified with copies of $\mathbb{C}$. Since this foliation is linear, it has a transverse measure given by Haar measure on a 2-torus transverse to the leaves of $\mathcal{F}$. Let D be the $\overline{\partial}$ operator along the leaves, acting on E. Then on a leaf $L = \operatorname{im}\{(z, w + z) : z \in \mathbb{C}\}$, a holomorphic section of E can be identified with a meromorphic function on $\mathbb{C}$ with all its poles simple and contained in $z + \Lambda_1$, for some z, and with zeros at points of Λ_2. Apply the foliation index theorem to deduce an existence result about such meromorphic functions in L^2.

3. Group C^*-Algebras, the Mishchenko-Fomenko Index Theorem, and Applications to Topology

3.1. The Mishchenko-Fomenko Index. The last two chapters have been about applications of von Neumann algebras to topology. In this chapter, we start to talk about applications of C^*-algebras. First we recall that a (complex) commutative C^*-algebra is always of the form $C_0(Y)$, where Y is a locally compact (Hausdorff) space, so that the study of (complex) commutative C^*-algebras is equivalent to the study of locally compact spaces. Real commutative C^*-algebras are only a bit more complicated; they correspond to locally compact spaces (associated to the complexification) together with an involutive homeomorphism (associated to the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$). These were called *real spaces* by Atiyah. We also recall that by Swan's Theorem (see for example [**47**], Theorem I.6.18), the sections of a vector bundle over a compact Hausdorff space X form a finitely generated projective module over $C(X)$, and conversely.

Definition 3.1. Let A be a C^*-algebra (over $\mathbb{R}$ or $\mathbb{C}$) with unit, and let X be a compact space. An *A-vector bundle* over X will mean a locally trivial bundle over X whose fibers are finitely generated projective (right) A-modules, with A-linear transition functions.

Example 3.2. If $A = \mathbb{R}$ or $\mathbb{C}$, an A-vector bundle is just a usual vector bundle. If $A = C(Y)$, an A-vector bundle over X is equivalent to an ordinary vector bundle over $X \times Y$. This is proved by the same method as Swan's Theorem, to which the statement reduces if X is just a point.

Definition 3.3. Let A be a C^*-algebra and let E_0, E_1 be A-vector bundles over a compact manifold M. An *A-elliptic operator*

$$D\colon C^\infty(M, E_0) \to C^\infty(M, E_1)$$

will mean an elliptic A-linear ψDO from sections of E_0 to sections of E_1. Such an operator extends to a bounded A-linear map on suitable Sobolev spaces (Hilbert A-modules) $\mathcal{H}_0$ and $\mathcal{H}_1$. One can show [**62**] that this map is an A-Fredholm operator, *i.e.*, one can find a decomposition

$$\mathcal{H}_0 = \mathcal{H}_0' \oplus \mathcal{H}_0'', \quad \mathcal{H}_1 = \mathcal{H}_1' \oplus \mathcal{H}_1'',$$

with $\mathcal{H}_0''$ and $\mathcal{H}_1''$ finitely generated projective, and

$$D\colon \mathcal{H}_0' \xrightarrow{\cong} \mathcal{H}_1', \quad D\colon \mathcal{H}_0'' \to \mathcal{H}_1''.$$

This means that "up to A-compact perturbation" the kernel and cokernel of D are finitely generated projective A-modules. The *Mishchenko-Fomenko index* of D (see [**62**]) is

$$\operatorname{Ind} D = [\mathcal{H}_0''] - [\mathcal{H}_1''],$$

computed in the group of formal differences of isomorphism classes of such modules, $K_0(A)$. (See Exercise 3.13 below.)

3.2. Flat C^*-Algebra Bundles and the Assembly Map. If X is a compact space and A is a C^*-algebra with unit, the group of formal differences of isomorphism classes of A-vector bundles over X is denoted $K^0(X; A)$. The following is analogous to Swan's Theorem.

Proposition 3.4. *If X is a compact space and A is a C^*-algebra with unit, then $K^0(X; A)$ is naturally isomorphic to $K_0(C(X) \otimes A)$.*

Sketch of proof. Suppose E is an A-vector bundle over X. Then the space $\Gamma(X, E)$ of continuous sections of E comes with commuting actions of $C(X)$ and of A. As such, it is a module for the algebraic tensor product; we will show from the local triviality that it is in fact a module for the C^*-tensor product. Now just as in the case of ordinary vector bundles, one shows that E is complemented, *i.e.*, that there is another A-vector bundle F such that $E \oplus F$ is a trivial bundle with fibers that are finitely generated free A-modules. Thus

$$\begin{aligned}\Gamma(X, E) \oplus \Gamma(X, F) &\cong \Gamma(X, E \oplus F) \\ &\cong \Gamma(X, X \times A^n) \cong C(X, A)^n \cong \big(C(X) \otimes A\big)^n\end{aligned}$$

for some n. Hence $\Gamma(X, E)$ is a finitely generated projective $(C(X) \otimes A)$-module. Now it's clear that the Grothendieck groups of A-vector bundles and of finitely generated projective $(C(X) \otimes A)$-modules coincide. □

DEFINITION 3.5. Let X be a compact space, $\widetilde{X} \to X$ a normal covering with covering group π. Let $C_r^*(\pi)$ be the reduced group C^*-algebra of π (the completion of the group ring in the operator norm for its action on $L^2(\pi)$). The *universal* $C_r^*(\pi)$*-bundle* over X is

$$\mathcal{V}_X = \widetilde{X} \times_\pi C_r^*(\pi) \to X.$$

This is clearly a $C_r^*(\pi)$-vector bundle over X. As such, by Proposition 3.4, it has a class $[\mathcal{V}_X] \in K^0(X; C_r^*(\pi))$, which is pulled back (via the classifying map $X \to B\pi$) from the class of

$$\mathcal{V} = E\pi \times_\pi C_r^*(\pi) \to B\pi$$

in $K^0(B\pi; C_r^*(\pi))$. Here $B\pi$ is the *classifying space* of π, a space (with the homotopy type of a CW complex) having π as its fundamental group, and with contractible universal cover $E\pi$. Such a space always exists and is unique up to homotopy equivalence. Furthermore, by obstruction theory, every normal covering with covering group π is pulled back from the "universal" π-covering $E\pi \to B\pi$. "Slant product" with $[\mathcal{V}]$ (a special case of the Kasparov product) defines the *assembly map*

$$\mathcal{A} \colon K_*(B\pi) \to K_*(C_r^*(\pi)).$$

(There is a slight abuse of notation here. $B\pi$ may not be compact, but it can always be approximated by finite CW complexes. So if there is no finite model for $B\pi$, $K_*(B\pi)$ is to be interpreted as the direct limit of $K_*(X)$ as X runs over the finite subcomplexes of $B\pi$. This direct limit is independent of the choice of a model for $B\pi$.)

Note that since the universal $C_r^*(\pi)$-bundle over X or $B\pi$ is canonically trivialized over the universal cover, it comes with a flat connection, that is, a notion of what it means for a section to be locally flat. (A locally flat section near x is one that, in a small evenly covered neighborhood—a neighborhood whose inverse image in the universal cover is of the form $U \times \pi$, lifts to a constant section $U \to C_r^*(\pi)$.)

3.3. Kasparov Theory and the Index Theorem. The formalism of *Kasparov theory* (see [**5**] or [**40**]) attaches, to an elliptic operator D on a manifold M, a K-homology class $[D] \in K_*(M)$. If M is compact, the *collapse map* $c \colon M \to \mathrm{pt}$ is proper and $\operatorname{Ind} D = c_*([D]) \in K_*(\mathrm{pt})$.

Now if E is an A-vector bundle over M and D is an elliptic operator over M, we can form "D with coefficients in E," an A-elliptic operator. The Mishchenko-Fomenko index of this operator is computed by pairing

$$[D] \in K_*(M) \quad \text{with} \quad [E] \in K^0(M; A).$$

In particular, if $\widetilde{M} \to M$ is a normal covering of M with covering group π, then we can form D with coefficients in $\mathcal{V}_X$, and its index is $\mathcal{A} \circ u_*([D])$, where $u \colon M \to B\pi$ is the classifying map for the covering.

CONJECTURE 3.6 (Novikov Conjecture). *The assembly map*

$$\mathcal{A} \colon K_*(B\pi) \to K_*(C_r^*(\pi))$$

is rationally injective for all groups π, and is injective for all torsion-free groups π.

This is quite different from the original form of Novikov's conjecture, though it implies it. Therefore Conjecture 3.6 is often called the *Strong* Novikov Conjecture. We will see the exact connection with the original form of the conjecture shortly. Stronger than Conjecture 3.6 is the *Baum-Connes Conjecture*, which gives a conjectural calculation of $K_*(C_r^*(\pi))$.[4] When π is torsion-free, the Baum-Connes Conjecture amounts to the statement that $\mathcal{A}$ is an isomorphism. There are no known counterexamples to Conjecture 3.6, or for that matter to the Baum-Connes Conjecture for discrete groups (though it is known to fail for some groupoids). Conjecture 3.6 is known for discrete subgroups of Lie groups ([**49**], [**48**]), amenable groups [**39**], hyperbolic groups [**50**], and many other classes of groups.

3.4. Applications.

(a) The L^2-Index Theorem and Integrality of the Trace. The connection with Atiyah's Theorem from Chapter 2 is as follows. Suppose D is an elliptic operator on a compact manifold M, and $\widetilde{M} \to M$ is a normal covering of M with covering group π. The group C^*-algebra $C_r^*(\pi)$ embeds in the group von Neumann algebra, and the trace τ then induces a homomorphism $\tau_* \colon K_0(C_r^*(\pi)) \to \mathbb{R}$. The image under τ_* of the index of D with coefficients in $C_r^*(\pi)$ can be identified with the L^2-index of $\widetilde{D}$, the lift of D to $\widetilde{M}$. Atiyah's Theorem thus becomes the assertion that the following diagram commutes:

$$\begin{tikzcd}
K_0(M) \arrow[r, "u_*"] \arrow[d, "c_*"] & K_0(B\pi) \arrow[d, "\mathcal{A}"] \\
K_0(\mathrm{pt}) = \mathbb{Z} \arrow[dr, hook] & K_0(C_r^*(\pi)) \arrow[d, "\tau_*"] \\
& \mathbb{R}.
\end{tikzcd}$$

This has the consequence, not obvious on its face, that τ_* takes only integral values on the image of the assembly map.[5] Thus if the assembly map is surjective, as when π is torsion-free and the Baum-Connes Conjecture holds for π, then $\tau_* \colon K_0(C_r^*(\pi)) \to \mathbb{R}$ takes only integral values. This in particular implies the *Kaplansky-Kadison Conjecture*, that $C_r^*(\pi)$ has no idempotents other than 0 or 1 [**94**]. The reason is that if $e = e^2 \in C_r^*(\pi)$, and if $0 \lneqq e \lneqq 1$, then e defines a class in $K_0(C_r^*(\pi))$ and $0 < \tau(e) < 1 = \tau(1)$, contradicting our integrality statement.

(b) Original Version of the Novikov Conjecture. Consider the signature operator D on a closed oriented manifold M^{4k}. This is constructed (see page 10) so that $\operatorname{Ind} D$ is the *signature* of M, *i.e.*, the signature of the form

$$\langle x, y \rangle = \langle x \cup y, [M] \rangle$$

[4]The precise statement is that an assembly map $\mathcal{A}_{\mathrm{BC}} \colon K_*^{\pi}(\mathcal{E}\pi) \to K_*(C_r^*(\pi))$ is an isomorphism. Here $\mathcal{E}\pi$ is a contractible CW complex on which G acts properly (though not necessarily freely), and K_*^{π} is equivariant K-homology. When G is torsion-free, $\mathcal{E}\pi = E\pi$, $K_*^{\pi}(\mathcal{E}\pi) = K_*^{\pi}(E\pi) = K_*(B\pi)$, and $\mathcal{A}_{\mathrm{BC}} = \mathcal{A}$. In general, one has a π-equivariant map $E\pi \to \mathcal{E}\pi$, and $\mathcal{A}$ factors through $\mathcal{A}_{\mathrm{BC}}$.

[5]Here we are using the fact that every element of $K_0(B\pi)$ lies in the image of K_0 of *some* manifold M with a map $M \to B\pi$. This can be deduced from "Conner-Floyd type" theorems about the relationship between K-homology and bordism. Of course in the case where $B\pi$ can be chosen to be a compact manifold, this fact is obvious.

on the middle cohomology $H^{2k}(M, \mathbb{R})$. The signature is obviously an oriented homotopy invariant, since it only depends on the structure of the cohomology ring (determined by the homotopy type) and on the choice of fundamental class $[M]$ (determined by the orientation). Hirzebruch's formula says $\operatorname{sign} M = \langle \mathcal{L}(M), [M] \rangle$, where $\mathcal{L}(M)$ is a power series in the rational Pontryagin classes, the Poincaré dual of $\mathrm{Ch}[D]$. Here $\mathrm{Ch}\colon K_0(M) \to H_*(M, \mathbb{Q})$ is the Chern character, a natural transformation of homology theories (and in fact a rational isomorphism). The unusual feature of Hirzebruch's formula is that the rational Pontryagin classes, and thus the $\mathcal{L}$-class, are *not* homotopy invariants of M; only the term in $\mathcal{L}(M)$ of degree equal to the dimension of M is a homotopy invariant. For example, it is known from surgery theory how to construct "fake" complex projective spaces homotopy equivalent to $\mathbb{CP}^m$, $m \geq 3$, with wildly varying Pontryagin classes.

If $u\colon M \to B\pi$ for some discrete group π (such as the fundamental group of M), $u_*(\mathrm{Ch}[D]) \in H_*(B\pi, \mathbb{Q})$ is called a *higher signature* of M, and Novikov conjectured that, like the ordinary signature (the case $\pi = 1$), it is an oriented homotopy invariant. The conjecture follows from injectivity of the assembly map, since Kasparov ([**49**], §9, Theorem 2) and Mishchenko ([**63**], [**61**]) showed that $\mathcal{A} \circ u_*([D])$ is an oriented homotopy invariant. Another proof of the homotopy invariance of $\mathcal{A} \circ u_*([D])$ may be found in [**46**]. For much more on the background and history of the Novikov Conjecture, see [**30**].

(c) Positive Scalar Curvature. An oriented Riemannian manifold M^n has a natural principal $SO(n)$-bundle attached to it, the (oriented) *orthonormal frame bundle*, $P \to M$. The fiber of P over any point $x \in M$ is, by definition, the set of oriented orthonormal bases for the tangent space T_xM, and $SO(n)$ acts simply transitively on this set. Now $SO(n)$ has a double cover $\mathrm{Spin}(n)$ (which if $n \geq 3$ is also the universal cover), and a lifting of $P \to M$ to a principal $\mathrm{Spin}(n)$-bundle $\widehat{P} \to M$ is called a *spin structure* on M. When M is connected, it is fairly easy to show that such a structure exists if and only if the second Stiefel-Whitney class $w_2(M)$ vanishes in $H^2(M, \mathbb{Z}/2)$, and that $H^1(M, \mathbb{Z}/2)$ acts simply transitively on the set of spin structures (compatible with a fixed choice of orientation); see [**54**], Chapter II, §2. If M^n is a closed spin manifold, then M carries a special first-order elliptic operator, the [$\mathrm{Cliff}_n(\mathbb{R})$-linear] *Dirac operator* D ([**54**], Chapter II, §7), with a class $[D] \in KO_n(M)$. The operator D depends on a choice of Riemannian metric, though its K-homology class is independent of the choice. Lichnerowicz [**55**] proved that

$$D^2 = \nabla^* \nabla + \frac{\kappa}{4}, \tag{2}$$

where κ is the scalar curvature of the metric. Thus if $\kappa > 0$, the spectrum of D is bounded away from 0 and $\operatorname{Ind} D = 0$ in

$$KO_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n \equiv 0 \bmod 4, \\ \mathbb{Z}/2, & n \equiv 1 \text{ or } 2 \bmod 8, \\ 0, & \text{otherwise.} \end{cases}$$

Gromov and Lawson [**35**] established the fundamental tools for proving a partial converse to this statement. Their work was completed by Stolz, who proved:

THEOREM 3.7 (Stolz [**91**]). *If M^n is a closed simply connected spin manifold with Dirac operator class $[D] \in KO_n(M)$, and if $n \geq 5$, then M admits a metric of positive scalar curvature if and only if* $\operatorname{Ind} D = 0$ *in* $KO_n(\mathrm{pt})$.

What if M is not simply connected? Then Gromov-Lawson ([**36**], [**37**]) and Schoen-Yau ([**88**], [**87**], [**89**]) showed there are other obstructions coming from the fundamental group, and Gromov-Lawson suggested that the "higher index" of D is responsible. Rosenberg ([**75**], [**76**], [**80**]) then pointed out that the Mishchenko-Fomenko Index Theorem is an ideal tool for verifying this.

THEOREM 3.8 (Rosenberg). *Suppose M is a closed spin manifold and $u\colon M \to B\pi$ classifies the universal cover of M. If M admits a metric of positive scalar curvature and if the (strong) Novikov Conjecture holds for π, then $u_*([D]) = 0$ in $KO_n(B\pi)$.*

SKETCH OF PROOF. Suppose M admits a metric of positive scalar curvature. Consider the Dirac operator $D_{\mathcal{V}}$ with coefficients in the universal $C^*_r(\pi)$-bundle $\mathcal{V}_M$. As we remarked earlier, the bundle $\mathcal{V}_M$ has a natural flat connection. If we use this connection to define $D_{\mathcal{V}}$, then Lichnerowicz's identity (2) will still hold with $D_{\mathcal{V}}$ in place of D, since there is no contribution from the curvature of the bundle. Thus $\kappa > 0$ implies $\operatorname{Ind} D_{\mathcal{V}} = \mathcal{A}(u_*([D])) = 0$. Thus if $\mathcal{A}$ is injective, we can conclude that $u_*([D]) = 0$ in $KO_n(B\pi)$. □

For some torsion-free groups, the converse is known to hold for $n \geq 5$, generalizing Theorem 3.7. See [**80**] for details.

CONJECTURE 3.9 (Gromov-Lawson). *A closed aspherical manifold cannot admit a metric of positive scalar curvature.*

Theorem 3.8 shows that the Strong Novikov Conjecture implies Conjecture 3.9, at least for spin manifolds.

For groups with torsion, the assembly map is usually not a monomorphism (see Exercise 3.14), so the converse of Theorem 3.8 is quite unlikely. However, for spin manifolds with finite fundamental group, it is possible (as conjectured in [**77**]) that vanishing of $\mathcal{A}(u_*([D])) = 0$ is necessary and sufficient for positive scalar curvature, at least once the dimension gets to be sufficiently large. Since not much is known about this, it is convenient to simplify the problem by "stabilizing."

DEFINITION 3.10. Fix a simply connected spin manifold J^8 of dimension 8 with $\widehat{A}$-genus 1. (Such a manifold is known to exist, and Joyce [**43**] constructed an explicit example with Spin(7) holonomy.) Taking a product with J does not change the KO-index of the Dirac operator. Say that a manifold M *stably admits a metric of positive scalar curvature* if there is a metric on $M \times J \times \cdots \times J$ with positive scalar curvature, for sufficiently many J factors. In support of this definition, we have:

PROPOSITION 3.11. *A simply connected closed manifold M^n of dimension $n \neq 3, 4$ stably admits a metric of positive scalar curvature if and only if it actually admits a metric of positive scalar curvature.*

SKETCH OF PROOF. We may as well assume $n \geq 5$, since if $n \leq 2$, then M is diffeomorphic to S^2 and certainly has a metric of positive scalar curvature. There are two cases to consider. If M admits a spin structure, then by Theorem 3.7, M

admits a metric of positive scalar curvature if and only if the index of D vanishes in KO_n. But if the index is non-zero in KO_n, then M does not even stably admit a metric of positive scalar curvature, since the KO_n-index of Dirac is the same for $M \times J \times \cdots \times J$ as it is for M. If M does not admit a spin structure, then Gromov and Lawson [**35**] showed that M always admits a metric of positive scalar curvature, and *a fortiori* it stably admits a metric of positive scalar curvature. $\square$

For finite fundamental group, the best general result is:

THEOREM 3.12 (Rosenberg-Stolz [**79**]). *Let M^n be a spin manifold with finite fundamental group π, with Dirac operator class $[D]$, and with classifying map $u\colon M \to B\pi$ for the universal cover. Then M stably admits a metric of positive scalar curvature if and only $\mathcal{A} \circ u_*([D]) = 0$ in $KO_n(C^*_r(\pi))$. (Of course, for π finite, $C^*_r(\pi) = \mathbb{R}[\pi]$.)*

This has been generalized by Stolz to those groups π for which the Baum-Connes assembly map $\mathcal{A}_{\mathrm{BC}}$ in KO is injective. This is a fairly large class including all discrete subgroups of Lie groups.

3.5. Exercises.

EXERCISE 3.13. **(Mishchenko-Fomenko)** Let A be a C^*-algebra. Suppose that a bounded A-linear map $D\colon H_0 \to H_1$ between two Hilbert A-modules is A-Fredholm, *i.e.*, has a decomposition as in Definition 3.3. Show that $\operatorname{Ind} D \in K_0(A)$ is well-defined, *i.e.*, does not depend on the choice of decomposition. On the other hand, show by example that it is not necessarily true that D has closed range, and hence it is not necessarily true that we can define $\operatorname{Ind} D$ as $[\ker D] - [\operatorname{coker} D]$.

EXERCISE 3.14. Let G be a finite group of order n. Show that

$$C^*_r(G) = \mathbb{C}G \cong \bigoplus_{\sigma \in \widehat{G}} M_{\dim(\sigma)}(\mathbb{C}), \quad \text{and} \quad K_0(C^*_r(G)) \cong \mathbb{Z}^c, \quad K_1(C^*_r(G)) = 0,$$

where $\widehat{G}$ is the set of irreducible representations of G and $c = \#(\widehat{G})$ is the number of conjugacy classes in G. (This is all for the *complex* group algebra.) Since $K_0(BG)$ is a torsion group, deduce that the assembly map $\mathcal{A} : K_*(BG) \to K_*(C^*_r(G))$ is identically zero in all degrees. (This is not necessarily the case for the assembly map on KO in degrees $1, 2, 5, 6 \mod 8$ for the *real* group ring if G is of even order—see Exercise 3.18 below and [**79**].) On the other hand, the Baum-Connes Conjecture is true for this case (for more or less trivial reasons—here $\mathcal{E}G = \mathrm{pt}$ and the definition of $K_0^G(\mathrm{pt})$ makes it coincide with $K_0(C^*_r(G))$).

Compute the trace map $\tau_*\colon K_0(C^*_r(G)) \to \mathbb{R}$ for this example, and show that it sends the generator of $K_0(C^*_r(G))$ attached to an irreducible representation σ to $\frac{\dim(\sigma)}{|G|}$. (Hint: The generator corresponds to a certain minimal idempotent in $\mathbb{C}G$. Write it down explicitly (as a linear combination of group elements), using the Schur orthogonality relations.) Deduce that $\tau_*\big(K_0(C^*_r(G))\big) = \frac{1}{|G|}\mathbb{Z}$, the rational numbers with denominator a divisor of $|G|$.

EXERCISE 3.15. Let Γ be the infinite dihedral group, the semidirect product $\mathbb{Z} \rtimes \{\pm 1\}$, where $\{\pm 1\}$ acts on $\mathbb{Z}$ by multiplication. Show by explicit calculation that $C^*_r(\Gamma)$ can be identified with the algebra

$$\big\{f \in C\big([0,1], M_2(\mathbb{C})\big) : f(0) \text{ and } f(1) \text{ are diagonal matrices}\big\}.$$

(To show this, identify $C_r^*(\Gamma)$ with the crossed product $C(S^1) \rtimes \{\pm 1\}$, where $\{\pm 1\}$ acts on $C_r^*(\mathbb{Z}) \cong C(S^1)$ by complex conjugation. The orbit space $S^1/\{\pm 1\}$ can be identified with an interval. Interior points of this interval correspond to irreducible representations of Γ of dimension 2, and over each endpoint there are two irreducible representations of Γ, each of dimension 1.)

Then show that the range of the trace map $\tau_*\colon K_0(C_r^*(\Gamma)) \to \mathbb{R}$ is the half-integers $\frac{1}{2}\mathbb{Z} = \{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \cdots\}$.

This example and others like it, along with Exercise 3.14, led to the conjecture ([**4**], p. 21) that for an arbitrary group G, $\tau_*\bigl(K_0(C_r^*(G))\bigr)$ is the subgroup of $\mathbb{Q}$ generated by the numbers $\frac{1}{|H|}$, where H is a finite subgroup of G. However, this conjecture has turned out to be false ([**83**], [**82**]), even with $K_0(C_r^*(G))$ (which in general is inaccessible) replaced by the more tractable image of the Baum-Connes map $\mathcal{A}_{\mathrm{BC}}$. However, it is shown in [**60**] that the range of the trace on the image of the Baum-Connes map $\mathcal{A}_{\mathrm{BC}}$ is contained in the sub*ring* of $\mathbb{Q}$ generated by the reciprocals of the orders of the finite subgroups. In particular, if the Baum-Connes conjecture holds for G, then the range of the trace lies in this subring.

Exercise 3.16. Suppose Γ is a discrete group and π is a subgroup of Γ of finite index. Then one has a commuting diagram

$$\begin{tikzcd}
 & K_0(B\pi) \arrow[r, "\mathcal{A}_\pi"] \arrow[dl, "\iota_*"'] \arrow[d, dotted] & K_0(C_r^*(\pi)) \arrow[r, "\tau_*"] \arrow[d, "\iota_*"] & \mathbb{R} \arrow[d, equal] \\
K_0(B\Gamma) \arrow[r] \arrow[rr, bend right, "\mathcal{A}_\Gamma"'] & K_0^\Gamma(\mathcal{E}\Gamma) \arrow[r, "\mathcal{A}_{\mathrm{BC}}"] & K_0(C_r^*(\Gamma)) \arrow[r, "\tau_*"] & \mathbb{R}.
\end{tikzcd}$$

Here ι_* is the map induced by the inclusion $\pi \hookrightarrow \Gamma$. But there is also a transfer map ι^* backwards from $K_0(C_r^*(\Gamma))$ to $K_0(C_r^*(\pi))$ which multiplies traces by the index $[\Gamma : \pi]$, since $C_r^*(\Gamma)$ is a free $C_r^*(\pi)$-module of rank $[\Gamma : \pi]$. Similarly, there is a compatible transfer map $\iota^*\colon K_0(B\Gamma) \to K_0(B\pi)$, and $\iota_* \circ \iota^*$ is an isomorphism on $K_0(B\Gamma)$ after inverting $[\Gamma : \pi]$. Suppose that the Baum-Connes Conjecture holds for both π and Γ, so that, in this diagram, $\mathcal{A}_\pi$ and $\mathcal{A}_{\mathrm{BC}}$ are isomorphisms. Then what does this imply about integrality of the trace on $K_0(C_r^*(\Gamma))$? Compare with the conjectures discussed in Exercise 3.15.

Exercise 3.17. Suppose Γ is a discrete group and $e = e^2 \in \mathbb{C}[\Gamma]$. Show that $\tau(e)$ must lie in $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$. Hint ([**9**]): Consider the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on $\mathbb{C}[\Gamma]$, as well as the positivity of τ. In fact, it is even proved in [**9**] that $\tau(e) \in \mathbb{Q}$, but this is much harder.

Exercise 3.18. Let $\pi = \mathbb{Z}/2$, a cyclic group of order 2, so that the *real* group C^*-algebra $\mathbb{R}[\pi]$ of π is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ and the classifying space $B\pi = \mathbb{RP}^\infty$. Show that the assembly map $\mathcal{A}\colon KO_1(B\pi) \to KO_1(\mathbb{R}[\pi]) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is surjective. Hints: For the summand corresponding to the trivial representation, you don't have to do any work, because of the commutative diagram

$$\begin{tikzcd} KO_*(\mathrm{pt}) \arrow[r, "\mathcal{A}_{\{1\}}", "\cong"'] \arrow[d] & KO_*(\mathbb{R}) \arrow[d] \\ KO_*(B\pi) \arrow[r, "\mathcal{A}_\pi"] & KO_*(\mathbb{R} \oplus \mathbb{R}). \end{tikzcd}$$

For the other summand, make use of the commutative diagram

$$\begin{tikzcd} KO_*(S^1) \arrow[r, "\mathcal{A}_{\mathbb{Z}}", "\cong"'] \arrow[d] & KO_*(C_r^*(\mathbb{Z})) \arrow[d] \\ KO_*(B\pi) \arrow[r, "\mathcal{A}_\pi"] & KO_*(\mathbb{R}[\pi]), \end{tikzcd}$$

where the vertical arrows are induced by the "reduction mod 2" map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$.

Exercise 3.19. Let M^n be a smooth compact manifold and let Y be some compact space. Suppose $D\colon x \mapsto D_x$ is a continuously varying family of elliptic operators on M, parameterized by Y. Show that D defines a $C(Y)$-elliptic operator over M, and thus has a $C(Y)$-index in the sense of Mishchenko and Fomenko. (This is the same as the "families index" of Atiyah and Singer.) Also show that if $\dim\ker D_x$ and $\dim\ker D_x^*$ remain constant, so that $x \mapsto \ker D_x$ and $x \mapsto \ker D_x^*$ define vector bundles $\ker D$ and $\ker D^*$ over X, then $\operatorname{Ind} D = [\ker D] - [\ker D^*]$ in $K_0(C(Y)) \cong K^0(Y)$. (The isomorphism here is given by Swan's Theorem.)

4. Other C^*-Algebras and Applications in Topology: Group Actions, Foliations, $\mathbb{Z}/k$-Indices, and Coarse Geometry

4.1. Crossed Products and Invariants of Group Actions. If a (locally compact) group G acts on a locally compact space X, one can form the *transformation group C^*-algebra* or *crossed product* $C^*(G, X)$ or $C_0(X) \rtimes G$. The definition is easiest to explain when G is discrete; then $C^*(G, X)$ is the universal C^*-algebra generated by a copy of $C_0(X)$ and unitaries u_g, $g \in G$, subject to the relations that

$$u_g u_h = u_{gh}, \quad u_g f u_g^* = g \cdot f \text{ for } g,\, h \in G,\ f \in C_0(X). \tag{3}$$

Here $g \cdot f(x) = f(g^{-1} \cdot x)$. In general, $C^*(G, X)$ is the C^*-completion of the twisted convolution algebra of $C_0(X)$-valued continuous functions of compact support on G, and its multiplier algebra still contains copies of $C_0(X)$ and of G satisfying relations (3). (In fact, products of an element of $C^*(G)$ and of an element of $C_0(X)$, in either order, lie in the crossed product and are dense in it.) When G acts freely and properly on X, $C^*(G, X)$ is strongly Morita equivalent to $C_0(G\backslash X)$.[6] It thus plays the role of the algebra of functions on $G\backslash X$, even when the latter is a "bad" space, and captures much of the equivariant topology, as we see from:

Theorem 4.1 (Green-Julg [**44**]). *If G is compact, there is a natural isomorphism*

$$K_*(C^*(G, X)) \cong K_G^{-*}(X).$$

[6](Strong) Morita equivalence (see [**71**]) is one of the most useful equivalence relations on the class of C^*-algebras. When A and B are separable C^*-algebras, it has a simple characterization [**8**]: A and B are strongly Morita equivalent if and only if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable, infinite-dimensional Hilbert space.

There are many other results relating the structure of $C^*(G, X)$ to the topology of the transformation group (G, X); the reader interested in this topic can see the surveys [**68**], [**69**], and [**66**] for an introduction and references. In this chapter we will only need the rather special cases where either G is compact or else G acts locally freely (*i.e.*, with finite isotropy groups).

DEFINITION 4.2. An n-dimensional *orbifold* X is a space covered by charts each homeomorphic to $\mathbb{R}^n/G$, where G is a finite group (which may vary from chart to chart) acting linearly on $\mathbb{R}^n$, and with compatible transition functions. A *smooth orbifold* is defined similarly, but with the transition functions required to lift to be C^∞ on the open subsets of Euclidean space. The most obvious kind of example is a quotient of a manifold by a locally linear action of a finite group. But not every orbifold, not even every compact smooth orbifold, is a quotient of a manifold by a finite group action. (The simplest counterexample or "bad orbifold" is the "teardrop" X, shown in Figure 3. Here the bottom half of the space is a hemisphere, and the top half is the quotient of a hemisphere by a cyclic group acting by rotations around the pole. If X were of the form M/G with M a manifold and G finite, then M would have to be S^2, and we run afoul of the fact that any nontrivial orientation-preserving diffeomorphism of S^2 of finite order has to have at least two fixed points, by the Lefschetz fixed-point theorem.)

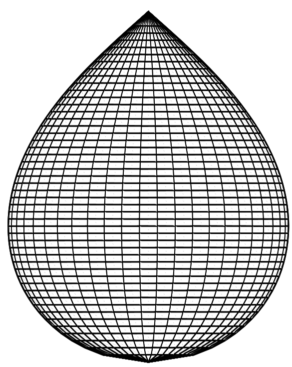

FIGURE 3. The teardrop

On a smooth orbifold X, we have a notion of Riemannian metric, which on a patch looking like $\mathbb{R}^n/G$, G finite, is simply a Riemannian metric on $\mathbb{R}^n$ invariant under the action of G. Similarly, once a Riemannian metric has been fixed, we have a notion of orthonormal frame at a point. As on a smooth manifold, these patch together to give the orthonormal frame bundle $\widetilde{X}$, and $O(n)$ acts locally freely on $\widetilde{X}$, with $\widetilde{X}/O(n)$ identifiable with X. $C^*_{\text{orb}}(X) = C^*(O(n), \widetilde{X})$ is called the *orbifold C^*-algebra* of X. This notion is due to Farsi [**28**]. (It depends on the orbifold structure, not just the homeomorphism class of X as a space.) Note that $C^*_{\text{orb}}(X)$ is strongly Morita equivalent to $C_0(X)$ when X is a manifold, or to $C^*(G, M)$ when X is the quotient of a manifold M by an action of a finite group G.

An elliptic operator D on a smooth orbifold X (which in each local chart $\mathbb{R}^n/G$, G a finite group, is a G-invariant elliptic operator on $\mathbb{R}^n$) defines a class

$[D] \in K^{-*}(C^*_{\mathrm{orb}}(X))$ (which we think of as $K^{\mathrm{orb}}_*(X)$). Note that if X is actually a manifold, this is just $K_*(X)$, by Morita invariance of Kasparov theory. If X is compact, then as in the manifold case, $\operatorname{Ind} D = c_*([D]) \in K_*(\mathrm{pt})$.

Applying the Kasparov formalism and working out all the terms, one can deduce ([**27**], [**28**], [**29**]) various index theorems for orbifolds, originally obtained by Kawasaki [**51**] by a different method.

4.2. Foliation C^*-Algebras and Applications.

Definition 4.3. Let M^n be a compact smooth manifold, $\mathcal{F}$ a foliation of M by leaves L^p of dimension p, codimension $q = n - p$. Then one can define a C^*-algebra $C^*(M, \mathcal{F})$ encoding the structure of the foliation. (This is the C^*-completion of the convolution algebra of functions, or more canonically, half-densities, on the holonomy groupoid.) When the foliation is a fibration $L \to M \to X$, where X is a compact q-manifold, then $C^*(M, \mathcal{F})$ is strongly Morita equivalent to $C(X)$. Since K-theory is Morita invariant, this justifies thinking of $K_*(C^*(M, \mathcal{F}))$ as $K^{-*}(M/\mathcal{F})$, the *K-theory of the space of leaves.* When the foliation comes from a locally free action of a Lie group G on M, then $C^*(M, \mathcal{F})$ is just the crossed product $C^*(G, M)$.

Introducing $C^*(M, \mathcal{F})$ makes it possible to extend the Connes index theorem for foliations. If D is an operator elliptic along the leaves, then in general $\operatorname{Ind} D$ is an element of the group $K_0(C^*(M, \mathcal{F}))$. If there is an invariant transverse measure μ, then one obtains Connes' real-valued index by composing with the map

$$\int d\mu \colon K_0(C^*(M, \mathcal{F})) \to \mathbb{R}.$$

Theorem 4.4 (Connes-Skandalis [**19**]). *Let $(M, \mathcal{F})$ be a compact (smooth) foliated manifold and let*

$$D \colon C^\infty(M, E_0) \to C^\infty(M, E_1)$$

be elliptic along the leaves. Then $\operatorname{Ind} D \in K_0(C^*(M, \mathcal{F}))$ *agrees with a "topological index"* $\operatorname{Ind}_{\mathrm{top}}(D)$ *computed from the characteristic classes of $\sigma(D)$, just as in the usual Atiyah-Singer index theorem.*

Example 4.5. The simplest example of this is when M splits as a product $Y \times L$, the foliation $\mathcal{F}$ is by slices $\{x\} \times L$, and D is given by a continuous family of elliptic operators D_x on L, parameterized by the points in Y, just as in Exercise 3.19. Then $C^*(M, \mathcal{F})$ is Morita equivalent to $C(Y)$, and $\operatorname{Ind} D$ as in the Theorem 4.4 is exactly the Atiyah-Singer "families index" in $K_0(C(Y)) = K^0(Y)$, which, as shown in Exercise 3.19, can also be viewed as a case of a Mishchenko-Fomenko index.

Corollary 4.6 (Connes-Skandalis [**19**], Corollary 4.15). *Let $(M, \mathcal{F})$ be a compact foliated manifold and let D be the Euler characteristic operator along the leaves. Then* $\operatorname{Ind} D$ *is the class of the zeros Z of a generic vector field along the fibers, counting signs appropriately.*[7] *(Compare the Poincaré-Hopf Theorem, which identifies the Euler characteristic of a compact manifold with the sum of the zeros of a generic vector field, counted with appropriate signs.)*

[7] Think of Z as a manifold transverse to the leaves of $\mathcal{F}$, and take the "push-forward" of the class of the trivial vector bundle over Z.

The advantage of Theorem 4.4 and of Corollary 4.6 over Theorem 2.4 and its corollaries is that we don't need to assume the existence of an invariant transverse measure, which is quite a strong hypothesis. However, if such a measure μ exists, the numerical index in the situation of Corollary 4.6 is simply $\mu(Z)$.

EXAMPLE 4.7. Let M be a compact Riemann surface of genus $g \geq 2$, so that its universal covering space $\widetilde{M}$ is the hyperbolic plane, and its fundamental group π is a discrete torsion-free cocompact subgroup of $G = PSL(2, \mathbb{R})$. Let $V = \widetilde{M} \times_\pi S^2$, where π acts on $S^2 = \mathbb{CP}^1$ by projective transformations (*i.e.*, the embedding $PSL(2, \mathbb{R}) \hookrightarrow PSL(2, \mathbb{C})$); V is an S^2-bundle over M. Foliate V by the images of $\widetilde{M} \times \{x\}$. In this case there is no invariant transverse measure, since π does not leave any measure on S^2 invariant. Nevertheless, $\operatorname{Ind} D$ is non-zero in $K_0(C^*(V, \mathcal{F}))$. (It is $-2(g-1) \cdot [S^2]$, where $[S^2]$ is the push-forward of the class of $S^2 \hookrightarrow V$ ([**19**], pp. 1173–1174).)

One case of Theorem 4.4 that is easier to understand is the case where the foliation $\mathcal{F}$ results from a locally free action of a simply connected solvable Lie group G on the compact manifold M. As explained before, we then have $C^*(M, \mathcal{F}) \cong C(M) \rtimes G$. However, because of the structure theory of simply connected solvable Lie groups, the crossed product by G is obtained by $\dim G$ successive crossed products by $\mathbb{R}$. However, when it comes to crossed products by $\mathbb{R}$, there is a remarkable result of Connes that can be used for computing the K-theory. For simplicity we state it only for complex K-theory, though there is a version for KO as well.

THEOREM 4.8 (Connes' "Thom Isomorphism" [**15**], [**70**], [**22**]). *Let A be a C^*-algebra equipped with a continuous action α of $\mathbb{R}$ by automorphisms. Then there are natural isomorphisms $K_0(A) \xrightarrow{\cong} K_1(A \rtimes_\alpha \mathbb{R})$ and $K_1(A) \xrightarrow{\cong} K_0(A \rtimes_\alpha \mathbb{R})$.*

Note. Homotop the action α of $\mathbb{R}$ on A to the trivial action by considering α_t, $\alpha_t(s) = \alpha(ts)$, $0 \leq t \leq 1$, so $\alpha_1 = \alpha$ and α_0 is the trivial action. One way of understanding the theorem is that it says that, from the point of view of K-theory, $K_*(A \rtimes_{\alpha_t} \mathbb{R})$ is independent of t, and thus

$$K_*(A \rtimes_\alpha \mathbb{R}) \cong K_*(A \rtimes_{\text{trivial}} \mathbb{R}) \cong K_*(A \otimes C^*(\mathbb{R})) \cong K_*(A \otimes C_0(\mathbb{R})),$$

which can be computed easily by Bott periodicity.

SKETCH OF PROOF. Connes' method of proof is to show that there is a unique family of maps $\phi^i_\alpha \colon K_i(A) \to K_{i+1}(A \rtimes_\alpha \mathbb{R})$, $i \in \mathbb{Z}/2$, defined for all C^*-algebras A equipped with an $\mathbb{R}$-action α, and satisfying compatibility with suspension, naturality, and reducing to the usual isomorphism $K_0(\mathbb{C}) \to K_1(\mathbb{R})$ when $A = \mathbb{C}$. Then these maps have to be isomorphisms, since Takesaki-Takai duality [**93**] gives an isomorphism $(A \rtimes_\alpha \mathbb{R}) \rtimes_{\widehat{\alpha}} \mathbb{R} \cong A \otimes \mathcal{K}$ (here $\widehat{\alpha}$ is the dual action of the Pontryagin dual $\widehat{\mathbb{R}} \cong \mathbb{R}$ of $\mathbb{R}$), and then by the axioms, $\phi^{i+1}_{\widehat{\alpha}} \circ \phi^i_\alpha \colon K_i(A) \to K_{i+2}(A) \cong K_i(A)$ must coincide with the Bott periodicity isomorphism. The only real problem is thus the existence and uniqueness. First, Connes shows that if e is a projection in A, there is an action α' exterior equivalent to α (in other words, related to it by a 1-cocycle with values in the unitary elements of the multiplier algebra) that leaves e fixed. Since exterior equivalent actions are opposite "corners" of an action β of $\mathbb{R}$ on $M_2(A)$, by Connes' "cocycle trick," the K-theory for their crossed products is the same.[8] So if there is a map ϕ^0_α with the correct properties, $\phi^0_\alpha([e])$ is determined via the commuting diagram

$$\begin{tikzcd}
K_0(A) \arrow[r, "\phi^0_\alpha"] \arrow[d, equal] & K_1(A \rtimes_\alpha \mathbb{R}) \arrow[d, "\cong"] \\
K_0(A) \arrow[r, "\phi^0_{\alpha'}"] & K_1(A \rtimes_{\alpha'} \mathbb{R}) \\
\mathbb{Z} = K_0(\mathbb{C}) \arrow[u, "{[1]\mapsto[e]}"] \arrow[r, "\phi^0_{\text{trivial}}", "\cong"'] & K_1(C^*(\mathbb{R})) = \mathbb{Z}. \arrow[u]
\end{tikzcd}$$

Here, the upward arrows at the bottom are induced by the inclusion $\mathbb{C} \cdot e \hookrightarrow A$. The axioms quickly reduce all other cases of uniqueness down to this one, so it remains only to prove existence. There are many arguments for this: see [**70**], [**22**], and §10.2.2 and §19.3.6 in [**5**]. The most elegant argument uses KK-theory, but even without this one can define ϕ^*_α to be the connecting map in the long exact K-theory sequence for the "Toeplitz extension"

$$0 \to (C_0(\mathbb{R}) \otimes A) \rtimes_{\tau\otimes\alpha} \mathbb{R} \to (C_0(\mathbb{R} \cup \{+\infty\}) \otimes A) \rtimes_{\tau\otimes\alpha} \mathbb{R} \to A \rtimes_\alpha \mathbb{R} \to 0.$$

Here τ is the translation action of $\mathbb{R}$ on $\mathbb{R} \cup \{+\infty\}$ fixing the point at infinity. But

$$(C_0(\mathbb{R}) \otimes A) \rtimes_{\tau\otimes\alpha} \mathbb{R} \cong (C_0(\mathbb{R}) \otimes A) \rtimes_{\tau\otimes\text{trivial}} \cong A \otimes \mathcal{K}$$

by Takai duality again, so the connecting map in K-theory becomes a natural map ϕ^*_α satisfying the correct axioms. □

Now we're ready to apply this to the foliation index theorem. Suppose the foliation $\mathcal{F}$ results from a locally free action of a simply connected even-dimensional solvable Lie group G on the compact manifold M. Then

$$C^*(M, \mathcal{F}) \cong C(M) \rtimes G,$$

and iterated applications of Theorem 4.8 set up an isomorphism

$$K_0(C^*(M, \mathcal{F})) \cong K^0(M \times \mathbb{R}^{\dim G}) \cong K^0(M),$$

the last isomorphism given by Bott periodicity. Under these isomorphisms, one can check that the index class of the leafwise Dirac operator goes first to the exterior product of the class of the trivial line bundle on M with the Bott class in $K^0(\mathbb{R}^{\dim G})$, and thus under Bott periodicity to the class of the trivial vector bundle on M.

Example 4.9. Let G, π, and M be as in Example 4.7, and consider the 2-dimensional subgroup H of G, the image in G of

$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a,\, b \in \mathbb{R}, \quad a > 0 \right\} \subset SL(2, \mathbb{R}).$$

[8]For α and α' to be exterior equivalent means that ${\alpha_t}'(a) = u_t \alpha_t(a) u_t^*$, for some map $t \mapsto u_t$ from $\mathbb{R}$ to the unitaries of the multiplier algebra of A such that for each $a \in A$, $t \mapsto u_t a$ and $t \mapsto a u_t$ are norm-continuous. Then one can manufacture an action of $\mathbb{R}$ on $M_2(A)$, the 2×2 matrices with entries in A, by the formula

$$\beta(t) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha_t(a_{11}) & \alpha_t(a_{12}) u_t^* \\ u_t \alpha_t(a_{21}) & {\alpha'}_t(a_{22}) \end{pmatrix}.$$

Then H acts freely on G (say on the left) and hence locally freely on $V = G/\pi$, the unit sphere bundle of M. So we have a foliation of V by orbits of H. This foliation does not have an invariant transverse measure, since such a measure would correspond to a π-invariant measure of $H\backslash G \cong S^1$, which does not exist. However, the discussion above computes the index of the leafwise Dirac operator on $(V, \mathcal{F})$ and shows it is non-zero.

4.3. C^*-Algebras and $\mathbb{Z}/k$-Index Theory.

Definition 4.10. A $\mathbb{Z}/k$-*manifold* is a smooth compact manifold with boundary, M^n, along with an identification of ∂M with a disjoint union of k copies of a fixed manifold βM^{n-1}. It is oriented if M is oriented, the boundary components have the induced orientation, and the identifications are orientation-preserving. See Figure 4 for an illustration.

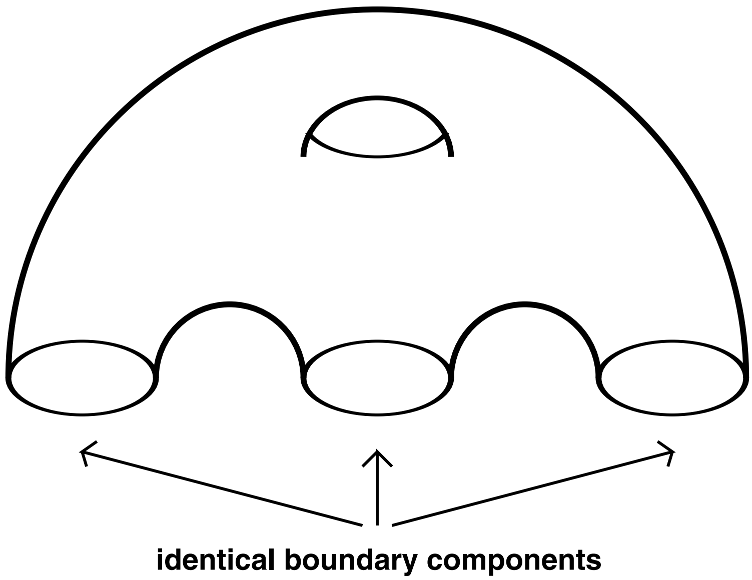

Figure 4. A $\mathbb{Z}/3$-manifold

One should really think of a $\mathbb{Z}/k$-manifold M as the singular space $M_\Sigma = M/\!\sim$ obtained by identifying all k of the boundary components with one another. This space is not a manifold (if $k > 2$), and does not satisfy Poincaré duality. The neighborhood of a point on βM is a cone on k copies of B^{n-1} joined along S^{n-2}, as illustrated in Figure 5. If M is an oriented $\mathbb{Z}/2$-manifold, then M_Σ is a manifold, but is not orientable, because of the way the two copies of βM have been glued together. (For instance, if M is a cylinder, so $\beta M = S^1$, then M_Σ is a Klein bottle.) So an oriented $\mathbb{Z}/k$-manifold of dimension $4n$ does not have a signature in the usual sense. But it does have a signature mod k, just as a non-orientable manifold has a signature for $\mathbb{Z}/2$-cohomology. (Since $+1 = -1$ in $\mathbb{Z}/2$, the mod 2 "signature" of a non-orientable manifold is simply the middle Betti number.) The signature of a $\mathbb{Z}/k$-manifold was defined by Sullivan [**92**], who showed that M_Σ has a fundamental

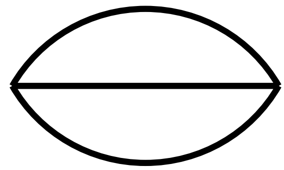

FIGURE 5. Link of a boundary point in M_Σ ($n = 2$, $k = 3$)

class in homology mod k, and there is a $\mathbb{Z}/k$-version of Hirzebruch's formula,

$$\operatorname{sign} M = \langle \mathcal{L}(M), [M] \rangle \in \mathbb{Z}/k.$$

This formula is a special case of an index theorem for elliptic operators on $\mathbb{Z}/k$-manifolds, due originally to Freed and Melrose [**31**], [**32**]. Other proofs were later given by Higson [**38**], Kaminker-Wojciechowski [**45**], and Zhang [**97**]. Higson's proof in particular made use of noncommutative C^*-algebras. The approach we will present here is due to the author [**78**]. For simplicity we'll deal with the "ordinary" (K_0) index of a complex elliptic operator D.

DEFINITION 4.11. A $\mathbb{Z}/k$-*elliptic operator* on a $\mathbb{Z}/k$-manifold M^n, $\partial M \cong \beta M \times \mathbb{Z}/k$, will mean an elliptic operator on M (in the usual sense) whose restriction to a collar neighborhood of the boundary (diffeomorphic to $\beta M \times [0, \varepsilon) \times \mathbb{Z}/k$) is the restriction of an $\mathbb{R} \times \mathbb{Z}/k$-invariant operator on $\beta M \times \mathbb{R} \times \mathbb{Z}/k$. Thus, near the boundary, the operator is entirely determined by what happens on βM.

We want to define a $\mathbb{Z}/k$-valued index for such an operator by using the philosophy of noncommutative geometry, that says we should use a noncommutative C^*-algebra to encode the equivalence relation on M (that identifies the k copies of βM with one another), instead of working on the singular quotient space M_Σ. We begin by following a trick introduced in [**38**] to get rid of the complications involved with analysis near the boundary. First we attach cylinders to the boundary, replacing M by the noncompact manifold $N = M \cup_{\partial M} \partial M \times [0, \infty)$, as shown in Figure 6. It's important to note that an operator as in Definition 4.11 has a canonical extension to N, because of the translation invariance in the direction normal to the boundary.

Now we introduce the C^*-algebra $C^*(M; \mathbb{Z}/k)$ of the equivalence relation on N that is trivial on M itself and that identifies the k cylinders with one another. A simple calculation shows that

$$\begin{aligned} C^*(M; \mathbb{Z}/k) \cong \{(f, g) : f \in C(M), \quad & g \in C_0(\beta M \times [0, \infty), M_k(\mathbb{C})), \\ & \text{with } g|_{\beta M \times \{0\}} \text{ diagonal, and } f|\partial M \text{ matching } g|_{\beta M \times \{0\}}\}. \end{aligned}$$

Furthermore, just as an elliptic operator on an ordinary manifold defines a class in K-homology, a $\mathbb{Z}/k$-elliptic operator D on M, as extended canonically to N, defines a class in $K^0(C^*(M; \mathbb{Z}/k))$. (This group should be viewed as the $\mathbb{Z}/k$-manifold K-homology of M.)

Similarly, we define a C^*-algebra $C^*(\mathrm{pt}; \mathbb{Z}/k)$ which is almost the same, except that M and βM are both replaced by a point. In other words,

$$C^*(\mathrm{pt}; \mathbb{Z}/k) = \{f \in C_0([0, \infty), M_k(\mathbb{C})) : f(0) \text{ a multiple of } I_k\}.$$

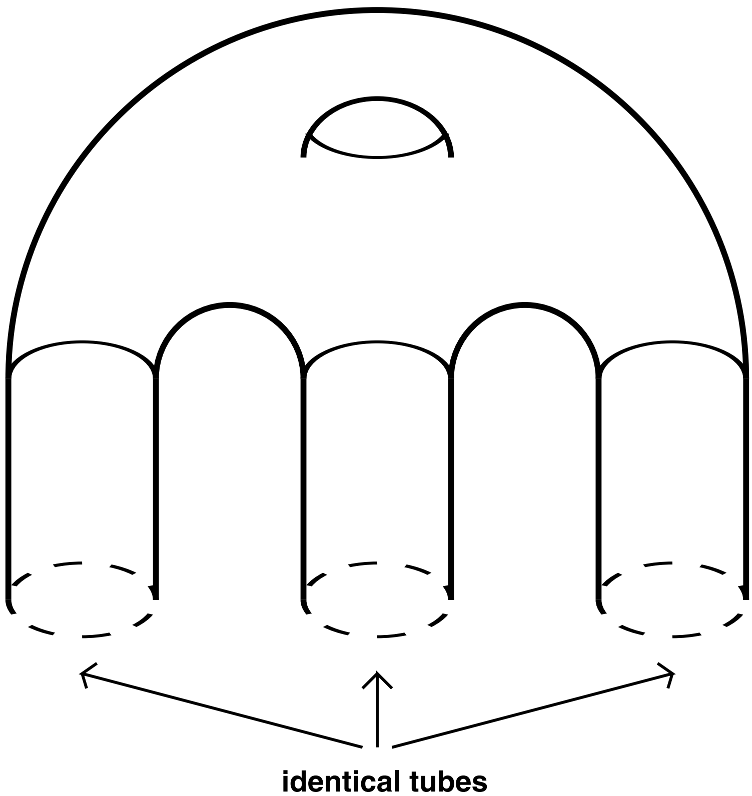

FIGURE 6. A $\mathbb{Z}/3$-manifold with infinite cylinders attached

This is simply the mapping cone of the inclusion of the scalars into $M_k(\mathbb{C})$ as multiples of the $k \times k$ identity matrix, for which the induced map on K-theory is multiplication by k on $\mathbb{Z}$, so $K^0(C^*(\mathrm{pt};\mathbb{Z}/k)) \cong \mathbb{Z}/k$.

Now the collapse map $c\colon (M, \beta M) \to (\mathrm{pt}, \mathrm{pt})$ induces a map on C^*-algebras in the other direction, $C^*(\mathrm{pt};\mathbb{Z}/k) \hookrightarrow C^*(M;\mathbb{Z}/k)$, and hence a map of K-homology groups

$$c_*\colon K^0(C^*(M;\mathbb{Z}/k)) \to K^0(C^*(\mathrm{pt};\mathbb{Z}/k)) \cong \mathbb{Z}/k.$$

The image of $[D]$ under this map is called the *analytic $\mathbb{Z}/k$-index* of D.

DEFINITION 4.12. **(the topological $\mathbb{Z}/k$-index)** Let $[\sigma(D)] \in K^*(T^*M)$ be the class of the principal symbol of the operator, where $K^*(T^*M)$ is the K-theory with compact supports of the cotangent bundle of M. Note that $[\sigma(D)]$ is invariant under the identifications on the boundary, *i.e.*, it comes by pullback from the quotient space T^*M_Σ (the image of T^*M with the k copies of $T^*M|_{\beta M}$ collapsed to one) under the collapse map $M \twoheadrightarrow M_\Sigma$. Following [**31**] we define the *topological $\mathbb{Z}/k$-index* $\mathrm{Ind}_{\mathrm{t}} D$ of D as follows. Start by choosing an embedding $\iota\colon (M, \partial M) \hookrightarrow (D^{2r}, S^{2r-1})$ of M into a ball of sufficiently large even dimension $2r$, for which ∂M embeds $\mathbb{Z}/k$-equivariantly into the boundary (if we identify S^{2r-1} with the unit sphere in $\mathbb{C}^r$, $\mathbb{Z}/k$ acting as usual by multiplication by roots of unity).

We take the push-forward map on complex K-theory

$$\iota_!\colon K^0(T^*M) \to \widetilde{K}^0(T^*D^{2r}) \cong \widetilde{K}^0(D^{2r})$$

and observe that $\iota_!([\sigma(D)])$ descends to $\widetilde{K}^0(M_k^{2r}) \cong K^0(\mathrm{pt}; \mathbb{Z}/k) \cong \mathbb{Z}/k$, M_k^{2r} being the Moore space obtained by dividing out by the $\mathbb{Z}/k$-action on the boundary of D^{2r}, and call the image the topological index of D, $\mathrm{Ind_t}(D)$.

THEOREM 4.13 ($\mathbb{Z}/k$-index theorem). *Let $(M, \phi\colon \partial M \xrightarrow{\cong} \beta M \times \mathbb{Z}/k)$ be a closed $\mathbb{Z}/k$-manifold, and let D be an elliptic operator on M in the sense of Definition 4.11. Then the analytic index of D in $K_i(\mathrm{pt}; \mathbb{Z}/k)$ coincides with the topological index* $\mathrm{Ind_t}(D)$.

SKETCH OF PROOF. The idea, based on the Kasparov-theoretic proof of the Atiyah-Singer Theorem ([**5**], Chapter IX, §24.5), is to write the class of D in $K^0(C^*(M; \mathbb{Z}/k))$ as a Kasparov product:

$$[D] = [\sigma(D)]\widehat{\otimes}_{C_0(T^*M_\Sigma)}\widehat{\alpha} \in K^0(C^*(M; \mathbb{Z}/k)),$$

where

$$\widehat{\alpha} \in KK\big(C_0(T^*M_\Sigma) \otimes C^*(M; \mathbb{Z}/k), \mathbb{C}\big)$$

is a canonical class constructed using the almost complex structure on T^*M and the Thom isomorphism, and we view $[\sigma(D)]$ as living in $K^0(T^*M_\Sigma)$.

But now, by associativity of the Kasparov product, we compute that

$$\mathrm{Ind}(D) = [c^*]\widehat{\otimes}_{C^*(M;\mathbb{Z}/k)}[D] = [\sigma(D)]\widehat{\otimes}_{C_0(T^*M_\Sigma)}\Big([c^*]\widehat{\otimes}_{C^*(M;\mathbb{Z}/k)}\widehat{\alpha}\Big).$$

So we just need to identify the right-hand side of this equation with $\mathrm{Ind_t}(D)$. However, by Definition 4.12, $\mathrm{Ind_t}(D) = \widehat{\iota_!}([\sigma(D)])$, where

$$\widehat{\iota_!}\colon K^0(T^*M_\Sigma) \to K^0(T^*D^{2r}_\Sigma) \cong K^0(M_k^{2r})$$

is the push-forward map on K-theory. And examination of the definition of $\widehat{\iota_!}$ shows it is precisely the Kasparov product with

$$[c^*]\widehat{\otimes}_{C^*(M;\mathbb{Z}/k)}\widehat{\alpha},$$

followed by a "Poincaré duality" isomorphism $K^0(C^*(\mathrm{pt}; \mathbb{Z}/k)) \xrightarrow{\cong} K_0(\mathrm{pt}; \mathbb{Z}/k)$. ☐

4.4. Roe C^*-Algebras and Coarse Geometry. Finally, we mention an application of C^*-algebras to the topology "at infinity" of noncompact spaces. Recall that we began Chapter 1 by talking about the differences between the spectral theory of the Laplacian on compact and on noncompact manifolds. The same points would have been equally valid for arbitrary elliptic operators.

Roe had the idea of introducing certain C^*-algebras attached to a noncompact manifold, but *depending on a choice of metric*, that can be used for doing index theory "at infinity."

DEFINITION 4.14. ([**72**], [**73**]) Let M be a complete Riemannian manifold (usually noncompact). Fix a suitable Hilbert space $\mathcal{H}$ (for example, $L^2(M, d\,\mathrm{vol})$) on which $C_0(M)$ acts non-degenerately, with no nonzero element of $C_0(M)$ acting by a compact operator. A bounded operator T on $\mathcal{H}$ is called *locally compact* if $\varphi T, T\varphi \in \mathcal{K}(\mathcal{H})$ for all $\varphi \in C_c(M)$, and *of finite propagation* if for some $R > 0$ (depending on T), $\varphi T \psi = 0$ for all $\varphi, \psi \in C_c(M)$, with $\mathrm{dist}(\mathrm{supp}\,\varphi, \mathrm{supp}\,\psi) > R$. Let $C^*_{\mathrm{Roe}}(M)$ be the C^*-algebra generated by the locally compact, finite propagation operators. One can show that this algebra is (up to isomorphism) independent of the choice of $\mathcal{H}$.

EXAMPLE 4.15. If M is compact, the finite propagation condition is always trivially satisfied, and $C^*_{\mathrm{Roe}}(M) = \mathcal{K}$, the compact operators. If $M = \mathbb{R}^n$ with the usual Euclidean metric, then $K_i(C^*_{\mathrm{Roe}}(M)) \cong \mathbb{Z}$ for $i \equiv n \mod 2$, and $K_i(C^*_{\mathrm{Roe}}(M)) = 0$ for $i \equiv n-1 \mod 2$. (See [**73**], p. 33 and p. 74.)

DEFINITION 4.16. Let X and Y be *proper metric spaces*, that is, metric spaces in which closed bounded sets are compact. Then a map $f\colon X \to Y$ is called a *coarse map* if it is proper (the inverse image of a pre-compact set is pre-compact) and if it is *uniformly expansive*, *i.e.*, for each $R > 0$, there exists $S > 0$ such that if $d_X(x, x') \leq R$, then $d_Y(f(x), f(x')) \leq S$. Note that this definition only involves the large-scale behavior of f; f need not be continuous, and we can always modify f any way we like on a compact set (as long as the image of that compact set remains bounded) without affecting this property. A *coarse equivalence* is a coarse map $f\colon X \to Y$ such that there exists a coarse map $g\colon Y \to X$ and there is a constant $K > 0$ with $d_X(x, g \circ f(x)) \leq K$ and $d_Y(y, f \circ g(y)) \leq K$ for all $x \in X$ and $y \in Y$.

EXAMPLE 4.17. The inclusion map $\mathbb{Z} \hookrightarrow \mathbb{R}$ (when $\mathbb{Z}$ and $\mathbb{R}$ are equipped with their standard metrics) is a coarse equivalence, with coarse inverse the "rounding down" map $x \mapsto \lfloor x \rfloor$. More generally, if M is a connected compact manifold with fundamental group π, and if $\widetilde{M}$ is the universal cover of M, then $\widetilde{M}$ is coarsely equivalent to $|\pi|$, the group π viewed as a metric space with respect to a word-length metric (defined by a choice of a finite generating set). The coarse equivalence is again obtained by fixing a basepoint $x_0 \in \widetilde{M}$ and a fundamental domain F for the action of π on $\widetilde{M}$, and defining $f\colon |\pi| \to \widetilde{M}$ by $g \mapsto g \cdot x_0$, $g\colon \widetilde{M} \to |\pi|$ by $x \mapsto g$ whenever $x \in g \cdot F$. (The previous example is the special case where $M = S^1$, $\widetilde{M} = \mathbb{R}$, $\pi = \mathbb{Z}$, $x_0 = 0$, and $F = [0, 1)$.)

PROPOSITION 4.18 (Roe [**73**], Lemma 3.5). *A coarse equivalence $X \to Y$ induces an isomorphism $C^*_{\mathrm{Roe}}(X) \to C^*_{\mathrm{Roe}}(Y)$.*

THEOREM 4.19 (Roe). *If M is a complete Riemannian manifold, there is a functorial "assembly map" $\mathcal{A}\colon K_*(M) \to K_*(C^*_{\mathrm{Roe}}(M))$. If D is a geometric elliptic operator on M (say the Dirac operator or the signature operator), it has a class in $K_0(M)$, and $\mathcal{A}([D])$ is its "coarse index." For noncompact spin manifolds, vanishing of $\mathcal{A}([D])$ (for the Dirac operator) is a necessary condition for there being a metric of uniformly positive scalar curvature in the quasi-isometry class of the original metric on M.*

There is a *Coarse Baum-Connes Conjecture* analogous to the usual Baum-Connes Conjecture, that the assembly map $\mathcal{A}\colon K_*(M) \to K_*(C^*_{\mathrm{Roe}}(M))$ is an isomorphism for M uniformly contractible. (The uniform contractibility assures that M has no "local topology"; without this, we certainly wouldn't expect an isomorphism, since $K_*(C^*_{\mathrm{Roe}}(M))$ only depends on the coarse equivalence class of M.)

Unfortunately, the Coarse Baum-Connes Conjecture is now known to fail in some cases. For one thing, it is known to fail for some uniformly contractible manifolds without bounded geometry [**20**]. This suggests that perhaps one should change the domain of the assembly map from $K_*(M)$ to its "coarsification"

$KX_*(M)$ ([**73**], pp. 14-15), which is the inductive limit of the $K_*(|\mathcal{U}|)$, the nerves of coverings $\mathcal{U}$ of X by pre-compact open sets, as the coverings become coarser and coarser. As one would hope, it turns out that $\mathcal{A}\colon K_*(M) \to K_*(C^*_{\text{Roe}}(M))$ factors through $KX_*(M)$, and that $K_*(M) \to KX_*(M)$ is an isomorphism when M is uniformly contractible and of bounded geometry. However, there is also an example of a manifold M of bounded geometry for which $KX_*(M) \to K_*(C^*_{\text{Roe}}(M))$ is not an isomorphism [**95**]. But it is still conceivable (though it seems increasingly unlikely) that the Coarse Baum-Connes Conjecture holds for all uniformly contractible manifolds with bounded geometry, or at least for all universal covers of compact manifolds.

The main interest of the Coarse Baum-Connes Conjecture, aside from its aesthetic appeal as a parallel to the usual Baum-Connes Conjecture, is its connection with the usual Novikov Conjecture (Conjecture 3.6). One has:

THEOREM 4.20. **(Principle of descent)** *The Coarse Baum-Connes Conjecture for $C^*_{\text{Roe}}(|\pi|)$, where π is a group, but viewed as a discrete metric space, implies the Novikov Conjecture for π.*

A sketch of proof can be found in [**73**], Chapter 8. Theorem 4.20 has been applied in [**96**] to prove the Novikov Conjecture for any group π for which $|\pi|$ admits a uniform embedding into a Hilbert space. This covers both amenable groups and hyperbolic groups.

4.5. Exercises.

EXERCISE 4.21. Consider the teardrop X shown in Figure 3, obtained by gluing together D^2/μ_n and D^2. (Here D^2 is the closed unit disk in $\mathbb{C}$, and μ_n is the cyclic group of n-th roots of unity, which acts on D^2 by rotations.) Compute the topology of the (oriented) orthonormal frame bundle P of X, which should be a closed 3-manifold, and describe the locally free action of $S^1 \cong SO(2)$ on P with $P/S^1 \cong X$. Show that $C^*_{\text{orb}}(X)$ is Morita equivalent to

$$A = \{f \in C(S^2, M_n(\mathbb{C})) : f(x_0) \text{ is diagonal}\},$$

where x_0 is a distinguished point on S^2. This fits into a short exact sequence

$$0 \to C_0(\mathbb{R}^2, M_n(\mathbb{C})) \to A \to \mathbb{C}^n \to 0.$$

Deduce that $K_0(C^*_{\text{orb}}(X))$ is free abelian of rank $n+1$, and that $K_1(C^*_{\text{orb}}(X)) = 0$. From this it follows by duality that $K^0(C^*_{\text{orb}}(X))$ is free abelian of rank $n+1$. Compute the class in $K^0(C^*_{\text{orb}}(X)) = K_0^{\text{orb}}(X)$ of the Euler characteristic operator D, and also its index $\operatorname{Ind} D$ in $K_0(\text{pt}) = \mathbb{Z}$.

EXERCISE 4.22. Suppose a foliation $\mathcal{F}$ results from a locally free action of a simply connected even-dimensional solvable Lie group G on a compact manifold M. Show that the index of the leafwise Euler characteristic operator is 0 in $C^*(M, \mathcal{F})$, both by an application of Corollary 4.6 and by a calculation using the Thom Isomorphism Theorem (Theorem 4.8), as was done above with the Dirac operator.

EXERCISE 4.23. Let M be a compact oriented surface of genus g and with $k > 1$ boundary components (all necessarily circles), as in Figure 4, which shows the case $g = 1$ and $k = 3$. Regard M as a $\mathbb{Z}/k$-manifold. Compute the $\mathbb{Z}/k$-index of the Euler characteristic operator on M.

EXERCISE 4.24. Construct complete Riemannian metrics g on $\mathbb{R}^2$ for which $K_*(C^*_{\mathrm{orb}}(X))$, $X = (\mathbb{R}^2, g)$ is not isomorphic to $K_*(\mathrm{pt})$, and give an example of an application to index theory on X. (Hint: The Coarse Baum-Connes Conjecture is valid for the open cone on a compact metrizable space Y. If Y is embedded in the unit sphere S^{n-1} in $\mathbb{R}^n$, the *open cone* on Y is by definition the union of the rays in $\mathbb{R}^n$ starting at the origin and passing through Y, equipped with the restriction of the Euclidean metric on $\mathbb{R}^n$.)

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Index

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On Novikov-Type Conjectures

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Contents

This paper is based on lectures given by the second author at the 2000 Summer conference at Mt. Holyoke. We also added a brief epilogue, essentially "What there wasn't time for." Although the focus of the conference was on noncommutative geometry, the topic discussed was conventional commutative motivations for the circle of ideas related to the Novikov and Baum-Connes conjectures. While the article is mainly expository, we present here a few new results (due to the two of us).

It is interesting to note that while the period from the 80's through the mid-90's has shown a remarkable convergence between index theory and surgery theory (or more generally, the classification of manifolds) largely motivated by the Novikov conjecture, most recently, a number of divergences has arisen. Possibly, these subjects are now diverging, but it also seems plausible that we are only now close to discovering truly deep phenomena and that the difference between these subjects is just one of these. Our belief is that, even after decades of mining this vein, the gold is not yet all gone.

As the reader might guess from the title, the focus of these notes is not quite on the Novikov conjecture itself, but rather on a collection of problems that are suggested by heuristics, analogies and careful consideration of consequences. Many of the related conjectures are false, or, as far as we know, not directly mathematically related to the original conjecture; this is a good thing: we learn about the subtleties of the original problem, the boundaries of the associated phenomenon, and get to learn about other realms of mathematics.

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Of course in such a subject of active research, there are necessarily many interesting developments in this field since the 2000 conference and the subsequent drafting of this article. We regret that we cannot incorporate all of these newer findings in this document.

1. Topology and K-theory

1.1. For the topologist, the Novikov conjecture is deeply embedded in one of the central projects of his field, that of classifying manifolds within a homotopy type up to homeomorphism or diffeomorphism. To put matters in perspective, let us begin by reviewing some early observations regarding this problem.

The first quite nontrivial point is that there are closed manifolds that are homotopy equivalent but not diffeomorphic (homeomorphism is much more difficult). It is quite easy to give examples which are manifolds with boundary: the punctured torus and the thrice punctured 2-sphere are homotopy equivalent but not diffeomorphic; their boundaries have different numbers of components.

The first class of examples without boundary are the lens spaces: quotients of the sphere by finite cyclic groups of isometries of the round metric. To be concrete, let S^{2n-1} be the unit sphere in $\mathbb{C}^n$ with coordinates $(u_1, \dots, u_n)$. For any n-tuple of primitive k-th roots of unity $e^{2\pi i a_r/k}$, one has a $\mathbb{Z}_k$ action by multiplying the r-th coordinate by the r-th root of unity. The quotient manifolds under these actions are homotopy equivalent (preserving the identification of fundamental group with $\mathbb{Z}_k$) iff the products of the rotation numbers $a_1 a_2 \cdots a_n$ are the same mod k. On the other hand, these manifolds are diffeomorphic iff they are isometric iff the sets of rotation numbers are the same (*i.e.* they agree after reordering). There are essentially two different proofs of this fact, both of which depend on the same sophisticated number-theoretic fact, the Franz independence lemma.

The first proof, due to de Rham, uses Reidemeister torsion. Since the cellular chain complex of a lens space is acyclic when tensored with $\mathbb{Q}[x]$ for x a primitive k-th root of unity, one gets a based (by cells) acyclic complex $0 \to C_{2n-1} \to \cdots \to C_0 \to 0$, which gives us a well-defined nonzero determinant element in $\mathbb{Q}[x]$ (now called the associated element of K_1). This quantity is well-defined up to multiplication by a root of unity (and a sign). One now has to check that these actually determine the rotation numbers, a fact verified by Franz's lemma. See [**Mil66**] and [**Coh73**]. The second proof came much later and is due to Atiyah and Bott [**AB68**]. It uses index-theoretic ideas critically, and implies more about the topology of lens spaces. We will return to it a bit later.

After de Rham's theorem, it was very natural to ask, following Hurewicz, whether all homotopy equivalent simply-connected manifolds are diffeomorphic. (It was not until Milnor's examples of exotic spheres that mathematicians really considered seriously the existence of different categories of manifolds.) However, very classical results can be used to disprove this claim as well. Consider a sphere bundle over the sphere S^4 where the fiber is quite high-dimensional. Since $\pi_3(O(n)) = \mathbb{Z}$ for large n, we can construct an infinite number of these bundles by explicit clutching operations; their total spaces are distinguished by p_1. On the other hand, if we could nullhomotop the clutching maps in $\pi_3(\text{Isometries}(S^{n+1}))$ pushed into $\pi_3(\text{Selfmaps}(S^{n+1}))$, we would show that the total space is homotopy equivalent to a product. A little thought shows that $\pi_3(\text{Selfmaps}(S^{n+1}))$ is the same as the third stable homotopy group of spheres, which is finite by Serre's thesis. Combining this information, one quickly concludes that there are infinitely many manifolds homotopy equivalent to $S^4 \times S^{n+1}$ for large n, distinguished by p_1.

Much of our picture of high-dimensional manifolds comes from filtering the various strands arising in the above examples, analyzing them separately, and recombining them.

1.2. Before considering the parts that are most directly connected to operator K-theory, it is worthwhile to discuss the connection between the classification of manifolds and algebraic K-theory.

The aforementioned Reidemeister torsion invariant is an invariant of complexes defined under an acyclicity hypothesis. It is a computationally feasible shadow of a more basic invariant of homotopy equivalences, namely Whitehead torsion.

Let X and Y be finite complexes and $f : X \to Y$ a homotopy equivalence. Then using the chain complex of the mapping cylinder of f rel X or its universal cover, one obtains as before a finite-dimensional acyclic chain complex of based $\mathbb{Z}\pi$ chain complexes. The torsion $\tau(f)$ of f is the element of $K_1(\mathbb{Z}\pi)$ determined by means of the determinant, up to the indeterminacy of basis, which is a sign and element of π (viewed as a 1×1 matrix over the group ring). The quotient $K_1(\mathbb{Z}\pi)/\pm\pi$ is denoted by $\mathrm{Wh}(\pi)$.

A geometric interpretation of the vanishing of $\tau(f)$ is the following: say that X and Y are stably diffeomorphic (or, more naturally for this discussion, PL homeomorphic) if their regular neighborhoods in Euclidean space are diffeomorphic. The quantity $\tau(f)$ vanishes iff f is homotopic to a diffeomorphism between thickenings of X and Y. A homotopy equivalence with vanishing torsion is called a *simple homotopy equivalence.* As before, we recommend [**Coh73, Mil66**] for Whitehead's theory of simple homotopy and [**RS72**] for the theory of regular neighborhoods.

Remark. If we require X and Y to be manifolds, then one can ask that the stabilization only allow taking products with disks. Doing such does change the notion; the entire difference however is that we have discarded the topological K-theory. Two manifolds will be stably diffeomorphic in this restricted sense iff they have the same stable tangent bundle (in KO, or KPL for the PL analogue) and are simple homotopy equivalent. The proof of this fact is no harder than the polyhedral result.

1.3. Much deeper are unstable results. The prime example is Smale's h-cobordism theorem (or the Barden-Mazur-Stallings extension thereof).

Theorem 1. *Let M^n be a closed manifold of dimension at least 5; then $\{W^{n+1}$: M is one of two components of the boundary of W, and W deform retracts to both$\}$/diffeomorphism (or PL homeomorphism or homeomorphism) is in $1-1$ correspondence with Wh(π).*

The various W in the theorem are called h-cobordisms. The significance of this theorem should be obvious: it provides a way to produce diffeomorphisms from homotopy data. As such, it stands behind almost all of the high-dimensional classification theorems.

1.4. The proof that $\mathrm{Wh}(0) = 0$ is an easy argument using linear algebra and the Euclidean algorithm. Thus, in the simply-connected case the h-cobordisms are products. In particular, every homotopy sphere is the union of two balls and an h-cobordism that runs between their boundaries. The h-cobordism theorem asserts that the h-cobordism is just an annulus; since the union of a ball and an annulus is a ball, one can show that every homotopy sphere can be obtained by glueing two balls together along their boundary. This result implies the Poincaré conjecture in high dimensions: every homotopy sphere is a PL sphere. Using versions for manifolds with boundary, one can quickly prove the following theorems.

THEOREM 2 (Zeeman unknotting theorem). *Every proper embedding*[1] *of* D^n *in* D^{n+k} *for* $k > 2$ *is PL or differentiably trivial.*

THEOREM 3 (Rothenberg-Sondow Theorem). *If* p *is a prime number, smooth* $\mathbb{Z}_p$ *actions on the disk whose fixed set is a disk of codimension exceeding* 2 *are determined by an element of* $Wh(\mathbb{Z}_p)$ *and the normal representation at a fixed point.*

In the topological setting, the actions in the Rothenberg-Sondow theorem are conjugate iff the normal representations are the same. However, all known proofs of this claim are surprisingly difficult. Although Whitehead torsion is a topological invariant for closed manifolds, the situation is much more complicated for problems involving group actions and stratified spaces. Unfortunately, this topic cannot be discussed here, but see [**Ste88, Qui88a, Wei94**].

The group $\mathrm{Wh}(\mathbb{Z}_p)$ is free abelian of rank $(p-3)/2$; it is detected by taking the determinant of a representative matrix and mapping the group ring $\mathbb{Z}[\mathbb{Z}_p]$ to the ring of integers in the cyclotomic field associated to the p-th roots of unity. According to Dirichlet's unit theorem, the group of units of this number ring has rank $(p-3)/2$.

1.5. In general, there have been great strides in calculating $\mathrm{Wh}(\pi)$ for π finite (see [**Oli88**]). We will see later that $\mathrm{Wh}(\pi)$ is conjecturally 0 for all torsion-free groups, and that there is even a conjectural picture of what $\mathrm{Wh}(\pi)$ "should" look like in general.

This picture looks even stronger when combined with higher algebraic K-theory. Remarkably, the best general lower bounds we have for higher algebraic K-theory are based on the ideas developed for application in operator K-theory, namely, cyclic homology. See [**BHM93**]. These results have implications for lower bounds on the size of the higher homotopy of diffeomorphism groups.

1.6. The h-cobordism theorem removes the possibility of any bundle theory, since bundles over an h-cobordism are determined by their restrictions to an end.[2]

[1]This means that the boundary is embedded in the boundary.

[2]This claim is correct only when we discuss stable bundle theory; there is room for unstable information from the way in which the two boundary components destabilize the "same" stabler tangent bundle of the interior. This information does actually arise in the topological setting, and reflects a relationship between the destabilization of bundle theory and algebraic K-theory. Note that the "local structure" around points in the interior of the h-cobordism is the product of $\mathbb{R}$ and the local structure at a boundary point; equivalently, in the case of manifolds, the dimension of a manifold is one more than the dimension of its boundary. Hence the "tangential data" on the interior is "stabler" than the data on the boundary.

A key to understanding the role of bundles, unstably, is provided by Wall's π-π theorem (as reformulated using work of Sullivan):

THEOREM 4. *Let M be a manifold with boundary such that $\partial M \to M$ induces an isomorphism of fundamental groups, and let $S(M) = \{(M', f) \mid f : (M', \partial M') \to (M, \partial M)$ is a simple homotopy equivalence of pairs$\}/\mathrm{Cat}$ isomorphism. There is then a classifying space depending on the category, denoted F/Cat, such that $S(M) = [M : F/\mathrm{Cat}]$. This $S(M)$ is called the* (Cat-) *structure set of M.*

If M is noncompact, one can analogously define $S^p(M)$, the proper structure set, using proper homotopy equivalences. This is all explained in [**Wal99, Bro72**].

1.7. Much is known about the various F/Cat. For the duration we shall assume that Cat = Top. In that case, first of all, one has a complete analysis of F/Cat (due mainly to Sullivan, with an assist by Kirby and Siebenmann):

$$\begin{aligned}[M : F/\mathrm{Top}] \otimes \mathbb{Z}_{(2)} \quad &\cong \quad H^4(M; \mathbb{Z}_{(2)}) \oplus H^8(M; \mathbb{Z}_{(2)}) \oplus H^{12}(M; \mathbb{Z}_{(2)}) \oplus \cdots \\ &\qquad \oplus H^2(M; \mathbb{Z}_2) \oplus H^6(M; \mathbb{Z}_2) \oplus H^{10}(M; \mathbb{Z}_2) \oplus \cdots\end{aligned}$$

at 2, and away from 2,

$$[M : F/\mathrm{Top}] \otimes \mathbb{Z}[1/2] \cong \mathrm{KO}^0(M) \otimes \mathbb{Z}[1/2].$$

In the second formula, a structure is associated to (the "Poincaré dual" of) the difference of the signature operators on domain and range. In fact, all the formulas turn out to work much better in Poincaré dual form; the π-π classification should then be given by $S(M^n) \cong H_n(M; \mathbb{L})$, where $\mathbb{L}$ is the spectrum whose homotopy type is determined by the above calculations in cohomology. The reason for this terminology will become clearer as we progress. The connection to the signature operator is hopefully suggestive as well. (For these topics, see [**MM79**] and [**RW**].)

1.8. The material in the previous subsections gives rise to a complete analysis of $S(M)$ for M closed and simply-connected. Let $\widehat{M}$ denote M with a little open ball removed. Then $S(M) = S(\widehat{M})$ by the Poincaré conjecture. The latter satisfies the hypotheses of the π-π theorem, and thus $S(M) \cong H_n(\widehat{M}; L)$. Concretely, up to finite indeterminacy, the structure set is determined by the differences between the Pontrjagin classes $p_i(M)$ for $4i < n$.

What about p_i when $4i = n$? The answer is that it is determined by the lower Pontrjagin classes. The reason is that the Hirzebruch signature theorem asserts that $\mathrm{sign}(M^{4i}) = \langle L_i(M), [M] \rangle$. Here, the quantity $\mathrm{sign}(M)$ is the signature of the inner product pairing on H^{2i} of the oriented manifold M, and L is a graded polynomial in the Pontrjagin classes of M. This formula has many remarkable consequences. For instance, Milnor used it to detect exotic spheres. However, for us, it first implies that a particular combination of Pontrjagin classes is homotopy invariant. As a second point, Hirzebruch's formula can be viewed as a simple application of the Atiyah-Singer index theorem [**APS76**].

1.9. For general non-simply connected manifolds, there may exist further restrictions on the variation of the Pontrjagin classes, and there may exist more manifolds with the same tangential data. We shall deal with each of these possibilities one at a time. Although the complete story must necessarily involve interesting finite-order invariants, we shall concentrate on the $\otimes\mathbb{Q}$ story which, at our current level of ignorance, seems to be closely tied to analysis. Said slightly differently, the whole known and even conjectured story with $\otimes\mathbb{Q}$ can be explained analytically. However, no one has any direct approach to obtaining isomorphisms between L-theory and operator K-theory, and as we shall explain in the epilogue, this connection seems unlikely.

1.10. The Novikov conjecture is the assertion that, if $f : M \to B\pi$ is a map, then the image of the Poincaré dual of the graded L-class of M in $\oplus H_{n-4k}(B\pi;\mathbb{Q})$ is an oriented homotopy invariant. Note that, for the homotopy equivalent manifold, one must use the obvious reference map to $B\pi$ obtained by composing the homotopy equivalence with f.

For π trivial, this statement is a consequence of the Hirzebruch signature theorem. In fact, the Novikov conjecture is known for an extremely large class of groups at present. We will describe some of this work in the next section.

1.11. It is worth noting that the cases for which the Novikov conjecture is known are the only combinations of Pontrjagin classes that can be homotopy invariant. This claim can be proven axiomatically from the simply-connected case together with cobordism of manifolds and the π-π theorem. However, we shall "take the high road," and use the surgery exact sequence, and work for simplicity in the topological category. In this venue, we assert that, for M a compact closed manifold of dimension at least 5, there is an exact sequence,

$$\cdots \to L_{n+1}(\pi_1) \to S(M) \to H_n(M,\mathbb{L}) \to L_n(\pi_1) \to \cdots$$

where the L are 4-periodic, purely algebraically defined groups, and covariantly functorial in $\pi_1 = \pi_1 M$.

If M has boundary and if one is working rel boundary then the same sequence holds. For manifolds with boundary, and for working not rel boundary, the sequence changes by the presence of relative homology groups and relative L-groups $L(\pi_1, \pi_1^\infty)$; the π-π theorem then reduces to the statement that $L(\pi,\pi) = 0$, which is perfectly obvious from the exact sequence of a pair (which indeed does hold in this setting).

We can do better by taking advantage of the periodicity.[3] Let M be an n-manifold, and define $S_k(M) = S(M \times D^j)$ for any j such that $n + j - k$ is divisible by 4. With that notation, the sequence becomes

$$\cdots \to L_{n+1}(\pi_1) \to S_n(M) \to H_n(M,\mathbb{L}) \to L_n(\pi_1) \to \cdots$$

(with obvious relative versions). With this notation, one can then say that the sequence is a covariantly functorial sequence of abelian groups and homomorphisms. The push-forward map on structures (elements of S-groups are called "structures")

[3]Periodicity is not quite true in the topological category: it can fail by a copy of $\mathbb{Z}$, if M is closed, and cannot fail if M has boundary. See [**Nic82**], and see also [**BFMW96**] for the geometric explanation and repair of this failure.

is closely related to the push-forward of elliptic operators of Atiyah and Singer [**AS68a**] although defined very differently.

The functoriality implies that one can define $S_n(X)$ for any CW complex X just by taking the direct limit of $S_n(X^k)$ as X^k runs though any ascending union of sub-CW-complexes whose union is X. (Note that homology and L-theory both commute with direct limits.) Consequently, the map $H_n(M, \mathbb{L}) \to L_n(\pi_1)$ factors through the map $H_n(B\pi_1, \mathbb{L}) \to L_n(\pi_1)$. The latter is called the *assembly map*. For π trivial, the classification of simply-connected manifolds explained in Subsection 1.8 implies that the assembly map for a trivial group is an isomorphism. (Hence the homology theory introduced in Subsection 1.7 has L-groups as its homotopy groups, explaining the source of the notation.) The groups $L_i(\{e\}) = \mathbb{Z}, 0, \mathbb{Z}_2, 0$ for $i = 0, 1, 2, 3 \bmod 4$, respectively—exactly the homotopy groups of F/Top mentioned above.

The commutativity of the diagram

$$\begin{tikzcd}
H_n(M, \mathbb{L}) \arrow[r] \arrow[d] & L_n(\pi_1) \arrow[d] \\
H_n(B\pi_1, \mathbb{L}) \arrow[r] & L_n(\pi_1)
\end{tikzcd}$$

quickly implies that the only possible restriction on the characteristic classes comes from the difference of the L-classes in $H_n(B\pi_1, \mathbb{L})$. Moreover, the homotopy invariance of the higher signatures is exactly equivalent to the rational injectivity of the assembly map.

1.12. A similar discussion applies to manifolds with boundary. We leave it to the reader, with references to [**Wei99, Wei90**] for the impatient reader. For instance, the π-π theorem implies that there are no homotopy invariant characteristic classes of π-π manifolds. The extended higher signature conjecture would have posited the proper homotopy invariance of the L-class in $H_n(B\pi_1, B\pi_1^\infty) = 0$.

1.13. Although rational injectivity of the assembly map is conjectured to be universal, surjectivity is not. The simplest example of this notion comes from the Hirzebruch signature formula. Note that the right-hand side of the formula

$$\mathrm{sign}(M) = \langle L(p^*(M)), [M] \rangle$$

is clearly multiplicative in coverings: if $N \to M$ is finite covering, the L-classes pull back, but the fundamental class is multiplied by the degree of the covering. This argument implies that, for closed manifolds, signature is multiplicative in coverings.

Note that, as a consequence, if M and M' are homotopy equivalent and cobordant by a cobordism V, then an obstruction to the homotopy equivalence being homotopic to a diffeomorphism can be obtained by gluing the boundary components of V together to obtain a Poincaré duality space, which might well not satisfy the multiplicativity of signature. In fact, this method can be extended further, as noted by [**Wal99**] and [**APS75a, APS76**], and underlies the proof of de Rham's theorem on lens spaces given in [**AB68**]. Suppose for simplicity that M has fundamental group π, and so does the cobordism V mentioned above. Then the cohomology of the universal cover of V has a π action on it. The *equivariant signature* of this quadratic form can be shown to be a multiple of the regular representation;

i.e. each character except for the one corresponding to the trivial element must vanish. Atiyah and Bott had computed these characters for the lens space situation in the course of their argument. The multiplicativity issue is equivalent to the vanishing of the average of these characters.

REMARK. The multiplicativity invariant can be used even if the fundamental group is infinite: one must use the von Neumann signature of the universal cover in place of ordinary signature. (The relevant multiplicativity is Atiyah's L^2 signature theorem.) This is the key point in the proof of the following flexibility theorem, perhaps one of the simplest general applications of analytic methods that does not yet have a purely topological proof:

THEOREM 5 ([**CW03**]). *If M^{4k+3} has non-torsion-free fundamental group, $k > 0$, then $S(M)$ is infinite. (This theorem fails in all other dimensions, except 0 and 1 when the hypothesis is vacuous – at least if the Poincaré conjecture is true.)*

1.14. The "trick" of the previous subsection can sometimes be turned into a method or an invariant, a so-called secondary invariant, even in situations where the manifolds are not (a priori known to be) cobordant. A first example arises in the situation of the previous subsection. If M is an odd-dimensional manifold with finite fundamental group, then some multiple sM of it bounds a manifold with the same fundamental group. One can then consider the signature of the universal cover of that manifold, multiplied by $1/s$ to correct for the initial multiplication. (Signature can be defined for manifolds with boundary just by throwing away the singular part of the inner product.)

A much deeper way of accomplishing the same task, which applies in some circumstances where the cobordism group is not torsion, is due to [**APS75a**], who defined a real-valued invariant of odd-dimensional manifolds with finite-dimensional unitary representations of their fundamental groups. If the image of the representation is a finite group, it reduces to what we just considered above, but in general it is much more subtle. Years ago, the first author conjectured that:

CONJECTURE 1. *If $\pi_1 M$ is torsion-free, then for any unitary representation ρ, the Atiyah-Patodi-Singer invariant is homotopy invariant; in general it is an invariant, up to a rational number.*

The second statement was proven in [**Wei88**] as an application of known cases of the Novikov conjecture. In the original paper, it was shown to follow from the Borel conjecture. Keswani [**Kes98**] proved it for a class of groups, such as amenable groups, assuming a version of the Baum-Connes conjecture.

Cheeger and Gromov [**CG85**] considered the von Neumann analogue of this discussion. Mathai [**Mat92**] made the analogous conjecture to the one above: that the Cheeger-Gromov invariant is homotopy invariant for manifolds with torsion-free fundamental group. Special cases are verified in [**Mat92, Kes00, CW03, Cha04**], in all cases using Novikov-like ideas. The flexibility result of the previous subsection is the converse to Mathai's conjecture.

Finally, we should mention the ideas of [**Lot92**] and [**Wei99**] which define "higher" versions of these secondary signature-type invariants in situations in which the Novikov conjectures provide for the existence of higher signatures to be definable (in a homotopy-invariant fashion). Unlike the classical secondary invariants, these ideas require some cohomological vanishing condition, rather like Reidemeister torsion. We will postpone further discussion of these subjects until Subsection 1.17.

1.15. We have seen that the entire classification of simply-connected manifolds follows essentially from the Poincaré conjecture (Smale's theorem) and the formal structure of surgery theory. The same result is true for any fundamental group: understanding any manifold with that fundamental group well enough will determine the classification theory for all. The Borel conjecture, or topological rigidity conjecture, is the following:

CONJECTURE 2. *If M is an aspherical manifold and $f : M' \to M$ is a homotopy equivalence, then f is homotopic to a homeomorphism.*

In fact, it is reasonable to extend the conjecture to manifolds with boundary and homotopy equivalences f that are already homeomorphisms on the boundary. (Similarly, one can deal with proper homotopy equivalences between noncompact aspherical manifolds that are assumed to be homeomorphisms in the complement of some unspecified compact set.)

Notice that the Borel conjecture implies that $\mathrm{Wh}(\pi) = 0$ for the fundamental group of an aspherical manifold (exercise!). Note also that, by enlarging our perspective to include the noncompact case, the aggregate of π to which the conjecture applies is the set of countable groups of finite cohomological dimension. (In fact, it is pretty obvious, by direct matrix considerations, that one can remove the countability, if one so desires!) Furthermore, by feeding the problem into the surgery exact sequence, one obtains in addition the statement that $A : H_n(B\pi_1, \mathbb{L}) \to L_n(\pi_1)$ is an isomorphism for all n. Indeed, the Borel conjecture (for all n) is equivalent to the truth of these two assertions.

Much is known about the Borel conjecture; so far, no one knows of any counterexample to the claim that these algebraic assertions hold for all finite groups. (Note that for all groups with torsion, the map A fails to be a surjection by the flexibility theorem for n divisible by 4.)

1.16. It is probably worthwhile to discuss the motivation for the Borel conjecture and its variants. Reportedly, Borel asked this question in response to the theorems of Bieberbach and Mostow about the classification of flat and hyperbolic manifolds, respectively. In the first case, an isomorphism of fundamental groups gives an affine diffeomorphism between the manifolds and in the second (assuming the dimension is at least three) it gives an isometry (which is unique). Since symmetric manifolds of noncompact type are all aspherical, Borel suggested that perhaps this condition should be the topological abstraction of a symmetric space, and that, without assuming a metric condition, one should instead try for a homeomorphism.

In light of this suggestion, it is worthwhile to consider the noncompact version. For noncompact hyperbolic manifolds of finite volume (the nonuniform hyperbolic lattices), Mostow's rigidity theorem remains true (although its failure in dimension 2 is even more dramatic: the homotopy type does not even determine the proper homotopy type of the manifold) as was proven by Prasad.

It might therefore seem reasonable (as was done in at least one ICM talk!) to suggest that Borel's conjecture could be extended to properly homotop a homotopy equivalence to a nonuniform lattice quotient to a homeomorphism. The following result that we proved with a theorem of Alex Lubotzky shows that this situation never occurs. (We shall give a different nonuniform rigidity theorem in Part 2.)

THEOREM 6 ([**CWb**]). *Suppose that* Γ *is a nonuniform irreducible arithmetic lattice in a semisimple Lie group* G*. Let* K *be the maximal compact subgroup of* G*. If* $\mathrm{rank}_{\mathbb{Q}}(\Gamma) > 2$*, then there is a non-properly-rigid finite-sheeted cover of* $\Gamma\backslash G/K$*.*

In Section Two we will explain why it is extremely likely that proper rigidity holds if $\mathrm{rank}_{\mathbb{Q}}(\Gamma) = 1$ or 2. (If $\mathrm{rank}_{\mathbb{Q}}(\Gamma) = 0$, then the lattice is cocompact by the well-known theorem of Borel and Harish-Chandra.)

REMARK. In fact, if the $\mathbb{R}$-rank is large enough and $\mathrm{rank}_{\mathbb{Q}}(\Gamma) > 2$, then one can construct infinite structure sets with nontrivial elements detected by Pontrjagin classes (*e.g.* for $\mathrm{SL}_n(\mathbb{Z})$ for n sufficiently large, using Borel's calculations). Unlike the elements constructed in Theorem 6, these elements do not die on passage to further finite-sheeted covers. Note that, for a product of three punctured surfaces, the proper rigidity conjecture is always false (for any cover), but is virtually true, in that any counterexample dies on passing to another finite cover!

1.17. We now shall consider a much more fruitful (but still false) conjecture suggested by the heuristic that led to the Borel conjecture: the "equivariant Borel conjecture" or "equivariant topological rigidity conjecture." Notice that Mostow rigidity actually immediately implies the following seeming strengthening of itself.

THEOREM 7. *Suppose that* M *and* N *are hyperbolic manifolds, and* $f : \pi_1 N \to \pi_1 M$ *is an isomorphism which commutes with the representation of a group* G *on* $Out(\pi)$ *induced by actions of* G *by isometries on* M *and* N*. Then there is a unique isometry between* M *and* N *(realizing* f*) which conjugates the* G*-actions to each other.*

Mostow rigidity is the case of this theorem when G is trivial; on the other hand, since the isometry between M and N realizing any given group isomorphism is unique, it must automatically intertwine any actions by isometries that agree on fundamental groups. So let us now make another conjecture:

CONJECTURE 3. *Suppose that* G *acts aspherically and tamely on a compact closed aspherical manifold* M*, and that* $f : N \to M$ *is an equivariant homotopy equivalence. Then* f *is homotopic to a homeomorphism.*

The condition that the action be tame means that one assumes that all components of all fixed point sets are, say, locally flatly embedded topological submanifolds, and the asphericality means that these components are all aspherical. This condition means that these spaces are the terminal objects in the category of spaces which are connected to a given one by equivariant 1-equivalences; *i.e.* one considers maps $X \to Y$ which induce isomorphisms $[K, X]^G \to [K, Y]^G$ for any G-1-complex, *i.e.* a 1-complex with a G-action. See [**May**].

In fact, this conjecture is false for several different reasons. However, it points us in the right direction. For one, its analytic analogue is the celebrated Baum-Connes conjecture. For a second, its "Novikov shadow" does seem to be true:

CONJECTURE 4 (Equivariant Novikov conjecture, see [**RW90**]). *Suppose that G acts tamely and aspherically on a finite-dimensional space X, and that $f : M \to X$ is an equivariant map. Then for any equivariant map $g : N \to M$ which is a homotopy equivalence, one has $f_*g_*(\Delta(N)) = f_*(\Delta(M))$ in $K_*^G(X)$, where Δ denotes the equivariant signature operator.*

The hypothesis that G acts tamely is point-set theoretic; smoothness is certainly more than enough. For example, this conjecture holds whenever X is a symmetric space of noncompact type and G is a compact group of isometries of X. It is also worth noting that one can often build equivariant maps from any M with the appropriate fundamental group to this X using harmonic map techniques; see [**RW90**].

An analysis of this conjecture (for the case of discrete G) is that, if G is the "orbifold fundamental group of X," *i.e.* $1 \to \pi_1 X \to \Gamma \to G \to 1$ is exact, then $KO_*^G(X) \otimes \mathbb{Z}[1/2]$ must inject into $L(\Gamma) \otimes \mathbb{Z}[1/2]$. In the next subsections we will discuss more refined estimates of $L(\Gamma)$. Working rationally, we see where we went wrong in our understanding of $L(\Gamma)$. Before our estimate was $KO_n(B\Gamma) \otimes \mathbb{Q} = KO_n(X/G) \otimes \mathbb{Q}$, which differs a great deal from $KO_n^G(X) \otimes \mathbb{Q}$ because of the fixed-point sets. If X is a point, we see that the representation theory of G enters, exactly as we saw before in our discussion of secondary invariants. In fact, most of the higher secondary invariants, when they are defined, take values in $\mathrm{KO}_{n+1}^G(EG, X) \otimes \mathbb{Q}$.

REMARK. The conjecture that these refined lower bounds for L-theory hold universally would imply the infiniteness of structure sets proven above using L^2 signatures.

1.18. Unfortunately, the equivariant Borel conjecture is false. The first source of counterexamples discovered was related to the Nil's of algebraic K-theory [**BHS64, Wal78a**]. Soon thereafter analogous counterexamples were discovered based on Cappell's Unils [**Cap74b**]. See [**CK91**] for a discussion of these examples. The full explanation requires an understanding of equivariant h-cobordism and classification theorems; we cannot describe these topics here, but recommend the surveys [**CWa, HW01, Wei94**].

Let us begin with the equivariant h-cobordism theorem. According to Steinberger and West's analysis (Quinn provided a more general version for all stratified spaces), one has an exact sequence:[4]

$$\begin{aligned} \cdots \to H_*(M/G; \mathrm{Wh}(G_m)) \to \mathrm{Wh}(G) \to \mathrm{Wh}^{\mathrm{top}}(M/G \text{ relsing}) \\ \to H_*(M/G; K_0(G_m)) \to K_0(G) \to K_0^{\mathrm{top}}(M/G \text{ relsing}) \\ \to H_*(M/G; K_{-1}(G_m)) \to \cdots \end{aligned}$$

where G_m is the isotropy of the point m. Here the "rel sing" means that we are considering h-cobordisms which are already products on the singular set; note that, unlike the smooth case, this condition does not give us a neighborhood of the set on which it is a product. It is precisely that which is measured by the homology term.

[4]For the development of of such homology groups, see [**Qui82**] and the appendix of [**Wei94**] for homology with coefficients in a cosheaf of spectra. Note that it is an analogue of generalized cohomology theories and of sheaf cohomology.

Perhaps the connection between a Whitehead group and K_0 seems odd. This interaction is analogous to (and actually stems from) a phenomenon studied by Siebenmann, arising from his thesis. Siebenmann discovered that the h-cobordism theorem can be extended from the situation of compact manifolds to a wide range of noncompact manifolds. A condition that renders the statements much simpler is "fundamental group tameness," which asserts that there is an ascending exhausting sequence of compact sets $K_1, K_2, \dots$ in W, such that the maps $M \backslash K_1 \leftarrow M \backslash K_2 \leftarrow M \backslash K_3 \leftarrow \cdots$ are all 1-equivalences. Let us assume that W has one end (so these complements are all connected). Then we denote the common fundamental group of the complements by $\pi_1^\infty W$. According to Siebenmann, there is a map

$$\mathrm{Wh}^p(W) \to \mathrm{Wh}(\pi_1 W, \pi_1^\infty W)$$

which thus fits into an exact sequence

$$\begin{aligned} \cdots \to \mathrm{Wh}(\pi_1^\infty W) \to \mathrm{Wh}(\pi_1 W) \to \mathrm{Wh}^p(W) \\ \to KO(\pi_1^\infty W) \to KO(\pi_1^\infty W) \to \cdots \end{aligned}$$

Note that when the fixed set consists of isolated points, this exact sequence for Wh^p of the orbit space of the free part is the same as the Steinberger-West sequence.

If W is the interior of a compact manifold with boundary, then the target of the boundary map $\mathrm{Wh}^p(W) \to K_0(\pi_1^\infty W)$ measures the obstruction to completing the h-cobordism as a manifold with corners. The homology term is analogous to a controlled K_0 or Wh; we will return to controlled algebra in Section Two. In the Whitehead story, it turns out that $\mathrm{Wh}^{\mathrm{top}}$ decomposes into a sum of terms, one for each stratum of M/G, each of the form $\mathrm{Wh}^{\mathrm{top}}(Z/H\,\mathrm{relsing})$ for some Z and some H. For surgery theory, this decomposition does not hold, and the strata interact in a much more interesting way.

In any case, we now consider some particularly simple equivariantly aspherical manifolds, and understand what is implied by the vanishing of $\mathrm{Wh}^{\mathrm{top}}$. Let $M = D^n$ with a linear action. Then the homology would be concentrated at the origin. The map $H_*(M/G; \mathrm{Wh}(G_m)) \to \mathrm{Wh}(G)$ is an isomorphism (and similarly for K_0), and indeed $\mathrm{Wh}^{\mathrm{top}}(M) = 0$. Now let us consider $M = S^1 \times D^n$. Again the homology is concentrated entirely on the singular part, and we have

$$H_*(S^1; \mathrm{Wh}(G)) \cong \mathrm{Wh}(G) \times K_0(G) \to \mathrm{Wh}(\mathbb{Z} \times G).$$

This map is indeed a split injection, but it is not an isomorphism. The cokernel is $\mathrm{Nil}(G) \times \mathrm{Nil}(G)$ according to the "fundamental theorem of algebraic K-theory" [**Bas68**]. These Nil groups are rather mysterious. A general theorem of Farrell shows that Nil is infinitely generated if it is nontrivial. Some calculations can be found in [**BM67**] and [**CdS95**].

Similarly, in L-theory, the equivariant Borel conjecture would imply calculational results about the L-theory of, say, linear groups. The map $H_*(M/G; L(G_m)) \to L(G)$ should be an isomorphism when M is aspherical. (Incidentally, away from 2, the left-hand side can be identified with $KO^G[1/2]$.) It is also not so hard to change the context of all of this discussion from finite quotients of aspherical manifolds to proper actions on contractible manifolds. However, these conjectures were already disproved by Cappell's results on the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$. Cappell showed that $L_2(\mathbb{Z}_2 * \mathbb{Z}_2)$ is not just a sum of copies of $L_2(\mathbb{Z}_2)$ concentrated at fixed points, but rather that there is another infinitely generated summand that is not in the image of the relevant homology group.

These conjectures can be somewhat rehabilitated by considering the properties of Nil and Unil. For instance, one can ask about other rings besides integral group rings. The conjectures then lose some of their geometric immediacy, but with $\mathbb{Q}$, for instance, they stand a chance of being true. (For instance, whenever 1/2 is in the coefficient ring, Cappell's Unils vanish identically, and there is no need for any further corrections to the isomorphism conjecture.) Just as the "fundamental theorem of algebraic K-theory" is true for all rings, it is very worthwhile to understand to what extent the purported calculations apply to general rings, even to the point of mere split injectivity. One might mention at this point the wonderful result of Bökstedt, Hsiang and Madsen affirming the algebraic K-theoretic version of injectivity of the usual assembly map for $\mathbb{Z}$, after tensoring with the rationals, for groups with finitely generated integral homology [**BHM93**]. Unfortunately, their method does not apply to other rings, and does not directly imply anything about the algebraic K-theoretic injectivity statement raised here.

1.19. There is another more geometric reason why the equivariant Borel conjecture fails; the reasons are orthogonal to the algebraic problems discussed in the previous subsection, but are, undoubtedly[5], related to pseudoisotopy theory. This second failure occurs when the gap hypothesis does not hold.[6] Again, the construction of the counterexamples and their classification would take us rather far afield, but it seems worth mentioning one simple example: a crystallographic group.

Suppose that one looks at an action of $\mathbb{Z}_p$ on $T^{p-3} \times T^p$, where the action on the first coordinates is trivial, and on the second set is by permutation. The fixed point set is a T^{p-2} in T^{2p-3} exactly at the edge of dimensions for which it is possible for homotopic embeddings not to be isotopic. In fact, it is quite easy to build nonisotopic embeddings: take a curve in $\pi_1(T^{2p-3})$ and push a small sheet of the T^{p-2} around that curve and then link this little sheet to the original T^{p-2} some number of times. This construction does not completely determine the curve. Using the opposite linking, one can replace a curve by its opposite; in addition, curves that can be homotoped into T^{p-2} do not change the isotopy class of the embedding. But in essence one produces isotopy classes of embeddings of T^{p-2} isotopic to the original embedding (see [**Shi99**]), one for each element in $\mathbb{Z}[\mathbb{Z}^{(p-1)*}]^{\mathbb{Z}_2}$.

Cappell and the second author showed [**CWa**] in this special case that an embedding is the fixed set of some $\mathbb{Z}_p$-action equivariantly homotopy equivalent to the affine action iff the new embedding is isotopic to its translate under the $\mathbb{Z}_p$ action. Furthermore, the action is unique up to conjugacy. Note that the counterexamples exist even rationally; they do display a nilpotency. Any particular action is conjugate to the affine action after passing to a cover. Passing to any large enough cover, we find that the curves used for modifying the embedding no

[5] This word brazenly advertises that we are aware of no direct connections.

[6] In fact, there are general theorems of Shirokova [**Shi99**] which assert that the equivariant Borel conjecture is always false (under very weak assumptions) whenever the gap hypothesis fails. This gap hypothesis is the bane of the classification theory of group actions; it assumes that, whenever M^H lies in M^K, either these fixed sets coincide or one is somewhat less than half the dimension of the other. Necessary in the establishment of the foundations of equivariant surgery theory (see Memoirs of the AMS by Dovermann and Petrie [**DP82**] and by Dovermann and Rothenberg [**DR88**]), the condition is needed to allow for surgeries performed inductively over the strata. It is important to realize that the gap hypothesis is usually assumed to make progress, not at all because it is natural or generally true.

longer go around, and thus do not change the embedding. More generally, there are connections to embedding theory, but they are not quite as precise as in this special case [**Wei99**].

1.20. Farrell and Jones have suggested another very general way to handle the failure of these algebraic assembly maps to be an isomorphism. Essentially the idea is the following: we could have been led to our previous isomorphism conjecture by a somewhat different line of reasoning.

Observing that the assembly map $A : H_*(B\pi; \mathbb{L}) \to L_*(\pi)$ is not an isomorphism for groups with torsion, we could have looked for "the universal version of an assembly map that does not oversimplify $L(\pi)$ for π finite." For each Γ we might consider $\underline{E}\Gamma$, which has an equivariant map from $E\Gamma \to \underline{E}\Gamma$, and build an assembly map $H_*(\underline{E}\Gamma/\Gamma; L(\Gamma_x)) \to L(\Gamma)$. (The various Γ_x will run over finite subgroups of Γ.) Now that we see that even this modified conjecture fails for groups like $\mathbb{Z} \times \pi$, for π finite, and $\mathbb{Z}_2 * \mathbb{Z}_2$, they suggest reiterating the process, but now with respect to the "virtually cyclic groups," *i.e.* the groups with cyclic subgroups of finite index. Thus one builds a more complicated classifying space E^Γ, on which Γ acts, and uses it for an assembly map. One should be somewhat careful, because the action cannot be proper to give us the requisite infinite groups as isotropy, but it is quite simple to build the correct thing simplicially. We refer the interested reader to [**FJ93, DL98**] for more information.

1.21. It is worth making one more remark before closing this general section (*i.e.* one devoid of information about the conjectures themselves) about the connection to index theory and the geometric implications thereof.

This article is devoted to analogues in index theory of the Novikov conjecture and the Borel conjecture, indeed in all of the versions discussed above and in the ones to be discussed in Section Two. There is no need for the Farrell-Jones isomorphism ideas, because for virtually cyclic groups the Baum-Connes conjecture is true, unlike its Borel cousin. The topological side of these issues, for the moment, has additional complications arising from deep arithmetic connections. (The L-theory of finite groups, for instance, has a beautiful and important arithmetic side not visible in the theory of C^*-algebras). On the other hand, the analysis has beautiful connections to representation theory (some discussion on this subject will spill off into the Epilogue) and other geometric applications through other operators besides the signature operator.

These other applications can also suggest a variety of problems and methods. Two of these merit at least a mention here, although we do not have the space to develop them fully. The first is the work that a number of people have done on the (generalized) Hopf conjecture: that for any aspherical $2n$-manifold, the Euler characteristic is 0 or of sign $(-1)^n$. Methods of L^2 index theory applied to the de Rham (and Dolbeault) complex have given positive results in some spaces with negative curvature and Kähler structure (see [**Gro91**]). The second problem, now known to be very closely tied to the Baum-Connes conjecture, is the characterization of the closed[7] manifolds which have metrics of positive scalar curvature.

[7]This connection extends rather further into the setting of noncompact manifolds, as we will discuss in Section 2. Other noncompact instances will be mentioned later.

The connection between index theory and the positive scalar curvature problem already appears in the Annals papers of Atiyah and Singer [**AS68b**]; they prove in the same paper the Hirzebruch signature formula using the index theorem and a vanishing theorem for the $\widehat{A}$-genus of a spin manifold with positive scalar curvature (based on a key calculation of Lichnerowicz). Combining ideas that are intimately related to (partial results on) the Novikov conjecture, one can obtain information on the non-simply connected case. We recommend the papers [**GL80**], [**GL83**], [**Ros83**], [**Ros86a**], [**Ros86b**], [**Sto92**] and the forthcoming monograph by Rosenberg and Stolz that shows that a "stable" version of the positive scalar curvature problem can be completely solved if the Baum-Connes conjecture were true, or even just the (easier) injectivity half.

2. K-theory and Topology

This section is an introduction to some of the topological methods that have been applied to the conjectures made in the previous section. Unlike that section, which explained how K-theory contributes to topology, this one studies contributions that topology makes to K-theory and L-theory.

2.1. The same general technique used to prove the h-cobordism theorem (handlebody theory) was subsequently applied by a number of researchers to a host of other problems, which in light of surgery theory imply solutions to Novikov and Borel conjectures in special cases. Here are some of those problems:

(a) Putting a boundary on a noncompact manifold: Suppose that W is a noncompact manifold. When is W the interior of a compact manifold with boundary? Aside from homotopical or homological conditions at infinity, the answer is regulated by $K_0(\pi_1^\infty W)$. This idea was Siebenmann's thesis [**Sie**]. A nice special case due earlier to [**BLL65**] is that, if W is simply-connected at infinity, then W can be compactified iff its integral homology is finitely generated. You might want to go back now and review the proper h-cobordism discussion from the previous section.

(b) Fibering over a circle: Without loss of generality, suppose that one has a surjection from $\pi_1 M \to \mathbb{Z}$. When is this map induced from a fiber bundle structure on M over the circle? The obvious necessary condition is that the induced infinite cyclic cover of M should have some finiteness properties. Ultimately, the result is determined by $\mathrm{Wh}(\pi_1 M)$. The history here is somewhat complicated; see Farrell's thesis for an analysis of the problem as a sequence of obstructions in terms of pieces of $\mathrm{Wh}(\pi_1 M)$; Siebenmann [**Sie70**] gave a complete "one step" analysis of the problem. [**Far72**] sketches a very simple proof based on Siebenmann's thesis.

Remark. If one lightens the demand on the fibration to be an approximate fibration (see, *e.g.* [**HTW91**]), then the obstruction to an approximate fibration over S^1 lies entirely in the Nil piece of $\mathrm{Wh}(\pi_1 M)$. In this form, a slightly strong form of the Borel conjecture can be stated as follows:

CONJECTURE 5. *Suppose that M is a manifold and V is the cover of M induced by a homomorphism $\pi_1 M \to \Gamma$, where Γ is a group with $B\Gamma$ a finite complex. Then there is an aspherical homology manifold Z and an approximate fibration $M \to Z$ iff V is homotopically finite, and an obstruction involving the various Nil($\pi_1 V$) vanishes.*

Note that this conjecture implies that $B\Gamma$ is automatically a Poincaré complex; this implication can be verified directly. Otherwise, the space V is never finitely dominated.[8] When V is simply-connected, this conjecture boils down to the Borel conjecture for the group Γ (see the discussion in Chapter 13 of [**Wei94**] and also the introduction to [**HTWW00**]). Also, as a result of [**WW88**], if one wants to avoid discussion of approximate fibrations, in the special case that Z is a manifold, one can decide to allow $T \times M$ to fiber over Z (for some torus factor), and then one can remove the Nil obstruction as well.

(c) Splitting theorems: Here one has a homotopy equivalence $f : M' \to M$ and a codimension one submanifold N of M; the problem is to homotop f to a map, still called f, such that f is transverse to N, and $f^{-1}(N)$ is homotopy equivalent to N (mapped to one another by f). The ultimate theorem in this direction is Cappell's splitting theorem, which applies whenever $\pi_1 N$ injects into $\pi_1 M$ and the normal bundle of N is trivial.

Earlier partial results are due to Wall, Farrell, Farrell-Hsiang, Lee and others.

Cappell [**Cap74b**] gave very useful conditions under which $\tau(f)$, the Whitehead torsion of the map, determines the obstruction. However, in general this claim does not hold: in [**Cap74a**] he gave infinitely many PL manifolds homotopy equivalent to $\mathbb{RP}^{4k+1} \# \mathbb{RP}^{4k+1}$ that are not connected sums. This example is responsible for some of the instances of non-rigid affine crystallographic group actions on Euclidean space discussed in the last section.

Note that the fibering theorem gives some situations in which one can analyze the splitting problem, and one can show in fact (using some surgery theory) that the splitting theorem and the fibering problem are equivalent for the class of groups that arise in the latter problem: The fibering problem reduces to an analysis of groups that act simplicially on the line, and the splitting theorem to those that act on some tree.

2.2. The translation of fibering and splitting theorems was done first by Shaneson in his thesis [**Sha69**]; see also Wall's book [**Wal99**]. This translation led to the first proofs of the Borel conjecture for tori by Hsiang-Shaneson and Wall (the same proof works verbatim for poly-$\mathbb{Z}$ groups); Farrell and Hsiang had earlier given a proof of the Novikov conjecture for the free abelian case. Cappell's paper [**Cap74b**] gives the Mayer-Vietoris sequence in L-theory for groups acting on trees associated to his splitting theorem. The corresponding theorems in algebraic K-theory are due to Waldhausen [**Wal78a**] and in operator K-theory to Pimsner [**Pim86**]. However, as is now extremely well known, most interesting groups do not act at all on trees. (See Serre [**Ser80**] for an early example.)

[8]According to Wall, this condition is equivalent to a chain complex condition.

2.3. The vanishing of algebraic K-groups and the Borel conjectures were next proved for the class of flat and almost flat manifolds in a very beautiful and influential paper of [**FH83**]. This paper combined a variant of Brauer's induction theory from classical representation theory, due to Dress [**Dre75**], with controlled topology methods. These methods could have been adapted (more easily, in fact) to index theory, but there never seemed to be a need for it. They were however applied successfully to crystallographic groups with torsion in algebraic K-theory by [**Qui88b**] and to L-theory by Yamasaki [**Yam87**]. These papers were very influential in formulating the cruder isomorphism conjectures mentioned in Subsection 1.17 of Section One. (A more perspicacious mathematical community could have done so on the basis of thinking carefully about proper actions on trees, which can be analyzed on the basis of the theorems of Waldhausen and Cappell.)

2.4. To develop controlled topology one must redo all of the classical topology problems such as those mentioned in Subsection 2.1 (for example, putting boundaries on open manifolds), but in addition keep track of the size of these constructions in some auxiliary space. Here is the classical example.

THEOREM 8 ([**CF79**]). *Suppose that M is a compact manifold. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that, if $f : N \to M$ is a δ-controlled homotopy equivalence, then it is ϵ-homotopic to a homeomorphism.*

Now for the definitions. A *δ-controlled homotopy equivalence* is a map $f : N \to M$, equipped with a map $g : M \to N$, so that the composites fg and gf are homotopic to the identity by homotopies H and H', such that the tracks (*i.e.* the images of $(H')^{-1}(p,t)$ as t varies, for any specific p) of all of these homotopies (perhaps pushed using f) in M have diameter less than δ. A similar definition holds for ϵ-homotopy.

The result stated (called the α-approximation theorem) is an example of a rigidity theorem. While it clarifies the idea of control, the following theorem of Quinn [**Qui79, Qui82**] separates the geometric problem from the control and also has obstructions, and thus gives a better feel for the subject.

THEOREM 9 (Controlled h-cobordism theorem). *Let X be a finite-dimensional ANR (e.g. a polyhedron). Then for all $\epsilon > 0$ there is a $\delta > 0$ such that, if $f : M^n \to X$ is a map with all "local fundamental groups $= \pi$" (e.g. if there is a map $M \to B\pi$ which when restricted to any fiber of f is an isomorphism on fundamental groups), $n > 4$, then any δ-h-cobordism with boundary M defines an element in $H_0(X; Wh(\pi))$; this element vanishes iff the h-cobordism is ϵ-homeomorphic over X to $M \times [0,1]$. Moreover, every element of this group arises from some h-cobordism.*

Notice two extremes: If X is a point, this result is the classical h-cobordism discussed in Section One, Subsection 1.3. If $M = X$ then this result gives a metric criterion (due to Chapman and Ferry) that can be used to produce product structures, since $\mathrm{Wh}(e) = K_0(e) = K_{-i}(e) = 0$. This result includes the celebrated result of Chapman that the Whitehead torsion of a homeomorphism vanishes. Note that we need all the negative K-groups because Wh is a spectrum, so all of its homotopy contributes to the homology groups. The main theorems of controlled topology assert that various types of controlled groups are actually groups that are parts of homology theories. See [**CF79, Qui79, Qui82, Qui88a, FP95, Wei94**] for more information.

2.5. We give a cheating application of the α-approximation theorem to rigidity phenomena. Suppose that $f : M \to T$ is a homotopy equivalence to a torus. Now pass to a large finite cover; the target is still a torus, which we identify with the original one by the obvious affine diffeomorphism. Now we have a new map which is an α-approximation. Thus, all sufficiently large covers of M are tori. Unfortunately, this proof is somewhat circular (at least for the original proof of α-approximation which used the classification of homotopy tori.) However, a slight modification of this argument shows that any embedding of the torus in another torus of codimension exceeding two, homotopic to an affine embedding, is isotopic to the affine embedding in all sufficiently large covers. It also can be used to show that any sufficiently large cover of a homotopy affine G-torus is affine (see [**Ste88**]). To reiterate, as we saw before, the counterexamples to equivariant Borel mentioned in Section One, Subsections 1.18 and 1.19, die on passage to covers. Controlled topology implies that they all do.

2.6. There are, by now, a number of other versions of control in the literature, which, while fun for the experts, can be somewhat bewildering to the beginner. Some of these are: bounded control [**Ped95, AM88, FP95, HTW91**], continuous control at infinity [**ACFP94, Ped00**], and foliated control [**FJ86, FJ87**]. These theorems have all enjoyed applications to rigidity and to the Novikov conjecture. They are also important in other topological problems. To give the flavor of one of these variants, let us discuss the bounded theory.

Definition 1. Let X be a metric space. A *space over* X is a space M equipped with a map $f : M \to X$. The map f need not be continuous, but it is usually important that it be proper. A map between spaces (M, f) and (N, g) over X is a continuous map $h : M \to N$ such that $d_X(f(m), gh(m)) < C$ for some C. Since C can vary, this construction forms a category.

If X has bounded diameter, this category is essentially equivalent to the usual category of spaces and maps. But things heat up a lot when X is as simple as the real line or Euclidean space. Note that it is easy to define the homotopy category over X, and thus notions of h-cobordism over X, homotopy equivalence and homeomorphism can all be defined "over X." Note also that the all-important fundamental group must be generalized in this setting. Without any additional hypothesis, this generalization can be complicated, but the following condition is often sufficient, especially for problems involving torsion-free groups:

Definition 2. The *fundamental group of* (M, f) *over* X is π if there are constants C and D, and a map $u : M \to B\pi$ such that, for all $x \in X$, the image of π_1 of the inverse image of the ball of radius C about x inside the inverse image of the ball of radius D is isomorphic to π via the map u.

For simplicity we will assume that this condition holds, unless otherwise stated.

Example. If M is a space with fundamental group Γ, then its universal cover is a space over Γ, where Γ is given its word metric. It is in fact simply-connected over Γ. (A typical "bad example" would be an irregular cover, *e.g.* the cover of M corresponding to finite subgroups (which plays an important role in understanding groups with torsion.)

Remark. The analogue in this setting of being contractible is being uniformly contractible; *i.e.* there is a function f such that, for all x in X and all C, the ball $B_x(C)$ is nullhomotopic in $B_x(f(C))$. Similarly, the notion of uniform asphericality requires that the map from the cover of $B_x(C)$ into the universal cover of $B_x(f(C))$ be nullhomotopic.[9]

These spaces are the terminal objects in the subcategory of spaces with the same "bounded 1-type" of a given one. By analogy, we shall be interested in their rigidity properties.

2.7. It is worth pondering the theory in some detail when $X = \mathbb{R}^n$. First, let us consider the Novikov conjecture in this setting:

Theorem 10. *Let M be a manifold with a proper map π to $\mathbb{R}^n$ and $f : N \to M$ be a homotopy equivalence over $\mathbb{R}^n$. Give N the structure of a space over $\mathbb{R}^n$ by using πf. Then* $\mathrm{sig}(\pi^{-1}(0)) = \mathrm{sig}(f^{-1}\pi^{-1}(0))$.

Here, we are assuming that 0 is a regular value of π and πf. This theorem readily implies Novikov's theorem about the topological invariance of rational Pontrjagin classes [**Nov66**]. See [**FW95**].

2.8. The following geometric result of Chapman is sufficient for proving a number of bounded Borel conjecture results. After one develops bounded Whitehead theory and surgery, it implies calculations of K-groups and L-groups, calculations which can be done independently algebraically (see [**PW89, FP95**]). Even after they are proven, the following theorem still feels "greater than the sum of its parts."

Theorem 11 ([**Cha81**]). *Suppose that $N \to M = V \times \mathbb{R}^n$ is a bounded homotopy equivalence, where V is some compact manifold. Then there is some manifold Z homotopy equivalent to $V \times T^n$ whose infinite abelian cover is N; moreover, the manifold Z is unique up to homeomorphism if we insist that it be "transfer invariant," i.e. homeomorphic to its own finite-sheeted covers that are induced from the torus.*

The K-theoretic version would be that $\mathrm{Wh}^{\mathrm{bdd}}(V \times \mathbb{R}^n) = K_{1-n}(\pi_1(V))$ and that $L_k^{\mathrm{bdd}}(V \times \mathbb{R}^n) = L_{k-n}^{-n}(V)$. In the second case, the dimension of the L-group is shifted (remember they are 4-periodic, so negative L-theory is nothing to fear), and the superscript "decoration" is also shifted. See [**Sha69, PR80, Ran92, Wei94, WW88**] for some discussion of this topic.

2.9. Now that we have a bounded version of the Borel conjecture, we can repair the aesthetic defect uncovered in I.18: we can give a topological rigidity analog of Mostow rigidity for nonuniform lattices. For noncompact arithmetic manifolds $M = \Gamma\backslash G/K$, where G is a real connected linear Lie group and K its maximal compact subgroup, the slight strengthening of Siegel's conjecture proven in [**Ji98**] provides the following picture from reduction theory. For each such M there is a compact polyhedron P and a Lipschitz map $\pi : M \to cP$ from M to the open cone on P such that (1) every point inverse deform retracts to an arithmetic manifold,

[9] We assume that the maps induced by inclusions of balls in one another are injections on fundamental groups.

(2) the map π respects the radial direction, and (3) all point inverses have uniformly bounded size. See Chang [**Cha01**] for further discussion.

In [**FJ98**] Farrell and Jones show topological rigidity of these arithmetic homogeneous spaces relative to the ends. On the contrary, if $\mathrm{rank}_{\mathbb{Q}}(\Gamma) > 2$, then M may not be properly rigid, as discussed in the remark of I.17. The following theorem asserts that M is topologically rigid in the category of continuous coarsely Lipschitz maps.

THEOREM 12. *Let $M = \Gamma\backslash G/K$ be a manifold for which Γ is an arithmetic lattice in a real connected linear Lie group G. Endow M with the associated metric. If $f : M' \to M$ is a bounded homotopy equivalence, then f is boundedly homotopic to a homeomorphism.*

To see that the reduction theory implies the vanishing of $S^{\mathrm{bdd}}(M)$, one appeals to the bounded surgery exact sequence:

$$H_{n+1}(M;\mathbf{L}(e)) \to L^{\mathrm{bdd}}_{n+1}(M) \to S^{\mathrm{bdd}}(M) \to H_n(M;\mathbf{L}(e)) \to L^{\mathrm{bdd}}_n(M).$$

We note that the radial direction of cP can be scaled to increase control arbitrarily, and that all the fundamental groups arising in the point inverses $\pi^{-1}(*)$ are K-flat by [**FJ98**]. These two ingredients give us an isomorphism $L^{\mathrm{bdd}}_*(M) \cong H_*(cP;\mathbf{L}(\pi^{-1}(*)))$. Given the Leray spectral sequence for π and the stalkwise equivalence of L-cosheaves, one also has the identification

$$H_*(M;\mathbf{L}(e)) \cong H_*(cP;\mathbf{L}(\pi^{-1}(*))).$$

These isomorphisms give the required vanishing of $S^{\mathrm{bdd}}(M)$.

REMARK. A C^*-algebraic analogue of this calculation is relevant to the question of whether M has a metric of positive scalar curvature in its natural coarse quasi-isometry class (see following subsection). The unresolved state of the Baum-Connes conjecture for lattices prevents us from repeating the above argument in that setting.

2.10. It is now well recognized that the original approach by [**GL83**] and [**SY79**] proving that no compact manifold of nonpositive sectional curvature can be endowed with a metric of positive scalar curvature is actually based on a restriction on the coarse quasi-isometry type of complete noncompact manifolds. Block and Weinberger [**BW99**] investigate the problem of complete metrics for noncompact symmetric spaces when no quasi-isometry conditions are imposed. In particular they show the following:

THEOREM 13. *Let G be a semisimple Lie group and consider the double quotient $M \equiv \Gamma\backslash G/K$ for Γ irreducible in G. Then M can be endowed with a complete metric of positive scalar curvature if and only if Γ is an arithmetic group with* $\mathrm{rank}_{\mathbb{Q}}\,\Gamma \geq 3$.

In fact, in the case of $\mathrm{rank}_{\mathbb{Q}}\,\Gamma \leq 2$, one cannot impose upon M a metric of uniformly positive scalar curvature even in the complement of a compact set. However, in the cases for which such complete metrics are constructed, they always exhibit the coarse quasi-isometry type of a ray.

In the context of the relative assembly map $A\colon H_*(B\pi, B\pi^\infty, \mathbb{L}) \to L(\pi,\pi^\infty)$ and the classifying map $f : (M, M_\infty) \to (B\pi, B\pi^\infty)$, one might expect that the obstruction for complete positive scalar curvature on a spin manifold M is given by the image $f_*[D_M]$ of the Dirac operator in $KO_n(B\pi, B\pi^\infty)$ instead of the signature class in $H_*(B\pi, B\pi^\infty, \mathbb{L})$. One could reasonably conjecture that a complete spin manifold with uniformly positive scalar curvature satisfies $f_*[D_M] = 0$ in $KO_n(B\pi, B\pi^\infty)$ if $\pi_1(M)$ and $\pi_1^\infty(M)$ are both torsion-free.

However, the standard methods in L-theory fail in the K-theoretic framework because there is no assembly map from $KO_n(B\pi, B\pi^\infty)$ to the relative K-theory of some appropriate pair of C^*-algebras which might reasonably be an isomorphism for torsion-free groups. For the case of $\operatorname{rank}_{\mathbb{Q}} \Gamma = 2$ considered in [**BW99**], an alternate route was found (the cases of $\operatorname{rank}_{\mathbb{Q}} \Gamma \leq 1$ are covered by [**GL83**]): the verification that $f_*[D_M]$ vanishes in $KO_{n-1}(B\pi^\infty)$ under the assumption that suitable Novikov-type conjectures hold for the group π^∞.

The results of [**BW99**] do not settle whether these uniformly positive curvature metrics on $\Gamma\backslash G/K$ can be chosen to be (a) quasi-isometric (*i.e.* uniformly bi-Lipschitz) to the original metric inherited from G or (b) of bounded geometry in the sense of having bounded curvature and volume. The first author proved the former negatively in [**Cha01**] by identifying a coarse obstruction of Dirac type in the group $K_*(C_*(M, \pi))$, where $C_*(M, \pi)$ is a generalized Roe algebra of locally compact operators on $\widetilde{M}$ whose propagation is controlled by the projection map $\pi : \widetilde{M} \to M$. This algebra encodes not only the coarse behavior of M but also its local geometry.

2.11. The principle of descent was first formulated explicitly in [**FW95**], although it appears, somewhat implicitly, in [**GL83, Kas88, FW91, Cha01, CP95**] as well. The paper of Gromov and Lawson is especially nice from this point of view, in that they explicitly suggest the use of a families form of a non-compact index theorem to deduce Gromov-Lawson conjecture type results (for manifolds of positive scalar curvature). The principle of descent is, in general, a vehicle for translating bounded Borel or Baum-Connes conjecture type results from the universal cover of a manifold to deduce Novikov conjecture type results for the manifold itself. This principle remains a powerful tool and is exploited in the most recent exciting advances in the subject (see, *e.g.* [**Yu00, Tu99, HR00**]).

The bounded Novikov conjecture states that, if X is uniformly contractible and M is a manifold over X, then $f_*(L(M) \cap [M]) \in H_*^{\ell f}(X; \mathbb{Q})$ is a bounded homotopy invariant. Equivalently, if Γ is the fundamental group of M, then the bounded assembly map

$$A^{\mathrm{bdd}} : H_*^{\ell f}(E\Gamma; \mathbb{L}(e)) \to \mathbb{L}^{\mathrm{bdd}}(E\Gamma)$$

is a split injection. Here $\mathbb{L}^{\mathrm{bdd}}(E\Gamma)$ is the spectrum whose 0-th space is given by a simplicial model for which the n-simplices are n-ad surgery problems on k-manifolds together with a proper coarse map to Γ with the word metric. The map is on the level of the space of sections of assembly maps associated to the fibration $E \times_\Gamma E \to B\Gamma$ to a twisted generalized cohomology.

Assuming the split injectivity of A^{bdd}, we consider the following commutative diagram:

$$\begin{tikzcd} H_*(B\Gamma;\mathbb{L}(e)) \arrow[rr, "A"] \arrow[d, "\cong"] & & \mathbb{L}_*(\mathbb{Z}\Gamma) \arrow[d, "FBT"] \\ H^0(B\Gamma; H_*^{\ell f}(\mathbb{R}^n;\mathbb{L}(e))) \arrow[rr, "A^{\mathrm{bdd}}"] & & H^0(B\Gamma;\mathbb{L}^{\mathrm{bdd}}(\mathbb{R}^n)) \end{tikzcd}$$

The right-hand "family bounded transfer map" is a composite

$$\mathbb{L}_*(\mathbb{Z}\Gamma) \to H^0(B\Gamma;\mathbb{L}^{\mathrm{bdd}}(E\Gamma)) \to H^0(B\Gamma;\mathbb{L}^{\mathrm{bdd}}(\mathbb{R}^n)).$$

The left-hand vertical isomorphism arises when $B\Gamma$ is a finite complex from Spanier-Whitehead duality and the proper homotopy equivalence of the map $E\Gamma \to \mathbb{R}^n$. It too is a family bounded transfer map, for which, at each point x of $B\Gamma$, one lifts a cycle to the universal cover $E\Gamma$ based at x. The splitting of A^{bdd} clearly induces a splitting of A, and the descent argument is complete.

The principle of descent is also instrumental in deducing the (analytic) Novikov Conjecture from the coarse Baum-Connes Conjecture. The latter states that, for any bounded geometry space X, the coarse assembly map $A_\infty : KX_*(X) \to K_*(C^*X)$ is an isomorphism. Here KX_* denotes the coarse homology theory corresponding to K-homology. See [**Roe96**] for more details.

Epilogue

In this epilogue, we would like to mention some issues that there was no time to discuss during those lectures. For the most part, the ideas discussed above provide quite close parallels (at least at the level of conjecture) between topology and index theory. There are several areas where the subjects have diverged that create new opportunity for further developments in one subject or the other.

E.1. In index theory there is a powerful computational calculus which builds two-way maps between relevant groups. So far, no analogous flexible theory has been developed in topology. Besides the sad conclusion that beautiful results like those of [**HK01, Tu00, Yu00**] are not yet known on the topological side (let alone in algebraic K-theory, other coefficient rings, twistings, etc.), even the simple curvature calculations of [**Yu00**] which give a rational counterexample to a coarse version of the Novikov conjecture for a metric space without bounded geometry cannot be copied in topology. Thus, we have very little information about how the epsilons in controlled topology depend on dimension.

E.2. In topology, however, there are a number of subtle arithmetic issues that don't arise in index theory. Some of these are associated to "decorations"[10] (see [**Sha69**]), Nil and Unil (see [**Bas68, Cap74b**]). The latter shows that very routine version of Baum-Connes type conjectures in topology are false even for the infinite dihedral group (which is crystallographic and hence amenable).

[10]Decorations are superscripts adorning L-groups, and modify their definitions by restricting or refining their precise definition using modules and maps which lie in subgroups of appropriate algebraic K-groups.

However, injectivity statements still have a reasonable chance of being correct. For instance, it seems that the object $\lim(L^{\mathrm{bdd}}(V))$, where V runs through the finite-dimensional subspaces of a Hilbert space (V included in W gives rise to a map between the bounded L-theories by taking the product, used in defining the directed system) should arise in geometrizing [**HK01**]. A dreamer could hope that one can do this geometrization topologically using a more complicated category (more arrows connecting subspaces to one another) for other Banach spaces. But, at the moment, one cannot even recover the analytically proven results for groups which embed uniformly in Hilbert space.

E.3. Nonetheless, there is the spectacular work of Farrell and Jones (see [**FJ98**]) which naturally led to and verified variant versions of this conjecture for discrete subgroups of linear Lie groups. The main difficulty in mimicking these methods seems to be the very strong transfer formulae in topology. The "gamma element" is precisely a transfer-projection element.

Philosophically speaking, the strong rigidity of the signature operator in families (a consequence of Hodge theory) makes its study more particular. Besides the difficulty in transferring the ideas of Farrell and Jones to index theory, this issue also arises in trying to develop a stratified index theory for operators on stratified spaces, parallel to [**Wei94**]. It would be interesting to see geometric examples of this phenomenon, for example, for some version of positive scalar curvature metrics on spaces with certain singularities. Of course, for particular classes of stratified spaces, and operators, one should be able to obtain such theories. See, for example, [**Hig90**] for a theory of operators on $\mathbb{Z}/k$-manifolds.

E.4. In the past year, a number of counterexamples to versions of Baum-Connes, in particular the coarse version, were obtained using metric spaces that contain expanding graphs (see, for example, [**HLS02**]). On the other hand, Kevin Whyte and the second author showed that some of these examples do not give counterexamples to the bounded version of the Borel conjecture. If, as seems likely, the isomorphism conjecture (or "stratified Borel conjecture" with rational coefficients) will be verified for hyperbolic groups, then the limit constructions of Gromov will also not lead to counterexamples to the topological versions of these problems.

E.5. Finally, it seems important to mention the circle of mathematical connections between index theory, cyclic homology, pseudoisotopy theory (i.e. algebraic K-theory of spaces), and Goodwillie's calculus of functors. The Goodwillie idea (see, for example, [**Goo90, GW99**]) gives a powerful method for analyzing situations where assembly maps are not isomorphisms.

The Borel conjecture is about the "linear part" of classification of manifolds. There are higher order "nonlinear terms" which are responsible for the counterexamples to the equivariant form. One should connect these ideas to families of operators and the work of Bismut and Lott [**BL95**]. In addition, it would be good to have a better understanding of an index-theoretic (or perhaps we should say operator-algebraic) viewpoint on spectral invariants like Ray-Singer torsion and of eta invariants (and their higher versions, see [**Lot92, Lot99, Wei88, Wei90, LP00b, LLK02**]).

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The Residue Index Theorem of Connes and Moscovici

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Contents

1. Introduction

Several years ago Alain Connes and Henri Moscovici discovered a quite general "local" index formula in noncommutative geometry [**12**] which, when applied to Dirac-type operators on compact manifolds, amounts to an interesting combination of two quite different approaches to index theory.

Atiyah and Bott noted that the index of an elliptic operator D may be expressed as a complex residue

$$\mathrm{Index}(D) = \mathrm{Res}_{s=0}\big(\Gamma(s)\,\mathrm{Trace}(\varepsilon(I+\Delta)^{-s})\big),$$

where $\Delta = D^2$ (see [**1**]). Rather surprisingly, the residue may be computed, at least in principle, as the integral of an explicit expression involving the coefficients of D, the metric g, and the derivatives of these functions. However the formulas can be very complicated.

In a different direction, Atiyah and Singer developed the crucial link between index theory and K-theory. They showed, for example, that an elliptic operator D on M determines a class

$$[D] \in K_0(M)$$

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in the K-homology of M (see [**2**] for one account of this). As it turned out, this was a major advance: when combined with the Bott periodicity theorem, the construction of $[D]$ leads quite directly to a proof of the index theorem.

When specialized to the case of elliptic operators on manifolds, the index formula of Connes and Moscovici associates to an elliptic operator D on M a cocycle for the group $HCP^*(C^\infty(M))$, the periodic cyclic cohomology of the algebra of smooth functions on M. In this respect the Connes-Moscovici formula calls to mind the construction of Atiyah and Singer, since cyclic cohomology is related to K-homology by a Chern character isomorphism. But the actual formula for the Connes-Moscovici cocycle involves only residues of zeta-type functions associated to D. In this respect it calls to mind the Atiyah-Bott formula.

The proper context for the Connes-Moscovici index formula is the noncommutative geometry of Connes [**7**], and in particular the theory of spectral triples. Connes and Moscovici have developed at length a particular case of the index formula which is relevant to the transverse geometry of foliations [**12, 13**]. This work, which involves elaborate use of Hopf algebras, has attracted considerable attention (see the survey articles [**8**] and [**26**] for overviews). At the same time, other instances of the index formula are beginning to be developed (see for example [**9**], which among other things gives a good account of the meaning of the term "local" in noncommutative geometry).

The original proof of the Connes-Moscovici formula, which is somewhat involved, reduces the local index formula to prior work on the transgression of the Chern character, and is therefore is actually spread over several papers [**12, 11, 10**]. Roughly speaking, the residues of zeta functions which appear in the formula are related by the Mellin transform to invariants attached to the heat semigroup $e^{-t\Delta}$. The heat semigroup figures prominently in the theory of the JLO cocycle in cyclic theory, and so previous work on this subject can now be brought to bear on the local index formula.

The main purpose of these notes is to present, in a self-contained way, a new and perhaps more accessible proof of the local index formula. But for the benefit of those who are just becoming acquainted with Connes' noncommutative geometry, we have also tried to provide some context for the formula by reviewing at the beginning of the notes some antecedent ideas in cyclic and Hochschild cohomology.

As for the proof of the theorem itself, in contrast to the original proof of Connes and Moscovici, we shall work directly with the complex powers Δ^{-z}. Our strategy is to find an elementary quantity $\langle a^0, [D, a^1], \ldots, [D, a^p]\rangle_z$ (see Definition 4.12), a sort of multiple zeta function, which is meromorphic in the argument z, and whose residue at $z = -\frac{p}{2}$ is the complicated combination of residues which appears in the Connes-Moscovici cocycle. The proof of the index formula can then be organized in a fairly conceptual way using the new quantities. The main steps are summarized in Theorems 5.5, 5.6, 7.1 and 7.12.

The "elementary quantity" $\langle a^0, [D, a^1], \ldots, [D, a^p]\rangle_z$ was obtained by emulating some computations of Quillen [**23**] on the structure of Chern character cocycles in cyclic theory. Quillen constructed a natural "connection form" Θ in a differential graded cochain algebra, along with a "curvature form" $K = d\Theta + \Theta^2$, for which the quantities

$$\Gamma(z)\operatorname{Trace}\big(K^{-z}\big) = \frac{\Gamma(z)}{2\pi i}\operatorname{Trace}\bigg(\int \lambda^{-z}(\lambda - K)^{-1}\, d\lambda\bigg)$$

have components $\langle 1, [D, a^1], \ldots, [D, a^p]\rangle_z$. Taking residues at $z = -\frac{p}{2}$ we get (at least formally)

$$\operatorname{Trace}\bigl(K^{\frac{p}{2}}\bigr) = \operatorname{Res}_{z=-\frac{p}{2}} \langle 1, [D, a^1], \ldots, [D, a^p]\rangle_z$$

Now, in the context of vector bundles with curvature form K, the pth component of the Chern character is a constant times $\operatorname{Trace}(K^{\frac{p}{2}})$. As a result, it is natural to guess that our elementary quantities $\langle \cdots \rangle_z$ are related to the Chern character and index theory, after taking residues. All this will be explained in a little more detail at the end of the notes, in Appendix B. Appendix A explains the relation between the Connes-Moscovici cocycle and the JLO cocycle, which was one of the original objects of Quillen's study and which, as we noted above, played an important role in the original approach to the index formula.

A final appendix presents a proof of Connes' Hochschild class formula. This is a straightforward development of the proof of the local index formula presented here. (Connes' Hochschild formula is introduced in Section 3 as motivation for the development of the local index formula.)

Obviously the whole of the present work is strongly influenced by the work of Connes and Moscovici. Moreover, in several places the computations which follow are very similar to ones they have carried out in their own work. I am very grateful to both of them for their encouragement and support. I also thank members of Penn State's Geometric Functional Analysis Seminar, especially Raphaël Ponge, for their advice, and for patiently listening to early versions of this work.

2. The Cyclic Chern Character

In this section we shall establish some notation and terminology related to Fredholm index theory and cyclic cohomology. For obvious reasons we shall follow Connes' approach to cyclic cohomology, which is described for example in his book [**7**, Chapter 3]. Along the way we shall make explicit choices of normalization constants.

2.1. Fredholm Index Problems. A linear operator $T\colon V \to W$ from one vector space to another is *Fredholm* if its kernel and cokernel are finite-dimensional, in which case the *index* of T is defined to be

$$\operatorname{Index}(T) = \dim\ker(T) - \dim\operatorname{coker}(T).$$

The index of a Fredholm operator has some important stability properties, which make it feasible in many circumstances to attempt a computation of the index even if computations of the kernel and cokernel, or even their dimensions, are beyond reach.

First, if $F\colon V \to W$ is any finite-rank operator then $T + F$ is also Fredholm, and moreover $\operatorname{Index}(T) = \operatorname{Index}(T + F)$. Second, if V and W are Hilbert spaces then the set of all bounded Fredholm operators from V to W is an open subset of the set of all bounded operators in the operator norm-topology, and moreover the index function is locally constant. In addition, if $K\colon V \to W$ is any compact operator between Hilbert spaces (which is to say that K is a norm-limit of finite-rank operators), then $T+K$ is Fredholm, and moreover $\operatorname{Index}(T) = \operatorname{Index}(T+K)$. In fact, an important theorem of Atkinson asserts that a bounded linear operator between Hilbert spaces is Fredholm if and only if it is invertible modulo compact operators. See for example [**15**].

The following situation occurs frequently in geometric problems which make contact with Fredholm index theory. One is presented with an associative algebra A of bounded operators on a Hilbert space H, and one is given a bounded self-adjoint operator $F\colon H \to H$ with the property that $F^2 = 1$, and for which, for every $a \in A$, the operator $[F, a] = Fa - aF$ is *compact*. This setup (or a small modification of it) was first studied by Atiyah [**2**], who made the following observation related to index theory and K-theory. Since $F^2 = 1$ the operator $P = \frac{1}{2}(F+1)$ is a projection on H (it is the orthogonal projection onto the $+1$ eigenspace of F). If u is any invertible element of A then the operator $PuP\colon PH \to PH$ is Fredholm. This is because the operator $Pu^{-1}P\colon PH \to PH$ is an inverse, modulo compact operators, and so Atkinson's theorem, cited above, applies.

A bit more generally, if $U = [u_{ij}]$ is an $n \times n$ invertible matrix over A then the matrix $PUP = [Pu_{ij}P]$, regarded as an operator on the direct sum of n copies of PH, is a Fredholm operator (for basically the same reason). Now the invertible matrices over A constitute generators for the (algebraic) K-theory group $K_1^{\mathrm{alg}}(A)$ (see [**22**] for details[1]). It is not hard to see that Atiyah's index construction gives rise to a homomorphism of groups

$$\mathrm{Index}_F\colon K_1^{\mathrm{alg}}(A) \to \mathbb{Z}.$$

If A is a reasonable[2] topological algebra, for instance a Banach algebra, so that *topological* K-groups are defined, then the index construction even descends to a homomorphism

$$\mathrm{Index}_F\colon K_1^{\mathrm{top}}(A) \to \mathbb{Z}.$$

In short, the data consisting of A and F together provides a supply of Fredholm operators, and one can investigate in various examples the possibility of determining the indices of these Fredholm operators.

2.1. EXAMPLE. Let A be the algebra of smooth, complex-valued functions on the unit circle S^1, let H be the Hilbert space $L^2(S^1)$, and let F be the *Hilbert transform* on the circle, which maps the trigonometric function $\exp(2\pi i n x)$ to $\exp(2\pi i n x)$ when $n \geq 0$ and to $-\exp(2\pi i n x)$ when $n < 0$. To see that the operators $[F, a]$ are compact, one can first make an explicit computation in the case where a is a trigonometric monomial $a(x) = \exp(2\pi i n x)$, with the result that $[F, a]$ is in fact a finite-rank operator. The general case follows by approximating a general $a \in A$ by a trigonometric polynomial. In this example one has the famous index formula

$$\mathrm{Index}(PuP) = -\frac{1}{2\pi i}\int_{S^1} u^{-1}du.$$

The right hand side is (minus) the winding number of the function $u\colon S^1 \to \mathbb{C}\setminus\{0\}$. (There is also a simple generalization to matrices $U = [u_{ij}]$.) The topological K_1-group here is $\mathbb{Z}$, and the index homomorphism is an isomorphism.

These notes are concerned with formulas for the Fredholm indices which arise from certain instances of Atiyah's construction. We are going to write down a bit more carefully the basic data for the construction, and then add a first additional hypothesis to narrow the scope of the problem just a little.

[1] Except to provide some background context, we shall not use K-theory in these notes.

[2] See the appendix of [**3**] for a discussion of some types of reasonable topological algebra.

2.2. Definition. Let A be an associative algebra over $\mathbb{C}$. An *odd Fredholm module* over A is a triple consisting of:

(a) a Hilbert space H,
(b) a representation of A as bounded operators on H, and
(c) a self-adjoint operator $F\colon H \to H$ such that $F^2 = 1$ and such that $[F, \pi(a)]$ is a compact operator, for every $a \in A$.

An *even Fredholm module* over A consists of the same data as above, together with a self-adjoint operator $\varepsilon\colon H \to H$ such that $\varepsilon^2 = 1$, such that ε commutes with each operator $\pi(a)$, and such that ε anticommutes with F.

Since ε is self-adjoint and since $\varepsilon^2 = 1$, the Hilbert space H decomposes as an orthogonal direct sum $H = H_0 \oplus H_1$ in such a way that $\varepsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The additional hypothesis imply that

$$F = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \quad \text{and} \quad \pi(a) = \begin{pmatrix} \pi_0(a) & 0 \\ 0 & \pi_1(a) \end{pmatrix}.$$

Even Fredholm modules often arise from geometric problems on even-dimensional manifolds — hence the terminology. They are actually closer to Atiyah's original constructions in [**2**] than are the odd Fredholm modules.

Associated to an even Fredholm module there is the following index construction. If p is an idempotent element of A then the operator

$$\pi_1(p) F \pi_0(p)\colon \pi_0(p)H_0 \to \pi_1(p)H_1$$

is Fredholm, since $\pi_0(p)F\pi_1(p)\colon \pi_1(p)H_1 \to \pi_0(p)H_0$ is an inverse, modulo compact operators. This construction passes easily to matrices, and we obtain a homomorphism

$$\mathrm{Index}_F\colon K_0^{\mathrm{alg}}(A) \to \mathbb{Z}.$$

which is the counterpart of the index homomorphism we previously constructed in the odd case.

2.3. Definition. A Fredholm module over A is *finitely summable* if there is some $d \geq 0$ such that for every integer $n \geq d$ every product of commutators

$$[F, \pi(a^0)][F, \pi(a^1)] \cdots [F, \pi(a^n)]$$

is a trace-class operator. (See [**25**] for a discussion of trace class operators.)

2.4. Example. The Fredholm module presented in Example 2.1 is finitely summable: one can take $d = 1$.

We are going to determine formulas in multilinear algebra for the indices of Fredholm operators associated to finitely summable Fredholm modules.

2.2. Cyclic Cocycles.

2.5. Definition. A $(p+1)$-linear functional $\phi\colon A^{p+1} \to \mathbb{C}$ is said to be *cyclic* if

$$\phi(a^0, a^1, \dots, a^p) = (-1)^p \phi(a^p, a^0, \dots, a^{p-1}),$$

for all $a^0, \dots, a^p$ in A.

2.6. Definition. The *coboundary* of a $(p+1)$-linear functional $\phi\colon A^{p+1} \to \mathbb{C}$ is the $(p+2)$-linear functional

$$b\phi(a^0,\dots,a^{p+1}) = \sum_{j=0}^{p}(-1)^j\phi(a^0,\dots,a^ja^{j+1},\dots,a^{p+1}) \\ + (-1)^{p+1}\phi(a^{p+1}a^0,\dots,a^p)$$

A $(p+1)$-multilinear functional ϕ is a *p-cocycle* if $b\phi = 0$.

It is easy to check that the coboundary of any coboundary is zero, or in other words $b^2 = 0$. Thus every coboundary is a cocycle and as a result we can form what are called the *Hochschild cohomology groups* of A: the pth Hochschild group is the quotient of the p-cocycles by the p-cocycles which are coboundaries. We will return to these groups in Section 3, but for the purposes of index theory we are much more interested in the special properties of *cyclic* cocycles.

2.7. Theorem (Connes). *Let ϕ be a $(p+1)$-linear functional on which is both cyclic and a cocycle.*

(a) *If p is odd, and if u is an invertible element of A then the quantity*

$$\langle\phi,u\rangle = \text{constant}\cdot\phi(u^{-1},u,\dots,u^{-1},u)$$

depends only on the class of u in the abelianization of $GL_1(A)$, and defines a homomorphism from the abelianization into $\mathbb{C}$.

(b) *If p is even and if e is an idempotent element of A then the quantity*

$$\langle\phi,e\rangle = \text{constant}\cdot\phi(e,e,\dots,e)$$

depends only on the equivalence class[3] *of e. If e_1 and e_2 are orthogonal, in the sense that $e_1e_2 = e_2e_1 = 0$, then*

$$\langle\phi,e_1+e_2\rangle = \langle\phi,e_1\rangle + \langle\phi,e_2\rangle. \qquad \square$$

2.8. Remark. We have inserted as yet unspecified constants into the formulas for the pairings $\langle\,,\rangle$. As we shall see, they are needed to make the pairings for various p consistent with one another, The constants will be made explicit in Theorem 2.27.

2.9. Example. The simplest non-trivial instances of the theorem occur when $p=1$ or $p=2$. For $p=1$ the explicit conditions on ϕ are

$$\begin{cases} \phi(a^0,a^1) = -\phi(a^1,a^0) \\ \phi(a^0a^1,a^2) - \phi(a^0,a^1a^2) + \phi(a^2a^0,a^1) = 0, \end{cases}$$

while for $p=2$ the conditions are

$$\begin{cases} \phi(a^0,a^1,a^2) = \phi(a^2,a^0,a^1) \\ \phi(a^0a^1,a^2,a^3) - \phi(a^0,a^1a^2,a^3) + \phi(a^0,a^1,a^2a^3) - \phi(a^3a^0,a^1,a^2) = 0. \end{cases}$$

The reader who has not done so before ought to try to tackle the theorem for himself or herself in these cases before consulting Connes' paper [**4**].

[3] Two idempotents e and f are *equivalent* if there are elements x and y of A such that $e = xy$ and $f = yx$. If A is for example a matrix algebra then two idempotent matrices are equivalent if and only if their ranges have the same dimension.

The pairings $\langle\,,\,\rangle$ defined by the theorem extend easily to invertible and idempotent matrices, and thereby define homomorphisms

$$\begin{aligned} \langle\phi,\ \rangle\colon K_1^{\mathrm{alg}}(A) &\to \mathbb{C} \qquad p \text{ odd} \\ \langle\phi,\ \rangle\colon K_0^{\mathrm{alg}}(A) &\to \mathbb{C} \qquad p \text{ even} \end{aligned}$$

The question now arises, can the index homomorphisms constructed in the previous section be recovered as instances of the above homomorphisms, for suitable cyclic cocycles ϕ? This was answered by Connes, as follows:

2.10. THEOREM. *Let (A, H, F) be a finitely summable, odd Fredholm module and let $n = 2k + 1$ be an odd integer such that for all $a^0, \ldots, a^n$ in A the product $[F, a^0] \cdots [F, a^n]$ is a trace-class operator. The formula*

$$\phi\big(a^0, \ldots, a^n\big) = \frac{1}{2}\operatorname{Trace}\big(F[F, a^0][F, a^1] \ldots [F, a^n]\big)$$

defines a cyclic n-cocycle on A. If u is an invertible element of A then

$$\phi(u, u^{-1}, \ldots, u, u^{-1}) = (-1)^{k+1} 2^{2k+1} \operatorname{Index}(PuP\colon PH \to PH),$$

where $P = \frac{1}{2}(F + 1)$. □

2.11. THEOREM. *Let (A, H, F) be a finitely summable, even Fredholm module, and let $n = 2k$ be an even integer such that for all $a^0, \ldots, a^n$ in A the product $[F, a^0] \cdots [F, a^n]$ is a trace-class operator. The formula*

$$\phi\big(a^0, \ldots, a^n\big) = \frac{1}{2}\operatorname{Trace}\big(\varepsilon F[F, a^0][F, a^1] \ldots [F, a^n]\big)$$

defines a cyclic n-cocycle on A. If e is an idempotent element of A then

$$\phi(e, e, \ldots, e) = (-1)^k \operatorname{Index}(eFe\colon eH_0 \to H_1).$$

□

The proofs of these results may be found in [**4**] or [**7**, IV.1] (but in the next section we shall at least verify that the formulas do indeed define cyclic cocycles).

2.3. Cyclic Cohomology. Throughout this section we shall assume that A is an associative algebra over $\mathbb{C}$ with a multiplicative identity 1. The definitions for algebras without an identity are a little different and will be considered later.

It is a remarkable fact that if ϕ is a cyclic multilinear functional then so is its coboundary $b\phi$.[4] As a result of this we can form the *cyclic cohomology groups* of A:

[4]Note that cyclicity for a $(p+1)$-linear functional ϕ has to do with invariance under the action of the cyclic group C_{p+1}, whereas cyclicity for $b\phi$ has to do with invariance under C_{p+2}, so to a certain extent b intertwines the actions of two different groups — this is what is so remarkable.

2.12. Definition. The pth cyclic cohomology group of a complex algebra A is the quotient $HC^p(A)$ of the cyclic p-cocycles by the cyclic p-cocycles which are cyclic coboundaries.

But we are interested in a small modification of the cyclic cohomology groups, called the *periodic* cyclic cohomology groups of A. There are only two such groups — an even one and an odd one. The even periodic group $HCP^{\text{even}}(A)$ in some sense combines all the $HC^{2k}(A)$ into one group, while the odd periodic group $HCP^{odd}(A)$ does the same for the $HC^{2k+1}(A)$. One reason for considering the periodic groups is that Connes' construction of the cyclic cocycle associated to a Fredholm module produces not one cyclic cocycle, but one for each sufficiently large integer n of the correct parity. As we shall see, the periodic cyclic cohomology groups provide a framework within which these different cocycles can be compared with one another.

The definition of $HCP^{\text{even}\,/\,\text{odd}}(A)$ is, at first sight, a little strange, but after we look at some examples it will come to seem more natural.

2.13. Definition. Let A be an associative algebra over $\mathbb{C}$ with a multiplicative identity element 1. If p is a non-negative integer, then denote by $C^p(A)$ the space of $(p+1)$-multilinear maps ϕ from A into $\mathbb{C}$ which have the property that if $a^j = 1$, for some $j \geq 1$, then $\phi(a^0, \ldots, a^p) = 0$. Define operators

$$b\colon C^p(A) \to C^{p+1}(A) \quad \text{and} \quad B\colon C^{p+1}(A) \to C^p(A),$$

by the formulas

$$b\phi(a^0, \ldots, a^{p+1}) = \sum_{j=0}^{p} (-1)^j \phi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{p+1}) \\ + (-1)^{p+1} \phi(a^{p+1} a^0, \ldots, a^p)$$

and

$$B\phi(a^0, \ldots, a^p) = \sum_{j=0}^{p} (-1)^{pj} \phi(1, a^j, a^{j+1}, \ldots, a^{j-1}).$$

2.14. Remark. The operator b is the same as the coboundary operator that we encountered in the previous section, except that we are now considering a slightly restricted class of multilinear maps on which b is defined (we should note that a simple computation shows b to be well defined as a map from $C^p(A)$ into $C^{p+1}(A)$). In what follows, we could in fact work with *all* multilinear functionals, rather than just those for which $\phi(a^0, \ldots, a^p) = 0$ when $a^j = 1$ for some $j \geq 1$ (although this would entail a small modification to the formula for the operator B; see [**21**]). The setup we are considering is a bit more standard, and allows for some slightly simpler formulas.

2.15. Lemma. *$b^2 = 0$, $B^2 = 0$ and $bB + Bb = 0$.* □

As a result of the lemma, we can assemble from the spaces $C^p(A)$ the following double complex, which is continued indefinitely to the left and to the top.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots \\
\cdots \arrow[r, "B"] & C^3(A) \arrow[u, "b"] \arrow[r, "B"] & C^2(A) \arrow[u, "b"] \arrow[r, "B"] & C^1(A) \arrow[u, "b"] \arrow[r, "B"] & C^0(A) \arrow[u, "b"] \\
\cdots \arrow[r, "B"] & C^2(A) \arrow[u, "b"] \arrow[r, "B"] & C^1(A) \arrow[u, "b"] \arrow[r, "B"] & C^0(A) \arrow[u, "b"] & \\
\cdots \arrow[r, "B"] & C^1(A) \arrow[u, "b"] \arrow[r, "B"] & C^0(A) \arrow[u, "b"] & & \\
\cdots \arrow[r, "B"] & C^0(A) \arrow[u, "b"] & & &
\end{tikzcd}
$$

2.16. Definition. The *periodic cyclic cohomology of* A is the cohomology of the totalization of this complex.

Thanks to the symmetry inherent in the complex, all even cohomology groups are the same, as are all the odd groups. As a result, one speaks of the *even* and *odd* periodic cyclic cohomology groups of A. A cocycle for the even group is a sequence

$$(\phi_0, \phi_2, \phi_4, \dots),$$

where $\phi_{2k} \in C^{2k}$, $\phi_{2k} = 0$ for all but finitely many k, and

$$b\phi_{2k} + B\phi_{2k+2} = 0$$

for all $k \geq 0$. Similarly a cocycle for the odd group is a sequence

$$(\phi_1, \phi_3, \phi_5, \dots),$$

where $\phi_{2k+1} \in C^{2k+1}$, $\phi_{2k+1} = 0$ for all but finitely many k, and

$$b\phi_{2k+1} + B\phi_{2k+3} = 0$$

for all $k \geq 0$ (and in addition $B\phi_1 = 0$).

2.17. Definition. We shall refer to cocycles of the above sort as (b, B)*-cocycles*. This will help us distinguish between these cocycles and the cyclic cocycles which we introduced in the last section.

Suppose now that ϕ_n is a cyclic n-cocycle, as in the last section, and suppose that ϕ_n has the property that $\phi(a^0, \dots, a^n) = 0$ when some a^j is equal to 1. Note that Connes' cocycles described in Theorems 2.10 and 2.11 have this property. By definition, $b\phi_n = 0$, and clearly $B\phi_n = 0$ too, since the definition of D involves the insertion of 1 as the first argument of ϕ_n. As a result, the sequence

$$(0, \dots, 0, \phi_n, 0, \dots),$$

obtained by placing ϕ_n in position n and 0 everywhere else, is a (b, B)-cocycle. In this way we shall from now on regard every cyclic cocycle as a (b, B)-cocycle.

2.18. Remark. It is known that every (b, B)-cocycle is cohomologous to a cyclic cocycle of some degree p (see [**21**]).

Let us now return to the cocycles which Connes constructed from a Fredholm module.

2.19. Theorem. *Let (A, H, F) be a finitely summable, odd Fredholm module and let n be an odd integer such that for all $a^0, \dots, a^n$ in A the product $[F, a^0] \cdots [F, a^n]$ is a trace-class operator. The formula*

$$\mathrm{ch}_n^F\big(a^0, \dots, a^n\big) = \frac{\Gamma(\frac{n}{2}+1)}{2 \cdot n!} \operatorname{Trace}\big(F[F, a^0][F, a^1] \dots [F, a^n]\big)$$

defines a cyclic n-cocycle on A whose periodic cyclic cohomology class is independent of n.

Proof. Define

$$\psi_{n+1}(a^0, \dots, a^{n+1}) = \frac{\Gamma(\frac{n}{2}+2)}{(n+2)!} \operatorname{Trace}\big(a^0 F[F, a^1][F, a^2] \dots [F, a^{n+1}]\big).$$

It is then straightforward to compute that $b\psi_{n+1} = -\mathrm{ch}_{n+2}^F$ while $B\psi_{n+1} = \mathrm{ch}_n^F$. Hence $\mathrm{ch}_n^F - \mathrm{ch}_{n+2}^F$ is a (b, B)-coboundary. □

2.20. Remarks. Obviously, the multiplicative factor $\frac{\Gamma(\frac{n}{2}+1)}{2n!}$ is chosen to guarantee that the class of ch_n^F in periodic cyclic cohomology is independent of n. Since $b^2 = 0$, the formula $b\psi_{n+1} = -\mathrm{ch}_{n+2}^F$ proves that ch_{n+2}^F is a cocycle (i.e. $b\,\mathrm{ch}_{n+2}^F = 0$). In addition, since it is clear from the definition of the operator B that the range of B consists entirely of cyclic multilinear functionals, the formula $B\psi_{n+1} = \mathrm{ch}_n^F$ proves that ch_n^F is cyclic.

2.21. Theorem. *Let (A, H, F) be a finitely summable, even Fredholm module and let n be an odd integer such that for all $a^0, \dots, a^n$ in A the product $[F, a^0] \cdots [F, a^n]$ is a trace-class operator. The formula*

$$\mathrm{ch}_n^F\big(a^0, \dots, a^n\big) = \frac{\Gamma(\frac{n}{2}+1)}{2 \cdot n!} \operatorname{Trace}\big(\varepsilon F[F, a^0][F, a^1] \dots [F, a^n]\big)$$

defines a cyclic n-cocycle on A whose periodic cyclic cohomology class is independent of n.

Proof. Define

$$\psi_{n+1}(a^0, \dots, a^{n+1}) = \frac{\Gamma(\frac{n}{2}+2)}{(n+2)!} \operatorname{Trace}\big(a^0 F[F, a^1][F, a^2] \dots [F, a^{n+1}]\big)$$

and proceed as before. □

2.22. Definition. The cocycle ch_n^F defined in Theorem 2.19 or 2.21 is the *cyclic Chern character* of the odd or even Fredholm module (A, H, F).

2.4. Comparison with de Rham Theory. Let M be a smooth, closed manifold and denote by $C^\infty(M)$ the algebra of smooth, complex-valued functions on M. For $p \geq 0$ denote by Ω_p the space of p-dimensional de Rham currents (dual to the space Ω^p of smooth p-forms).

Each current $c \in \Omega_p$ determines a cochain $\phi_c \in C^p(A)$ for the algebra $C^\infty(M)$ by the formula

$$\phi_c(f^0, \ldots, f^p) = \int_c f^0 df^1 \cdots df^p.$$

The following is a simple computation:

2.23. Lemma. *If $c \in \Omega_p$ is any p-current on M then*

$$b\phi_c = 0 \quad \textit{and} \quad B\phi_c = p \cdot \phi_{d^* c},$$

where $d^* \colon \Omega_p \to \Omega_{p-1}$ is the operator which is adjoint to the de Rham differential. □

The lemma implies that if we assemble the spaces Ω_p into a bicomplex, as follows,

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots \\
\cdots \arrow[r, "4d^*"] & \Omega_3 \arrow[u, "0"] \arrow[r, "3d^*"] & \Omega_2 \arrow[u, "0"] \arrow[r, "2d^*"] & \Omega_1 \arrow[u, "0"] \arrow[r, "d^*"] & \Omega_0 \arrow[u, "0"] \\
\cdots \arrow[r, "3d^*"] & \Omega_2 \arrow[u, "0"] \arrow[r, "2d^*"] & \Omega_1 \arrow[u, "0"] \arrow[r, "d^*"] & \Omega_0 \arrow[u, "0"] & \\
\cdots \arrow[r, "2d^*"] & \Omega_1 \arrow[u, "0"] \arrow[r, "d^*"] & \Omega_0 \arrow[u, "0"] & & \\
\cdots \arrow[r, "d^*"] & \Omega_0 \arrow[u, "0"] & & &
\end{tikzcd}$$

then the construction $c \mapsto \phi_c$ defines a map from this bicomplex to the bicomplex which computes periodic cyclic cohomology of $A = C^\infty(M)$.

A fundamental result of Connes [**4**, Theorem 46] asserts that this map of complexes is an isomorphism on cohomology:

2.24. Theorem. *The inclusion $c \mapsto \phi_c$ of the above double complex into the (b, B)-bicomplex induces isomorphisms*

$$HCP^{\mathrm{even}}_{\mathrm{cont}}(C^\infty(M)) \cong H_0(M) \oplus H_2(M) \oplus \cdots$$

and

$$HCP^{\mathrm{odd}}_{\mathrm{cont}}(C^\infty(M)) \cong H_1(M) \oplus H_3(M) \oplus \cdots$$

Here $HCP^*_{\mathrm{cont}}(C^\infty(M))$ denotes the periodic cyclic cohomology of M, computed from the bicomplex of continuous *multilinear functionals on $C^\infty(M)$.* □

It follows that an even/odd (b, B)-cocycle for $C^\infty(M)$ is something very like a family of closed currents on M of even/odd degrees.

Connes' theorem is proved by first identifying the (continuous) Hochschild cohomology of the algebra $A = C^\infty(M)$. Lemma 2.23 shows that there is a map of complexes

$$
\begin{tikzcd}
\Omega^0 \arrow[r, "0"] \arrow[d] & \Omega^1 \arrow[r, "0"] \arrow[d] & \cdots \arrow[r, "0"] & \Omega^n \arrow[r, "0"] \arrow[d] & 0 \arrow[r, "0"] \arrow[d] & \cdots \\
C^0(A) \arrow[r, "b"'] & C^1(A) \arrow[r, "b"'] & \cdots \arrow[r, "b"'] & C^n(A) \arrow[r, "b"'] & C^{n+1}(A) \arrow[r, "b"'] & \cdots
\end{tikzcd}
$$

in which the vertical maps come from the construction $c \mapsto \phi_c$. The following result is known as the Hochschild-Kostant-Rosenberg theorem (see [**21**]), although this precise formulation is due to Connes [**4**].

2.25. Theorem. *The above map induces an isomorphism from Ω_p to the pth continuous Hochschild cohomology group $HH^p_{\text{cont}}(C^\infty(M))$.*

Let us conclude this section with a few brief remarks about non-periodic cyclic cohomology groups. We already noted that every cyclic p-cocycle determines a (B, b)-cocycle (even or odd, according to the parity of p). In view of the Hochschild-Kostant-Rosenberg theorem, and in view of the fact that every cyclic p-cocycle is in particular a Hochschild p-cocycle, so that if A is any algebra then there is a natural map from pth cyclic cohomology group $HC^p(A)$, as given in Definition 2.12, into the Hochschild group $HH^p(A)$, it might be thought that when $A = C^\infty(M)$ the cyclic p-cocycles correspond to the summand $H_p(M)$ in Theorem 2.24. But this is not exactly right. It cannot be right because if a (b, B)-cocycle is cohomologous to a cyclic p-cocycle, it may be shown that it is also cohomologous to a cyclic $(p+2)$-cocycle, and to a cyclic $(p+4)$-cocycle, and so on. So when $A = C^\infty(M)$ a single cyclic p-cocycle can encode information not just about closed p-currents, but also about closed $(p-2)$-currents, closed $(p-4)$-currents, and so on. The precise relation, again discovered by Connes, is as follows:

2.26. Theorem. *Denote by $Z_p(M)$ the set of closed de Rham k-currents on M. There are isomorphisms*

$$HC^{2k}_{\text{cont}}(C^\infty(M)) \cong H_0(M) \oplus H_2(M) \oplus \cdots \oplus H_{2k-2}(M) \oplus Z_{2k}(M)$$

and

$$HC^{2k+1}_{\text{cont}}(C^\infty(M)) \cong H_1(M) \oplus H_3(M) \oplus \cdots \oplus H_{2k-1}(M) \oplus Z_{2k+1}(M).$$

Here $HC^*_{\text{cont}}(C^\infty(M))$ denotes the cyclic cohomology of M, computed from the complex of continuous *cyclic multilinear functionals on $C^\infty(M)$.* □

2.5. Pairings with K-Theory. The pairings described in Theorem 2.7 between cyclic cocycles and K-theory have the following counterparts in periodic cyclic theory.

2.27. Theorem (Connes). *Let A be an algebra with a multiplicative identity.*

(a) *If $\phi = (\phi_1, \phi_3, \dots)$ is an odd (b, B)-cocycle for A, and u is an invertible element of A, then the quantity*

$$\langle \phi, u \rangle = \frac{1}{\Gamma(\frac{1}{2})} \sum_{k=0}^{\infty} (-1)^{k+1} k!\, \phi_{2k+1}(u^{-1}, u, \dots, u^{-1}, u)$$

depends only on the class of u in the abelianization of $GL_1(A)$ and the periodic cyclic cohomology class of ϕ, and defines a homomorphism from the abelianization into $\mathbb{C}$.

(b) *If $\phi = (\phi_0, \phi_2, \dots)$ is an even (b, B)-cocycle for A, and e is an idempotent element of A, then the quantity*

$$\langle \phi, e \rangle = \phi_0(e) + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} \phi_{2k}(e - \frac{1}{2}, e, e, \dots, e).$$

depends only on the equivalence class of e and the periodic cyclic cohomology class of ϕ. Moreover if e_1 and e_2 are orthogonal idempotents in A, then

$$\langle \phi, e_1 + e_2 \rangle = \langle \phi, e_1 \rangle + \langle \phi, e_2 \rangle.$$

Proof. See [**16**] for the odd case and [**17**] for the even case. □

2.28. Example. Putting together Theorem 2.10 with the formula (a) in Theorem 2.27, we see that if (A, H, F) is a finitely summable, odd Fredholm module, and if u is an invertible element of A, then

$$\langle \mathrm{ch}_n^F, u \rangle = \mathrm{Index}(PuP \colon PH \to PH),$$

where P is the idempotent $P = \frac{1}{2}(F+1)$, and the odd integer n is large enough that the Chern character is defined. Similarly, if (A, H, F) is an even Fredholm module and if e is an idempotent element of A then

$$\langle \mathrm{ch}_n^F, e \rangle = \mathrm{Index}(eFe \colon eH_0 \to eH_1),$$

for all even n which again are large enough that the Chern character is defined.

2.29. Remarks. The pairings described in Theorem 2.27 extend easily to the algebraic K-theory groups $K_1^{\mathrm{alg}}(A)$ and $K_0^{\mathrm{alg}}(A)$.

2.6. Improper Cocycles and Coefficients. We are going to describe two extensions of the notion of (b, B)-cocycle which will be useful in these notes.

If V is a complex vector space and p is a non-negative integer, then let us denote by $C^p(A, V)$ space of $(p+1)$-multilinear maps ϕ from A into V for which $\phi(a^0, \dots, a^p) = 0$ whenever $a^j = 1$ for some $j \geq 1$.

Define operators

$$b \colon C^p(A, V) \to C^{p+1}(A, V) \quad \text{and} \quad B \colon C^{p+1}(A, V) \to C^p(A, V),$$

by the same formulas we used previously:

$$\begin{aligned} b\phi(a^0, \dots, a^{p+1}) = \sum_{j=0}^{p} (-1)^j \phi(a^0, \dots, a^j a^{j+1}, \dots, a^{p+1}) \\ + (-1)^{p+1} \phi(a^{p+1} a^0, \dots, a^p) \end{aligned}$$

and

$$B\phi(a^0, \dots, a^p) = \sum_{j=0}^{p} (-1)^{pj} \phi(1, a^j, a^{j+1}, \dots, a^{j-1}).$$

Then assemble the double complex

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots \\
\cdots \arrow[r, "B"] & C^3(A,V) \arrow[u, "b"] \arrow[r, "B"] & C^2(A,V) \arrow[u, "b"] \arrow[r, "B"] & C^1(A,V) \arrow[u, "b"] \arrow[r, "B"] & C^0(A,V) \arrow[u, "b"] \\
\cdots \arrow[r, "B"] & C^2(A,V) \arrow[u, "b"] \arrow[r, "B"] & C^1(A,V) \arrow[u, "b"] \arrow[r, "B"] & C^0(A,V) \arrow[u, "b"] & \\
\cdots \arrow[r, "B"] & C^1(A,V) \arrow[u, "b"] \arrow[r, "B"] & C^0(A,V) \arrow[u, "b"] & & \\
\cdots \arrow[r, "B"] & C^0(A,V) \arrow[u, "b"] & & &
\end{tikzcd}
$$

just as before.

2.30. Definition. The *periodic cyclic cohomology of A, with coefficients in V* is the cohomology of the totalization of this complex.

It is easy to check that the periodic cyclic cohomology of A with coefficients in V is just the space of homomorphisms into V from the periodic cyclic cohomology of A with coefficients in $\mathbb{C}$ (the latter is the "usual" periodic cyclic cohomology of A). Nevertheless the concept of coefficients will be a convenient one for us.

If we totalize the (b, B)-bicomplex (either the above one involving V or the previous one without V) by taking a direct *product* of cochain groups along the diagonals instead of a direct sum, then we obtain a complex with zero cohomology.

2.31. Definition. We shall refer to cocycles for this complex, consisting in the even case of sequences $(\phi_0, \phi_2, \phi_4, \dots)$, *all* of whose terms may be nonzero, as *improper* (b, B)-cocycles.

On its own, an improper periodic (b, B)-cocycle has no cohomological significance, but once again the concept will be a convenient one for us. For example in Section 5 we shall construct a fairly simple improper (b, B)-cocycle with coefficients in the space V of meromorphic functions on $\mathbb{C}$. By taking residues at $0 \in \mathbb{C}$ we shall obtain a linear map from V to $\mathbb{C}$, and applying this linear map to our cocycle we shall obtain in Section 5 a *proper* (b, B)-cocycle with coefficients in $\mathbb{C}$.

2.7. Nonunital Algebras. If the algebra A has no multiplicative unit then we define periodic cyclic cohomology as follows. Denote by $\widetilde{A}$ the algebra obtained by adjoining a unit to A and form the double complex

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots \\
\cdots \arrow[r, "B"] & C^3(\widetilde{A}) \arrow[u, "b"] \arrow[r, "B"] & C^2(\widetilde{A}) \arrow[u, "b"] \arrow[r, "B"] & C^1(\widetilde{A}) \arrow[u, "b"] \arrow[r, "B"] & C^0(A) \arrow[u, "b"] \\
\cdots \arrow[r, "B"] & C^2(\widetilde{A}) \arrow[u, "b"] \arrow[r, "B"] & C^1(\widetilde{A}) \arrow[u, "b"] \arrow[r, "B"] & C^0(A) \arrow[u, "b"] & \\
\cdots \arrow[r, "B"] & C^1(\widetilde{A}) \arrow[u, "b"] \arrow[r, "B"] & C^0(A) \arrow[u, "b"] & & \\
\cdots \arrow[r, "B"] & C^0(A) \arrow[u, "b"] & & &
\end{tikzcd}$$

in which the spaces $C^p(\widetilde{A})$ are, as before, the $(p+1)$-multilinear functionals ϕ from $\widetilde{A}$ into $\mathbb{C}$ with the property that $\phi(a^0, \dots, a^p) = 0$ whenever $a^j = 1$ for some $j \geq 1$, and in which $C^0(A)$ is the space of linear functionals on A (not on $\widetilde{A}$). If we interpret $b\colon C^0(A) \to C^1(\widetilde{A})$ using the formula

$$b\phi(a^0, a^1) = \phi(a^0 a^1) - \phi(a^1 a^0) = \phi(a^0 a^1 - a^1 a^0)$$

then the differential is well defined, since the commutator $a^0a^1 - a^1a^0$ always belongs to A, even when $a^0, a^1 \in \widetilde{A}$.

2.32. Definition. The periodic cyclic cohomology of A is the cohomology of the totalization of the above complex. A (b, B)-cocycle for A is a cocycle for the above complex.

2.33. Remark. The periodic cyclic cohomology of a non-unital algebra A is isomorphic to the kernel of the restriction map from $HCP^*(\tilde{A})$ to $HCP^*(\mathbb{C})$.

2.34. Definition. By a cyclic p-cocycle on A we shall mean a cyclic cocycle ϕ on $\widetilde{A}$ for which $\phi(a^0, \dots, a^p) = 0$ whenever $a^j = 1$ for some j.

3. The Hochschild Character

The purpose of this section is to provide some motivation for the development of the local index formula in cyclic cohomology by describing an antecedent formula in Hochschild cohomology.

3.1. Spectral Triples. Examples of Fredholm modules arising from geometry often involve the following structure.

3.1. Definition. A *spectral triple* is a triple (A, H, D) consisting of:

(a) An associative algebra A of bounded operators on a Hilbert space H, and
(b) An unbounded self-adjoint operator D on H such that
  (i) for every $a \in A$ the operators $a(D \pm i)^{-1}$ are compact, and
  (ii) for every $a \in A$, the operator $[D, a]$ is defined on $\operatorname{dom} D$ and extends to a bounded operator on H.

3.2. Remark. In item (b) we require that D be self-adjoint in the sense of unbounded operator theory. This means that D is defined on some dense domain $\operatorname{dom} D \subseteq H$, that $\langle Du, v\rangle = \langle u, D\rangle$, for all $u, v \in \operatorname{dom} D$, and that the operators $(D \pm i)$ map $\operatorname{dom} H$ bijectively onto H. In item (ii) we require that each $a \in A$ map $\operatorname{dom} D$ into itself.

3.3. Example. Let $A = C^\infty(S^1)$, let $H = L^2(S^1)$ and let $D = \frac{1}{2\pi i}\frac{d}{dx}$. The operator D, defined initially on the smooth functions in $H = L^2(S^1)$ (we are thinking now of S^1 as $\mathbb{R}/\mathbb{Z}$), has a unique extension to a self-adjoint operator on H, and the triple (A, H, D) incorporating this extension is a spectral triple in the sense of Definition 3.1.

3.4. Remark. If the algebra A has a unit, which acts as the identity operator on the Hilbert space H, then item (i) is equivalent to the assertion that $(D \pm i)^{-1}$ be compact operators, which is equivalent to the assertion that there exist an orthonormal basis for H consisting of eigenvectors v_j for D, with eigenvalues λ_j converging to ∞ in absolute value.

In a way which is similar to our treatment of Fredholm modules, we shall call a spectral triple *even* if the Hilbert space H is equipped with a self-adjoint grading operator ε, decomposing H as a direct sum $H = H_0 \oplus H_1$, such that ε maps the domain of D into itself, anticommutes with D, and commutes with every $a \in A$. Spectral triples without a grading operator will be referred to as *odd*.

Let (A, H, D) be a spectral triple, and assume that D is invertible. In the polar decomposition $D = |D|F$ of D the operator F is self-adjoint and satisfies $F^2 = 1$.

3.5. Lemma. *If (A, H, D) is a spectral triple (A, H, F) is a Fredholm module in the sense of Definition 2.2.* □

3.6. Example. The Fredholm module described in Example 2.1 is obtained in this way from the spectral triple of Example 3.3, after we make a small modification to D to make it invertible — for example by replacing $\frac{1}{2\pi i}\frac{d}{dx}$ with $\frac{1}{2\pi i}\frac{d}{dx} + \frac{1}{2}$.

3.2. The Residue Trace. We are going to develop for spectral triples a refinement of the notion of finite summability that we introduced for Fredholm modules. For this purpose we need to quickly review the following facts about compact operators and their singular values (see [**25**] for more details).

3.7. Definition. If K is a compact operator on a Hilbert space then the *singular value sequence* $\{\mu_j\}$ of K is defined by the formulas

$$\mu_j = \inf_{\dim(V)=j-1} \sup_{v \perp V} \frac{\|Kv\|}{\|v\|} \qquad j = 1, 2, \ldots.$$

The infimum is over all linear subspaces of H of dimension $j - 1$. Thus μ_1 is just the norm of K, μ_2 is the smallest possible norm obtained by restricting K to a codimension 1 subspace, and so on.

The trace ideal is easily described in terms of the sequence of singular values:

3.8. Lemma. *A compact operator K belongs to the trace ideal if and only if $\sum_j \mu_j < \infty$. Moreover if K is a positive, trace-class operator then* $\operatorname{Trace}(K) = \sum_j \mu_j$. □

Now, *any* self-adjoint trace-class operator can be written as a difference of positive, trace-class operators, $K = K^{(1)} - K^{(2)}$, and we therefore have a corresponding formula for the trace

$$\mathrm{Trace}(K) = \mathrm{Trace}(K^{(1)}) - \mathrm{Trace}(K^{(2)}) = \sum_j \mu_j^{(1)} - \sum_j \mu_j^{(2)}.$$

And since every trace-class operator is a linear combination of two self-adjoint, trace-class operators, we can go on to obtain a formula for the trace of a general trace-class operator.

We are going to define a new sort of trace by means of formulas like the one above.

3.9. Definition. Denote by $\mathcal{L}^{1,\infty}(H)$ the set of all compact operators K on H for which

$$\sum_{j=1}^{N} \mu_j = \mathcal{O}(\log N).$$

Thus every trace-class operator belongs to $\mathcal{L}^{1,\infty}(H)$ but operators in $\mathcal{L}^{1,\infty}(H)$ need not be trace class, since the sum $\sum_j \mu_j$ is permitted to diverge logarithmically.

3.10. Remark. The set $\mathcal{L}^{1,\infty}(H)$ is a two-sided ideal in $B(H)$.

Suppose now that we are given a linear subspace of $\mathcal{L}^{1,\infty}(H)$ consisting of operators for which the sequence of numbers

$$\frac{1}{\log N} \sum_{j=1}^{N} \mu_j$$

is not merely bounded but in fact convergent. It may be shown using fairly standard singular value inequalities that the functional Tr_ω which assigns to each positive operator in the subspace the limit of the sequence is *additive*:

$$\mathrm{Tr}_\omega(K^{(1)}) + \mathrm{Tr}_\omega(K^{(2)}) = \mathrm{Tr}_\omega(K^{(1)} + K^{(2)}).$$

As a result, if we assume that the linear subspace we are given is spanned by its positive elements, the prescription Tr_ω extends by linearity from positive operators to all operators and yields a linear functional.

A theorem of Dixmier [**14**] (see also [**7**, Section IV.2]) improves this construction by replacing limits with generalized limits, thereby obviating the need to assume that the sequence of partial sums $\frac{1}{\log N}\sum_{j=1}^{N} \mu_j$ is convergent:

3.11. Theorem. *There is a linear functional* $\mathrm{Tr}_\omega \colon \mathcal{L}^{1,\infty}(H) \to \mathbb{C}$ *with the following properties:*

(a) *If* $K \geq 0$ *then* $\mathrm{Tr}_\omega(K)$ *depends only on the singular value sequence* $\{\mu_j\}$, *and*
(b) *If* $K \geq 0$ *then* $\liminf_N \frac{1}{\log N}\sum_{j=1}^{N} \mu_j \leq \mathrm{Tr}_\omega(K) \leq \limsup_N \frac{1}{\log N}\sum_{j=1}^{N} \mu_j$. $\square$

It follows from (a) that $\mathrm{Tr}_\omega(K) = \mathrm{Tr}_\omega(U^*KU)$ for every positive $K \in \mathcal{L}^{1,\infty}(H)$ and every unitary operator U on H. Since the positive operators in $\mathcal{L}^{1,\infty}(H)$ span $\mathcal{L}^{1,\infty}(H)$, it follows that $\mathrm{Tr}_\omega(T) = \mathrm{Tr}_\omega(U^*TU)$, for every $T \in \mathcal{L}^{1,\infty}(H)$ and every unitary operator U. Replacing T by U^*T we get $\mathrm{Tr}_\omega(UT) = \mathrm{Tr}_\omega(TU)$, for every $T \in \mathcal{L}^{1,\infty}(H)$ and every unitary U. Since the unitary operators span all of $B(H)$, we finally conclude that

$$\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS),$$

for every $T \in \mathcal{L}^{1,\infty}(H)$ and every bounded operator S.

3.12. Remark. The *Dixmier trace* Tr_ω is not unique — it depends on a choice of suitable generalized limit for the sequence of partial sums $\frac{1}{\log N}\sum_{j=1}^{N}\mu_j$. But of course it is unique on (positive) operators for which this sequence is convergent, which turns out to be the case in many geometric examples.

3.3. The Hochschild Character Theorem. If (A, H, F) is a finitely summable Fredholm module then Connes' cyclic Chern character ch_n^F is defined for all large enough n of the correct parity (even or odd, according as the Fredholm module is even or odd). It is a cyclic n-cocycle, and in particular, disregarding its cyclicity, it is a Hochschild n-cocycle. We are going to present a formula for the Hochschild cohomology class of this cocycle, in certain cases.

3.13. Lemma. *Let (A, H, D) be a spectral triple, and let n be a positive integer. Assume that D is invertible and that*

$$a \cdot |D|^{-n} \in \mathcal{L}^{1,\infty}(H),$$

for every $a \in A$. Then the associated Fredholm module (A, H, F) has the property that the operators

$$[F, a^0][F, a^1]\cdots[F, a^n]$$

are trace class, for every $a^0, \dots, a^n \in A$. In particular, the Fredholm module (A, H, F) is finitely summable and if n has the correct parity, then the Chern character ch_n^F is defined. □

3.14. Definition. A spectral triple (A, H, D) is *regular* if there exists an algebra B of bounded operators on H such that

(a) $A \subseteq B$ and $[D, A] \subseteq B$, and
(b) if $b \in B$ then b maps the domain of $|D|$ (which is equal to the domain of D) into itself, and moreover $|D|B - B|D| \in B$.

3.15. Example. The spectral triple $(C^\infty(S^1), L^2(S^1), D)$ of Example 3.3 is regular.

3.16. Remark. We shall look at the notion of regularity in more detail in Section 4, when we discuss elliptic estimates.

We can now state Connes' Hochschild class formula:

3.17. Theorem. *Let (A, H, D) be a regular spectral triple. Assume that D is invertible and that for some positive integer n of the same parity as the triple, and every $a \in A$,*

$$a \cdot |D|^{-n} \in \mathcal{L}^{1,\infty}(H).$$

The Chern character ch_n^F of Definition 2.22 is cohomologous, as a Hochschild cocycle, to the cocycle

$$\Phi(a^0, \dots, a^n) = \frac{\Gamma(\frac{n}{2}+1)}{n \cdot n!}\,\mathrm{Trace}_\omega(\varepsilon a^0[D, a^1][D, a^2]\cdots[D, a^n]|D|^{-n}).$$

Here ε is 1 in the odd case, and the grading operator on H in the even case.

3.18. Remark. Actually this is a slight strengthening of what is actually provable. For the correct statement, see Appendix C.

3.4. A Simple Example. We shall prove Theorem 3.17 in Appendix C, as a by-product of our proof of the local index theorem (at the moment, it is probably not even obvious that the functional Φ given in the theorem is a Hochschild cocycle). Right now what we want to do is to compute a simple example of the Hochschild cocycle Φ.

We shall consider the spectral triple $(C^\infty(S^1), L^2(S^1), D)$, where D is the unique self-adjoint extension of the operator $\frac{1}{2\pi i}\frac{d}{dx} + \frac{1}{2}$ (recall that the term $\frac{1}{2}$ was added to guarantee that D is invertible).

The operator D is diagonalizable, with eigenfunctions $e_n(x) = \exp(2\pi i n x)$ and eigenvalues $n + \frac{1}{2}$, where $n \in \mathbb{Z}$. We see then that $|D|$ is given by the formula

$$|D| e_n = |n + \tfrac{1}{2}| e_n \qquad (n \in \mathbb{Z}).$$

As a result, $|D|^{-1} \in \mathcal{L}^{1,\infty}(H)$, and a brief computation shows that

$$\mathrm{Tr}_\omega(|D|^{-1}) = 2.$$

3.19. Lemma. *If $f \in C(S^1)$ then $\mathrm{Tr}_\omega(f \cdot |D|^{-1}) = 2\int_{S^1} f(x)\,dx$.*

Proof. The linear functional $f \mapsto \mathrm{Tr}_\omega f \cdot |D|^{-1}$ is positive, and so by the Riesz representation theorem it is given by integration against a Borel measure on S^1. But the functional, and hence the measure, is rotation invariant. This proves that the measure is a multiple of Lebesgue measure, and the computation $\mathrm{Tr}_\omega(|D|^{-1}) = 2$ fixes the multiple. □

With this computation in hand, we can now determine the cocycle Φ which appears in Theorem 3.17:

$$\Phi(f_0, f_1) = \mathrm{Trace}_\omega(f_0[D, f_1]|D|^{-1}) = \frac{1}{\pi i}\int_{S^1} f_0(x) f_1'(x)\,dx.$$

Now if $a^0, a^1 \in C^\infty(S^1)$, and if Ψ is any 1-cocycle which is in fact a Hochschild coboundary, then it is easily computed that $\Psi(a^0, a^1) = 0$. As a result, of Theorem 3.17 it therefore follows that

$$\Gamma(\tfrac{3}{2}) \cdot \tfrac{1}{2}\,\mathrm{Trace}(F[F, a^0][F, a^1]) \stackrel{def}{=} \mathrm{ch}_1^F(a^0, a^1) = \Gamma(\tfrac{3}{2})\Phi(a^0, a^1)$$

If we combine this with the Fredholm index formula in Theorem 2.10 we arrive at a proof of the well-known index formula

$$\mathrm{Index}(PuP) = -\tfrac{1}{4}\,\mathrm{Trace}(F[F, u^{-1}][F, u]) = -\tfrac{1}{2}\Phi(u^{-1}, u) = -\frac{1}{2\pi i}\int_{S^1} u^{-1} du$$

associated to an invertible element $u \in C^\infty(S^1)$, which we already mentioned in Example 2.1.

3.20. Remark. In this very simple example we have determined not only the Hochschild cocycle Φ but also the cyclic cocycle ch_1^F. This is an artifact of the low-dimensionality of the example: the natural map from the first cyclic cohomology group into the first Hochschild group happens always to be injective. In higher dimensional examples a determination of Φ will in general stop quite a bit short of a determination of ch_n^F.

3.5. Weyl's Theorem. The simple computation which we carried out above has a general counterpart which originates with a famous theorem of Weyl. We shall state the theorem in the context of Dirac-type operators, for which we refer the reader to Roe's introductory text [**24**] (this book also contains a proof of Weyl's theorem).

3.21. THEOREM. *Let M be a closed Riemannian manifold of dimension n, and let D be a Dirac-type operator on M, acting on the sections of some complex Hermitian vector bundle S over M. The operator D has a unique self-adjoint extension, and $|D|^{-n} \in \mathcal{L}^{1,\infty}(H)$. Moreover*

$$\mathrm{Tr}_\omega(|D|^{-n}) = \frac{\dim(S)}{(2\sqrt{\pi})^n} \frac{\mathrm{Vol}(M)}{\Gamma(\frac{n}{2}+1)}.$$

3.22. REMARK. If D is not invertible then we define $|D|^{-n}$ by, for example, $|D|^{-1} = |D+P|^{-n}$, where P is the orthogonal projection onto the kernel of D. (Incidentally, we might note that altering an operator in $\mathcal{L}^{1,\infty}(H)$ by any finite rank operator — or indeed any trace-class operator — has no effect on the Dixmier trace.)

The theorem may be extended, as follows:

3.23. THEOREM. *Let M be a closed Riemannian manifold of dimension n, and let D be a Dirac-type operator on M, acting on the sections of some complex Hermitian vector bundle S over M. The operator D has a unique self-adjoint extension, and $|D|^{-n} \in \mathcal{L}^{1,\infty}(H)$. If F is any endomorphism of S then*

$$\mathrm{Tr}_\omega(|D|^{-n}) = \frac{1}{(2\sqrt{\pi})^n\Gamma(\frac{n}{2}+1)} \int_M \mathrm{trace}(F(x))\, dx.$$

Thanks to the theorem, the Hochschild character Φ of Theorem 3.17 may be computed in the case where $A = C^\infty(M)$, $H = L^2(S)$, and D is a Dirac-type operator acting on sections of S (it may be shown that this constitutes an example of a regular spectral triple; compare Section 4). The commutators $[D,a]$ are endomorphisms of S, and so

$$\Phi(a^0,\dots,a^n) = \frac{1}{(2\sqrt{\pi})^n\Gamma(\frac{n}{2}+1)} \int_M \mathrm{trace}(\varepsilon a^0[D,a^1]\cdots[D,a^n])\, dx.$$

3.24. REMARK. In many cases the pointwise trace which appears here can be further computed. For example, if D is the Dirac operator associated to a Spin^c-structure on M then we obtain the simple formula

$$\Phi(a^0,\dots,a^n) = \frac{1}{(2\sqrt{\pi})^n\Gamma(\frac{n}{2}+1)} \int_M a^0 da^1 \cdots da^n.$$

In summary, we see that $\Phi(a^0,\dots,a^n)$ is an integral over M of an explicit expression involving the a^j and their derivatives. Unfortunately, in higher dimensions, this very precise information about Φ is not enough to deduce an index theorem, since it is impossible to recover the pairing between cyclic cocycles and idempotents or invertibles from the Hochschild cohomology class of the cyclic cocycle. For the purposes of index theory we need to obtain a similar formula for the cyclic cocycle ch_n^F itself, or for a cocycle which is cohomologous to it in cyclic or periodic theory. This is what the Connes-Moscovici formula achieves.

The formula involves in a crucial way a *residue trace* which in certain circumstances extends to a certain class of operators, including some unbounded operators, 2 times the Dixmier trace on $\mathcal{L}^{1,\infty}(H)$. We shall discuss this in detail in the next section, but we shall conclude here with a somewhat vague formulation of the local index formula, to give the reader some idea of the direction in which we are heading. The statement will be refined in the coming sections.

3.25. THEOREM. *Let (A, H, D) be a suitable even spectral triple*[5] *and let (A, H, F) be the associated Fredholm module. The Chern character ch_n^F is cohomologous, as a (b, B)-cocycle, to the cocycle $\phi = (\phi_0, \phi_2, \dots)$ given by the formulas*

$$\phi_p(a^0, \dots, a^p) = \sum_{k \geq 0} c_{pk} \operatorname{Res} \operatorname{Tr} \left(\varepsilon a^0 [D, a^1]^{(k_1)} \cdots [D, a^p]^{(k_p)} \Delta^{-\frac{p}{2} - |k|} \right).$$

The sum is over all multi-indices $(k_1, \dots, k_p)$ with non-negative integer entries, and the constants c_{pk} are given by the formula

$$c_{pk} = \frac{(-1)^k}{k!} \frac{\Gamma(k_1 + \cdots + k_p + \frac{p}{2})}{(k_1 + 1)(k_1 + k_2 + 2) \cdots (k_1 + \cdots + k_p + p)}.$$

The operators $X^{(k)}$ are defined inductively by $X^{(0)} = X$ and $X^{(k)} = [D^2, X^{(k-1)}]$.

3.26. REMARK. Note that when $p = n$ and $k = 0$ we obtain the term

$$\begin{aligned} &\frac{\Gamma(\frac{n}{2})}{n!} \operatorname{Res} \operatorname{Tr} \left(\varepsilon a^0 [D, a^1] \cdots [D, a^p] |D|^{-n} \right) \\ &\qquad = \frac{2\Gamma(\frac{n}{2})}{n!} \operatorname{Tr}_\omega \left(\varepsilon a^0 [D, a^1] \cdots [D, a^p] |D|^{-n} \right) \\ &\qquad\qquad = \frac{\Gamma(\frac{n}{2} + 1)}{n \cdot n!} \operatorname{Tr}_\omega \left(\varepsilon a^0 [D, a^1] \cdots [D, a^p] |D|^{-n} \right). \end{aligned}$$

Thus we recover precisely the Hochschild cocycle of Theorem 3.17. The relation between Theorem 3.17 and the local index formula will be further discussed in Appendix C.

4. Differential Operators and Zeta Functions

Apart from cyclic theory, the local index theorem requires a certain amount of Hilbert space operator theory. We shall develop the necessary topics in this section, beginning with a very rapid review of the basic theory of linear elliptic operators on manifolds.

4.1. Elliptic Operators on Manifolds. Let M be a smooth, closed manifold, and let S be a smooth vector bundle over M. Let us equip M with a smooth measure and S with an inner product, so that we can form the Hilbert space $L^2(M, S)$.

Let $\mathcal{D}$ be the algebra of linear differential operators on M acting on smooth sections of S. This is an associative algebra of operators and it is filtered by the usual notion of order of a differential operator: an operator X has order q or less if any local coordinate system it can be written in the form

$$X = \sum_{|\alpha| \leq q} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha},$$

[5] There is an analogous theorem in the odd case.

where α is a multi-index $(\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

If s is a non-negative integer then the space of order s or less operators is a finitely generated module over the ring $C^\infty(M)$. If $X_1, \ldots, X_N$ is a generating set then the Sobolev space $W^s(M,S)$ is defined to be the completion of $C^\infty(M,S)$ induced from the norm

$$\|\phi\|^2_{W^s(M,S)} = \sum_j \|X_j \phi\|^2_{L^2(M,S)}.$$

Different choices of generating set result in equivalent norms and the same space $W^s(M,S)$. Every differential operator of order q extends to a bounded linear operator from $W^s(M,S)$ to $W^{s-q}(M,S)$, for all $s \geq q$. The *Sobolev Embedding Theorem* implies that that $\cap_{s\geq 0} W^s(M,S) = C^\infty(M,S)$.

Now let Δ be a linear *elliptic* operator of order r. The reader unfamiliar with the definition of ellipticity can take the following *basic estimate* as the definition: if Δ is elliptic of order r, then there is some $\varepsilon > 0$ such that

$$\|\Delta\phi\|_{W^s(M,S)} + \|\phi\|_{L^2(M,S)} \geq \varepsilon \|\phi\|_{W^{s+r}(M,S)},$$

for every $\phi \in C^\infty(M,S)$.

Suppose now that Δ is also *positive*, which is to say that $\langle \Delta\phi, \phi\rangle_{L^2(M,S)} \geq 0$, for all $\phi \in C^\infty(M,S)$. It may be shown then that Δ is essentially self-adjoint on the domain $C^\infty(M,S)$, and for $s \geq 0$ we can define the linear spaces

$$H^s = \operatorname{dom}(\Delta^{\frac{s}{r}}) \subseteq H,$$

which are Hilbert spaces in the norm

$$\|\phi\|^2_{H^s} = \|\phi\|^2_{L^2} + \|\Delta^{\frac{r}{s}}\phi\|^2_{L^2}.$$

It follows from the basic estimate that $H^s = W^s(M,S)$.

Let us say that an operator Δ of order r is of *scalar type* if in every local coordinate system Δ can be written in the form

$$\Delta = \sum_{|\alpha|\leq r} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha},$$

where the $a_\alpha(x)$, for $|\alpha| = r$, are scalar multiples of the identity operator (acting on the fiber S_x of S). Good examples are the Laplace operators $\Delta = \nabla^*\nabla$ associated to affine connections on S, which are positive, elliptic of order 2, and of scalar type. Other examples are the squares of Dirac-type operators on Riemannian manifolds. If Δ is of scalar type then

$$\operatorname{order}([\Delta, X]) \leq \operatorname{order}(X) + \operatorname{order}(\Delta) - 1$$

(whereas the individual products $X\Delta$ and ΔX have order one greater, in general).

The following theorem, which is quite well known, will be fundamental to what follows in these notes. For a proof which is somewhat in the spirit of these notes see [**19**].

4.1. Theorem. *Let Δ be elliptic of order r, positive and of scalar type, and assume for simplicity that Δ is invertible as a Hilbert space operator. Let X be any differential operator. If* $\operatorname{Re}(z)$ *is sufficiently large then the operator $X\Delta^{-z}$ is of trace class. Moreover the function* $\zeta(z) = \operatorname{Trace}(X\Delta^{-z})$ *extends to a meromorphic function on* $\mathbb{C}$ *with only simple poles.*

4.2. Remarks. The assumption that Δ is of scalar type is not necessary, but it simplifies the proof. and covers the cases of interest. The meaning of the complex power Δ^{-z} will be clarified in the coming paragraphs.

4.2. Abstract Differential Operators. In this section we shall give abstract counterparts of the ideas presented in the previous section.

Let H be a complex Hilbert space. We shall assume as given an unbounded, positive, self-adjoint operator Δ on H. The operator Δ and its powers Δ^k are provided with definite domains $\mathrm{dom}(\Delta^k) \subseteq H$, which are dense subspaces of H. We shall denote by H^∞ the intersection of the domains of all the Δ^k:

$$H^\infty = \cap_{k=1}^\infty \mathrm{dom}(\Delta^k).$$

We shall assume as given an algebra $\mathcal{D}(\Delta)$ of linear operators on the vector space H^∞. We shall assume the following things about $\mathcal{D}(\Delta)$:[6]

(i) If $X \in \mathcal{D}(\Delta)$ then $[\Delta, X] \in \mathcal{D}(\Delta)$ (we shall not insist that Δ belongs to $\mathcal{D}(\Delta)$).

(ii) The algebra $\mathcal{D}(\Delta)$ is filtered,

$$\mathcal{D}(\Delta) = \cup_{q=0}^\infty \mathcal{D}_q(\Delta) \quad \text{(an increasing union).}$$

We shall write $\mathrm{order}(X) \le q$ to denote the relation $X \in \mathcal{D}_q(\Delta)$. Sometimes we shall use the term "differential order" to refer to this filtration. This is supposed to call to mind the standard example, in which $\mathrm{order}(X)$ is the order of X as a differential operator.

(iii) There is an integer $r > 0$ (the "order of Δ") such that

$$\mathrm{order}([\Delta, X]) \le \mathrm{order}(X) + r - 1$$

for every $X \in \mathcal{D}(\Delta)$.

To state the final assumption, we need to introduce the linear spaces

$$H^s = \mathrm{dom}(\Delta^{\frac{s}{r}}) \subseteq H$$

for $s \ge 0$. These are Hilbert spaces in their own right, in the norms

$$\|v\|_s^2 = \|v\|^2 + \|\Delta^{\frac{s}{r}} v\|^2.$$

The following key condition connects the algebraic hypotheses we have placed on $\mathcal{D}(\Delta)$ to operator theory:

(iv) If $X \in \mathcal{D}(\Delta)$ and if $\mathrm{order}(X) \le q$ then there is a constant $\varepsilon > 0$ such that

$$\|v\|_q + \|v\| \ge \varepsilon \|Xv\|, \quad \forall v \in H^\infty.$$

4.3. Example. The standard example is of course that in which Δ is a Laplace-type operator $\Delta = \nabla^*\nabla$, or Δ is the square of a Dirac-type operator, on a closed manifold M and $\mathcal{D}(\Delta)$ is the algebra of differential operators on M. We can obtain a slightly more complicated example by dropping the assumption that M is compact, and defining $\mathcal{D}(\Delta)$ to be the algebra of compactly supported differential operators on M (Δ is still a Laplacian or the square of a Dirac operator). Item (i) above was formulated with the non-compact case in mind.

4.4. Remark. In the standard example the "order" r of Δ is $r = 2$. But other orders are possible. For example Connes and Moscovici consider an important example in which $r = 4$.

[6] Various minor modifications of these axioms are certainly possible.

4.5. Remark. For the purposes of these notes we could get by with something a little weaker than condition (iv), namely this:

(iv′) If $X \in \mathcal{D}(\Delta)$ and if $\operatorname{order}(X) \le kr$ then there is a constant $\varepsilon > 0$ such that

$$\|\Delta^k v\| + \|v\| \ge \varepsilon \|Xv\|, \quad \forall v \in H^\infty.$$

The advantage of this condition is that it involves only integral powers of the operator Δ (in contrast the $\| \ \|_s$ involve fractional powers of Δ). Condition (iv′) is therefore in principle easier to verify. However in the main examples, for instance the one developed by Connes and Moscovici in [**12**], the stronger condition holds.

4.6. Definition. We shall refer to an algebra $\mathcal{D}(\Delta)$ (together with the distinguished operator Δ) satisfying the axioms (i)-(iv) above as an *algebra of generalized differential operators*.

4.7. Lemma. *If $X \in \mathcal{D}(\Delta)$, and if X has order q or less, then for every $s \ge 0$ the operator X extends to a bounded linear operator from H^{s+q} to H^s.*

Proof. If s is an integer multiple of the order r of Δ then the lemma follows immediately from the elliptic estimate above. The general case (which we shall not actually need) follows by interpolation. □

4.3. Zeta Functions. Let $\mathcal{D}(\Delta)$ be an algebra of generalized differential operators, as in the previous sections. We are going to define certain zeta-type functions associated with $\mathcal{D}(\Delta)$.

To simplify matters we shall now assume that the operator Δ is invertible. *This assumption will remain in force until Section 6*, where we shall first consider more general operators Δ.

The complex powers Δ^{-z} may be defined using the functional calculus. They are, among other things, well-defined operators on the vector space H^∞.

4.8. Definition. The algebra $\mathcal{D}(\Delta)$ has *finite analytic dimension* if there is some $d \ge 0$ with the property that if $X \in \mathcal{D}(\Delta)$ has order q or less, then, for every $z \in \mathbb{C}$ with real part greater than $\frac{q+d}{r}$, the operator $X\Delta^{-z}$ extends by continuity to a trace class operator on H (here r is the order of Δ, as described in Section 4.2).

4.9. Remark. The condition on $\operatorname{Re}(z)$ is meant to imply that the order of $X\Delta^{-z}$ is less than $-d$. (We have not yet assigned a notion of order to operators such as $X\Delta^{-z}$, but we shall do so in Definition 4.15.)

4.10. Definition. The smallest value $d \ge 0$ with the property described in Definition 4.8 will be called the *analytic dimension* of the algebra $\mathcal{D}(\Delta)$.

Assume that $\mathcal{D}(\Delta)$ has finite analytic dimension d. If $X \in \mathcal{D}(\Delta)$ and if $\operatorname{order}(X) \le q$ then the complex function $\operatorname{Trace}(X\Delta^{-z})$ is holomorphic in the right half-plane $\operatorname{Re}(z) > \frac{q+d}{r}$.

4.11. Definition. An algebra $\mathcal{D}(\Delta)$ of generalized differential operators which has finite analytic dimension has the *analytic continuation property* if for every $X \in \mathcal{D}(\Delta)$ the analytic function $\operatorname{Trace}(X\Delta^{-z})$, defined initially on a half-plane in $\mathbb{C}$, extends to a meromorphic function on the full complex plane.

Actually, for what follows it would be sufficient to assume that $\operatorname{Trace}(X\Delta^{-s})$ has an analytic continuation to $\mathbb{C}$ with only isolated singularities, which could perhaps be essential singularities.[7] The analytic continuation property is obviously an abstraction of Theorem 4.1 concerning elliptic operators on manifolds.

We are ready to present what is, in effect, the main definition of these notes, in which we describe the "elementary quantities" which were mentioned in the introduction. The reasoning which leads to this definition will be explained in Appendix B.

In order to accommodate the cyclic cohomology constructions to be carried out in Section 5 we shall now assume that the Hilbert space H is $\mathbb{Z}/2$-graded, that Δ has even grading-degree, and that the grading operator $\varepsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ multiplies $\mathcal{D}(\Delta)$ into itself. (The case $\varepsilon = I$, where the grading is trivial, is one important possibility.)

4.12. Definition. Let $\mathcal{D}(\Delta)$ be an algebra of generalized differential operators which has finite analytic dimension. Define, for $\mathrm{Re}(z) \gg 0$ and $X^0, \ldots, X^p \in \mathcal{D}(\Delta)$, the quantity

$$(4.1) \quad \langle X^0, X^1, \ldots, X^p \rangle_z = (-1)^p \frac{\Gamma(z)}{2\pi i} \operatorname{Trace}\left(\int \lambda^{-z} \varepsilon X^0 (\lambda - \Delta)^{-1} X^1 (\lambda - \Delta)^{-1} \cdots X^p (\lambda - \Delta)^{-1} \, d\lambda \right)$$

(the factors in the integral alternate between the X^j and copies of $(\lambda - \Delta)^{-1}$). The contour integral is evaluated down a vertical line in $\mathbb{C}$ which separates 0 and $\mathrm{Spectrum}(\Delta)$.

4.13. Remark. The operator $(\lambda - \Delta)^{-1}$ is bounded on all of the Hilbert spaces H^s, and moreover its norm on each of these spaces is bounded by $|\operatorname{Im}(\lambda)|^{-1}$. As a result, if

$$\operatorname{order}(X^0) + \cdots + \operatorname{order}(X^p) \leq q$$

and if the integrand in equation (4.1) is viewed as a bounded operator from H^{s+q} to H^s, then the integral converges absolutely in the operator norm whenever $\mathrm{Re}(z) + p > 0$. In particular, if $\mathrm{Re}(z) > 0$ then the integral (4.1) converges to a well defined operator on H^∞.

The following result establishes the traceability of the integral (4.1), when $\mathrm{Re}(z) \gg 0$.

4.14. Proposition. *Let $\mathcal{D}(\Delta)$ be an algebra of generalized differential operators and let $X^0, \ldots, X^p \in \mathcal{D}(\Delta)$. Assume that*

$$\operatorname{order}(X^0) + \cdots + \operatorname{order}(X^p) \leq q.$$

If $\mathcal{D}(\Delta)$ has finite analytic dimension d, and if $\mathrm{Re}(z) + p > \frac{1}{r}(q + d)$, then the integral in Equation (4.1) extends by continuity to a trace-class operator on H, and the quantity $\langle X^0, \ldots, X^p \rangle_z$ defined by Equation (4.1) is a holomorphic function of z in this half-plane. If in addition the algebra $\mathcal{D}(\Delta)$ has the analytic continuation property then the quantity $\langle X^0, \ldots, X^p \rangle_z$ extends to a meromorphic function on $\mathbb{C}$.

For the purpose of proving the proposition it is useful to develop a little more terminology, as follows.

[7]The exception to this is Appendix A, which is independent of the rest of the notes, where we shall assume at one point that the singularities are all simple poles.

4.15. Definition. Let $m \in \mathbb{R}$. We shall say that an operator $T\colon H^\infty \to H^\infty$ has *analytic order m or less* if, for every s,[8] T extends to a bounded operator from H^{m+s} to H^s.

4.16. Example. The resolvents $(\lambda - \Delta)^{-1}$ have analytic order $-r$ or less.

Let us note the following simple consequence of our definitions:

4.17. Lemma. *Let $\mathcal{D}(\Delta)$ have finite analytic dimension d. If T has analytic order less than $-d-q$, and if $X \in \mathcal{D}(\Delta)$ has order q, then XT is a trace-class operator.* □

4.18. Definition. Let T and T_α ($\alpha \in A$) be operators on H^∞. We shall write

$$T \approx \sum_{\alpha \in A} T_\alpha$$

if, for every $m \in \mathbb{R}$, there is a finite set $F \subseteq A$ with the property that if $F' \subseteq A$ is a finite subset containing F then T and $\sum_{\alpha \in F'} T_\alpha$ differ by an operator of analytic order m or less.

One should think of m as being large and negative. Thus $T \approx \sum_{\alpha \in A} T_\alpha$ if every sufficiently large finite partial sum agrees with T up to operators of large negative order.

4.19. Definition. If $Y \in \mathcal{D}(\Delta)$ then denote by $Y^{(k)}$ the k-fold commutator of Y with Δ. Thus $Y^{(0)} = Y$ and $Y^{(k)} = [\Delta, Y^{(k-1)}]$ for $k \geq 1$.

4.20. Lemma. *Let $Y \in \mathcal{D}(\Delta)$ and let h be a positive integer. For every positive integer k there is an asymptotic expansion*

$$\begin{aligned}[(\lambda-\Delta)^{-h}, Y] \approx hY^{(1)}(\lambda-\Delta)^{-(h+1)} + \frac{h(h+1)}{2} Y^{(2)}(\lambda-\Delta)^{-(h+2)} + \cdots \\ + \frac{h(h+1)\cdots(h+k)}{k!} Y^{(k)}(\lambda-\Delta)^{-(h+k)} + \cdots ,\end{aligned}$$

4.21. Remark. If $\operatorname{order}(Y) \leq q$ then, according to the axioms in Section 4.2, $\operatorname{order}(Y^{(p)}) \leq q + p(r-1)$. Therefore, thanks to the elliptic estimate of Section 4.2, the operator $Y^{(p)}(\lambda-\Delta)^{-(h+p)}$ has analytic order $q - hr - p$ or less. Hence the terms in the asymptotic expansion of the lemma are of decreasing analytic order.

Proof of Lemma 4.20. Let us write $L = \lambda - \Delta$ and observe that the k-fold iterated commutator of Y with L is $(-1)^k$ times $Y^{(k)}$, the k-fold iterated commutator of Y with Δ. Let us also write $z = -h$.

To prove the lemma we shall use Cauchy's formula,

$$\binom{z}{p} L^{z-p} = \frac{1}{2\pi i} \int w^z (w-L)^{-p-1}\, dw.$$

The integral (which is carried out along the same contour as the one in Definition 4.12) is norm-convergent in the operator norm on any $\mathcal{B}(H^s)$. Applying this

[8]Strictly speaking we should say "for every $s \geq 0$ such that $m + s \geq 0$," since we have not defined H^s for negative s.

formula in the case $p = 0$ we get

$$\begin{aligned}
[L^z, Y] &= \frac{1}{2\pi i}\int w^z \big[(w-L)^{-1}, Y\big]\,dw \\
&= -\frac{1}{2\pi i}\int w^z (w-L)^{-1} Y^{(1)} (w-L)^{-1}\,dw \\
&= -Y^{(1)}\frac{1}{2\pi i}\int w^z (w-L)^{-2}\,dw \\
&\qquad - \frac{1}{2\pi i}\int w^z \big[(w-L)^{-1}, Y^{(1)}\big](w-L)^{-1}\,dw \\
&= -\binom{z}{1} Y^{(1)} L^{z-1} + \frac{1}{2\pi i}\int w^z (w-L)^{-1} Y^{(2)} (w-L)^{-2}\,dw.
\end{aligned}$$

The integrals all converge in the operator norm of $\mathcal{B}(H^{s+q}, H^s)$ for any q large enough (and in fact any $q \geq \operatorname{order}(Y)$ would do). By carrying out a sequence of similar manipulations on the remainder integral we arrive at

$$\begin{aligned}
[L^z, Y] = -&\binom{z}{1} Y^{(1)} L^{-z-1} + \binom{z}{2} Y^{(2)} L^{-z-2} - \dots \\
&+ (-1)^p \binom{z}{p} Y^{(p)} L^{-z-p} + \frac{(-1)^p}{2\pi i}\int w^z (w-L)^{-1} Y^{(p)} (w-L)^{-p}\,dw.
\end{aligned}$$

Simple estimates on the remainder integral now prove the lemma. □

We are now almost ready to prove Proposition 4.14. In the proof we shall use asymptotic expansions, as in Definition 4.18. But we shall be considering operators which, like $(\lambda - \Delta)$, depend on a parameter λ. In this situation we shall amend Definition 4.18 by writing $T \approx \sum_\alpha T_\alpha$ if, for every $m \ll 0$, every sufficiently large finite partial sum agrees with T up to an operator of analytic order m or less, whose norm as an operator from H^{s+m} to H^s is $O(|\operatorname{Im}(\lambda)|^m)$. The reason for doing so is that we shall then be able to integrate with respect to λ, and obtain an asymptotic expansion for the integrated operator.

Proof of Proposition 4.14. The idea of the proof is to use Lemma 4.20 to move all the terms $(\lambda - \Delta)^{-1}$ in $X^0(\lambda-\Delta)^{-1}\cdots X^p(\lambda-\Delta)^{-1}$ to the right. If the operators X^j actually commuted with Δ then we would of course get

$$X^0(\lambda-\Delta)^{-1}\cdots X^p(\lambda-\Delta)^{-1} = X^0\cdots X^p(\lambda-\Delta)^{-(p+1)},$$

and after integrating and applying Cauchy's integral formula we could conclude without difficulty that

$$\langle X^0, \dots, X^p\rangle_z = \frac{\Gamma(z+p)}{p!}\operatorname{Trace}(\varepsilon X^0\cdots X^p \Delta^{-z-p})$$

(compare with the manipulations below). The proposition would then follow immediately from this formula. The general case is only a little more difficult: we shall see that the above formula gives the leading term in a sort of asymptotic expansion for $\langle X^0, \dots, X^p\rangle_z$.

It will be helpful to define quantities

$$c(k_1, \dots, k_j) = \frac{(k_1 + \dots + k_j + j)!}{k_1!\cdots k_j!(k_1+1)\cdots(k_1+\dots+k_j+j)},$$

which depend on non-negative integers $k_1,\dots,k_j$. These have the property that $c(k_1)=1$, for all k_1, and

$$c(k_1,\dots,k_j)=c(k_1,\dots,k_{j-1})\frac{(k_1+\cdots+k_{j-1}+j)\cdots(k_1+\cdots+k_j+j-1)}{k_j!}$$

(the numerator in the fraction is the product of the k_j successive integers from $(k_1+\cdots+k_{j-1}+j)$ to $(k_1+\cdots+k_j+j-1)$). Using this notation and Lemma 4.20 we obtain an asymptotic expansion

$$(\lambda-\Delta)^{-1}X^1\approx\sum_{k_1\geq 0}c(k_1){X^1}^{(k_1)}(\lambda-\Delta)^{-(k_1+1)},$$

and then

$$\begin{aligned}(\lambda-\Delta)^{-1}X^1(\lambda-\Delta)^{-1}X^2 &\approx\sum_{k_1\geq 0}c(k_1){X^1}^{(k_1)}(\lambda-\Delta)^{-(k_1+2)}X^2\\ &\approx\sum_{k_1,k_2\geq 0}c(k_1,k_2){X^1}^{(k_1)}\,{X^2}^{(k_2)}(\lambda-\Delta)^{-(k_1+k_2+2)},\end{aligned}$$

and finally

$$(\lambda-\Delta)^{-1}X^1\cdots(\lambda-\Delta)^{-1}X^p\approx\sum_{k\geq 0}c(k){X^1}^{(k_1)}\cdots{X^p}^{(k_p)}(\lambda-\Delta)^{-(|k|+p)},$$

where we have written $k=(k_1,\dots,k_p)$ and $|k|=k_1+\cdots+k_p$. It follows that

$$\begin{aligned}&\frac{(-1)^p\Gamma(z)}{2\pi i}\int\lambda^{-z}(\lambda-\Delta)^{-1}\cdots X^p(\lambda-\Delta)^{-1}\,d\lambda\\ &\qquad\approx\sum_{k\geq 0}c(k){X^1}^{(k_1)}\cdots{X^p}^{(k_p)}\frac{(-1)^p\Gamma(z)}{2\pi i}\int\lambda^{-z}(\lambda-\Delta)^{-(|k|+p+1)}\,d\lambda\\ &\qquad=\sum_{k\geq 0}c(k){X^1}^{(k_1)}\cdots{X^p}^{(k_p)}(-1)^p\Gamma(z)\binom{-z}{|k|+p}\Delta^{-z-|k|-p}.\end{aligned}$$

The terms of this asymptotic expansion have analytic order $q-k-r(\mathrm{Re}(z)+p)$ or less, and therefore if $\mathrm{Re}(z)+p>\frac{1}{r}(q+d)$ then the terms all have analytic order less than $-d$. This proves the first part of the proposition: after multiplying by εX^0, if $\mathrm{Re}(z)+p>\frac{1}{r}(q+d)$ then all the terms in the asymptotic expansion are trace class, and the integral extends to a trace class operator on H. To continue, it follows from the functional equation for $\Gamma(z)$ that

$$(-1)^p\Gamma(z)\binom{-z}{|k|+p}=(-1)^{|k|}\Gamma(z+p+|k|)\frac{1}{(|k|+p)!}.$$

So multiplying by εX^0 and taking traces we get

$$\begin{aligned}\langle X^0,\dots,X^p\rangle_z\approx\sum_{k\geq 0}&(-1)^{|k|}\Gamma(z+p+|k|)\frac{1}{(|k|+p)!}c(k)\\ &\times\mathrm{Trace}\left(\varepsilon X^0{X^1}^{(k_1)}\cdots{X^p}^{(k_p)}\Delta^{-z-|k|-p}\right),\end{aligned}$$

where the symbol $\approx$ means that, given any right half-plane in $\mathbb{C}$, any sufficiently large finite partial sum of the right hand side agrees with the left hand side modulo a function of z which is holomorphic in that half-plane. The second part of the proposition follows immediately from this asymptotic expansion and Definition 4.11. $\square$

4.22. Remark. In the coming sections we shall make use of a modest generalization of the first part of Proposition 4.14, in which the operators $X^0, \dots, X^p$ are chosen from the algebra generated by $\mathcal{D}(\Delta)$ and D (a square root of the operator Δ that we shall discuss next), with at least one X^j actually in $\mathcal{D}(\Delta)$ itself. The conclusion of the proposition and the proof are the same.

At the end of Section 7 we shall also need a version of Lemma 4.20 involving powers Δ^{-h} for non-integral h. Once again the formulation of the lemma, and the proof, are otherwise unchanged.

4.4. Square Root of the Laplacian. We shall now assume that a self-adjoint operator D is specified, for which $D^2 = \Delta$. If the Hilbert space H is nontrivially $\mathbb{Z}/2$-graded we shall also assume that the operator D has grading degree 1. We shall also assume that an algebra $A \subseteq \mathcal{D}(\Delta)$ is specified, consisting of operators of differential order zero (the operators in A are therefore bounded operators on H).

4.23. Example. In the standard example, D will be a Dirac-type operator and A will be the algebra of C^∞-functions on M.

Continuing the axioms listed in Section 4.2, we shall assume that

(v) If $a \in A \subseteq \mathcal{D}(\Delta)$ then $[D, a] \in \mathcal{D}(\Delta)$.

We shall also assume that

(vi) If $a \in A$ then $\operatorname{order}\big([D, a]\big) \leq \operatorname{order}(D) - 1$, where we set $\operatorname{order}(D) = \frac{r}{2}$.

4.5. Spectral Triples. In Section 5 we shall use the square root D to construct cyclic cocycles for the algebra A from the quantities $\langle X^0, \dots, X^p \rangle_z$. But first we shall conclude our discussion of analytic preliminaries by briefly discussing the relation between our algebras $\mathcal{D}(\Delta)$ and the notion of spectral triple.

4.24. Definition. A *spectral triple* is a triple (A, H, D), composed of a complex Hilbert space H, an algebra A of bounded operators on H, and a self-adjoint operator D on H with the following two properties:

(i) If $a \in A$ then the operator $a \cdot (1 + D^2)^{-1}$ is compact.
(ii) If $a \in A$ then $a \cdot \operatorname{dom}(D) \subseteq \operatorname{dom}(D)$ and the commutator $[D, a]$ extends to a bounded operator on H.

Various examples are listed in [**12**]; in the standard example A is the algebra of smooth functions on a complete Riemannian manifold M, D is a Dirac-type operator on M, and H is the Hilbert space of L^2-sections of the vector bundle on which D acts.

Let (A, H, D) be a spectral triple. Let $\Delta = D^2$, and as in Section 4.2 let us define

$$H^\infty = \cap_{k=1}^\infty \operatorname{dom}(\Delta^k) = \cap_{k=1}^\infty \operatorname{dom}(D^k).$$

Let us assume that A maps the space H^∞ into itself (this does not follow automatically). Having done so, let us define $\mathcal{D}(A, D)$ to be the smallest algebra of linear operators on H^∞ which contains A and $[D, A]$ and which is closed under the operation $X \mapsto [\Delta, X]$. Note that $\mathcal{D}(A, D)$ does not necessarily contain D.

Equip the algebra $\mathcal{D}(A, D)$ with the smallest filtration so that

(i) If $a \in A$ then $\operatorname{order}(a) = 0$ and $\operatorname{order}\big([D, a]\big) = 0$.
(ii) If $X \in \mathcal{D}(A, D)$ then $\operatorname{order}\big([\Delta, X]\big) \leq \operatorname{order}(X) + 1$.

The term "smallest" means here that we write $\operatorname{order}(X) \leq q$ if and only if the order of X is q or less in every filtration satisfying the above conditions (there is at least one such filtration). Having filtered $\mathcal{D}(A, D)$ in this way we obtain an example of the sort of algebra of generalized differential operators which was considered in Section 4.2.

Denote by δ the unbounded derivation of $\mathcal{B}(H)$ given by commutator with $|D|$. Thus the domain of δ is the set of all bounded operators T which map the domain of $|D|$ into itself, and for which the commutator extends to a bounded operator on H.

4.25. Definition. A spectral triple is *regular* if A and $[D, A]$ belong to $\cap_{n=1}^{\infty} \delta^n$.

We want to prove the following result.

4.26. Theorem. *Let (A, H, D) be a spectral triple with the property that every $a \in A$ maps H^∞ into itself. It is regular if and only if the algebra $\mathcal{D}(A, D)$ satisfies the basic estimate (iv) of Section 4.2.*

The proof is based on the following computation. Denote by B the algebra of operators on H^∞ generated by all the spaces $\delta^n[A]$ and $\delta^n\big[[D, A]\big]$, for all $n \geq 0$. According to the definition of regularity every operator in B extends to a bounded operator on H.

4.27. Lemma. *Assume that (A, H, D) is a regular spectral triple. Every operator in $\mathcal{D}$ of order k may be written as a finite sum of operators $b|D|^\ell$, where b belongs to the algebra B and where $\ell \leq k$.*

Proof. The spaces $\mathcal{D}_k$ of operators of order k or less in $\mathcal{D}(A, D)$ may be defined inductively as follows:

(a) $\mathcal{D}_0 =$ the algebra generated by $A + [D, A]$.
(b) $\mathcal{D}_1 = [\Delta, \mathcal{D}_0] + \mathcal{D}_0[\Delta, \mathcal{D}_0]$.
(c) $\mathcal{D}_k = \sum_{j=1}^{k-1} \mathcal{D}_j \cdot \mathcal{D}_{k-j} + [\Delta, \mathcal{D}_{k-1}] + \mathcal{D}_0[\Delta, \mathcal{D}_{k-1}]$.

Define $\mathcal{E}$, a space of operators on H^∞, to be the linear span of the operators of the form $b|D|^k$, where $k \geq 0$. The space $\mathcal{E}$ is an algebra since

$$b_1|D|^{k_1} \cdot b_2|D|^{k_2} = \sum_{j=0}^{k_1} \binom{k_1}{j} b_1 \delta^j(b_2)|D|^{k_1+k_2-j}.$$

Filter the algebra $\mathcal{E}$ by defining $\mathcal{E}_k$ to be the span of all operators $b|D|^\ell$ with $\ell \leq k$. The formula above shows that this does define a filtration of the algebra $\mathcal{E}$. Now the algebra $\mathcal{D}$ of differential operators is contained within $\mathcal{E}$, and the lemma amounts to the assertion that $\mathcal{D}_k \subseteq \mathcal{E}_k$. Clearly $\mathcal{D}_0 \subseteq \mathcal{E}_0$. Using the formula

$$[\Delta, b|D|^{k-1}] = [|D|^2, b|D|^{k-1}] = 2\delta(b)|D|^k + \delta^2(b)|D|^{k-1}$$

and our formula for $\mathcal{D}_k$ the inclusion $\mathcal{D}_k \subseteq \mathcal{E}_k$ is easily proved by induction. $\square$

Proof of Theorem 4.26, Part One. Suppose that (A, H, D) is regular. According to the lemma, to prove the basic estimate for $\mathcal{D}(A, D)$ it suffices to prove that if $k \geq \ell$ and if $X = b|D|^\ell$, where $b \in B$, then there exists $\varepsilon > 0$ such that

$$\|D^k v\| + \|v\| \geq \varepsilon \|Xv\|,$$

for every $v \in H^\infty$. But we have

$$\|Xv\| = \|b|D|^\ell v\| \le \|b\| \cdot \||D|^\ell v\| = \|b\| \cdot \|D^\ell v\|,$$

And since by spectral theory for every $\ell \le k$ we have that

$$\|D^\ell v\|^2 \le \|D^k v\|^2 + \|v\|^2 \le \big(\|D^k v\| + \|v\|\big)^2$$

it follows that

$$\|D^k v\| + \|v\| \ge \frac{1}{\|b\|+1}\|Xv\|,$$

as required. □

To prove the converse, we shall develop a *pseudodifferential calculus*, as follows.

4.28. Definition. Let (A, H, D) be a spectral triple for which $A \cdot H^\infty \subseteq H^\infty$, and for which the basic elliptic estimate holds. Fix an operator $K\colon H^\infty \to H^\infty$ of order $-\infty$ and such that $\Delta_1 = \Delta + K$ is invertible. A *basic pseudodifferential operator of order $k \in \mathbb{Z}$* is a linear operator $T\colon H^\infty \to H^\infty$ with the property that for every $\ell \in Z$ the operator T may be decomposed as

$$T = X\Delta_1^{\frac{m}{2}} + R,$$

where $X \in \mathcal{D}(A, D)$, $m \in \mathbb{Z}$, and $R\colon H^\infty \to H^\infty$, and where

$$\operatorname{order}(X) + m \le k \quad \text{and} \quad \operatorname{order}(R) \le \ell.$$

A *pseudodifferential operator of order $k \in \mathbb{Z}$* is a finite linear combination of basic pseudodifferential operators of order k.

4.29. Remarks. Every pseudodifferential operator is a sum of two basic operators (one with the integer m in Definition 4.28 even, and one with m odd). The class of pseudodifferential operators does not depend on the choice of operator K.

4.30. Lemma. *If T is a pseudodifferential operator and $z \in \mathbb{C}$ then*

$$[\Delta_1^z, T] \approx \sum_{j=1}^{\infty} \binom{z}{j} T^{(j)} \Delta_1^{z-j}.$$

Proof. See the proof of Lemma 4.20. □

4.31. Proposition. *The set of all pseudodifferential operators is a filtered algebra. If T is a pseudodifferential operator then so is $\delta(T)$, and moreover* $\operatorname{order}(\delta(T)) \le \operatorname{order}(T)$.

Proof. The set of pseudodifferential operators is a vector space. The formula

$$X\Delta_1^{\frac{m}{2}} \cdot Y\Delta_1^{\frac{n}{2}} \approx \sum_{j=0}^{\infty} \binom{\frac{m}{2}}{j} XY^{(j)} \Delta_1^{\frac{m+n}{2}-j}$$

shows that it is closed under multiplication. Finally,

$$\begin{aligned} \delta(T) = |D|T - T|D| &\approx \Delta_1^{\frac{1}{2}} T - T\Delta_1^{\frac{1}{2}} \\ &\approx \sum_{j=1}^{\infty} \binom{\frac{1}{2}}{j} T^{(j)} \Delta_1^{\frac{1}{2}-j}. \end{aligned}$$

This computation reduces the second part of the lemma to the assertion that if T is a pseudodifferential operator of order k then $T^{(1)} = [\Delta, T]$ is a pseudodifferential operator of order $k+1$ or less. This in turn follows from the definition of pseudodifferential operator and the fact that if X is a differential operator then the differential operator $[\Delta, X]$ has order at most one more than the order of X. $\square$

Proof of Theorem 4.26, Part Two. Suppose that (A, H, D) is a spectral triple for which $A \cdot H^\infty \subseteq H^\infty$ and for which the basic estimate holds. By the basic estimate, every pseudodifferential operator of order zero extends to a bounded operator on H. Since every operator in A or $[D, A]$ is pseudodifferential of order zero, and since $\delta(T)$ is pseudodifferential of order zero whenever T is, we see that if $b \in A$ or $b \in [D, A]$ then for every n the operator $\delta^n(b)$ extends to a bounded operator on H. Hence the spectral triple (A, H, D) is regular, as required. $\square$

4.32. Definition. A spectral triple (A, H, D) is *finitely summable* if there is a Schatten ideal $\mathcal{L}^p(H)$ (where $1 \le p < \infty$)) such that

$$a \cdot (1 + D^2)^{-\frac{1}{2}} \in \mathcal{L}^p(H)$$

for every $a \in A$.

If the spectral triple (A, H, D) is regular and finitely summable then for every $X \in \mathcal{D}(A, D)$ the zeta function $\mathrm{Trace}(X\Delta^{-\frac{z}{2}})$ is defined in a right half-plane in $\mathbb{C}$, and is holomorphic there. The following concept has been introduced by Connes and Moscovici [**12**, Definition II.1].

4.33. Definition. Let (A, H, D) be a regular and finitely summable spectral triple. It has *discrete dimension spectrum* if[9] there is a discrete subset F of $\mathbb{C}$ with the following property: for every operator X in $\mathcal{D}(A, D)$ if $\mathrm{order}(X) \le q$ then the zeta function $\mathrm{Trace}(X\Delta^{-\frac{z}{2}})$ extends to a meromorphic function on $\mathbb{C}$ with all poles contained in $F + q$. The *dimension spectrum* of (A, H, D) is then the smallest such set F.

4.34. Remark. The definition above extends without change to arbitrary algebras of generalized differential operators, and at one point (in Section 6) we shall use it in this context.

A final item of terminology: in Appendix A we shall make use of the following notion:

4.35. Definition. A regular and finitely summable spectral triple has *simple* dimension spectrum if it has discrete dimension spectrum and if all the zeta-type functions above have only simple poles.

5. The Residue Cocycle

In this section we shall assume as given an algebra $\mathcal{D}(\Delta)$, a square root D of Δ, and an algebra $A \subseteq \mathcal{D}(\Delta)$, as in the previous sections. We shall assume the finite analytic dimension and analytic continuation properties set forth in Definitions 4.8 and 4.11. We shall also assume that the Hilbert space H is nontrivially $\mathbb{Z}/2$-graded and therefore that the operator D has odd grading-degree. This is the "even-dimensional" case. The "odd-dimensional" case, where H has no grading, will be considered separately in Section 7.4.

[9]Connes and Moscovici add a technical condition concerning decay of zeta functions along vertical lines in $\mathbb{C}$; compare Appendix A.

5.1. Improper Cocycle. We are going to define a periodic cyclic cocycle $\Psi = (\Psi_0, \Psi_2, \dots)$ for the algebra A. The cocycle will be *improper*, in the sense that all the Ψ_p will be (typically) nonzero. Moreover the cocycle will assume values in the field of meromorphic functions on $\mathbb{C}$. But in the next section we shall convert it into a proper cocycle with values in $\mathbb{C}$ itself.

We are going to assemble Ψ from the quantities $\langle X^0, \dots, X^p \rangle_z$ defined in Section 4. In doing so we shall follow quite closely the construction of the JLO cocycle in entire cyclic cohomology (see [**20**] and [**17**]), which is assembled from the quantities

$$\langle X^0, \dots, X^p \rangle^{\mathrm{JLO}} = \mathrm{Trace}\left(\int_{\Sigma^p} \varepsilon X^0 e^{-t_0\Delta} \dots X^p e^{-t_p \Delta}\, dt \right) \tag{5.1}$$

(the integral is over the standard p-simplex). In Appendix A we shall compare our cocycle to the JLO cocycle. For now, let us note that the quantities in Equation (5.1) are scalars, while the quantities $\langle X^0, \dots, X^p \rangle_z$ are functions of the parameter z. But this difference is superficial, and the computations which follow in this section are more or less direct copies of computations already carried out for the JLO cocycle in [**20**] and [**17**].

We begin by establishing some "functional equations" for the quantities $\langle \cdots \rangle_z$. In order to keep the formulas reasonably compact, if $X \in \mathcal{D}(\Delta)$ then we shall write $(-1)^X$ to denote either $+1$ or -1, according as the $\mathbb{Z}/2$-grading degree of X is even or odd.

5.1. Lemma. *The meromorphic functions $\langle X^0, \dots, X^p \rangle_z$ satisfy the following functional equations:*

$$\langle X^0, \dots, X^{p-1}, X^p \rangle_{z+1} = \sum_{j=0}^{p} \langle X^0, \dots, X^{j-1}, 1, X^j, \dots, X^p \rangle_z \tag{5.2}$$

$$\langle X^0, \dots, X^{p-1}, X^p \rangle_z = (-1)^{X^p} \langle X^p, X^0, \dots, X^{p-1} \rangle_z \tag{5.3}$$

Proof. The first identity follows from the fact that

$$\begin{aligned} \frac{d}{d\lambda} &\left(\lambda^{-z} X^0 (\lambda - \Delta)^{-1} \cdots X^p (\lambda - \Delta)^{-1} \right) \\ &= (-z) \lambda^{-z-1} X^0 (\lambda - \Delta)^{-1} \cdots X^p (\lambda - \Delta)^{-1} \\ &\quad - \sum_{j=0}^{p} \lambda^{-z} X^0 (\lambda - \Delta)^{-1} \cdots X^j (\lambda - \Delta)^{-2} X^{j+1} \cdots X^p (\lambda - \Delta)^{-1} \end{aligned}$$

and the fact that the integral of the derivative is zero. As for the second identity, if $p \gg 0$ then the integrand in Equation (4.1) is a trace-class operator, and Equation (5.3) is an immediate consequence of the trace-property. In general we can repeatedly apply Equation (5.2) to reduce to the case where $p \gg 0$. □

5.2. Lemma.

$$\begin{aligned} \langle X^0, \dots, [D^2, X^j], \dots, X^p \rangle_z = \\ \langle X^0, \dots, X^{j-1} X^j, \dots X^p \rangle_z - \langle X^0, \dots, X^j X^{j+1}, \dots X^p \rangle_z \end{aligned} \tag{5.4}$$

PROOF. This follows from the identity

$$X^{j-1}(\lambda-\Delta)^{-1}[D^2,X^j](\lambda-\Delta)^{-1}X^{j+1} = X^{j-1}(\lambda-\Delta)^{-1}X^jX^{j+1} - X^{j-1}X^j(\lambda-\Delta)^{-1}X^{j+1}.$$

Note that when $j=p$, equation (5.4) should read as

$$\langle X^0,\dots,X^{p-1},[D^2,X^p]\rangle_z = \langle X^0,\dots,X^{p-1}X^p\rangle_z - (-1)^{X^p}\langle X^pX^0,\dots,X^{p-1}\rangle_z.$$

□

5.3. LEMMA.

$$\sum_{j=0}^{p}(-1)^{X^0\cdots X^{j-1}}\langle X^0,\dots,[D,X^j],\dots,X^p\rangle_z = 0 \tag{5.5}$$

PROOF. The identity is equivalent to the formula

$$\operatorname{Trace}\left(\varepsilon\Big[D,\int\lambda^{-z}X^0(\lambda-\Delta)^{-1}\cdots X^p(\lambda-\Delta)^{-1}\,d\lambda\Big]\right)=0,$$

which holds since the supertrace of any (graded) commutator is zero. □

With these preliminaries out of the way we can obtain very quickly the (improper) (b,B)-cocycle which is the main object of study in these notes.

5.4. DEFINITION. If p is a non-negative and even integer then define a $(p+1)$-multilinear functional on A with values in the meromorphic functions on $\mathbb{C}$ by the formula

$$\Psi_p(a^0,\dots,a^p) = \langle a^0,[D,a^1],\dots,[D,a^p]\rangle_{s-\frac{p}{2}}$$

5.5. THEOREM. *The even (b,B)-cochain $\Psi=(\Psi_0,\Psi_2,\Psi_4\cdots)$ is an (improper) (b,B)-cocycle.*

PROOF. First of all, it follows from the definition of B and Lemma 5.1 that

$$\begin{aligned}
B\Psi_{p+2}(a^0,\dots,a^{p+1}) &= \sum_{j=0}^{p+1}(-1)^j\langle 1,[D,a^j],\dots,[D,a^{j-1}]\rangle_{s-\frac{p+2}{2}}\\
&= \sum_{j=0}^{p+1}\langle [D,a^0],\dots,[D,a^{j-1}],1,[D,a^j],\dots,[D,a^{p+1}]\rangle_{s-\frac{p+2}{2}}\\
&= \langle [D,a^0],[D,a^1],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}.
\end{aligned}$$

Next, it follows from the definition of b and the Leibniz rule $[D,a^ja^{j+1}] = a^j[D,a^{j+1}]+[D,a^j]a^{j+1}$ that

$$\begin{aligned}
b\Psi_p(a^0,\dots,a^{p+1}) &= \big(\langle a^0a^1,[D,a^2],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}\\
&\qquad - \langle a^0,a^1[D,a^2],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}\big)\\
&\quad - \big(\langle a^0,[D,a^1]a^2,[D,a^3],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}\\
&\qquad - \langle a^0,[D,a^1],a^2[D,a^3],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}\big)\\
&\quad + \cdots\\
&\quad + \big(\langle a^0,[D,a^1],\dots,[D,a^p]a^{p+1}\rangle_{s-\frac{p}{2}}\\
&\qquad - \langle a^{p+1}a^0,[D,a^1],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}\big).
\end{aligned}$$

Applying Lemma 5.2 we get

$$b\Psi_p(a^0,\dots,a^{p+1}) = \sum_{j=1}^{p+1}(-1)^{j-1}\langle a^0,[D,a^1],\dots,[D^2,a^j],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}$$

Setting $X^0 = a^0$ and $X^j = [D,a^j]$ for $j \geq 1$, and applying Lemma 5.3 we get

$$\begin{aligned} &B\Psi_{p+2}(a^0,\dots,a^{p+1}) + b\Psi_p(a^0,\dots,a^{p+1}) \\ &\qquad = \sum_{j=0}^{p+1}(-1)^{X^0\cdots X^{j-1}}\langle X^0,\dots,[D,X^j],\dots,X^{p+1}\rangle_{s-\frac{p}{2}} = 0. \end{aligned}$$

□

5.2. Residue Cocycle. By taking residues at $s = 0$ we map the space of meromorphic functions on $\mathbb{C}$ to the scalar field $\mathbb{C}$, and we obtain from any (b,B)-cocycle with coefficients in the space of meromorphic functions a (b,B)-cocycle with coefficients in $\mathbb{C}$. This operation transforms the improper cocycle Ψ that we constructed in the last section into a *proper* cocycle $\operatorname{Res}_{s=0}\Psi$. Indeed, it follows from Proposition 4.14 that if p is greater than the analytic dimension d of $\mathcal{D}(\Delta)$ then the function

$$\Psi_p(a^0,\dots,a^p)_s = \langle a^0,[D,a^1],\dots,[D,a^p]\rangle_{s-\frac{p}{2}}$$

is holomorphic at $s = 0$.

The following proposition identifies the proper (b,B)-cocycle $\operatorname{Res}_{s=0}\Psi$ with the residue cocycle studied by Connes and Moscovici.

5.6. Theorem. *For all $p \geq 0$ and all $a^0,\dots,a^p \in A$,*

$$\begin{aligned} &\operatorname{Res}_{s=0}\Psi_p(a^0,\dots,a^p) \\ &\qquad = \sum_{k\geq 0} c_{p,k}\operatorname{Res}_{s=0}\operatorname{Tr}\Big(\varepsilon a^0[D,a^1]^{(k_1)}\cdots[D,a^p]^{(k_p)}\Delta^{-\frac{p}{2}-|k|-s}\Big). \end{aligned}$$

The sum is over all multi-indices $(k_1,\dots,k_p)$ with non-negative integer entries, and the constants c_{pk} are given by the formula

$$c_{pk} = \frac{(-1)^k}{k!}\frac{\Gamma(|k|+\frac{p}{2})}{(k_1+1)(k_1+k_2+2)\cdots(k_1+\cdots+k_p+p)}$$

5.7. Remarks. Before proving the theorem we need to make one or two comments about the above formula.

First, the constant $c_{00} = \Gamma(0)$ is not well defined since 0 is a pole of the Γ-function. To cope with this problem we replace the term $c_{00}\operatorname{Res}_{s=0}\big(\operatorname{Tr}(\varepsilon a^0\Delta^{-s})\big)$ with $\operatorname{Res}_{s=0}\big(\Gamma(s)\operatorname{Tr}(\varepsilon a^0\Delta^{-s})\big)$.

Second, it follows from Proposition 4.14 that if $|k|+p > d$ then the (p,k)-contribution to the above sum of residues is actually zero. Hence for every p the sum is in fact finite (and as we already noted above, the sum is 0 when $p > d$).

PROOF OF THEOREM 5.6. We showed in the proof of Proposition 4.14 that

$$\langle X^0, \ldots, X^p\rangle_z \approx \sum_{k\geq 0}(-1)^{|k|}\Gamma(z+p+|k|)\frac{c(k)}{(|k|+p)!} \times \operatorname{Trace}\left(\varepsilon X^0 {X^1}^{(k_1)}\cdots {X^p}^{(k_p)}\Delta^{-z-|k|-p}\right).$$

After we note that

$$c_{pk} = (-1)^{|k|}\Gamma\big(|k|+\frac{p}{2}\big)\frac{c(k)}{(p+|k|)!}$$

the proof of the theorem follows immediately from the asymptotic expansion upon setting $z = s - \frac{p}{2}$ and taking residues at $s = 0$. □

6. The Local Index Formula

The objective of this section is to compute the pairing between the periodic cyclic cocycle $\operatorname{Res}_{s=0}\Psi$ and idempotents in the algebra A (compare Section 2.5). We shall prove the following result.

6.1. THEOREM. *Let* $\operatorname{Res}_{s=0}\Psi$ *be the index cocycle associated to an algebra* $\mathcal{D}(\Delta)$ *of generalized differential operators with finite analytic dimension and the analytic continuation property, together with a square root* D *of* Δ *and a subalgebra* $A \subseteq \mathcal{D}(\Delta)$*. If* e *is an idempotent element of* A *then*

$$\langle \operatorname{Res}_{s=0}\Psi, e\rangle = \operatorname{Index}\big(eDe\colon eH_0 \to eH_1\big).$$

Theorem 6.1 will later be superseded by a more precise result at the level of cyclic cohomology, and we shall we shall only sketch one or two parts of the proof which will be dealt with in more detail later. Furthermore, to slightly simplify the analysis we shall assume that $\mathcal{D}(\Delta)$ has discrete dimension spectrum, in the sense of Definition 4.33.

6.1. Invertibility Hypothesis Removed.

Up to now we have been assuming that the self-adjoint operator Δ is invertible (in the sense of Hilbert space operator theory, meaning that Δ is a bijection from its domain to the Hilbert space H). We shall now remove this hypothesis.

To do so we shall begin with an operator D which is not necessarily invertible (with $D^2 = \Delta$). We shall assume that the axioms (i)-(vi) in Sections 4.2 and 4.4 hold. Fix a bounded self-adjoint operator K with the following properties:

(i) K commutes with D.
(ii) K has analytic order $-\infty$ (in other words, $K \cdot H \subseteq H^\infty$).
(iii) The operator $\Delta + K^2$ is invertible.

Having done so, let us construct the operator

$$D_K = \begin{pmatrix} D & K \\ K & -D\end{pmatrix}$$

acting on the Hilbert space $H \oplus H^{\mathrm{opp}}$, where H^{opp} is the $\mathbb{Z}/2$-graded Hilbert space H but with the grading reversed. It is invertible.

6.2. EXAMPLE. If D is a Fredholm operator then we can choose for K the projection onto the kernel of D.

Let $\Delta_K = (D_K)^2$ and denote by $\mathcal{D}(\Delta_K)$ the smallest algebra of operators on $H \oplus H^{\mathrm{opp}}$ which contains the 2×2 matrices over $\mathcal{D}(\Delta)$ and which is closed under multiplication by operators of analytic order $-\infty$.

The axioms (i)-(iv) of Section 4.2 are satisfied for the new algebra. Moreover, if we embed A into $\mathcal{D}(\Delta_K)$ as matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then the axioms (v) and (vi) in Section 4.4 are satisfied too.

6.3. Lemma. *Assume that the operators K_1 and K_2 both have the properties* (i)-(iii) *listed above. Then $\mathcal{D}(\Delta_{K_1}) = \mathcal{D}(\Delta_{K_1})$. Moreover the algebra has finite analytic dimension d and has the analytic continuation property with respect to Δ_{K_1} if and only if it has the same with respect to Δ_{K_2}. If these properties do hold then the quantities $\langle X^0, \ldots, X^p\rangle_z$ associated to Δ_{K_1} and Δ_{K_2} differ by a function which is analytic in the half-plane* $\mathrm{Re}(z) > -p$.

Proof. It is clear that $\mathcal{D}(\Delta_{K_1}) = \mathcal{D}(\Delta_{K_2})$. To investigate the analytic continuation property it suffices to consider the case where K_1 is a fixed function of Δ, in which case K_1 and K_2 commute. Let us write

$$X\Delta^{-z} = \frac{1}{2\pi i}\int \lambda^{-z} X(\lambda - \Delta)^{-1}\, d\lambda$$

for $\mathrm{Re}(z) > 0$. Observe now that

$$(\lambda - \Delta_{K_1})^{-1} - (\lambda - \Delta_{K_2})^{-1} \approx M(\lambda - \Delta_{K_1})^{-2} - M(\lambda - \Delta_{K_1})^{-3} + \cdots,$$

where $M = \Delta_{K_1} - \Delta_{K_2}$ (this is an asymptotic expansion in the sense described prior to the proof of Proposition 4.14). Integrating and taking traces we see that

$$\text{(6.1)}\qquad \mathrm{Trace}\big(X\Delta_{K_1}^{-z}\big) - \mathrm{Trace}\big(X\Delta_{K_2}^{-z}\big) \approx \sum_{k\geq 1} (-1)^{k-1}\binom{-z}{k}\mathrm{Trace}\big(XM\Delta_{K_1}^{-z-k}\big),$$

which shows that the difference $\mathrm{Trace}\big(X\Delta_{K_1}^{-z}\big) - \mathrm{Trace}\big(X\Delta_{K_2}^{-z}\big)$ has an analytic continuation to an entire function. Therefore Δ_{K_1} has the analytic continuation property if and only if Δ_{K_2} does (and moreover the analytic dimensions are equal).

The remaining part of the lemma follows from the asymptotic formula

$$\begin{aligned}\langle X^0, \ldots, X^p\rangle_z \approx \sum_{k\geq 0} (-1)^{|k|}\Gamma(z + p + |k|)\frac{1}{(|k| + p)!}c(k) \\ \times \mathrm{Trace}\left(\varepsilon X^0 {X^1}^{(k_1)} \cdots {X^p}^{(k_p)}\Delta^{-z-|k|-p}\right)\end{aligned}$$

that we proved earlier. □

6.4. Definition. The *residue cocycle* associated to the possibly non-invertible operator D is the residue cocycle $\mathrm{Res}_{s=0}\Psi$ associated to the invertible operator D_K, as above.

Lemma 6.3 shows that if $p > 0$ then the residue cocycle given by Definition 6.4 is independent of the choice of the operator K. In fact this is true when $p = 0$ too. Indeed, Equation (6.1) shows that not only is the difference $\mathrm{Trace}(\varepsilon a^0\Delta_{K_1}^{-s}) - \mathrm{Trace}(\varepsilon a^0\Delta_{K_2}^{-s})$ analytic at $s = 0$, but it vanishes there too. Therefore

$$\begin{aligned}\mathrm{Res}_{s=0}\Psi_0^{K_1}(a^0) - \mathrm{Res}_{s=0}\Psi_0^{K_2}(a^0) \\ = \mathrm{Res}_{s=0}\Gamma(s)\left(\mathrm{Trace}(\varepsilon a^0\Delta_{K_1}^{-s}) - \mathrm{Trace}(\varepsilon a^0\Delta_{K_2}^{-s})\right) = 0.\end{aligned}$$

6.5. Example. If D happens to be invertible already then we obtain the same residue cocycle as before.

6.6. Example. In the case where D is Fredholm, the residue cocycle is given by the same formula that we saw in Theorem 5.6:

$$\begin{aligned}\operatorname{Res}_{s=0}\Psi_p(a^0,\dots,a^p) \\ = \sum_{k\geq 0} c_{p,k}\operatorname{Res}_{s=0}\operatorname{Tr}\left(\varepsilon a^0[D,a^1]^{(k_1)}\cdots[D,a^p]^{(k_p)}\Delta^{-\frac{p}{2}-|k|-s}\right).\end{aligned}$$

The complex powers Δ^{-z} are defined to be zero on the kernel of D (which is also the kernel of Δ). When $p=0$ the residue cocycle is

$$\operatorname{Res}_{s=0}\big(\Gamma(s)\operatorname{Trace}(\varepsilon a^o\Delta^{-s})\big)+\operatorname{Trace}(\varepsilon a^0 P),$$

where the complex power Δ^{-s} is defined as above and P is the orthogonal projection onto the kernel of D.

6.2. Proof of the Index Theorem. Let us fix an idempotent $e\in A$ and define a family of operators by the formula

$$D_t = D + t[e,[D,e]], \qquad t\in[0,1].$$

Note that $D_0=D$ while $D_1=eDe+e^\perp De^\perp$, so that in particular D_1 commutes with e. Denote by Ψ^t the improper cocycle associated to D_t (via the mechanism just described in the last section which involves the incorporation of some order $-\infty$ operator K_t, which we shall assume depends smoothly on t).

6.7. Lemma. *Define an improper (b,B)-cochain Θ^t by the formula*

$$\begin{aligned}\Theta^t_p(a^0,\dots,a^p) = \\ \sum_{j=0}^{p}(-1)^{j-1}\langle a^0,\dots,[D_{K_t},a^j],\dot D_{K_t},[D_{K_t},a^{j+1}],\dots,[D_{K_t},a^p]\rangle_{s-\frac{p+1}{2}},\end{aligned}$$

where $\dot D=\frac{d}{dt}D_{K_t}\in\mathcal{D}(\Delta_K)$. Then

$$B\Theta^t_{p+1}+b\Theta^t_{p-1}+\frac{d}{dt}\Psi^t_p=0.$$

□

The lemma, which is nothing more than an elaborate computation, can be proved by following the steps taken in Section 7.1 below (compare Remark 7.9).

Proof of Theorem 6.1. It follows from the asymptotic expansion method used to prove Lemma 6.3 that each Ψ^t and each Θ^t is meromorphic. Since we are assuming that $\mathcal{D}(\Delta)$ has discrete dimension spectrum the poles of all these functions are located within the same discrete set in $\mathbb{C}$. As a result, the integral $\int_0^1\Theta^t\,dt$ is clearly meromorphic too. Since

$$B\int_0^1\Theta^t_{p+1}\,dt+b\int_0^1\Theta^t_{p-1}\,dt=\Psi^0-\Psi^1,$$

it follows by taking residues that $\operatorname{Res}_{s=0}\Psi^0$ and $\operatorname{Res}_{s=1}\Psi^1$ are cohomologous. As a result, we can compute the pairing $\langle\operatorname{Res}_{s=0}\Psi,e\rangle$ using Ψ^1 in place of Ψ^0. If we choose the operator K_1 to commute with not only D_1 but also e, then D_{K_1}

commutes with e and the explicit formula for the pairing $\langle \mathrm{Res}_{s=0} \Psi, e\rangle$ given in Theorem 2.27 simplifies, as follows:

$$\begin{aligned}\langle \mathrm{Res}_{s=0} \Psi^1, e\rangle &= \mathrm{Res}_{s=0} \Psi^1_0(e) + \sum_{k\geq 1} (-1)^k \frac{(2k)!}{k!} \mathrm{Res}_{s=0} \Psi^1_{2k}(e - \frac{1}{2}, e, \dots, e)\\ &= \mathrm{Res}_{s=0} \Psi^1_0(e)\end{aligned}$$

(all the higher terms vanish since they involve commutators $[D_{K_1}, e]$). We conclude that

$$\begin{aligned}\langle \mathrm{Res}_{s=0} \Psi^1, e\rangle &= \mathrm{Res}_{s=0} \Psi^1_0(e)\\ &= \mathrm{Res}_{s=0} \Big(\Gamma(s)\, \mathrm{Trace}(\varepsilon e\, (\Delta_{K_1})^{-s})\Big)\\ &= \mathrm{Index}\big(eDe \colon eH_0 \to eH_1\big),\end{aligned}$$

as required (the last step is the index computation made by Atiyah and Bott that we mentioned in the introduction).

□

6.8. Remark. The proof of the corresponding odd index formula (involving the odd pairing in Theorem 2.27) is not quite so simple, but could presumably be accomplished following the argument developed by Getzler in [**16**] for the JLO cocycle.

7. The Local Index Theorem in Cyclic Cohomology

Our goal in this section is to identify the cohomology class of the residue cocycle $\mathrm{Res}_{s=0} \Psi$ with the cohomology class of the Chern character cocycle ch^F_n associated to the operator $F = D|D|^{-1}$ (see Section 2.1). Here n is any even integer greater than or equal to the analytic dimension d. It follows from the definition of analytic dimension and some simple manipulations that

$$[F, a^0] \cdots [F, a^n] \in \mathcal{L}^1(H),$$

for such n, so that the Chern character cocycle is well-defined.

We shall reach the goal in two steps. First we shall identify the cohomology class of $\mathrm{Res}_{s=0} \Psi$ with the class of a certain specific cyclic cocycle, which involves no residues. Secondly we shall show that this cyclic cocycle is cohomologous to the Chern character ch^F_n.

To begin, we shall return to our assumption that D is invertible, and then deal with the general case at the end of the section.

7.1. Reduction to a Cyclic Cocycle. The following result summarizes step one.

7.1. Theorem. *Fix an even integer n strictly greater than $d-1$. The multilinear functional*

$$(a^0, \dots, a^n) \mapsto \frac{1}{2} \sum_{j=0}^{n} (-1)^{j+1} \langle [D, a^0], \dots, [D, a^j], D, [D, a^{j+1}], \dots, [D, a^n]\rangle_{-\frac{n}{2}}.$$

is a cyclic n-cocycle which, when considered as a (b, B)-cocycle, is cohomologous to the residue cocycle $\mathrm{Res}_{s=0} \Psi$.

7.2. Remark. It follows from Proposition 4.14 that the quantities

$$\langle [D,a^0],\dots,[D,a^j],D,[D,a^{j+1}],\dots,[D,a^n]\rangle_z$$

which appear in the theorem are holomorphic in the half-plane $\mathrm{Re}(z) > -\frac{n}{2} + \frac{1}{r}(d-(n+1))$. Therefore it makes sense to evaluate them at $z = -\frac{n}{2}$, as we have done. Appearances might suggest otherwise, because the term $\Gamma(z)$ which appears in the definition of $\langle \dots \rangle_z$ has poles at the non-positive integers (and in particular at $z = -\frac{n}{2}$ if n is even). However these poles are canceled by zeroes of the contour integral in the given half-plane.

Theorem 7.1 and its proof have a simple conceptual explanation, which we shall give in a little while (after Lemma 7.8). However, a certain amount of elementary, if laborious, computation is also involved in the proof, and we shall get to work on this first. For this purpose it is useful to introduce the following notation.

7.3. Definition. If $X^0,\dots,X^p$ are operators in the algebra generated by $\mathcal{D}(\Delta)$, 1 and D, and if at least one belongs to $\mathcal{D}(\Delta)$ then define

$$\langle\langle X^0,\dots,X^p\rangle\rangle_z = \sum_{k=0}^{p} (-1)^{X^0\cdots X^k}\langle X^0,\dots,X^k,D,X^{k+1},\dots,X^p\rangle_z,$$

which is a meromorphic function of $z\in\mathbb{C}$.

The new notation allows us to write a compact formula for the cyclic cocycle appearing in Theorem 7.1:

$$(a^0,\dots,a^n)\mapsto \tfrac{1}{2}\langle\langle [D,a^0],\dots,[D,a^n]\rangle\rangle_{-\frac{n}{2}}.$$

We shall now list some properties of the quantities $\langle\langle \cdots \rangle\rangle_z$ which are analogous to the properties of the quantities $\langle \cdots \rangle_z$ that we verified in Section 5. The following lemma may be proved using the formulas in Lemmas 5.1 and 5.2.

7.4. Lemma. *The quantity $\langle\langle X^0,\dots,X^p\rangle\rangle_z$ satisfies the following identities:*

$$\langle\langle X^0,\dots,X^p\rangle\rangle_z = \langle\langle X^p,X^0,\dots,X^{p-1}\rangle\rangle_z \tag{7.1}$$

$$\sum_{j=0}^{p}\langle\langle X^0,\dots,X^j,1,X^{j+1},\dots,X^p\rangle\rangle_{z+1} = \langle\langle X^0,\dots,X^p\rangle\rangle_z \tag{7.2}$$

In addition,

$$\begin{aligned}&\langle\langle X^0,\dots,X^{j-1}X^j,\dots,X^p\rangle\rangle_z - \langle\langle X^0,\dots,X^jX^{j+1},\dots,X^p\rangle\rangle_z\\ &\quad = \langle\langle X^0,\dots,[D^2,X^j],\dots,X^p\rangle\rangle_z - (-1)^{X^0\cdots X^{j-1}}\langle X^0,\dots,[D,X^j],\dots,X^p\rangle_z.\end{aligned} \tag{7.3}$$

(In both instances within this last formula the commutators are graded commutators.) □

7.5. Remark. When $j=p$, equation (7.3) should be read as

$$\begin{aligned}&\langle\langle X^0,\dots,X^{p-1}X^p\rangle\rangle_z - \langle\langle X^pX^0,\dots,X^{p-1}\rangle\rangle_z\\ &\quad = \langle\langle X^0,\dots,X^{p-1},[D^2,X^p]\rangle\rangle_z - (-1)^{X^0\cdots X^{p-1}}\langle X^0,\dots,X^{p-1},[D,X^p]\rangle_z.\end{aligned}$$

We shall also need a version of Lemma 5.3, as follows.

7.6. Lemma.

$$\sum_{j=0}^{p}(-1)^{X^0\cdots X^{j-1}}\langle\!\langle X^0,\dots,[D,X^j],\dots,X^p\rangle\!\rangle_z = 2\sum_{k=0}^{p}\langle X^0,\dots,X^{k-1},D^2,X^k,\dots,X^p\rangle_z. \tag{7.4}$$

□

Proof. This follows from Lemma 5.3. Note that $[D,D]=2D^2$, which helps explain the factor of 2 in the formula. □

The formula in Lemma 7.6 can be simplified by means of the following computation:

7.7. Lemma.

$$\sum_{j=0}^{p}\langle X^0,\dots,X^j,D^2,X^{j+1},\dots,X^p\rangle_z = (z+p)\langle X^0,\dots,X^p\rangle_z$$

Proof. If we substitute into the integral which defines $\langle X^0,\dots,D^2,\dots,X^p\rangle_z$ the formula

$$D^2 = \lambda - (\lambda-\Delta)$$

we obtain the (supertrace of the) terms

$$(-1)^{p+1}\frac{\Gamma(z)}{2\pi i}\int \lambda^{-z+1}X^0(\lambda-\Delta)^{-1}\cdots 1(\lambda-\Delta)^{-1}\cdots X^p(\lambda-\Delta)^{-1}\,d\lambda$$
$$-(-1)^{p+1}\frac{\Gamma(z)}{2\pi i}\int \lambda^{-z}X^0(\lambda-\Delta)^{-1}\cdots X^p(\lambda-\Delta)^{-1}\,d\lambda$$

Using the functional equation $\Gamma(z)=(z-1)\Gamma(z-1)$ we therefore obtain the quantity

$$(z-1)\langle X^0,\dots,X^j,1,X^{j+1},\dots,X^p\rangle_{z-1} + \langle X^0,\dots,X^p\rangle_z$$

(the change in the sign preceding the second bracket comes from the fact that the bracket contains one fewer term, and the fact that $(-1)^{p+1}=-(-1)^p$). Adding up the terms for each j, and using Lemma 5.1 we therefore obtain

$$\begin{aligned}\sum_{j=0}^{p}\langle X^0,\dots,X^j,D^2,X^{j+1},\dots,X^p\rangle_z &= (z-1)\langle X^0,\dots,X^p\rangle_z + (p+1)\langle X^0,\dots,X^p\rangle_z\\ &= (z+p)\langle X^0,\dots,X^p\rangle_z\end{aligned}$$

as required. □

Putting together the last two lemmas we obtain the formula

$$\sum_{j=0}^{p}(-1)^{X^0\cdots X^{j-1}}\langle\!\langle X^0,\dots,[D,X^j],\dots,X^p\rangle\!\rangle_z = 2(z+p)\langle X^0,\dots,X^p\rangle_z. \tag{7.5}$$

With this in hand we can proceed to the following computation:

7.8. Lemma. *Define multilinear functionals Θ_p on A, with values in the space of meromorphic functions on $\mathbb{C}$, by the formulas*

$$\Theta_p(a^0, \dots, a^p) = \langle\!\langle a^0, [D, a^1], \dots, [D, a^p] \rangle\!\rangle_{s - \frac{p+1}{2}}.$$

Then

$$B\Theta_{p+1}(a^0, \dots, a^p) = \langle\!\langle [D, a^0], \dots, [D, a^p] \rangle\!\rangle_{s - \frac{p}{2}}.$$

and in addition

$$b\Theta_{p-1}(a^0, \dots, a^p) + B\Theta_{p+1}(a^0, \dots, a^p) = 2s\Psi_p(a^0, \dots, a^p)$$

for all $s \in \mathbb{C}$ and all $a^0, \dots, a^p \in A$.

Proof. The formula for $B\Theta_{p+1}(a^0, \dots, a^p)$ is a consequence of Lemma 7.4. The computation of $b\Theta_{p-1}(a^0, \dots, a^p)$ is a little more cumbersome, although still elementary. It proceeds as follows. First we use the Leibniz rule to write

$$\begin{aligned} b\Theta_{p-1}(a^0, \dots, a^p) &= \sum_{j=0}^{p-1} (-1)^j \langle\!\langle a^0, \dots, [D, a^j a^{j+1}], \dots, [D, a^p] \rangle\!\rangle_{s - \frac{p}{2}} \\ &\qquad + (-1)^p \langle\!\langle a^p a^0, [D, a^1], \dots, [D, a^{p-1}] \rangle\!\rangle_{s - \frac{p}{2}} \\ &= \langle\!\langle a^0 a^1, [D, a^2], \dots, [D, a^p] \rangle\!\rangle_{s - \frac{p}{2}} \\ &\quad + \sum_{j=1}^{p-1} (-1)^j \langle\!\langle a^0, \dots, a^j [D, a^{j+1}], \dots, [D, a^p] \rangle\!\rangle_{s - \frac{p}{2}} \\ &\quad + \sum_{j=1}^{p-1} (-1)^j \langle\!\langle a^0, \dots, [D, a^j] a^{j+1}, \dots, [D, a^p] \rangle\!\rangle_{s - \frac{p}{2}} \\ &\quad + (-1)^p \langle\!\langle a^p a^0, [D, a^1], \dots, [D, a^{p-1}] \rangle\!\rangle_{s - \frac{p}{2}} \end{aligned}$$

Next we rearrange the terms to obtain the formula

$$\begin{aligned} b\Theta_{p-1}(a^0, \dots, a^p) = &\Big(\langle\!\langle a^0 a^1, [D, a^2], \dots, [D, a^p] \rangle\!\rangle_{s - \frac{p}{2}} \\ &\qquad - \langle\!\langle a^0, a^1 [D, a^2], \dots, [D, a^p] \rangle\!\rangle_{s - \frac{p}{2}} \Big) \\ &+ \sum_{j=1}^{p-2} (-1)^j \Big(\langle\!\langle a^0, \dots, [D, a^j] a^{j+1}, \dots, [D, a^p] \rangle\!\rangle_{s - \frac{p}{2}} \\ &\qquad - \langle\!\langle a^0, \dots, a^{j+1} [D, a^{j+2}], \dots, [D, a^p] \rangle\!\rangle_{s - \frac{p}{2}} \Big) \\ &+ (-1)^{p-1} \Big(\langle\!\langle a^0, [D, a^1], \dots, [D, a^{p-1}] a^p \rangle\!\rangle_{s - \frac{p}{2}} \\ &\qquad - \langle\!\langle a^p a^0, [D, a^1], \dots, [D, a^{p-1}] \rangle\!\rangle_{s - \frac{p}{2}} \Big). \end{aligned}$$

We can now apply Lemma 7.6:

$$\begin{aligned}
b\Theta_{p-1}(a^0,\dots,a^p) = \Big(&\langle\langle a^0,[D^2,a^1],[D,a^2],\dots,[D,a^p]\rangle\rangle_{s-\frac{p}{2}} \\
&+ \langle a^0,[D,a^1],[D,a^2],\dots,[D,a^p]\rangle_{s-\frac{p}{2}}\Big) \\
&+\sum_{j=1}^{p-2}(-1)^j\Big(\langle\langle a^0,[D,a^1],\dots,[D^2,a^{j+1}],\dots,[D,a^p]\rangle\rangle_{s-\frac{p}{2}} \\
&\qquad + (-1)^j\,\langle a^0,[D,a^1],\dots,[D,a^{j+1}],\dots,[D,a^p]\rangle_{s-\frac{p}{2}}\Big) \\
&+(-1)^{p-1}\Big(\langle\langle a^0,[D,a^1],\dots,[D,a^{p-1}],[D^2,a^p]\rangle\rangle_{s-\frac{p}{2}} \\
&\qquad + (-1)^{p-1}\langle a^0,[D,a^1],\dots,[D,a^{p-1}],[D,a^p]\rangle_{s-\frac{p}{2}}\Big).
\end{aligned}$$

Collecting terms we get

$$\begin{aligned}
b\Theta_{p-1}(a^0,\dots,a^p) = &\sum_{k=1}^{p}(-1)^{k-1}\langle\langle a^0,[D,a^1],\dots,[D^2,a^k],\dots,[D,a^p]\rangle\rangle_{s-\frac{p}{2}} \\
&+ p\langle a^0,[D,a^1],\dots,[D,a^p]\rangle_{s-\frac{p}{2}}.
\end{aligned}$$

All that remains now is to add together $b\Theta$ and $B\Theta$, and apply Equation (7.5) to the result. Writing $a^0 = X^0$ and $[D,a^j] = X^j$ for $j = 1,\dots,p$ we get

$$\begin{aligned}
&b\Theta_{p-1}(a^0,\dots,a^p) + B\Theta_{p+1}(a^0,\dots,a^p) \\
&\quad = \sum_{j=0}^{p}(-1)^{X^0\cdots X^{j-1}}\langle\langle X^0,\dots,[D,X^j],\dots,X^p\rangle\rangle_{s-\frac{p}{2}} - p\langle X^0,\dots,X^p\rangle_{s-\frac{p}{2}} \\
&\quad = 2(s-\frac{p}{2}+p)\langle X^0,\dots,X^p\rangle_{s-\frac{p}{2}} - p\langle X^0,\dots,X^p\rangle_{s-\frac{p}{2}} \\
&\quad = 2s\langle X^0,\dots,X^p\rangle_{s-\frac{p}{2}},
\end{aligned}$$

as the lemma requires. □

7.9. Remark. The statement of Lemma 7.8 can be explained as follows. If we replace D by tD and Δ by $t^2\Delta$ in the definitions of $\langle\cdots\rangle_z$ and Ψ_p, so as to obtain a new improper (b,B)-cocycle $\Psi^t = (\Psi^t_0,\Psi^t_2,\dots)$, then it is easy to check from the definitions that

$$\Psi^t_p(a^0,\dots,a^p) = t^{-2s}\Psi_p(a^0,\dots,a^p).$$

Now, we expect that as t varies, the cohomology class of the cocycle Ψ^t should not change. And indeed, by borrowing known formulas from the theory of the JLO cocycle (see for example [**17**], or [**18**, Section 10.2], or Section 6 below) we can construct a (b,B)-cochain Θ such that

$$B\Theta + b\Theta + \frac{d}{dt}\Psi^t = 0.$$

This is the same Θ as that which appears in the lemma.

The proof of Theorem 7.1 is now very straightforward:

Proof of Theorem 7.1. According to Lemma 7.8 the (b,B)-cochain

$$\big(\mathrm{Res}_{s=0}\big(\frac{1}{2s}\Theta_1\big),\mathrm{Res}_{s=0}\big(\frac{1}{2s}\Theta_3\big),\dots,\mathrm{Res}_{s=0}\big(\frac{1}{2s}\Theta_{n-1}\big),0,0,\dots\big)$$

cobounds the difference of $\operatorname{Res}_{s=0}\Psi$ and the cyclic n-cocycle $\operatorname{Res}_{s=0}\big(\frac{1}{2s}B\Theta_{n+1}\big)$. Since

$$\operatorname{Res}_{s=0}\Big(\frac{1}{2s}B\Theta_{n+1}\Big)(a^0,\dots,a^n)=\frac{1}{2}\langle\!\langle [D,a^0],\dots,[D,a^n]\rangle\!\rangle_{-\frac{n}{2}}$$

the theorem is proved. □

7.2. Computation with the Cyclic Cocycle. We turn now to the second step. We are going to alter D by means of the following homotopy:

$$D_t = D|D|^{-t} \qquad (0\le t\le 1)$$

(the same strategy is employed by Connes and Moscovici in [**10**]). We shall similarly replace Δ with $\Delta_t = D_t^2$, and we shall use Δ_t in place of Δ in the definitions of $\langle\cdots\rangle_z$ and of $\langle\!\langle\cdots\rangle\!\rangle_z$.

To simplify the notation we shall drop the subscript t in the following computation and denote by $\dot D = -D_t\cdot\log|D|$ the derivative of the operator D_t with respect to t.

7.10. Lemma. *Define a multilinear functional on A, with values in the analytic functions on the half-plane* $\operatorname{Re}(z)+n>\frac{d-1}{2}$*, by the formula*

$$\Phi^t_n(a^0,\dots,a^n)=\langle\!\langle a^0\dot D,[D,a^1],\dots,[D,a^n]\rangle\!\rangle_z.$$

Then $B\Phi^t_n$ is a cyclic $(n-1)$-cochain and

$$\begin{aligned} &bB\Phi^t_n(a^0,\dots,a^n)\\ &\qquad=\frac{d}{dt}\langle\!\langle[D,a^0],\dots,[D,a^n]\rangle\!\rangle_z+(2z+n)\sum_{j=0}^{n}\langle\dot D,[D,a^j],\dots,[D,a^{j-1}]\rangle_z.\end{aligned}$$

7.11. Remark. Observe that the operator $\log|D|$ has analytic order δ or less, for every $\delta>0$. As a result, the proof of Proposition 4.14 shows that the quantity is a holomorphic function of z in the half-plane $\operatorname{Re}(z)+n>\frac{d-1}{2}$. But we shall not be concerned with any possible meromorphic continuation to $\mathbb{C}$.

Proof. Let us take advantage of the fact that $bB+Bb=0$ and compute $Bb\Phi^t$ instead (fewer minus signs and wrap-around terms are involved).

A straightforward application of the definitions in Section 2 shows that the quantity $Bb\Phi^t_n(a^0,\dots,a^n)$ is the sum, from $j=0$ to $j=n$, of the following terms:

$$\begin{aligned} &-\langle\!\langle\dot D,[D,a^ja^{j+1}],[D,a^{j+2}],\dots,[D,a^{j-1}]\rangle\!\rangle_z\\ &\quad+\langle\!\langle\dot D,[D,a^j],[D,a^{j+1}a^{j+2}],\dots,[D,a^{j-1}]\rangle\!\rangle_z\\ &\qquad-\cdots\\ &\qquad\quad+(-1)^n\langle\!\langle\dot D,[D,a^j],[D,a^{j+1}],\dots,[D,a^{j-2}a^{j-1}]\rangle\!\rangle_z.\end{aligned}$$

If we add the term $\langle\!\langle\dot Da^j,[D,a^{j+1}],\dots,[D,a^{j-1}\rangle\!\rangle_z$ to the beginning of this expression, and also the terms

$$\begin{aligned} &-\langle\!\langle a^{j-1}\dot D,[D,a^ja^{j+1}],[D,a^{j+2}],\dots,[D,a^{j-2}]\rangle\!\rangle_z\\ &\quad-\langle\!\langle[\dot D,a^j],[D,a^{j+1}],\dots,[D,a^{j-1}]\rangle\!\rangle_z\end{aligned}$$

at the end, then, after summing over all j, we have added zero in total. But we can now invoke Leibniz's rule to expand $[D,a^ka^{k+1}]$ and apply part (iii) of Lemma 7.4

to obtain the quantity

$$
\begin{aligned}
&\Big(\langle\!\langle \dot D, [D^2, a^j], [D, a^{j+2}], \dots, [D, a^{j-1}]\rangle\!\rangle_z \\
&\quad - \langle\!\langle \dot D, [D, a^j], [D^2, a^{j+1}], \dots, [D, a^{j-1}]\rangle\!\rangle_z \\
&\qquad + \dots \\
&\qquad\quad + (-1)^n \langle\!\langle \dot D, [D, a^j], [D, a^{j+1}], \dots, [D^2, a^{j-1}]\rangle\!\rangle_z\Big) \\
&+ (n+1)\langle \dot D, [D, a^j], [D, a^{j+1}], \dots, [D, a^{j-1}]\rangle_z \\
&\qquad\qquad - \langle\!\langle [\dot D, a^j], [D, a^{j+1}], \dots, [D, a^{j-1}]\rangle\!\rangle_z
\end{aligned}
$$

(summed over j, as before). Applying Equation 7.5 we arrive at the following formula:

$$
\begin{aligned}
Bb\Phi_n^t(a^0, \dots, a^n) = {} & \textstyle\sum_{j=0}^n \langle\!\langle [D, \dot D], [D, a^j], [D, a^{j+2}], \dots, [D, a^{j-1}]\rangle\!\rangle_z \\
& - (2z + (n+1))\textstyle\sum_{j=0}^n \langle \dot D, [D, a^j], \dots, [D, a^{j-1}]\rangle_z \\
& \quad - \textstyle\sum_{j=0}^n \langle\!\langle [\dot D, a^j], [D, a^{j+1}], \dots, [D, a^{j-1}]\rangle\!\rangle_z .
\end{aligned}
$$

To complete the proof we write

$$
\langle\!\langle [D, a^0], \dots, [D, a^p]\rangle\!\rangle_z = \textstyle\sum_{j=0}^n \langle D, [D, a^j], \dots, [D, a^{j-1}]\rangle_z
$$

and differentiate with respect to t, bearing in mind the definition of the quantities $\langle \dots \rangle_z$ and the fact that $\frac{d}{dt}(\lambda - \Delta)^{-1} = (\lambda - \Delta)^{-1}[D, \dot D](\lambda - \Delta)^{-1}$. We obtain

$$
\begin{aligned}
\tfrac{d}{dt}\langle\!\langle [D, a^0], \dots, [D, a^p]\rangle\!\rangle_z = {} & -\textstyle\sum_{j=0}^n \langle\!\langle [D, \dot D], [D, a^j], [D, a^{j+2}], \dots, [D, a^{j-1}]\rangle\!\rangle_z \\
& + \textstyle\sum_{j=0}^n \langle \dot D, [D, a^j], \dots, [D, a^{j-1}]\rangle_z \\
& \quad + \textstyle\sum_{j=0}^n \langle\!\langle [\dot D, a^j], [D, a^{j+1}], \dots, [D, a^{j-1}]\rangle\!\rangle_z .
\end{aligned}
$$

This proves the lemma. □

We can now complete the second step, and with it the proof of the Connes-Moscovici Residue Index Theorem:

7.12. Theorem (Connes and Moscovici). *The residue cocycle* $\operatorname{Res}_{s=0}\Psi$ *is cohomologous, as a* (b, B)*-cocycle, to the Chern character cocycle of Connes.*

Proof. Thanks to Theorem 7.1 it suffices to show that the cyclic cocycle

$$
\frac{1}{2}\langle\!\langle [D, a^0], \dots, [D, a^n]\rangle\!\rangle_{-\frac{n}{2}} \tag{7.6}
$$

is cohomologous to the Chern character. To do this we use the homotopy D_t above. Thanks to Lemma 7.10 the coboundary of the cyclic cochain

$$
\int_0^1 B\Psi_n^t(a^0, \dots, a^{n-1})\, dt
$$

is the difference of the cocycles (7.6) associated to $D_0 = D$ and $D_1 = F$. For D_1 we have $D_1^2 = \Delta_1 = I$ and so

$$
\begin{aligned}
\frac{1}{2}\langle\!\langle [D_1, a^0], \dots, [D_1, a^n] \rangle\!\rangle_z \\
&= \frac{1}{2}\sum_{j=1}^{n}(-1)^{j+1}\frac{(-1)^{n+1}\Gamma(z)}{2\pi i} \times \\
&\qquad \operatorname{Trace}\left(\int \lambda^{-z}\varepsilon[F,a^0]\cdots[F,a^j]F\cdots[F,a^n](\lambda - I)^{-(n+2)}\, d\lambda\right).
\end{aligned}
$$

Since F anticommutes with each operator $[F, a^j]$ this simplifies to

$$
\frac{1}{2}\sum_{j=1}^{n}\frac{(-1)^{n+1}\Gamma(z)}{2\pi i}\operatorname{Trace}\left(\int \lambda^{-z}\varepsilon F[F,a^0]\cdots[F,a^n](\lambda - I)^{-(n+2)}\, d\lambda\right).
$$

The terms in the sum are now all the same, and after applying Cauchy's formula we get

$$
\frac{n+1}{2}(-1)^{n+1}\Gamma(z)\cdot \operatorname{Trace}\left(\varepsilon F[F,a^0]\cdots[F,a^n]\right)\cdot \binom{-z}{n+1}.
$$

Using the functional equation for the Γ-function, this reduces to

$$
\frac{\Gamma(z+n+1)}{2\cdot n!}\operatorname{Trace}\left(\varepsilon F[F,a^0]\cdots[F,a^n]\right)
$$

and evaluating at $z = -\frac{n}{2}$ we obtain the Chern character of Connes. $\square$

7.3. Invertibility Hypothesis Removed. In the case where D is non-invertible we employ the device introduced in Section 6.1, and associate to D the residue cocycle for the operator D_K.

Now Connes' Chern character cocycle is defined for a not necessarily invertible operator D by forming first D_K, then $F_K = D_K|D_K|^{-1}$, then $\operatorname{ch}_n^{F_K}$. See [**4**, Part I]. The following result therefore follows immediately from our calculations in the invertible case.

7.13. Theorem. *For any operator D, invertible or not, the class in periodic cyclic cohomology of the residue cocycle* $\operatorname{Res}_{s=0}\Psi$ *is equal to the class of the Chern character cocycle of Connes.* $\square$

7.4. The Odd-Dimensional Case. We shall briefly indicate the changes which must be made to deal with the "odd" degree case, consisting of a self-adjoint operator D on a trivially graded Hilbert space H.

The basic definition of the quantity $\langle \cdots \rangle_z$ is unchanged, except of course that now we set $\varepsilon = I$, and so we could omit ε from Equation (4.1). The formula

$$
\Psi^p(a^0, \dots, a^p) = \langle a^0, [D,a^1], \dots, [D,a^p]\rangle_{s-\frac{p}{2}}
$$

now defines an *odd*, improper cocycle, with values in the meromorphic functions on $\mathbb{C}$. The proof of this is almost the same as the proof of Theorem 5.5. We obtain the formula

$$\begin{aligned}(7.7)\quad B\Psi_{p+2}(a^0,\dots,a^{p+1}) + b\Psi_p(a^0,\dots,a^{p+1}) \\ = \langle [D,a^0],[D,a^1],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}} \\ + \sum_{j=1}^{p+1}(-1)^{j-1}\langle a^0,[D,a^1],\dots,[D^2,a^j],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}},\end{aligned}$$

as in that proof, but instead of appealing to Lemma 5.3 we now note that

$$[D^2,a^j] = D[D,a^j] + [D,a^j]D$$

and

$$\langle \cdots,[D,a^{j-1}],D[D,a^j],\dots\rangle_z = \langle\cdots,[D,a^{j-1}]D,[D,a^j],\dots\rangle_z.$$

Using these relations the right hand side of Equation (7.7) telescopes to 0. The computation of $\operatorname{Res}_{s=0}\Psi$ is unchanged from the proof of Theorem 5.6, except for the omission of ε.

With similar modifications to the proofs of Lemmas 7.8 and 7.10 we obtain without difficulty the odd version of Theorem 7.12.

Appendix A. Comparison with the JLO Cocycle

In this appendix we shall use the residue theorem and the Mellin transform of complex analysis to compare the residue cocycle with the JLO cocycle.

The JLO cocycle, discovered by Jaffe, Lesniewski and Osterwalder [**20**], was developed in the context of spectral triples, as in Section 4.5, and accordingly we shall begin with such a spectral triple (A,H,D). Since we are going to compare the JLO cocycle with the residue cocycle we shall assume that (A,H,D) has the additional properties considered in Section 4.5 (although the theory of the JLO cocycle itself can be developed in greater generality). Thus we shall assume that our spectral triple is regular, is finitely summable, and has discrete dimension spectrum. We shall also make an additional assumption later on in this section.

We shall consider only the even, $\mathbb{Z}/2$-graded case here, but the odd case can be developed in exactly the same way.

A.1. Definition. If $X^0,\dots,X^p$ are bounded operators on H, and if $t>0$, let us define

$$\langle X^0,\dots,X^p\rangle_t^{\mathrm{JLO}} = t^{\frac{p}{2}}\operatorname{Trace}\left(\int_{\Sigma^p}\varepsilon X^0e^{-u_0t\Delta}\dots X^pe^{-u_pt\Delta}\,du\right).$$

The integral is over the standard p-simplex

$$\Sigma^p = \left\{\ (u_0,\dots,u_p)\,\middle|\, u_j\geq 0\ \&\ u_0+\dots+u_p=1\right\}.$$

The *JLO cocycle* is the improper (b,B)-cocycle

$$(a^0,\dots,a^p)\mapsto\langle a^0,[D,a^1],\dots,[D,a^p]\rangle_t^{\mathrm{JLO}},$$

which should be thought of here as a cocycle with coefficients in the space of functions of $t>0$.

Strictly speaking the "traditional" JLO cocycle is given by the above formula for the particular value $t=1$. Our formula for $\langle a^0,[D,a^1],\dots,[D,a^p]\rangle_t^{\mathrm{JLO}}$ corresponds to the traditional cocycle associated to the operator $t^{\frac12}D$. It will be quite convenient to think of the JLO cocycle as a function of $t>0$.

Of course, it is a basic result that the JLO cocycle really *is* a cocycle. See [**20**] or [**17**].

The proper context for the JLO cocycle is Connes' entire cyclic cohomology [**5, 7**] for Banach algebras. We shall not describe this theory here, except to say that there is a natural map

$$HCP^*(A) \to HCP^*_{\text{entire}}(A),$$

and that the arguments which follow show that the image of the residue cocycle in entire cyclic cohomology is the JLO cocycle.[10]

The following formula (which is essentially due to Connes [**6**, Equation (17)]) exhibits the connection between the JLO cocycle and the cocycle that we constructed in Section 5.

A.2. Lemma. *If $p > 0$ and if $X^0, \ldots, X^p$ are generalized differential operators in $\mathcal{D}(A, D)$, then*

$$\langle X^0, \ldots, X^p \rangle_t^{\text{JLO}} \\ = t^{-\frac{p}{2}} \frac{(-1)^p}{2\pi i} \operatorname{Trace}\left(\int e^{-t\lambda} \varepsilon X^0 (\lambda - \Delta)^{-1} X^1 \cdots X^p (\lambda - \Delta)^{-1}\, d\lambda \right).$$

As in Section 4.3 the contour integral should be evaluated along a (downward pointing) vertical line in the complex plane which separates 0 from the spectrum of Δ. The hypotheses guarantee the absolute convergence of the integral, in the norm-topology. The formula in the lemma is also correct for $p = 0$, but in this case the integral has to be suitably interpreted since it does not converge in the ordinary sense.

Proof of the Lemma. For simplicity let us assume that the operators X^j are bounded (this is the only case of the lemma that we shall use below).

By Cauchy's Theorem, we may replace the contour of integration along which the contour integral is computed by the imaginary axis in $\mathbb{C}$ (traversed upward). Having done so we obtain the formula

$$\text{(A.1)} \quad \frac{(-1)^p}{2\pi i} \int e^{-t\lambda} X^0 (\lambda - \Delta)^{-1} X^1 \cdots X^p (\lambda - \Delta)^{-1}\, d\lambda \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itv} X^0 (iv + \Delta)^{-1} \cdots X^p (iv + \Delta)^{-1}\, dv.$$

Note that this has the appearance of an inverse Fourier transform. As for the JLO cocycle, if we define functions g^j from $\mathbb{R}$ into the bounded operators on H by

$$u \mapsto \begin{cases} X^j e^{-u\Delta} & \text{if } u \geq 0 \\ 0 & \text{if } u < 0, \end{cases}$$

then we obtain the formula

$$\text{(A.2)} \quad t^{\frac{p}{2}} \int_{\Sigma^p} X^0 e^{-u_0 t\Delta} \ldots X^p e^{-u_p t\Delta}\, du \\ = t^{-\frac{p}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g^0(t - u_1) g^1(u_1 - u_2) \cdots g^{n-1}(u_{n-1} - u_n) g^n(u_n)\, du_1 \ldots du_n,$$

[10] This result can be improved somewhat. Entire cyclic cohomology is defined for locally convex algebras, and one can identify the JLO cocycle and the residue cocycle in the entire cyclic cohomology of various completions of A.

which has the form of a convolution product, evaluated at t.

Suppose now that $f^0, \ldots, f^n$ are Schwartz-class functions from $\mathbb{R}$ into a Banach algebra B. Define their Fourier transforms in the obvious way, by the formulas

$$\widehat{f^j}(v) = \int_{-\infty}^{\infty} e^{-iuv} f(u)\, du.$$

Then, just as in ordinary Fourier theory, one has the formula

$$(f^0 \star \cdots \star f^n)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itv} \widehat{f^0}(v) \cdots \widehat{f^n}(v)\, du. \tag{A.3}$$

Returning to the case at hand, let $B = B(H)$ and let $f^j_\delta(u)$ be the convolution product of a C^∞, compactly supported bump function $\delta^{-1}\phi(\delta^{-1}x)$ with the function g^j. Applying the formula (A.3) to the functions f^j_δ and then taking the limit as $\delta \to 0$ we obtain the equality of (A.1) and (A.2), which proves the lemma. □

A.3. Lemma. *If $p \geq 0$ and $a^0, \ldots, a^p \in A$, then there is some $\alpha > 0$ such that*

$$\left| \langle a^0, [D, a^1], \ldots, [D, a^p] \rangle_t^{\mathrm{JLO}} \right| = O(e^{-\alpha t})$$

as $t \to \infty$. In addition if $k > \frac{d-p}{2}$ then

$$\left| \langle a^0, [D, a^1], \ldots, [D, a^p] \rangle_t^{\mathrm{JLO}} \right| = O\big(t^{-k}\big)$$

as $t \to 0$, □

Proof. See [**18**, Equation (10.47)] for the first relation, and [**18**, Equation (10.43)] for the second. □

The following proposition now shows that the improper cocycle which we considered in Section 5 is the Mellin transform of the JLO cocycle.

For the rest of this section let us fix a real number $k > \frac{d-p}{2}$.

A.4. Proposition. *If $p \geq 0$ and $a^0, \ldots, a^p \in A$, and if $\mathrm{Re}(s) > k$ then*

$$\langle a^0, [D, a^1], \ldots, [D, a^p] \rangle_{s-\frac{p}{2}} = \int_0^\infty \langle a^0, [D, a^1], \ldots, [D, a^p] \rangle_t^{\mathrm{JLO}} t^s \, \frac{dt}{t}.$$

Proof. By Lemma A.3 the integral is absolutely convergent as long as $\mathrm{Re}(s) > k$. The identity follows from Lemma A.2 and the formula

$$\Gamma(z)\lambda^{-z} = \int_0^\infty e^{-t\lambda} t^z \, \frac{dt}{t}$$

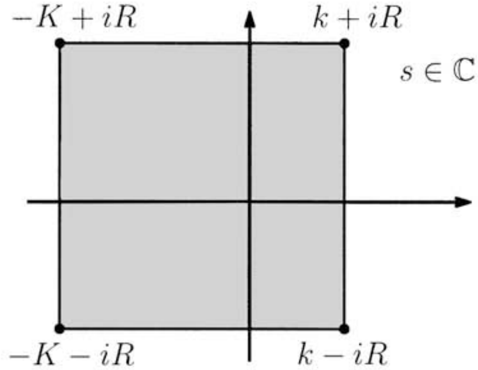
which is valid for all $\lambda > 0$ and for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. (In the case $p = 0$ Lemma A.2 does not apply, but then the proposition is a direct consequence of the displayed formula). □

Having established this basic relation, we are now going to apply the inversion formula for the Mellin transform to obtain an asymptotic formula for the JLO cocycle. In order to do so we shall need to make an additional analytic assumption, as follows: *the function $\langle a^0, [D, a^1], \ldots, [D, a^p] \rangle_z$ has only finitely many poles in each vertical strip $\alpha < \mathrm{Re}(z) < \beta$, and in each such strip and for every N one has*

$$\Big| \langle a^0, [D, a^1], \ldots, [D, a^p] \rangle_z \Big| = O(|z|^{-N})$$

as $|z| \to \infty$. Note that a similar assumption is made by Connes and Moscovici in [**12**].

Consider the rectangular contour in the complex plane which is indicated in the figure.



Here R is a large positive number (we shall take the limit as $R \to \infty$). The real numbers $k > \frac{d-p}{2}$ and K should be chosen so that there are no poles on the vertical lines $\mathrm{Re}(s) = k$ and $\mathrm{Re}(s) = -K$.

Let us integrate the function $\langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{s-\frac{p}{2}} t^{-s}$ around this contour.

The function $\langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{s-\frac{p}{2}}$ is holomorphic in the region $\mathrm{Re}(s) > \frac{d_1-p}{2}$, and converges rapidly to zero along each vertical line there. Therefore, by the inversion formula for the Mellin transform,

$$\text{(A.4)} \quad \lim_{R\to\infty} \frac{1}{2\pi i} \int_{k-iR}^{k+iR} \langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{s-\frac{p}{2}} t^{-s}\, ds = \langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_t^{\mathrm{JLO}}$$

(the contour integral is along the vertical line from $k - iR$ to $k + iR$).

Turning our attention to the left vertical side of the contour, we note that

$$\int_{-K-iR}^{-K+iR} \langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{s-\frac{p}{2}} t^{-s}\, dz = t^K \int_{-R}^{-R} \langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{-K-\frac{p}{2}+ir} t^{-ir}\, dr\,.$$

By hypothesis, the quantity $\langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{-K-\frac{p}{2}+ir}$ is an integrable function of $r \in \mathbb{R}$. Taking the limit as $R \to \infty$, the integral on the right-hand side (not including the term t^K) is the Fourier transform, evaluated at $\log(t)$, of an integrable function of r. It is therefore a bounded function of t. Hence

$$\text{(A.5)} \quad \lim_{R\to\infty} \int_{-K-iR}^{-K+iR} \langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{s-\frac{p}{2}} t^{-s}\, ds = O(t^K)$$

as $t \to 0$.

Since the horizontal components of the contour contribute zero to the contour integral, in the limit as $R \to \infty$, it follows from the Residue Theorem that

$$\text{(A.6)} \quad \langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_t^{\mathrm{JLO}} = \sum_{-K<\mathrm{Re}(w)<k} \mathrm{Res}_{s=w}\left(\langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{s-\frac{p}{2}} t^{-s}\right) + O(t^K)$$

as $t \to 0$.

A.5. THEOREM. *Assume that the spectral triple (A, H, D) is regular and has simple dimension spectrum. Then for all $p \geq 0$ and all $a^0, \ldots, a^p \in A$ the quantity $\langle a^0, [D, a^1], \ldots, [D, a^p] \rangle_t^{\mathrm{JLO}}$ has an asymptotic expansion in powers of t as t decreases to zero. The residue cocycle* $\operatorname{Res}_{s=0} \Psi(a^0, \ldots, a^p)$ *is the coefficient of the constant term in this asymptotic expansion.* □

Appendix B. Complex Powers in a Differential Algebra

In this appendix we shall try to sketch out a more conceptual view of the improper cocycle which was constructed in Section 5. This involves Quillen's cochain picture of cyclic cohomology [**23**], and in fact it was Quillen's account of the JLO cocycle from this perspective which first led to the formula for the quantity $\langle X^0, \ldots, X^p \rangle_z$ given in Definition 4.12.

We shall not attempt to carefully reconstruct the results of Sections 5 and 7 from the cochain perspective, and in fact for the sake of brevity we shall disregard analytic niceties altogether. Our purpose is only to set the main definitions of these notes against a background which may (or may not) make them seem more natural.

With this limited aim in mind we shall assume, as we did in the body of the notes, that the operator Δ is invertible. We shall also consider only the even case, in which the Hilbert space H on which Δ acts is $\mathbb{Z}/2$-graded.

As we did when we looked at cyclic cohomology in Section 2, let us fix an algebra A. But let us now also fix a second algebra L. For $n \geq 0$, denote by $\operatorname{Hom}^n(A, L)$ the vector space of n-linear maps from A to L. By a 0-linear map from A to L we shall mean a linear map from $\mathbb{C}$ to L, or in other words just an element of L. Let $\operatorname{Hom}^{**}(A, L)$ be the direct product

$$\operatorname{Hom}^{**}(A, L) = \prod_{n=0}^{\infty} \operatorname{Hom}^n(A, L).$$

Thus an element ϕ of $\operatorname{Hom}^{**}(A, L)$ is a sequence of multilinear maps from A to L. We shall denote by $\phi(a^1, \ldots, a^n)$ the value of the n-th component of ϕ on the n-tuple $(a^1, \ldots, a^n)$.

The vector space $\operatorname{Hom}^{**}(A, L)$ is $\mathbb{Z}/2$-graded in the following way: an element ϕ is even (resp. odd) if $\phi(a^1, \ldots, a^n) = 0$ for all odd n (resp. for all even n). We shall denote by $\deg_M(\phi) \in \{0, 1\}$ the grading-degree of ϕ. (The letter "M" stands for "multilinear;" a second grading-degree will be introduced below.)

B.1. LEMMA. *If $\phi, \psi \in \operatorname{Hom}^{**}(A, L)$, then define*

$$\phi \vee \psi(a^1, \ldots, a^n) = \sum_{p+q=n} \phi(a^1, \ldots, a^p) \psi(a^{p+1}, \ldots, a^n)$$

and

$$d_M \phi(a^1, \ldots, a^{n+1}) = \sum_{i=1}^{n} (-1)^{i+1} \phi(a^1, \ldots, a^i a^{i+1}, \ldots, a^{n+1}).$$

The vector space $\operatorname{Hom}^{**}(A, L)$*, so equipped with a multiplication and differential, is a $\mathbb{Z}/2$-graded differential algebra.* □

Let us now suppose that the algebra L is $\mathbb{Z}/2$-graded. If $\phi \in \operatorname{Hom}^{**}(A, L)$ then let us write $\deg_L(\phi) = 0$ if $\phi(a^1, \ldots, a^n)$ belongs to the degree-zero part of L for every n and every n-tuple $(a^1, \ldots, a^n)$. Similarly, if $\phi \in \operatorname{Hom}^{**}(A, L)$ then let us write $\deg_L(\phi) = 1$ if $\phi(a^1, \dots, a^n)$ belongs to the degree-one part of L for every n and every n-tuple $(a^1, \dots, a^n)$. This is a new $\mathbb{Z}/2$-grading on the vector space $\operatorname{Hom}^{**}(A, L)$. The formula

$$\deg(\phi) = \deg_M(\phi) + \deg_L(\phi)$$

defines a third $\mathbb{Z}/2$-grading—the one we are really interested in. Using this last $\mathbb{Z}/2$-grading, we have the following result:

B.2. Lemma. *If $\phi, \psi \in \operatorname{Hom}^{**}(A, L)$, then define*

$$\phi \diamond \psi = (-1)^{\deg_M(\phi)\deg_L(\psi)} \phi \vee \psi$$

and

$$d\phi = (-1)^{\deg_L(\phi)} d'\phi$$

These new operations once again provide $\operatorname{Hom}^{**}(A, L)$ with the structure of a $\mathbb{Z}/2$-graded differential algebra (for the total $\mathbb{Z}/2$-grading $\deg(\phi) = \deg_M(\phi) + \deg_L(\phi)$). □

We shall now specialize to the following situation: A will be, as in Section 5, an algebra of differential order zero and grading degree zero operators contained within an algebra $\mathcal{D}(\Delta)$ of generalized differential operators, and L will be the algebra of all operators on the $\mathbb{Z}/2$-graded vector space $H^\infty \subseteq H$.

Denote by ρ the inclusion of A into L. This is of course a 1-linear map from A to L, and we can therefore think of ρ as an element of $\operatorname{Hom}^{**}(A, L)$ (all of whose n-linear components are zero, except for $n = 1$).

Denote by D a square root of Δ, as in Section 4.4. Think of D as a 0-linear map from A to L, and therefore as an element of $\operatorname{Hom}^{**}(A, L)$ too. Combining D and ρ let us define the "superconnection form"

$$\theta = D - \rho \in \operatorname{Hom}^{**}(A, L)$$

This has odd $\mathbb{Z}/2$-grading degree (that is, $\deg(\theta) = 1$). Let K be its "curvature:"

$$K = d\theta + \theta^2,$$

which has even $\mathbb{Z}/2$-grading degree. Using the formulas in Lemma B.2 the element K may be calculated, as follows:

B.3. Lemma. *One has*

$$K = \Delta - E \in \operatorname{Hom}^{**}(A, L),$$

where $E \colon A \to L$ is the 1-linear map defined by the formula

$$E(a) = [D, \rho(a)]. \qquad \square$$

In all of the above we are following Quillen, who then proceeds to make the following definition, which is motivated by the well-known Banach algebra formula

$$e^{b-a} = \sum_{n=0}^{\infty} \int_{\Sigma^n} e^{-t_0 a} b e^{-t_1 a} \cdots b e^{-t_n a}\, dt.$$

B.4. Definition. Denote by $e^{-K} \in \operatorname{Hom}^{**}(A, L)$ the element

$$e^{-K} = \sum_{n=0}^{\infty} \int_{\Sigma^n} e^{-t_0 \Delta} E e^{-t_1 \Delta} \dots E e^{-t_n \Delta}\, dt.$$

The n-th term in the sum is an n-linear map from A to L, and the series should be regarded as defining an element of $\mathrm{Hom}^{**}(A, L)$ whose n-linear component is this term. As Quillen observes in [**23**, Section 8], the exponential e^{-K} defined in this way has the following two crucial properties:

B.5. LEMMA (Bianchi Identity). $d(e^{-K}) + [e^{-K}, \theta] = 0$. □

B.6. LEMMA (Differential Equation). *Suppose that δ is a derivation of $\mathrm{Hom}^{**}(A, L)$ into a bimodule. Then*

$$\delta(e^{-K}) = -\delta(K)e^{-K},$$

modulo (limits of) commutators. □

Both lemmas follow from the "Duhamel formula"

$$\delta(e^{-K}) = \int_0^1 e^{-tK}\delta(K)e^{-(1-t)K}\,dt,$$

which is familiar from semigroup theory and which may be verified for the notion of exponential now being considered. (Once more, we remind the reader that we are disregarding analytic details.)

Suppose we now introduce the "supertrace" $\mathrm{Trace}_\varepsilon(X) = \mathrm{Trace}(\varepsilon X)$ (which is of course defined only on a subalgebra of L). Quillen reinterprets the Bianchi Identity and the Differential Equation above as coboundary computations in a complex which computes periodic cyclic cohomology (using improper cocycles, in our terminology here). As a result he is able to recover the following basic fact about the JLO cocycle — namely that it really is a cocycle:

B.7. THEOREM (Quillen). *The formula*

$$\Phi_{2n}(a^0, \ldots, a^{2n}) = \int_{\Sigma^n} \mathrm{Trace}\left(\varepsilon a^0 e^{-t_0\Delta}[D, a^1]e^{-t_1\Delta}[D, a^2]\cdots[D, a^n]e^{-t_n\Delta}\right)\,dt$$

defines a (b, B)-cocycle. □

The details of the argument are not important here. What *is* important is that using the Bianchi Identity and a Differential Equation one can construct cocycles for cyclic cohomology from elements of the algebra $\mathrm{Hom}^{**}(A, L)$. With this in mind, let us consider other functions of the curvature operator K, beginning with resolvents.

B.8. LEMMA. *If $\lambda \notin \mathrm{Spectrum}(\Delta)$ then the element $(\lambda - K) \in \mathrm{Hom}^{**}(A, L)$ is invertible.*

PROOF. Since $(\lambda - K) = (\lambda - \Delta) + E$ we can write

$$\begin{aligned}(\lambda - K)^{-1} &= (\lambda - \Delta)^{-1} - (\lambda - \Delta)^{-1}E(\lambda - \Delta)^{-1} \\ &\quad + (\lambda - \Delta)^{-1}E(\lambda - \Delta)^{-1}E(\lambda - \Delta)^{-1} - \cdots\end{aligned}$$

This is a series whose nth term is an n-linear map from A to L, and so the sum has an obvious meaning in $\mathrm{Hom}^{**}(A, L)$. One can then check that the sum defines $(\lambda - K)^{-1}$, as required. □

With resolvents in hand, we can construct other functions of K using formulas modeled on the holomorphic functional calculus.

B.9. Definition. For any complex z with positive real part define $K^{-z} \in \mathrm{Hom}^{**}(A, L)$ by the formula

$$K^{-z} = \frac{1}{2\pi i} \int \lambda^{-z} (\lambda - K)^{-1} \, d\lambda,$$

in which the integral is a contour integral along a downward vertical line in $\mathbb{C}$ separating 0 from Spectrum(Δ).

The assumption that $\mathrm{Re}(z) > 0$ guarantees convergence of the integral (in each component within $\mathrm{Hom}^{**}(A, L)$ the integral converges in the pointwise norm topology of n-linear maps from A to the algebra of bounded operators on H; the limit is also an operator from H^∞ to H^∞, as required). The complex powers K^{-z} so defined satisfy the following key identities:

B.10. Lemma (Bianchi Identity). $d(K^{-z}) + [K^{-z}, \theta] = 0$. □

B.11. Lemma (Differential Equation). *If δ is a derivation of* $\mathrm{Hom}^{**}(A, L)$ *into a bimodule, then*

$$\delta(K^{-z}) = -z \delta(K) K^{-z-1},$$

modulo (limits of) commutators. □

These follow from the derivation formula

$$\delta(K^{-z}) = \frac{1}{2\pi i} \int \lambda^{-z} (\lambda - K)^{-1} \delta(K) (\lambda - K)^{-1} \, d\lambda.$$

In order to simplify the Differential Equation it is convenient to introduce the Gamma function, using which we can write

$$\delta\Big(\Gamma(z) K^{-z}\Big) = -\delta(K) \Gamma(z+1) K^{-(z+1)}$$

(modulo limits of commutators, as before). Except for the appearance of $z+1$ in place of z in the right hand side of the equation, this is exactly the same as the differential equation for e^{-K}. Meanwhile, even after introducing the Gamma function we still have available the Bianchi identity:

$$d\Big(\Gamma(z) K^{-z}\Big) + \Big[\Gamma(z) K^{-z}, \theta\Big] = 0.$$

The degree n component of $\Gamma(z) K^{-z}$ is the multilinear function

$$(a^1, \ldots, a^n) \mapsto \frac{(-1)^n}{2\pi i} \Gamma(z) \int \lambda^{-z} (\lambda - \Delta)^{-1} [D, a^1] \cdots [D, a^n] (\lambda - \Delta)^{-1} \, d\lambda.$$

Quillen's approach to JLO therefore suggests (and in fact upon closer inspection proves) the following result:

B.12. Theorem. *If we define*

$$\Psi_p^s(a^0, \ldots, a^p) = \frac{(-1)^p \Gamma(s - \frac{p}{2})}{2\pi i} \operatorname{Trace}\Big(\int \lambda^{\frac{p}{2} - s} \varepsilon a^0 (\lambda - \Delta)^{-1} [D, a^1] \cdots [D, a^p] (\lambda - \Delta)^{-1} \, d\lambda \Big),$$

then $b\Psi_p^s + B\Psi_{p+2}^s = 0$. □

This is, of course, precisely the conclusion that we reached in Section 5.

Appendix C. Proof of the Hochschild Character Theorem

In this final appendix we shall prove Connes' Hochschild character theorem by appealing to some of the computations that we made in Section 7.

C.1. Definition. A *Hochschild n-cycle* over an algebra A is an element of the $(n+1)$-fold tensor product $A \otimes \cdots \otimes A$ which is mapped to zero by the differential

$$b(a^0 \otimes \cdots a^n) = \sum_{j=0}^{n-1} (-1)^j a^0 \otimes \cdots \otimes a^j a^{j+1} \otimes \cdots a^n \\ + (-1)^n a^n a^0 \otimes a^1 \otimes \cdots \otimes a^{n-1}.$$

C.2. Remark. Obviously, two Hochschild n-cochains which differ by a Hochschild coboundary will agree when evaluated on any Hochschild cycle. The converse is not quite true.

C.3. Theorem. *Let (A, H, D) be a regular spectral triple. Assume that D is invertible and that for some positive integer n of the same parity as the triple, and every $a \in A$,*

$$a \cdot |D|^{-n} \in \mathcal{L}^{1,\infty}(H).$$

The Chern character ch_n^F of Definition 2.22 and the cochain

$$\Phi(a^0, \ldots, a^n) = \frac{\Gamma(\frac{n}{2}+1)}{n \cdot n!} \operatorname{Trace}_\omega(\varepsilon a^0 [D, a^1][D, a^2] \cdots [D, a^n] |D|^{-n})$$

are equal when evaluated on any Hochschild cycle $\sum_i a_i^0 \otimes \cdots \otimes a_i^n$. Here ε is 1 in the odd case, and the grading operator on H in the even case.

Proof. We showed in Lemma 7.8 that

$$b\Theta_{n-1}(a^0, \ldots, a^n) + B\Theta_{n+1}(a^0, \ldots, a^n) = 2s\Psi_n(a^0, \ldots, a^n),$$

at least for all s whose real part is large enough that all the terms are defined (since we are no longer assuming any sort of analytic continuation property this is now an issue). It follows that $B\Theta_{n+1}$ and $2s\Psi_n$ agree on any Hochschild cycle. Now, it is not hard to compute that $2s\Psi_n$ is defined when $\operatorname{Re}(s) > 0$, and

$$\lim_{s \to 0} 2s\Psi_n(a^0, \ldots, a^n) = \frac{\Gamma(\frac{n}{2}+1)}{n \cdot n!} \operatorname{Trace}_\omega(\varepsilon a^0 [D, a^1][D, a^2] \cdots [D, a^n] |D|^{-n}).$$

On the other hand $B\Theta_{n+1}(a^0, \ldots, a^n)$ is defined when $\operatorname{Re}(s) \geq 0$ (compare Remark 7.2). Since the computations in Section 7.2 show that $B\Theta_{n+1}$ is cohomologous, even as a cyclic cocycle, to the Chern character ch_n^F, the theorem is proved. □

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The Riemann Hypothesis: Arithmetic and Geometry

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Contents

1. Introduction

This paper describes basic properties of the Riemann zeta function and its generalizations, presents various formulations of the Riemann hypothesis, and indicates various geometric analogies. It briefly discusses the approach of A. Connes to a "spectral" interpretation of the Riemann zeros via noncommuative geometry, which is treated in detail by Paula Tretkoff [**33**] in this volume.

The origin of the Riemann hypothesis was as an arithmetic question concerning the asymptotic distribution of prime numbers. In the last century profound geometric analogues were discovered, and some of them proved. In particular there are striking analogies in the subject of spectral geometry, which is the study of global geometric properties of a manifold encoded in the eigenvalues of various geometrically natural operators acting on functions on the manifold. This has led to the search for a "geometric" and/or "spectral" interpretation of the zeros of the Riemann zeta function.

One should note that a geometric or spectral interpretation of the zeta zeros by itself is not enough to prove the Riemann hypothesis; the essence of the problem seems to lie in a suitable "positivity property" which must be established. A hope

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is that there exists such an interpretation in which the positivity will be a natural (and provable) consequence of the internal structure of the "geometric" object.

2. Basics

The Riemann zeta function is an analytic device that encodes information about the ring of integers $\mathbb{Z}$. In particular, it relates to the multiplicative action of $\mathbb{Z}$ on the additive group $\mathbb{Z}$. In its most elementary form, the Riemann zeta function can be defined by the well-known series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

where the domain of convergence is the half-plane $\{s : \Re(s) > 1\}$. This series was studied well before Riemann, and in particular Euler observed that it can be rewritten in the product form

$$\begin{aligned}\zeta(s) &= \prod_{p \text{ prime}} (1 + p^{-s} + p^{-2s} + \dots) \\ &= \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.\end{aligned}$$

The zeta function can be extended to a meromorphic function on the entire complex plane. More specifically, if we define the completed zeta function $\hat{\zeta}(s)$ by

$$\hat{\zeta}(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s),$$

then we have the following.

THEOREM 2.1. *The completed zeta function $\hat{\zeta}$ has an analytic continuation to the entire complex plane except for simple poles at $s = 0, 1$. Furthermore, this function $\hat{\zeta}$ satisfies the functional equation*

$$\hat{\zeta}(s) = \hat{\zeta}(1-s).$$

PROOF. With a suitable change of variables, the integral definition of Γ gives

$$\Gamma(\frac{s}{2}) = n^s \pi^{\frac{s}{2}} \int_0^{\infty} e^{-\pi n^2 x} x^{\frac{s}{2}-1} dx, \tag{1}$$

for every $n \in \mathbb{Z}^+$. Rearranging (1) and summing over $n \in \mathbb{Z}^+$, one can show that for all $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$\begin{aligned}\hat{\zeta}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) &= \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 x} x^{\frac{s}{2}-1} dx \\ &= \frac{1}{2}\int_0^{\infty} (\theta(x) - 1) x^{\frac{s}{2}} \frac{dx}{x},\end{aligned} \tag{2}$$

where $\theta(x) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x}$. This θ-function satisfies the functional equation

$$\theta(x^{-1}) = \sqrt{x}\theta(x).$$

Now, the integral in (2) can be split as

$$\int_0^{\infty} (\theta(x) - 1) x^{\frac{s}{2}} \frac{dx}{x} = \int_0^{1} (\theta(x) - 1) x^{\frac{s}{2}} \frac{dx}{x} + \int_1^{\infty} (\theta(x) - 1) x^{\frac{s}{2}} \frac{dx}{x}.$$

Applying the change of variables $x \mapsto x^{-1}$ in the first of these, we obtain

$$\hat{\zeta}(s) = \frac{1}{s(s-1)} + \frac{1}{2}\int_1^\infty (\theta(x)-1)(x^{\frac{1-s}{2}} + x^{\frac{s}{2}})\frac{dx}{x}. \tag{3}$$

This integral is uniformly convergent on $\{s : \Re(s) > \sigma\}$ for any $\sigma \in \mathbb{R}$, and thus is an entire function of s. Therefore, (3) exhibits the meromorphic continuation of $\hat{\zeta}$, and it clearly satisfies the functional equation.

□

It is now natural to define the entire function

$$\xi(s) = \frac{1}{2}s(s-1)\hat{\zeta}(s).$$

The factor of $\frac{1}{2}$ here was introduced by Riemann and has stuck. Hadamard showed that ξ has the product expansion

$$\xi(s) = \prod_\rho (1 - \frac{s}{\rho})e^{\frac{s}{\rho}},$$

where the product is over the zeros of ξ.

The location of the zeros of ξ is of great importance in number theoretic applications of the zeta function. Euler's product formula easily shows that every zero ρ has $\Re(\rho) \leq 1$, and the functional equation then gives that all zeros lie in the closed strip $\{s : 0 \leq \Re(\rho) \leq 1\}$. In fact, it can be shown that all zeros lie within the open strip $\{s : 0 < \Re(\rho) < 1\}$, although this is a non-trivial result.

Since $\xi(s)$ is real-valued for real values of s, it is clear that we have

$$\xi(\bar{s}) = \overline{\xi(s)}.$$

Thus if ρ is a zero of ξ, so are $\bar{\rho}$, $1-\rho$ and $1-\bar{\rho}$. Consequently, zeros on the line $\Re(s) = \frac{1}{2}$ occur in conjugate pairs, and zeros off this line occur in quadruples.

The Riemann hypothesis is now stated simply as follows.

Conjecture. *All zeros of $\xi(s)$ lie on the line $\Re(s) = \frac{1}{2}$.*

Riemann confirmed the position of many of the zeros of $\xi(s)$ to be on this critical line by hand, by making use of the symmetry from the functional equation. For if the approximate location of a zero close to the critical line is known, one can consider a small contour C around the zero which is symmetric about the critical line. By estimating the integral

$$\frac{1}{2\pi \mathrm{i}}\int_C \frac{\xi'(s)}{\xi(s)}ds,$$

one can determine the number of zeros (including multiplicity) enclosed within the curve C. If only one such zero exists, symmetry dictates that it must lie on the critical line. To date, no double zeros have been found on the critical line.

The Riemann hypothesis can be reformulated in a number theoretic context as follows. If we define

$$\pi(x) = \sum_{\substack{p \leq x \\ p \text{ prime}}} 1$$

as usual, then the Riemann hypothesis is known to be equivalent to the veracity of the following error term in the Prime Number Theorem:

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x^{\frac{1}{2}}(\log x)^2).$$

Note that it is a theorem that

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x \exp(-2\sqrt{\log x})).$$

Although Riemann's zeta function was the original object of interest, it is only one of a much larger set of *L-functions* with similar properties. These functions arise in many applications, and natural generalizations of the Riemann hypothesis appear to hold for all of them as well. For example:

- Dirichlet L-functions:

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s},$$

  where $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ is a character on $(\mathbb{Z}/q\mathbb{Z})^\times$. The character χ is extended to a q-periodic function on $\mathbb{Z}$, where we set $\chi(n) = 0$ for all n with $\gcd(n, q) \neq 1$.
- L-functions of cuspidal automorphic representations of $GL(N)$, *cf* [**24**], [**16**]. In the case of $GL(2)$ this includes such exotic objects as L-functions attached to Maass cusp forms.

The latter generalize to the "$GL(N)$ case" the "$GL(1)$ case" of the multiplicative group of a field acting on the additive group, which are just the Dirichlet L-functions. The resulting L-functions all have a Dirichlet series representation, which converges for $\Re(s) > 1$. When multiplied by appropriate Gamma-function factors and exponentials one obtains a "completed L-function", which analytically continues to $\mathbb{C}$, except for possible poles at $s = 0, 1$ and which satisfies a functional equation relating values at s to values at $1 - s$ of another such L-function. The generalized Riemann hypothesis asserts that all zeros of such L-functions lie on the line $\Re(s) = 1/2$.

This generalization appears to be the most natural context in which to study the Riemann hypothesis. In fact, from a number theoretic point of view, the Riemann zeta function cannot really be segregated from the above generalizations. It seems plausible that a proof of the original Riemann hypothesis will not be found without proving it in these more general circumstances.

At this point, it is also worth noting that much can be achieved in practical situations without the specificity of a proof of the Riemann hypothesis for particular cases. Many results, originally proven under the assumption of some generalized Riemann hypothesis, have more recently been fully proven by using results describing the behaviour of the Riemann hypothesis "on average" across certain families of L-functions. Two such examples are:

- Vinogradov:

  Every sufficiently large odd number can be written as a sum of three primes (a relative of Goldbach's conjecture).
- Cogdell, Piatetski-Shapiro, Sarnak:

  Hilbert's eleventh problem. Given a quadratic form F over a number field K, which elements of K are represented as values of F?

3. The Explicit Formula

Riemann's original memoir included a formula relating zeros of the zeta function to prime numbers. Early classical forms of the " explicit formula" of prime number theory were found by Guinand [**19**] and [**20**]. However it was A. Weil [**35**], [**36**], [**37**], who put the "explicit formula" in an elegant form that connects the arithmetic context of the Riemann hypothesis with objects that appear geometric in nature. This type of connection is central to most of the modern approaches to the Riemann hypothesis.

To state the explicit formula, we require the Mellin transform. For a function $f : (0, \infty) \to \mathbb{C}$, the Mellin transform $\mathcal{M}[f]$ of f is defined by

$$\mathcal{M}[f](s) = \int_0^\infty f(x) x^s \frac{dx}{x} \qquad (s \in \mathbb{C}).$$

This is the Fourier transform on the multiplicative group $\mathbb{R}_{>0}$; if we put $g(u) = f(e^u)$, we see that

$$\begin{aligned} \mathcal{M}[f](s) &= \int_{-\infty}^\infty f(e^u) e^{us} \frac{d(e^u)}{e^u} \\ &= \int_{-\infty}^\infty g(u) e^{-iu(is)} du = \hat{g}(is), \end{aligned}$$

where $\hat{g}$ is the Fourier transform of g on the additive group $\mathbb{R}$.

The convolution operation associated with the Mellin transform is

$$f * g(x) = \int_0^\infty f(\frac{x}{y}) g(y) \frac{dy}{y},$$

so that

$$\mathcal{M}[f * g](s) = \mathcal{M}[f](s) \mathcal{M}[g](s).$$

We also have an involution

$$\tilde{f}(x) = \frac{1}{x} f(\frac{1}{x}),$$

giving

$$\mathcal{M}[\tilde{f}](s) = \mathcal{M}[f](1 - s).$$

The "explicit formula" is a family of assertions, for a set of "test functions". We consider the family of *nice test functions* to consist of $f : (0, \infty) \to \mathbb{C}$ such that f is piecewise C^2, compactly supported and has the averaging property at discontinuities:

$$f(x) = \frac{1}{2}[\lim_{t \to x^+} f(t) + \lim_{t \to x^-} f(t)].$$

The "spectral side" $W_{spec}(f)$ of the explicit formula consists of three terms

$$W_{spec}(f) := W^{(2)}(f) - W^{(1)}(f) + W^{(0)}(f),$$

in which

$$\begin{aligned} W^{(2)}(f) &= \mathcal{M}[f](1), \\ W^{(1)}(f) &= \sum_{\rho \text{ zeros of } \xi} \mathcal{M}[f](\rho). \\ W^{(0)}(f) &= \mathcal{M}[f](0). \end{aligned}$$

The "arithmetic" side of the explicit formula consists of terms corresponding to the finite primes p plus the "infinite prime" (the real place),

$$W_{arith}(f) := W_\infty(f) + \sum_{p \text{ prime}} W_p(f)$$

in which

$$W_p(f) := \log p \left(\sum_{n=1}^{\infty} f(p^n) + \tilde{f}(p^n) \right),$$

and for $p = \infty$,

$$W_\infty(f) := (\gamma + \log p) f(1) + \int_1^\infty [f(x) + \tilde{f}(x) - \frac{2}{x^2} f(1)] \frac{x dx}{x^2 - 1}.$$

THEOREM 3.1 (Explicit Formula). *For any nice test function $f : (0, \infty) \to \mathbb{C}$ there holds*

$$W_{spec}(f) = W_{arith}(f).$$

The "explicit formula" has a formal resemblance to a fixed point formula of Atiyah-Bott-Lefschetz type; Here the "spectral side" has the form of a generalized Euler characteristic, in which the term $W^{(j)}(f)$ should measure the contribution of the trace of an operator on the j-th cohomology group of an unknown object, while the "arithmetic side" would be viewed as contributions coming from the fixed points of a map on an unknown object. This resemblance has been noted by many authors, starting with Andre Weil, whose first proof of the Riemann hypothesis in the one-variable function field case was based on exactly this interpretation.

The statement of the explicit formula takes on the form

"spectral term" = "arithmetic term".

Note that the "spectral" side of the "explicit formula" is expressed in terms of the Mellin transform of f, while the terms on the "arithmetic" side are expressed directly in terms of values of f; the proof below shows that the "arithmetic" terms do have an expression in terms of the Mellin transform of f.

Using the explicit formula, Weil was able to reformulate the Riemann hypothesis as a positivity statement.

THEOREM 3.2 (Weil's Positivity Statement). *The Riemann hypothesis is equivalent to*

$$W^{(1)}(f * \bar{\tilde{f}}) \geq 0,$$

for all nice test functions f.

REMARK 3.3. For two "nice" functions f and g, we can define the intersection product

$$\langle f_1, f_2 \rangle := W^{(1)}(f_1 * \bar{\tilde{f}}_2).$$

The conjectural Castelnuovo inequality states that

$$\mathcal{M}[f](0)\mathcal{M}[f](1) \geq \frac{1}{2} W_{spec}(f * \bar{\tilde{f}}).$$

Weil's positivity statement above follows from it.

The "explicit formula" was originally given by Weil [**35**] in terms of the Fourier transform; the Mellin transform version given here can be found in Patterson [**30**], who proves it for a wide class of test functions; one needs the test functions to have Mellin transforms $\mathcal{M}[f](s)$ that are holomorphic in a region $-\epsilon < \Re(s) < 1 + \epsilon$. A proof for a certain explicit set of test functions is given in [**5**], [**4**]; however these test functions fall outside the class above. There are a number of proofs of the "explicit formula", all based on similar ideas, which we indicate below.

Sketch of proof of explicit formula. Consider the logarithmic derivative of the completed zeta function $\hat{\zeta}(s)$,

$$\frac{\hat{\zeta}'(s)}{\hat{\zeta}(s)} = \frac{d}{ds}[\log \hat{\zeta}(s)].$$

We assume the function $\mathcal{M}[f](s)$ extends to an analytic function in the closed strip $-\epsilon < \Re(s) < 1+\epsilon$ and has rapid enough decay vertically. We evaluate in two ways the contour integral

$$-\frac{1}{2\pi \mathrm{i}} \int_{\Box_T} \mathcal{M}[f](s) \frac{\hat{\zeta}'(s)}{\hat{\zeta}(s)} ds, \tag{4}$$

around a closed box $\Box_T$ on the vertical lines $\Re(s) = 1 + \frac{1}{2}\epsilon$, and $\Re(s) = -\frac{1}{2}\epsilon$, going from height $-iT$ to height $+iT$, oriented counterclockwise, and then letting the height of the box $T \to \infty$. Firstly, taking the logarithm of the Hadamard factorization for $\xi(s)$ gives

$$\log \hat{\zeta}(s) = \log 2 - \log s - \log(s-1) + {\sum_{\rho \text{ zeros of } \xi}}' \left(\log(1 - \frac{s}{\rho}) \right),$$

where the prime indicates the zeros must be summed in pairs $\rho, 1-\rho$. Differentiating,

$$\frac{d}{ds}[\log \hat{\zeta}(s)] = -\frac{1}{s} - \frac{1}{s-1} + {\sum_{\rho \text{ zeros of } \xi}}' \frac{1}{s-\rho}.$$

Adding up the residues of the poles these contribute in the box (as $T \to \infty$) gives the geometric term; the terms $W^{(0)}(f)$ and $W^{(2)}(f)$ come from the poles of $\hat{\zeta}(s)$ at $s = 0$ and $s = 1$, respectively.

Secondly, the Euler product form gives

$$\frac{\hat{\zeta}'(s)}{\hat{\zeta}(s)} = \frac{d}{ds} \left(\log \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) - \sum_{p \text{ a prime}} \log(1 - p^{-s}) \right).$$

The derivative of the sum is

$$-\frac{1}{2}\log \pi + \frac{1}{2} \frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} - \sum_{p \text{ a prime}} \frac{(\log p) p^{-s}}{1 - p^{-s}}.$$

This is substituted in the right vertical side of the box integral and evaluated for each term separately. The integral, with $\Re(s) > 1$, evaluates to

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}[f](s) \left(\sum_{n=1}^{\infty} (\log p) p^{-ns} \right) ds = (\log p) \sum_{n=1}^{\infty} f(p^n),$$

since each term separately is an inverse Mellin transform. For the left vertical side integral with $\Re(s) < 0$, we use the functional equation and obtain similarly the contribution $(\log p) \sum_{n=1}^{\infty} \tilde{f}(p^n)$. The horizontal sides contribute zero in the limit $T \to \infty$.

The contour integral for the Gamma function term requires delicate care to convert the answer to the form for $W_\infty(f)$ given above; see [**5**], [**4**], [**30**]. $\Box$

For spectral and trace formula interpretations of the "explicit formula", see Goldfeld [**17**], [**18**] Haran [**21**], [**22**], Hejhal [**23**], as well as the recent work of Connes [**10**], [**11**], [**9**]. A number of other interesting viewpoints on the "explicit formula" appear in Burnol [**6**] and Deninger [**12**].

4. The Function Field Case

We now consider the Riemann hypothesis for function fields over finite fields, or equivalently, for zeta functions attached to complete nonsingular projective varieties. For function fields of one variable the Riemann hypothesis was formulated in E. Artin's 1923 thesis, in analogy with the number field case. It was proved for genus one function fields by H. Hasse in 1931, and it was then proved for all one-variable function fields by A. Weil in the 1940's. Weil's key idea was to introduce an underlying geometric object—a projective variety—which allows the translation of the problem to a problem in algebraic geometry. Finally, in 1973 Deligne proved the Riemann hypothesis for the zeta functions of complete nonsingular projective varieties of any dimension.

Let $\mathbb{F}_q$ be the finite field with $q = p^k$ elements for some prime p and some $k \in \mathbb{Z}^+$, and let K be a function field in one variable T over $\mathbb{F}_q$. Let O_K denote the ring of integers of K. We exclude one prime from O_K which we define to be the "prime at infinity". For instance, if $K = \mathbb{F}_q(T)$, then we can let $O_K = \mathbb{F}_q[T]$, the ring of polynomials in T, where the prime $\frac{1}{T}$ is excluded as the prime at infinity.

We now define a zeta function for K by

$$\zeta_K(s) = \sum_{I \in I_K} (N(I))^{-s},$$

where I_K denotes the set of ideals contained in the ring of integers O_K, and $N(I) = \#(O_K/I)$ is the norm of I. In our example above, all ideals $I = (f)$ are principal, generated by a monic polynomial $f(T)$, with norm

$$N((f)) = q^{-\deg f}.$$

Therefore we have

$$\begin{aligned}\zeta_K(s) &= \sum_{\substack{f \text{ monic} \\ \text{polynomials over } \mathbb{F}_q}} q^{-(\deg f)s} \\ &= \sum_{n=1}^{\infty} q^n q^{-ns},\end{aligned}$$

since there are q^n monic polynomials of degree n over $\mathbb{F}_q$. Thus,

$$\zeta_K(s) = \frac{1}{1 - q^{1-s}}.$$

We complete this to a function $\hat{\zeta}_K(s)$ by including a term corresponding to the prime at infinity. In the present example, we obtain

$$\hat{\zeta}_K(s) = \left(\frac{1}{1 - q^{1-s}}\right)\left(\frac{1}{1 - q^{-s}}\right).$$

Now $\hat{\zeta}(s)$ satisfies the functional equation

$$q^{-s}\hat{\zeta}_K(s) = q^{-(1-s)}\hat{\zeta}_K(1-s),$$

Here the additional non-vanishing factor q^{-s} plays the role of a "conductor," analogous to the conductor term appearing in the functional equation of a Dirichlet L-function.

Weil made several celebrated conjectures about these zeta functions, all of which are now proven. A major implication of the Weil conjectures is that $\hat{\zeta}(s)$ can be expressed in terms of L-functions arising from the structure of an underlying geometric object, namely a non-singular projective variety V having K as a function field. Essentially, $\hat{\zeta}_K$ can be written as a quotient of L-functions arising from the cohomology of V:

$$\hat{\zeta}_K(s) = \frac{L(s, H^1)}{L(s, H^0)L(s, H^2)}.$$

For instance, consider the line

$$X_1 + X_2 = 1$$

over $\mathbb{F}_q$. We projectivize to obtain

$$V(\mathbb{F}_q) = \mathbb{P}^1(\mathbb{F}_q) \ : \ X_1 + X_2 = X_3,$$

with $(X_1, X_2, X_3) \neq (0, 0, 0)$, subject to the usual equivalence relation

$$(\lambda X_1, \lambda X_2, \lambda X_3) \sim (X_1, X_2, X_3), \text{ for all } \lambda \in \mathbb{F}_q^\times.$$

Thus V consists of q points of the form $(x, 1-x, 1)$ for $x \in \mathbb{F}_q$, plus the point at infinity $(1, -1, 0)$. We now extend this space to the projective space $V(\overline{\mathbb{F}_q})$ over the algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$.

Such a projective variety has a natural dynamical system induced by the Frobenius automorphism,

$$\begin{array}{rcl} \mathrm{Frob} : V & \longrightarrow & V \\ (x_1, x_2, x_3) & \longmapsto & (x_1^q, x_2^q, x_3^q). \end{array}$$

Associated to this dynamical system is a dynamical zeta-function, defined by

$$\hat{\zeta}_{\mathrm{dyn}}(T) = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \#\mathrm{Fix}(\mathrm{Frob}^n)\right),$$

where $\#\mathrm{Fix}(\mathrm{Frob}^n)$ is the number of fixed points of Frob^n. For instance, the example above yields

$$\begin{aligned} \hat{\zeta}_{\mathrm{dyn}}(T) &= \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n}(q^n + 1)\right) \\ &= \exp\left(-\log(1 - Tq) - \log(1 - q)\right) \\ &= \frac{1}{(1-q)(1-Tq)}. \end{aligned}$$

Therefore, putting $T = q^{-s}$, we get

$$\hat{\zeta}_{\mathrm{arith}}(s) = \hat{\zeta}_{\mathrm{dyn}}(q^{-s}).$$

This connection is remarkable. As a heuristic analogy, a similar situation arises in statistical mechanics. Associated to a one-dimensional system, Ruelle defined a (two-variable) statistical mechanics zeta function by

$$\zeta(T,s) = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} p_n(s)\right).$$

Here, $p_n(s)$ is the partition function of a finite system Σ_n of "size" n , given by

$$p_n(s) = \sum_{\sigma \in \Sigma_n \text{ states}} e^{-sH(\sigma)},$$

where H is the Hamiltonian function. Here Σ_n could represent a system on the line with periodic boundary conditions of period n.

An analogous result in statistical mechanics to the number theoretic statement above is the following (Lagarias [**26**, Theorem 3.1]).

THEOREM 4.1. *For "homogeneously expanding maps" on* $[0,1]$,

$$\zeta(1,s) = \zeta(\beta^{-s}, 0),$$

where $\beta = \exp(\text{entropy})$.

Here a homogeneously expanding map $f : [0,1] \to [0,1]$ is a (possibly discontinuous) piecewise C^1-map all of whose pieces are linear with slopes $\pm\beta$ with $\beta > 1$; an example is $f(x) = \beta x \pmod 1$. For the function field zeta function above, the Frobenius automorphism acts like a uniformly expanding map with entropy $\log q$.

Can Weil's ideas be extended to the number field case of the Riemann hypothesis? This suggests, in particular, three questions. Firstly, what is the "geometrical" or "dynamical" zeta function which should be considered in the number field case? Secondly, how is this geometrical object related to the arithmetic zeta function? Thirdly, what property in the number field case is the analogue of the Castelnuovo positivity property that provides a "geometric" explanation of the truth of the Riemann hypothesis in the function field case? For possible ideas in these directions, see Connes [**10**], [**11**], [**9**], and Deninger [**13**], [**14**].

5. The Number Field Case and Non-commutative Geometry

Polya and Hilbert postulated the following idea as a possible approach to the Riemann hypothesis. Suppose some geometric considerations can lead to the construction of a Hilbert space $\mathcal{H}$ and an unbounded operator D, such that

$$\text{Spectrum}(D) = \{\rho : \xi(\rho) = 0\}.$$

One might then hope to be able to understand the location of the zeros using operator theory—ideally, by showing that

$$(D - \frac{1}{2})^* = -(D - \frac{1}{2}).$$

This hope is plausible, at least in philosophy, for several reasons. First, there is some analogy with the work of Selberg. Selberg's work considered the Laplace operator, which has the form

$$\Delta = (D - \frac{1}{2})^2.$$

The behaviour of "primes" (prime geodesics) is related to the spectrum of this operator via Selberg's trace formula. Second, work of Montgomery indicates that the distribution of the zeros of $\xi(s)$ compares well with results on the distribution of eigenvalues of random matrices; see [**2**], [**25**]. This has been strikingly supported by numerical computations of Odlyzko [**29**]. This suggests that spectral considerations lie beneath the theory of the zeta function.

Connes' recent idea is that the philosophy of Polya and Hilbert might be realized using non-commutative geometry. In the function field case above, a space is produced from the action of the Frobenius automorphism on the underlying variety V. Spectral methods in this context yield Weil's proof of the Riemann Hypothesis for function fields. In the number field case, Connes' proposal is that the appropriate space is generated by the action of the multiplicative group $k^\times$ of the number field on the adèle space[1] A. The space $A/k^\times$ is extremely badly behaved from the classical point of view, but the hope is that it may be handled effectively as a non-commutative space. A development of the ideas behind Connes' work is the topic of Paula Tretkoff's paper [**33**] in this volume. This approach can be said to give a "spectral" interpretation of the zeta zeros, but so far the positivity aspect remains elusive.

6. Equivalent Forms of the Riemann Hypothesis

To conclude, we present four equivalents of the Riemann hypothesis. These demonstrate connections of the Riemann hypothesis with other areas of mathematics, and some of them have a geometric flavor.

6.1. Ergodicity of Horocycle Flows. Consider the group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ acting on the hyperbolic plane $\mathbb{H}$ in the familiar way, so that the group is generated by the isometries

$$z \mapsto z+1$$

$$\text{and} \qquad z \mapsto -\frac{1}{z}$$

of the upper half-plane model of $\mathbb{H}$. Let us denote by h_t the horocycle in the upper half-plane having constant imaginary part $y = t$. We look at the projection of this horocycle onto the quotient space $\mathbb{H}/\Gamma$. Since h_t is invariant under the mapping $z \mapsto z+1$, this is a periodic horocycle, and we can restrict our attention to the segment of h_t lying within the vertical strip $\{z : 0 \le z \le 1\}$. Let γ_t denote the image of this segment in $\mathbb{H}/\Gamma$. The length of γ_t is $\frac{1}{t}$, and in particular, $\mathrm{length}(\gamma_t) \to \infty$ as $t \to 0$. Furthermore, γ_t satisfies the following ergodic property as $t \to 0$.

THEOREM 6.1. *For any "nice" open set S in $\mathbb{H}/\Gamma$*

$$\frac{\mathrm{length}(\gamma_t \cap S)}{\mathrm{length}(\gamma_t)} \longrightarrow \frac{\mathrm{vol}(S)}{\mathrm{vol}(\mathbb{H}/\Gamma)}$$

as $t \to 0$.

Here $\mathrm{vol}(\mathbb{H}/\Gamma) = \frac{\pi}{3}$, and "nice" can be taken to be that the boundary $\partial(S) = \bar{S}\backslash S$ has finite 1-dimensional Hausdorff measure, *cf.* Verjovsky [**34**]. A connection on the rate of convergence to ergodicity was noted by Zagier [**38**, pp. 279–280]. The Riemann hypothesis is equivalent to the following bound on the rate of convergence of the above (Sarnak [**32**, p. 738]); here a "smooth test function" is needed.

[1] More generally, we should be looking at the action of $\mathrm{GL}_n(k)$ on $M_n(A)$.

THEOREM 6.2. *The Riemann hypothesis holds if and only if, for any "nice" test function $f \in C_c^\infty(S\mathbb{H}/\Gamma)$, where $S\mathbb{H}/\Gamma$ is the unit tangent bundle over $\mathbb{H}/\Gamma$, for $t \to 0$ there holds*

$$\frac{1}{t}\int_{\gamma_t} f(z)d\nu_t z = \frac{\int_{\mathbb{H}/\Gamma} f(z)d\mu z}{\mathrm{vol}(\mathbb{H}/\Gamma)} + O(t^{\frac{3}{4}+\epsilon}),$$

for any $\epsilon > 0$. Here ν_t is the arc-length measure on the horocycle at height t and μ is Poincaré measure on $(S\mathbb{H}/\Gamma)$, which gives it volume $2\pi\mathrm{vol}(\mathbb{H}/\Gamma)$.

A subtlety in this criterion is that if a test function were used that is not sufficiently smooth, then slower rates of convergence can hold even if the Riemann hypothesis is valid. See Verjovsky [**34**] for an example involving the characteristic function of an open set S lifted to the unit tangent bundle.

6.2. Brownian motion. Gnedenko and Kolmogorov observed that the Riemann zeta function arises naturally in relation to Brownian motion, see [**3**]. Consider, for example, the case of "pinned Brownian motion" on $\mathbb{R}$. This is a standard Brownian motion on the line $B_t \in \mathbb{R}$, $t \geq 0$, started at $B_0 = 0$ and conditioned on the property $B_1 = 0$. Now let

$$Z = \max_{0\leq t\leq 1} B_t - \min_{0\leq t\leq 1} B_t$$

be the length of the range of B_t. Then the expectation of Z^s is known to be

$$E[Z^s] = \xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s).$$

In another Brownian system, we obtain the following equivalent of the Riemann hypothesis [**1**].

THEOREM 6.3 (Balazard, Saias, Yor). *Consider two-dimensional Brownian motion in the (x, y)-plane, starting at $(0, 0)$. Let $(\frac{1}{2}, W)$ be the first point of contact with the vertical line $x = \frac{1}{2}$. Then the Riemann hypothesis is equivalent to*

$$E\left[\log|\zeta(W)|\right] = 0.$$

This statement is actually a restatement of the following integral.

THEOREM 6.4. *The Riemann hypothesis is equivalent to*

$$\int_0^\infty \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2}dt = 0.$$

Note that it is known unconditionally ([**1**], [**7**, Theorem 1.5]) that

$$\frac{1}{2\pi}\int_{\Re(s)=1/2}\frac{\log|\zeta(s)|}{|s|^2}|ds| = \frac{1}{\pi}\int_0^\infty \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2}dt = \sum_{\substack{\rho \text{ zeros of } \xi\\ \Re(\rho)>\frac{1}{2}}} \log\frac{\rho}{1-\rho}.$$

6.3. Li's Positivity Criterion. Define for $n \geq 0$ the Li coefficient

$$\lambda_n := \frac{1}{(n-1)!}\frac{d^n}{ds^n}\left(s^n\log\xi(s)\right)|_{s=1}.$$

Note that these quantities are given at $s = 1$, which can be computed in the absolute convergence region $\Re(s) > 1$ of the Euler product, as a limit $s \to 1^+$. These coefficients have a power series interpretation as:

$$\frac{1}{(z-1)^2}\frac{\xi'(\frac{1}{1-z})}{\xi(\frac{1}{1-z})} = \sum_{n=0}^{\infty} \lambda_{n+1} z^n.$$

The Riemann hypothesis can be rephrased as the positivity of these coefficients ([**28**]).

Theorem 6.5 (Li). *The Riemann hypothesis is equivalent to*

$$\lambda_n \geq 0 \quad \textit{for all} \quad n \geq 1.$$

In fact, this criterion is related to Weil's explicit formula, since it can be shown ([**5**]) that

$$\lambda_n = W^{(1)}(\phi_n * \tilde{\bar{\phi}}_n)$$

for a certain sequence of test functions (ϕ_n); this sequence of test functions falls outside the class of test functions considered in §3 but the "explicit formula" can be justified for them, in a slightly modified form.

6.4. An Elementary Formulation. Because the Riemann hypothesis is such a fundamental question, it seems appropriate to give a completely elementary statement of it. Lagarias [**27**] establishes the following result.

Theorem 6.6. *The Riemann hypothesis is equivalent to the following assertion: Let $H_n = \sum_{j=1}^{n} \frac{1}{j}$ be the nth harmonic number, and let $\sigma(n) = \sum_{d|n} d$ be the sum of the divisors of n. Then for each $n \geq 1$,*

$$\sigma(n) \leq H_n + \exp(H_n)\log(H_n),$$

and equality holds only for $n = 1$.

This assertion represents an encoded form of a necessary and sufficient condition for the Riemann hypothesis due to Guy Robin [**31**].

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Noncommutative Geometry and Number Theory

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Contents

Introduction

In almost every branch of mathematics we use the ring of rational integers, yet in looking beyond the formal structure of this ring we often encounter great gaps in our understanding. The need to find new insights into the ring of integers is, in particular, brought home to us by our inability to decide the validity of the classical Riemann hypothesis, which can be thought of as a question on the distribution of prime numbers. Inspired by ideas from noncommutative geometry, Alain Connes [**8**], [**10**], [**9**] has in recent years proposed a set-up within which to approach the Riemann Hypothesis. The following chapters provide an introduction to these ideas of Alain Connes and are intended to aid in a serious study of his papers and in the analysis of the details of his proofs, which for the most part we do not reproduce here. We also avoid reproducing too much of the classical material, and choose instead to survey, without proofs, basic facts about the Riemann Hypothesis needed directly for understanding Connes's papers. These chapters should be read therefore with a standard textbook on the Riemann zeta function at hand—for example, the book of Harold M. Edwards [**15**], which also includes a translation

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of Riemann's original paper. For the function field case, the reader can consult André Weil's book [**39**]. A good introduction to the Riemann zeta function and the function field case can also be found in Samuel J. Patterson's study [**30**]. A concise and informative survey of the Riemann Hypothesis, from which we quote several times, is given by Enrico Bombieri on the Clay Mathematics Institute website [**3**] (see also the updated report of Peter Sarnak on that same website). Some advanced notions from number theory are referred to as motivation for Connes's approach, but little knowledge of number theory is assumed for the discussion of the results of his papers. Although Connes's papers apply to arbitrary global fields, we most often restrict our attention to the field of rational numbers, as this still brings out the main points and limits the technicalities.

There are some similarities between Alain Connes's work in [**8**], [**10**], [**9**] and work of Shai Haran in [**21**], [**23**], [**22**]. We do not pursue here the relation to Shai Haran's papers, although we refer to them several times.

1. The objects of study

1.1. The Riemann zeta function. Riemann formulated his famous hypothesis in 1859 in a foundational paper [**31**], just 8 pages in length, on the number of primes less than a given magnitude. The paper centers on the study of a function $\zeta(s)$, now called the Riemann zeta function, which has the formal expression,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

the right hand side of which converges for $\Re(s) > 1$. This function, in fact, predates Riemann. In a paper [**17**], published in 1748, Euler observed a connection with primes via the formal product expansion,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}, \tag{2}$$

valid for $\Re(s) > 1$. Equation (2) is a direct result of the unique factorization, up to permutation of factors, of a positive rational integer into a product of prime numbers. The contents of Euler's paper are described in [**15**]. It seems that Euler was aware of the asymptotic formula,

$$\sum_{p<x} \frac{1}{p} \sim \log(\log x), \qquad (x \to \infty)$$

where the sum on the left hand side is over the primes p less than the real number x.

The additive structure of the integers leads to considering negative as well as positive integers and to the definition of the usual absolute value $|\cdot|$ on the ring $\mathbb{Z}$ of rational integers, defined by

$$|n| = \operatorname{sg}(n)\, n, \qquad n \in \mathbb{Z}.$$

The p-adic valuations are already implicit in the unique factorization of positive integers into primes. Namely, for every prime p and every integer n one can write

$$n = p^{\operatorname{ord}_p(n)} n'$$

where n' is an integer not divisible by p. The p-adic absolute value of n is then defined to be

$$|n|_p = p^{-\mathrm{ord}_p(n)}.$$

We denote by $\mathbb{Q}$ the field of fractions of $\mathbb{Z}$, namely the field of rational numbers, and by $M_{\mathbb{Q}}$ the set of valuations just introduced, extended to $\mathbb{Q}$ in the obvious way, and indexed by ∞ and by the primes p. We write, for $x \in \mathbb{Q}$,

$$|x|_v = |x|, \qquad v = \infty \in M_{\mathbb{Q}}$$

and

$$|x|_v = |x|_p, \qquad v = p \in M_{\mathbb{Q}}.$$

The following important observation is obvious from the definitions.

PRODUCT FORMULA: *For every $x \in \mathbb{Q}$, $x \neq 0$, we have*

$$\prod_{v \in M_{\mathbb{Q}}} |x|_v = 1.$$

Riemann derived a formula for $\sum n^{-s}$ valid for all $s \in \mathbb{C}$. For $\Re(s) > 0$, the Γ-function is defined by

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx.$$

The function $\Gamma(s)$ has an analytic continuation to all $s \in \mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$, with residue $(-1)^m m!$ at $-m$, $m \geq 0$. This can be seen using the formula

$$\Gamma(s) = \lim_{N \to \infty} \frac{1 \cdot 2 \cdot \cdots \cdot N}{s(s+1)\cdots(s+N-1)} (N+1)^{s-1}.$$

Moreover, we have $s\Gamma(s) = \Gamma(s+1)$, and at the positive integers $m > 0$, we have $\Gamma(m) = (m-1)!$. Riemann observed that, for $\Re(s) > 1$,

$$\Gamma(\frac{s}{2})\pi^{-s/2} \sum_{n=1}^\infty n^{-s} = \int_0^\infty \sum_{n=1}^\infty \exp(-n^2 \pi x) x^{s/2} \frac{dx}{x}. \tag{3}$$

Moreover, he noticed that the function on the right hand side is unchanged by the substitution $s \mapsto 1 - s$ and that one may rewrite the integral in (3) as

$$\int_1^\infty \sum_{n=1}^\infty \exp(-n^2\pi x) \left(x^{s/2} + x^{(1-s)/2} \right) \frac{dx}{x} \quad - \quad \frac{1}{s(1-s)}, \tag{4}$$

which converges for all $s \in \mathbb{C}$ and has simple poles at $s = 1$ and $s = 0$. This shows that $\zeta(s) = \sum n^{-s}$, $\Re(s) > 1$, can be analytically continued to a function $\zeta(s)$ on all of $s \in \mathbb{C}$ with a simple pole at $s = 1$ (the pole at $s = 0$ in (4) being accounted for by $\Gamma(\frac{s}{2})$).

Riemann defined, for $t \in \mathbb{C}$ given by $s = \frac{1}{2} + it$, the function

$$\xi(t) = \frac{1}{2} s(s-1)\Gamma(\frac{s}{2})\pi^{-s/2}\zeta(s), \tag{5}$$

for which we have the following important result.

THEOREM 1. (i) *Let Z be the set of zeros of $\xi(t)$. We have a product expansion of the form*

$$\xi(t) = \frac{1}{2}\pi^{-s/2} e^{bs} \prod_{\rho \in Z} (1 - \frac{s}{\rho}) e^{s/\rho}, \qquad s = \frac{1}{2} + it, \tag{6}$$

where $b = \log 2\pi - 1 - \frac{1}{2}\gamma$ *and* $\gamma = -\Gamma'(1) = 0.577...$ *is Euler's constant. Moreover* $\xi(t)$ *is an entire function, and the set* Z *is contained in*

$$\{s \in \mathbb{C} \mid 0 \leq \Re(s) \leq 1\} = \{t \in \mathbb{C} \mid -\frac{i}{2} \leq \Im(t) \leq \frac{i}{2}\}$$

(ii) *The function* $\xi(t)$ *satisfies the* Functional Equation

$$\xi(t) = \xi(-t). \tag{7}$$

Moreover the set Z *is closed under complex conjugation.*

(iii) *By equation (5), the poles of* $\Gamma(s)$ *at the non-positive integers give rise to zeros of* $\zeta(s)$ *at the negative even integers. These are called the* Trivial Zeros. *The remaining zeros of* $\zeta(s)$ *are at the elements of* Z. *They are called the* Non-trivial Zeros.

The proof of the product formula of part (i) of Theorem 1 was sketched by Riemann and proved rigorously by Hadamard [**19**] in 1893. The rest of Theorem 1 is due to Riemann.

We can now state the central unsolved problem about the zeta function, namely to decide whether the following hypothesis is valid.

Riemann Hypothesis: *The zeros of* $\xi(t)$ *are real. Equivalently, the* non-trivial *zeros of* $\zeta(s)$ *lie on the line* $\Re(s) = \frac{1}{2}$.

Riemann verified this hypothesis by hand for the first zeros and commented that

> "Without doubt it would be desirable to have a rigorous proof of this proposition, however I have left this research aside for the time being after some quick unsuccessful attempts, because it appears unnecessary for the immediate goal of my study."
>
> *(translated from German)*

Riemann's "immediate goal" was to find a formula for the number of primes less than a given positive real number x. However, Riemann's great contribution was not so much his concrete results on this question, but rather the methods of his paper, particularly his realization of the existence of a relation between the location of the zeros of $\zeta(s)$ and the distribution of the primes. One illustration of this phenomenon is the two types of product formulae, one for $\zeta(s)$ as a product over the primes as in (2) and the other as a product over the zeros Z of the related function $\xi(t)$ as in Theorem 1 (i).

Prior to Riemann, and following some ideas of Euler from 1737, Tchebychev had initiated the study of the distribution of primes by analytic methods around 1850, by studying the function

$$\pi(x) = \mathrm{Card}\{p \,\mathrm{prime}, p \leq x\}.$$

He introduced the function

$$J(x) = \frac{1}{2}\left(\sum_{p^n < x} \frac{1}{n} + \sum_{p^n \leq x} \frac{1}{n}\right) = \pi(x) + \frac{1}{2}\pi(\sqrt[2]{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \dots$$

and showed using the Euler product (2) that

$$\frac{1}{s}\log\zeta(s) = \int_1^\infty J(x) x^{-s} \frac{dx}{x}, \qquad \Re(s) > 1. \tag{8}$$

Riemann later used this formula and the calculus of residues to compute $J(x)$, and hence $\pi(x)$, in terms of the singularities of $\log\zeta(s)$, which occur at the zeros and poles of $\zeta(s)$. Building on work of Tchebychev and Gauss, Riemann made a rigorous study of (8), its inversion, and its relation to $\pi(x)$.

This work was developed after Riemann and culminated in the independent proof in 1896 by Hadamard [**20**] and de la Vallée-Poussin [**34**] of an asymptotic formula for $\pi(x)$.

Prime Number Theorem: *As $x \to \infty$, we have*

$$\pi(x) \quad\sim\quad \mathrm{Li}(x) = \int_0^x \frac{dt}{\log t} \quad\sim\quad \frac{x}{\log x}.$$

where the integral on the right hand side is understood in the sense of a Cauchy principal value, that is

$$\int_0^x \frac{dt}{\log t} = \lim_{\varepsilon\to 0}\left(\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t}\right).$$

Moreover, it became clear that the better one understood the location of the zeros of $\zeta(s)$, the better one would understand this approximation to $\pi(x)$. For example, the Prime Number Theorem is equivalent to the statement that there are no zeros of $\zeta(s)$ on the line $\Re(s) = 1$ and the Riemann Hypothesis is equivalent to the statement that, for every $\varepsilon > 0$, the relative error in the Prime Number Theorem is less than $x^{-1/2+\varepsilon}$ for all sufficiently large x.

Riemann also studied $N(T)$, $T > 0$—the number of zero of $\xi(t)$ between 0 and T—and sketched a proof of the fact that

$$N(T) \sim \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}.$$

1.2. Local fields. The prime numbers are intimately related to finite fields. If F is a finite field, then it is necessarily of prime characteristic $p > 1$, that is, p is the minimal non-zero integer for which the identity $p1_F = 0$ is true in F, where 1_F is the multiplicative unit element in F. By a result of Wedderburn, a finite field must also be commutative and has the structure of a vector space of dimension $f \geq 1$ over its prime ring $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, the field of p elements. The number of elements of F is then $q = p^f$ and it is isomorphic to the field $\mathbb{F}_q$ of roots of the equation $X^q = X$.

An arbitrary field with the discrete topology is locally compact. (In general, a metric space is called locally compact if every point has a neighborhood which is compact.) From the topological viewpoint, the interesting locally compact fields should not be discrete. This leads to considering, for discrete fields like $\mathbb{Q}$, their embeddings into closely associated locally compact non-discrete fields or rings.

The set $M_{\mathbb{Q}}$ of valuations of the field of fractions $\mathbb{Q}$ of the ring $\mathbb{Z}$ defines a family of metric spaces $(\mathbb{Q}, d_v)$, where $d_v(x,y) = |x-y|_v$, $v \in M_{\mathbb{Q}}$. To each such pair $(\mathbb{Q}, d_v)$, we can associate the corresponding completions with respect to the topologies induced by the metrics. Each of these completions also has the structure of a field. We denote by $\mathbb{Q}_p$ the field given by the completion of $\mathbb{Q}$ with respect to the metric d_p, for p a prime number. The completion of $\mathbb{Q}$ with respect to $d_\infty(x,y) = |x-y|$ is the field $\mathbb{R}$ of real numbers. The fields $\mathbb{Q}_p$ and $\mathbb{R}$ are examples of commutative locally compact non-discrete fields. We can consider $\mathbb{Q}$ as a subfield of its completions $\mathbb{Q}_p$ and $\mathbb{R}$, thereby enriching the ambient topological structure and allowing the application of techniques from classical analysis. Let K^* denote the group of non-zero elements of a field K. As we shall see in Chapter 3, consideration of the actions of $\mathbb{Q}_p^*$ on $\mathbb{Q}_p$ and of $\mathbb{R}^*$ on $\mathbb{R}$ already lead to some interesting trace formulae and can be seen as a first step in analyzing "locally" the ringstructure of $\mathbb{Z}$ from an operator-theoretic viewpoint.

In general, a local field K is a commutative field which carries a topology with respect to which the field operations are continuous and as a metric space it is complete, non-discrete and locally compact. (Often, one does not assume commutativity in the definition of a local field). Any locally compact Hausdorff topological group has a unique (up to scalars) non-zero left invariant measure which is finite on compact sets. If the group is abelian, this measure is also right invariant. It is called the Haar measure. The action of $K^* = K \setminus \{0\}$ on K by multiplication,

$$(\lambda, x) \mapsto \lambda x, \qquad \lambda \in K^*,\ x \in K,$$

induces a scaling of the Haar measure on the additive group K and hence a homomorphism of multiplicative groups

$$\begin{aligned} K^* \to &\quad \mathbb{R}^*_{>0} \\ \lambda \mapsto &\quad |\lambda|, \end{aligned}$$

where $\mathbb{R}^*_{>0}$ is the positive real numbers. Let

$$\mathrm{Mod}(K) = \{|\lambda| \in \mathbb{R}^*_{>0}, \lambda \in K^*\}.$$

Then $\mathrm{Mod}(K)$ is a closed subgroup of $\mathbb{R}^*_{>0}$. There are two classes of local fields, as follows (see [**38**], §I-4, Theorem 5 and Theorem 8).

(i) Archimedean local fields: $\mathrm{Mod}(K) = \mathbb{R}^*_{>0}$, in which case $K = \mathbb{R}$ or $\mathbb{C}$.

(ii) Non-archimedean local fields: $\mathrm{Mod}(K) \neq \mathbb{R}^*_{>0}$, in which case one has

$$\mathrm{Mod}(K) = q^{\mathbb{Z}},$$

where $q = p^d$ for some prime p and some positive integer d. Moreover,

$$R = \{x \in K \mid |x| \leq 1\}$$

is the unique maximal compact subring of K. It is a local ring with unique maximal ideal

$$\mathcal{P} = \{x \in K \mid |x| < 1\},$$

with $R/\mathcal{P} \simeq \mathbb{F}_q$, the finite field with q elements. (Notice that if K' is an extension of K of degree d then for $a \in K$ its modulus with respect to K' is the dth power of its modulus with respect to K.) If the non-archimedean local field has characteristic $p > 1$, then it is isomorphic to a field of formal power series in one indeterminate with coefficients in a finite field. If the non-archimedean field is of characteristic zero, then it is a finite algebraic extension of $\mathbb{Q}_p$.

For p prime and $x \in \mathbb{Q}_p$, there exists an integer r such that x can be written in the form

$$x = \sum_{i=0}^{\infty} a_{r+i} p^{r+i},$$

with $0 \leq a_{r+i} \leq p-1$, $i \geq 0$, and $a_r \neq 0$. We then have $|x|_p = p^{-r}$ and $x \in \mathbb{Z}_p$ if and only if $a_j = 0$ for $j < 0$. The maximal compact subring of $\mathbb{Q}_p$ is the ring of p-adic integers $\mathbb{Z}_p$ given by

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\},$$

which contains the rational integers $\mathbb{Z}$ as a subring. The unique maximal ideal of $\mathbb{Q}_p$ is

$$\mathcal{P} = \{x \in \mathbb{Z}_p \mid |x|_p < 1\},$$

so that $\mathbb{Z}_p/\mathcal{P} \simeq \mathbb{Z}/p\mathbb{Z} \simeq \mathbb{F}_p$.

1.3. Global fields and their adele rings. A *global field* K can be defined as a discrete cocompact subfield of a (non-discrete) locally compact semi-simple commutative ring $A = A_K$, called its adele ring [**38**]. There are two classes of global fields, as follows.

(i) Global fields of characteristic 0: These are the number fields, that is, the commutative fields which are finite dimensional vector spaces over $\mathbb{Q}$, this dimension being usually referred to as the degree of the field over $\mathbb{Q}$.

(ii) Global fields of characteristic p, where p is prime: These are the finitely generated extensions of $\mathbb{F}_p$ of transcendence degree 1 over $\mathbb{F}_p$. If K is such a field, then there is a $T \in K$, transcendental over $\mathbb{F}_p$, such that K is a finite algebraic extension of $\mathbb{F}_p(T)$. The field $\mathbb{F}_q$, where $q = p^f$ for some f, is included in K as its maximal finite subfield, called the field of constants. The field K may also be realized as a function field of a non-singular algebraic curveover $\mathbb{F}_q$. An analogue of the Riemann Hypothesis exists for such fields and is discussed in Chapter 2.

The field $\mathbb{Q}$ of rational numbers is a global field of characteristic 0 with a set of inequivalent valuations $v \in M_{\mathbb{Q}}$, $v = \infty$ or $v = p$, prime, as in §1.1. As in §1.2, associated local fields are given by the completions

$$\mathbb{Q} \hookrightarrow \mathbb{R}, \quad \text{for} \quad v = \infty$$

and

$$\mathbb{Q} \hookrightarrow \mathbb{Q}_p, \quad \text{for} \quad v = p.$$

The adele ring $A = A_{\mathbb{Q}}$ combines all these local fields in a way that gives them equal status and combines their topologies into an overall locally compact topology. It is given by the restricted product

$$A = \mathbb{R} \times \prod_p{}' \mathbb{Q}_p$$

with respect to $\mathbb{Z}_p$. This means that the elements of A are infinite vectors indexed by $M_{\mathbb{Q}}$,

$$x = (x_\infty, x_2, x_3, \ldots)$$

with $x_\infty \in \mathbb{R}$, $x_p \in \mathbb{Q}_p$ and $x_p \in \mathbb{Z}_p$ for all but finitely many primes p. Addition and multiplication are componentwise. The embeddings of $\mathbb{Q}$ into its completions with respect to the elements of $M_{\mathbb{Q}}$ induce a diagonal embedding of $\mathbb{Q}$ into A whose image is called the principal adeles. Therefore, $a \in \mathbb{Q}$ corresponds to the principal adele

$$a = (a, a, a, \ldots).$$

A basis for the topology on A is given by the restricted products $U_\infty \times \prod'_p U_p$ where U_∞ is open in $\mathbb{R}$ and U_p is open in $\mathbb{Q}_p$ with $U_p = \mathbb{Z}_p$ for all but finitely many primes p. The quotient $A/\mathbb{Q}$ of the adeles by the principal adeles is compact. The reason for the restricted product is precisely to enable the definition of a non-trivial locally compact topology on A extending the locally compact topologies of the factors.

The group $J = J_\mathbb{Q}$ of ideles of $\mathbb{Q}$ is the restricted product

$$J = \mathbb{R}^* \times \prod_p{}' \mathbb{Q}_p^*$$

with respect to $\mathbb{Z}_p^* := \{x \in \mathbb{Q}_p \mid |x|_p = 1\}$. These are the invertible elements of A. The elements of J are infinite vectors indexed by $M_\mathbb{Q}$,

$$u = (u_\infty, u_2, u_3, \ldots)$$

with $u_\infty \in \mathbb{R}, u_\infty \neq 0$, $u_p \in \mathbb{Q}_p, u_p \neq 0$ and $|u_p|_p = 1$ for all but finitely many primes p. Multiplication is componentwise. The topology on J is induced by the inclusion

$$\begin{aligned} J &\hookrightarrow A \times A \\ u &\mapsto (u, u^{-1}) \end{aligned}$$

The group $\mathbb{Q}^*$ of non-zero rational numbers injects diagonally into J and their image is called the principal ideles. Therefore $q \in \mathbb{Q}^*$ corresponds to the principal idele

$$q = (q, q, q, \ldots).$$

The group J carries the norm given by

$$\begin{array}{rcl} |\ \ | : J & \to & \mathbb{R}^*_{>0} \\ u = (u_v)_{v \in M_\mathbb{Q}} & \mapsto & \displaystyle\prod_{v \in M_\mathbb{Q}} |u_v|_v \ = \ |u_\infty| \prod_{p \text{ prime}} |u_p|_p. \end{array}$$

The Product Formula of §1.1 implies that $\mathbb{Q}^*$ is contained in Ker$|\ \ |$, the elements of J of norm 1, that is, for $u \in \mathbb{Q}$, $u \neq 0$ we have

$$|u| = |u|_\infty \prod_p |u|_p = 1.$$

The action of J on A, therefore, embodies simultaneously the actions of $\mathbb{Q}_p^*$ on $\mathbb{Q}_p$ and $\mathbb{R}^*$ on $\mathbb{R}$. However, the roles of the individual valuations $v \in M_\mathbb{Q}$ remain independent, so one cannot hope to gain much additional insight into the structure of the ring $\mathbb{Z}$ in this way.

The *Idele Class Group* is the quotient $C = J/\mathbb{Q}^*$ of the ideles by the principal ideles. By the Product Formula, the norm defined on the ideles induces a well-defined norm on this quotient, which we also denote by

$$|\ \ | : C \to \mathbb{R}^*_{>0}.$$

Let J^1 be the subgroup of J given by the kernel of the norm map. The principal ideles $\mathbb{Q}^*$ form a discrete subgroup of J^1 and the quotient group $C^1 = J^1/\mathbb{Q}^*$ is compact. The idele class group C is the direct product of C^1 and $\mathbb{R}^*_{>0}$ and as such is called a quasi-compact group. For proofs of these facts, see [**38**], Chapter 4. A (continuous) homomorphism

$$\chi : C \to \mathbb{C}^*$$

is called a quasi-character. The quasi-characters form a group under pointwise multiplication. A quasi-character is called principal if it is trivial on C^1, and the principal quasi-characters form a subgroup. The norm map on C gives a non-trivial homomorphism of C onto $\mathbb{R}^*_{>0}$ and the principal quasi-characters are of the form $u \mapsto |u|^t$, $t \in \mathbb{C}$, $u \in C$. Every quasi-character admits a factorization

$$\chi(u) = \chi_0(u)|u|^t, \qquad t \in \mathbb{C}, \quad u \in C,$$

with $\chi_0 : C \to U(1)$ a *unitary* character on C, that is, a homomorphism onto the group of complex numbers with absolute value 1. Every quasi-character χ on C can be considered as a homomorphism $\chi : J \to \mathbb{C}$ which is trivial on $\mathbb{Q}^*$. For every $v \in M_{\mathbb{Q}}$, there is a natural embedding of $\mathbb{Q}_v^*$ (where $\mathbb{Q}_\infty = \mathbb{R}$) into J by sending $x \in \mathbb{Q}_v^*$ to the idele $(u_w)_{w \in M_{\mathbb{Q}}}$ with $u_w = 1$ for all $w \neq v$ and $u_v = x$. Thereby, the quasi-character χ induces a homomorphism χ_v on $\mathbb{Q}_v$ with $\chi_p(\mathbb{Z}_p^*) = \{1\}$ for almost all primes p. The finite set S of p for which $\chi_p(\mathbb{Z}_p^*) \neq \{1\}$ is called the set of ramified primes. We may then write $\chi = \chi_\infty \prod_p \chi_p$. At an unramified prime $p \notin S$, the local factor χ_p is determined by its value at p. The L-function with non-principal quasi-character (or Grössencharacter) $\chi(u) = \chi_0(u)|u|^t$ is defined in the region $\Re(s) > 1 - \tau$, where $\tau = \Re(t)$, by

$$L(\chi, s) = \prod_{p \notin S} \left(1 - \chi_p(p)p^{-s}\right)^{-1}.$$

For $\chi \neq 1$ this function has an analytic continuation to all of $\mathbb{C}$, also denoted $L(\chi, s)$. For more details, see [**38**], Chapter 7. Notice that the Riemann zeta function corresponds to the case where χ is trivial and S is empty. We have the following generalization of the Riemann Hypothesis for unitary characters χ_0 on C (which has a corresponding version for all global fields, not just $\mathbb{Q}$).

Generalized Riemann Hypothesis: *We have $L(\chi_0, s) = 0$ for $\Re(s) \in]0, 1[$ if and only if $\Re(s) = \frac{1}{2}$.*

1.4. Connes's dynamical system. There are natural symmetry groups which arise for global fields. In the characteristic zero case, given a finite extension K of $\mathbb{Q}$, the associated symmetry group is the group of field automorphisms of K which leave $\mathbb{Q}$ fixed. This symmetry group is called the Galois group of K over $\mathbb{Q}$. In the characteristic p case, the symmetry groups come from Frobenius automorphisms of the corresponding variety over $\mathbb{F}_q$ given by raising coordinates on the variety to the q-th power. For a global field K of finite characteristic p, the corresponding idele class group C_K turns out to be canonically isomorphic to the Weil group W_K generated by all automorphisms leaving K fixed, and induced on a certain field extension of K by powers of the Frobenius. Therefore the natural symmetry group is in fact the idele class group which in turn has an interpretation as a Galois group.

The following proposal of Weil, made in 1951, is a central motivation for Connes's approach [**36**]

> "The search for an interpretation of C_K when K is a number field, which is in any way analogous to the interpretation as a Galois group when K is a function field, seems to me to constitute one of the fundamental problems of number theory nowadays; it is

possible that such an interpretation holds the key to the Riemann hypothesis."

(translated from French)

For a local field, the corresponding Weil group W_K is again generated by powers of the Frobenius automorphism of an extension of K. By the main result of local class field theory the group W_K is isomorphic to K^*, which therefore locally plays the role of the idele class group.

Class field theory relates the arithmetic of a number field, or of a local field, to the Galois extensions of the field. For a local field, by the remarks above, class field theory tells us that the group K^* plays a central role in this relation. This group acts naturally on the space consisting of the elements of K, and understanding the action of multiplication on the additive structure of K, that is, the map,

$$\begin{aligned} K^* \times K &\to K \\ (\lambda, x) &\mapsto \lambda x \end{aligned}$$

is certainly a basic part of understanding the arithmetic of a local field.

In the light of the situation for global fields of characteristic $p > 0$, one can view the analogue of the action of W_K for the field $\mathbb{Q}$ as being the passage to the quotient by $\mathbb{Q}^*$ of the map,

$$\begin{aligned} J \times A &\to A \\ (u, x) &\mapsto ux, \end{aligned}$$

that is

$$\begin{aligned} C \times X &\to X \\ ([u], [x]) &\mapsto [u][x]. \end{aligned}$$

Here C is the idele class group as above and X is the space of cosets $X = A/\mathbb{Q}^*$. The notation [ ] means the class modulo the multiplicative action of $\mathbb{Q}^*$ and will be mainly dropped in future. Therefore $[a] = [b]$ in X for $a, b \in A$ if and only if there is a $q \in \mathbb{Q}^*$ such that $a = qb$. The space X is very singular (not even Hausdorff).

Connes proposes to study the dynamical system (X, C) using the following guidelines.

- Relate the spectral geometry of the action (X, C) to the zeros of $\zeta(s)$.
- Relate the non-commutative geometry of the orbits of (X, C) to the valuations $M_{\mathbb{Q}}$ of $\mathbb{Q}$.
- Show that the consequent relation of the zeros of $\zeta(s)$ to the primes of $\mathbb{Z}$ is fine enough to prove the Riemann Hypothesis.

1.5. Weil's Explicit Formula. A very crude relation between the zeros of $\zeta(s)$ and the primes appeared already in §1.1 when we compared the product formula over Z for $\xi(t)$ in Theorem 1(i) with the Euler product formula (2) for $\zeta(s)$. Namely, we have, for $\Re(s) > 1$,

$$s(s-1)\Gamma(\frac{s}{2}) \prod_{p \text{ prime}} (1 - p^{-s})^{-1} = \exp(bs) \prod_{\rho \in Z} (1 - \frac{s}{\rho}) \exp(\frac{s}{\rho}).$$

An important refinement of ideas going back to Riemann's paper led Weil to develop his "Explicit Formula". Roughly speaking, the idea is to take logarithms and then

Mellin transforms in the last displayed formula. For a function $f : \mathbb{R}_+ \to \mathbb{C}$ we define the (formal) Mellin transform to be

$$M(f,z) = \int_0^\infty f(t)t^z \frac{dt}{t}.$$

Then Weil's formula is (formally) as follows:

$$\begin{aligned}(9) \quad M(f,0) - \sum_{\rho\in Z} M(f,\rho) + M(f,1) \;\; &= \\ (\log 4\pi + \gamma) f(1) \;\; &+ \sum_{m=1}^{\infty} \sum_{p \text{ prime}} (\log p)\{f(p^m) + p^{-m} f(p^{-m})\} \\ &+ \int_1^\infty \{f(x) + x^{-1} f(x^{-1}) - \frac{2}{x} f(1)\} \frac{dx}{x - x^{-1}}.\end{aligned}$$

Here,

$$\sum_{\rho\in Z} M(f,\rho) := \lim_{T\to\infty} \sum_{|\Im(\rho)|<T} M(f,\rho).$$

Weil also observed that the Riemann Hypothesis is equivalent to the positivity of

$$R(f) := \sum_{\rho\in Z} M(f,\rho)$$

for functions of the form

$$f(x) = \int_0^\infty g(xy)\overline{g(y)}dy. \tag{10}$$

This translates into the negativity of the left hand side of (9) for such f which also satisfy

$$M(f,0) = \int_0^\infty f(x)\frac{dx}{x} = 0, \quad M(f,1) = \int_0^\infty f(x)dx = 0.$$

Indeed, for f as in (10), we have

$$M(f,\rho) = M(g,\rho - 1/2)\overline{M(g, -(\overline{\rho} - 1/2))},$$

so that RH implies the positivity of $R(f)$. Conversely, we have enough functions $M(f,z)$ to localize the zeros of $\zeta(s)$. To make this rigorous and not just formal, we must impose some conditions on the class of functions $f : \mathbb{R}_+ \to \mathbb{C}$. We require that f be continuous and continuously differentiable except at finitely many points where f and f' have discontinuities of the first kind. At these discontinuities f and f' are defined as the average of their left and right values. Also, there is a $\delta > 0$ such that $f(x) = O(x^\delta)$ as $x \to 0^+$ and $f(x) = O(x^{-1-\delta})$ as $x \to +\infty$. Then $M(f,z)$ is analytic for $-\delta < \Re(z) < 1 + \delta$ (see [**3**]).

As suggested in §1.4, for the action (X, C) the zeros of $\zeta(s)$ should have a spectral interpretation. Inspection of (9) suggests that these eigenvalues should appear with a negative sign to match the term $-\sum_{\rho\in Z} M(f,\rho)$. This is, in fact, a feature of Connes's approach. We discuss this in Section 3.2. For some related comments on the comparison of (X, C) with Hamiltonian flows in quantum chaos, see [**9**].

2. The Riemann Hypothesis for curves over finite fields

There is an analogue of the Riemann Hypothesis (RH) for certain zeta functions attached to curves defined over finite fields. This analogue, introduced by E. Artin (1924) and checked by him for a few curves of genus 1, is known as the function field case after the function field of the curve. F. Schmidt (1931) showed that the zeta function for curves is rational and has a functional equation. Hasse (1934), using ideas from algebraic geometry together with some analytic methods, proved the analogue of RH for all genus 1 curves. In the early 1940's, Weil formulated an approach to RH for arbitrary curves defined over finite fields (see [**35**]). Subsequently, Weil developed the methods from algebraic geometry needed to execute his approach, and he published complete proofs in his 1948 book [**39**]. A more elementary proof was developed by Stepanov (1969), and this was further simplified by Bombieri (1972) [**2**]. Weil pioneered the study of zeta functions for arbitrary varieties over finite fields and developed some conjectures about these functions, in particular connecting topological data for these varieties to counting rational points on them over finite extensions of the base field. Included in these Weil Conjectures is a generalization of RH and its interpretation as a statement about the eigenvalues of the Frobenius automorphism acting on the cohomology of a variety. Dwork (1960) used p-adic analysis to establish the rationality of the zeta function for arbitrary varieties. Various cohomology theories relevant to the Weil Conjectures were developed, in particular by M. Artin, Grothendieck, Serre and Verdier. The complete proof of the Weil Conjectures was finally obtained in 1973 by Deligne [**11**], [**12**].

The successful solution of the analogue of RH for function fields provides strong encouragement for believing the validity of RH in the as yet unsolved number field case. It is still, however, an *open problem* to prove RH in the same generality for function fields using the program proposed by Alain Connes: it is anticipated that doing so would give much new information about the program in the number field case.

2.1. The zeta function of a curve over a finite field. Let p be a prime number and $\mathbb{F}_q$ the field of $q = p^d$, $d \geq 1$, elements. The map $\alpha \mapsto \alpha^q$ is the identity on $\mathbb{F}_q$. There are d automorphisms of $\mathbb{F}_q$ leaving $\mathbb{F}_p$ fixed, namely $\alpha \mapsto \alpha^{p^i}$, $\alpha \in \mathbb{F}_q$, $i = 0, \ldots, d-1$. The field $\mathbb{F}_q$ is the finite extension of $\mathbb{F}_p$ of degree d. Let K be a field extension of $\mathbb{F}_q$ with transcendence degree equal to 1. Then K is a finite algebraic extension of $\mathbb{F}_p(T)$ where T is transcendental over $\mathbb{F}_p$, and is a global field of characteristic p, as described in Section 1.3. We can also write $K = \mathbb{F}_q(x, y)$ where x is transcendental over $\mathbb{F}_q$ and there is an irreducible polynomial $F = F(x, y) \in \mathbb{F}_q[x, y]$ such that $F(x, y) = 0$. This equation defines a plane curve defined over $\mathbb{F}_q$ which has a smooth model Σ whose meromorphic function field is K.

Let $\overline{\mathbb{F}}_q$ denote the algebraic closure of $\mathbb{F}_q$, and $\overline{\Sigma} = \Sigma(\overline{\mathbb{F}}_q)$ denote the curve over $\overline{\mathbb{F}}_q$ given by the points of Σ rational over $\overline{\mathbb{F}}_q$. The field of functions of $\overline{\Sigma}$ is $\overline{K} = \overline{\mathbb{F}}_q(x, y)$. The Frobenius automorphism of $\overline{K}$ is given by the map $u \mapsto u^q$, $u \in \overline{K}$. This induces the map $(x, y) \mapsto (x^q, y^q)$ on the solutions $(x, y) \in \overline{\mathbb{F}}_q^{\,2}$ of $F(x, y) = 0$, which in turn defines a Frobenius map Fr on $\overline{\Sigma}$. By linearity over the integers, the map Fr extends to the additive group of finite formal sums of points on $\overline{\Sigma}$.

For every integer $j \geq 1$, the automorphism group of $\mathbb{F}_{q^j}$ over $\mathbb{F}_q$ is generated by the Frobenius map whose j-th power fixes the elements of $\mathbb{F}_{q^j}$. In the same way, the fixed points in $\overline{\Sigma}$ of the j-th iterate Fr^j are the points $\Sigma(\mathbb{F}_{q^j})$ of $\overline{\Sigma}$ rational over $\mathbb{F}_{q^j}$.

We introduce some definitions.

Definition 1. The divisor group $\mathrm{Div}(\Sigma)$ of Σ over $\mathbb{F}_q$ is the formal additive group of *finite* sums

$$\mathrm{Div}(\Sigma) = \{\mathcal{A} = \sum_i a_i P_i,\ a_i \in \mathbb{Z},\ P_i \in \Sigma(\mathbb{F}_{q^{d_i}}) \text{ some } d_i \in \mathbb{N},\ \mathrm{Fr}(\mathcal{A}) = \mathcal{A}\}$$

invariant under the Frobenius automorphism Fr on $\overline{\Sigma}$. A divisor $\mathcal{A} = \sum_i a_i P_i$ is said to be effective (written $\mathcal{A} > 0$) if $a_i > 0$ for all i.

For two divisors $\mathcal{A}$ and $\mathcal{B}$ we write $\mathcal{A} > \mathcal{B}$ when $\mathcal{A} - \mathcal{B}$ is effective.

Definition 2. If $\mathcal{A} = \sum_i a_i P_i \in \mathrm{Div}(\Sigma)$, then the degree $d(\mathcal{A})$ of $\mathcal{A}$ is $\sum_i a_i$. The norm $N(\mathcal{A})$ of $\mathcal{A}$ is $q^{d(\mathcal{A})}$.

Notice that for $\mathcal{A}, \mathcal{B} \in \mathrm{Div}(\Sigma)$, we have

$$N(\mathcal{A} + \mathcal{B}) = N(\mathcal{A})N(\mathcal{B}). \tag{11}$$

Definition 3. An effective divisor $\mathcal{A} \in \mathrm{Div}(\Sigma)$ is prime if it cannot be written as the sum of two effective divisors in $\mathrm{Div}(\Sigma)$.

The effective divisors are the analogues of the positive integers and the prime divisors are the analogues of the prime integers. Every effective divisor can be uniquely decomposed (up to permutation of the summands) into a sum of prime divisors.

Definition 4. The zeta function of the curve Σ with function field K is given by

$$\zeta_K(s) = \zeta(s, \Sigma) = \sum_{\mathcal{A}>0} \frac{1}{N(\mathcal{A})^s} = \prod_{\mathcal{P}} (1 - N(\mathcal{P})^{-s})^{-1}, \qquad \Re(s) > 1, \tag{12}$$

where the product is over the prime divisors $\mathcal{P}$ in $\mathrm{Div}(\Sigma)$.

The Euler product decomposition on the right of (12) is a consequence of (11) and the uniqueness of the prime decomposition of effective divisors.

It is useful to use the change of variables $u = q^{-s}$, and write the zeta function as

$$Z(u) = \zeta_K(s) = \prod_{\mathcal{P}} (1 - u^{d(\mathcal{P})})^{-1}. \tag{13}$$

We then have

$$u \frac{d}{du} \log Z(u) = \sum_{j=1}^{\infty} \left(\sum_{d(\mathcal{P})|j} d(\mathcal{P}) \right) u^j. \tag{14}$$

The quantity $\sum_{d(\mathcal{P})|j} d(\mathcal{P})$ in (14) equals the number of points of $\Sigma(\mathbb{F}_{q^j})$.

Theorem 2. *The zeta function has an analytic continuation to the whole complex plane* $\mathbb{C}$ *and may be written*

$$\zeta_K(s) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}, \tag{15}$$

for a certain polynomial P *of degree* $2g$*, where* g *is the genus of* Σ*. Moreover, the polynomial* P *satisfies*

$$q^{gs}P(q^{-s}) = q^{g(1-s)}P(q^{s-1}). \tag{16}$$

The genus of $\overline{\Sigma}$ is the dimension over $\overline{\mathbb{F}}_q$ of the space of sections of the canonical sheaf of $\overline{\Sigma}$. From (16) we see that the zeta function satisfies the *functional equation*

$$q^{(g-1)s}\zeta_K(s) = q^{(g-1)(1-s)}\zeta_K(1-s).$$

The proof of Theorem 2 uses the Riemann-Roch Theorem for $\overline{\Sigma}$ (see, for example, [**16**], Chapter V, §5).

One can write the polynomial P as a product over the 'multiset' Z (points with multiplicities) of reciprocal zeroes of P:

$$P(u) = \prod_{\rho\in Z}(1-\rho u). \tag{17}$$

This gives an analogue of the Hadamard product formula of (6),

$$\zeta_K(s) = \left(\prod_{\rho\in Z}(1-\rho u)\right)(1-u)^{-1}(1-qu)^{-1}. \tag{18}$$

From the Euler product we know that $\zeta_K(s) \neq 0$ for $\Re(s) > 1$ and therefore $1 \leq |\rho| \leq q$. Moreover, (16) implies the symmetry of Z under $\rho \mapsto q/\rho$.

2.2. The Riemann Hypothesis, the explicit formula and positivity. We begin with a statement of the Riemann hypothesis for function fields.

The Riemann Hypothesis for curves over finite fields: *The zeros of* $\zeta_K(s)$ *lie on* $\Re(s) = \frac{1}{2}$*, or equivalently each* $\rho \in Z$ *has* $|\rho| = q^{\frac{1}{2}}$*.*

The proof of this statement is a theorem due in its full generality to André Weil who proved it in 1942, see [**39**]. It is equivalent to the positivity of a certain functional that we describe below. Following the treatment in [**30**], we note that as in the number field case, we have a formal relation between prime divisors and zeros of ζ_K. Namely, we see from (13) and (18) that

$$\prod_{\mathcal{P}}(1-u^{d(\mathcal{P})})^{-1} = \left(\prod_{\rho\in Z}(1-\rho u)\right)(1-u)^{-1}(1-qu)^{-1}.$$

Taking logarithmic derivatives as in (14) we obtain

$$\sum_{\mathcal{P}} d(\mathcal{P})u^{d(\mathcal{P})}(1-u^{d(\mathcal{P})})^{-1} = -\sum_{\rho\in Z}\rho u(1-\rho u)^{-1} + u(1-u)^{-1} + qu(1-qu)^{-1}.$$

Comparing coefficients of u^j we have

$$\sum_{d(\mathcal{P})|j} d(\mathcal{P}) = 1 - \sum_{\rho\in Z}\rho^j + q^j. \tag{19}$$

The left hand side of this formula is the number of points of $\Sigma(\mathbb{F}_{q^j})$, that is, the number of fixed points of the jth power of the Frobenius map acting on $\Sigma(\overline{\mathbb{F}_q})$. For instance, for the projective line $\Sigma = \mathbb{P}_1$, the left hand side is $1 + q^j$.

Multiplying (19) by $q^{-j/2}$ and using the functional equation, we obtain, after some manipulation, the following identity:

$$q^{-|j|/2} \sum_{d(\mathcal{P})|j} d(\mathcal{P}) = q^{j/2} + q^{-j/2} - \sum_{\rho \in Z} (\rho/q^{\frac{1}{2}})^j \,. \tag{20}$$

Let h be a function $h\colon \mathbb{Z} \to \mathbb{C}$. Define the discrete Mellin transform of h by

$$M^d(h,z) = \sum_{j \in \mathbb{Z}} h(j) z^j.$$

Suppose that h is such that its discrete Mellin transform converges in the interval $q^{-1/2} \leq |z| \leq q^{1/2}$. Multiplying (20) by $h(j)$ and summing over j gives the following:

Explicit Formula for curves over finite fields:

$$\begin{aligned}(21)\quad M^d(h, q^{\frac{1}{2}}) - \sum_{\rho \in Z} M^d(h, \rho/q^{\frac{1}{2}}) + M^d(h, q^{-\frac{1}{2}}) = \\ (2-2g)h(0) + \sum_{\ell \in \mathbb{Z}\setminus\{0\}} \sum_{\mathcal{P}} d(\mathcal{P}) q^{-d(\mathcal{P})|\ell|/2} h(d(\mathcal{P})\ell),\end{aligned}$$

where g is the genus of Σ.

Just as for the Riemann zeta function (see Section 1.5), the Riemann Hypothesis is equivalent to a positivity statement. Consider the natural involution and convolution on the algebra of functions $h : \mathbb{Z} \to \mathbb{C}$. The involution is given by,

$$h^*(j) = \overline{h(-j)}, \qquad j \in \mathbb{Z}, \tag{22}$$

and the convolution of two functions h_1 and h_2 by

$$h_1 * h_2(j) = \sum_{j_1 + j_2 = j} h_1(j_1) h_2(j_2), \qquad j \in \mathbb{Z}.$$

The Mellin transform takes this convolution into a product, namely

$$M^d(h_1 * h_2, z) = M^d(h_1, z) M^d(h_2, z).$$

Define a Hermitian form $R(h_1, h_2)$ by

$$R(h_1, h_2) = \sum_{\rho \in Z} M^d(h_1 * h_2^*, \rho/q^{\frac{1}{2}}). \tag{23}$$

By (21) we have

$$\begin{aligned}(24)\quad R(h_1,h_2) = M^d(h_1 * h_2^*, q^{\frac{1}{2}}) + M^d(h_1 * h_2^*, q^{-\frac{1}{2}}) + (2g-2)\,(h_1 * h_2^*)\,(0) \\ - \sum_{\mathcal{P}} \sum_{\ell \in \mathbb{Z}\setminus\{0\}} d(\mathcal{P}) q^{-d(\mathcal{P})|\ell|/2}\,(h_1 * h_2^*)\,(d(\mathcal{P})\ell).\end{aligned}$$

Theorem 3. *The Riemann hypothesis holds for $\zeta_K(s)$ if and only if R is positive semidefinite.*

PROOF. Suppose that the Riemann hypothesis holds for $\zeta_K(s)$. Then for any $\rho \in Z$, we have $|\rho/q^{\frac{1}{2}}| = 1$. Therefore,

$$\begin{aligned} M^d(h * h^*, \rho/q^{\frac{1}{2}}) &= M^d(h, \rho/q^{\frac{1}{2}}) M^d(h^*, \rho/q^{\frac{1}{2}}) \\ &= M^d(h, \rho/q^{\frac{1}{2}}) \overline{M^d(h, (\bar{\rho}/q^{\frac{1}{2}})^{-1})} \\ &= |M^d(h, \rho/q^{\frac{1}{2}})|^2. \end{aligned}$$

It follows that the form R is a finite sum of positive semidefinite forms and hence is itself positive semi-definite. The argument can be reversed by making an artful choice of the function h in terms of a presumed zero of ζ away from the critical line. Namely, suppose that R is positive semidefinite and that RH is false for $\zeta_K(s)$. Then there exists a $\rho_0 \in Z$ with $\rho_1 := q/\bar{\rho}_0 \neq \rho_0$. On the other hand, as the polynomial P in Theorem 2 has real coefficients, we have $\rho_1 \in Z$. Now choose, as we may, a polynomial F with

$$F(\rho_0) = i, \quad F(\rho_1) = -i, \quad F(\rho) = 0, \qquad \rho \in Z \setminus \{\rho_0, \rho_1\}.$$

There exists a function $h : \mathbb{Z} \to \mathbb{C}$ with

$$\begin{aligned} R(h, h) &= \sum_{\rho \in Z} M^d(h, \rho/q^{\frac{1}{2}}) \overline{M^d(h, (\bar{\rho}/q^{\frac{1}{2}})^{-1})} \\ &= \sum_{\rho \in Z} F(\rho) \overline{F(q/\bar{\rho})} \\ &= F(\rho_0)\overline{F(\rho_1)} + F(\rho_1)\overline{F(\rho_0)} = -2. \end{aligned}$$

This is absurd, so that RH must hold. □

To prove the positivity of R, Weil used ideas from algebraic geometry. The corresponding "geometric" ideas are lacking in the number field case, and Connes's papers propose a set-up within which such "non-commutative" geometric ideas may emerge.

The theory of étale l-adic cohomology, a cohomology theory for $\overline{\Sigma}$ with coefficients in $\mathbb{Q}_\ell$ where ℓ is a prime not equal to p, allows the explicit formula of (24) to be interpreted as a trace formula. Recall the formula given in (19),

$$\operatorname{Card}(\Sigma(\mathbb{F}_{q^j})) = 1 - \sum_{\rho \in Z} \rho^j + q^j. \tag{25}$$

This can be viewed as a Lefschetz fixed point formula in the context of finite fields. The classical Lefschetz fixed point formula applies to a complex variety V. Let F be an action on V. Then the number of fixed points of F is, by this formula,

$$\operatorname{Card}(\text{fixed points of } F) = \sum_j (-1)^j \operatorname{Tr}(F^* \mid H^j(V)),$$

where F^* is the induced action of F on cohomology.

The dimensions of the cohomology groups $H^*(\overline{\Sigma}, \mathbb{Q}_\ell)$ are given by

$$\dim(H^j(\overline{\Sigma}, \mathbb{Q}_l)) = \begin{cases} 0, & j > 2, \\ 1, & j = 0, 2, \\ 2g, & j = 1. \end{cases}$$

Recall that the Frobenius gives a map on $\overline{\Sigma}$, and that the points of $\Sigma(\mathbb{F}_{q^j})$ are the fixed points of Fr^j acting on the curve over the algebraic closure. By a version of the Lefschetz fixed point theorem, one identifies this with an alternating sum of traces on cohomology groups,

$$\begin{aligned}\mathrm{Card}(\Sigma(\mathbb{F}_{q^j})) &= \mathrm{Card}(\text{fixed points of } \mathrm{Fr}^j) = 1 - \sum_{\rho\in Z}\rho^j + q^j \\ &= \mathrm{Tr}((\mathrm{Fr}^*)^j \mid H^0(\overline{\Sigma},\mathbb{Q}_\ell)) - \mathrm{Tr}((\mathrm{Fr}^*)^j \mid H^1(\overline{\Sigma},\mathbb{Q}_\ell)) + \mathrm{Tr}((\mathrm{Fr}^*)^j \mid H^2(\overline{\Sigma},\mathbb{Q}_\ell)).\end{aligned}$$

The first and last terms of the right hand side together sum to $1+q^j$, so that the zeros of the zeta function appear as the eigenvalues of Fr^* acting on H^1 and we have

$$\mathrm{Tr}((\mathrm{Fr}^*)^j \mid H^1(\overline{\Sigma},\mathbb{Q}_\ell)) = \sum_{\rho\in Z}\rho^j.$$

To show the positivity of the form in (23), Weil essentially works with $\overline{\Sigma}/S_g$, the symmetric product of $\overline{\Sigma}$ with itself g times. This is closely related to the Jacobian J of the curve $\overline{\Sigma}$. The Frobenius map on $\overline{\Sigma}$ induces an endomorphism Fr of J which is invertible over $\mathbb{Q}$. There is a standard involution $e \to e'$ on the endomorphisms e of J, called the Rosati involution, for which $\mathrm{Fr}\circ\mathrm{Fr}' = q$. Working over $\mathbb{Q}(q^{1/2})$ in this endomorphism algebra, one can reinterpret Theorem 3 as a statement about the positivity of the Rosati involution, which in turn follows from the Castelnuovo–Severi inequality for surfaces; see [**30**] §5.17 and [**32**].

2.3. Weil's proof by the theory of correspondences. In this section we briefly discuss Weil's proof of the Riemann Hypothesis for curves over finite fields using the theory of correspondences. Full details are given in his book [**39**]. Useful additional references are [**16**], [**32**]. Once again the Riemann Hypothesis is reduced to a positivity statement which follows from the Castelnuovo–Severi inequality for surfaces.

A correspondence on a curve is the graph of a multi-valued map of a curve Σ to itself, and it may also be viewed as a divisor on the surface $\Sigma\times\Sigma$. Recall the formula given in (19), which yields the explicit formula, and is given by

$$\sum_{\rho\in Z}\rho^j = 1 + q^j - \mathrm{Card}(\Sigma(\mathbb{F}_{q^j})), \qquad j\ge 1. \tag{26}$$

If $\mathrm{Fr}:\overline{\Sigma}\to\overline{\Sigma}$ is the Frobenius map, then for every integer $j\ge 1$, the map Fr^j is single-valued and of degree q^j. Let F^j be the graph of Fr^j on $\overline{\Sigma}\times\overline{\Sigma}$. Then, for any point P on $\overline{\Sigma}$, we have

$$m(F^j) := \mathrm{Card}\left(F^j \cap \left(\{P\}\times\overline{\Sigma}\right)\right) = 1, \tag{27}$$

and

$$d(F^j) := \mathrm{Card}\left(F^j \cap \left(\overline{\Sigma}\times\{P\}\right)\right) = q^j. \tag{28}$$

Moreover, the graph F^j intersects the diagonal Δ of $\overline{\Sigma}\times\overline{\Sigma}$ in $N_j = \mathrm{Card}(\Sigma(\mathbb{F}_{q^j}))$ points.

There is an intersection form $(D_1,D_2)\to D_1\cdot D_2$ defined on divisors on the surface $\overline{\Sigma}\times\overline{\Sigma}$, or correspondences on the curve $\overline{\Sigma}$, which remains well-defined on divisor classes, two divisors being in the same class if their difference is the divisor of a function on $\overline{\Sigma}\times\overline{\Sigma}$. The divisors form a ring with multiplication $\circ$ induced by composition of maps, and has a transpose $D\to D^t$ given by composition with permutation of the two factors of $\overline{\Sigma} \times \overline{\Sigma}$. For a divisor D on $\overline{\Sigma} \times \overline{\Sigma}$, and a point P on $\overline{\Sigma}$, we generalize (27) and (28) to arbitrary divisors by setting

$$m(D) = D \cdot \left(\{P\} \times \overline{\Sigma}\right) \text{ and } d(D) = D \cdot \left(\overline{\Sigma} \times \{P\}\right).$$

Weil defines the trace on divisors, invariant on a given class, by

$$\operatorname{Trace}(D) = m(D) + d(D) - D \cdot \Delta,$$

where as above Δ is the diagonal of $\overline{\Sigma} \times \overline{\Sigma}$. The Riemann hypothesis is then equivalent to the positivity statement

$$\operatorname{Trace}(D \circ D^t) > 0, \tag{29}$$

for D not equivalent to a divisor of the form $\{P\} \times \overline{\Sigma}$ or $\overline{\Sigma} \times \{P\}$, with P a point of $\overline{\Sigma}$. Indeed, if $D = m\Delta + nF$ for integers m and n, we have

$$D \circ D^t = \left(m^2 + qn^2\right) \Delta + mn \left(F + F^t\right),$$

using $\Delta = \Delta^t$ and $F \circ F^t = q\Delta$. As the self-intersection number of Δ is $2 - 2g$, the trace of Δ equals $2g$ and, as we saw,

$$\operatorname{Trace}(F) = \operatorname{Trace}(F^t) = 1 + q - N_1.$$

The positivity statement in (29) implies that the form

$$\operatorname{Trace}(D \circ D^t)(m, n) = 2gm^2 + 2\left(1 + q - N_1\right) mn + 2gqn^2,$$

is positive definite for $g \neq 0$ (it is identically zero for $g = 0$). Therefore, it has negative discriminant, so that

$$\left(1 + q - N_1\right)^2 - (2g)(2gq) < 0,$$

that is

$$|N_1 - 1 - q| < 2gq^{1/2}.$$

The same argument applied to $\mathbb{F}_{q^j}$ yields, for all $j \geq 1$,

$$|N_j - 1 - q^j| < 2gq^{j/2}.$$

Applying (25), we deduce that for all $j \geq 1$,

$$\Big|\sum_{\rho \in Z} \rho^j\Big| \leq 2gq^{j/2}. \tag{30}$$

From (17) we have

$$\log P(u) = -\sum_{\rho \in Z} \sum_{j=1}^{\infty} j\rho^j u^j = -\sum_{j=1}^{\infty} j \left(\sum_{\rho \in Z} \rho^j\right) u^j. \tag{31}$$

Because of (30), we see that the series in (31) converges absolutely for $|u| < q^{-1/2}$, and this means that we have $|\rho| \leq q^{1/2}$ for all $\rho \in Z$. Moreover, (16) shows if $\rho \in Z$ then $q\rho^{-1} \in Z$, so that we deduce that $|\rho| = q^{1/2}$ for all $\rho \in Z$. Therefore the Riemann Hypothesis is proven.

Weil showed that the positivity of (29) is a consequence of the negative semidefiniteness of the intersection form on divisors of degree zero of a projective embedding of the surface $\overline{\Sigma} \times \overline{\Sigma}$, which in turn follows from the Castelnuovo–Severi inequality. For higher dimensional varieties rather than curves (where RH is replaced by the Weil Conjectures), no proof along these lines has been found: Deligne's proof uses in an essential way a deformation of the variety in a family.

2.4. Finding a theory over the complex numbers. Connes aims to construct a theory over $\mathbb{C}$ rather than over $\mathbb{Q}_\ell$. As before, let K be the function field of a smooth curve Σ over a finite field $\mathbb{F}_q$, and K_{un} the field generated over K by the roots of unity of order prime to p. Let K_{ab} be the maximal abelian extension of K. Then the elements of $\mathrm{Gal}(K_{\mathrm{ab}}/K)$ inducing powers of the Frobenius on K_{un} form a group called the Weil group W_K. The *Main Theorem of Class Field Theory* for function fields tells us that W_K is isomorphic to the idele class group C_K of K. Therefore, in this set-up one can view the idele class group as capturing the Frobenius action. In a formulation using operator algebras over $\mathbb{C}$, the elements of the zero set Z of the zeta function should ideally appear as eigenvalues of $C = C_K$ acting via a representation

$$W : C \to \mathcal{L}(H)$$

where $\mathcal{L}(H)$ is the algebra of bounded operators in a complex Hilbert space H. Moreover, guided by the Lefschetz formula (alternating sum), the Hilbert space H should appear via its negative $\ominus H$. The problem in the function field case is to replace the ℓ-adic cohomology groups by an object defined over $\mathbb{C}$ which exists also for number fields.

We summarise these goals in the following dictionary:

Zeta Function	**Classical Geometry**	**Noncommutative Geometry**
Spectral interpretation of zeros	Spectrum of Fr^* on $H^1(\Sigma, \mathbb{Q}_\ell)$	Spectrum of C on H
Functional equation	Riemann-Roch Theorem	Appropriate symmetry
Explicit formula	Lefschetz formula	Geometric trace formula
Riemann hypothesis	Castelnuovo positivity	Positivity of Weil functional

3. The local trace formula and the Pólya-Hilbert space

We now turn to a more direct study of Connes's approach. In the previous chapters, we motivated the study of the dynamical system (X, C) of Connes, where $C = J/K^*$ is the idele class group of a global field K and $X = A/K^*$ is the space of adele cosets. Once again we, for simplicity, restrict ourselves mainly to the case $K = \mathbb{Q}$, although the discussion goes through for arbitrary global fields.

Classical spaces, their topology, their differentiable and conformal structure, can be understood from the study of their associated (commutative!) algebras.

A main goal of noncommutative geometry is to extend these structures to a wider class of examples by developing their analogues on noncommutative algebras whose corresponding spaces are non-existent or hard to study. Connes proposes that, although global fields, their adele rings and idele class groups are commutative, the Riemann Hypothesis (RH) itself should be viewed as part of "noncommutative number theory".

We saw in Chapter 1 that a natural first step in this program is to study, in the case where K is a local field, the action on K of the group K^* of its non-zero elements. Connes develops a rigorous trace formula for the action (K^*, K). This trace formula, which we discuss in §3.1, turns out to provide positive support for his approach to the Riemann Hypothesis for global fields.

Indeed, Connes conjectures a trace formula for the action (C, X) for global fields which is a sum of the contributions of the local trace formulae and this conjecture turns out to be equivalent to RH. In §3.2, we discuss Connes's interesting and rigorous interpretation of the non-trivial zeros of the L-functions with Grossencharacter for a global field (of which the Riemann zeta function is a special case) in terms of the action of the idele class group on a certain Hilbert space. We call this the Polya-Hilbert space after earlier suggestions by Polya and Hilbert that there should be a spectral interpretation of the non-trivial zeros of the Riemann zeta function. This is not sufficient to prove Connes's conjectured global trace formula, but it provides evidence for the rich information on RH contained in the action (C, X). Although we do not give proofs of the results of §3.2, we end in §3.3 with an explanation of why the non-trivial zeros of the L-functions turn up in the spectral formula of §3.2. Full proofs are given in [**9**].

We conclude this section by noting, as pointed out to us by Peter Sarnak, that Connes gives a spectral interpretation only of the zeros on the line $s = 1/2$ (and hence in fact he has a spectral interpretation only assuming RH). This is because he uses certain Sobolev spaces which only pick up zeros on this line.

3.1. The local trace formula. For the time being, we assume that K is a local field, and we look at the action of K^* by multiplication on K. For simplicity, we work with the case $K = \mathbb{R}$ or $K = \mathbb{Q}_p$ for p a prime number. These are the local fields obtained by completing the field $\mathbb{Q}$ at the elements of $M_{\mathbb{Q}}$, as in Chapter 1.

Let $H = L^2(K)$ where the L^2-norm is with respect to the additive Haar measure dx on K. Let U be the regular representation of K^*, the non-zero elements of K, on H. Therefore, for $\lambda \in K^*$ and $x \in K$, we have

$$(U(\lambda)\xi)(x) = \xi(\lambda^{-1}x).$$

The operator $U(\lambda)$ is not of trace class; and, in order to associate to it a trace class operator, one averages it over a test function $h \in \mathcal{S}(K^*)$ with compact support. If $h(1) \neq 0$, it is necessary to further modify the operator, as we shall see.

Define the operator in H given by

$$U(h) = \int_{K^*} h(\lambda)U(\lambda)d^*\lambda,$$

where $d^*\lambda$ denotes the multiplicative Haar measure on K^*, normalized by requiring that

$$\int_{|\lambda|\in[1,\Lambda]} d^*\lambda \sim \log \Lambda, \qquad \Lambda \to \infty.$$

To the operator $U(h)$ we associate the Schwartz kernel $k(x,y)$ on K^2 for which

$$(U(h)\xi)(x) = \int_K k(x,y)\xi(y)dy.$$

Let $\delta = \delta(x)$ denote the Dirac delta distribution on K. We have

$$k(x,y) = \int_{K^*} h(\lambda^{-1})\delta(y-\lambda x)d^*\lambda.$$

The associated distributional trace is given by

$$\mathrm{Tr}_D(U(h)) = \int_K k(x,x)dx = \int_{K^*} h(\lambda^{-1}) \left(\int_K \int_K \delta(x-y)\delta(y-\lambda x)\, dx\, dy \right) d^*\lambda.$$

When $\lambda \neq 1$, the distribution $\delta(x-y)$ has support on the line $x=y$ on K^2, whereas $\delta(x-\lambda y)$ has support on the transverse line $x = \lambda y$. As explained in [**9**], the integral

$$\int_K \int_K \delta(x-y)\delta(x-\lambda y)\, dx\, dy = |1-\lambda|^{-1}$$

is well-defined and equals, as expected, the displayed value. When $h(1) = 0$, so that the non-transverse case $\lambda = 1$ is cancelled out, we deduce that

$$\mathrm{Tr}_D(U(h)) = \int_{K^*} \frac{h(\lambda^{-1})}{|1-\lambda|} d^*\lambda.$$

In [**9**], the case $h(1) \neq 1$ is dealt with by introducing a cut-off. For $\Lambda > 0$, let P_Λ be the projection onto those functions $\xi = \xi(x) \in H$ supported on $|x| \leq \Lambda$. We define the corresponding projection in Fourier space by

$$\widehat{P}_\Lambda = F P_\Lambda F^{-1},$$

where

$$(F\xi)(x) = \widehat{\xi}(x) = \int_K \xi(y)\alpha(xy)dy,$$

for α a fixed nontrivial character of the additive group K. The cut-off at Λ is defined by the trace class operator

$$R_\Lambda = \widehat{P}_\Lambda P_\Lambda.$$

As R_Λ is trace class, the operator $R_\Lambda U(h)$ is also. By using standard formulae from symbol calculus, Connes derives the identity

$$\begin{aligned}\mathrm{Tr}(R_\Lambda U(h)) &= \int_{K^*} h(\lambda^{-1}) \int_K \int_{|x|\leq\Lambda;|\xi|\leq\Lambda} \delta(x+u-\lambda x)\alpha(u\xi)\, dx\, d\xi\, du\, d^*\lambda \\ &= \int_{K^*} h(\lambda^{-1}) \int_{|x|\leq\Lambda;|\xi|\leq\Lambda} \alpha((\lambda-1)x\xi)\, dx\, d\xi\, d^*\lambda.\end{aligned}$$

Having fixed the character α, we normalize the additive Haar measure on K to be self-dual. Then, there is a constant $\rho > 0$ such that

$$\int_{1\leq|\lambda|\leq\Lambda} \frac{d\lambda}{|\lambda|} \sim \rho \log \Lambda, \qquad \Lambda \to \infty,$$

so that

$$d^*\lambda = \rho^{-1}\frac{d\lambda}{|\lambda|}.$$

We have therefore

$$\int_{K^*} h(\lambda^{-1})\alpha((\lambda-1)x\xi)d^*\lambda = \rho^{-1}\int_K \frac{h((u+1)^{-1})}{|u+1|}\alpha(ux\xi)du.$$

We deduce that

$$\mathrm{Tr}(R_\Lambda U(h)) = \rho^{-1}\int_{|x|,|\xi|\le\Lambda} \widehat{g}(x\xi)\,dx\,d\xi,$$

where $g \in C_c^\infty(K)$ is given by

$$g(u) = \frac{h((u+1)^{-1})}{|u+1|}.$$

Let $v = x\xi$. Then $dx\,d\xi = dv\frac{dx}{|x|}$; and, for $|v| \le \Lambda^2$, we have

$$\rho^{-1}\int_{\frac{|v|}{\Lambda}\le|x|\le\Lambda} \frac{dx}{|x|} = \rho^{-1}\int_{1\le|y|\le\frac{\Lambda^2}{|v|}} \frac{dy}{|y|} = 2\log'\Lambda - \log|v|,$$

where we define

$$2\log'\Lambda = \int_{|\lambda|\in[\Lambda^{-1},\Lambda]} d^*\lambda.$$

It follows that

$$\mathrm{Tr}(R_\Lambda U(h)) = \int_{|v|\le\Lambda^2} \widehat{g}(v)(2\log'\Lambda - \log|v|)dv.$$

Using the fact that $g \in C_c^\infty(K)$, we deduce from this the asymptotic formula, as $\Lambda \to \infty$,

$$\begin{aligned}\mathrm{Tr}(R_\Lambda U(h)) =& 2g(0)\log'\Lambda - \int_K \widehat{g}(v)\log|v|dv + o(1)\\ =& 2h(1)\log'\Lambda - \int_K \frac{h((x+1)^{-1})}{|x+1|}\left(\int_K \log|v|\alpha(xv)dv\right)dx + o(1).\end{aligned}$$

In [**9**] it is shown that the distribution given by pairing with

$$-\int_K \log|v|\alpha(xv)dv$$

differs by a multiple $c_v\delta(x)$ of the delta function at $x = 0$ from the distribution on K defined by

$$P(f) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in\mathrm{Mod}(K)}} \left(\rho^{-1}\int_{|x|\ge\varepsilon} f(x)\frac{dx}{|x|} + f(0)\log\varepsilon\right).$$

As $\log|v|$ vanishes at $v = 1$, on replacing x by $x-1$, the constant c_v is determined by the condition that

$$L(f) = c_v f(0) + P(f)$$

is the unique distribution on K extending $\rho^{-1}\frac{1}{|u-1|}$, for $u \ne 1$, whose Fourier transform vanishes at 1, that is, $\widehat{L}(1) = 0$. If $k(u) = |u|^{-1}h(u^{-1})$, we have

$$\begin{aligned}L(k) =& -\int_K \frac{h((x+1)^{-1})}{|x+1|}\left(\int_K \log|v|\alpha(xv)dv\right)dx\\ =& -\int_K \frac{h(u^{-1})}{|u|}\left(\int_K \log|v|\alpha((u-1)v)dv\right)du.\end{aligned}$$

Therefore, outside of $x = 0$, the distribution

$$-\int_K \log|v|\alpha(xv)dv$$

agrees with $\rho^{-1}\frac{1}{|x|}$. If $h(1) = 0$, we find, as expected from our computation above of the distributional trace in this case, that

$$\begin{aligned}\operatorname{Tr}(R_\Lambda U(h)) =& \rho^{-1}\int_K \frac{h((x+1)^{-1})}{|x+1|}\frac{dx}{|x|} + o(1) \\ =& \int_{K^*} \frac{h(\lambda^{-1})}{|1-\lambda|} d^*\lambda + o(1), \qquad (h(1) = 0);\end{aligned}$$

and taking the limit as $\Lambda \to \infty$, we have

$$\operatorname{Tr}(U(h)) = \int_{K^*} \frac{h(\lambda^{-1})}{|1-\lambda|} d^*\lambda, \qquad (h(1) = 0).$$

Returning to the general case, $h(1) \neq 0$, we see that

$$\begin{aligned}\operatorname{Tr}(R_\Lambda U(h)) =& h(1)(2\log'\Lambda + c_v) + \lim_{\varepsilon\to 0}\left(\rho^{-1}\int_{|u-1|\geq\varepsilon} \frac{h(u^{-1})}{|u-1|}\frac{du}{|u|} + h(1)\log\varepsilon\right) \\ =& h(1)(2\log'\Lambda + c_v) + \lim_{\varepsilon\to 0}\left(\int_{|u-1|\geq\varepsilon} \frac{h(u^{-1})}{|u-1|} d^*u + h(1)\log\varepsilon\right)\end{aligned}$$

In [**9**], it is shown, for example, that for $v = \infty$ we have $c_v = \log(2\pi) + \gamma$, where γ is Euler's constant. We have, therefore, the following result.

THEOREM 4. *If $h \in \mathcal{S}_c(K^*)$, then $R_\Lambda U(h)$ is a trace class operator in $L^2(K)$; and as $\Lambda \to \infty$, we have the asymptotic formula*

$$\operatorname{Tr}(R_\Lambda U(h)) = 2h(1)\log'\Lambda + \int' \frac{h(\lambda^{-1})}{|1-\lambda|} d^*\lambda + o(1),$$

where $2\log'\Lambda = \int_{\lambda\in K^*, |\lambda|\in[\Lambda^{-1},\Lambda]} d^*\lambda$ and $\int'$ is the pairing of $h(u^{-1})/|u|$ with the unique distribution extending $\rho^{-1}du/|1-u|$ whose Fourier transform vanishes at 1.

For $K = \mathbb{R}$, we have

$$\begin{aligned}\operatorname{Tr}(R_\Lambda U(h)) = & h(1)(2\log\Lambda + \log(2\pi) + \gamma) \\ & + \lim_{\varepsilon\to 0}\left(\int_{|u-1|\geq\varepsilon} \frac{h(u^{-1})}{|u-1|} d^*u + h(1)(\log\varepsilon)\right) + o(1).\end{aligned} \tag{32}$$

3.2. The global case and the Pólya-Hilbert space. Connes conjectures that an analogue of the formula in Theorem 4 holds for the action of C on X. However, a major obstacle is to know whether it makes sense to talk about measurable functions on X. By analogy with functions on a manifold, one may try to think of A as a universal cover of X. One then views functions on X as averages of functions on A over the "universal covering group" $\mathbb{Q}^*$. (This is the map E below). Connes shows nonetheless that a "Pólya-Hilbert space" related to the action (C, X) allows a spectral interpretation of the non-trivial zeros of the Riemann zeta function.

We work with the case $K = \mathbb{Q}$. Let $\mathcal{S}(A)_0$ denote the subspace of $\mathcal{S}(A)$ given by

$$\mathcal{S}(A)_0 = \{f \in \mathcal{S}(A) : f(0) = \int f dx = 0\}. \tag{33}$$

Let E be the "averaging over $\mathbb{Q}^*$" operator which to $f \in \mathcal{S}(A)_0$ associates the element of $\mathcal{S}(C)$ given by

$$E(f)(u) = |u|^{1/2} \sum_{q \in \mathbb{Q}^*} f(qu). \tag{34}$$

For $\delta \geq 0$, let $L^2(X)_{0,\delta}$ be the completion of $\mathcal{S}(A)_0$ with respect to the norm given by

$$||f||_\delta^2 = \int_C |E(f)(u)|^2 (1 + \log^2 |u|)^{\delta/2} d^* u, \tag{35}$$

$$\text{for} \quad \int_{|u| \in [1,\Lambda]} d^* u \simeq \log \Lambda, \qquad \Lambda \to \infty.$$

If $g(x) = f(qx)$ for some fixed $q \in \mathbb{Q}^*$, then $||g||_\delta = ||f||_\delta$; and so one sees that this norm respects, in this sense, the passage to the quotient $A/\mathbb{Q}^*$. We define $L^2(X)_\delta$ by the short exact sequence

$$0 \to L^2(X)_{0,\delta} \to L^2(X)_\delta \to \mathbb{C} \oplus \mathbb{C}(1) \to 0. \tag{36}$$

When $\delta = 0$, we write $L^2(X)_0$ and $L^2(X)$ for the first two terms. Here $\mathbb{C}$ is the trivial C-module and $\mathbb{C}(1)$ is the C-module for which $u \in C$ acts by $|u|$, where $|\cdot|$ is the norm on C. Indeed, by the definition of $\mathcal{S}(A)_0$ in (33), we see that its two-dimensional supplement in $\mathcal{S}(A)$ is the C-module $\mathbb{C} \oplus \mathbb{C}(1)$.

Multiplication of C on A induces a representation of C on $L^2(X)_\delta$ given by

$$(U(\lambda)\xi)(x) = \xi(\lambda^{-1}x). \tag{37}$$

We introduce a Hilbert space H_δ via another short exact sequence,

$$0 \to L^2(X)_{0,\delta} \to L^2(C)_\delta \to H_\delta \to 0, \tag{38}$$

where the inclusion of $L^2(X)_{0,\delta}$ into $L^2(C)_\delta$ is effected by the isometry E. Here $L^2(C)_\delta$ is the completion with respect to the weighted Haar measure as in (35), where we write $L^2(C)$ when $\delta = 0$. The spectral interpretation on H_δ of the critical zeros of the L-functions in [**8**] relies on taking $\delta > 0$. Indeed, this is needed to control the growth of the functions on the non-compact quotient X; ultimately this parameter is eliminated from the conjectural trace formula by using cut-offs. It is important here to use the measure $|u|d^*u$ (implicit in (35)) instead of the additive Haar measure dx, this difference being a veritable one for global fields, where one has $dx = \lim_{\varepsilon \to 0} \varepsilon |x|^{1+\varepsilon} d^*x$.

The regular representation V of C on $L^2_\delta(C)$ descends to H_δ (it commutes with E up to a phase as an easy calculation shows), and we denote it by W. Connes describes (H_δ, W) as the Pólya-Hilbert space with group action for his approach to the Riemann hypothesis. He proves in [**8**] and [**9**] the remarkable result given in Theorem 5 relating the trace of this action to the zeros on the critical line of the L-functions with Grössencharacter discussed in Chapter 1.

The norm $|\ \ |$ on the abelian locally compact group C has kernel

$$C^1 = \{u \in C : |u| = 1\};$$

and, as $K = \mathbb{Q}$, it has image $\mathbb{R}^*_{>0}$. Now the compact group C^1 acts on H_δ, which splits with respect to this action into a canonical direct sum of pairwise orthogonal subspaces H_{δ,χ^1} where χ^1 runs through the discrete abelian group $\widehat{C^1}$ of characters of C^1. One can restrict the action (H_δ, W) to an action (H_{δ,χ^1}, W) for any $\chi^1 \in \widehat{C^1}$, and we have

$$H_{\delta,\chi^1} = \{\xi \in H_\delta : W(u)\xi = \chi^1(u)\xi \quad \text{for all } u \in C^1\}.$$

Choose a (non-canonical) decomposition $C = C^1 \times N$ with $N \simeq \mathbb{R}^*_{>0}$. For $\chi^1 \in \widehat{C^1}$, there is a unique extension to a quasi-character χ of C, vanishing on N. The choice of χ is unimportant in what follows, since if $\chi'(u) = \chi(u)|u|^{i\tau}$, $u \in C$, we have $L(\chi', s) = L(\chi, s + i\tau)$.

THEOREM 5. *For any Schwartz function $h \in \mathcal{S}(C)$ the operator*

$$W(h) = \int_C W(u)h(u)d^*u,$$

in H_δ is trace class, and its trace is given by

$$\mathrm{Trace}(W(h)) = \sum_{\substack{L(\chi,\frac{1}{2}+i\sigma)=0,\\ \sigma\in\mathbb{R}}} \widehat{h}(\chi, i\sigma)$$

where the sum is over the characters of C^1 with χ being the unique extension to a quasi-character on C vanishing on N. The multiplicity of the zero is counted as the largest integer $n < \frac{1}{2}(1+\delta)$ with n at most the multiplicity of $\frac{1}{2} + i\sigma$ as a zero of $L(\chi, s)$. Also, we define

$$\widehat{h}(\chi, s) := \int_C h(u)\chi(u)|u|^s d^*u.$$

Now, the action of C is free on $L^2(C)_\delta$, so that the short exact sequence (38) tells us that the trace of the action of C on H_δ should be, up to a correction due to a regularization, the negative of the trace of the action of C on $L^2(X)_{0,\delta}$. From (36), we see that the regularized trace of the action of C on $L^2(X)_\delta$ should involve the sum of the corresponding trace on $L^2(X)_{0,\delta}$ and the trace on $\mathbb{C} \oplus \mathbb{C}(1)$. Therefore the regularized trace of the action of C on $L^2(X)_\delta$ should involve the trace of the action on $\mathbb{C} \oplus \mathbb{C}(1)$ *minus* the trace of this action on H_δ. This *minus* sign is crucial for the comparison with the Weil distribution.

When $\chi = 1$ in Theorem 5, the corresponding L-function is the Riemann zeta function $\zeta(s)$; and the proof of the theorem shows that

$$\mathrm{Trace}(W(h)_{|H_{\delta,1}}) = \sum_{\substack{\zeta(\frac{1}{2}+is)=0,\\ s\in\mathbb{R}}} \widehat{h}(is),$$

where $\widehat{h}(is) = \widehat{h}(1, is)$, with the same convention for the multiplicity. There is a conjecture that the zeros of the Riemann zeta function are all simple, in which case we could take $n = 1$ and any $\delta > 1$.

3.3. A computation. The following result formulated by Weil in [**37**], and appearing in Tate's thesis, shows how the operation E in (34) brings in the zeros of the L-functions in the critical strip and provides the key to the proof of Theorem 5 and the appearance of the non-trivial zeros of the $L(\chi, s)$. It indicates that the non-trivial zeros of the L-functions should "span" H_δ, as they are "orthogonal" to the image of E.

Proposition 1. *Let χ be a unitary character on C. For any $\rho \in \mathbb{C}$ with $\Re(\rho) \in (\frac{1}{2}, \frac{1}{2})$, we have*

$$\int_C E(\xi)(u)\chi(u)|u|^\rho d^*u = 0, \qquad \text{for all } \xi \in \mathcal{S}(A)_0,$$

precisely when $L(\chi, \frac{1}{2} + \rho) = 0$.

We discuss the classical zeta function, the other cases being similar. Let p be a finite prime, and let $f \in \mathcal{S}(\mathbb{Q}_p)$. Let

$$\Delta_{p,s}(f) = \int_{\mathbb{Q}_p^*} f(x)|x|^s d^*x, \qquad \Re(s) > 0,$$

and also

$$\Delta'_{p,s}(f) = \int_{\mathbb{Q}_p^*} (f(x) - f(p^{-1}x)\,|x|^s d^*x,$$

where

$$\langle 1_{\mathbb{Z}_p}, \Delta'_{p,s}\rangle = \int_{\mathbb{Z}_p^*} d^*x.$$

We have

$$\Delta'_{p,s} = (1 - p^{-s})\Delta_{p,s} \tag{39}$$

For $p = \infty$, define

$$\Delta_{\infty,s}(f) = \Delta'_{\infty,s}(f) = \int_{\mathbb{R}^*} f(x)|x|^s ds.$$

One can define these functions globally by multiplying over all places. For $\sigma = \Re(s) > 1$ and $f \in \mathcal{S}(A)$, the following integral converges absolutely:

$$\int_J f(u)|u|^s d^*u.$$

Moreover, from (39), for $\sigma = \Re(s) > 1$,

$$\int_J f(u)|u|^s d^*u = \Delta_s(f) = \zeta(s)\Delta'_s(f) \neq 0,$$

where

$$\Delta_s = \Delta_{\infty,s} \times \prod_p \Delta_{p,s}$$

and

$$\Delta'_s = \Delta'_{\infty,s} \times \prod_p \Delta'_{p,s}.$$

In fact, for a somewhat larger range of convergence we have the following.

Lemma 1. *If $\sigma = \Re(s) > 0$, then there is a constant $c \neq 0$ with, for all $f \in \mathcal{S}(A)_0$,*

$$\int_J f(x)|x|^s d^*x = c\int_C E(f)(u)|u|^{s-\frac{1}{2}} d^*u = c\zeta(s)\Delta'_s(f).$$

4. The Weil distribution and the global trace formula

In this chapter, we discuss the conjectural part of Connes's approach and in particular a new (G)RH equivalent. Roughly speaking, the (generalized) Riemann Hypothesis for a global field K would follow if a global asymptotic trace formula for the action (C_K, X_K), where $X_K = A_K/K^*$, can be shown to be a sum of local contributions of trace formulae as in Theorem 4 for the actions (K_v^*, K_v), where v ranges over a set M_K of inequivalent valuations of K, and K_v is the completion of the field K with respect to the metric induced by the valuation v . If one restricts to a finite set S of places, this works; but the error term depends on S in a way that has not yet been controlled.

4.1. Global trace formula. Unless stated otherwise, we let $K = \mathbb{Q}$ from now on, and drop the K-indices, although our discussion goes through for arbitrary global fields. In Chapter 3, we introduced for $\delta \geq 0$ the Hilbert spaces $L^2(X)_{0,\delta}$, $L^2(X)_\delta$ and $L^2(C)_\delta$, and an inclusion of $L^2(X)_{0,\delta}$ in $L^2(C)_\delta$, effected by the isometry,

$$(Ef)(u) = |u|^{\frac{1}{2}} \sum_{q \in \mathbb{Q}^*} f(qu), \qquad u \in C.$$

We also saw that for $\sigma = \Re(s) > 0$, there is a constant $c \neq 0$ with, for all $f \in \mathcal{S}(A)_0$,

$$\int_C (Ef)(u)|u|^{s-\frac{1}{2}} d^*u = c\zeta(s)\Delta_s'(f).$$

As $\Delta_s' \neq 0$, if for $\Re(s) \in (0,1)$ the left hand side of this formula vanishes for all $f \in \mathcal{S}(A)_0$, then $\zeta(s) = 0$. We therefore see a link between orthogonality to the image of E and the non-trivial zeros of $\zeta(s)$. As in Chapter 1, let χ be a unitary quasi-character on C and $L(\chi, s)$ the associated L-function. Adapting the argument of Chapter 3 for the case $\chi = 1$ to arbitrary χ, we have

$$\int_C (Ef)(u)\chi(u)|u|^{s-\frac{1}{2}} d^*u = cL(\chi, s)\Delta_s'(f).$$

To study certain spectral aspects of the general L-functions of $\mathbb{Q}$, of which the Riemann zeta function is a special case, Connes works mostly in the Hilbert space $L^2(C)$ (so that $\delta = 0$). As in the local case, one has again to regularize using a cut-off at $\Lambda > 0$. Let S_Λ be the orthogonal projection onto

$$S_\Lambda = \{\xi \in L^2(C) : \xi(u) = 0, \quad |u| \notin [\Lambda^{-1}, \Lambda]\}.$$

Let

$$V : C \to \mathcal{L}(L^2(C))$$

be the regular representation

$$(V(\lambda)\xi)(u) = \xi(\lambda^{-1}u), \qquad \lambda, u \in C.$$

Let the measure $d^*\lambda$ on C be normalized so that

$$\int_{|\lambda| \in [1,\Lambda]} d^*\lambda \sim \log \Lambda, \qquad \Lambda \to \infty,$$

and let

$$2\log' \Lambda = \int_{|\lambda| \in [\Lambda^{-1},\Lambda]} d^*\lambda.$$

For $h \in \mathcal{S}_c(C)$ with compact support, let

$$V(h) = \int_C h(\lambda)V(\lambda)d^*\lambda.$$

Then the operator $S_\Lambda V(h)$ is trace class and

$$\mathrm{Trace}(S_\Lambda V(h)) = 2h(1)\log' \Lambda,$$

as the action of C on C has no fixed points. The cut-off in $L^2_0(X)$ is much less straightforward.

In the function field case, Connes proposes working with a family of closed subspaces $B_{\Lambda,0}$ of $L^2_0(X)$ such that $E(B_{\Lambda,0}) \subset S_\Lambda$. These subspaces are given by

$$B_{\Lambda,0} = \{f \in \mathcal{S}(A)_0 : f(x) = 0,\ \widehat{f}(x) = 0,\ |x| > \Lambda\},$$

where $|\quad|$ is the obvious extension to A of the norm on J, and this set is non-empty. Therefore, if $Q_{\Lambda,0}$ is the orthogonal projection onto $B_{\Lambda,0}$ and

$$Q'_{\Lambda,0} = EQ_{\Lambda,0}E^{-1},$$

then we have the inequality of projections

$$Q'_{\Lambda,0} \leq S_\Lambda.$$

For all $\Lambda > 0$, the following distribution, Δ_Λ, is therefore positive:

$$\Delta_\Lambda = \mathrm{Trace}((S_\Lambda - Q'_{\Lambda,0})V(h)), \qquad h \in \mathcal{S}(C).$$

The positivity of Δ_Λ signifies that for $h \in \mathcal{S}(C)$,

$$\Delta_\Lambda(h * h^*) \geq 0$$

where $h^*(u) = \overline{h(u^{-1})}$. Therefore, the limiting distribution

$$\Lambda_\infty = \lim_{\Lambda\to\infty} \Delta_\Lambda$$

is also positive.

However, in the number field case, the analogous set to $B_{\Lambda,0}$ is empty. To save this situation, one is guided by the physical fact that there exist signals with finite support in the time variable and also in the dual frequency variable, such as a musical signal. That is because the relative positions of P_Λ and $\widehat{P}_\Lambda$, as defined in the local case, can be analyzed. To do this, Connes appeals to the work of Landau, Pollak and Slepian [**28**], [**29**], [**27**]. We will also often quote in what follows the related and useful discussion in [**22**, §3].

Consider the case $K = \mathbb{Q}$. Recall that $\mathbb{Q}$ has one infinite place given by the usual Euclidean absolute value and that the corresponding completion is the field of real numbers. As in Chapter 3, at the infinite place we define P_Λ to be the orthogonal projection onto the subspace,

$$P_\Lambda = \{\xi \in L^2(\mathbb{R}) : \xi(x) = 0, \text{ for all } x, |x| > \Lambda\},$$

and $\widehat{P}_\Lambda = FP_\Lambda F^{-1}$, where F is the Fourier transform associated to the character $\alpha(x) = \exp(-2\pi i x)$.

For $\xi \in P_\Lambda$, the function $F\xi(y)$ is an analytic function and is, therefore, never supported on $|y| \leq \Lambda$. Therefore the spaces P_Λ and $\widehat{P}_\Lambda$ have zero intersection, which explains our claim above that, for $K = \mathbb{Q}$, the analogous space to $B_{\Lambda,0}$ would be trivial.

The projections P_Λ and $\widehat{P}_\Lambda$ commute with the second order differential operator on $\mathbb{R}$ given by

$$H_\Lambda \psi(x) = -\partial((\Lambda^2 - x^2)\partial)\psi(x) + (2\pi\Lambda x)^2\psi(x),$$

where $\partial = \frac{d}{dx}$, as can be checked by a straightforward calculation. The operator H_Λ has a discrete simple spectrum, which we may index by integers $n \geq 0$, on functions with support in $[-\Lambda, \Lambda]$. The corresponding eigenfunctions ψ_n are called the prolate spheroidal wave-functions, which can be taken as real-valued.

Since $P_\Lambda \widehat{P}_\Lambda P_\Lambda$ (that is, the operator $\widehat{P}_\Lambda$ restricted to $[-\Lambda, \Lambda]$) commutes with H_Λ, the prolate spheroidal functions are also eigenfunctions of $P_\Lambda \widehat{P}_\Lambda P_\Lambda$. By [**28**], [**29**], and [**27**] (see also [**22**]), one knows also that their eigenvalues are close to zero for $n \geq 4\Lambda^2 + O(\log \Lambda)$. Moreover, the projections P_Λ and $\widehat{P}_\Lambda$ in $L^2(\mathbb{R})$ have relative angles close to zero outside the space spanned by the eigenfunctions ψ_n, $n \leq 4\Lambda^2$ (see [**9**]). These results motivate Connes to substitute for the zero space $P_\Lambda \cap \widehat{P}_\Lambda$ the subspace $B_{\infty,\Lambda}$ of P_Λ given by the linear span in $L^2(\mathbb{R})$ of the ψ_n, $n \leq 4\Lambda^2$.

We may write the adeles A over $\mathbb{Q}$ as the direct product $A = \mathbb{R} \times A_f$ where A_f, the finite adeles, is the restricted product over the finite primes, p, of $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$. Let $R = \prod_p \mathbb{Z}_p$, and let 1_R be the characteristic function on R. Let $W_f = \prod_p \mathbb{Z}_p^*$ be the units of R and $W = \{\pm 1\} \times W_f$. Consider the elements $f \in \mathcal{S}(A) = \mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(A_f)$ satisfying

$$f(wa) = f(a), \qquad a \in A, w \in W.$$

Then f may be written as a finite linear combination

$$f(a) = \sum_{j=1}^{m} f_j(q_j a_\infty) \otimes 1_R(q_j a_f), \qquad a = (a_\infty, a_f), \quad a_\infty \in \mathbb{R}, a_f \in A_f,$$

where the $f_j \in \mathcal{S}(\mathbb{R})$ are even and $q_j \in \mathbb{Q}^*$, $q_j > 0$. Let

$$B^1_{\Lambda,0} = \{f \in \mathcal{S}(A)_0 : f = f_\infty \otimes 1_R, f_\infty \in B_{\infty,\Lambda}, f_\infty \text{ even}\}.$$

As in Section 3.2, let C^1 be the compact subgroup of the idele class group C of $\mathbb{Q}$ given by the kernel of the norm map on C. We have a (non-canonical) isomorphism between C and $C^1 \times N$, where $N \simeq \mathbb{R}^*_{>0}$. We can extend any character χ^1 of C^1 to a quasi-character χ of C, vanishing on N. As in [**9**], §8, Lemma 1 we may, in a similar way to the case $\chi = 1$ above, define subspaces $B^\chi_{\Lambda,0}$, where χ is the extension of a character χ^1 of C^1, by writing $A = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times A_{f,S}$, where S is the finite set of ramified primes for χ (see Section 1.3) and $A_{f,S}$ is the finite adeles with the restricted product over the finite primes not in S. Let $B_{\Lambda,0} = \oplus_{\chi^1 \in \widehat{C}^1} B^\chi_{\Lambda,0}$, and $Q_{\Lambda,0}$ be the corresponding orthogonal projection, with $Q'_{\Lambda,0} = E Q_{\Lambda,0} E^{-1}$.

Again, for all $\Lambda > 0$, we introduce the positive distribution

$$\Delta_\Lambda = \mathrm{Trace}((S_\Lambda - Q'_{\Lambda,0})V(h)), \qquad h \in \mathcal{S}(C).$$

Connes observes that the above considerations show that the limiting distribution

$$\Lambda_\infty = \lim_{\Lambda \to \infty} \Delta_\Lambda$$

is positive, just as it was in the function field case.

If we let Q_Λ be the projection in $L^2(X)$ onto $B_{\Lambda,0} \oplus \mathbb{C} \oplus \mathbb{C}(1)$, then Connes conjectures the following *global* analogue of the local geometric trace formula. We state it only for the global field $K = \mathbb{Q}$, but the statement for an arbitrary global field is analogous.

Conjecture 1. *For $h \in \mathcal{S}_c(C)$, we have as $\Lambda \to \infty$,*

$$\text{Trace}(Q_\Lambda U(h)) = 2h(1)\log' \Lambda + \sum_{p \text{ prime}} \int'_{\mathbb{Q}_p^*} \frac{h(u^{-1})}{|1-u|} d^*u + \int'_{\mathbb{R}^*} \frac{h(u^{-1})}{|1-u|} d^*u + o(1), \tag{40}$$

where

$$2\log' \Lambda = \int_{|\lambda| \in [\Lambda^{-1}, \Lambda]} d^*u \sim 2 \log \Lambda.$$

This means that one *conjectures* that the global trace formula in $L^2(X)$ behaves like a sum of local formulae. Observe that,

$$\text{Trace}(Q_\Lambda U(h)) = \int_C h(u)(1 + |u|) d^*u + \text{Trace}(Q_{\Lambda,0} U(h)). \tag{41}$$

Using (41), and noting the phase shift by $1/2$ in E, we have as a consequence of Conjecture 1 that, as $\Lambda \to \infty$,

$$\begin{aligned}\Delta_\Lambda(h) = &\int_C h(u)(|u|^{1/2} + |u|^{-1/2}) d^*u \\ &- \sum_{p \text{ prime}} \int'_{\mathbb{Q}_p^*} \frac{h(u)}{|1-u|} |u|^{1/2} d^*u - \int'_{\mathbb{R}^*} \frac{h(u)}{|1-u|} |u|^{1/2} d^*u + o(1).\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta_\infty(h) = &\int_C h(u)(|u|^{1/2} + |u|^{-1/2}) d^*u \\ &- \sum_{p \text{ prime}} \int'_{\mathbb{Q}_p^*} \frac{h(u)}{|1-u|} |u|^{1/2} d^*u - \int'_{\mathbb{R}^*} \frac{h(u)}{|1-u|} |u|^{1/2} d^*u.\end{aligned}$$

In Chapter 1, we gave a version of Weil's explicit formula relating the zeros of $\zeta(s)$ to the rational primes p by comparing the Hadamard product formula with Euler's product formula. An adelic version of this explicit formula was developed by Weil which invokes a relation between the zeros of all the $L(\chi, s)$, where χ ranges over the Grössencharacters of $\mathbb{Q}$, and the rational primes p. In Appendix II of [**9**], Connes reworks (for an arbitrary global field) this version of Weil's explicit formula and shows thereby that the right hand side of the above formula gives

$$\Delta_\infty(h) = \sum_{\substack{L(\chi, \frac{1}{2}+\rho)=0, \\ \Re(\rho) \in (-\frac{1}{2}, \frac{1}{2})}} \int_C h(u)\chi(u)|u|^\rho d^*u.$$

Conjecture 1 would show Δ_∞ to be positive by construction. To prove GRH one would then apply the following result of Weil.

Theorem 6. *The Generalized Riemann Hypothesis (GRH) is equivalent to the positivity of Δ_∞.*

This is an adelic version of RH and of Weil's result on the equivalence with the positivity of R discussed in Section 1.5. Replacing $h(u)$ by $f(u) = |u|^{-1/2}h(u^{-1})$, we may write Weil's explicit formula as

$$\int_{\mathbb{R}^*}' \frac{h(u^{-1})}{|1-u|} d^*u + \sum_{p \text{ prime}} \int_{\mathbb{Q}_p^*}' \frac{h(u^{-1})}{|1-u|} d^*u =$$
$$\int_C h(u) d^*u \; - \sum_{\substack{L(\chi,\rho)=0, \\ 0<\Re(\rho)<1}} \int_C h(u)\chi(u)|u|^\rho d^*u + \int_C h(u)|u| d^*u.$$

Therefore, Connes *conjectures* that

$$\Delta_\infty(f) = \int_C h(u) d^*u - \sum_{v \in M_{\mathbb{Q}}} \int_{\mathbb{Q}_v^*}' \frac{h(u^{-1})}{|1-u|} d^*u + \int_C h(u)|u| d^*u,$$

and this *would imply* by the explicit formula that

$$\Delta_\infty(f) = \sum_{\substack{L(\chi,\rho)=0, \\ 0<\Re(\rho)<1}} \int_C h(u)\chi(u)|u|^\rho d^*u,$$

which *would be* positive.

In some sense the formulae, for $\Lambda > 0$,

$$\Delta_\Lambda(h) = \mathrm{Trace}((S_\Lambda - Q'_{\Lambda,0})V(h))$$

are "calculating the trace in H". Connes is able to prove the following version of the above explicit formula which shows that the action of C on H by W picks up the zeros of the L-functions as an absorption spectrum in $L^2(X)$ with the non-critical zeros as resonances. A full proof in the function field case is given in Part VIII, Lemma 3 of [**9**] and the necessary modifications for the number field case are indicated in the subsequent discussion in that paper of the analysis of the relative position of the projections P_Λ and $\widehat{P}_\Lambda$.

THEOREM 7. *Let $h \in \mathcal{S}_c(C)$, then*

$$\Delta_\infty(h) = \sum_{\chi,\rho} N(\chi, \frac{1}{2} + \rho) \int_{z \in i\mathbb{R}} \widehat{h}(\chi, z) d\mu_\rho(z),$$

where the sum is over the pairs (χ^1, ρ) of characters χ^1 of C^1, with χ being the unique quasi-character on C vanishing on N, and over the zeros ρ of $L(\chi, \frac{1}{2} + \rho)$ with $\Re(\rho) \in (-\frac{1}{2}, \frac{1}{2})$. The number $N(\chi, \frac{1}{2} + \rho)$ is the multiplicity of the zero, the measure $d\mu_\rho(z)$ is the harmonic measure with respect to $i\mathbb{R} \subset \mathbb{C}$ and

$$\widehat{h}(\chi, z) = \int_C h(u)\chi(u)|u|^z d^*u.$$

The measure $d\mu_\rho(z)$ is a probability measure on the line $i\mathbb{R}$ which coincides with the Dirac mass at $\rho \in i\mathbb{R}$. Transforming the area to the right of $i\mathbb{R} \subset \mathbb{C}$ to the interior $|z| < 1$ of the unit circle, so that ρ is mapped to u with $|u| = 1$, we may write this measure as $P_z(u)du$ where P_z is the Poisson kernel,

$$P_z(u) = \frac{1 - |z|^2}{|u - z|^2}.$$

4.2. Analogy with the Guillemin trace formula. One can summarize Connes's approach to the Riemann Hypothesis (RH) as a program for the derivation of a conjectured explicit formula à la Weil. Whereas in Weil's set-up the explicit formula is known and the open problem is to prove a positivity result, in Connes's set-up the positivity is part of the Pólya-Hilbert space (H, W) construction and the problem is to prove the corresponding explicit formula. In his original paper [**8**], Connes found a striking analogy between his conjectured global trace formula for the action U of C on $L^2(X)$ and the distributional trace formula à la Guillemin for flows on manifolds.

Let M be a C^∞ manifold and v a smooth vector field on M with isolated zeros. We have the associated flow $F_t = \exp(tv)$, $t \in \mathbb{R}$, with its action on smooth functions,

$$(U(t)\xi)(x) = \xi(F_t(x)), \qquad \xi \in C^\infty(M), \quad x \in M, \quad t \in \mathbb{R}.$$

For $h \in C_c^\infty(\mathbb{R})$ with $h(0) = 0$, let

$$U(h) = \int_{\mathbb{R}} h(t)U(t)dt.$$

If $U(h)$ has kernel $k(x, y)$, that is,

$$(U(h)\xi)(x) = \int_{\mathbb{R}} k(x, y)\xi(y)dy,$$

then the distributional trace of $U(h)$ is defined as

$$\mathrm{Trace}_D(U(h)) = \int_{\mathbb{R}} k(x, x)dx = \rho(h).$$

That is, ρ is viewed as a distribution. The Guillemin trace formula tells us that

$$\begin{aligned}
\mathrm{Trace}_D(U(h)) &= \sum_{\gamma} \int_{I_\gamma} \frac{h(u)}{|1-(F_u)_*|} d^*u \\
&= \sum_{x, v_x=0} \int \frac{h(t)}{|1-(F_t)_*|} dt \; + \sum_{\gamma \text{ periodic}} \sum_{T=T_\gamma^m} T_\gamma^* \frac{1}{|1-(F_T)_*|} h(T).
\end{aligned}$$

Here $(F)_*$ is the Poincaré return map: it is the restriction of $d(\exp(Tv))$, where T is a period, to the normal of the orbit, and at a zero of the vector field it is the map induced on the tangent space by the flow. Therefore $(F_u)_*$ is the restriction of the tangent map to F_u to the transverse space of the orbits. In the formula (42), one considers the zeros as periodic orbits, while $I_\gamma \subset \mathbb{R}$ is the isotropy group of any $x \in \gamma$ and d^*u is the unique Haar measure $d\mu$ of total mass 1. Notice the resemblance to Connes's global trace formula when $h(0) = 0$. Assume $1 - (F_u)_*$ is invertible. Then $|1 - (F_u)_*| = \det(1 - (F_u)_*)$. With the assumption $h(1) = 0$, we can write the global trace formula for $U : C \to \mathcal{L}^2(X)$, $h \in \mathcal{S}_c(C)$ as

$$\mathrm{Trace}_D(U(h)) = \sum_{v \in M_{\mathbb{Q}}} \int_{\mathbb{Q}_v^*} \frac{h(u^{-1})}{|1-u|_v} d^*u,$$

where for the evaluation of h, we can embed $u \in \mathbb{Q}_v$ in J in the obvious way with 1's in every place but the v-th place. Notice that the action of J on A has fixed points coming from the elements of A with zero components. In [**8**], Connes shows the following.

LEMMA 2. *For $x \in X = A/\mathbb{Q}^*$, $x \neq 0$, the isotropy group I_x of x in $C = J/\mathbb{Q}^*$ is cocompact if and only if there exists exactly one $v \in M_{\mathbb{Q}}$ with $\widetilde{x}_v = 0$, where $\widetilde{x} = (\widetilde{x}_v)_{v \in M_{\mathbb{Q}}}$ is a lift of x to J.*

PROOF. For $v_1 \neq v_2$ in $M_{\mathbb{Q}}$, the map $|\ | : \mathbb{Q}_{v_1}^* \times \mathbb{Q}_{v_2}^* \to \mathbb{R}_+^*$ is not proper. □

Assume that only fixed points of the C-action as in the Lemma contribute to the trace formula. Then Connes has the following *heuristic.* For $v \in M_{\mathbb{Q}}$, let

$$\widetilde{H}_v = \{\widetilde{x} \in A : \widetilde{x} = (\widetilde{x}_v), \widetilde{x}_v = 0, \widetilde{x}_u \neq 0, u \neq v\},$$

and

$$H_v = \{[\widetilde{x}] \in X : \widetilde{x} \in \widetilde{H}_v\}.$$

Let N_x be the "normal space" to $x \in H_v$, that is,

$$N_x \simeq X/H_v \simeq A/\widetilde{H}_v \simeq \mathbb{Q}_v,$$

so that $\mathbb{Q}_v$ can be viewed as the "transverse space" to H_v. Let $j \in I_x$, the isotropy group of $x \in H_v$, and let $\widetilde{j}$ be a lift of j to J. Then $\widetilde{j}$ acts on A linearly and fixes $\widetilde{x}$. The induced action on the transverse space N_x, the Poincaré return map in this situation, is just the multiplication map,

$$\begin{array}{ll} \mathbb{Q}_v^* \times \mathbb{Q}_v & \to \mathbb{Q}_v \\ (\lambda, a) & \mapsto \lambda a. \end{array}$$

By *analogy* with the Guillemin trace formula, the corresponding contribution to the trace formula should be $\int_{\mathbb{Q}_v^*} \frac{h(\lambda^{-1})}{|1-\lambda|_v} d^*\lambda$, for $h \in \mathcal{S}(C)$ with $h(1) = 0$. This is the contribution form the *local* trace formula. The cut-offs were giving the regularizations. When $v = p$, prime, the period of the orbit is the covolume of the isotropy group $\mathbb{Q}_p^*$, and this equals $\log p$, since the image of the p-adic norm on $\mathbb{Q}_p^*$ is $p^{\mathbb{Z}}$.

Connes proposes that Weil's explicit formula incorporates a noncommutative number theoretic analogue for (X, C) of Guillemin's trace formula for flows on manifolds.

5. Related aspects of noncommutative number theory (With Appendix by Peter Sarnak)

In the previous chapters, we have tried to present in a direct and elementary way the essentials of Connes's proposed approach to the Riemann Hypothesis. Connes has presented in his papers [**8**], [**10**], [**9**] more sophisticated motivations which are of interest in a broader context as they invite a new interaction between operator algebras and number theory. They all relate in some measure to older work of Bost–Connes [**5**], an overview of which is the focus of the present chapter.

We have added, with his permission, an Appendix authored by Peter Sarnak, which is a reproduction of a letter he wrote to Enrico Bombieri regarding the appearance in Connes's set-up of symplectic symmetry, in the sense of work of Katz–Sarnak [**26**], for families of Dirichlet L-functions with quadratic characters.

5.1. Von Neumann Algebras and Galois Theory. An additional motivation for Connes's approach, that is described in detail in [**10**], arises from his observation that certain features of Galois theory related to the idele class group resemble those of the classification of factors of von Neumann algebras. For *local fields*, the role of the idele class group is played by the group of non-zero elements of the field, which by local class field theory has a Galois interpretation as the Weil group. For *global fields K of characteristic* $p > 0$ we have an isomorphism between the idele class group and the Weil group W_K for the global field K (see Section 2.4). The subfields K' of K_{un} with $[K' : K] < \infty$ are classified by the subgroups

$$\{1\} \neq \Gamma \subset \text{Mod}(K) = q^{\mathbb{Z}} \subset \mathbb{R}_+^*.$$

Define

$$\theta_\lambda(\mu) = \mu^\lambda, \quad \lambda \in \Gamma,$$

for μ an ℓ-th root of unity, $(\ell, p) = 1$. Then,

$$K' = \{x \in K_{\text{un}} : \theta(x) = x, \text{ for all } \lambda \in \Gamma\}.$$

The Galois groups of infinite extensions are constructed as projective limits of the finite groups attached to finite extensions. When K is a global field of characteristic 0, the main result of class field theory says that there is an isomorphism between $\text{Gal}(K_{\text{ab}}/K)$, where K_{ab} is the maximal abelian extension of K, and the quotient C/D of the idele class group of K by the connected component D of the identity in C.

When $K = \mathbb{Q}$, this translates into an isomorphism between C and $\mathbb{R}_+^* \times \prod_p \mathbb{Z}_p^*$, and moreover $D = \mathbb{R}_+^*$, so that $\text{Gal}(\mathbb{Q}_{\text{ab}}/\mathbb{Q})$ is isomorphic to $\prod_p \mathbb{Z}_p^*$. Indeed, for a global field of characteristic 0, the connected component D is always non-trivial due to the archimedean places.

Can operator algebras enable us to do Galois theory "with the infinite place", as proposed by Weil (see Section 1.4)? Von Neumann algebras appear as the commutants of unitary representations in Hilbert space; the central simple ones are called factors, and the approximately finite dimensional ones are the weak closure of the union of increasing sequences of finite dimensional algebras. As in Galois theory, one has a correspondence between virtual subgroups Γ of $\mathbb{R}_+^*$ (ergodic actions of $\mathbb{R}_+^*$) and the factors M. The non-simple approximately finite dimensional factors are $M_\infty(\mathbb{C})$, the operators in Hilbert space, which is of Type I_∞ and $R_{0,1} = R \otimes M_\infty(\mathbb{C})$, which is of Type II_∞ and has trace $\tau = \tau_0 \otimes \text{Tr}_{M_\infty(\mathbb{C})}$, where R is the unique approximately finite factor with finite trace τ_0. By the theory of von Neumann algebras, there exists up to conjugacy a unique 1-parameter group $\theta_\lambda \in \text{Aut}(R_{0,1})$, $\lambda \in \mathbb{R}_+^*$ with

$$\tau(\theta_\lambda(a)) = \lambda\tau(a), \quad a \in \text{Dom}(\tau), \lambda \in \mathbb{R}_+^*$$

If Γ is a virtual subgroup of $\mathbb{R}_+^*$ and α the corresponding ergodic action of $\mathbb{R}_+^*$ on an abelian algebra A, then

$$R_\Gamma = \{x \in R_{0,1} \otimes A : (\theta_\lambda \otimes \alpha_\lambda)x = x, \text{ for all } \lambda \in \mathbb{R}_+^*\}$$

is the corresponding factor. For the background on the material from the theory of von Neumann algebras, see [**7**]. A direction for further research, proposed by Connes, is to develop this analogy with Galois theory.

We note the following corollary of the work of Bost-Connes [**5**], see also §5.4.

THEOREM 8. *Let A be the adele ring of $\mathbb{Q}$ and $L^\infty(A)$ the essentially bounded functions on A with the supremum norm. Then $L^\infty(A) \rtimes \mathbb{Q}^*$, the crossed product with $\mathbb{Q}^*$ for multiplication on A, is isomorphic to $R_{0,1}$. Moreover, the restriction of the action of C on $A/\mathbb{Q}^*$ corresponds to the action θ_λ on $R_{0,1}$.*

The space $X = A/\mathbb{Q}^*$ is the orbit space associated to $L^\infty(A) \rtimes \mathbb{Q}^*$. The main result of [**5**] was to construct a dynamical system, with natural symmetry group

$W = \mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}}/\mathbb{Q})$, and partition function the Riemann zeta function at whose pole at $s = 1$ there was a phase transition. This phase transition corresponded to a passage from a family of Type I_∞ factor equilibrium states indexed by W, in the region $s > 1$, to a unique Type III_1 factor equilibrium state in the region $0 < s \leq 1$. Bost and Connes show that the corresponding Type III_1 factor for the critical strip $0 < s \leq 1$ has a Type II_∞ factor in its continuous decomposition given by $L^\infty(A) \rtimes \mathbb{Q}^*$. This provides a motivation for studying the action (X, C). The II_∞ nature of the von Neumann algebra associated to X points away from a study of this space using classical measure theory. Extensions of these results to arbitrary global fields are due to Harari-Leichtnam [**24**], Arledge-Laca-Raeburn [**1**] and the author [**6**]. This work is also related to earlier work of Julia [**25**] and others, which aims at enriching our knowledge of the Riemann zeta function by creating a dictionary between its properties and phenomena in statistical mechanics. The starting point of these approaches is the observation that, just as the zeta functions encode arithmetic information, the partition functions of quantum statistical mechanical systems encode their large-scale thermodynamical properties. The first step is therefore to construct a quantum dynamical system with partition function the Riemann zeta function. In order for the quantum dynamical system to reflect the arithmetic of the primes, it must also capture some sort of interaction between them. This last feature translates in the statistical mechanical language into the phenomenon of spontaneous symmetry breaking at a critical temperature with respect to a natural symmetry group. In the region of high temperature there is a unique equilibrium state, as the system is in disorder and is symmetric with respect to the action of the symmetry group. In the region of low temperature, a phase transition occurs and the symmetry is broken. This symmetry group acts transitively on a family of possible extremal equilibrium states.

In the following sections of this chapter, we give an overview of the construction of [**5**], emphasizing even more than in that paper the intervention of adeles and ideles (see also [**6**]). The symmetry group of the system is a Galois group, in fact, the Galois group over the rational number field of its maximal abelian extension.

5.2. The problem studied by Bost–Connes. We recall a few basic notions from the C^*-algebraic formulation of quantum statistical mechanics. For the background, see [**7**]. Recall that a C^*-algebra B is an algebra over the complex numbers $\mathbb{C}$ with an adjoint $x \mapsto x^*$, $x \in B$, that is, an anti-linear map with $x^{**} = x$, $(xy)^* = y^*x^*$, $x, y \in B$, and a norm $\| \cdot \|$ with respect to which B is complete and addition and multiplication are continuous operations. One requires in addition that $\|xx^*\| = \|x\|^2$ for all $x \in B$. All our C^*-algebras will be assumed unital. The most basic example of a noncommutative C^*-algebra is $B = M_N(\mathbb{C})$ for $N \geq 2$ an integer. The C^*-algebra plays the role of the "space" on which the system evolves, the evolution itself being described by a 1-parameter group of C^*-automorphisms $\sigma : \mathbb{R} \mapsto \mathrm{Aut}(B)$. The quantum dynamical system is therefore the pair (B, σ_t). It is customary to use the inverse temperature $\beta = 1/kT$ rather than the temperature T, where k is Boltzmann's constant. One has a notion due to Kubo-Martin-Schwinger (KMS) of an equilibrium state at inverse temperature β. Recall that a state φ on a C^*-algebra B is a positive linear functional on B satisfying $\varphi(1) = 1$. It is the generalization of a probability distribution.

Definition 5. Let (B, σ_t) be a dynamical system, and φ a state on B. Then φ is an equilibrium state at inverse temperature β, or KMS_β-state, if for each $x, y \in B$

there is a function $F_{x,y}(z)$, bounded and holomorphic in the band $0 < \Im(z) < \beta$ and continuous on its closure, such that for all $t \in \mathbb{R}$,

$$F_{x,y}(t) = \varphi(x\sigma_t(y)), \qquad F_{x,y}(t + \sqrt{-1}\beta) = \varphi(\sigma_t(y)x). \tag{42}$$

In the case where $B = M_N(\mathbb{C})$, every 1-parameter group σ_t of automorphisms of B can be written in the form,

$$\sigma_t(x) = e^{itH} x e^{-itH}, \qquad x \in B, \qquad t \in \mathbb{R},$$

for a self-adjoint matrix $H = H^*$. For $H \geq 0$ and for all $\beta > 0$, there is a unique KMS_β equilibrium state for (B, σ_t) given by

$$\phi_\beta(x) = \mathrm{Trace}(xe^{-\beta H})/\mathrm{Trace}(e^{-\beta H}), \qquad x \in M_N(\mathbb{C}). \tag{43}$$

This has the form of a classical "Gibbs state" and is easily seen to satisfy the KMS_β condition of Definition 5. The KMS_β states can therefore be seen as generalizations of Gibbs states. The normalization constant $\mathrm{Trace}(e^{-\beta H})$ is known as the partition function of the system. A symmetry group G of the dynamical system (B, σ_t) is a subgroup of $\mathrm{Aut}(B)$ commuting with σ:

$$g \circ \sigma_t = \sigma_t \circ g, \qquad g \in G, t \in \mathbb{R}.$$

Consider now a system (B, σ_t) with interaction. Guided by quantum statistical mechanics, one hopes to see the following features. When the temperature is high, so that β is small, the system is in disorder, there is no interaction between its constituents, and the state of the system does not see the action of the symmetry group G: the KMS_β-state is unique. As the temperature is lowered, the constituents of the system begin to interact. At a critical temperature β_0, a phase transition occurs; and the symmetry is broken. The symmetry group G then permutes transitively a family of extremal KMS_β- states generating the possible states of the system after phase transition: the KMS_β-state is no longer unique. This phase transition phenomenon is known as spontaneous symmetry breaking at the critical inverse temperature β_0. The partition function should have a pole at β_0. For a fuller explanation, see [**5**]. The problem solved by Bost and Connes was the following.

PROBLEM 1. Construct a dynamical system (B, σ_t) with partition function the zeta function $\zeta(\beta)$ of Riemann, where $\beta > 0$ is the inverse temperature, having spontaneous symmetry breaking at the pole $\beta = 1$ of the zeta function with respect to a natural symmetry group.

As mentioned in §5.1, the symmetry group is the unit group of the ideles, given by $W = \prod_p \mathbb{Z}_p^*$, where the product is over the primes p and $\mathbb{Z}_p^* = \{u_p \in \mathbb{Q}_p : |u_p|_p = 1\}$. We use, as before, the normalization $|p|_p = p^{-1}$. This is the same as the Galois group $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$. Here $\mathbb{Q}^{ab}$ is the maximal abelian extension of the rational number field $\mathbb{Q}$, which in turn is isomorphic to its maximal cyclotomic extension, that is, the extension obtained by adjoining to $\mathbb{Q}$ all the roots of unity. The interaction detected in the phase transition comes about from the interaction between the primes coming from considering at once all the embeddings of the nonzero rational numbers $\mathbb{Q}^*$ into the completions $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the prime valuations $|\cdot|_p$. The natural generalization of this problem to the number field case was solved in [**6**] and is the following.

PROBLEM 2. Given a number field K, construct a dynamical system (B, σ_t) with partition function the Dedekind zeta function $\zeta_K(\beta)$, where $\beta > 0$ is the inverse temperature, having spontaneous symmetry breaking at the pole $\beta = 1$ of the Dedekind function with respect to a natural symmetry group.

Recall that the Dedekind zeta function is given by

$$\zeta_K(s) = \sum_{\mathcal{C}\subset\mathcal{O}} \frac{1}{N(\mathcal{C})^s}, \qquad \Re(s) > 1. \tag{44}$$

Here $\mathcal{O}$ is the ring of integers of K, and the summation is over the ideals $\mathcal{C}$ of K contained in $\mathcal{O}$. The symmetry group is the unit group of the finite ideles of K.

For a generalization to the function field case, see [**24**]. We restrict ourselves in what follows to the case of the rational numbers, that is, to a discussion of Problem 1.

5.3. Construction of the C^*-algebra. We give a different construction of the C^*-algebra of [**5**] from that found in their original paper. It is essentially equivalent to the construction of [**1**], except that we work with adeles and ideles, and turns out to be especially useful for the generalization to the number field case in [**6**] . Let A_f denote the finite adeles of $\mathbb{Q}$, that is the restricted product of $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$. Recall that this restricted product consists of the infinite vectors $(a_p)_p$, indexed by the primes p, such that $a_p \in \mathbb{Q}_p$ with $a_p \in \mathbb{Z}_p$ for almost all primes p. The (finite) adeles form a ring under componentwise addition and multiplication. The (finite) ideles $\mathcal{J}$ are the invertible elements of the adeles. They form a group under componentwise multiplication. Let $\mathbb{Z}_p^*$ be those elements of $u_p \in \mathbb{Z}_p$ with $|u_p|_p = 1$. Notice that an idele $(u_p)_p$ has $u_p \in \mathbb{Q}_p^*$ with $u_p \in \mathbb{Z}_p^*$ for almost all primes p. Let

$$R = \prod_p \mathbb{Z}_p, \qquad I = \mathcal{J} \cap R, \qquad W = \prod_p \mathbb{Z}_p^*.$$

Further, let $\mathcal{I}$ denote the semigroup of integral ideals of $\mathbb{Z}$. It is the semigroup of $\mathbb{Z}$-modules of the form $m\mathbb{Z}$ where $m \in \mathbb{Z}$. Notice that I, as above, is also a semigroup. We have a natural short exact sequence,

$$1 \to W \to I \to \mathcal{I} \to 1. \tag{45}$$

The map $I \to \mathcal{I}$ in this short exact sequence is given as follows. To $(u_p)_p \in I$ associate the ideal $\prod_p p^{\mathrm{ord}_p(u_p)}$, where $\mathrm{ord}_p(u_p)$ is determined by the formula $|u_p|_p = p^{-\mathrm{ord}_p(u_p)}$. It is clear that this map is surjective with kernel W, that is, that the above sequence is indeed short exact. By the Strong Approximation Theorem, we have

$$\mathbb{Q}/\mathbb{Z} \simeq A_f/R \simeq \oplus_p \mathbb{Q}_p/\mathbb{Z}_p; \tag{46}$$

and we have therefore a natural action of I on $\mathbb{Q}/\mathbb{Z}$ by multiplication in A_f/R and transport of structure. We use here that $I \cdot R \subset R$. Mostly we shall work in A_f/R rather than $\mathbb{Q}/\mathbb{Z}$. We have the following straightforward Lemma (see [**6**]).

LEMMA 3. *For $a = (a_p)_p \in I$ and $y \in A_f/R$, the equation*

$$ax = y$$

has $n(a) := \prod_p p^{\mathrm{ord}_p(a_p)}$ solutions in $x \in A_f/R$. Denote these solutions by

$$[x : ax = y].$$

In the above lemma, it is important to bear in mind that we are computing modulo R. Now, let $\mathbb{C}[A_f/R] := \mathrm{span}\{\delta_x : x \in A_f/R\}$ be the group algebra of A_f/R over $\mathbb{C}$, so that $\delta_x \delta_{x'} = \delta_{x+x'}$ for $x, x' \in A_f/R$. We have (see for comparison [**1**]),

LEMMA 4. *The formula*

$$\alpha_a(\delta_y) = \frac{1}{n(a)} \sum_{[x:ax=y]} \delta_x,$$

for $a \in I$, defines an action of I by endomorphisms of $C^*(A_f/R)$.

The endomorphism α_a for $a \in I$ is a one-sided inverse of the map $\delta_x \mapsto \delta_{ax}$ for $x \in A_f/R$, so it is like a semigroup "division". The C^*-algebra can be thought of as the operator norm closure of $\mathbb{C}[A_f/R]$ in its natural left regular representation in $l^2(A_f/R)$. We now appeal to the notion of semigroup crossed product developed by Laca and Raeburn and used in [**1**], applying it to our situation. A covariant representation of $(C^*(A_f/R), I, \alpha)$ is a pair (π, V) where

$$\pi : C^*(A_f/R) \to B(\mathcal{H})$$

is a unital representation and

$$V : I \to B(\mathcal{H})$$

is an isometric representation in the bounded operators in a Hilbert space $\mathcal{H}$. The pair (π, V) is required to satisfy

$$\pi(\alpha_a(f)) = V_a \pi(f) V_a^*, \qquad a \in I, \quad f \in C^*(A_f/R).$$

Notice that the V_a are not in general unitary. Such a representation is given by (λ, L) on $l^2(A_f/R)$ with orthonormal basis $\{e_x : x \in A_f/R\}$, where λ is the left regular representation of $C^*(A_f/R)$ on $l^2(A_f/R)$ and

$$L_a e_y = \frac{1}{\sqrt{n(a)}} \sum_{[x:ax=y]} e_x.$$

The universal covariant representation, through which all other covariant representations factor, is called the (semigroup) crossed product $C^*(A_f/R) \rtimes_\alpha I$. This algebra is the universal C^*-algebra generated by the symbols $\{e(x) : x \in A_f/R\}$ and $\{\mu_a : a \in I\}$ subject to the relations

$$\mu_a^* \mu_a = 1, \quad \mu_a \mu_b = \mu_{ab}, \qquad a, b \in I, \tag{47}$$

$$e(0) = 1, \quad e(x)^* = e(-x), \quad e(x)e(y) = e(x+y), \qquad x, y \in A_f/R, \tag{48}$$

$$\frac{1}{n(a)} \sum_{[x:ax=y]} e(x) = \mu_a e(y) \mu_a^*, \qquad a \in I, y \in A_f/R. \tag{49}$$

The relations in (47) reflect a multiplicative structure, those in (48) an additive structure, and those in (49) how these multiplicative and additive structures are related via the crossed product action. Julia [**25**] observed that by using only the multiplicative structure of the integers, one cannot hope to capture an interaction between the different primes. When $u \in W$ then μ_u is a unitary, so that $\mu_u^* \mu_u = \mu_u \mu_u^* = 1$, and we have for all $x \in A_f/R$,

$$\mu_u e(x) \mu_u^* = e(u^{-1}x), \qquad \mu_u^* e(x) \mu_u = e(ux). \tag{50}$$

Therefore we have a natural action of W as inner automorphisms of $C^*(A_f/R) \rtimes_\alpha I$ using (50).

To recover the C^*-algebra of [**5**] we must split the short exact sequence (45). The ideals in $\mathcal{I}$ are all of the form $m\mathbb{Z}$ for some $m \in \mathbb{Z}$. This generator m is determined up to sign. Consider the image of $|m|$ in I under the diagonal embedding $q \mapsto (q)_p$ of $\mathbb{Q}^*$ into I, where the p-th component of $(q)_p$ is the image of q in $\mathbb{Q}_p^*$ under the natural embedding of $\mathbb{Q}^*$ in $\mathbb{Q}_p^*$. The map

$$+ : m\mathbb{Z} \mapsto (|m|)_p \tag{51}$$

defines a splitting of (45). Let I_+ denote the image and define B to be the semigroup crossed product $C^*(A_f/R) \rtimes_\alpha I_+$ with the restricted action α from I to I_+. By transport of structure using (46), this algebra is easily seen to be isomorphic to a semigroup crossed product of $C^*(\mathbb{Q}/\mathbb{Z})$ by $\mathbb{N}_+$, where $\mathbb{N}_+$ denotes the positive natural numbers. This is the algebra constructed in [**5**] (see also [**1**]). From now on, we use the symbols $\{e(x) : x \in \mathbb{Q}/\mathbb{Z}\}$ and $\{\mu_a : a \in \mathbb{N}_+\}$. It is essential to split the short exact sequence in this way in order to obtain the symmetry breaking phenomenon. In particular, this replacement of I by I_+ now means that the group W acts by outer automorphisms. For $x \in B$, one has that $\mu_u^* x \mu_u$ is still in B (computing in the larger algebra $C^*(A_f/R) \rtimes_\alpha I$), but now this defines an outer action of W. This coincides with the definition of W as the symmetry group as in [**5**].

5.4. The Theorem of Bost–Connes. Using the abstract description of the C^*-algebra B of §5.3, to define the time evolution σ of our dynamical system (B, σ) it suffices to define it on the symbols $\{e(x) : x \in \mathbb{Q}/\mathbb{Z}\}$ and $\{\mu_a : a \in \mathbb{N}_+\}$. For $t \in \mathbb{R}$, let σ_t be the automorphism of B defined by

$$\sigma_t(\mu_m) = m^{it}\mu_m, \quad m \in \mathbb{N}_+, \qquad \sigma_t(e(x)) = e(x), \quad x \in \mathbb{Q}/\mathbb{Z}. \tag{52}$$

By (47) and (50), we clearly have that the action of W commutes with this 1-parameter group σ_t. Hence W will permute the extremal KMS$_\beta$-states of (B, σ_t). To describe the KMS$_\beta$-states for $\beta > 1$, we shall represent (B, σ_t) on a Hilbert space. Namely, following [**5**], let $\mathcal{H}$ be the Hilbert space $l^2(\mathbb{N}_+)$ with canonical orthonormal basis $\{\varepsilon_m, m \in \mathbb{N}_+\}$. For each $u \in W$, one has a representation π_u of B in $B(\mathcal{H})$ given by

$$\pi_u(\mu_m)\varepsilon_n = \varepsilon_{mn},\ m, n \in \mathbb{N}_+$$

$$\pi_u(e(x))\varepsilon_n = \exp(2i\pi n u \circ x)\varepsilon_n,\ n \in \mathbb{N}_+,\ x \in \mathbb{Q}/\mathbb{Z}. \tag{53}$$

Here $u \circ x$ for $u \in W$ and $x \in \mathbb{Q}/\mathbb{Z}$ is the multiplication induced by transport of structure using (46). One verifies easily that (53) does indeed give a C^*-algebra representation of B. Let H be the unbounded operator in $\mathcal{H}$ whose action on the canonical basis is given by

$$H\varepsilon_n = (\log n)\varepsilon_n, \quad n \in \mathbb{N}_+. \tag{54}$$

Then clearly, for each $u \in W$, we have

$$\pi_u(\sigma_t(x)) = e^{itH}\pi_u(x)e^{-itH}, \qquad t \in \mathbb{R},\ x \in B.$$

Notice that, for $\beta > 1$,

$$\mathrm{Trace}(e^{-\beta H}) = \sum_{n=1}^{\infty}\langle e^{-\beta H}\varepsilon_n, \varepsilon_n\rangle = \sum_{n=1}^{\infty} n^{-\beta}\langle\varepsilon_n, \varepsilon_n\rangle = \sum_{n=1}^{\infty} n^{-\beta},$$

so that the Riemann zeta function appears as a partition function of Gibbs state type. We can now state the main result of [**5**].

THEOREM 9 (Bost-Connes). *The dynamical system (B, σ_t) has symmetry group W. The action of $u \in W$ is given by $[u] \in \mathrm{Aut}(B)$ where*

$$[u] : e(y) \mapsto e(u \circ y), \quad y \in \mathbb{Q}/\mathbb{Z}, \qquad [u] : \mu_a \mapsto \mu_a, \quad a \in \mathbb{N}.$$

This action commutes with σ,

$$[u] \circ \sigma_t = \sigma_t \circ [u], \qquad u \in W, \quad t \in \mathbb{R}.$$

Moreover,

(1) *for $0 < \beta \leq 1$, there is a unique* KMS_β *state. (It is a factor state of Type* III_1 *with associated factor the Araki-Woods factor R_∞.)*
(2) *for $\beta > 1$ and $u \in W$, the state*

$$\phi_{\beta,u}(x) = \zeta(\beta)^{-1}\mathrm{Trace}(\pi_u(x)e^{-\beta H}), \qquad x \in B$$

is a KMS_β *state for (B, σ_t). (It is a factor state of Type* I_∞*). The action of W on B induces an action on these* KMS_β *states which permutes them transitively, and the map $u \mapsto \phi_{\beta,u}$ is a homomorphism of the compact group W onto the space $\mathcal{E}_\beta$ of extremal points of the simplex of* KMS_β *states for (B, σ_t).*
(3) *the ζ-function of Riemann is the partition function of (B, σ_t).*

Part (1) of the above theorem is difficult, and the reader is referred to [**5**] for complete details, as for a full proof of (2). That for $\beta > 1$ the KMS_β states given in part (2) fulfil Definition 5 of §5.2 is a straightforward exercise. Notice that they have the form of Gibbs equilibrium states.

Theorem 9 solves Problem 1 of §5.2. More information is contained in its proof however. As mentioned already, given the existence of the Artin isomorphism in class field theory for the rationals, one can recover the Galois action of W explicitly. Despite the progress in [**6**], it is still an open problem to exhibit this Galois action in terms of an analogue of (B, σ_t) in a completely satisfactory way for general number fields. Another feature occurs in the analysis of the proof of part (1) of Theorem 9. One can treat the infinite places in a similar way to that already described for the finite places, so working with the (full) adeles A and (full) ideles J. The ring of adeles A of $\mathbb{Q}$ consists of the infinite vectors $(a_\infty, a_p)_p$ indexed by the archimedean place and the primes p of $\mathbb{Q}$ with $a_p \in \mathbb{Z}_p$ for all but finitely many p. The group J of ideles consists of the infinite vectors $(u_\infty, u_p)_p$ with $u_\infty \in \mathbb{R}, u_\infty \neq 0$ and $u_p \in \mathbb{Q}_p, u_p \neq 0$ and $|u_p|_p = 1$ for all but finitely many primes p. There is a norm $|\cdot|$ defined on J given by $|u| = |u_\infty|_\infty \prod_p |u_p|_p$. We have natural diagonal embeddings of $\mathbb{Q}$ in A and $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ in J induced by the embeddings of $\mathbb{Q}$ into its completions. Notice that by the product formula $\mathbb{Q}^* \subset \mathrm{Ker}|\cdot|$. We define an equivalence relation on A by $a \equiv b$ if and only if there exists a $q \in \mathbb{Q}^*$ with $a = qb$. With respect to this equivalence, we form the coset space $X = A/\mathbb{Q}^*$. The ideles J act on A by componentwise multiplication, which induces an action of $C = J/\mathbb{Q}^*$ on X. Notice that this action has fixed points. For example, whenever an adele a has $a_p = 0$, it is a fixed point of the embedding of $\mathbb{Q}_p^*$ into J (to $q_p \in \mathbb{Q}_p^*$ one assigns the idele with 1 in every place except the pth place.) On the other hand, every Type III_1 factor has a continuous decomposition, that is it can be written as a crossed product of $\mathbb{R}$ with a Type II_∞ factor. Connes has observed that the von-Neumann algebra of Type III_1 in the region $0 < \beta \leq 1$ of Theorem 9 has in its continuous decomposition the Type II_∞ factor given by the crossed product of $L^\infty(A)$ by the action of $\mathbb{Q}^*$ by multiplication. The associated von Neumann algebra has orbit space $X = A/\mathbb{Q}^*$. As we have seen, the pair (X, C) plays a fundamental role in Connes's proposed approach to the Riemann hypothesis in [**8**] and can be thought of as playing the role for number fields of the curve and Frobenius for the proof of the Riemann hypothesis in the case of curves over finite fields.

5.5. Appendix by Peter Sarnak. In this appendix we reproduce, with his permission, the text of a letter written by Peter Sarnak in June, 2001 and addressed to Enrico Bombieri.

Below is the symplectic pairing that I mentioned to you in Zurich. There is nothing deep about it or the analysis that goes with it. Still, its existence is consistent with various themes. To put things in context, recall that the phenomenological and analytic results on the high order zeroes of a given L-function and the low zeroes for families of L-functions suggest that there is a natural spectral interpretation for the zeroes as well as a symmetry group associated with a family [KaSa] (N. Katz and P. Sarnak, Bulletin of the AMS, **36** (1999), 1–26). In particular, for Dirichlet L-functions $L(s, \chi)$, $\chi^2 = 1$, the symmetry predicted in [KaSa] is a symplectic one, *i.e.* $\mathrm{Sp}(\infty)$. So we expect that there is a suitable spectral interpretation: the linear transformation whose spectrum corresponds to the zeroes of $L(s, \chi)$ should correspond to a symplectic form. It should be emphasized that this by itself does not put the zeroes on the line. Such a symplectic pairing is a symmetry which is central to understanding this family of L-functions. On the other hand, the existence of an invariant unitary (or Hermitian) pairing for the operator, as suggested by Hilbert and Pólya would of course put zeroes on the line. However, I think the existence of the latter is not very likely. In the analogous function field settings there are spectral interpretations of the zeroes and invariant bilinear pairings due to Grothendieck. The known proofs of the Riemann Hypothesis (that is, the Weil Conjectures in this setting) do not proceed with any magical unitary structures but rather with families and their monodromy, high tensor power representations of the latter and positivity [De] (P. Deligne, Publ, IHES, **48** (1974), 273–308).

One can look for symplectic pairing in the well-known spectral interpretation of the zeroes of $\zeta(2s)$ in the eigenvalue problem for $X = \mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}$. Indeed the resonances (or scattering frequencies) through the theory of Eisenstein series for X are at the zeroes of $\xi(2s)$, ξ being the completed zeta function of Riemann. Lax and Phillips have constructed an operator B (see [LaPh] (P. Lax, R. Phillips, Bulletin of the AMS, **2** (1980), 261–295)) whose spectrum consists of the Maass cusp forms on X together with the zeroes of $\xi(2s)$. The problem with finding a symplectic pairing for the part of the spectrum corresponding to $\xi(2s)$ is that I don't know of any geometric way of isolating this part of the spectrum of B. The space $L^2(X)$ as it stands is too big. Nevertheless, this spectral interpretation of the zeroes of $\xi(2s)$ is important since it can be used to give reasonable zero-free regions for $\zeta(s)$; see [GLSa](S. Gelbart, E. Lapid, P. Sarnak, A new method for lower bounds for L-functions, C.R. Acad. Sci. Paris, 339 (2004), 91–94). Moreover, this spectral proof of the non-vanishing of $\zeta(s)$ on $\Re(s) = 1$ extends to much more general L-functions where the method of Hadamard and de la Vallée Poussin does not work (at least with our present knowledge)[Sh](F. Shahidi, Perspect. Math., **10** (1990), 415–437, Academic Press).

Assuming the Riemann hypothesis for $L(s,\chi)$, Connes [Co] (A. Connes, Selecta Math. (N.S.), **5** (1999), No. 1, 29–106) gives a spectral interpretation of the zeroes. I recall this construction below. Like the spectral interpretation as resonances, Connes' space is defined very indirectly – as the annihilator of a complicated space of functions (it is very close to the space considered by Beurling [Be] (A. Beurling, Proc. Nat. Acad. Sci., USA, **41** (1955), 312–314)). We can look for invariant pairings for his operator. Note that for an even dimensional space, a necessary condition that a transformation A, of determinant equal to 1, preserve a standard symplectic form or orthogonal pairing is that its eigenvalues (as a set) be invariant under $\lambda \to \lambda^{-1}$. In fact, if the eigenvalues are also distinct, then this is also a sufficient condition. On the other hand, if A is not diagonalizable then there are other obstructions (besides the "functional equation" $\lambda \to \lambda^{-1}$) for preserving such pairings.

The set-up in [Co] is as follows: Let χ be a non-trivial Dirichlet character of conductor q and with $\chi(-1) = 1$ (one can easily include all χ). For $f \in \mathcal{S}(\mathbb{R})$ and even and $x > 0$ set

$$\Theta_f(x) := \left(\frac{x}{\sqrt{q}}\right)^{1/2} \sum_{n=1}^{\infty} f\left(\frac{nx}{q}\right) \chi(n). \tag{1}$$

According to Poisson Summation and Gauss sums we have

$$\Theta_f\left(\frac{1}{x}\right) = \Theta_{\hat{f}}(x). \tag{2}$$

Hence $\Theta_f(x)$ is rapidly decreasing as $x \to 0$ or $x \to \infty$. Consider the vector space W of distributions D on $(0,\infty)$ (with respect to the multiplicative group) with suitable growth conditions at 0 and ∞ for which

$$D(\Theta_f) = \int_0^\infty D(x)\Theta_f(x)\frac{dx}{x} = 0, \tag{3}$$

for all f as above.

For $y > 0$, let U_y be the translation on the space of distributions, $U_y D(x) = D(yx)$. Clearly, U_y leaves the subspace W invariant and yields a representation of $\mathbb{R}^*_{>0}$. By (2), if $D \in W$ then so is $RD := D(1/x)$. This R acts as an involution on W. To see which characters (*i.e.* eigenvalues of U_y) x^s, $s \in \mathbb{C}$, of $\mathbb{R}^*$ are in W, consider

$$\int_0^\infty \Theta_f(x) x^s \frac{dx}{x} = q^{s/2} L(s+\frac{1}{2},\chi) \int_0^\infty f(y) y^{s+1/2}\frac{dy}{y}. \tag{4}$$

Now $f \in \mathcal{S}(\mathbb{R})$ and is even, hence

$$I = \int_0^\infty f(x) x^{s+\frac{1}{2}} \frac{dx}{x} = \int_0^1 f(x) x^{s+\frac{1}{2}} \frac{dx}{x} + g(s),$$

where $g(s)$ is entire. Moreover, for $N \geq 0$,

$$I = \int_0^1 \sum_{n=0}^{N} a_{2n} x^{2n} x^{s+\frac{1}{2}} \frac{dx}{x} + \text{a holomorphic function in } \Re(s) > -N+1,$$

$$= \sum_{n=0}^{\infty} \frac{a_{2n}}{2n - \frac{1}{2} + s} + \text{a holomorphic function in } \Re(s) > -N+1. \tag{5}$$

Thus, for general such f, I has a simple pole at $s = \frac{1}{2} - 2n$.

According to (4) and (5) we have that $x^s(\Theta_f) = 0$ for all f iff

$$s = i\gamma \text{ where } \rho = \frac{1}{2} + i\gamma \text{ is a } \textbf{nontrivial} \text{ zero of } L(s,\chi). \tag{6}$$

If the multiplicity of the zero of $L(s,\chi)$ at $\rho = \frac{1}{2} + i\gamma$ is $m_\gamma \geq 1$, then differentiating (4) $m_\gamma - 1$ times shows that

$$x^{i\gamma},\, (\log x)^{i\gamma}, \ldots, (\log x)^{m_\gamma - 1} x^{i\gamma} \tag{7}$$

are in W.

The involution R of W ensures that $x^{i\gamma} \in W$ iff $x^{-i\gamma} \in W$ (and similarly with multiplicities). Of special interest is $\gamma = 0$. We have from (2) that for j odd,

$$\int_0^\infty (\log x)^j\, \Theta_f(x) \frac{dx}{x} = -\int_0^\infty (\log x)^j\, \Theta_{\widehat{f}}(x) \frac{dx}{x}. \tag{8}$$

Hence, if $f = \widehat{f}$ and j is odd,

$$\int_0^\infty (\log x)^j\, \Theta_f(x) \frac{dx}{x} = 0. \tag{9}$$

If $f = -\widehat{f}$ then

$$\int_0^\infty f(x) x^{1/2} \frac{dx}{x} = 0. \tag{10}$$

So from (4) we see that if $f = -\widehat{f}$ then

$$\int_0^\infty \Theta_f(x)\, (\log x)^j\, \frac{dx}{x} = 0, \qquad \text{for } j = 0, \ldots, m_0. \tag{11}$$

Combining (9) and (11) we see that

$$\begin{aligned} W_0 &= \operatorname{span}\{1, \log x, (\log x)^2, \ldots\} \cap W \\ &= \operatorname{span}\{1, \log x, \ldots, (\log x)^{m_0 - 1}\} \end{aligned} \tag{12}$$

is even dimensional.

Hence m_0 is even (of course this also follows from the functional equation for $L(s,\chi)$).

In order to continue, we need to specify the precise space of distributions that we are working with. To allow for zeroes ρ of $L(s,\chi)$ with $\Re(\rho) \neq \frac{1}{2}$, one needs to allow spaces of distributions which have exponential growth at infinity. This can be done and one can proceed as we do here; however to avoid such definitions we will assume the Riemann Hypothesis for $L(s,\chi)$ (anyway, this is not the issue as far as the symplectic pairing goes). This way we can work with the familiar tempered distributions. We change variable, setting $x = e^t$ so that our distributions $D(t)$ satisfy

$$\int_{-\infty}^\infty D(t) \Theta_f(e^t) dt = 0. \tag{13}$$

The group U_y now acts by translations $\tau \in \mathbb{R}$,

$$U_\tau D(t) = D(t + \tau). \tag{14}$$

If now V is the space of such tempered distributions satisfying the annihilation condition (13) for all f (which is a topological vector space), then for $D \in V$, its Fourier transform $\widehat{D}(\xi)$ is supported in $\{\gamma \mid \xi(\frac{1}{2} + i\gamma, \chi) = 0\}$. Since $\widehat{D}$ is also tempered, it is easy to describe $\widehat{D}$ and hence the space V. It consists of all tempered distributions D of the form

$$D(t) = \sum_{\gamma} \sum_{j=0}^{m_\gamma - 1} a_{j,\gamma}(D) t^j e^{i\gamma t}. \tag{15}$$

The representation (15) is unique and the series converges as a tempered distribution, *i.e.*

$$\sum_{|\gamma| \le T} \sum_{j=0}^{m_\gamma - 1} |a_{j,\gamma}(D)| \ll T^A \tag{16}$$

for some A depending on D.

The action (14) on V gives a group of transformations whose spectrum consists of the numbers $e^{i\gamma\tau}$ with multiplicity m_γ. The subspaces V_γ of V given by

$$V_\gamma = \mathrm{span}\{e^{i\gamma t}, te^{i\gamma t}, \dots, t^{m_\gamma - 1} e^{i\gamma t}\} \tag{17}$$

are U_τ-invariant, the action taking the form

$$e^{i\gamma\tau} \begin{pmatrix} 1 & \tau & \tau^2 & \dots & \tau^{m_\gamma - 1} \\ & 1 & 2\tau & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 1 \end{pmatrix}, \tag{18}$$

in the apparent basis. Thus U_τ is not diagonalizable if $m_\gamma > 1$ for some γ. The span of the subspaces V_γ is dense in V. So the action U_τ on V gives a spectral interpretation of the nontrivial zeroes of $L(s,\chi)$. While one expects for these $L(s,\chi)$'s that all their zeroes are simple, there are more general L-functions (*e.g.* those of elliptic curves of rank bigger than 1) which have multiple zeroes. Thus the possibility of multiple zeroes especially at $s = \frac{1}{2}$ must be entertained and it is instructive to do so. In any case, since multiple zeroes mean that this action U_τ is not diagonalizable, we infer that there cannot be any direct unitarity that goes along with it.

There is however a symplectic pairing on V preserved by U_τ. It is borrowed from $\mathrm{sym}^\nu \rho$, where ρ is the standard two dimensional representation of SL_2 (via (18) above) when ν is odd. We pair V_γ with $V_{-\gamma}$ for $\gamma > 0$ and separate the <u>even</u> dimensional space V_0.

For $D, E \in V$ set

$$[D,E] := \sum_{j=0}^{m_0 - 1} \frac{(-1)^j a_{j,0}(D) a_{m_0-1-j,0}(E)}{\binom{m_0-1}{j}} + \sum_{\gamma > 0} \gamma e^{-\gamma^2} \sum_{j=0} m_\gamma - 1 \frac{(-1)^j}{\binom{m_\gamma - 1}{j}} \cdot \left(a_{j,\gamma}(D) a_{m_\gamma-1-j,-\gamma}(E) - a_{m_\gamma-1-j,-\gamma}(D) a_{j,\gamma}(E) \right). \tag{19}$$

There is nothing special about the factor $\gamma e^{-\gamma^2}$ —it is put there for convergence.

The bilinear pairing $[\ ,\ ]$ on $V \times V$ is symplectic and U_τ-invariant. That is,

1. $[D,E] = -[E,D]$
2. It is non-degenerate: for $D \neq 0$ there is an E such that $[D,E] \neq 0$
3. $[U_\tau D, U_\tau E] = [D,E]$ for $\tau \in \mathbb{R}$.

The verification of these is straightforward. Note that if $m_0 > 0$ and even, one checks that the transformations (18) cannot preserve a symmetric pairing. Thus the symplectic feature is intrinsic to this spectral interpretation of the zeroes of $L(s,\chi)$.

It would be of some interest to carry out the above adelically as in Connes' papers and also for other (say GL_2) L-functions, especially where, for example, an orthogonal rather than symplectic invariance is expected [KaSa]. Another point is that it would be nice to define the pairing $[\ ,\ ]$ directly without the Fourier Transform (*i.e.* without first diagonalizing to Jordan form). If we assume RH as we have done as well as that the zeroes of $L(s,\chi)$ are simple, then such a definition is possible. Set $H(t) = e^{-t^2/2}$; then for D and E in V, we have that $H * D(t)$ and $\frac{d}{dt}\left(H * E\right)(t)$ are almost periodic functions on $\mathbb{R}$. Up to a constant factor we have that

$$[D,E] = M\left(\left(H * D\right)(t)\frac{d}{dt}\left(H * E\right)(t)\right). \tag{20}$$

Here for an almost periodic function $f(t)$ on $\mathbb{R}$, $M(f)$ is its mean-value given by

$$M(f) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)dx.$$

5.5.1. *Section added by Peter Sarnak on February, 2002.* The Hilbert-Pólya idea that there is a naturally defined *self-adjoint* operator whose eigenvalues are simply related to the zeroes of an L-function seems far-fetched. However, Luo and Sarnak [LuSa] (W. Luo and P. Sarnak, Quantum Variance for Hecke eigenforms, Ann. Sci. Ecole Norm. Sup. (4) 37 (2004), 91–94) have recently constructed a self-adjoint *non-negative* operator A on

$$L_0^2(X) = \{\psi \in L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}) \mid \int_X \psi(z)dv(z) = 0\}$$

whose eigenvalues are essentially the central critical values $L(\frac{1}{2},\varphi)$ as φ varies over the (Hecke-eigen) Maass cusp forms for X. In particular, this gives a spectral proof that $L(\frac{1}{2},\varphi) \geq 0$. The fact that $L(\frac{1}{2},\varphi)$ cannot be negative (which is an immediate consequence of RH for $L(s,\varphi)$) is known and was proved by theta function methods (see [KatSa], S. Katok and P. Sarnak, Israel Math. Jnl., **84** (1993) 193–227, and also [Wal], L. Waldspurger, J. Math. Pures et Appl., **60** (1981), 365–384). The operator A comes from polarizing the quadratic form $B(\psi)$ on $L_0^2(X)$ which appears as the main term in the Shnirelman sums for the measures $\varphi_j^2(z)dv(z)$, where φ_j is an orthonormal basis of Maass cusp forms for $L^2(X)$. Denote by λ_j the (Laplace) eigenvalue of φ_j. It is known, see page 688 in [Se](A. Selberg, Collected Papers, Vol. I (1989), Springer-Verlag), that

$$\sum_{\lambda_j\leq\lambda} 1 \sim \frac{\lambda}{12}, \qquad \lambda\to\infty.$$

The quadratic form $B(\psi)$ comes from the following: For $\psi \in L_0^2(X)$ fixed,

$$\sum_{\lambda_j\leq\lambda} |\langle\varphi_j^2,\psi\rangle|^2 \sim B(\psi)\sqrt{\lambda}$$

as $\lambda\to\infty$.

Incidentally, the family of L-functions $L(s, \varphi)$, as φ varies as above, has an orthogonal $O(\infty)$ symmetry in the sense of [KaSa], see also [Ke-Sn] (J. Keating and N. Snaith, Comm. Math. Phys., **214** (2000), 91–110).

Notes added in proof: (1) In his paper "On a representation of the idele class group related to primes and zeros of L-functions" (Duke Math. J. vol. 127, no.3, pp519–595 (2005)), Ralf Meyer gives another approach to a spectral interpretation for the poles and zeros of the L-function of a global field K. His construction is motivated by the work of Alain Connes. As we remarked before, Connes gives a spectral interpretation only of the zeros on the line $s = 1/2$ (and hence in fact he has a spectral interpretation only assuming RH). Meyer uses natural spaces of functions on the adele ring and the idele class group of K to construct a virtual representation of the idele class group of K whose character is equal to a variant of the Weil distribution that occurs in Weil's explicit formula. Thereby, Meyer takes a bigger space of functions and thus captures all the zeros, giving an unconditional spectral interpretation of all the zeros which by itself is not related to RH (we thank Peter Sarnak for this comment).

(2) For further progress on the analogue of Problem 1, §5.2, for imaginary quadratic fields see: A. Connes, M. Marcolli: "From Physics to Number Theory via Noncommutative Geometry" available on arXiv:math.NT/0404128 v1 6 Apr 2004, as well as: A. Connes, M. Marcolli, N. Ramachandran: "KMS states and complex multiplication" available on arXiv: math.OA/0501424.

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