Contemporary Aspects of Complex Analysis, Differential Geometry and Mathematical Physics

Proceedings of the 7th International Workshop on Complex Structures and Vector Fields

edited by Stancho Dimiev
Kouei Sekigawa
Contemporary Aspects of Complex Analysis, Differential Geometry and Mathematical Physics
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PREFACE

This volume is dedicated to Professor Kouei Sekigawa for his 60th birthday, and to Professor Kahlheintz Spallek for his 70th birthday.

The 7th International Workshop on Complex Structures and Vector Fields was held at the Plovdiv University (Plovdiv, Bulgaria) from August 31 to September 4, 2004. This time, there were many participants from Japan, France, Germany, Iran, Bulgaria, and many valuable contributions were made on Differential Geometry, Complex Analysis, Mathematical Physics and the Applications through constant communications among the participants in this Workshop. The present volume is organized mainly by the articles presented by the participants based on their communications in this Workshop. It is also worthy to emphasize that many new young participants contributed their ambitious and magnificent works to this volume.

The Editors would like to express here their gratitude to Professor T. Oguro for his Constant outstanding cooperation and efforts in the arrangement of this volume, and also to Professor S. Manoff for his help.

Lastly, Profs. K. Sekigawa and K. Spallek were celebrated also as ones of the founders of this Workshop.

For the Editors, S. Dimiev and S. Manoff
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   Four-dimensional Walker metrics and symplectic manifolds

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GEODESICS AND TRAJECTORIES FOR KÄHLER MAGNETIC FIELDS

DEDICATED TO PROFESSOR AKIHiko MORIMOTO ON THE OCCASION OF HIS 77TH BIRTHDAY

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We give a survey on constructing real surfaces associated with trajectories for Kähler magnetic fields and on comparing sectors and crescents on these surfaces.

Keywords: Magnetic fields, trajectories, geodesics, magnetic exponential maps, Jacobi fields, sectors, ruled real surfaces, crescents and bow-shapes

1. Introduction

In his papers the author has been making an attempt at investigating Kähler manifolds from Riemannian geometric point of view. It is needless to say that geodesics are quite important objects in Riemannian geometry. As a generalization of geodesics in connection with complex structure, we introduced the notion of trajectories for Kähler magnetic fields. A closed 2-form on a Kähler manifold \((M, J, \langle \cdot, \cdot \rangle)\) is called a Kähler magnetic field if it is a constant multiple of the Kähler form \(\mathbb{B}_J\) on \(M\). We call a smooth curve \(\gamma\) parameterized by its arc length a trajectory for a Kähler magnetic field \(\mathbb{B}_\kappa = \kappa \mathbb{B}_J\) with a constant \(\kappa\) if it satisfies \(\nabla_\gamma \dot{\gamma} = \kappa J \dot{\gamma}\). When \(\kappa = 0\), which is the case without the force of magnetic fields, a trajectory is a geodesic. Since trajectories for a Kähler magnetic field are determined by their initial vectors, they introduce a dynamics on the unit tangent bundle.

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just like geodesics introduce the geodesic flow. In this aspect the author hopes trajectories for Kähler magnetic fields would give us Riemannian geometric information on Kähler manifolds.

On a complex space form $M^n(c;\mathbb{C})$ of constant holomorphic sectional curvature $c$, which is a model space when we study Kähler magnetic fields, a trajectory for a Kähler magnetic field $B_\kappa$ is so-called a circle of curvature $|\kappa|$. On a complex projective space $\mathbb{C}P^n(c)$, it is closed of length $2\pi/\sqrt{\kappa^2 + c}$, and on a complex Euclidean space $\mathbb{C}^n$, it is also closed of length $2\pi/|\kappa|$ except the case $\kappa = 0$. On a complex hyperbolic space $\mathbb{C}H^n(c)$, the features depend on the strength of the magnetic field: It is closed when $|\kappa| > \sqrt{c}$ and of length $2\pi/\sqrt{\kappa^2 + c}$, and is open when $|\kappa| \leq \sqrt{c}$. In each case, every trajectory lies on a totally geodesic complex line $M^1(c;\mathbb{C})$, and this Riemann surface is a natural object for this trajectory. In this paper we consider how to construct such surfaces for trajectories on a complete Kähler manifold of variable curvature. In Riemannian geometry it is a basic idea to compare the geometry of an arbitrary Riemannian manifold with geometry of a space of constant curvature. Once we construct a real 2-dimensional object associated with a trajectory we study some comparison theorems on trajectories by tracing those comparison theorems on geodesics.

2. Variations of trajectories

One of the most important tools in Riemannian geometry is an exponential map, which connects a Riemannian manifold with its tangent space. We first generalize this notion for Kähler magnetic fields. For a unit tangent vector $u \in TM$ we denote by $\gamma_{u,\kappa}$ a trajectory for a Kähler magnetic field $B_\kappa$ with initial vector $u$. Given a point $x \in M$ we define the magnetic exponential map $B_\kappa \exp_x : T_x M \rightarrow M$ of the tangent space $T_x M$ at $x$ for $B_\kappa$ by

$$B_\kappa \exp_x (v) = \begin{cases} \gamma_v/\|v\|\kappa(\|v\|), & \text{if } v \neq 0_x, \\ x, & \text{if } v = 0_x, \end{cases}$$

where $0_x$ is the origin of $T_x M$.

In order to study its differential we consider vector fields corresponding to Jacobi fields. We say a vector field $Y$ along a trajectory $\gamma$ for $B_\kappa$ to be a normal magnetic Jacobi field for $B_\kappa$ if it satisfies

i) $\nabla_\gamma \nabla_\gamma Y - \kappa J \nabla_\gamma Y + R(Y, \dot{\gamma}) \dot{\gamma} = 0$,

ii) $\nabla_\gamma Y \perp \dot{\gamma}$,
where $R$ denotes the curvature tensor of $M$. Every normal magnetic Jacobi field is obtained by some variation of trajectories for $\mathbb{B}_\kappa$.

Here we study the map $\text{Proj}_nD(\mathbb{B}_\kappa \exp_x (ru))$ of $T_{ru} \{ v \in T_x M \mid \|v\| = r \}$ onto $\{ w \in T_{ru} M \mid \langle w, \dot{\gamma}_u(r) \rangle = 0 \}$. For a vector field $X$ along a trajectory $\gamma$ for $\mathbb{B}_\kappa$, we denote by $X^\perp$ the component of $X$ orthogonal to $\dot{\gamma}$, that is, $X^\perp = X - \langle X, \dot{\gamma} \rangle \dot{\gamma}$. A real number $t_0$ is said to be a $\mathbb{B}_\kappa$-conjugate value for $\gamma(0)$ along $\gamma$ if there exists a nontrivial normal magnetic Jacobi field $Y$ for $\mathbb{B}_\kappa$ along $\gamma$ with $Y^\perp(0) = 0$ and $Y^\perp(t_0) = 0$. We denote by $t_c(\gamma(0); \gamma, \kappa)$ the minimum positive $\mathbb{B}_\kappa$-conjugate value of $\gamma(0)$ along $\gamma$. For a complex space form $M^n(c; \mathbb{C})$ we see this value is given by

$$t_c(\kappa, c) = \begin{cases} \pi/\sqrt{\kappa^2 + c}, & \text{if } \kappa^2 + c > 0, \\ \infty, & \text{if } \kappa^2 + c \leq 0. \end{cases}$$

A comparison theorem on normal magnetic Jacobi fields is established as follows (see [2]):

**Proposition 2.1.** Let $\gamma$ and $\hat{\gamma}$ be trajectories for Kähler magnetic fields $\mathbb{B}_\kappa$ on Kähler manifolds $M$ and $\tilde{M}$ respectively. Suppose their dimensions satisfy $\dim(M) \geq \dim(\tilde{M})$, and their sectional curvatures along trajectories satisfy $\min_{v \perp \dot{\gamma}(t)} \text{Riem}(\gamma(t), v) \geq \max_{\dot{\gamma}(t)} \text{Riem}(\dot{\gamma}(t), \dot{\gamma})$ for $0 \leq t < t_c(\gamma(0); \gamma, \kappa)$. We then have the following properties.

1) $t_c(\gamma(0); \gamma, \kappa) \leq t_c(\hat{\gamma}(0); \hat{\gamma}, \kappa)$.

2) If normal magnetic Jacobi fields $Y$ and $\tilde{Y}$ along $\gamma$ and $\hat{\gamma}$ satisfy

$$Y^\perp(0) = \tilde{Y}^\perp(0) = 0, \quad \|\nabla_\gamma Y^\perp(0)\| = \|\nabla_\hat{\gamma} \tilde{Y}^\perp(0)\|,$$

then for every $t$ with $0 \leq t < t_c(\gamma(0); \gamma, \kappa)$ we find

$$\langle \nabla_{\dot{\gamma}} Y^\perp(t), Y^\perp(t) \rangle/\|Y^\perp(t)\|^2 \leq \langle \nabla_{\dot{\hat{\gamma}}} \tilde{Y}^\perp(t), \tilde{Y}^\perp(t) \rangle/\|\tilde{Y}^\perp(t)\|^2, \quad (2.1)$$

$$\|Y^\perp(t)\| \leq \|\tilde{Y}^\perp(t)\|. \quad (2.2)$$

If an equality holds in one of these inequalities (2.1) and (2.2), then for all $\tau$ with $0 \leq \tau \leq t$ we see

i) these inequalities at $\tau$ turn to equalities,

ii) $\text{Riem}(\gamma(\tau), Y(\tau)) = \text{Riem}(\dot{\gamma}(\tau), \tilde{Y}(\tau)).$

This proposition guarantees immersion property of magnetic exponential maps. As sectional curvatures of $\mathbb{C}H^n(c)$ satisfy $c \leq \text{Riem}_{\mathbb{C}H^n(c)}(\leq c/4)$, we have the following:

**Corollary 2.1.** On a Kähler manifold $M$ whose sectional curvatures satisfy $\text{Riem}_M \leq c \leq 0$, there are no $\mathbb{B}_\kappa$-conjugate points if $|\kappa| \leq \sqrt{|c|}$. In
other words, $\mathbb{B}_\kappa \exp_x(T_xM)$ for $|\kappa| \leq \sqrt{|c|}$ is an immersed surface without singularities for all $x \in M$.

We should note that on $\mathbb{C}H^n(c)$, which does not satisfy the assumption of Corollary 2.1, there are no $\mathbb{B}_\kappa$-conjugate points if $|\kappa| \leq \sqrt{|c|}$. We would like to relax the assumption on sectional curvatures to the assumption on holomorphic sectional curvatures. From this point of view we need to take real 2-dimensional objects associated with trajectories.

As a candidate for a real 2-dimensional object associated with a Kähler magnetic field, we take an image of complex line through a magnetic exponential map. For a unit tangent vector $u \in T_xM$ and a real number $\theta$ with $0 < \theta < 2\pi$, we put

$$S_\kappa(u, r, \theta) = \mathbb{B}_\kappa \exp_x \left( \{ tw_s \mid 0 \leq s \leq \theta, \ 0 \leq t \leq r \} \right),$$

where $w_s = \cos su + \sin sJu$. We call it a $\mathbb{B}_\kappa$-sector of radius $r$ and vertical angle $\theta$ if $r \leq t_c(x; \gamma_{w_s}, \kappa)$ for all $s$. For a $\mathbb{B}_\kappa$-sector $S_\kappa(u, r, \theta)$ we call a curve $\rho_S$ given by $s \mapsto \mathbb{B}_\kappa \exp_x(rw_s)$ the arc of this sector.

![Figure 1. a sector and its arc](image)

On a complex space form $M^n(c; \mathbb{C})$, a $\mathbb{B}_\kappa$-sector of radius $r$ ($\leq t_c(\kappa, c)$) and vertical angle $2\pi$ is an intersection of a geodesic ball of radius $\ell(r; \kappa, c)$ and a totally geodesic complex line $M^1(c; \mathbb{C})$. Here the radius $\ell(r; \kappa, c)$ is given by the following relationship:

$$\begin{align*}
\sqrt{\kappa^2 + c \sin(\sqrt{c} \ell(r; \kappa, c)/2)} &= \sqrt{c} \sin(\sqrt{\kappa^2 + c r/2}), \quad &\text{when } c > 0, \\
\ell(r; \kappa, 0) &= (2/|\kappa|) \sin(|\kappa|r/2), \quad &\text{when } c = 0, \\
\sqrt{|c| - \kappa^2} \sinh(\sqrt{|c|} \ell(r; \kappa, c)/2) &= \sqrt{|c|} \sinh(\sqrt{|c| - \kappa^2} r/2), \quad &\text{when } c < 0 \text{ and } \kappa^2 < |c|, \\
2 \sinh(\sqrt{|c|} \ell(r; \kappa, c)/2) &= \sqrt{|c|} r, \quad &\text{when } c < 0 \text{ and } \kappa^2 = |c|, \\
\sqrt{\kappa^2 + c} \sinh(\sqrt{|c|} \ell(r; \kappa, c)/2) &= \sqrt{|c|} \sin(\sqrt{\kappa^2 + c r/2}), \quad &\text{when } c < 0 \text{ and } \kappa^2 > |c|. 
\end{align*}$$
Thus sectors are candidates for real 2-dimensional objects associated with trajectories. We here consider lengths of arcs for sectors.

**Example 2.1.** On $M^n(c; C)$, the length $L(r, \theta; \kappa, c)$ of arc for a $B_\kappa$-sector of radius $r$ and vertex angle $\theta$ is given as follows:

$$L(r, \theta; \kappa, c) = \begin{cases} 
(\theta / \sqrt{c}) \sin \sqrt{c} \ell(r; \kappa, c), & \text{when } c > 0, \\
\theta \ell(r; \kappa, 0), & \text{when } c = 0, \\
(\theta / \sqrt{|c|}) \sinh \sqrt{|c|} \ell(r; \kappa, c), & \text{when } c < 0.
\end{cases}$$

We now give a comparison theorem on sectors. For each trajectory $\gamma$ composing a sector, it corresponds a normal magnetic Jacobi field $Y$ with initial condition that $Y(0) = 0$ and $\nabla_\gamma Y(0)$ is parallel to $J\gamma(0)$. On $M^n(c, C)$, the square of its norm $A^2(t; \kappa, c)$ in the case $||\nabla_\gamma Y(0)|| = 1$ is given as follows:

$$A^2(t; \kappa, c) = \begin{cases} 
\frac{\kappa^2}{(\kappa^2 + c)^2} (1 - \cos \sqrt{\kappa^2 + ct}^2 + \frac{1}{\kappa^2 + c} \sin^2 \sqrt{\kappa^2 + ct}, & \text{when } c \geq 0 \text{ or when } c < 0 \text{ and } \kappa^2 > |c|, \\
(|c|/4)t^4 + t^2, & \text{when } c < 0 \text{ and } \kappa^2 = |c|, \\
\frac{\kappa^2}{(\kappa^2 + c)^2} (\cosh |c| - \kappa^2 t - 1)^2 - \frac{1}{\kappa^2 + c} \sinh^2 \sqrt{|c| - \kappa^2 t}, & \text{when } c < 0 \text{ and } \kappa^2 < |c|.
\end{cases}$$

This norm is related to the lengths of arcs for $B_\kappa$-sectors on $M^n(c, C)$ in the following manner:

$$L(r, \theta; \kappa, c) = \theta \Lambda(r; \kappa, c).$$

Thus in order to study lengths of arcs for $B_\kappa$-sectors, we need to investigate the norm of a normal magnetic Jacobi field $Y$ along a trajectory $\gamma$ with initial condition $Y(0) = 0$ and $\nabla_\gamma Y(0) = J\gamma(0)$. For a normal magnetic Jacobi field $Y$ for $B_\kappa$ along $\gamma$, we have $\langle Y, \gamma \rangle' = \kappa(Y, J\gamma)$, hence we obtain the following rough estimate by Proposition 2.1 (see [2]).

**Proposition 2.2.** Let $M$ be a Kähler manifold whose sectional curvature satisfies $\text{Riem}_M \geq c$. A normal magnetic Jacobi field $Y$ for $B_\kappa$ along $\gamma$ on $M$ with initial condition $Y(0) = 0$, $||\nabla_\gamma Y(0)|| = \lambda$ satisfies

$$||Y(t)|| \leq \lambda \Lambda(t; \kappa, c) \quad \text{for} \quad 0 \leq t \leq t_c(\gamma(0); \gamma, \kappa).$$
The equality holds if and only if $Y^\#(\tau)$ is parallel to $J\gamma(\tau)$ and the holomorphic sectional curvature $HRiem(\gamma(\tau))$ of the complex line spanned by $\gamma(\tau)$ is equal to $c$ for all $0 \leq \tau \leq t$.

**Corollary 2.2.** On a Kähler manifold $M$ with $Riem_M \geq c$, the length $\text{length}(\rho_S)$ of arc of a $\mathbb{B}_\kappa$-sector $S$ of radius $r$ and vertex angle $\theta$ is not greater than $L(r, \theta; \kappa, c)$. The equality $\text{length}(\rho_S) = L(r, \theta; \kappa, c)$ holds if and only if $S$ is totally geodesic, holomorphic and of constant (holomorphic sectional) curvature $c$.

**Corollary 2.3.** If a $\mathbb{B}_\kappa$-sector $S$ of radius $r$ on a Kähler manifold $M$ with $Riem_M \geq c$ has an arc of length $L(r, \theta; \kappa, c)$, then its vertex angle is not less than $\theta$.

In these corollaries there are no benefit in taking complex lines in a tangent space. In order to get a sharp estimate (from below) we need to refine our comparison theorem on magnetic Jacobi fields.

### 3. Variations of magnetic fields

The advantages in considering trajectories of Kähler magnetic fields are that we can get information on complex structure and that we have a parameter of forces of magnetic fields. In this sense we should say that one of the most natural way to construct a surface is to vary the forces of Kähler magnetic fields. But for now the author could not obtain a canonical variation of forces. In this section we just glance a surface obtained by a variation of forces on a complex space form. Here we should note that if $\gamma$ is a trajectory for a Kähler magnetic field $\mathbb{B}_\kappa$ then the curve $\tilde{\gamma}$ defined by $\tilde{\gamma}(s) = \gamma(\lambda s)$ with some constant $\lambda$ satisfies $\nabla_{\tilde{\gamma}} \tilde{\gamma}' = \lambda \kappa J \tilde{\gamma}'$.

**Example 3.1.** On $\mathbb{C}P^n(c)$, for a trajectory $\gamma_{u, \kappa}$ for $\mathbb{B}_\kappa$, we have a surface

$$\Gamma_{u, \kappa} : (-\infty, \infty) \times \left[ -\pi/\sqrt{\kappa^2 + c}, \pi/\sqrt{\kappa^2 + c} \right] \to \mathbb{C}P^n(c)$$

with a singularity at $\gamma(0)$ defined by

$$\Gamma_{u, \kappa}(s, t) = \gamma_{u, \kappa}\left( \sqrt{(\kappa^2 + c)/(s^2\kappa^2 + c)} \right).$$

Its image is a totally geodesic complex line $\mathbb{C}P^1(c)$.

**Example 3.2.** Similarly on $\mathbb{C}^n$, for a trajectory $\gamma_{u, \kappa}$ ($\kappa \neq 0$) we have a surface $\Gamma_{u, \kappa} : (0, \infty) \times [-\pi/|\kappa|, \pi/|\kappa|] \to \mathbb{C}^n$ with a singularity at $\gamma_{u, \kappa}(0)$ defined by $\Gamma_{u, \kappa}(s, t) = \gamma_{u, \kappa}(t/s)$. Its image is a totally geodesic half line $\{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$. 
Example 3.3. The situation is a bit different for a trajectory $\gamma_{u,\kappa}$ on $\mathbb{C}H^n(c)$. When $|\kappa| > \sqrt{|c|}$, we have a surface

$$\Gamma_{u,\kappa} : (|c|/|\kappa|, \infty) \times \left[ -\pi/\sqrt{\kappa^2 + c}, \pi/\sqrt{\kappa^2 + c} \right] \to \mathbb{C}H^n(c)$$

defined by (3.1). When $|\kappa| < \sqrt{|c|}$, we also have a surface

$$\Gamma_{u,\kappa} : (-\sqrt{|c|}/|\kappa|, \sqrt{|c|}/|\kappa|) \times (-\infty, \infty) \to \mathbb{C}H^n(c)$$

defined by (3.1). These are natural surfaces from the viewpoint of congruency of Kähler magnetic flows (see [1]). When $\kappa = \pm \sqrt{|c|}$, which is the case of horocycle trajectories, there are no such natural surfaces in this sense, but we can define a surface $\Gamma_u : [-1, 1] \times (-\infty, \infty) \to \mathbb{C}H^n(c)$ by $\Gamma_u(s, t) = \gamma_{u,s\sqrt{|c|}}(t)$. This is a one-side deformation of $\gamma_{u,\kappa}$. All these surfaces have a singularity at $\gamma_{u,\kappa}(0)$.

Geodesics on a complex Euclidean space and horocycle trajectories on a complex hyperbolic space are exceptional trajectories. Since the definition of such surfaces also deeply depends on lengths of trajectories, it seems difficult to generalize them on arbitrary Kähler manifolds.

4. Ruled real surfaces associated with trajectories

Another candidate to get a totally geodesic complex line $M^1(c; \mathbb{C})$ from a trajectory on $M^n(c; \mathbb{C})$ is a ruled real surface. For given a trajectory $\gamma$ we attach each point $\gamma(s)$ a geodesic $\sigma_s$ with initial vector $\dot{\sigma}_s(0) = \text{sgn}(\kappa)J\gamma(s)$. On a complex space form it has a "center" where all these geodesics meet, though some cases it is imaginary. If we denote by $t_f(\kappa, c)$ the radius of this "circle" $\gamma$ for a Kähler magnetic field $B_\kappa$ on $M^n(c; \mathbb{C})$, we find
In this section, we study ruled real surfaces on general Kähler manifolds and give a precise definition of \( tf(c; \kappa) \). Let \( \gamma \) be a trajectory for a non-trivial Kähler magnetic field \( B_\kappa \) on a complete Kähler manifold \( M \). We shall call a map \( \alpha : \mathbb{R}^2 \rightarrow M \) a ruled real surface associated with \( \gamma \) if it satisfies the following conditions:

i) \( \alpha(s, 0) = \gamma(s) \),

ii) for each \( s \) the map \( \sigma_s(\cdot) = \alpha(s, \cdot) \) is a geodesic with initial vector \( \text{sgn}(\kappa)J\gamma(s) \).

We then find the Jacobi field \( Y_s \) along a geodesic \( \sigma_s \) induced by this variation of normal geodesics satisfies the following properties:

1) \( Y_{s_0}(0) = \dot{\gamma}(s_0) \),

2) \( \frac{1}{2} \frac{d}{dt} \| Y_{s_0}(t) \|^2 \bigg|_{t=0} = -|\kappa| \),

3) \( \nabla_{\dot{\sigma}_{s_0}} Y_{s_0}(0) \) is parallel to \( \dot{\gamma}(s_0) \).

Hence we trivially find that the curvature vector \( R(JY_s, Y_s)Y_s \) associated with this ruled real surface shows its holomorphic property: A ruled real surface \( \alpha \) associated with a trajectory for a Kähler magnetic field on \( M \) is a complex line if and only if the vector \( R(J \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t}) \frac{\partial \alpha}{\partial t} \) is parallel to \( J \frac{\partial \alpha}{\partial t} \) at each point. In this case it is totally geodesic.

It is a natural way to study Jacobi fields obtained from ruled real surfaces associated with trajectories. We shall say that a Jacobi field \( Y \) along

\[
t_f(\kappa, c) = \begin{cases} 
\frac{1}{\sqrt{c}} \tan^{-1} \frac{\sqrt{c}}{|\kappa|}, & \text{if } c > 0, \\
\frac{1}{|\kappa|}, & \text{if } c = 0, \\
\frac{1}{2\sqrt{|c|}} \log \frac{|\kappa|+|c|}{|\kappa|-|c|}, & \text{if } c < 0 \text{ and } |\kappa| > \sqrt{|c|}, \\
\infty, & \text{if } c < 0 \text{ and } |\kappa| \leq \sqrt{|c|},
\end{cases}
\]
a normal geodesic $\sigma$ is associated with $\mathbb{B}_\kappa$ if it is determined by initial conditions

$$Y(0) = -\text{sgn}(\kappa)J\dot{\sigma}(0) \quad \text{and} \quad \nabla_\phi Y(0) = \kappa J\dot{\sigma}(0).$$

We call a point $\sigma(t_0)$ a $\mathbb{B}_\kappa$-trajectory focal point of $\sigma(0)$ if this Jacobi field vanishes at $t_0$, and call the value $t_0$ a $\mathbb{B}_\kappa$-trajectory focal value of $\sigma(0)$. The minimum positive $\mathbb{B}_\kappa$-trajectory focal value is said to be the first $\mathbb{B}_\kappa$-trajectory focal value of $\sigma(0)$, and is denoted by $t_f(\sigma(0); \sigma, \kappa)$. In case there are no $\mathbb{B}_\kappa$-trajectory focal points of $\sigma(0)$ we put $t_f(\sigma(0); \sigma, \kappa) = \infty$. On a complex space form $M^n(c; \mathbb{C})$ the first $\mathbb{B}_\kappa$-trajectory focal value is given by $t_f(\kappa, c).

$$\text{focal point}$$

Figure 4. $\mathbb{B}_\kappa$-trajectory focal point

A comparison theorem on Jacobi fields associated with a Kähler magnetic field is established as follows.

**Theorem 4.1.** Let $M$ be a Kähler manifold and $\sigma$ be a normal geodesic on $M$. Suppose that the sectional curvature of 2-planes spanned by $\dot{\sigma}(t)$ and a vector orthogonal to $\dot{\sigma}(t)$ is not greater than $c$ for $0 \leq t \leq t_f(c; \kappa)$. Then the following results hold:

1. $t_f(\sigma(0); \sigma, \kappa) \geq t_f(c; \kappa)$.
2. Let $Y$ and $\hat{Y}$ be Jacobi fields along normal geodesics $\sigma$ and $\hat{\sigma}$ which are associated with $\mathbb{B}_\kappa$ on $M$ and $M^n(c; \mathbb{C})$ respectively. We then find for $0 \leq t \leq t_f(c; \kappa)$ that
   i) $\langle \nabla_\phi Y(t), Y(t) \rangle/\|Y(t)\|^2 \geq \langle \nabla_\phi \hat{Y}(t), \hat{Y}(t) \rangle/\|\hat{Y}(t)\|^2$;
   ii) $\|Y(t)\| \geq \|\hat{Y}(t)\|$.

In one of these inequalities, an equality holds if and only if $Y(\tau)$ is parallel to $J\dot{\sigma}(\tau)$ and $\text{HRiem}(\hat{\sigma}(\tau)) = c$ for all $0 \leq \tau \leq t$. In this case, these inequalities turn to equalities for all $0 \leq \tau \leq t$.

Our proof is quite elementary. If we restrict ourselves to the case $\kappa > 0$, we know $\hat{Y}$ is given as $\hat{Y} = -gJ\dot{\hat{\sigma}}$ with some nonnegative function $g$ on
the interval \([0, t_f(c; \kappa)]\). If we denote as \(Y = hE\) with some nonnegative function \(h\) and a unit vector field \(E\), we see

\[
h'' + h(\langle R(E, \dot{\sigma}) \dot{\sigma}, E \rangle - \|\nabla_{\dot{\sigma}} E \|^2) = 0,
\]
\[
h'(0) = g'(0) = -\kappa,
\]
\[
(h'g - hg')(0) = 0.
\]

Therefore we have

\[
(h'g - hg)' = hg(\|\nabla_{\dot{\sigma}} E \|^2 - \langle R(E, \dot{\sigma}) \dot{\sigma}, E \rangle + c)
\]
\[
\geq hg(c - \langle R(E, \dot{\sigma}) \dot{\sigma}, E \rangle) \geq 0
\]

and obtain our result.

Since our argument is quite elementary, we can not relax the assumption on sectional curvature to the assumption on holomorphic sectional curvature. Our argument on this theorem shows a bit more on the immersed Riemann surface. We denote by \(-t_n(c; \kappa)\) the maximum negative \(B_\kappa\)-trajectory focal value on \(M^n(c; \mathbb{C})\). We find

\[
t_n(c; \kappa) = \begin{cases} \frac{1}{\sqrt{c}}(\pi - \tan^{-1}\frac{\sqrt{c}}{|\kappa|}), & \text{if } c > 0, \\ \infty, & \text{if } c \leq 0. \end{cases}
\]

For about the maximum negative \(B_\kappa\)-trajectory focal value \(-t_n(\sigma(0); \sigma, \kappa)\) along a geodesic \(\sigma\) we have the following:

**Proposition 4.1.** Let \(M\) be a Kähler manifold and \(\sigma\) be a normal geodesic on \(M\). If \(\min_{\sigma(0) \perp \dot{\sigma}(t)} \text{Riem}(\dot{\sigma}(t), \nu) \leq c\) for \(-t_n(c; \kappa) < t \leq 0\), then we have \(t_n(\sigma(0); \sigma, \kappa) \geq t_n(c; \kappa)\).

Combining these comparison results on \(B_\kappa\)-trajectory focal values we obtain the following:

**Corollary 4.1.** On a Kähler manifold \(M\) whose sectional curvature satisfies \(\text{Riem} \leq c\) with some nonpositive constant \(c\), every ruled real surface associated with a trajectory for \(B_\kappa\) \((|\kappa| \leq \sqrt{|c|})\) is an immersed surface without singularities.

5. A comparison on crescents

In order to get information on ruled real surfaces associated with trajectories for Kähler magnetic fields, it is a way to estimate the distance between
two points on a trajectory. For a ruled real surface $\alpha$ associated with a trajectory segment $\gamma : [0, r] \to M$ for $B_\kappa$, we consider a subset

$$\mathcal{A} = \{ \alpha(s, t) \mid 0 \leq s \leq r, -t_n(\gamma(s); \kappa) \leq t \leq t_f(\gamma(s); \kappa) \}.$$  

We take a set $C(\gamma)$ of all curves on $\mathcal{A}$ of the following form which join $\gamma(0)$ and $\gamma(r)$:

$$\rho_\tau(s) = \alpha(s, \tau(s)), \quad 0 \leq s \leq r,$$

where $\tau : [0, r] \to \mathbb{R}$ is a smooth function satisfying $0 \leq \tau(s) \leq t_f(\gamma(s); \kappa)$ and $\tau(0) = \tau(r) = 0$. We take the infimum of lengths of such curves and set

$$d_\mathcal{A}(\gamma(0), \gamma(r)) = \inf \{ \text{length}(\rho_\tau) \mid \rho_\tau \in C(\gamma) \}.$$  

Clearly we see $d_\mathcal{A}(\gamma(0), \gamma(r)) \leq r$.

First we consider this quantity for trajectories on a complex space form. Since it only depends on the length of a trajectory segment and the strength of Kähler magnetic field, we denote by $d(r; \kappa, c)$ this quantity for a trajectory segment of length $r$ for $B_\kappa$ on a complex space form $M^n(c; \mathbb{C})$.

**Example 5.1.** On a complex Euclidean space $\mathbb{C}^n$, we see

$$d(r; \kappa, 0) = \begin{cases} \ell(r; \kappa, 0), & \text{if } 0 \leq r \leq \pi/|\kappa|, \\ 2/|\kappa|, & \text{if } \pi/|\kappa| \leq r < 2\pi/|\kappa|. \end{cases}$$  

When $0 < r < \pi/|\kappa|$, there exists a unique geodesic segment $\rho(s) = \alpha(s, \tau(s))$ on $\mathbb{C}^n$ which attains this quantity. In this case, the set $\mathcal{A}_\rho = \{ \alpha(s, t) \mid 0 \leq s \leq r, 0 \leq t \leq \tau(s) \}$ for this geodesic segment $\rho$ is contained in a totally geodesic $\mathbb{C}$.

**Example 5.2.** On a complex projective space $\mathbb{C}P^n(c)$ the following equalities hold:

$$\begin{cases} d(r; \kappa, c) = \ell(r; \kappa, c), & \text{if } 0 \leq r \leq \pi/\sqrt{\kappa^2 + c}, \\ \sqrt{\kappa^2 + c \sin(\sqrt{c}d(r; \kappa, c)/2)} = \sqrt{c}, & \text{if } \pi/\sqrt{\kappa^2 + c} \leq r < 2\pi/\sqrt{\kappa^2 + c}. \end{cases}$$  

When $0 < r < \pi/\sqrt{\kappa^2 + c}$, there exists a unique geodesic segment $\rho$ on $\mathbb{C}P^n(c)$ which attains $d(r; \kappa, c)$. The set $\mathcal{A}_\rho$ is contained in a totally geodesic $\mathbb{C}P^1(c)$.
Example 5.3. On a complex hyperbolic space $\mathbb{CH}^n(c)$ the following equalities hold:

\[
\begin{aligned}
    d(r; \kappa, c) &= \ell(r; \kappa, c), \\
    \sqrt{\kappa^2 + c} \sinh \left( \frac{\sqrt{|c|} d(r; \kappa, c)}{2} \right) &= \sqrt{|c|}, \\
    \text{when } \kappa^2 \leq |c| \text{ or when } \kappa^2 > |c| \text{ and } 0 < r < \pi/\sqrt{\kappa^2 + c}, \\
    \sqrt{\kappa^2 + c} \sinh \left( \frac{\sqrt{|c|} d(r; \kappa, c)}{2} \right) &= \sqrt{|c|}, \\
    \text{if } \kappa^2 > |c| \text{ and } \pi/\sqrt{\kappa^2 + c} \leq r < 2\pi/\sqrt{\kappa^2 + c}.
\end{aligned}
\]

Except the last case, there exists a unique geodesic segment $\rho$ on $\mathbb{CH}^n(c)$ which attains $d(r; \kappa, c)$. The set $\mathcal{A}_\rho$ is contained in a totally geodesic $\mathbb{CH}^1(c)$.

We next study the general case. By applying the comparison theorem on Jacobi fields associated with a trajectory, we obtain the following estimate.

Theorem 5.1. Let $\gamma$ be a trajectory segment of length $r$ for a Kähler magnetic field $B_\kappa$ on a Kähler manifold $M$ whose sectional curvatures satisfy $\text{Riem}_M \leq c$. If $0 < r < \pi/\sqrt{\kappa^2 + c}$, then we have $d_\mathcal{A}(\gamma(0), \gamma(r)) \geq d(r; \kappa, c)$. Here we regard $\pi/\sqrt{\kappa^2 + c}$ to be infinity when $\kappa^2 + c < 0$.

Now we study the case that the equality holds. For a curve $\rho(s) = \alpha(s, \tau(s)) \in C(\gamma)$, we call the pair $C = (\gamma, \tau)$ a crescent and the set $\mathcal{A}_\rho = \{ \alpha(s, t) | 0 \leq s \leq r, 0 \leq t \leq \tau(s) \}$ its represented shape. When the length of $\rho$ attains $d_\mathcal{A}(\gamma(0), \gamma(r))$, we call $C$ a bow-shape.

![Figure 5. a bow shape and a crescent](image)

Proposition 5.1. Under the assumption of Theorem 5.1, if $d_\mathcal{A}(\gamma(0), \gamma(r)) = d(r; \kappa, c)$, we have the following:

1. there is a curve $\rho$ with $d_\mathcal{A}(\gamma(0), \gamma(r)) = \text{length}(\rho)$,
2. the represented shape $\mathcal{A}_\rho$ of a crescent for this $\rho$ is totally geodesic and holomorphic in $M$, and is of constant holomorphic sectional curvature $c$. 

By standing another point of view we see Theorem 5.1 assures the following:

**Corollary 5.1.** Suppose sectional curvatures of $M$ satisfy $\text{Riem}_M \leq c$. If a trajectory segment $\gamma$ for $B_\kappa$ with length($\gamma$) = $r$ ($< \pi/\sqrt{\kappa^2 + c}$) satisfies $d_A(\gamma(0), \gamma(r)) \leq d(r'; \kappa, c)$, then we have $r \leq r'$.

**References**

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In this paper we give a survey on ordinary helices which are integral curves of Killing vector fields on symmetric spaces of rank one. On a real space form $\mathbb{R}^n$, $S^n$ or $H^n$, all ordinary helices are generated by some Killing vector fields, and they are congruent each other if they have the same curvatures. But the situation is not the same for other symmetric spaces of rank one. Even a complex hyperbolic space admits bounded ordinary helices. We also make mention of an example of closed ordinary helices in a complex projective plane with 6 self-intersection points.

*Keywords*: Ordinary helix, symmetric space of rank one, Killing vector field, Killing ordinary helix

1. Introduction

A smooth curve parameterized by its arclength on a Riemannian manifold $M$ is said to be a *Killing helix* if it is an integral curve of some Killing vector field on $M$. In other words, a Killing helix is an orbit of one-parameter subgroup of the isometry group of $M$. This clearly satisfies Frenet formula of some proper order $d$ with constant curvatures: There exist an orthonormal
system \( \{ V_1 = \gamma, V_2, \ldots, V_d \} \) of vector fields along \( \gamma \) and positive constants \( \kappa_1, \ldots, \kappa_{d-1} \) satisfying the system
\[
\nabla_{\gamma} V_j (s) = -\kappa_{j-1} V_{j-1} (s) + \kappa_j V_{j+1} (s), \quad 1 \leq j \leq d
\]
(1.1)
of ordinary differential equations with \( V_0 = V_{d+1} \equiv 0 \) and \( \kappa_0 = \kappa_d = 0 \). We call a smooth curve which is parameterized by its arclength and satisfies such differential equations a helix of proper order \( d \). A smooth curve is called a helix of order \( d \) if it is a helix of proper order \( r \) (\( \leq d \)).

A smooth curve satisfying Frenet formula of order 2 is said to be a circle. On symmetric spaces of rank one circles are necessarily Killing helices. We therefore interested in Killing helices satisfying Frenet formula of proper order 3:
\[
\left\{ \begin{array}{l}
\nabla_{\gamma} \gamma = \kappa_1 V_2, \\
\nabla_{\gamma} V_2 = -\kappa_1 \gamma + \kappa_2 V_3, \\
\nabla_{\gamma} V_3 = -\kappa_2 V_2.
\end{array} \right.
\]

We call such Killing helices as Killing ordinary helices. It is well-known that on a Euclidean 3-space \( \mathbb{R}^3 \) every ordinary helix is an open unbounded curve without self-intersection points. Needless to say every Killing ordinary helix on an arbitrary Riemannian manifold does not have self-intersection points.

We say two ordinary helices \( \gamma_1, \gamma_2 \) on a Riemannian manifold \( M \) are congruent if there exist an isometry \( \varphi \) of \( M \) and constant \( s_0 \) satisfying \( \gamma_2 (s) = \varphi \circ \gamma_1 (s+s_0) \) for every \( s \). It is known that on a real space form, which is either a standard sphere \( S^n \), a Euclidean space \( \mathbb{R}^n \) or a real hyperbolic space \( H^n \), all helices are Killing and they are congruent each other if and only if they have the same curvatures. However on other symmetric spaces of rank one it is not true. We introduce the notion of "torsion" to study ordinary helices on symmetric spaces of rank one and investigate some geometric properties on them.

2. Killing ordinary helices on nonflat complex space forms
For a helix \( \gamma \) of proper order \( d \) on an \( n \)-dimensional Kähler manifold \( M \) with complex structure \( J \) and Riemannian metric \( \langle \cdot, \cdot \rangle \), we define functions \( \tau_{ij} (s) = \langle V_i (s), JV_j (s) \rangle \) for \( 1 \leq i < j \leq d \) by using its Frenet frame \( \{ V_1 = \gamma, V_2, \ldots, V_d \} \) and call them its complex torsions. In this section we study Killing ordinary helices on a nonflat complex space form, which is either a complex projective space or a complex hyperbolic space. In the study of helices in a nonflat complex space form their complex torsions and
curvatures play an important role. Applying othonormalization of Gram-Schmidt to \( \{ V_1(s), JV_1(s), \ldots, V_d(s), JV_d(s) \} \) we take \( \mathbb{C} \)-independent orthonormal vectors \( \{ V_1(s) = V_1(s), V_2(s), \ldots, V_r(s), JV_r(s) \} \) \( (1 \leq r(s) \leq d) \). On a complex space form \( M^n(c; \mathbb{C}) \), for given two \( \mathbb{C} \)-dependent orthonormal systems \( \{ v_1, \ldots, v_r \} \) of \( T_xM^n(c; \mathbb{C}) \) and \( \{ w_1, \ldots, w_r \} \) of \( T_yM^n(c; \mathbb{C}) \), we have a holomorphic isometry \( \varphi_+ \) and an anti-holomorphic isometry \( \varphi_- \) satisfying \( \varphi_+(x) = y \) and \( (d\varphi_{\pm})_x(v_i) = w_i, \ i = 1, \ldots, r \). We hence obtain the following:

**Theorem 2.1.**

(1) A helix on a nonflat complex space form is Killing if and only if all of its complex torsions are constant functions.

(2) Two Killing helices \( \gamma_1, \gamma_2 \) on a nonflat complex space form are congruent if and only if their curvatures \( \kappa \) satisfy \( \kappa_1^{(1)} = \kappa_2^{(2)} \) for every \( i \) and complex torsions \( \tau_{ij}^{(\ell)} \) satisfy either \( \tau_{ij}^{(1)} = \tau_{ij}^{(2)} \) or \( \tau_{ij}^{(1)} = -\tau_{ij}^{(2)} \) for every \( i < j \).

On an arbitrary Kähler manifold \( M \) circles necessarily have constant complex torsion because \( \tau_{12} = \langle \nabla_1 V_1, JV_2 \rangle + \langle V_1, J\nabla_2 V_2 \rangle = 0 \). In particular, on a complex space form every circles are Killing. But the situation is not the same for ordinary helices. We call a helix on a Kähler manifold holomorphic if all its complex torsions are constant functions. Since complex torsions for an ordinary helix on a Kähler manifold satisfy the system of ordinary differential equations:

\[
\begin{align*}
\tau_{12}' &= \kappa_2 \tau_{13}, \\
\tau_{13}' &= -\kappa_2 \tau_{12} + \kappa_1 \tau_{23}, \\
\tau_{23}' &= -\kappa_1 \tau_{13},
\end{align*}
\]

derived from (1.1), we find they are constant if and only if \( \tau_{13}(s_0) = 0 \) and \( \kappa_1 \tau_{23}(s_0) = \kappa_2 \tau_{12}(s_0) \) hold at some point \( s_0 \). Thus, to study ordinary helices of curvature \( \kappa_1, \kappa_2 \) with constant complex torsions we are enough to study initial frames \( (v_1, v_2, v_3) \) satisfying

\[
\kappa_1 \langle v_2, JV_3 \rangle = \kappa_2 \langle v_1, JV_2 \rangle, \quad \langle v_1, JV_3 \rangle = 0.
\]

Since \( \langle v_1, v_3 \rangle = \langle v_1, JV_3 \rangle = 0 \) we see \( \langle v_1, JV_2 \rangle^2 + \langle v_2, JV_3 \rangle^2 \leq \|v_2\|^2 = 1 \). On the other hand, for given real numbers \( a_1, a_2 \) with \( a_1^2 + a_2^2 \leq 1 \), we find
the vectors
\[ v_1 = (1, 0, \ldots, 0), \]
\[ v_2 = (-ia_1, \sqrt{1-a_1^2}, 0, \ldots, 0), \]
\[ v_3 = (0, -ia_2/\sqrt{1-a_2^2}, \sqrt{1-a_1^2-a_2^2}/\sqrt{1-a_1^2}, 0, \ldots, 0) \]
under an identification of the tangent space \( T_x M \) with \( \mathbb{C}^n \) satisfies
\[ \langle v_1, Jv_2 \rangle = a_1, \quad \langle v_1, Jv_3 \rangle = 0, \quad \langle v_2, Jv_3 \rangle = a_2. \]
We hence establish the following ([7]):

**Proposition 2.1.** On a Kähler manifold \( M \) of complex dimension greater than 2 the following hold:
1. Every holomorphic ordinary helix on \( M \) satisfies \( \kappa_1 \tau_{23} = \kappa_2 \tau_{12}, \tau_{13} = 0 \) and \( |\tau_{12}| \leq \kappa_1/\sqrt{\kappa_1^2 + \kappa_2^2} \).
2. Conversely, for given positive constants \( k_1, k_2 \) and a constant \( \tau \) with \( |\tau| \leq k_1/\sqrt{k_1^2 + k_2^2} \), there exists a holomorphic ordinary helix on \( M \) with curvatures \( \kappa_i = k_i \) \( (i = 1, 2) \) and complex torsion \( \tau_{12} = \tau \).
3. When \( |\tau| > k_1/\sqrt{k_1^2 + k_2^2} \), we have no such a holomorphic ordinary helix on \( M \).

**Proposition 2.2.** For ordinary helices on a Kähler surface \( M \) the following hold:
1. Each holomorphic ordinary helix on \( M^2(c; \mathbb{C}) \) satisfies either
\[ \tau_{12} = \kappa_1/\sqrt{\kappa_1^2 + \kappa_2^2}, \quad \tau_{13} = 0, \quad \tau_{23} = \kappa_2/\sqrt{\kappa_1^2 + \kappa_2^2} \quad \text{(2.2)} \]
or
\[ \tau_{12} = -\kappa_1/\sqrt{\kappa_1^2 + \kappa_2^2}, \quad \tau_{13} = 0, \quad \tau_{23} = -\kappa_2/\sqrt{\kappa_1^2 + \kappa_2^2}. \quad \text{(2.3)} \]
2. Conversely, for given positive constants \( k_1, k_2 \) there exist a holomorphic ordinary helix on \( M \) with curvatures \( \kappa_1, \kappa_2 \) satisfying (2.2) and a Killing ordinary helix on \( M \) with curvatures \( \kappa_1, \kappa_2 \) satisfying (2.3).

Coming back to the case of a nonflat complex space form \( M^n(c; \mathbb{C}) \), we have by these propositions and Theorem 2.1 the following:

**Corollary 2.1.**
1. On nonflat \( M^n(c; \mathbb{C}) \), \( n \geq 3 \) there exists a unique Killing ordinary helix up to isometries with given curvatures \( \kappa_1, \kappa_2 \) and given complex torsion \( \tau_{12} \) if \( |\tau_{12}| \leq \kappa_1/\sqrt{\kappa_1^2 + \kappa_2^2} \).
2. On nonflat \( M^2(c; \mathbb{C}) \) there exists a unique Killing ordinary helix up to isometries with given curvatures \( \kappa_1, \kappa_2 \).
We denote by $\mathcal{M}_d(M)$ the set of all congruence classes of Killing helices of proper order $d$. For a complex space form $M^n(c; \mathbb{C})$, it is clear that the moduli space $\mathcal{M}_1(M^n(c; \mathbb{C}))$ of geodesics consists of one point, and the moduli space $\mathcal{M}_2(M^n(c; \mathbb{C}))$ of circles is bijective to a band $(0, \infty) \times [0, 1]$. Corollary 2.1 shows the following:

**Corollary 2.2.** The moduli space $\mathcal{M}_3(M^n(c; \mathbb{C}))$ of Killing ordinary helices on a nonflat complex space form is bijective to the following set:

$$\left\{ (\kappa_1, \kappa_2, \tau) \mid \kappa_i > 0, i = 1, 2, \ 0 \leq \tau \leq \kappa_1/\sqrt{\kappa_1^2 + \kappa_2^2} \right\}$$

$$\subset (0, \infty) \times (0, \infty) \times [0, 1], \quad n \geq 3,$$

$$\quad (0, \infty) \times (0, \infty), \quad n = 2.$$

3. Bounded Killing ordinary helices on a complex hyperbolic plane

It is interesting to study some geometric properties of Killing helices. For example we have the following problems:

1) Under what conditions are they closed?

2) On a noncompact manifold, under what conditions are they unbounded?

For circles on a complex space form we already gave complete answers in [5] and [2]. But for Killing ordinary helices only the following is known.

**Theorem 3.1.** On a complex hyperbolic plane $\mathbb{C}H^2(c)$ of constant holomorphic sectional curvature $c$, a Killing ordinary helix is bounded if and only if its curvatures $\kappa_1, \kappa_2$ satisfy one of the following conditions:

i) $\kappa_2 < \sqrt{3|m|}/6,$

ii) $\kappa_2 = |c|/4$ and $\kappa_1 > 5\sqrt{2|m|}/8,$

iii) $0 < \kappa_2 < \sqrt{|c|}/4$ and

$$\kappa_1^2 > \frac{32\kappa_2^4 - 20|c|\kappa_2^2 + c^2 + \sqrt{|c|(|c| - 12\kappa_2^2)^{3/2}}}{2(|c| - 16\kappa_2^2)},$$

iv) $\sqrt{|c|}/4 < \kappa_2 < \sqrt{3|m|}/6$ and

$$\frac{-32\kappa_2^4 + 20|c|\kappa_2^2 - c^2 - \sqrt{|c|(|c| - 12\kappa_2^2)^{3/2}}}{2(16\kappa_2^2 - |c|)} < \kappa_1^2$$

$$< \frac{-32\kappa_2^4 + 20|c|\kappa_2^2 - c^2 - \sqrt{|c|(|c| - 12\kappa_2^2)^{3/2}}}{2(16\kappa_2^2 - |c|)}.$$

In particular, a Killing ordinary helix on $\mathbb{C}H^2(c)$ is not bounded if $\kappa_2^2 \geq |c|/12$ or $\kappa_1^2 + \kappa_2^2 \leq 3|m|/4.$
In order to attack this problem we use the fibration $H^5_1 \rightarrow \mathbb{C}H^2$ of anti-de Sitter space. When $c = -4$, by a standard inclusion $H^5_1 = \{(z_0, z_1, z_2) \in \mathbb{C}^3 \mid -|z_0|^2 + |z_1|^2 + |z_2|^2 = -1\} \subset \mathbb{C}^3$ we see Riemannian connections $\nabla$ on $\mathbb{C}H^2(-4)$ and $\nabla$ on $\mathbb{C}^3$ satisfy the relationship

$$\nabla_X Y = \nabla_X Y + \langle X, Y \rangle N - \langle X, JY \rangle JN,$$

for tangent vector fields $X, Y$ on $\mathbb{C}H^n(-4)$. Here, $N$ is a unit normal vector field of $H^5_1$ in $\mathbb{C}^3$, and on $\mathbb{C}^3$ the vector fields are induced by horizontal lift of $X, Y$. Thus a horizontal lift $\tilde{\gamma}$ of an ordinary helix of curvature $\kappa_1, \kappa_2$ and the complex torsion $\tau_{12}$ satisfy the ordinary differential equation

$$\tilde{\gamma}'(4) + (\kappa_1^2 + \kappa_2^2 - 1)\tilde{\gamma}'' + \sqrt{-1}\kappa_1 \tau_{12} \tilde{\gamma}' - \kappa_2 \tilde{\gamma} = 0$$

(3.1)

if it is regarded as a curve on $\mathbb{C}^3$. As we are enough to study the case $\tau_{12} = \kappa_1/\sqrt{\kappa_1^2 + \kappa_2^2}$, we find the characteristic polynomial for (3.1) turns to

$$\lambda^2 + (\kappa_1^2 + \kappa_2^2 - 1)\lambda^2 + \sqrt{-1}\kappa_1 \tau_{12} \lambda - \kappa_2^2$$

$$= \left(\lambda - \sqrt{-(\kappa_1^2 + \kappa_2^2)}\right)\left(\lambda^3 + \sqrt{-(\kappa_1^2 + \kappa_2^2)} \lambda^2 - \lambda - \frac{-1\kappa_2}{\sqrt{\kappa_1^2 + \kappa_2^2}}\right).$$

Since $\gamma$ is bounded if and only if this polynomial has 4 different pure imaginary zeros, we get our result by elementary argument.

On $\mathcal{M}_3(\mathbb{C}H^2)$, which is bijective to the set $(0, \infty) \times (0, \infty)$ of pairs of curvatures, we consider a topology induced from canonical topology on $\mathbb{R}^2$. Theorem 3.1 shows the following:

**Proposition 3.1.** The set of all congruence classes of bounded Killing ordinary helices is an open subset of $\mathcal{M}_3(\mathbb{C}H^2)$, and its closure has a unique cusp at a point $(5\sqrt{2}\kappa/8, \sqrt{\kappa}/4)$ with respect to the identification.

**Corollary 3.1.** On $\mathbb{C}H^n$ ($n \geq 2$), there exist infinitely many bounded Killing ordinary helices and infinitely many unbounded Killing ordinary helices.

4. Killing ordinary helices on other symmetric spaces of rank one

We here make mention of Killing ordinary helices on nonflat quaternionic space form, on a Cayley plane and on a Cayley hyperbolic plane. In [1], the first author introduced structure torsion fields for helices on a quaternionic Kähler manifold, on a Cayley plane and on a Cayley hyperbolic plane.
The definition of structure torsion fields is quite similar to that of complex torsions for helices on a Kähler manifold. For example, for a helix $\gamma$ with Frenet frame $\{V_1, \ldots, V_d\}$ on a quaternionic Kähler manifold $M$, structure torsion field $\tau_{ij}$ is locally give by

$$\tau_{ij} = \langle V_i, J_1 V_j \rangle J_1 E + \langle V_i, J_2 V_j \rangle J_2 E + \langle V_i, J_3 V_j \rangle J_3 E$$

with a local basis $\{J_1, J_2, J_3\}$ of quaternionic Kähler structure on $M$ and a parallel vector field $E$ along $\gamma$ (see [1] for more detail).

We denote by $K$ either the field of quaternionic numbers $\mathbb{H}$ or the Cayley algebra $\mathbb{O}$. We shall say that a Riemannian manifold is a nonflat $K$-space form if it is a quaternionic projective space or a quaternionic hyperbolic space when $K = \mathbb{H}$ and is a Cayley plane or a Cayley hyperbolic plane when $K = \mathbb{O}$. We denote a $K$-space form by $M^n(c; K)$. Corresponding to Proposition 2.1, we see the following properties on helices on a nonflat $K$-space form.

1. A helix is Killing if and only if all of its structure torsion fields are parallel.

2. Two Killing helices $\gamma_1, \gamma_2$ are congruent if and only if their curvatures $\kappa_i^{(1)}$ satisfy $\kappa_i^{(1)} = \kappa_i^{(2)}$ for every $i$ and structure torsion fields $\tau_{ij}^{(2)}$ satisfy that $\{\tau_{ij}^{(1)}(s_0)\}$ and $\{\tau_{ij}^{(2)}(0)\}$ are isomorphic for some $s_0$.

Just like Killing ordinary helices on a Kähler manifold, for Killing ordinary helices on a $K$-space form, their structure torsion fields are determined only by $\tau_{12}$. Thus we find the moduli space $M_3(M^n(c; K))$ of Killing ordinary helices on a nonflat $M^n(c; K)$ is bijective to the following sets:

$$\left\{ \begin{array}{l}
\{ (\kappa_1, \kappa_2, \tau) \mid \kappa_i > 0, i = 1, 2, 0 \leq \tau \leq \kappa_1 / \sqrt{\kappa_1^2 + \kappa_2^2} \} \\
\quad \subset (0, \infty) \times (0, \infty) \times [0, 1], \\
\quad (0, \infty) \times (0, \infty), \\
\end{array} \right\}
$$

Here $\tau$ corresponds to the structure torsion, which is given as the norm $\|\tau_{12}(0)\| = \|\text{Proj}_{V_j(0)}(V_i(0))\|$ of structure torsion field $\tau_{12}$ at $s = 0$, where $\text{Proj}_{V_j(0)}$ denotes the projection of the tangent space onto the $K$-subspace spanned by $V_j(0)$.

As a consequence we find the following:

**Proposition 4.1.** Let $M$ be a Riemannian symmetric space of rank one which is not a real space form, and $(\kappa_1, \kappa_2)$ be a pair of positive numbers. On $M$ there exist infinitely many congruency classes of ordinary helices which are not Killing ordinary helices whose curvatures are $\kappa_1$, $\kappa_2$. 
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We also get that all the results in section 3 hold for Killing ordinary helices on a quaternionic hyperbolic space and on a Cayley hyperbolic plane.

5. Characterization of real space forms by Killing ordinary helices

In this section we study homogeneous Riemannian manifolds by use of Killing helices. For a Riemannian homogeneous manifold $M = G/K$ with isometry group $G$ and the compact isotropy subgroup $K$, we denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be an $Ad(K)$-invariant decomposition of $\mathfrak{g}$. By using the map $U : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ given by the equality

$$
\langle U(X, Y), Z \rangle = \frac{1}{2} \{ \langle [Z, X], Y \rangle + \langle [Z, Y], X \rangle \}
$$

for every $X, Y, Z \in \mathfrak{p}$, we define $\Lambda_p(X) \in \mathfrak{so}(\mathfrak{p})$ by $\Lambda_p(X)(Y) = \frac{1}{2} [X, Y]_\mathfrak{p} + U(X, Y)$ for $Y \in \mathfrak{p}$. We decompose $X \in \mathfrak{g}$ as $X = X_\mathfrak{k} + X_\mathfrak{p}$ with $X_\mathfrak{k} \in \mathfrak{k}$ and $X_\mathfrak{p} \in \mathfrak{p}$, and define a linear map $\Lambda : \mathfrak{g} \to \mathfrak{so}(\mathfrak{p})$ by $\Lambda(X) = ad_{X_\mathfrak{k}} + \Lambda_p(X_\mathfrak{p})$.

We denote the projection by $\varpi : G \to G/K$. For a pair $(X, Y)$ or a triplet $(X, Y, Z)$ of orthonormal vectors in $\mathfrak{p}$ and $H \in \mathfrak{k}$ we consider an orbit $\gamma(t) = \varpi(exp t(H + X)).$

(1) $\gamma$ is a circle on $G/K$ of curvature $\kappa$ with initial frame $(X, Y)$ if and only if the following hold:

$$
\begin{align*}
\Lambda(H + X)(X) &= [H, X] + \Lambda_p(X)(X) = \kappa Y, \\
\Lambda(H + X)(Y) &= [H, Y] + \Lambda_p(X)(Y) = -\kappa X.
\end{align*}
$$

(2) $\gamma$ is an ordinary helix on $G/K$ of curvatures $\kappa_1$, $\kappa_2$ with initial frame $(X, Y, Z)$ if and only if the following equalities hold:

$$
\begin{align*}
\Lambda(H + X)(X) &= [H, X] + \Lambda_p(X)(X) = \kappa Y, \\
\Lambda(H + X)(Y) &= [H, Y] + \Lambda_p(X)(Y) = -\kappa X + \tau Z, \\
\Lambda(H + X)(Z) &= [H, Z] + \Lambda_p(X)(Z) = -\tau Y.
\end{align*}
$$

For a pair $(X, Y)$ or a triplet $(X, Y, Z)$ of orthonormal vectors in $\mathfrak{p}$, if we have Killing circles of initial frame $(X, Y)$ and $(X, -Y)$ or Killing ordinary helices of initial frame $(X, Y, Z)$ and $(X, -Y, Z)$, then we find there is $H \in \mathfrak{k}$ with $[H, X] = Y$. When such a property holds for all the pair or all the triplet, this leads us to $Ad(K)$ acts transitively on the unit sphere $S(\mathfrak{p})$ in $\mathfrak{p}$. Thus we obtain the following with Propositions 2.1, 2.2 and 4.1.
Theorem 5.1. Let $M$ be a Riemannian homogeneous manifold.

(1) If there exists positive $\kappa$ such that every circle of curvature $\kappa$ on $M$ is Killing, then $M$ is either a Euclidean space or a Riemannian globally symmetric space of rank one.

(2) If there exist positive $\kappa_1, \kappa_2$ such that every ordinary helix of curvatures $\kappa, \kappa_2$ on $M$ is a Killing ordinary helix, then $M$ is a real space form.

6. A construction of a closed non-Killing ordinary helix on $\mathbb{CP}^2(c)$

As we see in previous sections there exists infinitely many non-Killing ordinary helices on symmetric spaces of rank one which are neither spheres nor real hyperboloid spaces. But only few results are known on geometries of such non-Killing ordinary helices. In this section, we give an example of non-Killing ordinary helices on a complex projective plane.

We consider a quotient of a 2-dimensional flat torus $N = (S^1 \times S^1)/\phi$. Here the identification $\phi$ is given by $\phi((e^{i\theta}, (a_1, a_2))) = (-e^{i\theta}, (-a_1, -a_2))$ by representing $S^1 \times S^1$ as 

$$\{ z \in \mathbb{C} \mid |z| = 1 \} \times \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1^2 + a_2^2 = 1 \}.$$ 

We define an embedding $f : N \to \mathbb{CP}^2$ by

$$f([e^{i\theta}, (a_1, a_2)]) = \varpi \left( \frac{1}{3} (e^{-\frac{2i\theta}{3}} + 2a_1 e^{\frac{i\theta}{3}}), \frac{\sqrt{2}}{3} (e^{-\frac{2i\theta}{3}} - a_1 e^{\frac{i\theta}{3}}), \frac{2}{\sqrt{6}} a_2 e^{\frac{i\theta}{3}} \right),$$

where $\varpi : S^5 \to \mathbb{CP}^2$ is the Hopf fibration. If we define a Riemannian metric on $S^1 \times S^1$ as $(u_1, u_2), (v_1, v_2)) = \frac{8}{\sqrt{6}} \langle u_1, v_1 \rangle_{S^1} + \frac{8}{\sqrt{6}} \langle u_2, v_2 \rangle_{S^1}$ by use of the canonical metric $\langle \cdot, \cdot \rangle_{S^1}$ on $S^1$, then $f : N \to \mathbb{CP}^2(c)$ turns to be a parallel isometric embedding with respect to the induced metric on $N$.

By direct computation with the aid of Gauss formula we find for a circle of curvature $\sqrt{c}/4$ on $N$ the curve $f \circ \gamma$ is a helix of curvatures $\sqrt{3c}/4$ and $\sqrt{6c}/4$ on $\mathbb{CP}^2(c)$. The complex torsions of $f \circ \gamma$ are described as

$$\tau_{12}(s) = \sqrt{\frac{2}{3}} \cos 3 \left( \frac{1}{2} s + \psi_0 \right), \quad \tau_{13}(s) = -\sin 3 \left( \frac{1}{2} s + \psi_0 \right),$$

$$\tau_{23}(s) = -\frac{1}{\sqrt{3}} \cos 3 \left( \frac{1}{2} s + \psi_0 \right),$$

where $\psi_0$ is the angle between $\dot{\gamma}(0)$ and the vector tangent to the first component of $N$. As the complex torsions of $f \circ \gamma$ are not constants, we find it is not Killing. For the universal Riemannian covering $p : \mathbb{R}^2 \to N$, we can choose a fundamental region for $N$ in $\mathbb{R}^2$ as
\[ \mathcal{F} = [0, 4\sqrt{2}\pi/(3\sqrt{c})] \times [0, 2\sqrt{6}\pi/(3\sqrt{c})]. \] Two points \((x_1, x_2)\) and \((y_1, y_2)\) on \(\mathbb{R}^2\) satisfy \(p((x_1, x_2)) = p((y_1, y_2))\) if and only if they satisfy one of the following conditions:

\[
\begin{align*}
\text{i)} \quad & x_1 - y_1 = 4\sqrt{2}m_1\pi/(3\sqrt{c}), \\
& x_2 - y_2 = 4\sqrt{6}m_2\pi/(3\sqrt{c}), \\
\text{ii)} \quad & x_1 - y_1 = 2\sqrt{2}(2m_1 + 1)\pi/(3\sqrt{c}), \\
& x_2 - y_2 = 2\sqrt{6}(2m_2 + 1)\pi/(3\sqrt{c}),
\end{align*}
\]

for some \(m_1, m_2 \in \mathbb{Z}\).

Since a lift of a circle of curvature \(\sqrt{c}/4\) on \(N\) onto \(\mathbb{R}^2\) is a circle of radius \(4/\sqrt{c}\) in the sense of Euclidean geometry, we obtain the following (see [6] for more detail):

**Proposition 6.1.** For a circle of curvature \(\sqrt{c}/4\) on \(N\) the curve \(f \circ \gamma\) given by use of isometric embedding \(f\) is a non-Killing ordinary helix of curvatures \(\sqrt{3c}/4\) and \(\sqrt{6c}/4\) on \(\mathbb{C}P^2(c)\). It is a closed curve of length \(8\pi/\sqrt{c}\) and has 6 self-intersection points.

In connection with submanifolds we have the following Killing ordinary helix. Let \(g : \mathbb{C}P^1(c/2) \cong S^2(c/2) \rightarrow \mathbb{C}P^2(c)\) denote the second Veronese embedding given by

\[ g([z_0, z_1]) = [z_0^2, \sqrt{2}z_0z_1, z_1^2] \]

with homogeneous coordinates. For a circle of curvature \(\sqrt{2c}/4\) on \(\mathbb{C}P^1(c/2)\), the curve \(g \circ \gamma\) is a Killing ordinary helix of curvatures \(\sqrt{6c}/4\) and \(\sqrt{3c}/2\). This ordinary helix is closed with length \(4\sqrt{2}/(5c)\pi\) and of course does not have self-intersection points (see [3]).

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REAL ANALYTICITY OF THE ALMOST KÄHLER MANIFOLDS

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The result in [KL] and a canonical construction will be used to obtain the real analyticity of the almost Kähler manifolds.

Keywords: almost complex manifold, real analytic manifold.

1. Introduction and some definitions

Let \( M \) be a \( C^\infty \)-smooth paracompact \( 2n \)-dimensional manifold and \( \omega \) be a closed nondegenerate differential two-form on \( M \). Then the couple \( (M, \omega) \) is called a symplectic manifold and the two-form \( \omega \) is called a symplectic form on the manifold \( M \). Let \( J \) be an antinvolutive automorphism of the tangent bundle on \( M \). Then for every point \( p \in M \) the restriction \( J_p \) of \( J \) on the fibre \( T_p M \) acts as an antinvolutive automorphism too, i.e. \( J_p : T_p M \to T_p M \) and \( J^2 = -Id \). The couple \( (M, J) \) is called an almost complex manifold and \( J \) is called an almost complex structure on \( M \). Let \( g \) be a riemannian metric on \( M \) and let \( h \) be defined by the equality \( h(X, Y) = 1/2 (g(X, JY) + g(JX, Y)) \) for each two vector fields \( X, Y \) defined on an open set in \( M \). Then \( h \) is called an almost hermitian metric on the almost complex manifold \( (M, J) \) and \( (M, J, h) \) is called an almost hermitian manifold. The two-form \( \Omega \) on an almost hermitian manifold \( (M, J, h) \) defined by the equality \( \Omega(X, Y) = h(X, JY) \) where \( X, Y \) are vector fields defined on an open set in \( M \) is called a fundamental form of

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the almost hermitian manifold. An almost hermitian manifold is called an almost Kähler manifold, if the fundamental form \( \Omega \) is a closed two-form, i.e. if \( d\Omega = 0 \). Then \( \Omega \) will be a symplectic form on \( M \) and \( (M, \Omega) \) will be a symplectic manifold too. So each almost Kähler manifold is a symplectic one.

Let us note that every almost complex manifold \( (M, J) \) and every almost symplectic manifold \( (M, \omega) \) is an even dimensional orientable smooth differentiable manifold. Also the nondegenerate \( 2n \)-form \( \omega^n \) coincide with the volume form of the differentiable manifold \( M \) up to multiplication with a constant.

**Definition 1.** We say that an almost complex manifold \( (M, J) \) is \( C^\infty \)-equivalent to an almost complex manifold \( (\tilde{M}, \tilde{J}) \) if there exists a diffeomorphism \( \varphi : M \to \tilde{M} \) which differential \( d\varphi \) commute with the automorphisms \( J^* \) on \( T^*M \) and \( \tilde{J}^* \) on \( T^*\tilde{M} \) respectively, i.e. if the diagram

\[
\begin{array}{ccc}
J^*(T^*M) & \longrightarrow & T^*M \\
\downarrow d\varphi & & \downarrow d\varphi \\
\tilde{J}^*(T^*\tilde{M}) & \longrightarrow & T^*\tilde{M}
\end{array}
\]

is commutative.

Here the automorphism \( J^* \) is defined on \( T^*M \) by the commutative diagram

\[
\begin{array}{ccc}
J_p(T_pM) & \longrightarrow & T_pM \\
\downarrow & & \downarrow \\
J^*_p(T^*_pM) & \longrightarrow & T^*_pM
\end{array}
\]

where \( J_p \) is the restriction of the given almost complex structure on the fiber \( T_pM \) and the two downarrow mappings are the same mapping of the natural duality between the fibers over \( p \) of the tangent and the cotangent bundles on \( M \).

**Definition 2.** We say that the almost hermitian manifold \( (M, J, h) \) is \( C^\infty \)-equivalent to the almost hermitian manifold \( (\tilde{M}, \tilde{J}, \tilde{h}) \) if the almost complex manifolds \( (M, J) \) and \( (\tilde{M}, \tilde{J}) \) are \( C^\infty \)-equivalent and if the jacobian \( \varphi^* \) maps the almost hermitian metric \( h \) on the almost hermitian metric \( \tilde{h} \), where \( \varphi \) is the diffeomorphism from definition 1.
2. Canonical construction

We consider now how to obtain canonically an hermitian metric $\tilde{h}$ and an almost complex structure $J$ on a manifold $\tilde{M}$, diffeomorphic to a given symplectic manifold $(M, w)$ in such a way that the symplectic form $\omega$ correspond to the fundamental form $\tilde{\Omega}$ of the almost hermitian manifold $(M, J, \tilde{h})$. Moreover, so obtained almost hermitian manifold will be an almost Kähler one as its fundamental form is a closed two form.

We follow the construction of a polarization of a symplectic form, given in [Bl], §4.2. The base of considerations is the procedure of polarization of a non-singular matrix $A$ into product of orthogonal matrix $J$ and a positive definite symmetric matrix $G$. Let us denote as usual by $O(n)$ the group of orthogonal $n \times n$ matrix and by $H(n)$ the group of positive definite symmetric $n \times n$ matrix.

Let us consider the map $\varphi : O(n) \times H(n) \to GL(n)$ defined by $\varphi(F, G) = FG$ — the product of the matrices $F$ and $G$. If $O(n)$, $H(n)$ and $GL(n)$ are equipped with their natural real analytic structures, the map $\varphi$ is a real analytic morphism of $O(n) \times H(n)$ in $GL(n)$. To prove that $\varphi$ is a one-to-one mapping, for an element $X \in T_{(F,G)}O(n) \times H(n)$ we consider the curve $(Fe^{tA}, G + tB)$ in $O(n) \times H(n)$, where $A$ is skew-symmetric and $B$ is symmetric matrix and such that for $t = 0$ the tangent vector to the curve at $(F, G)$ to be $X$. Then

$$d\varphi(X) = \lim_{t \to 0} \frac{Fe^{tA}(G + tB) - FG}{t} = \lim_{t \to 0} \left(Fe^{tA} - \frac{E}{t} G + Fe^{tA}B\right) = FAG + FG.$$  

So if $d\varphi(X) = 0 \implies F(AG + B) = 0$ and $AG + B = 0$. Then the symmetric matrix $-B = AG$ is presented as a product of the skew-symmetric matrix $A$ and the positive definite symmetric matrix $G$. This implies that $A = 0$, as we will see below and consequently, $B = 0$.

To see that a skew-symmetric matrix $A$ such that $AG$ is symmetric for some matrix $G \in H(n)$ is equal to zero matrix, let us consider first the mapping $\sigma_G : gl(n, \mathbb{R}) \to gl(n, \mathbb{R})$ defined by the equality $\sigma_G(A) = GAG^{-1}$ for some positive definite symmetric matrix $G \in H(n)$. If we denote by $\lambda_l$, $l = 1, \ldots, n$ the eigenvectors of the matrix $G$, they will be positive numbers and also it will exist an orthogonal matrix $P \in O(n)$ such that $\Delta = PGP^{-1}$ to be a diagonal matrix. Then as is easily to see, $\sigma_G = \sigma_P^{-1} \circ \sigma_\Delta \circ \sigma_P$ and consequently $\sigma_\Delta$ and $\sigma_G$ will have the same eigenvalues. But it is easy to
see the eigenvalues of the matrix

$$\sigma_\Delta(A)_{i\ell} = (\Delta A \Delta^{-1})_{i\ell} = \sum_{j,k} \lambda_i \delta_{ij} a_{jk} \delta_{kl} \frac{1}{\lambda_l} = \frac{\lambda_i}{\lambda_l} a_{i\ell}$$

from where follows that the matrix $\sigma_G(A)$ has $n^2$ positive numbers $\frac{\lambda_i}{\lambda_l}$ as eigenvalues.

Now $AG$ symmetric means that $AG = G^t A^t$, but $A$ anti-symmetric means that $A^t = -A$, so $AG = -G^t A$. Therefore $\sigma_G(A) = GAG^{-1} = G^t AG^{-1}$ and then from the note for eigenvalues of $\sigma_G(A)$ follows that $A = 0$.

The so written decomposition of a non-singular matrix is proved by C. Chevalley in [Ch, 1946, p. 14-16] as a continuous mapping and by Y. Hatekayama [Ht, 19621 as an analytic mapping.

We shall use the following two theorems:

**Theorem A.** (theorem for real analyticity of the mapping $\varphi^{-1}$, Theorem 4.2 from [Bl]) Polarization as a map from $GL(n, R) \to O(n) \times H(n)$ gives an analytic diffeomorphism between these manifolds with respect to the usual analytic structures.

**Theorem B.** (theorem for existence, Theorem 4.3 from [Bl]) Let $(M^{2n}, \omega)$ be a symplectic manifold. Then there exists a Riemannian metric $g$ and an almost complex structure $J$ such that

$$g(X, JY) = \omega(X, Y).$$

In order to prove this theorem it would be chosen any Riemannian metric $k$ on $M$ and a local $k$-orthonormal basis $\{X_1, \ldots, X_{2n}\}$. Then to consider the matrix $A$, where $A_{ij} = \omega(X_i, X_j)$. The matrix $A$ will be a non-singular skew-symmetric $2n \times 2n$ matrix for each point $p$ of an open set $U$ where the vector fields $\{X_1, \ldots, X_{2n}\}$ are defined. This is so because the two-form $\omega$ as a symplectic form is a nondegenerate differential two-form. For the matrix $A$ we construct a polarization, $A = FG$ with some orthogonal matrix $F$ and some positive definite symmetric matrix $G \in H(n)$. Then the metric $g$ is defined locally on $U$ by the equalities $g(X_i, X_j) = H_{ij}$ and the almost complex structure $J$ is defined locally by the equalities $JX_i = F^j_i X_j$. The rest of the proof of the theorem is to check the gluing of the so defined metric and almost complex structure on the common definition set of any two $k$-orthogonal basis $\{X_1, \ldots, X_{2n}\}$ and $\{Y_1, \ldots, Y_{2n}\}$ which is so because of the construction made above and because of the uniqueness of the polar decomposition.
We remark also that if \((J, h)\) is a starting structure for the polarization of the fundamental form \(\omega(X, Y) = h(X, JY)\), then the result of polarization is the same metric \(h\) and almost complex structure \(J\).

### 3. Real analyticity of the almost Kähler manifolds

The following main theorem will be proved:

**Theorem 1.** Let \((M, J, h)\) be a \(C^\infty\)-smooth almost Kähler manifold. Then there exists a real analytic manifold \(\tilde{M}\), a real analytic almost complex structure \(\tilde{J}\) and a real analytic almost hermitian metric \(\tilde{h}\) on \(\tilde{M}\) such that \((\tilde{M}, \tilde{J}, \tilde{h})\) is \(C^\infty\)-equivalent to \((M, J, h)\). Moreover, the almost hermitian manifold \((\tilde{M}, \tilde{J}, \tilde{h})\) can be chosen to be an almost Kähler manifold. Such a manifold \((\tilde{M}, \tilde{J}, \tilde{h})\) is unique up to a real analytic isomorphism.

**Remark.** More precisely, the uniqueness of the real analytic almost hermitian manifold \((\tilde{M}, \tilde{J}, \tilde{h})\), \(C^\infty\)-equivalent to the given one \((M, J, h)\), means that if for the almost hermitian manifold \((M, J, h)\) exists another real analytic almost hermitian manifold \((\tilde{M}, \tilde{J}, \tilde{h})\), \(C^\infty\)-equivalent to \((M, J)\), then these real analytic almost complex manifolds \((\tilde{M}, \tilde{J})\) and \((\tilde{M}, \tilde{J})\) are real analytically isomorphic, i.e. there exists a real analytic isomorphism \(\psi : \tilde{M} \to \tilde{M}\) with differential \(d\psi\) commuting with the authomorphisms \(\tilde{J}^*\) on \(T^*\tilde{M}\) and \(\tilde{J}^*\) on \(T^*\tilde{M}\). Also the metrics \(h\) and \(\tilde{h}\) correspond each other by the the isomorphism \(\psi\).

**Proof of Theorem 1.** We recall the main result in [KL]. This is the theorem that each symplectic manifold \(M\) is \(C^\infty\) isomorphic to a real analytic symplectic manifold, unique up to real analytic isomorphism.

**Existence.** Let us consider the fundamental two-form \(\omega\) of the given almost Kähler manifold \((M, J, h)\), \(\omega(X, Y) := h(X, JY)\) for any two vector fields \(X, Y\). This is a non-degenerate closed two-form on \(M\) and \((M, \omega)\) is a symplectic manifold. By the theorem for existence in [KL] follows that there exists a diffeomorph \(\varphi\) from \(M\) onto a real analytic manifold \(\tilde{M}\) and a symplectic real analytic form \(\tilde{\omega}\) on \(\tilde{M}\) such that \(\varphi^*\tilde{\omega} = \omega\).

Now we apply Theorem B above to obtain a Riemannian metric \(\tilde{g}\) and almost complex structure \(\tilde{J}\) with a fundamental form \(g(X, JY)\) coinciding with \(\tilde{\omega}\).

If now we decompose by polarization the fundamental form \(\omega\) on the given manifold \((M, J, h)\), we obtain the same almost complex structure and metrics \(h\), as is remarked after the proof of Theorem B above. So,
using the commutativity of the diagrams in Definition 1 for tangent and cotangent bundle of the manifolds \( M \) and \( \tilde{M} \) above and for the differential of the morphism \( \varphi \), we obtain that the almost complex structure \( \tilde{J} \) and the metric \( g = \tilde{h} \) correspond to the initial almost complex structure \( J \) and metric \( h \) on \( M \). So each almost Kähler manifold \((M, J, h)\) is diffeomorphic to a real analytic almost Kähler manifold \((\tilde{M}, \tilde{J}, \tilde{h})\).

**Uniqueness.** Let now \((\tilde{M}, \tilde{J}, \tilde{h})\) be another real analytic almost Kähler manifold, diffeomorphic to the given one \((M, J, h)\). Now applying the uniqueness part in [KL], we can find a real analytic diffeomorphism from \((\tilde{M}, \tilde{\omega})\) to \((\overline{M}, \overline{\omega})\). Then reasoning as in the part of existence we obtain that the constructed by polarization almost complex structure \( \tilde{J} \) and metric \( \tilde{h} \) correspond to the almost complex structure \( J \) and to the hermitian metric \( h \). So the uniqueness of the real analytic Kähler manifold up to real analytic isomorphism, corresponding to the given smooth almost Kähler manifold is obtained. \( \square \)

**References**


In this paper, we characterize parallel immersions of rank one Riemannian symmetric spaces into a real space form by using the notion of isotropic immersions and inequalities related to mean curvatures.

1. Introduction

Let \( f : M \to \bar{M} \) be an isometric immersion of a Riemannian manifold \( M \) into a Riemannian manifold \( \bar{M} \) with metric \( \langle \cdot, \cdot \rangle \) and \( \sigma \) be the second fundamental form of \( f \). We recall the notion of an isotropic immersion (cf. [6]): The immersion \( f \) is said to be isotropic if for all \( x \in M \) and tangent vectors \( X(\neq 0) \) to \( M \) at \( x \), \( \|\sigma(X,X)\|/\|X\|^2 \) is constant. Then we find a function \( \lambda \) on \( M \) defined by \( x(\in M) \mapsto \|\sigma(X,X)\|/\|X\|^2 \), so that the immersion \( f \) is also said to be \( \lambda \)-isotropic. Note that totally umbilic immersions are isotropic, but not vice versa.

It is known that all parallel immersions \( f \) of rank one Riemannian symmetric spaces into a real space form are equivariant, so that \( f \) are isotropic. But there exist many isotropic immersions of these spaces into a real space form, which are not parallel (cf. [4], [9], [10]). For example, the fourth standard minimal immersion \( f : S^3(1/8) \to S^{24}(1) \) is isotropic but not parallel.

The main purpose of this paper is to give a sufficient condition that isotropic immersions of rank one Riemannian symmetric spaces into a real space form are parallel by using inequalities related to mean curvatures.

From this point of view, in Section 3, we characterize parallel immersions
of a sphere into a real space form. In Sections 4 and 5, we characterize parallel immersions of other rank one Riemannian symmetric spaces into a real space form.

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2. Terminology

We use the following terminology in this paper:

An \( n \)-dimensional real space form \( M^n(c; R) \) of constant sectional curvature \( c \) is congruent to one of a standard sphere \( S^n(c) \), a Euclidean space \( R^n \) and a real hyperbolic space \( RH^n(c) \), according as \( c \) is positive, zero or negative.

A complex \( n \)-dimensional complex space form \( M^n(c; C) \) of constant holomorphic sectional curvature \( c \) is congruent to one of a complex projective space \( CP^n(c) \), a complex Euclidean space \( C^n(= R^{2n}) \) and a complex hyperbolic space \( CH^n(c) \), according as \( c \) is positive, zero or negative.

A quaternionic \( n \)-dimensional quaternionic space form \( M^n(c; Q) \) of constant quaternionic sectional curvature \( c \) is congruent to one of a quaternionic projective space \( QP^n(c) \), a quaternionic Euclidean space \( Q^n(= R^{4n}) \) and a quaternionic hyperbolic space \( QH^n(c) \), according as \( c \) is positive, zero or negative.

We here introduce the following lemma (cf. [8]):

Lemma 2.1. Let \( f \) be an isometric immersion of an \( n \)-dimensional Riemannian manifold \( M^n \) into a real space form \( M^m(c; R) \) of constant sectional curvature \( \tilde{c} \) and \( \sigma \) be the second fundamental form. Then we get the following equation:

\[
\frac{1}{2} \Delta \| \sigma \|^2 = \| \nabla' \sigma \|^2 - \tilde{c} n^2 H^2 + \tilde{c} n \| \sigma \|^2 + \sum_{i,j,k=1}^{n} \left< D_{e_i} \left( D_{e_j} (\sigma(e_k,e_k)) \right), \sigma(e_i,e_j) \right> \\
+ \sum_{i,j,k,l=1}^{n} \left[ 2 \langle \sigma(e_k,e_j), \sigma(e_i,e_l) \rangle \langle \sigma(e_i,e_k), \sigma(e_i,e_j) \rangle \\
- 2 \langle \sigma(e_k,e_j), \sigma(e_k,e_l) \rangle \langle \sigma(e_i,e_l), \sigma(e_i,e_j) \rangle \\
+ \langle \sigma(e_k,e_k), \sigma(e_i,e_l) \rangle \langle \sigma(e_i,e_j), \sigma(e_i,e_j) \rangle \\
- \langle \sigma(e_i,e_j), \sigma(e_i,e_k) \rangle \langle \sigma(e_i,e_k), \sigma(e_i,e_j) \rangle \right],
\]
where we denote by $\Delta$ the Laplacian on $M^n$, by $H$ the mean curvature of $f$, by $D$ the normal connection on the normal bundle $NM^n$, and by $\{e_1, \ldots, e_n\}$ a local field of orthonormal frames on $M^n$.

3. ISOTROPIC IMMERSIONS OF A SPHERE INTO A REAL SPACE FORM

Our aim here is to prove the following:

**Theorem 3.1.** Let $f$ be a $\lambda$-isotropic immersion of an $n(\geq 2)$-dimensional sphere $S^n(c)$ of constant sectional curvature $c$ into an $m$-dimensional real space form $M^m(\bar{c}; R)$ of constant sectional curvature $\bar{c}$. Let $\Delta$ denote the Laplacian on $S^n(c)$. Suppose that the mean curvature vector field $\mathfrak{h}$ and the mean curvature $H = \|\mathfrak{h}\|$ satisfy the following two inequalities:

(i) $H^2 \leq (2(n + 1)c/n) - \bar{c}$,
(ii) $0 \leq (1 - n)\Delta H^2 + n(\mathfrak{h}, \Delta \mathfrak{h})$.

Then $f$ is a parallel immersion and is equivalent to one of the following:

(I) $f$ is a totally umbilic immersion of $S^n(c)$ into $\bar{M}^m(\bar{c}; R)$, where $c \geq \bar{c}$ and $H^2 \equiv c - \bar{c}$.

(II) $f = f_2 \circ f_1 : S^n(c) \xrightarrow{f_1} S^{(n^2 + 3n - 2)/2(2(n + 1)c/n)} \xrightarrow{f_2} \bar{M}^m(\bar{c}; R)$, where $f_1$ is the second standard minimal immersion, $f_2$ is a totally umbilic immersion, $2(n + 1)c/n \geq \bar{c}$ and $H^2 \equiv (2(n + 1)c/n) - \bar{c}$.

**Proof.** Since $f$ is $\lambda$-isotropic, we get the following equation by the Gauss equation:

$$\langle \sigma(X, Y), \sigma(Z, W) \rangle = \frac{c - \bar{c}}{3} (2\langle X, Y \rangle \langle Z, W \rangle - \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle)$$

$$+ \frac{\lambda^2}{3} (\langle X, Y \rangle \langle Z, W \rangle + \langle X, Z \rangle \langle Y, W \rangle + \langle X, W \rangle \langle Y, Z \rangle),$$

where $X, Y, Z$ and $W$ are vector fields on $S^n(c)$.

We compute the equation of Lemma 2.1 by the above equation and get the following equation:

$$\frac{1}{2} \Delta \|\sigma\|^2 = \|\nabla' \sigma\|^2 - \frac{n^3(n - 1)}{n + 2} (H^2 - c + \bar{c}) (H^2 - \frac{2(n + 1)c}{n} c + \bar{c})$$

$$+ n((1 - n)\Delta H^2 + n(\mathfrak{h}, \Delta \mathfrak{h})).$$

By the assumption and Hopf's lemma, we find that the immersion $f$ is parallel (i.e. $\nabla' \sigma \equiv 0$). Therefore, we get the conclusion (cf. [7]).
As an immediate consequence of Theorem 3.1, we find the following:

**Corollary 3.1.** Let $f$ be an isotropic immersion of an $n(\geq 2)$-dimensional sphere $S^n(c)$ of constant sectional curvature $c$ into an $m$-dimensional real space form $\tilde{M}^m(\tilde{c}; R)$ of constant sectional curvature $\tilde{c}$. Suppose that the mean curvature vector field $\mathbf{h}$ and the mean curvature $H$ satisfy the following:

(i) $H^2 \leq (2(n + 1)c/n) - \tilde{c}$,

(ii) $\mathbf{h}$ is parallel.

Then $f$ is a parallel immersion and is equivalent to one of the following:

(I) $f$ is a totally umbilic immersion of $S^n(c)$ into $\tilde{M}^m(\tilde{c}; R)$, where $c \geq \tilde{c}$ and $H^2 \equiv c - \tilde{c}$.

(II) $f = f_2 \circ f_1 : S^n(c) \xrightarrow{f_1} S^{(n^2 + 3n - 2)/2}(2(n + 1)c/n) \xrightarrow{f_2} \tilde{M}^m(\tilde{c}; R)$, where $f_1$ is the second standard minimal immersion, $f_2$ is a totally umbilic immersion, $2(n + 1)c/n \geq \tilde{c}$ and $H^2 \equiv (2(n + 1)c/n) - \tilde{c}$.

**Remark 3.1.** We shall show that Theorem 3.1 is no longer true if we omit the assumption (ii):

We here recall the following isometric immersion given by S. Maeda (cf. [5]). Let $M^n$ be an $n(\geq 2)$-dimensional compact isotropy-irreducible Riemannian homogeneous space and $X_1$ (resp. $X_2$) : $M^n \rightarrow S^{N_1}(c_1)$ (resp. $S^{N_2}(c_2)$) be a minimal $\lambda_1$- (resp. $\lambda_2$-)isotropic immersion with respect to some eigenvalue $\mu_1$ (resp. $\mu_2$) of the Laplacian on $M^n$, where $\mu_1 \neq \mu_2$.

Using these minimal immersions, for $t \in (0, \pi/2)$, we define the following minimal immersion:

(a) $\chi_t(= (\chi_1, \chi_2)) : M^n \rightarrow S^{N_1}(\frac{c_1}{\cos^2 t}) \times S^{N_2}(\frac{c_2}{\sin^2 t})$.

Here the differential mapping $(\chi_t)_* \mathbf{X}$ of $\chi_t$ is given by, for each $X \in TM^n$, $(\chi_t)_* X = (\cos t \cdot (\chi_1)_* X, \sin t \cdot (\chi_2)_* X)$.

The above product space of spheres in (a) can be imbedded into a sphere as a Clifford hypersurface:

(b) $S^{N_1}(\frac{c_1}{\cos^2 t}) \times S^{N_2}(\frac{c_2}{\sin^2 t}) \rightarrow S^N(\tilde{c})$,

where $N = N_1 + N_2 + 1$ and $1/\tilde{c} = \cos^2 t/c_1 + \sin^2 t/c_2$.

Combining (a) with (b), we obtain the following isometric immersion:

(3.1) $f_t : M^n \rightarrow S^{N_1}(\frac{c_1}{\cos^2 t}) \times S^{N_2}(\frac{c_2}{\sin^2 t}) \rightarrow S^N(\tilde{c})$.

The above $f_t$ has the following properties:
(1) \( f_t \) is pseudo umbilic but not totally umbilic,
(2) the mean curvature \( H_t \) is constant and is given by
\[
H_t = \frac{|c_1 - c_2|}{\sqrt{(c_1/\cos^2 t) + (c_2/\sin^2 t)}},
\]
(3) \( f_t \) is a \( \lambda_t \)-isotropic immersion and \( \lambda_t \) is given by
\[
\lambda_t = \sqrt{\cos^4 t \lambda_1^2 + \sin^4 t \lambda_2^2 + \frac{(c_1 - c_2)^2}{(c_1/\cos^2 t) + (c_2/\sin^2 t)}},
\]
(4) the mean curvature vector field \( h_t \) is not parallel. In fact, its length
is given by \( \|Dh_t\|^2 = n(c_1 - c_2)^2 \cos^2 t \sin^2 t \).

In the following, let our submanifold \( M^n \) be a standard sphere
\( S^n(n/(2(n + 1))) \). Let \( X_1 : S^n(n/(2(n + 1))) \to S^{(n^2 + 3n - 2)/2}(1) \) be
the first standard minimal immersion and \( X_2 : S^n(n/(2(n + 1))) \to S^n(n/(2(n + 1))) \) be an identity mapping. If we put \( \cos t = 1/\sqrt{n+1} \)
and \( \sin t = \sqrt{n/(n+1)} \), then we get the isotropic immersion:
\[
f : S^n(n/(2(n + 1))) \to S^{(n^2 + 3n - 2)/2}(n + 1) \times S^n(1/2)
\to S^{(n^2 + 5n)/2}((n + 1)/(2n + 3)).
\]

We find that this isotropic immersion \( f \) satisfies the assumption (i) but
not the assumption (ii) in Theorem 3.1. In fact,
(i) \( H^2 - \frac{2(n + 1)}{n} c + \tilde{c} = \frac{(n + 2)^2}{2(2n + 3)(n + 1)^2} - 1 + \frac{n + 1}{2n + 3} = -\frac{n(n + 2)}{2(n + 1)^2} < 0, \)
(ii) \( (1 - n) \Delta H^2 + n\langle \eta, \Delta \eta \rangle = n\langle \eta, \Delta \eta \rangle = -n\|D\eta\|^2 = -\frac{n^3(n + 2)^2}{4(n + 1)^4} < 0. \)

This shows that Theorem 3.1 can not hold without the assumption (ii).

**Remark 3.2.** We consider that an isometric immersion \( f \) of a real space
form \( M(c; R) \) of constant sectional curvature \( c \) into a real space form
\( \overline{M}(\tilde{c}; R) \) of constant sectional curvature \( \tilde{c} \). In general, if \( f \) satisfies the
following inequality:
\[
H^2 - c + \tilde{c} \leq 0,
\]
then \( f \) is totally umbilic by the Gauss equation. So, we consider naturally
the following problem:
"Does there exist some \( \varepsilon(> 0) \) such that if \( f \) satisfies \( H^2 - c + \tilde{c} \leq \varepsilon, \)
then \( f \) is totally umbilic ?"
We give a negative answer to this problem by the following proposition:

**Proposition 3.1.** For all \( \epsilon > 0 \), there exists an isometric immersion \( f \) of a sphere \( S^n(c) \) into \( \mathbb{S}^N(\tilde{c}) \), which has the following properties:

1. \( f \) is not a totally umbilic immersion and \( H^2 - c + \tilde{c} \equiv \epsilon \),
2. \( f \) is an isotropic immersion,
3. the mean curvature vector field is not parallel.

**Proof.** We use also the above immersion (3.1). Let \( \mathcal{X}_I : S^n(n/(2(n+1))) \to S^n(n/(2(n+1))) \) be an identity mapping. We choose a natural number \( k \in \{2,3,\ldots\} \) such that \( \epsilon < (k-1)(k+n)/(2(n+1)) \). Let \( \mathcal{X}_2 : S^n(n/(2(n+1))) \to S^{n(k)}(c(k)) \) be the \( k \)-th standard minimal immersion, where \( n(k) = (k+n-2)!/(k^2 + k(n-1) - 2(n+1)\epsilon) \) and \( c(k) = k(k+n-1)/(2(n+1)) \).

If we put \( \cos t = \sqrt{1 - (2(n+1)\epsilon/((k-1)(k+n)))} \) and \( \sin t = \sqrt{2(n+1)\epsilon/((k-1)(k+n))} \), then we get the isotropic immersion:

\[
f : S^n(c) \to S^n(c/\cos^2 t) \times S^{n(k)}(c(k)/\sin^2 t) \to S^{n+n(k)+1}(\tilde{c}),
\]

where \( c = n/(2(n+1)) \) and \( \tilde{c} = kn(k+n-1)/(2(n+1)\{k^2 + k(n-1) - 2(n+1)\epsilon\}) \). Straightforward computation yields that \( H^2 - c + \tilde{c} = \epsilon \).

4. Isotropic immersions of a complex space form into a real space form

In this section, we consider the cases of submanifolds being a complex space form and a quaternionic space form.

**Theorem 4.1.** Let \( f \) be a \( \lambda \)-isotropic immersion of a complex \( n(\geq 2) \)-dimensional complex space form \( M^n(4c; C) \) of constant holomorphic sectional curvature \( 4c \) into an \( m \)-dimensional real space form \( \mathcal{M}^m(\tilde{c}; R) \) of constant sectional curvature \( \tilde{c} \). Suppose that the mean curvature \( H \) satisfies the following inequality:

\[
H^2 \leq (2(n+1)c/n) - \tilde{c}.
\]

Then \( f \) is a parallel immersion and is equivalent to one of the following:

1. \( f \) is a totally geodesic immersion of \( C^n(= R^{2n}) \) into \( R^m \), where \( H \equiv 0 \).
(II) \( f \) is a totally umbilic immersion of \( C^n(= R^{2n}) \) into \( RH^m(\bar{c}) \), where \( H^2 \equiv -\bar{c} \).

(III) \( f = f_2 \circ f_1 : CP^n(4c) \xrightarrow{f_1} S^{n^2+2n-1}(2(n+1)c/n) \xrightarrow{f_2} \bar{M}^m(\bar{c}; R) \), where \( f_1 \) is the first standard minimal immersion, \( f_2 \) is a totally umbilic immersion, \( 2(n+1)c/n \geq \bar{c} \) and \( H^2 \equiv (2(n+1)c/n) - \bar{c} \).

**Proof.** Let \( J \) be the complex structure on \( M^n(4c; C) \). Then the curvature tensor \( R \) of \( M^n(4c; C) \) is given by

\[
R(X,Y)Z = c\{(Y,Z)X - (X,Z)Y + (JY,Z)JX - (JX,Z)JY + 2(X,JY)JZ\}
\]

for all vector fields \( X, Y \) and \( Z \) on \( M^n(4c; C) \).

This, together with the condition that \( f \) is \( \lambda \)-isotropic, shows the following:

\[
\langle \sigma(X,Y), \sigma(Z,W) \rangle = \frac{\lambda^2 + 2(c-\bar{c})}{3} \langle X,Y \rangle \langle Z,W \rangle
\]

\[
+ \frac{\lambda^2 - (c-\bar{c})}{3} \{ \langle X,W \rangle \langle Y,Z \rangle + \langle X,Z \rangle \langle Y,W \rangle \}
\]

\[
+ c \{ \langle JX,W \rangle \langle JY,Z \rangle + \langle JX,Z \rangle \langle JY,W \rangle \}
\]

for all vector fields \( X, Y, Z \) and \( W \) on \( M^n(4c; C) \).

The equation (4.1) yields the following:

\[
H^2 = \frac{(n+1)\lambda^2 + 2(n+1)c - (2n-1)\bar{c}}{3n},
\]

(4.2)

\[
\|\sigma(X,JX)\|^2 = \frac{\lambda^2 - 4c + \bar{c}}{3},
\]

(4.3)

where \( X \) is an arbitrary unit vector field on \( M^n(4c; C) \).

It follows from (4.2) and (4.3) that

\[
H^2 - \frac{2(n+1)}{n}c + \bar{c} = \frac{n+1}{n} \|\sigma(X,JX)\|^2 \geq 0,
\]

where \( X \) is an arbitrary unit vector field on \( M^n(4c; C) \).

Therefore, by the assumption, we find \( H^2 \equiv (2(n+1)c/n) - \bar{c} \) and get the following equation:

\[
\sigma(X,JX) = 0
\]

(4.4)

for all vector fields \( X \) on \( M^n(4c; C) \).

It follows from (4.4) that \( \sigma(X,Y) = \sigma(JX,JY) \) for all vector fields \( X \) and \( Y \) on \( M^n(4c; C) \). Consequently we find that the immersion \( f \) is parallel (cf. [4]) and get the conclusion. \( \square \)
Using the same discussion as above, we get the following theorem:

**Theorem 4.2.** Let \( f \) be an isotropic immersion of a quaternionic \( n(\geq 2) \)-dimensional quaternionic space form \( M^n(4c; Q) \) of constant quaternionic sectional curvature \( 4c \) into an \( m \)-dimensional real space form \( \overline{M}^m(\overline{c}; R) \) of constant sectional curvature \( \overline{c} \). Suppose that the mean curvature satisfies the following inequality:

\[
H^2 \leq (2(n + 1)c/n) - \overline{c}.
\]

Then \( f \) is a parallel immersion and is equivalent to one of the following:

(I) \( f \) is a totally geodesic immersion of \( Q^n(= R^{4n}) \) into \( R^m \), where \( H = 0 \).

(II) \( f \) is a totally umbilic immersion of \( Q^n(= R^{4n}) \) into \( RH^m(\overline{c}) \), where \( H^2 \equiv -\overline{c} \).

(III) \( f = f_2 \circ f_1 : QP^n(4c) \xrightarrow{f_1} S^{2n^2 + 3n - 1}(2(n + 1)c/n) \xrightarrow{f_2} \overline{M}^m(\overline{c}; R) \), where \( f_1 \) is the first standard minimal immersion, \( f_2 \) is a totally umbilic immersion, \( 2(n + 1)c/n \geq \overline{c} \) and \( H^2 \equiv (2(n + 1)c/n) - \overline{c} \).

5. Isotropic immersions of Cayley projective plane into a real space form

Using a discussion similar to the proof of Theorem 3.1, we give a sufficient condition that an isotropic immersion of Cayley projective plane into a real space form is parallel without proof:

**Theorem 5.1.** Let \( f \) be an isotropic immersion of Cayley projective plane \( CayP^2(c) \) of maximal sectional curvature \( c \) into an \( m \)-dimensional real space form \( \overline{M}^m(\overline{c}; R) \) of constant sectional curvature \( \overline{c} \). Let \( \Delta \) denote the Laplacian on CayP^2(c). Suppose that the mean curvature vector field \( \mathfrak{h} \) and the mean curvature \( H \) satisfy the following two inequalities:

(i) \( 8H^2 \leq 9c - 8\overline{c} \),

(ii) \( 0 \leq 16(\mathfrak{h}, \Delta \mathfrak{h}) - 15\Delta H^2 \).

Then \( f \) is a parallel immersion and is equivalent to the following:

\[
f = f_2 \circ f_1 : CayP^2(c) \xrightarrow{f_1} S^{25}(3c/4) \xrightarrow{f_2} \overline{M}^m(\overline{c}; R),
\]

where \( f_1 \) is the first standard minimal immersion, \( f_2 \) is a totally umbilic immersion, \( 3c/4 \geq \overline{c} \) and \( 8H^2 \equiv 6c - 8\overline{c} \) (< \( 9c - 8\overline{c} \)).
Corollary 5.1. Let \( f \) be an isotropic immersion of Cayley projective plane \( \text{CayP}^2(c) \) of maximal sectional curvature \( c \) into an \( m \)-dimensional real space form \( \widetilde{M}^m(\tilde{c}; R) \) of constant sectional curvature \( \tilde{c} \). Suppose that the mean curvature vector field \( \mathfrak{h} \) and the mean curvature \( H \) satisfy the following:

(i) \( 8H^2 \leq 9c - 8\tilde{c} \),
(ii) \( \mathfrak{h} \) is parallel.

Then \( f \) is a parallel immersion and is equivalent to the following:

\[
f = f_2 \circ f_1 : \text{CayP}^2(c) \xrightarrow{f_1} S^{25}(3c/4) \xrightarrow{f_2} \widetilde{M}^m(\tilde{c}; R),
\]

where \( f_1 \) is the first standard minimal immersion, \( f_2 \) is a totally umbilic immersion, \( 3c/4 \geq \tilde{c} \) and \( 8H^2 = 6c - 8\tilde{c} \).

Remark 5.1. Theorem 5.1 is also no longer true if we omit the assumption (ii). Our discussion is indebted to Remark 3.1. Let \( \mathcal{X}_1 : \text{CayP}^2(1) \to S^{25}(3/4) \) be the first standard minimal immersion and \( \mathcal{X}_2 : \text{CayP}^2(1) \to S^{323}(13/8) \) the second standard minimal immersion. If we put \( \cos t = \sqrt{10/11} \) and \( \sin t = 1/\sqrt{11} \), then we get the following isotropic immersion

\[
f : \text{CayP}^2(1) \to S^{25}(33/40) \times S^{323}(143/8) \to S^{349}(429/544).
\]

We find that the above isotropic immersion \( f \) satisfies the assumption (i) but not the assumption (ii) in Theorem 5.1. In fact,

(i) \( 8H^2 - 9c + 8\tilde{c} = -26/11 < 0 \),
(ii) \( 16(\mathfrak{h}, \Delta \mathfrak{h}) - 15\Delta H^2 = 16(\mathfrak{h}, \Delta \mathfrak{h}) = -16\|D\mathfrak{h}\|^2 = -1960/121 < 0 \).

This shows that Theorem 5.1 can not hold without the assumption (ii).

Remark 5.2. It is interesting to compare Theorems 3.1 and 5.1 with Theorems 4.1 and 4.2. The assumptions of Theorem 4.1 and 4.2 do not need an inequality (ii). Moreover, in Theorems 4.1 and 4.2, our submanifold is not necessarily compact.

References


In this paper the interconnection between holomorphic and analytic functions is studied in the case of the algebra of bi-complex numbers. It is proved that each holomorphic bi-complex function is bi-complex analytic. In the general case of associative complex algebras this is not true.

1. Introduction

The main purpose of this talk is to remember the old theory of holomorphicity on commutative and associative real and complex algebras and to indicate some possible interconnections with the contemporary theory of many complex variables. According to the classical Frobenius theorem the set of complex associative algebras without divisors of zero reduces to two algebras only, namely, the field of complex numbers and the non-commutative algebra of quaternions. The geometric applications of complex numbers and quaternions are well known. However, there are complex algebras with a very regular distribution of their divisors of zero that permit to develop rich holomorphic structures related with abundants, but classifiable, singularities closely related with the mentioned divisor of zero. It is interesting to search geometric applications in this direction.
2. Even dimensional real anti-cyclic numbers

This notion is introduced in the book [1], namely \( x = x^0 + x^1 j + \cdots + x^{2n-1} j^{2n-1} \) is called a \( 2n \)-dimensional anti-cyclic number if \( j^{2n} = -1 \) and \( j^r j^s = j^{s+r} \), \( r, s = 0, 1, \ldots, n - 1 \), \( j \) being a symbol. In the case \( n = 1 \) we obtain an algebra isomorphic to the field of complex numbers.

For simplicity we shall consider only the cases \( n = 2 \) and \( n = 4 \).

A natural matrix representation of this algebra is obtained as follows

\[
j^1 \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad j^2 \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix},
\]

\[
j^3 \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad j^4 \rightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
\]

According to this setting we obtain the following matrix representation

\[
\alpha = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^3 & x^0 & x^1 & x^2 \\ -x^2 & -x^3 & x^0 & x^1 \\ -x^1 & -x^2 & -x^3 & x^0 \end{pmatrix}.
\]

We see that in the left side of the equality above is an anti-cyclic matrix that motivate the proposed terminology. Clearly, the obtained matrix representation is a generalization of the matrix representation of complex numbers

\[
\begin{pmatrix} x^0 & x^1 \\ -x^1 & x^0 \end{pmatrix} = x^0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x^1 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.
\]

The algebra \( \mathbb{R} [1, j, j^2, j^3] \) admits a complexification denoted by \( \mathbb{C} [1, j] \). Rewriting \( \alpha \) as follows

\[\alpha = (x^0 + x^2 j^2) + j(x^1 + x^3 j^2),\]

we set

\[z := x^0 + x^2 j^2, \quad w := x^1 + x^3 j^2, \quad j^2 = i, \quad i \in \mathbb{C}.
\]

Thus, we obtain \( \alpha = z + jw \), with \( z, w \in \mathbb{C} \), \( j^2 = i \). It is easy to see that the algebras \( \mathbb{R} [1, j, j^2, j^3] \) and \( \mathbb{C} [1, j] \) are isomorphic.

Formally, the complex algebra \( \mathbb{C} [1, j] \) can be considered as a complex analogous of the classical algebra of the double real numbers defined by the
condition \( j^2 = 1 \) with real \( z \) and \( w \) [3], [4], [5]. The elements of \( \mathbb{C}[1,j] \) will be called bi-complex numbers. This is an algebra of zero divisors, defined by the condition \( z^2 - iw^2 = 0 \), i.e. \( \alpha = z + jw \) is a zero divisor iff \( z^2 - iw^2 = 0 \), which is not difficult to prove. So the set of zero-divisors in \( \mathbb{C}[1,j] \) can be considered as a complex analytic set in \( \mathbb{C} \times \mathbb{C} \).

3. Abstract derivative of Scheffers

By \( A \) is denoted an arbitrary finite-dimensional, associative, and commutative algebra over the field \( \mathbb{K} = \mathbb{R}, \mathbb{C} \). If \( \{e_1, e_2, \ldots, e_n\} \) is a base of \( A \), and by \( \Gamma = \{\gamma^q_{ij}\} \) the structural tensor of \( A \) is denoted, where by \( e_i e_j = \gamma^q_{ij} e_q \), \( i, j, q = 1, 2, \ldots, n \), the multiplicative structure of the algebra \( A \) is defined.

We shall consider mapping \( F, x \rightarrow F(x) : A \rightarrow A, x \in A, \ F(x) \in A \). According to Scheffers [2] a mapping \( F', x \rightarrow F'(x) : A \rightarrow A, x \in A, \ F'(x) \in A \), is called a derivative of \( F \) if

\[
dF(x) = F'(x)dx.
\]

Having in mind that \( F'(x) = F'^k(x)dx_k, \ k = 1, 2, \ldots, n \), and \( dF = (\partial F^q/\partial x^l)dx^l e_q \), we receive

\[
\frac{\partial F^q}{\partial x^l} = F'^k \gamma^q_{kl}.
\]

4. Holomorphic functions on \( \mathbb{C}[1,j] \)

We shall consider mappings of the type \( f : \mathbb{C}[1,j] \rightarrow \mathbb{C}[1,j] \). This mapping will be called a bi-complex function of bi-complex variable. It is represented as \( f(\alpha) = f^0(z,w) + jf^1(z,w) \), where \( f^0(z,w) \) is called an even part of the bi-complex function \( f \), and respectively, \( f^1(z,w) \) is called an odd part of \( f \). According to the definition of Scheffers \( df = f'dx \), where \( f' \) is a bi-complex function (a mapping of the type \( \mathbb{C}[1,j] \rightarrow \mathbb{C}[1,j] \)). So \( f' \) is a represented by its even and odd parts, \( f'(\alpha) = f^0(z,w) + jf^1(z,w) \).

With this in mind we obtain

\[
df(\alpha) = (f^0(z,w) + jf^1(z,w))(dz + jdw)
= f^0(z,w)dz + if^1(z,w)dw + j(f^1(z,w)dz + f^0 dw)).
\]

From this and \( df = (\partial f/\partial z)dz + (\partial f/\partial w)dw \), one deduce the following system of complex partial differential equations with respect to the even and the odd parts of \( f \)

\[
\partial f^0/\partial z = \partial f^1/\partial w, \quad \partial f^0/\partial w = i\partial f^1/\partial z.
\]
We say that each bi-complex function $f$ of bi-complex variable is holomorphic if it satisfies the above written system of partial differential equations.

**Example 1.** One proves that each polynomial $P(z+jw)$ of the bi-complex variable $z+jw$ is a holomorphic bi-complex function. Especially, this holds for $P(\alpha) = \alpha = z+jw$.

**Example 2.** Each polynomial of the type $P(z-jw)$, especially $\alpha^* = z-jw$, is not a holomorphic function in the mentioned sense.

### 5. $\partial^*$-type interpretation

Considering $\alpha^* = z-jw$ as a conjugate of $\alpha = z+jw$, we calculate that

$$2dz = d\alpha + d\alpha^* \quad \text{and} \quad 2jdw = d\alpha - d\alpha^*.$$ 

We introduce the following differential operators

$$\frac{\partial}{\partial \alpha} := \frac{1}{2}(\frac{\partial}{\partial z} - j\frac{\partial}{\partial w}) \quad \text{and} \quad \frac{\partial}{\partial \alpha^*} := \frac{1}{2}(\frac{\partial}{\partial z} + j\frac{\partial}{\partial w}),$$

with the help of which we receive the following representation for the differential $df$, namely,

$$df = \frac{\partial f}{\partial \alpha}d\alpha + \frac{\partial f}{\partial \alpha^*}d\alpha^*.$$ 

Expressing $f = f(\alpha)$ by its even and odd parts we obtain

$$\frac{\partial f}{\partial \alpha^*} = \frac{1}{2}(\frac{\partial f^0}{\partial z} + \frac{\partial f^1}{\partial z} + j\frac{\partial f^0}{\partial w} + j\frac{\partial f^1}{\partial w}).$$

Now, it is clear that the system of partial differential equations written in previous paragraph is equivalent to the following unique equation

$$\frac{\partial f}{\partial \alpha^*} = 0.$$ 

### 6. Cauchy's integral formula

Let $\alpha, \beta \in \mathbb{C}[1,j]$, $\alpha = z+jw$, $z,w \in \mathbb{C}$, $\beta = u+jv$, $d\beta = du+jdv$, $u,v \in \mathbb{C}$. We shall consider open disks in $\mathbb{C}[1,j]$

$$\Delta(\beta_0, r) = \Delta(u_0, r) + j\Delta(v_0, r) := \{u+jv : |u-u_0| < r, |v-v_0| < r\},$$

$u_0, v_0 \in \mathbb{C}, \beta_0 = u_0+jv_0$.

The number $r$ is called radius of the disk $\Delta(\beta_0, r)$. 
Close disks are defined in the same way, replacing the strong inequalities above by non-strong ones. So disks in $\mathbb{C}[1,j]$ are bi-disks in $\mathbb{C} \times \mathbb{C}$.

The boundary $\partial \Delta(\beta_0, r)$ of the disk $\Delta(\beta_0, r)$ is a 3-dimensional manifold, namely, a solid torus centered at $\beta_0$, described by 3 real parameters $(r^*, s, t)$, $r^*$ called thickness, and $s, t$ — angular parameters. We shall always consider the thickness $r^*$ small enough with respect to the radius $r$. The thickness is always positive. In the case $r^* = 0$ the boundary $\partial \Delta(\beta_0, r)$ degenerates as a circle.

If $f(\beta)$ denotes a smooth bi-complex function we consider the following 1-form

$$\varphi(\beta) = \frac{f(\beta)}{\beta - \alpha} d\beta,$$

defined on the subset $\Gamma_\varepsilon = \Gamma(\beta_0, \alpha, r, \varepsilon) = \Delta(\beta_0, r) - \Delta(\alpha, \varepsilon)$ of $\mathbb{C}[1,j]$ for $\varepsilon$ small enough, i.e. $\beta \in \Gamma(\beta_0, \alpha, r, \varepsilon)$, $\beta \notin \Delta(\alpha, \varepsilon)$.

Applying the Stokes formula for the 1-form $\varphi(\beta)$ we receive

$$\int \int \int_{\partial \Gamma_\varepsilon} \frac{f(\beta)}{\beta - \alpha} d\beta = \int \int \int_{\Gamma_\varepsilon} d\left(\frac{f(\beta)}{\beta - \alpha}\right) \wedge d\beta.$$ 

Let us remark that integral of a bi-complex function is defined by integrating its even and odd parts. Using the formula

$$d\left(\frac{f(\beta)}{\beta - \alpha}\right) = \frac{\partial}{\partial \beta} \left(\frac{f(\beta)}{\beta - \alpha}\right) d\beta + \frac{\partial}{\partial \beta^*} \left(\frac{f(\beta)}{\beta - \alpha}\right) d\beta^*,$$

we obtain

$$d\left(\frac{f(\beta)}{\beta - \alpha}\right) \wedge d\beta = -\frac{\partial}{\partial \beta^*} \left(\frac{f(\beta)}{\beta - \alpha}\right) d\beta \wedge d\beta^*.$$ 

In view of $\partial \Gamma_\varepsilon = \partial \Delta(\beta_0, r) \cup \partial \Delta(\alpha, \varepsilon)$ we receive

$$\int \int \int_{\partial \Delta(\beta_0, r)} \frac{f(\beta)}{\beta - \alpha} d\beta - \int \int \int_{\partial \Delta(\alpha, \varepsilon)} \frac{f(\beta)}{\beta - \alpha} d\beta = -\int \int \int_{\Gamma_\varepsilon} \frac{\partial}{\partial \beta^*} \left(\frac{f(\beta)}{\beta - \alpha}\right) d\beta \wedge d\beta^*,$$

since $1/((\beta - \alpha))$ is a holomorphic bi-complex function over $\Gamma_\varepsilon$. The second integral above can be calculated. Indeed, setting $u - z = \varepsilon e^{it}$, $v - w = \varepsilon e^{is}$, $1 < s, t < 2\pi$, we obtain

$$I = \int \int \int_{\partial \Delta(\beta_0, r)} \frac{f(\beta)}{\beta - \alpha} d\beta = \int dr \int_0^{2\pi} \int_0^{2\pi} \frac{f(\alpha + \varepsilon(e^{it} + je^{is}))}{\varepsilon(e^{it} + je^{is})} \cdot \varepsilon i d(e^{it} + je^{is}),$$

and for $t = s$, after taking the limit $\varepsilon \to 0$ under the integral, we receive

$$I = 2\pi i (1 + j)r^* f(\alpha).$$
On the other hand, $\Gamma_\varepsilon$ is replaced by $\partial \Delta(\beta_0, r)$ and finally we obtain the Cauchy's formula for smooth functions $f$, namely,

$$(1 + j)^rf(\alpha) = \frac{1}{2\pi i} \iiint_{\partial \Delta(\beta_0, r)} \frac{f(\beta)}{\beta - \alpha} d\beta + \frac{1}{2\pi i} \iiint_{\Gamma_\varepsilon} \frac{\partial}{\partial \beta^*} (\frac{f(\beta)}{\beta - \alpha}) d\beta \wedge d\beta^*.$$ 

**Corollary.** In the case of holomorphic bi-complex function $f$ we have

$$\frac{\partial f(\beta)}{\partial \beta^*} = 0.$$ 

This implies the Cauchy's integral formula for holomorphic bi-complex functions

$$(1 + j)^rf(\alpha) = \frac{1}{2\pi i} \iiint_{\partial \Delta(\beta_0, r)} \frac{f(\beta)}{\beta - \alpha} d\beta.$$ 

**Remark.** Having in mind a holomorphic bi-complex function defined over the bounded domain $D$

$$f(z + jw) = f^0(z, w) + jf^1(z, w),$$

one can apply the ordinary Cauchy's formula for holomorphic functions of two complex variables

$$f(z + jw) = \frac{1}{(2\pi i)^2} \iint_{\partial D} \frac{f(u + jv)}{(u - z)(v - w)} du dv.$$ 

7. Bi-complex analytic functions

We say that the bi-complex function $f(z + jw)$ is a bi-complex analytic function over $D$ if locally it is developable in a convergent series of a bi-complex variables, i.e.

$$f(z + jw) = \sum_n a_n (z + jw)^n \quad \text{with} \quad a_n \in \mathbb{C}[1, j]. \quad (*)$$

Analogously, we can define complex analytic bi-complex function $f$ developing respectively the even and the odd parts of $f$ as functions of two complex variables

$$f(z + jw) = \sum_{n,m} a_{nm} z^n w^m \quad \text{with} \quad a_{nm} \in \mathbb{C}[1, j]. \quad (**)$$
Theorem. Each bi-complex-analytic bi-complex function is holomorphic bi-complex function, i.e. it satisfy the system of partial differential equations

\[ \frac{\partial f^0}{\partial z} = \frac{\partial f^1}{\partial w}, \quad \frac{\partial f^0}{\partial w} = i \frac{\partial f^1}{\partial z}. \]

Inversely, each holomorphic bi-complex function \( f(z + jw) \) is a bi-complex-analytic bi-complex function, i.e. it is developable in a convergent series of bi-complex variables (\(*\)). For the analytic bi-complex functions the second assertion is not true.

Proof. For the proof it is to remark that the first assertion is directly verifiable that for a partial sum of a bi-complex-analytic series and that one can take a limit in the mentioned system of partial differential equations, having in mind some precise definition of convergency [6].

The inverse statement can be obtained with the help of the mentioned above Cauchy’s integral formula for \( \beta = u + jv \) which is not a zero divisor, i.e. the kernel

\[ \frac{1}{\beta - \alpha} = \frac{1}{\beta} \cdot \frac{1}{1 - \frac{\alpha}{\beta}} = \frac{1}{u + iv} \cdot \frac{1}{1 - \frac{z + jw}{u + iv}}. \]

The inverse assertion is not valid for analytic bi-complex function as it is shown by the following simple example: \( f(z + jw) = z - jw \) is analytic bi-complex function but it does not satisfy the mentioned system of complex partial differential equations.

8. Singularities

By \( J(f) \) is denoted the Jacobian of \( f(z + jw) \)

\[ J(f) = \begin{pmatrix} \frac{\partial f^0}{\partial z} & \frac{\partial f^1}{\partial z} \\ \frac{\partial f^0}{\partial w} & \frac{\partial f^1}{\partial w} \end{pmatrix}. \]

As corollary, for bi-complex holomorphic functions we obtain

\[ \det J(f) = (\frac{\partial f^0}{\partial z})^2 - i(\frac{\partial f^1}{\partial z})^2 = i\{ (\frac{\partial f^0}{\partial w})^2 - i(\frac{\partial f^1}{\partial w})^2 \}. \]

By definition a singular point \( z_0 + jw_0 \) of the bi-complex function \( f \) is a point for which \( \det J(f) \) vanishes.
9. Generalizations

An analogous treatment for the algebra of 8-dimensional anti-cyclic numbers, including a complexification and a terta-complex holomorphicity determined by a system of complex partial differential equations in developed in [7].

References

1. V.V. Vishnevski, A.P. Shirokov and V.V. Shurigin, Spaces over algebras, (Kazan, Kazan University Press, 1985) (in Russian).
This paper considers a generalization of the existing concept of parallel (with respect to a given connection) geometric objects and its possible usage as a suggesting rule in searching for adequate field equations and local conservation laws in theoretical physics. The generalization tries to represent mathematically the two-sided (or dual) nature of the physical objects, the change and the conservation. The physical objects are presented mathematically by sections $\Psi$ of vector bundles, the admissible changes $D\Psi$ are described as a result of the action of appropriate differential operators $D$ on these sections, and the conservation proprieties are accounted for by the requirement that suitable projections of $D\Psi$ on $\Psi$ and on other appropriate sections must be zero. It is shown that the most important equations of theoretical physics obey this rule. Extended forms of Maxwell and Yang-Mills equations are also considered.

1. Definition of the General Parallelism Concept

We begin with the algebraic structure to be used further in the bundle picture. The basic concepts used are the tensor product $\otimes$ of two linear spaces (we shall use the same term linear space for a vector space over a field, and for a module over a ring, and from the context it will be clear which case is considered) and bilinear map.

Let $(U_1, V_1), (U_2, V_2)$ and $(U_3, V_3)$ be three couples of linear spaces. Let $\Phi : U_1 \times U_2 \to U_3$ and $\varphi : V_1 \times V_2 \to V_3$ be two bilinear maps. Then we can form the elements $(u_1 \otimes v_1) \in U_1 \otimes V_1$ and $(u_2 \otimes v_2) \in U_2 \otimes V_2$, and
apply the given bilinear maps as follows:

\[(\Phi, \varphi)(u_1 \otimes v_1, u_2 \otimes v_2) = \Phi(u_1, u_2) \otimes \varphi(v_1, v_2).\]

The obtained element is in \(U_3 \otimes V_3\).

We give now the corresponding bundle picture. Let \(M\) be a smooth \(n\)-dimensional real manifold. We assume that the following vector bundles over \(M\) are constructed: \(\xi_i, \eta_i\), with standard fibers \(U_i, V_i\) and sets of sections \(\text{Sec}(\xi_i), \text{Sec}(\eta_i), i = 1, 2, 3\).

Assume the two bundle maps are given: \((\Phi, id_M) : \xi_1 \times \xi_2 \to \xi_3\) and \((\varphi, id_M) : \eta_1 \times \eta_2 \to \eta_3\). Then if \(\sigma_1\) and \(\sigma_2\) are sections of \(\xi_1\) and \(\xi_2\) respectively, and \(\tau_1\) and \(\tau_2\) are sections of \(\eta_1\) and \(\eta_2\) respectively, we can form an element of \(\text{Sec}(\xi_3 \otimes \eta_3)\):

\[(\Phi, \varphi)(\sigma_1 \otimes \tau_1, \sigma_2 \otimes \tau_2) = \Phi(\sigma_1, \sigma_2) \otimes \varphi(\tau_1, \tau_2) \quad (1)\]

Let now one more vector bundle \(\tilde{\xi}\) is given and \(\sigma_2 \in \text{Sec}(\xi_2)\) is obtained by the action of the differential operator \(D : \text{Sec}(\tilde{\xi}) \to \text{Sec}(\xi_2)\) on a section \(\tilde{\sigma}\) of \(\tilde{\xi}\), so we can form the section (instead of \(\sigma_1\) we write just \(\sigma\)) \(\Phi(\sigma, D\tilde{\sigma}) \otimes \varphi(\tau_1, \tau_2) \in \text{Sec}(\xi_3 \otimes \eta_3)\). We give now the following

**Definition.** The section \(\tilde{\sigma}\) will be called \((\Phi, \varphi; D)\)-**parallel** with respect to \(\sigma\) if

\[(\Phi, \varphi; D)(\sigma \otimes \tau_1, \tilde{\sigma} \otimes \tau_2) = (\Phi, \varphi)(\sigma \otimes \tau_1, D\tilde{\sigma} \otimes \tau_2) \quad (2)\]

\[= \Phi(\sigma, D\tilde{\sigma}) \otimes \varphi(\tau_1, \tau_2) = 0.\]

This relation \((2)\) we call the **GENERAL PARALLELISM RULE** (GPR), the map \(\Phi\) "projects" the "changes" \(D\tilde{\sigma}\) of the section \(\tilde{\sigma}\) on the "known" section \(\sigma\), and \(\varphi\) "works" usually on the (local) bases of the bundles where \(\sigma\) and \(D\tilde{\sigma}\) take values.

Here is a simple and well known example. Let \(\xi_1\) be the bundle of exterior \(p\)-forms on \(M\) with the available differential operator exterior derivative \(d : \Lambda^p(M) \to \Lambda^{p+1}(M)\). In the case of physically important example of gauge fields, i.e. Lie algebra valued differential forms, with "\(\Phi = \) exterior product" and "\(\varphi = \) Lie bracket \([, ,]\)" , the GPR \((2)\) looks as follows:

\[(\wedge, [ , ]; d)(\alpha^i \otimes E_i, \beta^j \otimes E_j) = (\wedge, [ , ])(\alpha^i \otimes E_i, d\beta^j \otimes E_j) \]

\[= \alpha^i \wedge d\beta^j \otimes [E_i, E_j] = 0,\]

where \(\alpha \in \Lambda(M), \beta \in \Lambda^p(M), \{E_i\} \) is a basis of the corresponding Lie algebra.
Further we are going to consider particular cases of the (GPR) (2) with explicitly defined differential operators whenever they participate in the definition of the section of interest.

2. The GPR in Action

2.1. Classical mechanics

We begin studying the potential strength of the GPR in the frame of classical mechanics with very simple examples.

2.1.1. Integral invariance relations

These relations have been introduced and studied from the point of view of applications in mechanics by Lichnerowicz [1]. The definition is: a p-form on $M$ is called an integral invariance relation for the vector field $X$ on $M$ if $\iota(X)\alpha = 0$.

We specify the bundles over the real finite dimensional manifold $M$:

$\xi_1 = TM$; $\xi_2 = \Lambda^p(T^*M)$, $\eta_1 = \eta_2 = \eta_3 = M \times \mathbb{R}$, denote

$\text{Sec}(M \times \mathbb{R}) \equiv C^\infty(M)$,

$\Phi=$ substitution operator, denoted by $\iota(X), X \in \text{Sec}(TM)$;

$\varphi=$ point-wise product of functions, there is no $\xi$.

We denote by 1 the function $f(x) = 1, x \in M$. Consider the sections $X \otimes 1 \in \text{Sec}(TM \otimes M \times \mathbb{R}); \alpha \otimes 1 \in \text{Sec}(\Lambda^p(T^*M) \otimes M \times \mathbb{R})$. Then the GPR leads to

$$(\Phi, \varphi)(X \otimes 1, \alpha \otimes 1) = \iota(X)\alpha \otimes 1 = \iota(X)\alpha = 0.$$  \hfill (3)

We introduce now the differential operator $d$: if $\alpha$ is an exact 1-form, so that $\xi = M \times \mathbb{R}$, $\alpha = df, f \in C^\infty(M)$, the above relation becomes

$$(\Phi, \varphi; d)(X \otimes 1, f \otimes 1) = (\Phi, \varphi)(X \otimes 1, df \otimes 1) = \iota(X)df = X(f) = 0,$$

i.e. the derivative of $f$ along the vector field $X$ is equal to zero. So, we obtain the well known relation, defining the first integrals $f$ of the dynamical system determined by the vector field $X$. In this sense the first integrals of $X$ may be called $(\Phi, \varphi, d)$-parallel with respect to $X$, where $\Phi$ and $\varphi$ are defined above.

2.1.2. Absolute and relative integral invariants

These quantities have been introduced and studied in mechanics by Cartan [2]. By definition, a p-form $\alpha$ is called an absolute integral invariant of
the vector field $X$ if $i(X)\alpha = 0$ and $i(X)d\alpha = 0$. And $\alpha$ is called a relative integral invariant of the field $X$ if $i(X)d\alpha = 0$. So, in our terminology, we can call the relative integral invariants of $X$ $(\Phi, \varphi; d)$-parallel with respect to $X$, and the absolute integral invariants of $X$ have additionally $(\Phi, \varphi)$-parallelism with respect to $X$, with $(\Phi, \varphi)$ as defined above. A special case is when $p = n$, and $\omega \in \Lambda^n(M)$ is a volume form on $M$.

2.1.3. Symplectic mechanics

Symplectic manifolds are even dimensional and have a distinguished non-degenerate closed 2-form $\omega$, $d\omega = 0$. This structure may be defined in terms of the GPR in the following way. Choose $\xi_1 = \eta_1 = \eta_2 = M \times \mathbb{R}$, $\xi = \Lambda^2(M)$, $\xi_2 = \Lambda^3(M)$, and $d$ as a differential operator. Consider now the section $1 \otimes 1 \in \text{Sec}(M \times \mathbb{R}) \times \text{Sec}(M \times \mathbb{R})$ and the section $\omega \otimes 1 \in \text{Sec}(\Lambda^2(M) \otimes \text{Sec}(M \times \mathbb{R}))$, with $\omega$ - nondegenerate. The map $\Phi$ is the product $f.\omega$ and the map $\varphi$ is the product $f.g$ of functions. So, we have

$$(\Phi, \varphi; d)(1 \otimes 1, \omega \otimes 1) = (\Phi, \varphi)(1 \otimes 1, d\omega \otimes 1) = 1.d\omega \otimes 1 = d\omega = 0.$$ 

Hence, the relation $d\omega = 0$ is equivalent to the requirement $\omega$ to be $(\Phi, \varphi; d)$-parallel with respect to the section $1 \in \text{Sec}(M \times \mathbb{R})$.

The hamiltonian vector fields $X$ are defined by the condition $L_X\omega = 0$, which is equivalent to $di(X)\omega = 0$, where $L_X$ is the Lie derivative. So in terms of the GPR we obtain

$$(\Phi, \varphi; d)(1 \otimes 1, i(X)\omega \otimes 1) = (\Phi, \varphi)(1 \otimes 1, di(X)\omega \otimes 1) = L_X\omega \otimes 1 = 0,$$

so, we say that $X$ is hamiltonian if $i(X)\omega$ is $(\Phi, \varphi; d)$-parallel with respect to $1$.

The induced Poisson structure $\{f, g\} = \omega(X_f, X_g)$ is given in our terms by setting $\Phi = \omega^{-1}$, where $\omega^{-1} = id_{TM}$, $\varphi$ is the point-wise product of functions. We get

$$(\Phi, \varphi)(df \otimes 1, dg \otimes 1) = \omega^{-1}(df, dg) \otimes 1.$$ 

A closed 1-form $\alpha$, $d\alpha = 0$, is a first integral of the hamiltonian system $Z$ if $i(Z)\alpha = 0$, so, in our terms we say that the first integrals $\alpha$ are $(i, \varphi)$-parallel with respect to $Z$: $(i, \varphi)(Z \otimes 1, \alpha \otimes 1 = i(Z)\alpha \otimes 1 = 0$. The well known property that the Poisson bracket of two first integrals $\alpha$ and $\beta$ of $Z$ is again a first integral of $Z$ may be formulated as follows: the function $\omega^{-1}(\alpha, \beta)$ is $(i, \varphi; d)$-parallel with respect to $Z$. 

2.2. Frobenius integrability theorems and linear connections

2.2.1. Frobenius integrability theorems

Let $\Delta = (X_1, \ldots, X_r)$ be a differential system on $M$, i.e. the vector fields $X_i, i = 1, \ldots, r$ define a locally stable submodule of $\text{Sec}(TM)$ and at every point $p \in M$ the subspace $\Delta^r_p \subset T_p(M)$ has dimension $r$. Then $\Delta^r$ is called integrable if $[X_i, X_j] \in \Delta, i, j = 1, \ldots, r$. Denote by $\Delta_p^{n-r} \subset T_p(M)$ the complimentary subspace: $\Delta_p^r \oplus \Delta_p^{n-r} = T_p(M)$, and let $\pi : T_p(M) \to \Delta_p^{n-r}$ be the corresponding projection. So, the corresponding Frobenius integrability condition means $\pi([X_i, X_j]) = 0, i, j = 1, \ldots, r$.

In terms of the GPR we set $\Phi = \pi \circ [\cdot]$ and $\varphi$ again the point-wise product of functions. The integrability condition now is

$$(\Phi, \varphi)(X_i \otimes 1, X_j \otimes 1) = \pi([X_i, X_j]) \otimes 1 = 0, \quad i, j = 1, \ldots, r.$$ 

In the dual formulation we have the Pfaff system $\Delta^* = (\alpha_1, \ldots, \alpha_{n-r})$ of 1-forms. Then $\Delta^*$ is integrable if $d\alpha \wedge \alpha_1 \wedge \cdots \wedge \alpha_{n-r} = 0, \alpha \in \Delta^*$. In terms of GPR we set $\varphi$ the same as above, $\Phi = \wedge$ and $d$ as differential operator.

$$(\Phi, \varphi; d)(\alpha_1 \wedge \cdots \wedge \alpha_{n-r} \otimes 1, \alpha \otimes 1) = d\alpha \wedge \alpha_1 \wedge \cdots \wedge \alpha_{n-r} \otimes 1 = 0.$$ 

2.2.2. Linear connections

The concept of a linear connection in a vector bundle has proved to be of great importance in geometry and physics. In fact, it allows to differentiate sections of vector bundles along vector fields, which is a basic operation in differential geometry, and in theoretical physics the physical fields are represented mainly by sections of vector bundles. We recall now how one comes to it.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Then we can find its differential $df$. The map $f \to df$ is $\mathbb{R}$-linear: $d(\kappa f) = \kappa df, \kappa \in \mathbb{R}$, and it has the derivative property $d(fg) = fdg + gdff$. These two properties are characteristic ones, and they are carried to the bundle situation as follows.

Let $\xi$ be a vector bundle over $M$. We always have the trivial bundle $\xi_o = M \times \mathbb{R}$. Consider now $f \in C^\infty(M)$ as a section of $\xi_o$. We note that $\text{Sec}(\xi_o) = C^\infty(M)$ is a module over itself, so we can form $df$ with the above two characteristic properties. The new object $df$ lives in the space $\Lambda^1(M)$ of 1-forms on $M$, so it defines a linear map $df : \text{Sec}(TM) \to \text{Sec}(\xi_o), df(X) = X(f)$. Hence, we have a map $\nabla$ from $\text{Sec}(\xi_o)$ to the 1-forms with values in $\text{Sec}(\xi_o)$, and this map has the above two characteristic
properties. We say that $\nabla$ defines a linear connection in the vector bundle $\xi_o$.

In the general case the sections $\text{Sec}(\xi)$ of the vector bundle $\xi$ form a module over $C^\infty(M)$. So, a linear connection $\nabla$ in $\xi$ is a $\mathbb{R}$-linear map $\nabla: \text{Sec}(\xi) \to \Lambda^1(M, \xi)$. In other words, $\nabla$ sends a section $\sigma \in \text{Sec}(\xi)$ to a 1-form $\nabla\sigma$ valued in $\text{Sec}(\xi)$ in such a way, that

$$\nabla(k \sigma) = k \nabla(\sigma), \quad \nabla(f \sigma) = df \otimes \sigma + f \nabla(\sigma),$$

where $k \in \mathbb{R}$ and $f \in C^\infty(M)$. If $X \in \text{Sec}(TM)$ then we have the composition $i(X) \circ \nabla$, so that

$$i(X) \circ \nabla(f \sigma) = X(f) \sigma + f \nabla_X(\sigma),$$

where $\nabla_X(\sigma) \in \text{Sec}(\xi)$.

In terms of the GPR we put $\xi_1 = TM$, $\tilde{\xi}$ is the given vector bundle $\xi$, $\xi_2 = \Lambda^1(M) \otimes \tilde{\xi}$, and $\eta_1 = \eta_2 = \eta_3 = \xi_o$. Also, $\Phi(X, \nabla\sigma) = \nabla_X\sigma$ and $\varphi(f, g) = f.g$. Hence, choosing $f = g = 1$ we obtain

$$(\Phi, \varphi; \nabla)(X \otimes 1, \sigma \otimes 1) = (\Phi, \varphi)(X \otimes 1, \nabla\sigma \otimes 1) = \nabla_X\sigma \otimes 1 = \nabla_X\sigma, \quad (5)$$

and the section $\sigma$ is called $\nabla$-parallel with respect to $X$ if $\nabla_X\sigma = 0$.

2.2.3. Covariant exterior derivative

The space of $\xi$-valued $p$-forms $\Lambda^p(M, \xi)$ on $M$ is isomorphic to $\Lambda^p(M) \otimes \text{Sec}(\xi)$. So, if $(\sigma_1, \ldots, \sigma_r)$ is a local basis of $\text{Sec}(\xi)$, every $\Psi \in \Lambda^p(M, \xi)$ is represented by $\psi^i \otimes \sigma_i$, $i = 1, \ldots, r$, where $\psi^i \in \Lambda^p(M)$. Clearly the space $\Lambda(M, \xi) = \sum_{p=0}^\infty \Lambda^p(M, \xi)$, where $\Lambda^0(M, \xi) = \text{Sec}(\xi)$, is a $\Lambda(M) = \sum_{p=0}^\infty \Lambda^p(M)$-module: $\alpha \Psi = \alpha \wedge \Psi = (\alpha \wedge \psi^i) \otimes \sigma_i$.

A linear connection $\nabla$ in $\xi$ generates covariant exterior derivative $D: \Lambda^p(M, \xi) \to \Lambda^{p+1}(M, \xi)$ in $\Lambda(M, \xi)$ according to the rule

$$D\Psi = D(\psi^i \otimes \sigma_i) = d\psi^i \otimes \sigma_i + (-1)^p \psi^i \wedge \nabla(\sigma_i)$$

$$= (d\psi^i + (-1)^p \psi^j \wedge \Gamma^i_{\mu j} dx^\mu) \otimes \sigma_i = (D\Psi)^i \otimes \sigma_i.$$

We may call now a $\xi$-valued $p$-form $\Psi$ $\nabla$-parallel if $D\Psi = 0$, and $(X, \nabla)$-parallel if $i(X)D\Psi = 0$. This definition extends in a natural way to $q$-vectors with $q \leq p$. Actually, the substitution operator $i(X)$ extends to (decomposable) $q$-vectors $X_1 \wedge X_2 \wedge \cdots \wedge X_q$ as follows:

$$i(X_1 \wedge X_2 \wedge \cdots \wedge X_q)\Psi = i(X_q) \circ i(X)_{q-1} \circ \cdots \circ i(X_1)\Psi,$$
and extends to nondecomposable $q$-vectors by linearity. Hence, if $\Theta$ is a section of $\Lambda_q(TM)$, i.e. $\Theta$ is a $q$-vector, we may call $\Psi \in \Lambda^p(M, \xi) \langle \nabla, \Theta \rangle$-parallel if $i(\Theta)D\Psi = 0$.

Denote now by $L_{\xi}$ the vector bundle of (linear) homomorphisms $(\Pi, id) : \xi \rightarrow \xi$, and let $\Pi \in \text{Sec}(L_{\xi})$. Let $\chi \in \text{Sec}(\Lambda_q(TM) \otimes L_{\xi})$ may be represented as $\Theta \otimes \Pi$. Let $\Psi$ be a $\xi$-valued form. The map $\Phi$ will act as: $\Phi(\Theta, \Psi) = i(\Theta)(\psi^i \otimes \sigma_i)$, and the map $\varphi$ will act as: $\varphi(\Pi, \sigma_i) = \Pi(\sigma_i)$. So, if $\nabla(\sigma_k) = \Gamma_{\mu k}^j dx^\mu \otimes \sigma_j$, we may call $\Psi (\Theta, \nabla)$-parallel if

$$(\Phi, \varphi; D)(\Theta \otimes \Pi, \Psi = \psi^i \otimes \sigma_i) = (\Phi, \varphi)(\Theta \otimes \Pi, (D\Psi)^i \otimes \sigma_i) = i(\Theta)(D\Psi)^i \otimes \Pi(\sigma_i) = 0.$$  

If we have isomorphisms $\otimes^p TM \sim \otimes^p T^*M$, $p = 1, 2, \ldots$, defined in some natural way (e.g. through a metric tensor field), then to any $p$-form $\alpha$ corresponds unique $p$-vector $\vec{\alpha}$. In this case we may talk about autoparallel objects with respect to a (point-wise) bilinear map $\varphi : (\xi \times \xi) \rightarrow \eta$, where $\eta$ is also a vector bundle over $M$. So, $\Psi = \alpha^k \otimes \sigma_k$ may be called $(i, \varphi; \nabla)$-autoparallel with respect to the isomorphism "~" if

$$(i, \varphi; \nabla)(\vec{\alpha}^k \otimes \sigma_k, \alpha^m \otimes \sigma_m) = i(\vec{\alpha}^k)\, d\alpha^m \otimes \varphi(\sigma_k, \sigma_m) + (-1)^p i(\vec{\alpha}^k)(\alpha^j \wedge \Gamma_{\mu j}^m \, dx^\mu) \otimes \varphi(\sigma_k, \sigma_m) = 0.$$  

Although the above examples do not, of course, give a complete list of the possible applications of the GPR (2), they will serve as a good basis for the physical applications we are going to consider further.

### 3. Physical applications of GPR

#### 3.1. Autoparallel vector fields and 1-forms

In nonrelativistic and relativistic mechanics the vector fields $X$ on a manifold $M$ are the local representatives (velocity vectors) of the evolution trajectories for point-like objects. The condition that a particle is free is mathematically represented by the requirement that the corresponding vector field $X$ is autoparallel with respect to a given connection $\nabla$ (covariant derivative) in $TM$:

$$i(X)\nabla X = 0, \quad \text{or in components,} \quad X^a \nabla_\sigma X^\mu + \Gamma^\mu_{\sigma \nu} X^a X^\nu = 0.$$  

In view of the physical interpretation of $X$ as velocity vector field the usual latter used instead of $X$ is $u$. The above equation (8) presents a system of nonlinear partial differential equations for the components $X^\mu$, or $u^\mu$.  


When reduced to 1-dimensional submanifold which is parametrised locally by the appropriately chosen parameter \( s \), (8) gives a system of ordinary differential equations:

\[
\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\sigma\nu} \frac{dx^\nu}{ds} \frac{dx^\nu}{ds} = 0,
\]

(9)

and (9) are known as ODE defining the geodesic (with respect to \( \Gamma \)) lines in \( M \). When \( M \) is riemannian with metric tensor \( g \) and the corresponding Levi-Civita connection, i.e. \( \nabla g = 0 \) and \( \Gamma^\mu_{\nu\sigma} = \Gamma^\mu_{\sigma\nu} \), then the solutions of (9) give the extreme (shortest or longest) distance \( \int_a^b ds \) between the two points \( a, b \in M \), so (9) are equivalent to

\[
\delta \left( \int_a^b ds \right) = \delta \left( \int_a^b \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}} \right) = 0.
\]

A system of particles that move along the solutions to (9) with \( g \)-the Minkowski metric and \( g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} > 0 \), is said to form an inertial frame of reference.

It is interesting to note that the system (8) has (3+1)-soliton-like (even spatially finite) solutions on Minkowski space-time. In fact, in canonical coordinates \((x^1, x^2, x^3, x^4 = x, y, z, \xi)\) let \( u^\mu = (0, 0, \pm \frac{v}{c}, f) \) be the components of \( u \), where \( 0 < v = \text{const} < c \), and \( c \) is the velocity of light, so \( \frac{v}{c} < 1 \) and \( u^\nu u_\nu = \left(1 - \frac{v^2}{c^2}\right) f^2 > 0 \). Then every function \( f \) of the kind

\[
f(x, y, z, \xi) = f \left(x, y, \alpha(z \mp \frac{v}{c} \xi)\right), \ \alpha = \text{const},
\]

for example

\[
\alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}},
\]

defines a solution to (8). If \( u_\sigma u^\sigma = 0 \) then equations (8) \((X = u)\), are equivalent to \( u^\mu (du)_{\mu\nu} = 0 \), where \( d \) is the exterior derivative. In fact, since the connection used is riemannian, we have \( 0 = \nabla_\mu \frac{1}{2} (u^\nu u_\nu) = u^\nu \nabla_\mu u_\nu \), so the relation \( u^\nu \nabla_\nu u_\mu - u^\nu \nabla_\mu u_\nu = 0 \) holds and is obviously equal to \( u^\mu (du)_{\mu\nu} = 0 \). The soliton-like solution is defined by \( u = (0, 0, \pm f, f) \) where the function \( f \) is of the form

\[
f(x, y, z, \xi) = f(x, y, z \mp \xi).
\]

Clearly, for every autoparallel vector field \( u \) (or one-form \( u \)) there exists a canonical coordinate system on the Minkowski space-time, in which \( u \) takes such a simple form: \( u^\mu = (0, 0, \alpha f, f) \), \( \alpha = \text{const} \). The dependence of
$f$ on the three spatial coordinates $(x, y, z)$ is arbitrary, so it is allowed to be chosen soliton-like and, even, finite. Let now $\rho$ be the mass-energy density function, so that $\nabla_\sigma (\rho u^\sigma) = 0$ gives the mass-energy conservation, i.e. the function $\rho$ defines those properties of our physical system which identify the system during its evolution. In this way the tensor conservation law

$$\nabla_\sigma (\rho u^\sigma u^\mu) = (\nabla_\sigma \rho u^\sigma) u^\mu + \rho u^\sigma \nabla_\sigma u^\mu = 0$$

describes the two aspects of the physical system: its dynamics through equations (8) and its mass-energy conservation properties.

The properties described give a connection between free point-like objects and (3+1) soliton-like autoparallel vector fields on Minkowski space-time. Moreover, they suggest that extended free objects with more complicated space-time dynamical structure may be described by some appropriately generalized concept of autoparallel mathematical objects.

### 3.2. Electrodynamics

#### 3.2.1. Maxwell equations

The Maxwell equations $dF = 0$, $d\star F = 0$ in their 4-dimensional formulation on Minkowski space-time $(M, \eta)$, $\text{sign}(\eta) = (-, -, -, +)$ and the Hodge $\star$ is defined by $\eta$, make use of the exterior derivative as a differential operator. The field has, in general, 2 components $(F, \star F)$, so the interesting bundle is $\Lambda^2(M) \otimes V$, where $V$ is a real 2-dimensional vector space. Hence the adequate mathematical field will look like $\Omega = F \otimes e_1 + \star F \otimes e_2$, where $(e_1, e_2)$ is a basis of $V$. The exterior derivative acts on $\Omega$ as:

$$d\Omega = dF \otimes e_1 + d\star F \otimes e_2,$$

and the equation $d\Omega = 0$ gives the vacuum Maxwell equations.

In order to interpret in terms of the above given general view (GPR) on parallel objects with respect to given sections of vector bundles and differential operators we consider the sections (see the above introduced notation) $(1 \times 1, \Omega \times 1)$ and the differential operator $d$. Hence, the GPR acts as follows:

$$(\Phi, \varphi; d)(1 \otimes 1, \Omega \otimes 1) = (\Phi, \varphi)(1 \otimes 1, d\Omega \otimes 1) = (d\Omega \otimes 1).$$

The corresponding $(\Phi, \varphi; d)$-parallelism leads to $d\Omega = 0$. In presence of electric $j$ and magnetic $m$ currents, considered as 3-forms, the parallelism condition does not hold and on the right-hand side we'll have non-zero term, so the full condition is

$$(\Phi, \varphi)(1 \otimes 1, (dF \otimes e_1 + d\star F \otimes e_2) \otimes 1) = (\Phi, \varphi)(1 \otimes 1, (m \otimes e_1 + j \otimes e_2) \otimes 1)$$

(10)
The case \( m = 0, F = dA \) is, obviously a special case.

### 3.2.2. Extended Maxwell equations

The extended Maxwell equations (on Minkowski space-time) in vacuum \(^3\) read:

\[
F \wedge \ast dF = 0, \quad (\ast F) \wedge (\ast d \ast F) = 0, \quad F \wedge (\ast d \ast F) + (\ast F) \wedge (\ast d F) = 0 \quad (11)
\]

They may be expressed through the GPR in the following way. On \((M, \eta)\) we have the bijection between \(\Lambda_2(TM)\) and \(\Lambda^2(T^*M)\) defined by \(\eta\), which we denote by \(F \leftrightarrow \tilde{F}\). So, equations (11) are equivalent to

\[
i(\tilde{F})dF = 0, \quad i(\ast\tilde{F})d \ast F = 0, \quad i(\tilde{F})d \ast F + i(\ast\tilde{F})dF = 0.
\]

We consider the sections \(\tilde{\Omega} = \tilde{F} \otimes e_1 + \ast\tilde{F} \otimes e_2\) and \(\Omega = F \otimes e_1 + \ast F \otimes e_2\) with the differential operator \(d\). The maps \(\Phi\) and \(\varphi\) are defined as: \(\Phi\) is the substitution operator \(i\), and \(\varphi = \vee\) is the symmetrized tensor product in \(V\). So we obtain

\[
(\Phi, \varphi; d)(\tilde{F} \otimes e_1 + \ast\tilde{F} \otimes e_2, F \otimes e_1 + \ast F \otimes e_2) = i(\tilde{F})dF \otimes e_1 \vee e_1 + i(\ast\tilde{F})d F \otimes e_2 \vee e_2 + (i(\tilde{F})d \ast F + i(\ast\tilde{F})dF) \otimes e_1 \vee e_2 = 0.
\]

Equations (12) may be written down also as \(((i, \vee)\tilde{\Omega})d\Omega = 0\).

Equations (12) are physically interpreted as describing locally the intrinsic energy-momentum exchange between the two components \(F\) and \(\ast F\): the equations \(i(\tilde{F})dF = 0\) and \(i(\ast\tilde{F})d \ast F = 0\) say that every component keeps locally its energy-momentum, and the third equation \(i(\tilde{F})d \ast F + i(\ast\tilde{F})dF = 0\) says (in accordance with the first two) that if \(F\) transfers energy-momentum to \(\ast F\), then \(\ast F\) transfers the same quantity of energy-momentum to \(F\).

If the field exchanges (loses or gains) energy-momentum with some external systems, Extended Electrodynamics describes the potential abilities of the external systems to gain or lose energy-momentum from the field by means of 4 one-forms (currents) \(J_a, a = 1, 2, 3, 4\), and explicitly the exchange is given by

\[
i(\tilde{F})dF = i(\tilde{J}_1)F, \quad i(\ast\tilde{F})d \ast F = i(\tilde{J}_2)F, \quad i(\tilde{F})d \ast F + i(\ast\tilde{F})dF = i(\tilde{J}_3)F + i(\tilde{J}_4) \ast F.
\]
It is additionally assumed that every couple \((J_a, J_b)\) defines a completely integrable Pfaff system, i.e. the following equations hold:

\[
dJ_a \wedge J_a \wedge J_b = 0, \quad a, b = 1, \ldots, 4. \tag{14}
\]

The system (14) gives 12 equations for the currents \(J_a\).

### 3.3. Yang-Mills equations

In this case the field is a connection, represented locally by its connection form \(\omega \in \Lambda^1(M) \otimes g\), where \(g\) is the Lie algebra of the corresponding Lie group \(G\). If \(D\) is the corresponding covariant derivative, and \(\Omega = D\omega\) is the curvature, then Yang-Mills equations read \(D \ast \Omega = 0\). The formal difference with the Maxwell case is that \(G\) may NOT be commutative, and may have, in general, arbitrary finite dimension. So, the two sections are \(1 \otimes 1\) and \(* \Omega \otimes 1\), the maps \(\Phi\) and \(\varphi\) are product of functions and the differential operator is \(D\). So, we may write

\[
(\Phi, \varphi; D)(1 \otimes 1, * \Omega \otimes 1) = D \ast \Omega \otimes 1 = 0. \tag{15}
\]

Of course, equations (13) are always coupled to the Bianchi identity \(D\Omega = 0\).

#### 3.3.1. Extended Yang-Mills equations

In this case the field is an arbitrary 2-form \(\Psi\) on \((M, \eta)\) with values in a Lie algebra \(g\), \(\dim(g) = r\). If \(\{E_i\}, i = 1, 2, \ldots, r\) is a basis of \(g\) we have \(\Psi = \psi^i \otimes E_i\) and \(\tilde{\Psi} = \tilde{\psi}^i \otimes E_i\). The map \(\Phi\) is the substitution operator, the map \(\varphi\) is the corresponding Lie product \([,]\), and the differential operator is the exterior covariant derivative with respect to a given connection \(\omega\): 

\[D\Psi = d\Psi + [\omega, \Psi].\]

The GPR gives

\[
(\Phi, \varphi; D)(\tilde{\psi}^i \otimes E_i, \psi^j \otimes E_j) = i(\tilde{\psi}^i)(d\psi^m + \omega^j \wedge \psi^k C_{jk}^m) \otimes [E_m, E_i] = 0, \tag{16}
\]

where \(C_{jk}^m\) are the corresponding structure constants. If the connection is the trivial one, then \(\omega = 0\) and \(D \rightarrow d\), so, this equation reduces to

\[
i(\tilde{\psi}^i) d\psi^i \otimes C_{ij}^k \otimes E_k = 0. \tag{17}
\]

If, in addition, instead of \([,\) we assume for \(\varphi\) some bilinear map \(f: g \times g \rightarrow g\), such that in the basis \(\{E_i\}\) \(f\) is given by \(f(E_i, E_i) = E_i\), and \(f(E_i, E_j) = 0\) for \(i \neq j\) the last relation reads

\[
i(\tilde{\psi}^i) d\psi^i \otimes E_i = 0, \quad i = 1, 2, \ldots, r. \tag{18}
\]
The last equations define the components $\psi^i$ as independent 2-forms (of course $\psi^i$ may be arbitrary $p$-forms). If the bilinear $\varphi$ map is chosen to be the symmetrized tensor product $\vee : g \times g \to g \vee g$, we obtain
\[
i(\psi^i) d\psi^j \otimes E_i \vee E_j = 0, \quad i \leq j = 1, \ldots, r. \tag{19}
\]
Equations (19) may be used to model bilinear interaction among the components of $\Psi$. If the terms $i(\psi^i) d\psi^j \otimes E_i \vee E_j$ have the physical sense of energy-momentum exchange we may say that every component $\psi^i$ gets locally as much energy-momentum from $\psi^j$ as it gives to it. In fact, from (19) we have for every $i, j = 1, 2, \ldots, r$
\[
i(\ddot{\psi}^i) d\psi^i = 0, \quad \text{and} \quad i(\ddot{\psi}^i) d\psi^i + i(\ddot{\psi}^j) d\psi^j = 0. 
\]
Clearly, these equations may be considered as a natural generalization of equations (12).

3.4. **General Relativity**

In General Relativity the field function of interest is in a definite sense identified with a pseudometric $g$ on a 4-dimensional manifold, and only those $g$ are considered as appropriate to describe the real gravitational fields which satisfy the equations $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ are the components of the Ricci tensor. The main mathematical object which detects possible gravity is the Riemann curvature tensor $R_{\alpha\mu,\beta\nu}$, which is a second order nonlinear differential operator $R : g \to R(g)$. The map $\Phi$ is just a contraction:
\[
\Phi : (g_{\alpha\beta}, R_{\alpha\mu,\beta\nu}) = g^{\alpha\beta} R_{\alpha\mu,\beta\nu} = R_{\mu\nu}
\]
and is obviously bilinear. The map $\varphi$ is a product of functions, so the GPR gives
\[
(\Phi, \varphi; R)(g \otimes 1, g \otimes 1) = \Phi(g, R(g)) \otimes 1 = \text{Ric}(R(g)) \otimes 1 = 0. \tag{20}
\]
In presence of matter fields $\Psi^\alpha$, $\alpha = 1, 2, \ldots, r$, the system of equations is
\[
R_{\mu\nu} - \kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) = 0.
\]
It is easily obtained through the GPR if we modify the differential operator $R_{\alpha\mu,\beta\nu}$ to
\[
R_{\alpha\mu,\beta\nu} - \frac{\kappa}{2} (T_{\alpha\beta} g_{\mu\nu} + T_{\mu\nu} g_{\alpha\beta} - T_{\alpha\nu} g_{\mu\beta} - T_{\mu\beta} g_{\alpha\nu}) + \frac{\kappa}{3} (g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\nu} g_{\mu\beta}) T,
\]
where $\kappa$ is the gravitational constant, $T_{\mu\nu}(\Psi^\alpha) = T_{\nu\mu}(\Psi^\alpha)$ is the corresponding stress energy momentum tensor, and $T = g^{\mu\nu} T_{\mu\nu}$.
3.5. Schrödinger equation

The object of interest in this case is a map $\Psi : \mathbb{R}^4 \to \mathbb{C}$, and $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ is parametrized by the canonical coordinates $(x, y, z; t)$, where $t$ is the (absolute) time "coordinate". The operator $D$ used here is

$$D = i\hbar \frac{\partial}{\partial t} - H,$$

where $H$ is the corresponding Hamiltonian. The maps $\Phi$ and $\varphi$ are products of functions, so the GPR gives

$$(\Phi, \varphi; D)(1 \otimes 1, \Psi \otimes 1) = \left(1 \otimes \left(i\hbar \frac{\partial \Psi}{\partial t} - H\Psi\right)\right) \otimes 1 = 0. \quad (21)$$

3.6. Dirac equation

The original free Dirac equation on the Minkowski space-time $(M, \eta)$ makes use of the following objects: $\mathbb{C}^4$ — the canonical 4-dimensional complex vector space, $L_{\mathbb{C}^4}$ — the space of $\mathbb{C}$-linear maps $\mathbb{C}^4 \to \mathbb{C}^4$, $\Psi \in \text{Sec}(M \times \mathbb{C}^4)$, $\gamma \in \text{Sec}(T^*M \otimes L_{\mathbb{C}^4})$, and the usual differential $d : \psi^i \otimes e_i \to d\psi^i \otimes e_i$, where $\{e_i\}$, $i = 1, 2, 3, 4$, is a basis of $\mathbb{C}^4$. We identify further $L_{\mathbb{C}^4}$ with $(\mathbb{C}^4)^* \otimes \mathbb{C}^4$ and if $\{\varepsilon^i\}$ is a basis of $(\mathbb{C}^4)^*$, dual to $\{e_i\}$, we have the basis $\varepsilon^i \otimes e_j$ of $L_{\mathbb{C}^4}$. Hence, we may write

$$\gamma = \gamma^j_{\mu_1} dx^\mu \otimes (\varepsilon^i \otimes e_j),$$

and

$$\gamma(\Psi) = \gamma^j_{\mu_1} dx^\mu \otimes (\varepsilon^i \otimes e_j)(\psi^k \otimes e_k)$$

$$= \gamma^j_{\mu_1} dx^\mu \otimes \psi^k (\varepsilon^i, e_k) e_j = \gamma^j_{\mu_1} dx^\mu \otimes \psi^k \delta^i_k e_j = \gamma^j_{\mu_1} dx^\mu \otimes \varepsilon^i \otimes e_j.$$

The 4 matrices $\gamma^j_{\mu_1}$ satisfy $\gamma^j_{\mu_1} \gamma^j_{\mu_2} + \gamma^j_{\mu_1} \gamma^j_{\mu_2} = 2\eta_{\mu_1\mu_2} id_{\mathbb{C}^4}$, so they are nondegenerate: $\det(\gamma^j_{\mu_1}) \neq 0$, $\mu = 1, 2, 3, 4$, and we can find $(\gamma^j_{\mu_1})^{-1}$ and introduce $\gamma^{-1}$ by

$$\gamma^{-1} = ((\gamma^j_{\mu_1})^{-1})^j_{\mu_2} dx^\mu \otimes (\varepsilon^i \otimes e_j).$$

We introduce now the differential operators $D^\pm : \text{Sec}(M \times \mathbb{C}^4) \to \text{Sec}(T^*M \otimes \mathbb{C}^4)$ through the formula: $D^\pm = i\text{id} \pm \frac{1}{2} m \gamma^{-1}$, $i = \sqrt{-1}$, $m \in \mathbb{R}$. The corresponding maps are: $\Phi = \eta$, $\varphi : L_{\mathbb{C}^4} \times \mathbb{C}^4 \to \mathbb{C}^4$ given by $\varphi(\alpha^* \otimes \beta, \rho) = (\alpha^*, \rho) \beta$. We obtain

$$(\Phi, \varphi; D^\pm)(\gamma, \Psi) = (\Phi, \varphi)(\gamma^j_{\mu_1} dx^\mu \otimes (\varepsilon^i \otimes e_j), i \frac{\partial \psi^k}{\partial x^\nu} dx^\nu \otimes e_k$$

$$\pm \frac{1}{2} m(\gamma^j_{\mu_1})^{-1}_{\nu} dx^\nu \otimes (\varepsilon^r \otimes e_s) \psi^m e_m).$$
In terms of extended parallelism we can say that the Dirac equation is equivalent to the requirement the section \( Q E \) Sec\((A_4 \times C_4)\) to be parallel with respect to the given \( y E \) Sec\((M \times L_{C_4})\). Finally, in presence of external electromagnetic field \( A = A_\mu dx^\mu \) the differential operators \( D^* \) modify to \( D^* = (\text{id} - eA) \pm \frac{1}{2} m\gamma^{-1} \), where \( e \) is the electron charge.

4. Conclusion

It was shown that the GPR, defined by relation (2), naturally generalizes the geometrical concept of parallel transport, and that it may be successfully used as a unified tool to represent formally important equations in theoretical physics. If \( Q, \) is the object of interest then the GPR specifies the following things: the change \( D@ \) of \( @, \) the object \( Q \) with respect to which we consider the change, and the projection of the change \( DQ, \) on \( Q. \) When \( @ = Q \) we may speak about autoparallel objects, and in this case, as well as when the differential operator \( D \) depends on \( @ \) and its derivatives, we obtain nonlinear equation(s).

It was further shown that Maxwell vacuum equations appear as d-parallel, i.e. without specifying any projection procedure. This determines their linear nature and leads to the lack of spatial soliton-like solutions. The extended Maxwell equations (11) are naturally cast in the form of autoparallel (nonlinear) equations, and, as it was shown in our former works, they admit photon-like (3+1) spatially finite and spatially localized solutions, and some of them admit naturally defined spin properties. The Yang-Mills and the Extended Yang-Mills equations were also described in this way. The Einstein equations of General Relativity also admit such a formulation.

In quantum physics the Schrödinger equation admits the "parallel" formulation in a very simple way. A bit more complicated was to put the Dirac equation in this formulation, and this is due to the more complicated
mathematical structure of \( \gamma = \gamma_{\mu}^i dx^\mu \otimes (\varepsilon^i \otimes e_j) \).

These important examples make us think that the introduced in this paper extended concept for \((\Phi, \varphi; D)-parallel\) objects as a natural generalization of the existing geometrical concept for \(\nabla\)-parallel objects, may be successfully used in various directions, in particular, in searching for appropriate nonlinearizations of the existing linear equations in theoretical and mathematical physics.

References

We consider the action of $SO(N, C)$-action on simply connected minimal surfaces in the $N$-dimensional Euclidean space $\mathbb{R}^N$. If the minimal surface is holomorphic with respect to an appropriate orthogonal complex structure on $\mathbb{R}^N$, then $SO(N, C)$ preserves the holomorphicity. In this paper, we shall study minimal surfaces appearing as bubbles under the action of $SO(N, C)$ and prove that if $SO(N, C)$ preserves the stableness of the minimal surface, then the minimal surface is holomorphic with respect to an orthogonal complex structure on $\mathbb{R}^N$.

1. Introduction

Let $S$ be a branched minimal immersion of the unit disk $D$ into $\mathbb{R}^N$. Then we have the integral representation of $S$ by

$$2\text{Re} \int \frac{\partial S}{\partial z} \, dz.$$ 

Let $SO(N, C)$ be the complex orthogonal group acting on $\mathbb{C}^N$. Then an element $A$ of $SO(N, C)$ acts on $S$ as follows:

$$A \cdot S = 2\text{Re} \int A \frac{\partial S}{\partial z} \, dz.$$ 

Since

$$\left\langle \frac{\partial A \cdot S}{\partial z}, \frac{\partial A \cdot S}{\partial z} \right\rangle = \left\langle A \frac{\partial S}{\partial z}, A \frac{\partial S}{\partial z} \right\rangle = \left\langle \frac{\partial S}{\partial z}, \frac{\partial S}{\partial z} \right\rangle = 0,$$

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A · S is a weakly conformal harmonic map, that is, a branched minimal immersion.

Remark 1.1. See [1] about the action of $SO(N, \mathbb{C})$ on non simply connected minimal surfaces.

Let $T$ be a $2m$-dimensional subspace of $\mathbb{R}^N$. If we associate an orthogonal complex structure $J$ to $T$, then we call $T$ with $J$ a complex subspace of $\mathbb{R}^N$. We call $S$ to be holomorphic if the image of $S$ is contained in $T$ and $S$ is holomorphic in $T$. Holomorphic maps into a complex subspace are typical examples as stable minimal surfaces. Using a Lawson’s result [2], we see that $SO(N, \mathbb{C})$ preserves the holomorphicity.

In this paper, we study the bubbling off with respect to minimal surfaces under the action of $SO(N, \mathbb{C})$ and obtain the characterization of holomorphic maps in stable minimal surfaces as follows.

**Theorem 1.1.** If $A · S$ is stable for all $A \in SO(N, \mathbb{C})$, then $S$ is holomorphic.

2. Complex subspaces in $\mathbb{R}^N$

Let $\mathbb{R}^N$ be the $N$-dimensional Euclidean space with the innerproduct $\langle \ , \ \rangle$ and $\mathbb{C}^N$ the complexification of $\mathbb{R}^N$ with the complex linear extension $\langle \ , \ \rangle$. Let $SO(N, \mathbb{C})$ be the complex orthogonal group defined by

$$\{ A \in GL(N, \mathbb{C}) : \langle AX, AY \rangle = \langle X, Y \rangle, \ X, Y \in \mathbb{C}^N \}.$$ 

Let $T$ be a linear subspace of $\mathbb{C}^N$. Then we call $T$ to be isotropic if $\langle \ , \ \rangle$ vanishes on $T$. Let $T$ be an $s$-dimensional isotropic subspace in $\mathbb{C}^N$. Then there is an orthonormal vectors $e_1, \ldots, e_{2s}$ such that

$$\frac{1}{\sqrt{2}}(e_1 - ie_2), \ldots, \frac{1}{\sqrt{2}}(e_{2s-1} - ie_{2s})$$

is a unitary basis of $T$. Then we obtain the $2s$-dimensional linear subspace spanned by $e_1, \ldots, e_{2s}$ in $\mathbb{R}^N$ which has the orthogonal complex structure $J$ given by $Je_1 = e_2, \ldots, Je_{2s-1} = e_{2s}$. Note that $J$ is independent on the choice of $e_1, \ldots, e_{2s}$. Thus we can construct the complex $s$-dimensional complex subspace of $\mathbb{R}^N$ from an $s$-dimensional isotropic subspace of $\mathbb{C}^N$. Conversely, it is easy to see that an $s$-dimensional complex subspace of $\mathbb{R}^N$ gives the $s$-dimensional isotropic subspace of $\mathbb{C}^N$.

**Lemma 2.1.** There exists a one to one correspondence between $s$-dimensional complex subspaces of $\mathbb{R}^N$ and $s$-dimensional isotropic subspaces of $\mathbb{C}^N$. 

Let $G(2, N)$ be the Grassmannian manifold which consists of oriented $2$-planes in $\mathbb{R}^N$ and $CP^{N-1}$ the $(N-1)$-dimensional complex projective space. Furthermore let $Q_{N-2}$ denote the complex quadratic:

$$\{ [X] \in CP^{N-1} : \langle X, X \rangle = 0 \}.$$ 

Then we identify $G(2, N)$ with $Q_{N-2}$ by $T \mapsto [e_1 + ie_2]$, where $e_1$, $e_2$ is an oriented orthonormal basis of $T$. The following is well known.

**Lemma 2.2.** Let $T$ be a subspace $\mathbb{C}^N$. Then $[T] \subset Q_{N-2}$ if and only if $T$ is isotropic. We call $[T]$ a linear subspace of $Q_{N-2}$.

For an oriented surface $M$ in $\mathbb{R}^N$, we can define a generalized Gauss map $\psi$ of $M$ into $Q_{N-2}$ by corresponding $e_1 + ie_2$ to an oriented orthonormal basis $\{e_1, -e_2\}$ of the tangent space of $M$. Lawson [2] proved the following.

**Proposition 2.1.** Let $S$ be a minimal branched immersion of Riemann surface $M$ into $\mathbb{R}^N$. Then $S$ is holomorphic if and only if $S(M)$ is contained in a linear subspace of $Q_{N-2}$.

Since $[AX] \in Q_{N-2}$ if $[X] \in Q_{N-2}$, $SO(N, \mathbb{C})$ acts on $Q_{N-2}$. Furthermore if $[T]$ is a linear subspace of $Q_{N-2}$, then so is $[AT]$. This action is denoted by $A[T]$.

**Remark 2.1.** When $N = 3$ holds, $Q_1$ is $S^2(1)$ and $SO(3, \mathbb{C})$ acts on $S^2(1)$ as conformal transformations.

**Lemma 2.3.** Let $S$ be a minimal branched immersion of the unit disk $D$ into $\mathbb{R}^N$. Then if $S$ is holomorphic, then so is $A \cdot S$.

**Proof.** Since the Gauss map of $D$ into $Q_{N-2}$ is given by $[\frac{\partial S}{\partial z}]$, if $[\frac{\partial S}{\partial z}]$ is contained a linear subspace $[T]$ of $Q_{N-2}$, then

$$\left[ \frac{\partial A \cdot S}{\partial z} \right] = A \left[ \frac{\partial S}{\partial z} \right] \subset A[T].$$

Proposition 2.1 completes Lemma 2.3. \hfill $\Box$

### 3. The definition of s-isotropic minimal surfaces

Let $S$ be a branched minimal immersion of a Riemann surface $M$ into $\mathbb{R}^N$. Then we call $S$ to be $s$-isotropic if

$$\frac{\partial S}{\partial z} \wedge \cdots \wedge \frac{\partial^s S}{\partial z^s} \neq 0,$$

$$\langle \frac{\partial S}{\partial z}, \frac{\partial S}{\partial z} \rangle = \cdots = \langle \frac{\partial^s S}{\partial z^s}, \frac{\partial^s S}{\partial z^s} \rangle = 0.$$
Lemma 3.1. This definition only depends on the Gauss map.

Proof. Let \( \ell(z) \) be a local holomorphic section of \( \left[ \frac{\partial S}{\partial z} \right] \), that is,
\[
\frac{\partial S}{\partial z} = f(z) \ell(z),
\]
where \( f(z) \) is a holomorphic function. Then it is easy to see that \( S \) is \( s \)-isotropic if and only if
\[
\ell \wedge \frac{\partial \ell}{\partial z} \wedge \cdots \wedge \frac{\partial^{s-1} \ell}{\partial z^{s-1}} \neq 0,
\]
\[
\langle \ell, \ell \rangle = \langle \frac{\partial \ell}{\partial z}, \frac{\partial \ell}{\partial z} \rangle = \cdots = \langle \frac{\partial^{s-1} \ell}{\partial z^{s-1}}, \frac{\partial^{s-1} \ell}{\partial z^{s-1}} \rangle = 0. \]
\[\square\]

Lemma 3.2. Let \( S \) be \( s \)-isotropic. Then
\[
\left\langle \frac{\partial^p \ell}{\partial z^p}, \frac{\partial^q \ell}{\partial z^q} \right\rangle = 0, \quad 0 \leq p + q \leq 2s - 1.
\]

Proof. See [3]. \[\square\]

Lemma 3.3. Let \( S \) be \( s \)-isotropic. Then if
\[
\frac{\partial^s \ell}{\partial z^s} \wedge \ell \wedge \frac{\partial \ell}{\partial z} \wedge \cdots \wedge \frac{\partial^{s-1} \ell}{\partial z^{s-1}} \wedge \ell \wedge \frac{\partial \ell}{\partial z} \wedge \cdots \wedge \frac{\partial^{s-1} \ell}{\partial z^{s-1}} = 0,
\]
then there exist functions \( a_1, \ldots, a_{s-1} \) such that
\[
\frac{\partial^s \ell}{\partial z^s} = \sum_{t=0}^{s-1} a_t \frac{\partial^t \ell}{\partial z^t}.
\]
In particular, \( [\ell] \) is contained a linear subspace of \( Q_{N-2} \).

Proof. There exist \( a_1, \ldots, a_{s-1}, b_1, \ldots, b_{s-1} \) such that
\[
\frac{\partial^s \ell}{\partial z^s} = \sum_{t=0}^{s-1} a_t \frac{\partial^t \ell}{\partial z^t} + \sum_{t=0}^{s-1} b_t \frac{\partial^t \ell}{\partial z^t}.
\]
The isotropic condition implies \( b_t = 0 \). \[\square\]

Lemma 3.3 and Proposition 2.1 imply the following.

Lemma 3.4. Let \( S \) be an \( s \)-isotropic branched minimal immersion of a Riemann surface \( M \) into \( \mathbb{R}^N \) which is not holomorphic and not \( (s + 1) \)-isotropic. Then for generic points of \( M \), we obtain
\[
\frac{\partial^s \ell}{\partial z^s} \wedge \ell \wedge \cdots \wedge \frac{\partial^{s-1} \ell}{\partial z^{s-1}} \neq 0, \quad \langle \frac{\partial^s \ell}{\partial z^s}, \frac{\partial^s \ell}{\partial z^s} \rangle \neq 0.
\]
Let $p$ be a point given in Lemma 3.4 and $z$ a local complex coordinate such that $z(p) = 0$. Then we have the following.

**Lemma 3.5.** There exists a holomorphic function $\eta_t$ such that $\eta_t(0) \neq 0$ and

$$\langle \ell(z), \frac{\partial^t \ell}{\partial z^t}(0) \rangle = z^{2s-t} \eta_t, \quad 0 \leq t \leq 2s - 1.$$

**Proof.** We shall prove Lemma 3.5 by the induction on $t$. Lemma 3.2 implies

$$\langle \ell(0), \ell(0) \rangle = \cdots = \langle \frac{\partial^{2s-1} \ell}{\partial z^{2s-1}}(0), \ell(0) \rangle = 0$$

and hence

$$\langle \ell(z), \ell(0) \rangle = z^{2s} \xi_0.$$ 

We should prove $\xi_0(0) \neq 0$. Let's assume $\xi_0(0) = 0$. Then

$$\langle \frac{\partial^{2s} \ell}{\partial z^{2s}}(0), \ell(0) \rangle = 0.$$

By the induction, we will prove

$$\langle \frac{\partial^k \ell}{\partial z^k}(0), \frac{\partial^t \ell}{\partial z^t}(0) \rangle = 0, \quad 0 \leq k + t \leq 2s.$$

It is enough to prove the above for $k + t = 2s$. Since

$$\langle \frac{\partial^{2s-1} \ell}{\partial z^{2s-1}}, \frac{\partial \ell}{\partial z}(0) \rangle = \frac{\partial}{\partial z} \langle \frac{\partial^{2s-1} \ell}{\partial z^{2s-1}}(0), \ell(0) \rangle - \langle \frac{\partial^{2s} \ell}{\partial z^{2s}}(0), \ell(0) \rangle = 0,$$

if, for all $0 \leq p \leq k$,

$$\langle \frac{\partial^{2s-p} \ell}{\partial z^{2s-p}}, \frac{\partial^p \ell}{\partial z^p}(0) \rangle = 0,$$

then we get the following.

$$\langle \frac{\partial^{2s-k-1} \ell}{\partial z^{2s-k-1}}, \frac{\partial^{k+1} \ell}{\partial z^{k+1}}(0) \rangle = \frac{\partial}{\partial z} \langle \frac{\partial^{2s-k-1} \ell}{\partial z^{2s-k-1}}, \frac{\partial^k \ell}{\partial z^k}(0) \rangle - \langle \frac{\partial^{2s-k} \ell}{\partial z^{2s-k}}, \frac{\partial^k \ell}{\partial z^k}(0) \rangle = 0.$$

Consequently we obtain

$$\langle \frac{\partial^s \ell}{\partial z^s}, \frac{\partial^s \ell}{\partial z^s}(0) \rangle = 0,$$
which is a contradiction and we obtain \( \xi_0(0) \neq 0 \). Next we assume that, for all \( 0 \leq k \leq t-1 \), there exists a holomorphic function \( \eta_k \) such that \( \eta_k(0) \neq 0 \) and
\[
\left\langle \ell, \frac{\partial^k \ell}{\partial z^k}(0) \right\rangle = z^{2s-k} \eta_k.
\]
Then Lemma 3.2 says that there exists a holomorphic function \( \eta_k \) such that
\[
\left\langle \ell, \frac{\partial^t \ell}{\partial z^t}(0) \right\rangle = z^{2s-t} \eta_t.
\]
It is enough to see \( \eta_t(0) \neq 0 \), which is proved by the same induction as proving \( \eta_0(0) \neq 0 \).

**Lemma 3.6.** The isotropy condition guarantees an orthonormal vectors \( e_1, \ldots, e_{2s} \) such that \( e_{2t-1} + i e_{2t} \ (1 \leq t \leq s) \) span
\[
\left\{ \ell(0), \ldots, \frac{\partial^{t-1} \ell}{\partial z^{t-1}}(0) \right\} \cap \left\{ \ell(0), \ldots, \frac{\partial^{t-2} \ell}{\partial z^{t-2}}(0) \right\}^\perp.
\]

Let \( \xi \) ba a non-zero component of \( \frac{\partial^{s} \xi}{\partial z^s}(0) \) orthogonal to
\[
\ell(0), \ldots, \frac{\partial^{s-1} \xi}{\partial z^{s-1}}(0).
\]
Since \( \xi \) is orthogonal to \( e_1, \ldots, e_{2s}, \text{Re} \xi \) and \( \text{Im} \xi \) are so. We choose a plane \( W \) which contained \( \text{Re} \xi \) and \( \text{Im} \xi \) and denote by \( e_{2s+1}, e_{2s+2} \) an orthonormal basis of \( W \). Adding vectors \( e_{2s+3}, \ldots, e_N \) of \( \mathbb{R}^N \), we obtain an orthonormal basis \( e_1, \ldots, e_N \) of \( \mathbb{R}^N \). Using this orthogonal coordinate system, we put
\[
\ell(z) = (\phi_1(z), \ldots, \phi_N(z))
\]
and hence
\[
\phi_{2t-1} + i \phi_{2t} = \langle \ell, e_{2t-1} + i e_{2t} \rangle, \quad 1 \leq t \leq s + 1.
\]
Then we obtain the following.

**Lemma 3.7.** There is a holomorphic function \( \xi_t \) such that \( \xi_t(0) \neq 0 \) and
\[
\phi_{2t-1} + i \phi_{2t} = z^{2s-t+1} \xi_t \ 	ext{for} \ 1 \leq t \leq s.
\]

**Proof.** Therte exist \( a_1, \ldots, a_t \) such that
\[
e_{2t-1} + i e_{2t} = a_1 \ell(0) + \cdots + a_t \frac{\partial^{t-1} \ell}{\partial z^{t-1}}(0)
\]
and \( a_t \neq 0 \). So Lemma 3.5 completes Lemma 3.7.

On the other hand, since there exist \( b_1, \ldots, b_t \) such that for \( 1 \leq t \leq s \),
\[
\frac{\partial^{t-1} \ell}{\partial z^{t-1}}(0) = b_1(e_1 + i e_2) + \cdots + b_t(e_{2t-1} + i e_{2t})
\]
and $b_t \neq 0$, we obtain holomorphic function $A_{2t-1}$ and $A_{2t}$ such that
\[
\langle \ell(z), e_{2t-1} \rangle = z^{t-1}A_{2t-1}, \quad \langle \ell(z), e_{2t} \rangle = z^{t-1}A_{2t},
\]

Thus we obtain the following.

**Lemma 3.8.** There exist holomorphic functions $A_{2t-1}$ and $A_{2t}$ such that $A_{2t-1}(0) \neq 0$, $A_{2t}(0) \neq 0$ and

\[
\phi_{2t-1} = z^{t-1}A_{2t-1}, \quad \phi_{2t} = z^{t-1}A_{2t}, \quad t \leq s
\]

We put $f(z) = \phi_1(z) - i\phi_2(z)$. Lemma 3.5 implies $\phi_1(0) + i\phi_2(0) = 0$ and hence $f(0) = 2\phi_1(0) \neq 0$. Since

\[
\phi_1^2 + \cdots + \phi_N^2 = 0,
\]
we obtain

\[
\phi_1 = \frac{1}{2} f(1 - \frac{1}{f^2}(\phi_3^2 + \cdots + \phi_N^2)),
\]

\[
\phi_2 = \frac{1}{2} if(1 + \frac{1}{f^2}(\phi_3^2 + \cdots + \phi_N^2)).
\]

We put

\[
\Phi_1 = \frac{\phi_1}{f}, \ldots, \Phi_N = \frac{\phi_N}{f}
\]

(see [4]), then $\ell = f(\Phi_1, \ldots, \Phi_N)$.

**Lemma 3.9.** There exists a holomorphic function $B$ such that $B(0) \neq 0$ and $\Phi_3^2 + \cdots + \Phi_N^2 = z^{2s}B$.

**Proof.** If $\Phi_3^2 + \cdots + \Phi_N^2 = z^{2s+1}B'$, then $\Phi_1^2 + \Phi_2^2 = -z^{2s+1}B'$. On the other hand,

\[
\Phi_1^2 + \Phi_2^2 = (\Phi_1 + i\Phi_2)(\Phi_1 - i\Phi_2) = z^{2s}C, \quad C(0) \neq 0.
\]

This is a contradiction. \hfill \Box

4. A deformation by $SO(N, \mathbb{C})$

We may consider a $\ell(z)$ as $\Phi(z) = (\Phi_1(z), \ldots, \Phi_N(z))$ in the section 2. Let $A(\tau)$ be a one parameter subgroup of $SO(N, \mathbb{C})$ defined by
\[
\begin{pmatrix}
A_1(\tau) & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & A_2(\tau) & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_s(\tau) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
\end{pmatrix}
\]

(1)

where

\[
A_t(\tau) = \begin{pmatrix}
\cosh(\alpha_t \tau) & i \sinh(\alpha_t \tau) \\
-i \sinh(\alpha_t \tau) & \cosh(\alpha_t \tau)
\end{pmatrix}
\]

(2)

and \(\alpha_1 = 1\). We put \(\Psi = \Phi_3^2 + \cdots + \Phi_N^2\). Then we obtain

\[
\Phi_1 = \frac{1}{2}(1 - \Psi), \quad \Phi_2 = \frac{i}{2}(1 + \Psi)
\]

and hence

\[
A_1(\tau) = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{-\tau} - e^{\tau} \Psi \\ ie^{-\tau} + ie^{\tau} \Psi \end{pmatrix}
\]

(3)

By Lemma 3.7, \(\Phi_{2t} = i\Phi_{2t-1} - iz^{2s-t+1}\xi_t\) and hence

\[
A_t(\tau) \begin{pmatrix} \Phi_{2t-1} \\ \Phi_{2t} \end{pmatrix} = \begin{pmatrix} e^{-\alpha_t \tau} \Phi_{2t-1} + \sinh(\alpha_t \tau)z^{2s-t+1}\xi_t \\ ie^{-\alpha_t \tau} \Phi_{2t-1} - i \cosh(\alpha_t \tau)z^{2s-t+1}\xi_t \end{pmatrix}.
\]

(4)

We may consider that \(\Phi\) is defined on \(|z| \leq 1\). Let \(D(e^{\tau/s})\) be a disk of \(\mathbb{C}\) defined by \(|z| \leq e^{\tau/s}\). Then we can define a map \(\Phi^\tau(z)\) of \(D(e^{\tau/s})\) into \(\mathbb{C}^N\) such that

\[
\Phi^\tau(z) = e^{\tau}A(\tau)\Phi\left(\frac{z}{e^{\tau/s}}\right).
\]

For each any domain \(D \subset \mathbb{C}\), choosing a large \(\tau\), we see \(D \subset D(e^{\tau/s})\). So \(\Phi^\tau\) may be a map of \(D\) into \(\mathbb{C}^N\) for large \(\tau\).

**Theorem 4.1.** If we set \(\alpha_t = (s - t + 1)/s\), then \(\Phi^\tau\) converges to a map \(\Phi^\infty\) of \(\mathbb{C}\) into \(\mathbb{C}^{2s}\) defined by

\[
\begin{pmatrix}
\frac{1}{2} - \frac{1}{2}z^{2s}\Gamma_1, \frac{i}{2} + \frac{i}{2}z^{2s}\Gamma_1, \ldots, z^{t-1}\Gamma_t + \frac{1}{2}z^{2s-t+1}\Gamma_t, \ldots, \\
i z^{t-1}\Gamma_t' - \frac{i}{2}z^{2s-t+1}\Gamma_t, \ldots, z^s\Pi_1, z^s\Pi_2, 0, \ldots, 0
\end{pmatrix},
\]
where $\Gamma_t, \Gamma'_t$ are not zero constant, $\Pi_1, \Pi_2$ are constant and satisfy $\Pi_1^2 + \Pi_2^2 \neq 0$. Furthermore $\Phi^\infty$ gives a map $[\Phi^\infty]$ of $CP^1$ into $CP^{2s+2}$ and

$$[\Phi^\infty(0)] = [(1, i, 0, \ldots, 0)], \quad [\Phi^\infty(\infty)] = [(-1, i, 0, \ldots, 0)].$$

In particular, the minimal surface defined by $2\text{Re}\int \Phi^\infty dz$ is $s$-isotropic and is not holomorphic in $\mathbb{R}^{2s+2}$. 

5. A proof of Theorem

Let $M$ be a Riemann surface and $\Phi$ a holomorphic map of $M$ into $Q_{N-2}$. If $\Phi$ is the Gauss map of a minimal surface in $\mathbb{R}^N$, then we obtain the second variational formula for the area functional. Note [5,6,7] that the second variational formula depends only on $\Phi$. Micallef [6] proved the following

**Theorem 5.1.** Let $\Phi$ be a holomorphic map of $CP^1$ into $Q_{N-2}$. Then if a minimal surface $M = CP^1$--finite points with the Gauss map $\Phi$ is stable, then $M$ is holomorphic in $\mathbb{R}^N$.

We here prove our Theorem 1.1. Assume that $S$ is not holomorphic in $\mathbb{R}^N$. Then we obtain a point in Lemma 3.4. Considering the same one parameter subgroup as in the section 4, we have a minimal surface given by $2\text{Re}\int \Phi^\infty dz$, which is not holomorphic. By Theorem 5.1, a minimal surface given by $2\text{Re}\int \Phi^\infty dz$ is not stable and hence there exists $D$ such that the minimal surface restricted to $D$ is not stable. Since $\Phi^\tau$ converges to $\Phi^\infty$ on $D$, for large $\tau$. This fact says that the minimal surface given by $2\text{Re}\int \Phi^\tau dz$ for large $\tau$ is not stable. Its Gauss map is given by $[A(\tau)\Phi \left(\frac{z}{e^{\tau/s}}\right)]$. Again a minimal surface, whose Gauss map is $[A(\tau)\Phi \left(\frac{z}{e^{\tau/s}}\right)]$, is not stable. Namely $A(\tau) \cdot S$ is not stable.

**References**

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MULTIPLE SCALE PROCEDURE IN LAPLACE TRANSFORM SPACE FOR SOLUTION OF WEAKLY NONLINEAR WAVE EQUATION

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In this paper we formulate a Laplace-transform multiple scale expansion procedure to develop asymptotic solution of weakly non-linear partial differential equation. The method is applied to some general nonlinear wave and diffusion equations.

Keywords: Multiple scale, Asymptotic, Laplace transform.

1. Introduction

A common boundary value problem associated with the wave equations is the signaling problem, in which the boundary condition prescribed at the origin $x = 0$ propagated into quiescent region $x > 0$.

For example, the linear telegraph equation $u_{tt} - u_{xx} + au_t + bu = 0$ which governs the one dimensional propagation of waves $u(x,t)$ in electric transmission lines is a classic equation treated in standard texts on partial differential equations [1], [5], [6]. However, in many application areas such as water waves, elastic and acoustic waves, the governing hyperbolic systems are non-linear [7].

Developing uniformly valid perturbation solutions to hyperbolic systems of the non-linear problem has been a difficult task. When the boundary condition is zero, a multi-scale perturbation scheme based on slow time scale $T = \epsilon t$ has been developed [2]. For signaling problem in which the initial condition are zero and one wishes to study the propagation of boundary
data, a perturbation scheme based on long distance scale $X = \varepsilon x$ can be used.

There is no difficulty in formally applying finite difference method to non-linear equations and this can be done by linearized equations [8]. But due to computational errors the approximation is not satisfactory.

In this paper we are concerned with the asymptotic solution in region away from the boundary of signaling problems for weakly non-linear partial differential equations of the form (1). By using multiple space scales and imposing certain boundedness conditions on the Laplace transformed equation. We show how to determine the long distance behavior of the solutions of such weakly non-linear equations.

We also show that the same technique can be applied to certain types of non-linear differential equations. Multiple scale perturbation method for wave equations are considered in a number of papers and books [8]. A general treatment using characteristic coordinates of initial boundary value problems involving multiple time and space scales may be found in [3]. In [2] multiple scale method in Fourier transform space is given for initial value problems. The main advantages of the method outlined in this paper are its relative simplicity and extensibility to equations which are not hyperbolic. Asymptotic solutions are developed without using characteristic coordinates. We also give examples of fairly general type of non-linearities that can be easily dealt with using our method (section 3).

2. The Procedure

Consider the signaling problem associated with a weakly nonlinear hyperbolic system

$$u_{tt} = u_{xx} - \varepsilon f(u_t, u_x), \quad t, x > 0, \tag{1}$$

$$u(x, 0) = 0; \quad u_t(x, 0) = 0; \quad u(0, t) = a(t), \quad t, x > 0. \tag{2}$$

Where the subscripts denote partial differentiation, $0 < \varepsilon \ll 1$, and $a(t)$ is a function prescribed for $t > 0$, with $a(0+) = 0$. The function $f$ and $a$ are assumed to have Laplace transform in $t$.

We are concerned with evolution of the boundary conditions in the far field, at large distances from the origin. Using a multiple approach [9], we assume that the solution is bounded and depends explicitly on two independent length scales, a short scale $x$ and a long scale defined by $X = \varepsilon x$. The solution of the boundary value problem is then sought in the form
of a generalized asymptotic expansion:
\[ u = u^0(x, X, t) + \varepsilon u^1(x, X, t) + \cdots, \quad (3) \]
the space derivatives in (1) are replaced by
\[ u_x \to u_x + \varepsilon u_x, \]
\[ u_{xx} \to u_{xx} + 2\varepsilon u_{xx} + \varepsilon^2 u_{xxx}. \]
Introducing these derivatives and expansion (3) into (1) and equating like powers of \( \varepsilon \), the following equations are generated
\[ u^0_t = u^0_{xx}, \quad (4) \]
\[ u^0_t(x, X, 0) = 0, u^0_t(x, X, 0) = 0, u^0(0,0,t) = a(t), \quad (5) \]
\[ u^1_t = u^1_{xx} + 2u^1_{xx} - f(u^1_x, u^1_x), \quad (6) \]
\[ u^1_t(x, X, 0) = u^1_t(x, X, 0) = u^0(0,0,t) = 0, \quad (7) \]
\[ u^2_t = u^2_{xx} + 2u^2_{xx} - u^0_{xx} - u^1_x \frac{\partial f}{\partial u_t} - (u^1_x + u^0_x) \frac{\partial f}{\partial u_x}, \quad (8) \]
\[ u^2(x, X, 0) = u^2_t(x, X, 0) = u^2(0,0,t) = 0. \quad (9) \]
Now consider the Laplace transform \( \mathcal{L} \) in \( t \) of (4)-(6):
\[ U^0_{xx} - s^2 U^0 = 0, \quad (10) \]
\[ U^1_{xx} - s^2 U^1 = -2U^0_{xx} + F(sU^0, U^0_x). \quad (11) \]
Where \( U(x, X, s) = \mathcal{L}(u) = \int_0^\infty u(x, X, t)e^{-st}dt \), and \( F \) is the Laplace transform of \( f \) with respect to \( t \). This is now a system of ordinary differential equations.

The bounded solution of (10) has the form
\[ U^0 = B(X, s)e^{-sx}, \quad (12) \]
where \( B \) depends on the slow space variable \( X \). Our aim is to explicitly determine this dependence by suppressing secular terms in the solution of the \( O(\varepsilon) \) equation (11).

Using (12) in (11) we obtain
\[ U^1_{xx} - s^2 U^1 = 2sB_X(X, s)e^{-sx} + F(sBe^{-sx}, -sBe^{-sx}). \quad (13) \]
The bounded solution of the homogeneous part of (13) has the form \( C(X, s)e^{-sx} \) if the right side of (11) has an \( e^{-sx} \) term, then solution for \( U^1 \) will have a term \( xe^{-sx} \) which will make \( \frac{U^1}{U^0} \) unbounded as \( x \to \infty \). To suppress such secular terms arising in the solution of (13) its right side
should not contain $e^{-sx}$ terms. Imposing the requirement help us determine $B(X, s)$. We multiply (13) by $e^{sx}$, integrate with respect to $x$ from 0 to $M$, divide by $M$, and take the limit as $M \to \infty$. The left hand side of the resulting expression will vanish after integrating parts since $U^1$ and its derivatives are assumed to be bounded.

The resulting equation is:

$$2sB_X(X, s) + \lim_{M \to \infty} \frac{1}{M} \int_0^M e^{sx} F(sBe^{-sx}, -sBe^{-sx})dx = 0. \quad (14)$$

The integrand in (14), in fact, is independent of $x$. To prove this, let

$$\mathcal{L}^{-1}(sB) = \phi(X, t) \quad (15)$$

If the Laplace transform inversion is applied to (14), the integrand can be written as a convolution integral. The following Laplace inversion formulas will be used:

$$\mathcal{L}^{-1}(G(s)e^{-sx}) = H(t-x)g(t-x), \quad \mathcal{L}^{-1}(e^{sx}) = \delta(t+x),$$

where $H$ is the heaviside unit step function. Upon Laplace inversion, (14) becomes,

$$2\phi_X(X, t) + \frac{1}{M} \int_0^M \int_0^t \delta(t-\tau + x) f(H(\tau-x)\phi(X, \tau-x), -H(\tau-x)\phi(X, \tau-x))d\tau dx = 0. \quad (16)$$

which simplifies into

$$2\phi_X(X, t) + \frac{1}{M} \int_0^M f(H(t)\phi(X, t), -H(t)\phi(X, t))dx = 0. \quad (17)$$

The integrand is thus independent of $x$ and the resulting equation for the slow variation of $\phi(X, \tau)$ is

$$2\phi_X + f(\phi, -\phi) = 0, t > 0, \phi(0, t) = \frac{da(t)}{dt}. \quad (18)$$

This is an ordinary differential equation in $X$ for $\phi$. For large classes of nonlinearities and general boundary conditions, this initial value problem can be solved. The leading approximation to the original BVP (1) – (2) is

$$U^0(x, X, t) = \mathcal{L}^{-1}\left(\frac{1}{s} \mathcal{L}(\phi)e^{-sx}\right) = H(t-x) \int_0^{t-x} \phi(X, \tau)d\tau. \quad (19)$$

In the next few sections we will give examples of non-linear boundary value problems on the semi-infinite line whose first order solutions can be obtained by the Laplace transform-multiple scale procedure.
3. Examples of non-linear boundary value problems for wave equations

3.1. Non linearities of the form \( f(u_x) \)

Consider the weakly non-linear boundary value problem

\[
\begin{align*}
    u_{tt} &= u_{xx} - \varepsilon (u_x)^n; \quad x > 0, t > 0, \\
    u(0, t) &= a(t), \quad u(x, 0) = u_t(x, 0) = 0.
\end{align*}
\]

Here \( n \) is positive integer \( \neq 1 \). (when \( n = 1 \), an exact solution can be found).

Assume that \( L[a(t)] = A(s) \) and that \( a(0+) = 0 \). The \( O(1) \) problem then has the solution (12). In this case, the \( O(\varepsilon) \) equation lead to

\[
2\phi_x + (-1)^n \phi^n = 0,
\]

so that

\[
\phi = (v_n X + C(t))^{\frac{1}{1-n}},
\]

where

\[
v_n = (-1)^n \frac{n-1}{2},
\]

and \( C(t) \) is a function to be determined.

Since \( B(X = 0) = A(s) \), one can deduce that

\[
C(t) = \left( \frac{da}{dt} \right)^{1-n}
\]

The \( O(1) \) solution is then

\[
u^0(x, X, t) = H(t - x) \int_0^{t-x} \frac{d\tau}{\left( \left( \frac{dx}{d\tau} \right)^{1-n} + (-1)^n \frac{n-1}{2} X \right)^{\frac{1}{n-1}}},
\]

In the limiting case, \( \varepsilon \to 0 \), we regain the solution of the linear problem:

\[
u^0(x, t) = H(t - x) \int_0^{t-x} \frac{d\tau}{\left( \frac{da}{dt} \right)^{-1}} = H(t - x)a(t - x).
\]

Using a similar procedure, one can show that a nonlinearity of the form \( f(u_t, u_x) = (u_t)^n \), the \( O(1) \) solution is

\[
u^0(x, X, t) = H(t - x) \int_0^{t-x} \frac{d\tau}{\left( \left( \frac{dx}{d\tau} \right)^{1-n} + (-1)^n \frac{n-1}{2} X \right)^{\frac{1}{n-1}}}.
\]
3.2. Non linearity of the form \( f(u_t) \)

Consider the boundary value problem for the wave equation with Van der Pol nonlinearity:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} - \varepsilon (\beta u^3_t - u_t); x, t > 0, \\
u(0, t) &= a(t), u(x, 0) = u_t(x, 0) = 0.
\end{align*}
\]

This equation has been used by Myerscough [10] to model Wind-induced galloping oscillations of over head transmission lines.

The equation corresponding to (18) for this problem is the Bernoulli differential equation

\[
2\phi_X + \beta \phi^3 - \phi = 0,
\]

which can be solved by substituting \( \psi = \phi^{-2} \).

The \( O(1) \) solution of the problem is

\[
u^0(x, X, t) = H(t - x) \int_0^{t-x} \frac{1}{\sqrt{\beta + e^{-x}(a_{\tau}^{-2} - \beta)}} d\tau.
\]

The particular case \( a(t) = \sin(\omega t) \) has been solved in [4] using characteristic coordinates. From the above solution, for this particular condition, we obtain

\[
u^0(x, X, t) = \frac{1}{\sqrt{\beta(1 - e^{-x})}} \arcsin(\sqrt{\frac{\beta(1 - e^{-x})}{\beta^2\omega^2 + e^{-x}(1 - \beta^2\omega^2)}} \sin\omega(t - x)), \quad t > x.
\]

3.3. Nonlinearity of the form \( \frac{\partial}{\partial t} f(u), \frac{\partial}{\partial x} f(u) \)

Consider nonlinear hyperbolic equations of the form

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial}{\partial t} f(u),
\]

with initial and boundary conditions (2).

In this case it is readily seen that the equation (14) determining the slowly varying coefficient \( B \) is modified into

\[
2sB_X(X, s) + \lim_{M \to \infty} \frac{1}{M} \int_0^M s e^{sx} F(Be^{-sx}) dx.
\]

If \( L^{-1}(B) = \Psi(X, t) \), then inverting the above equation gives

\[-2\Phi_X = f(\Phi); \Phi(0, t) = a(t).\]
For example, consider
\[ u_{tt} = u_{xx} - \varepsilon (u^n)_t; \quad x, t > 0, \]
where \( n \neq 1 \). With the initial-boundary conditions (2).
In this case we get
\[ \Psi^{-n+1} = \frac{n-1}{2} X + a(t)^{-n+1}, \]
and
\[ u^0(x, X, t) = H(t - x)(\frac{n-1}{2} X + a(t - x)^{-n+1})^{\frac{1}{1-n}}. \]
Nonlinearitise of the form \( \frac{\partial}{\partial x} f(u) \) can be similarly dealt with.

References
ORTHOGONAL COMPOSITIONS IN
A FOUR-DIMENSIONAL WEYL SPACE

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Special compositions in an n-dimensional Weyl space are studied in [6], [7], [10], [11] and [2]. Compositions, generated by nets in an n-dimensional Weyl space are introduced in [12] and [13]. This paper is devoted to the study of special orthogonal compositions in a four-dimensional Weyl space. In the second paragraph there are found geometry characteristics of orthogonal Cartesian and quasichebyshevian compositions about the Weyl connection and about the Levi-Civita connection. The form of space, containing these compositions, is defined. In the third paragraph there are given the curvature properties on a four-dimensional Weyl space, containing special orthogonal compositions. There are found invariant tensors and compositions about a conformal transformation on a Weyl space.

1. Preliminaries

Let $W_n$ be an n-dimensional Weyl space with a metric tensor $g_{ik}$ and its inverse tensor $g^{kj}$, i.e. $g_{ik}g^{kj} = \delta_i^j$, $i, j, k = 1, 2, \ldots, n$, where $\delta_i^j$ are the Kronecker symbols. As it is well-known [5], the Weyl connection $\nabla$ with components $\Gamma^k_{ij}$ is determined by the equation:

$$\Gamma^k_{ij} = \left\{ \begin{array}{c} k \\ i \\ j \end{array} \right\} - (\omega_i \delta^k_j + \omega_j \delta^k_i - g_{ij} g^{ks} \omega_s),$$

where $\left\{ \begin{array}{cc} k \\ i \\ j \end{array} \right\}$ are the Cristoffel symbols, determined by the metric tensor $g_{ij}$, and $\omega_k$ is the complementary vector (1-form) of $W_n$. The following equations are valid:

$$\nabla_k g_{ij} = 2\omega_k g_{ij}, \quad \nabla_k g^{ij} = -2\omega_k g^{ij}.$$ 

We can assert that when standardizing a metric tensor $g_{ij}$ by the equation $\not g_{ij} = \lambda^2 g_{ij}$ ($\lambda$ is an arbitrary differentiation function on $W_n$), the one-form $\omega_k$ changes according to the rule: $\not \omega_k = \omega_k + \partial_k \ln \lambda.$
According to [5], the curvature tensor $R_{ij}.k$ on the Weyl connection $\nabla$ satisfies the equation:

(1.3) \[ R_{ij}.k = n(\nabla_j \omega_i - \nabla_i \omega_j). \]

Let $K_{ij}.k$ be a curvature tensor, determined by the Levi-Civita connection $\tilde{\nabla}$ of $g_{ij}$. In the paper [1] is found the relation between the curvature tensors $R_{ij}.k$ and $K_{ij}.k$, i.e.

(1.4) \[ R_{ij}.k = K_{ij}.k + \left( \tilde{\nabla}_j \omega_i - \tilde{\nabla}_i \omega_j \right) \delta_k^s + M_{jk} \delta_i^s - M_{ik} \delta_j^s + (g_{jk} M_{il} - g_{ik} M_{jl}) g^{ls}, \]

where $M_{jk} = \tilde{\nabla}_j \omega_k + \omega_j \omega_k - \frac{1}{2} g_{jk} g^{ls} \omega_s \omega_l$. Further in this paper we will consider a four-dimensional Weyl space $W_4$. Then from (1.1) and (1.3), using the first Bianchi identity for $n = 4$, we have:

(1.5) \[ R_{ij} - R_{ji} = 4(\nabla_i \omega_j - \nabla_j \omega_i) = 4(\tilde{\nabla}_i \omega_j - \tilde{\nabla}_j \omega_i). \]

Following [6], we will consider the composition $X_2 \times Y_2 \in W_4$, where $X_2$ (dim $X_2 = 2$) and $Y_2$ (dim $Y_2 = 2$) are its fundamental manifolds. Then through each point $p \in W_4$ goes exactly one two-dimensional surface, from the fundamental manifolds $X_2$ and $Y_2$, respectively. These surfaces are called positions, and they are denoted with $P(X_2)$ and $P(Y_2)$ [6].

Following [7] and [11], $W_4$ assumes the existence of a composition $X_2 \times Y_2$, if on $W_4$ is assigned a tensor field $a^k_i$ of the type (1, 1) for which the following equations are fulfilled:

(1.6) \[ a^k_i a^j_k = \delta^i_j, \]
(1.7) \[ N^k_{ij} = a^j_s \nabla_s a^k_i - a^i_s \nabla_s a^k_j - a^k_s (\nabla_i a^s_j - \nabla_j a^s_i) = 0, \]

where $N^k_{ij}$ is the Nijenhuis tensor of the structure $a^k_i$. We can assert that a tensor $N^k_{ij}$ has the same structure (1.7) about the Levi-Civita connection $\tilde{\nabla}$. The tensor field $a^k_i$ is called affinor of the composition. Moreover, it can be expressed as follows:

(1.8) \[ N^k_{ij} = a^j_s \partial_s a^k_i - a^i_s \partial_s a^k_j - a^k_s (\partial_i a^s_j - \partial_j a^s_i). \]

According to [3], the tensor field $a^k_i$ is called structure of an almost product on $W_4$, and in case when $N^k_{ij} = 0$ it is called structure of a product.

In the paper [12] the nets in $W_n$ are studied and there are found the derivative equations of the directional vectors of a given net. We will consider a net $(v^1, v^2, v^3, v^4) \in W_4$, defined by independent tangent vector fields $v^k_s$.
(s = 1, 2, 3, 4) of the curves of the net. We determine the inverse covectors \( \tilde{v}_k \) (s = 1, 2, 3, 4) of the vectors \( v^k \), respectively, by the equations:

\[
(1.9) \quad v^k \tilde{v}_k = \delta^s_i \iff v^i \tilde{v}_s = \delta^s_i, \quad \text{(summing by index } k). \]

In the paper [13] there is defined an affinor of the composition in \( W_n \) through the directional vectors and covectors of a given net. Using (1.9), the affinor \( a_k^i \) of composition \( X_2 \times Y_2 \in W_4 \) has the form:

\[
(1.10) \quad a_k^i = v^k_i v^1 + \frac{1}{2} v^2_i v^3 - \frac{1}{4} v^4_i v^3 - \delta^k_i = \delta^k_i - 2(\frac{1}{3} v^3_i + \frac{1}{4} v^4_i). \]

There follows immediately that \( a_k^i \) satisfies (1.6) and has the properties:

\[
(1.11) \quad a_1^1 v^k = v^k, \quad a_2^2 v^k = v^k, \quad a_3^3 v^k = -v^s, \quad a_4^4 v^k = -v^s. \]

The equation (1.11) indicates that in every point \( p \in W_4 \) the sections \( (\tilde{u}, u) \) and \( (u, \tilde{u}) \) are the tangents respective of the positions \( P(X_2) \) and \( P(Y_2) \), which go through the point \( p \in W_4 \). In this case, the composition \( X_2 \times Y_2 \in W_4 \) is called composition, associated to the net \( (u, u, u, u) \).

Further we will study compositions in \( W_4 \), associated to a normalized net, i.e. for the directional vectors to the net are valid the following equations:

\[
(1.12) \quad g_{ij} v^i v^j = 1, \quad g_{ij} = \cos \omega_{ks}, \quad k \neq s, \quad k, s = 1, 2, 3, 4, \]

where \( \alpha = \alpha \) are the angles between the tangent vectors \( v^i \) and \( v^i \).

In the paper [12], the prolonged covariant differentiation \( \nabla \) of the satellite \( A \) with weight \( \{p\} \) in the Weyl space is defined by the equality:

\[
(1.14) \quad \nabla_i A = \nabla_i A - p \omega_i A. \]

There are also found derivative equations for the directional vectors and covectors of a given net. For the net \( (v, v, v, v) \in W_4 \) they have the form:

\[
(1.15) \quad \nabla_i v^s = T_i^m v^s, \quad \nabla_i v_i = -T_i^m v^m, \]

where \( k = 1, 2, 3, 4 \) and has a summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \). Having in mind that the affinor \( a_k^i \) has summation by index \( m = 1, 2, 3, 4 \).
Using the third equality of (1.10) and (1.15), we receive:

\[
\nabla_s a^k_j = 2 \left[ \left( \frac{3}{1} T_s v_j + \frac{3}{2} T_s v_j \right) v^k + \left( \frac{4}{1} T_s v_j + \frac{4}{2} T_s v_j \right) v^k \right] - \left( \frac{1}{3} T_s v_j + \frac{1}{4} T_s v_j \right) v^k = \left( \frac{2}{3} T_s v_j + \frac{2}{4} T_s v_j \right) v^k \]

(1.17)

Having in mind structure (1.7) on tensor \( N_{ij}^k \), and taking into account (1.11), the equations (1.10), (1.16) and (1.17), for the Nijenhuis tensor we obtain:

(1.18)\[
\frac{1}{4} N_{ij}^k = A_{ij} v^k + B_{ij} v^k + C_{ij} v^k + D_{ij} v^k,
\]

where

(1.19)\[
A_{ij} = \left( v_i v_j - v_j v_i \right) \left( T_s v^s - T_s v^s \right), \quad B_{ij} = \left( v_i v_j - v_j v_i \right) \left( T_s v^s - T_s v^s \right),
\]

\[
C_{ij} = \left( \frac{3}{1} T_s v^s v_i - \frac{3}{4} T_s v^s v_i \right) v_j + \left( \frac{3}{2} T_s v^s v_i - \frac{3}{2} T_s v^s v_i \right) v_j \]

\[
- \left( \frac{3}{1} T_s v^s v_j - \frac{3}{4} T_s v^s v_j \right) v_i + \left( \frac{3}{2} T_s v^s v_j - \frac{3}{2} T_s v^s v_j \right) v_i,
\]

\[
D_{ij} = \left( \frac{4}{1} T_s v^s v_i - \frac{4}{3} T_s v^s v_i \right) v_j + \left( \frac{4}{2} T_s v^s v_i - \frac{4}{2} T_s v^s v_i \right) v_j \]

\[
- \left( \frac{4}{1} T_s v^s v_j - \frac{4}{3} T_s v^s v_j \right) v_i + \left( \frac{4}{2} T_s v^s v_j - \frac{4}{2} T_s v^s v_j \right) v_i.
\]

**Theorem 1.1.** The Weyl space \( W_4 \) is a space of the composition \( W_4(X_2 \times Y_2) \), associated to the net \( (v_1, v_2, v_3, v_4) \) if and only if:

(1.20)\[
T_s v^s = \frac{1}{4} T_s v^s, \quad T_s v^s = \frac{2}{3} T_s v^s, \quad T_s v^s = \frac{3}{2} T_s v^s, \quad T_s v^s = \frac{4}{1} T_s v^s.
\]

**Proof.** From (1.18), (1.19), and the linear independence of the vectors \( v^k \), \( s = 1, 2, 3, 4 \) follows that \( N_{ij}^k = 0 \) \( \iff \) \( A_{ij} = B_{ij} = C_{ij} = D_{ij} = 0 \). It is clear that \( A_{ij} = 0 \) and \( B_{ij} = 0 \) exactly when the following equations are realized:

\[
T_s v^s = \frac{1}{4} T_s v^s, \quad T_s v^s = \frac{2}{3} T_s v^s.
\]

Let \( C_{ij} = 0 \). Then after contracting by the vectors \( v^k, s = 1, 2, 3, 4 \), we obtain \( T_s v^s = \frac{3}{1} T_s v^s \). By analogy, if \( D_{ij} = 0 \) it follows that \( T_s v^s = \frac{4}{2} T_s v^s \).
Conversely, if the last two equations of (1.20) are realized, after suitable transformations we have \( C_{ij} = D_{ij} = 0 \). In this way we conclude that the tensor \( N_{ij}^k = 0 \) exactly when (1.20) are fulfilled.

\[ \square \]

2. Orthogonal compositions in a four-dimensional Weyl space

**Definition 2.1.** [10] The composition \( X_2 \times Y_2 \in W_4 \) is orthogonal, if in each point \( p \in W_4 \) the tangent sections \((v_1, v_2)\) and \((v_3, v_4)\) are orthogonal.

It is well-known that the two sections \((v_1, v_2)\) and \((v_3, v_4)\) are orthogonal, if an arbitrary vector on one of them is orthogonal of an arbitrary vector on the other. Then, according to equation (1.13), there are valid:

\[
\begin{align*}
g_{ij} v^i v^j &= g_{ij} v^1 v^3 = g_{ij} v^2 v^4 = g_{ij} v^1 v^3 = 0, \\
g_{ij} v^i v^j &= \cos \alpha, \quad g_{ij} v^i v^j = \cos \beta,
\end{align*}
\]

where \( \alpha = \alpha_{12}, \beta = \alpha_{34} \).

In the paper [10] it is proven, that a necessary and sufficient condition for one composition in \( W_n \) to be orthogonal is the existence of a tensor of type \((0, 2)\) \( a_{ij} = \tilde{g}_{ij} = a^k g_{kj} \), which satisfies the condition:

\[
\tilde{g}_{ij} = \tilde{g}_{ji}.
\]

Tensor \( \tilde{g}_{ij} \) is called associated tensor on a metric tensor \( g_{ij} \). According to (1.6), (2.2) is equivalent to:

\[
a^k a_j g_{ks} = g_{ij}.
\]

Using (1.6), (2.2), (2.3), \( g_{ik} g^{kj} = \delta_i^j \) and \( \tilde{g}_{ik} \tilde{g}^{kj} = \delta_i^j \), there immediately follow the equations:

\[
\tilde{g}^{ij} = a^i_k g^{kj} = a^j_k g^{ki}, \quad a^i_k a^j_s g_{ks} = g^{ij}.
\]

**Lemma 2.1.** Let \( W_4 \) be a space on an orthogonal composition, determined by the normalized net \((v_1, v_2, v_3, v_4)\). Then the metric tensor \( g_{ij} \) and the tensor
\( \tilde{g}_{ij} \), an associated to \( g_{ij} \), satisfy the following conditions:

\begin{align}
(2.5) \quad g_{ki}^{v_k} &= v_i + \cos \alpha^2 v_i, \quad g_{ki}^{v_k} = \cos \alpha^2 v_i + v_i, \\
&= \frac{3}{2} v_i + \cos \beta v_i, \quad g_{ki}^{v_k} = \cos \beta v_i + \frac{3}{2} v_i,
\end{align}

\begin{align}
(2.6) \quad g_{ij} &= v_i v_j + v_i v_j + v_i v_j + v_i v_j \\
&+ \cos \alpha \left( \frac{1}{2} v_i v_j + \frac{1}{2} v_i v_j \right) + \cos \beta \left( \frac{3}{4} v_i v_j + \frac{4}{3} v_i v_j \right),
\end{align}

\begin{align}
(2.7) \quad \tilde{g}_{ij} &= v_i v_j + v_i v_j - v_i v_j - v_i v_j \\
&+ \cos \alpha \left( \frac{1}{2} v_i v_j + \frac{1}{2} v_i v_j \right) - \cos \beta \left( \frac{3}{4} v_i v_j + \frac{4}{3} v_i v_j \right).
\end{align}

**Proof.** Equations (2.5) follow from (1.10) and (2.2). Equation (2.2), taking into account the second and third equation of (1.10), is equivalent to:

\[
g_{ji} - 2(g_{ki}^{v_k} v_j + g_{ki}^{v_k} v_j) = 2(g_{ki}^{v_k} v_j + g_{ki}^{v_k} v_j) - g_{ji},
\]

whence, using (2.5), we receive (2.6). Equation (2.7) immediately follows from (2.6) and (1.11).

We can assert that, according to Lemma 2.1, the tensor \( g_{ij} \) determines a Riemannian metric, and the tensor \( \tilde{g}_{ij} \) associated to it determines a pseudo-Riemannian metric with signature \((2,2)\). Then, we can consider the space \( W_4 \) with a Riemannian metric \( g_{ij} \), which originates the Levi-Civita connection \( \nabla \) and the structure of an almost product \( a_i^k \). In this case, we denote this space with \( W_4(g_{ij}, \nabla, a_i^k) = M^4 \). Following [3], [4] and [8], and because of (1.6), (2.2) and (2.3), we ascertain that \( M^4 \) is a Riemannian almost product manifold. Having in mind that the existence of an orthogonal composition in \( W_4 \) is possible for \( N_{ij}^k = 0 \), then \( M^4 \) is a Riemannian product manifold, i.e. the structure \( a_i^k \) is integrating. Later in this paper we consider the Weyl space \( W_4 \), which assumes the existence of an orthogonal composition. In this case, \( W_4(g_{ij}, \nabla, \omega_k) \) will be called a four-dimensional Weyl product space, i.e. \( a_i^k \) is the product structure.

In the paper [8] are examined \( 2n \)-dimensional Riemannian manifolds \( M^{2n} \) with the structure of an almost product \( P, P^2 = I_{2n}, \) tr \( P = 0 \), and there is given one classification of the fundamental four classes on these manifolds. It is clear from (1.11), for the considered manifold \( M^4 \), the structure \( P \) has a matrix \( (a_i^k) \) and tr \( P = a_k^k = 0 \). Hence \( M^4 \) is a Riemannian product manifold with the product structure \( a_i^k \). The fundamental
third classes, to which $M^4$ can belong in relation to the local base $\{\frac{\partial}{\partial x^i}\}$, have the following characteristics [8]:

1. Class $w_0$ of Riemannian $P$-manifolds:

\[(2.8) \quad F_{ijk} = 0 \Leftrightarrow \tilde{\nabla}_i a^k_j = 0.\]

2. Class $w_1$:

\[(2.9) \quad F_{ijk} = \frac{1}{4} (g_{ij} \alpha_k + g_{ik} \alpha_j - \tilde{g}_{ij} \tilde{\alpha}_k - \tilde{g}_{ik} \tilde{\alpha}_j).\]

3. Class $w_2$:

\[(2.10) \quad N^k_{ij} = 0, \quad \alpha_k = 0 ,\]

where $F_{ijk} = \tilde{\nabla}_i a^k_j g_{sk}$, $\alpha_k = g^{ij} F_{ijk} = \tilde{\nabla}_s a^k_j$ is a one-form, associated with the tensor $F_{ijk}$ and $\tilde{\alpha}_k = a^k_s \alpha_s$. The tensor $F_{ijk}$ of type $\langle 0,3 \rangle$ has the following properties:

\[(2.11) \quad F_{ijk} = F_{ikj}, \quad a^s_j a^k_i F_{isl} = -F_{ijk}.\]

**Remark.** We immediately verify that if $M^4 \in w_1$, then $N^k_{ij} = 0$.

Taking into account (1.1) and (2.4) for the covariant differentiations of $a^k_j$ about $\nabla$ and $\tilde{\nabla}$, we receive the following dependence:

\[(2.12) \quad \nabla_i a^k_j = \tilde{\nabla}_i a^k_j + \omega_j a^k_i - \tilde{\omega}_j a^k_i - g_{ij} g^{km} \tilde{\omega}_m + \tilde{g}_{ij} g^{km} \omega_m,\]

where $\tilde{\omega}_j = a^k_j \omega_s$. We denote that $P_{ijs} = \nabla_i a^k_j g_{ks}$. Then from (2.12), we have:

\[(2.13) \quad P_{ijs} = F_{ijs} + \omega_j \tilde{g}_{is} - \tilde{\omega}_j g_{is} - g_{ij} \tilde{\omega}_s + \tilde{g}_{ij} \omega_s.\]

Having in mind (2.2) and the properties (2.11) for $F_{ijk}$, then from (2.13) follows that the tensor $P_{ijk}$ has the properties:

\[(2.14) \quad P_{ijk} = P_{ikj}, \quad a^s_j a^k_i P_{isl} = -P_{ijk}.\]

In the paper [10] are given the following contentions:

**Theorem A.** [10] *Every orthogonal geodesic (respectively Chebyshevian) composition in $W_n$ is Cartesian.*

Consequently, in the case, when the composition $X_2 \times Y_2 \in W_4$ is orthogonal, from the three fundamental compositions: geodesic, Chebyshevian and Cartesian, which are connected with the concept parallel translation of the sections $(v, v)$ and $(v, v)$, the essential one is the Cartesian. A
composition \( X_2 \times Y_2 \in W_4 \) is *Cartesian*, if the sections \((v_1, v_2, v_3, v_4)\) and \((w_1, w_2, w_3, w_4)\) are translated parallelly of every curve of \( P(X_2) \) and \( P(Y_2) \). An invariant characteristic of a Cartesian composition is \( \nabla_t a_j^k = 0 \), where \( \nabla' \) is an arbitrary symmetric connection on \( W_4 \). Then in \( W_4 \) it is possible for the composition to be Cartesian about the Weyl connection \( \nabla \) or the connection \( \tilde{\nabla} \), i.e.

\[
(2.15) \quad \nabla_t a_j^k = 0 \quad \text{or} \quad \tilde{\nabla}_t a_j^k = 0.
\]

In [7] are introduced special compositions in \( n \)-dimensional spaces \( M^n \) with a symmetric linear connection, which are characterized by the concept quasi parallel translation of the tangent sections on two positions of the composition.

**Definition 2.2.** [7] Let \( M^n \) be a space with a symmetric linear connection \( \nabla' \). The section \( \beta = (v_1^i, v_2^i, \ldots, v_m^i) \in T_p M^n \), \( m < n \) is translated quasi parallelly of curve \( \gamma(t) \in M^n \), if for arbitrary vector \( \xi^k \in \beta \) the following condition is realized:

\[
(2.16) \quad \delta \xi^k = f \frac{dx^k}{dt} + A^i_s v^i_s, \quad s = 1, 2, \ldots, m,
\]

where \( f \) and \( A^i_s \) are arbitrary differentiation functions, \( \delta \) is the symbol of absolute differentiation, and \( \frac{dx^k}{dt} \) is a tangent vector of \( \gamma \).

The fundamental compositions, joined by quasi parallel translation of the tangent sections on the two positions are: quasicartesian, quasigeodesic and quasichebyshhevian.

**Definition 2.3.** [7] The composition \( X_2 \times Y_2 \in W_4 \) is called quasicartesian, if the tangent sections \((v_1, v_2, v_3, v_4)\) and \((w_1, w_2, w_3, w_4)\) are translated quasi parallelly of every curve of \( P(X_2) \) and \( P(Y_2) \).

**Definition 2.4.** [7] The composition \( X_2 \times Y_2 \in W_4 \) is called quasichebyshhevian, if the tangent section \((v_1, v_2)\) \{respectively \((v_3, v_4)\)\} is translated quasi parallelly of every curve of \( P(Y_2) \) \{respectively of \( P(X_2) \)\}.

In [7] it is substantiated that every quasigeodesic composition is geodesic. The vectors \( \tau_j \) and \( \sigma_j \) of a quasi parallel translation in \( W_4 \) are defined by the equations:

\[
(2.17) \quad \tau_j = \frac{1}{4} a_j^s \nabla_k a_s^k, \\
(2.18) \quad \sigma_j = \frac{1}{4} a_j^s \tilde{\nabla}_k a_s^k = \frac{1}{4} \tilde{\alpha}_j.
\]
The invariant characteristics of quasicartesian and quasichebyshevian compositions $X_2 \times Y_2 \in W_4$ respectively about the connections $\nabla$ or $\tilde{\nabla}$ have the form [7]:

1. Quasicartesian:

$$\nabla'_i a^k_j - \tau'_s (\delta^k_i a^s_j - a^k_i \delta^s_j) = 0; \quad (2.19)$$

2. Quasichebyshevian:

$$B'^{ik}_{ij} = a^s_i \nabla'_j a^k_s - a^s_j \nabla'_s a^k_i - 2\tau'_s (\delta^s_i \delta^k_j - a^s_i a^k_j) = 0, \quad (2.20)$$

where $\nabla' = \nabla$, $\tau'_s = \tau_s$, $B'^{ik}_{ij} = B^{ik}_{ij}$ or $\nabla' = \tilde{\nabla}$, $\tau'_s = \sigma_s$, $B'^{ik}_{ij} = \tilde{B}^{ik}_{ij}$.

We denote:

$$A^{ik}_{ij} = \nabla_i a^k_j + a^k_i \tau_j - \delta^k_i \tau_j - g_{ij} g^{km} \tau_m + \tilde{g}_{ij} g^{km} \tau_m, \quad \tilde{\tau}_j = a^s_j \tau_j. \quad (2.21)$$

From (2.12), using (2.17) and (2.18) follow the equations:

$$\tau_j = \sigma_j - \omega_j, \quad \tilde{\tau}_j = \tilde{\sigma}_j - \tilde{\omega}_j, \quad (2.22)$$

Then, after a transformation of (2.12), on account of (2.22), we determine the truthfulness of the following:

**Lemma 2.2.** The tensor $A^{ik}_{ij}$ on $W_4$ is invariant at the transformation (1.1) of the connection $\tilde{\nabla}$ in $\nabla$, i.e.

$$A^{ik}_{ij} = \tilde{A}^{ik}_{ij}, \quad (2.23)$$

where $\tilde{A}^{ik}_{ij}$ has the form (2.21) about the connection $\tilde{\nabla}$ and the respective vector $\sigma_j$.

Having in mind (2.20) and (2.21), immediately we obtain:

$$B^{ik}_{ij} = a^s_i A^{k s}_{js} - a^s_j A^{k s}_{si} \quad \text{and} \quad \tilde{B}^{ik}_{ij} = a^s_i \tilde{A}^{k s}_{js} - a^s_j \tilde{A}^{k s}_{si}. \quad (2.24)$$

Then, according to Lemma 2.2, we find:

$$B^{ik}_{ij} = \tilde{B}^{ik}_{ij}. \quad (2.24)$$

According to the equations (2.20) and (2.24), there follows the truthfulness of:

**Theorem 2.1.** An orthogonal composition $X_2 \times Y_2 \in W_4$ is quasichebyshevian about the connection $\nabla$ exactly when it is quasichebyshevian about the connection $\tilde{\nabla}$.

**Theorem 2.2.** An orthogonal composition $X_2 \times Y_2 \in W_4$ is quasichebyshevian exactly when the Riemannian product manifold $M^4$ with structure $a^k_i$ belongs to class $\omega_1$, i.e. equation (2.9) is valid.
Proof. Let $X_2 \times Y_2 \in W_4$ be a quasichebyshevian composition. From (2.20) and (2.24) follows that $B^k_{ij} = \bar{B}^k_{ij} = 0$. After contracting the last equation by $g_{kl}$, on account of (2.20) and $F_{jst} = \bar{\nabla}_j a^k_{st} g_{kl}$, we have:

$$\bar{B}^k_{ij} g_{kl} = a^i_{t} F_{jst} - a^j_{t} F_{sit} - 2(g_{jl} \sigma_i - \bar{g}_{jl} \bar{\sigma}_i) = 0.$$

After alternating over the indexes $i$ and $l$ in the last equation, and taking into account the properties (2.11), we obtain:

$$(2.25) \quad a^i_{t} F_{jst} = g_{jl} \sigma_i - g_{jl} \sigma_i - \bar{g}_{jl} \bar{\sigma}_i - \bar{g}_{jl} \bar{\sigma}_l.$$

After contracting (2.25) by $a^k_{t}$, taking into account (2.18), follows the equation (2.9), i.e. the Riemannian manifold $M^4 \in w_1$.

Conversely, let $M^4 \in w_1$, i.e. equation (2.9) is valid. From (2.18), $F_{jkl} g^{ls} = \bar{\nabla}_j a^k_{ls}$, Lemma 2.2, (2.9) and (2.24), we obtain $A^k_{ij} = 0$ and $B^k_{ij} = 0$, i.e. according to (2.20) the composition $X_2 \times Y_2 \in W_4$ is quasichebyshevian.

Corollary 2.1. The invariant characteristic of an orthogonal quasichebyshevian composition $X_2 \times Y_2 \in W_4$ about the Weyl connection $\nabla$ is $A^k_{ij} = 0$. The complementary vector $\omega_j$ of $W_4$ is determined by:

$$\omega_j = \frac{1}{4} \bar{\alpha}_j - \tau_j.$$

Theorem 2.3. If the orthogonal composition $X_2 \times Y_2 \in W_4$ is quasicartesian about the connection $\nabla$ (respectively $\bar{\nabla}$), then it is Cartesian about the connection $\nabla$ (respectively $\bar{\nabla}$).

Proof. Let $X_2 \times Y_2 \in W_4$ be orthogonal quasicartesian about $\nabla$, i.e. according to (2.19) the following equation is realized:

$$(2.26) \quad \nabla_i a^k_j = \delta_i^k \bar{\tau}_j - a^k_i \bar{\tau}_j.$$

After contracting (2.26) by $g_{kl}$, we receive $P_{ijkl} = g_{ul} \bar{\tau}_j - \bar{g}_{ul} \tau_j$. Using the properties (2.14), from the last equation follows $g_{ul} \bar{\tau}_j - \bar{g}_{ul} \tau_j = g_{ij} \bar{\tau}_i - \bar{g}_{ij} \tau_i$. Then, from $g_{ij} g^{ul} = \delta^i_j$ and $\bar{g}_{ij} g^{ul} = a^i_j$, it follows that $4 \bar{\tau}_j = 0$, $\tau_j = 0$ and $\nabla_i a^k_j = 0$. According to (2.15), the composition is Cartesian about the connection $\nabla$. According to (2.19), in the case when the composition is quasicartesian about $\bar{\nabla}$, we have $F_{ijk} = g_{ik} \bar{\sigma}_j - \bar{g}_{ik} \sigma_j$. By analogy, since $F_{ijk}$ has the properties (2.11), it follows that $\sigma_j = \bar{\sigma}_j = 0$ and $\bar{\nabla}_i a^k_j = 0$, i.e. the composition $X_2 \times Y_2 \in W_4$ is Cartesian about the connection $\bar{\nabla}$. □
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Having in mind Theorem A, Theorem 2.1 and Theorem 2.3, we determine that fundamental orthogonal compositions in $W_4$ are Cartesian and quasichebyshevian about $\nabla$ or $\tilde{\nabla}$.

From Lemma 2.2, Theorem 2.1, Theorem 2.2 and (2.22) follows truthfulness of the contentions:

**Corollary 2.2.** If the orthogonal composition $X_2 \times Y_2 \in W_4$ is Cartesian about $\nabla$ (respectively $\tilde{\nabla}$), then it is quasichebyshevian about $\tilde{\nabla}$ (respectively $\nabla$). The complementary vector $\omega_j$ of the Weyl space $W_4$ is determined by:

\[(2.27) \quad \omega_j = \frac{1}{4} \tilde{\alpha}_j \quad \text{(respectively } \omega_j = -\tau_j)\].

Let $\eta$ be a conformal transformation of the Weyl spaces $W_n(g_{ij}, \omega_k)$ into $\bar{W}_n(\bar{g}_{ij}, \bar{\omega}_k)$. Then, following [5], in the corresponding points of these spaces we have:

\[(2.28) \quad \bar{g}_{ij} = g_{ij}, \quad \bar{\omega}_i = \omega_i - p_i,\]

where covector $p_i$ is called vector of the conformal transformation.

The components $\Gamma_{ij}^k$ and $\bar{\Gamma}_{ij}^k$ of the Weyl connections $\nabla$ and $\bar{\nabla}$, respectively, satisfy the equality:

\[(2.29) \quad \bar{\Gamma}_{ij}^k = \Gamma_{ij}^k + \delta_i^k p_j + \delta_j^k p_i - g_{ij} g^{ks} p_s.\]

In the paper [2] it is proven, that if in $W_n$ is given the structure of an almost product $a_i^k$, then the Nijenhuis tensor for structure $a_i^k$ is an invariant about the conformal transformation $\eta$, i.e.

\[(2.30) \quad N_{ij}^k = \bar{N}_{ij}^k.\]

Therefore if $W_n$ is a space of a composition with affinor $a_i^k$, then the corresponding space $\bar{W}_n$ is also a space of a composition with affinor $a_i^k$.

Further we will consider a conformal transformation $\eta$ of $W_4$ into $\bar{W}_4$, in which orthogonal compositions exist. It is clear that, because of (2.28), the orthogonality of a composition in $W_4$ is preserved at $\eta$. Using (2.29), for the covariant differentiations of $a_i^k$ about $\nabla$ and $\bar{\nabla}$, we obtain the following dependence:

\[(2.31) \quad \bar{\nabla}_i a_j^k = \nabla_i a_j^k + \delta_i^k \bar{p}_j - a_i^k p_j + g_{ij} g^{km} \bar{p}_m - \bar{g}_{ij} g^{km} p_m,\]

where $\bar{p}_j = a_j^k p_s$.

**Lemma 2.3.** The tensor $A_{ij}^k$ determined by (2.21) is invariant about the conformal transformation $\eta$ of $W_4$ into $\bar{W}_4$, i.e.

\[(2.32) \quad A_{ij}^k = \bar{A}_{ij}^k.\]
The corresponding complementary vectors \( \omega_j \) and \( \bar{\omega}_j \) and the corresponding vectors of quasi parallel translation \( \tau_j \) and \( \bar{\tau}_j \) satisfy the equation:

\[
(2.33) \quad \omega_j + \tau_j = \bar{\omega}_j + \bar{\tau}_j.
\]

**Proof.** From the equation (2.31), taking into account (2.17), we receive 
\[
p_j = \omega_j - \bar{\omega}_j = \bar{\tau}_j - \tau_j,
\]
from where follows (2.33). After a transformation of (2.31), using (2.21) and 
\[
p_j = \bar{\tau}_j - \tau_j,
\]
we obtain (2.32).

Since \( B^k_{ij} = a^s_i A^k_{js} - a^s_j A^k_{si} \) and \( a^s_i = \bar{a}^s_i \), then from the equation (2.32) follows that the tensor \( B^k_{ij} \) is also invariant about the conformal transformation \( \eta \), i.e.

\[
(2.34) \quad B^k_{ij} = \bar{B}^k_{ij}.
\]

It is easy to examine that from (2.34) follows (2.32). From (2.34) immediately follows the truthfulness of:

**Theorem 2.4.** Every orthogonal quasichebyshevian composition \( X_2 \times Y_2 \in W_4 \) is invariant about the conformal transformation \( \eta \) of \( W_4 \) into \( \bar{W}_4 \).

According to Theorem 2.4 and the characteristic (2.20) of a quasichebyshevian composition, it follows that if \( X_2 \times Y_2 \in W_4 \) is a quasichebyshevian composition, then the conformal corresponding composition \( X_2 \times Y_2 \in \bar{W}_4 \) is quasichebyshevian, and in particular it can be Cartesian, i.e. \( \nabla_i a^k_j = 0 \) and \( \bar{\tau}_j = 0 \). Thus, on account of \( p_j = \bar{\tau}_j - \tau_j \), (2.17) and (2.34), we receive the following contention:

**Corollary 2.3.** An orthogonal composition \( X_2 \times Y_2 \in W_4 \) is transformed into a Cartesian composition \( X_2 \times Y_2 \in \bar{W}_4 \) by the conformal transformation \( \eta \) exactly when it is quasichebyshevian. The vector of conformal transformation \( p_j \) is defined by the equation:

\[
(2.35) \quad p_j = -\tau_j = -\frac{1}{4} a^s_j \nabla_k a^k_s.
\]

Using the characteristics of the considered up to here Cartesian and quasichebyshevian compositions on \( W_4 \) about the connections \( \nabla, \bar{\nabla} \) and \( \nabla \), we determine that have a fundamental role the vectors \( \omega_j \), \( \alpha_j \), \( \tau_j \) and \( p_j \), which are connected with equations (2.22), (2.27) and (2.35) for the different cases. Using (1.15) and (1.17), first we will define the vector \( \bar{\tau}_j = a^s_j \bar{\tau}_s \) by the coefficients on the derivative equations. From (2.17)
follows that \( \tilde{\tau}_j = \frac{1}{4} \nabla_k a_j^k \). Then, from (1.16) and (1.17), we obtain:

\[
\tilde{\tau}_j = \frac{1}{2} \left[ (T_k v^k + \frac{4}{3} T_k v^k) v_j + (T_k v^k + \frac{2}{3} T_k v^k) v_j^2 \right.
\]

\[
- \left( \frac{T_k v^k + \frac{2}{3} T_k v^k} \right) v_j + \left( \frac{T_k v^k + \frac{2}{3} T_k v^k} \right) v_j^4 \right].
\]

(2.36)

From (2.18) we have \( \tilde{\sigma}_j = \frac{1}{4} \alpha_j \), and from (2.22) follows the equation:

\[
(2.37) \quad \alpha_j = 4(\tilde{\tau}_j + \tilde{\omega}_j).
\]

The equation (2.35) is equivalent to:

\[
p_j = \frac{1}{2} \left[ (T_k v^k + \frac{4}{3} T_k v^k) v_j + (T_k v^k + \frac{2}{3} T_k v^k) v_j^2 \right.
\]

\[
+ \left( \frac{T_k v^k + \frac{2}{3} T_k v^k} \right) v_j + \left( \frac{T_k v^k + \frac{2}{3} T_k v^k} \right) v_j^4 \right].
\]

(2.38)

According to Corollary 2.2, in the case when the composition \( X_2 \times Y_2 \in W_4 \) is Cartesian about \( \nabla \) follows that \( \tau_j = \tilde{\tau}_j = 0 \). Then, from (2.36), the linear independence of \( v_j \), \( s = 1, 2, 3, 4 \) and (1.17), we receive that the composition \( X_2 \times Y_2 \in W_4 \) is Cartesian about \( \nabla \) exactly when:

\[
(2.39) \quad T_1 = T_2 = T_3 = T_4 = 0 \quad \text{and} \quad \omega_j = \frac{1}{4} \tilde{\alpha}_j.
\]

Then (2.39) is an invariant characteristic of a Cartesian composition about \( \nabla \).

3. Properties of the curvature tensor on a four-dimensional Weyl product space

Let \( W_4 \) be a Weyl product space. Following the results from paragraph 2, we determine that the geometry of \( W_4 \) is defined depending on the following possibilities:

1. \( W_4 \) is a space of a quasichebyshevian composition about the connection \( \nabla \) and the composition is Cartesian about \( \tilde{\nabla} \), i.e. \( M^4 \in w_0 \).

2. \( W_4 \) is a space of a Cartesian composition about the connection \( \nabla \) and the composition is quasichebyshevian about \( \tilde{\nabla} \), i.e. \( M^4 \in w_1 \).

3. \( W_4 \) is a space of a quasichebyshevian composition about the connection \( \nabla \) and \( \tilde{\nabla} \) (but the composition isn’t Cartesian about \( \nabla \) and \( \tilde{\nabla} \)). In this case \( M^4 \in w_1 \).

In this paragraph we will find the geometry properties of the curvature tensors respectively on the connections \( \nabla \) and \( \tilde{\nabla} \) for the first two cases, which are indicated above.
Let \( R_{ijk} \), \( R_{jk} \) and \( R \) (respectively \( K_{ijk} \), \( K_{jk} \) and \( K \)) be correspondingly a curvature tensor, a Ricci tensor and a scalar curvature of the Weyl connection \( \nabla \) (respectively the Levi-Civita connection \( \tilde{\nabla} \)). Then, from (1.1), (1.4) and (1.5) we obtain the following dependence:

\[
R_{ijk}g_{sl} = K_{ijkl} + \frac{1}{4}(R_{ji} - R_{ij})g_{kl} + M_{jk}g_{sl} - M_{sk}g_{jl} + M_{sl}g_{jk} - M_{jl}g_{sk},
\]

\[
M_{jk} = \nabla_j \omega_k - \omega_j \omega_k + \frac{1}{2}g_{jk}g^{sm}\omega_s \omega_m,
\]

where \( K_{ijkl} = K_{ijk}g_{sl} \) is a curvature tensor of type \((0,4)\) of \( \tilde{\nabla} \). We note that in \( W_n \) a Ricci tensor \( R_{jk} \) isn’t symmetric and a tensor \( R_{ijk}g_{sl} \) isn’t a curvature tensor of type \((0,4)\).

Now, let us consider the first case, i.e. when \( M^4 \) is a Riemannian \( P \)-manifold \( (\tilde{\nabla}_i a^k_j = 0) \).

In the paper [9] it is proven, that for every Riemannian \( P \)-manifold \( M^{2n} \) a curvature tensor \( K(x, y, z, w) \) of type \((0, 4)\) has the following properties:

\[
K(x, y, Pz, Pw) = K(x, y, z, w),
\]

where \( x, y, z, w \in T_p M^{2n} \). Moreover, a tensor \( \tilde{K}(x, y, z, w) = K(x, y, z, Pw) \) is also a curvature tensor and it is called an associated tensor of \( K(x, y, z, w) \). The corresponding Ricci tensor and scalar curvature of an associated tensor \( \tilde{K}(x, y, z, w) \) will be denoted by \( \tilde{K}(y, z) \) and \( \tilde{K} \). An important role in the geometry of Riemannian \( P \)-manifolds is performed by the following tensors:

\[
\pi_1(x, y, z, w) = g(y, z)g(x, w) - g(x, z)g(y, w); \\
\pi_2(x, y, z, w) = g(y, Pz)g(x, Pw) - g(x, Pz)g(y, Pw); \\
\pi_3(x, y, z, w) = g(y, z)g(x, Pw) - g(x, z)g(y, Pw) + g(y, Pz)g(x, w) - g(x, Pz)g(y, w).
\]

Every two-dimensional section \( \beta \) in \( T_p M^{2n} \) holds a Riemannian section curvature \( \nu(\beta, p) \) and an associated section curvature \( \tilde{\nu}(\beta, p) \), which are defined by the equations:

\[
\nu(\beta, p) = \nu(x, y) = \frac{K(x, y, y, x)}{\pi_1(x, y, y, x)}; \quad \tilde{\nu}(\beta, p) = \tilde{\nu}(x, y) = \frac{\tilde{K}(x, y, y, x)}{\pi_1(x, y, y, x)},
\]

where \( (x, y) \) is a base of the section \( \beta \). One section \( \beta \) is called totally real, if \( \beta \perp P\beta \). In this case, the curvatures \( \nu \) and \( \tilde{\nu} \) are called totally real section curvatures.
In [9] is proven the following theorem:

**Theorem B.** [9] If $M^4$ is a Riemannian $P$-manifold, then $M^4$ is with constant totally real section curvatures $\nu = \nu(p), \tilde{\nu} = \tilde{\nu}(p)$. Then the curvature tensor, the Ricci tensor and the scalar curvatures satisfy the conditions:

\begin{align*}
(3.6) \quad K(x, y, z, w) &= \nu[\pi_1(x, y, z, w) + \pi_2(x, y, z, w)] + \tilde{\nu}\pi_3(x, y, z, w), \\
(3.7) \quad K(y, z) &= \frac{1}{4}[Kg(y, z) + \tilde{K}g(y, Pz)], \quad \nu = \frac{K}{8}, \quad \tilde{\nu} = \frac{\tilde{K}}{8}.
\end{align*}

From equations (3.4) in relation to the local base $\{\frac{\partial}{\partial x_i}\}$ follows:

\begin{align*}
(\pi_1)_{ijks} &= gjkgis - gijkjs; \\
(\pi_2)_{ijks} &= \tilde{g}jk\tilde{g}is - \tilde{g}ijk\tilde{g}js; \\
(\pi_3)_{ijks} &= gjk\tilde{g}is - gijk\tilde{g}js + \tilde{g}jk\tilde{g}is - \tilde{g}ijk\tilde{g}js.
\end{align*}

In the first case considered, that $M^4$ is a Riemannian $P$-manifold and the matrix of the structure $P$ is $(a^k_i)$. Then, using (3.4) and (3.8), in relation to the local base, equations (3.6) and (3.7) accept the form:

\begin{align*}
(3.9) \quad K_{ijkl} &= \frac{K}{8}((\pi_1)_{ijkl} + (\pi_2)_{ijkl}) + \frac{\tilde{K}}{8}(\pi_3)_{ijkl}, \\
(3.10) \quad K_{jk} &= \frac{1}{4}[Kg_{jk} + \tilde{K}\tilde{g}_{jk}].
\end{align*}

**Theorem 3.1.** Let $W_4$ be a Weyl product space and the orthogonal composition $X_2 \times Y_2 \in W_4$ is Cartesian about the connection $\nabla$ ($M^4$ is a Riemannian $P$-manifold). Then the curvature tensor of the Weyl connection $\nabla$ has the following structure:

\begin{align*}
(3.11) \quad R_{ijks} &= \frac{1}{8}(g_{ij}S_{ks} - g_{ik}S_{js} + g_{is}S_{jk} - g_{js}S_{ik} - 2(R_{ij} - R_{ji})g_{ks} \\
&\quad - (R + 2\text{tr}M)(\pi_1)_{ijks} + (R - 6\text{tr}M)(\pi_2)_{ijks}),
\end{align*}

where $S_{jk} = 3R_{jk} + R_{kj}$ and $\text{tr} M = g^{jk}M_{jk} = g^{jk}\nabla_j\omega_k + g^{jk}\omega_j\omega_k$.

**Proof.** From the condition of the Theorem follows that $M^4$ is a Riemannian $P$-manifold, i.e. (3.9) and (3.10) are realized. After contracting (3.1) by $g^{il}$, we obtain:

\begin{equation}
(3.12) \quad \frac{1}{4}(3R_{jk} + R_{kj}) = K_{jk} + 2M_{jk} + g_{jk}\text{tr} M.
\end{equation}

Since $R = g^{jk}R_{jk}$ and $K = g^{jk}K_{jk}$, then from (3.12) follows:

\begin{equation}
(3.13) \quad K = R - 6\text{tr} M.
\end{equation}
Consecutively we perform the following transformations: We replace $K_{jk}$ from (3.10) in (3.12), from where, on account of (3.13), we define $M_{jk}$. Then we replace the expression for the tensor $M_{jk}$ and $K_{ijkl}$ from (3.9) in (3.1). After grouping in the last equation, we receive (3.11).

**Remark.** Having in mind Corollary 2.2, the Weyl space $W_4$ in Theorem 3.1 is a space of a quasichebyshevian composition and $\omega_j = -\tau_j$. Then, the equation (3.11) is a geometry characteristic of $W_4$.

Now, let us consider the second case, i.e. $W_4$ is a space of a Cartesian composition about the connection $\nabla$ and the composition $X_2 \times Y_2 \in W_4$ is quasichebyshevian about the Levi-Civita connection $\nabla$ ($M^4 \in w_1$). In this case $\nabla_j a_k^s = 0$. Therefore we have:

\[(3.14) \quad \nabla_i \nabla_j a_k^s - \nabla_j \nabla_i a_k^s = 0.\]

According to [5], from (3.14) follows:

\[(3.15) \quad R_{ijk.} a_i^s = R_{ijl.} a_i^s.\]

From (1.2) and $\nabla_j a_k^s = 0$, we obtain the equation:

\[(3.16) \quad \nabla_j \tilde{g}_{ks} = 2\omega_j \tilde{g}_{ks}.\]

We find the second covariant differentiation of (3.16) and after an alternation we have:

\[(3.17) \quad -R_{ijk.} \tilde{g}_{ls} - R_{ijl.} \tilde{g}_{kl} = 2\tilde{g}_{ks} (\nabla_i \omega_j - \nabla_j \omega_l).\]

We contract (3.17) with $g^{ks}$ and taking into account $g^{ks} \tilde{g}_{ks} = a_k^s = 0$, we find the property:

\[(3.18) \quad R_{ijl.} a_i^s = 0.\]

According to the first Bianci identity and (3.18), from (3.15) immediately follows the equation:

\[(3.19) \quad R_{jkl.} a_k^l = R_{kl.} a_j^l.\]

On account of (1.6), (3.15), (3.18) and (3.19) follows the truthfulness of:

**Theorem 3.2.** Let $W_4$ be a space of an orthogonal Cartesian composition about the Weyl connection $\nabla$. Then the curvature tensor and the Ricci tensor of connection $\nabla$ hold the following properties:

\[(3.20) \quad a_k^m a_i^s R_{jm.} = R_{ijk.}; \quad R_{ijl.} a_i^s = 0; \quad a_i^s a_j^l R_{sl} = R_{ji}.\]
In the paper [2] is introduced a Weyl tensor $W_{ijk}^l$ in an $n$-dimensional Weyl space $W_n$ and it is proven that a tensor $W_{ijk}^l$ is invariant of a conformal transformation $W_n \rightarrow \tilde{W}_n$. For $n = 4$ the Weyl tensor has the following form:

\begin{equation}
W_{ijk}^l = R_{ijk}^l - \frac{1}{8} \left[ g^{lm}(g_{jk}S_{im} - g_{ik}S_{jm}) + S_{sjk}^l - S_{ikj}^l \right. \left. - 2(R_{ij} - R_{ji}) \delta_i^l \delta_j - \frac{4R}{3} (g_{jk} \delta_i^l - g_{ik} \delta_j^l) \right],
\end{equation}

where $S_{jk} = 3R_{jk} + R_{kj}$. In [2] (Theorem 2.2 and Theorem 2.3) it is proven, that if $W_n (n > 3)$ has a zero Weyl tensor, then $W_n$ is conformal equivalent on flat $\tilde{W}_n$.

From (2.31) after contracting by $g_{ls}$, and using (3.8), we obtain:

\begin{equation}
W_{ijk}^l g_{ls} = R_{ijk}^l g_{ls} - \frac{1}{8} \left[ g_{jk}S_{is} - g_{ik}S_{js} + S_{sjk} g_{is} - S_{ikj} g_{js} \right. \left. - 2(R_{ij} - R_{ji}) g_{ks} - \frac{4R}{3} (\pi_1)_{ijks} \right].
\end{equation}

According to Theorem 3.1 and (3.22), there follows truthfulness of:

**Theorem 3.3.** Let $W_4$ be a Weyl product space and the orthogonal composition $X_2 \times Y_2 \in W_4$ be Cartesian about the connection $\tilde{\nabla}$. Then a Weyl tensor on $W_4$ has the form:

\begin{equation}
W_{ijk}^l g_{ls} = \frac{(R - 6trM)}{24} \{(\pi_1)_{ijks} + 3(\pi_2)_{ijks}\},
\end{equation}

where $trM = g^{jk}M_{jk} = g^{jk}\nabla_j \omega_k + g^{jk}\omega_j \omega_k$ and $R - 6trM = K$ is the scalar curvature of the Riemannian $P$-manifold $M^4$.

From (3.9), (3.13) and (3.23) follows:

**Corollary 3.1.** Let $W_4$ be a Weyl product space and the orthogonal composition $X_2 \times Y_2 \in W_4$ be Cartesian about the connection $\tilde{\nabla}$. Then $W_4$ is conformal equivalent on flat $\tilde{W}_4$ if and only if:

\begin{equation}
R = 6 \text{tr} M \Leftrightarrow K = 0.
\end{equation}

In this case a curvature tensor of a Riemannian $P$-manifold $M^4$ has the form:

\begin{equation}
K_{ijkl} = \frac{\tilde{K}}{8} (\pi_3)_{ijkl}.
\end{equation}

Let $k_{12}$ and $k_{34}$ be the section curvatures respectively of the tangent sections $(v_1, v_2)$ and $(v_3, v_4)$ of the composition. Then, using (2.1), from (3.5)
for $k_{12}$ and $k_{34}$ we have:

$$
(3.26) \quad k_{12} = \frac{K_{ijkl} v^i v^j v^k v^l}{\sin^2 \alpha}, \quad k_{34} = \frac{K_{ijkl} v^i v^j v^k v^l}{\sin^2 \beta}.
$$

If the composition $X_2 \times Y_2 \in W_4$ is Cartesian about $\widetilde{\nabla}$, then for a curvature tensor $K_{ijkl}$ the equation (3.9) is valid. From (3.9) and (3.26), following Lemma 2.1, we find:

$$
(3.27) \quad k_{12} = \frac{1}{4} (K + \widetilde{K}), \quad k_{34} = \frac{1}{4} (K - \widetilde{K})
$$

$$
\implies K = 2(k_{12} + k_{34}), \quad \widetilde{K} = 2(k_{12} - k_{34}).
$$

According to Corollary 3.1, in the case when $W_4$ is conformal equivalent on flat $\overline{W}_4$, i.e. $K = 0$, we have:

$$
(3.28) \quad k_{12} = -k_{34}, \quad \widetilde{K} = 4k_{12}.
$$

Using (3.25) and (3.28), it follows that $K_{ijkl} = \frac{1}{2} k_{12}(\pi_3)_{ijkl}$ and over $W_4$ the Weyl tensor $W_{ijkl}$ is equal to zero, i.e. $W_{ijkl} = 0$.

References


ON A FIBRE BUNDLE FORMULATION OF CLASSICAL AND STATISTICAL MECHANICS

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Some elements of classical mechanics and classical statistical mechanics are formulated in terms of fibre bundles. In the bundle approach the dynamical and distribution functions are replaced by liftings of paths in a suitably chosen bundle. Their time evolution is described by appropriate linear transports along paths in it or, equivalently, by corresponding invariant bundle equations of motion. In particular, the bundle version of the Liouville equation is derived.

Keywords: Classical mechanics, Statistical mechanics, Fibre Bundles, Liouville equation

1. Introduction

In the series of papers [1,2,3,4,5,6,7], we have reformulated nonrelativistic and relativistic quantum mechanics in terms of fibre bundles. In the present work, we want to try to apply some ideas and methods from these papers to classical mechanics and classical statistical mechanics. However, as a whole this is scarcely possible because these theories are more or less primary related to the theory of space (space-time) which is taken as a base of the corresponding bundle(s) in the bundle approach and, consequently, it has to be determined by other theory. For this reason, the fibre bundle formalism is only partially applicable to some elements of classical mechan-
ics and classical statistical mechanics. and, possibly, will hardly result into a consistent theory.

The presented in [1, sec. 4.1] motivations for applying the theory of fibre bundles to non-relativistic quantum mechanics can mutatis mutandis be transferred in the case of classical and statistical mechanics. In particular, in this way the observer-dependence of the time evolution and some quantities is explicitly introduced in the theory.

A different geometrical approach to the statistical mechanics, based on the projective geometry, can be found in [8]. Some links between classical mechanics and connection theory are given in [9].

The organization of this paper is the following. In Sect. 2 are recalled some facts of classical Hamiltonian mechanics and is fixed our notation. In Sect. 3, we give a fibre bundle description of (explicitly time-independent) dynamical functions, representing the observables in classical mechanics. In this approach they are represented by liftings of paths in a suitably chosen bundle. We show that their time evolution is governed by a kind of linear (possibly parallel) transport along paths in this bundle or, equivalently, via the corresponding bundle equation of motion derived here. Sect. 4 is devoted to the bundle (analogue of the) Liouville equation, the equation on which classical statistical mechanics rests. In the bundle description, we replace the distribution function by a lifting of paths in the same bundle appearing in Sect. 3. In it we derive the bundle version of the Liouville equation which turns to be the equation for (linear) transportation of that lifting with respect to a suitable linear transport along paths. Sect. 5 closes the paper with some remarks.

2. Hamilton description of classical mechanics (review)

The states of a dynamical system in classical mechanics [10] are described via its generalized coordinates \( q = (q^1, \ldots, q^N) \in \mathbb{R}^N \) and momenta \( p = (p_1, \ldots, p_N) \in \mathbb{R}^N \) with \( N \in \mathbb{N} \) being the number of system's degree of freedom. The quantities characterizing a dynamical system, the so-called dynamical functions or variables, are described by \( C^1 \) functions in \( \mathbb{R}^F = \{ f: F \to \mathbb{R} \} \) with \( F \) being the system's phase space. The Poisson bracket of \( f, g \in \mathbb{R}^F \) is [10, § 8.4]

\[
[f, g]_p := \sum_{i=1}^N \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right)
\]  

(2.1)
which is an element of $\mathbb{R}^F$. The subset of $\mathbb{R}^F$ consisting of $C^1$ functions and endowed with the operations addition, multiplication (with real numbers) and forming of Poisson brackets is called the *dynamical algebra* and will be denoted by $\mathcal{D}$ [11, Section 1.2]. The set $\mathcal{D}$ is closed with respect to the mentioned operations and is a special kind of Lie algebra, the Poisson bracket playing a role of a Lie bracket.

If $h(q, p; t)$ is the system's Hamiltonian, the system evolves in time $t \in \mathbb{R}$ according to the (canonical) Hamilton equations [10, chapter 7, § 8.5]

\[
\dot{q}^i = \frac{\partial h(q, p; t)}{\partial p_i} = [q^i, h]_P, \quad \dot{p}_i = -\frac{\partial h(q, p; t)}{\partial q^i} = [p_i, h]_P, \quad (2.2)
\]

where $i = 1, \ldots, N$ and the dot means full derivative with respect to the time, e.g. $\dot{q}^i := dq^i/dt$. The system's state is completely known for every instant $t$ if for some $t_0 \in \mathbb{R}$ are fixed the initial values $(q, p)|_{t=t_0} = (q_0, p_0) \in \mathcal{F}$ with $\mathcal{F} = \mathbb{R}^{2N}$ being the system's phase space.

If $g$ is a dynamical function depending on time, $g \in \mathbb{R}^F \times \mathbb{R}$, then its full time derivative is [10, equation (8.58)]

\[
\frac{dg}{dt} := \dot{g} = [g, h]_P + \frac{\partial g}{\partial t}. \quad (2.3)
\]

To any dynamical function $f \in \mathcal{D}$ there corresponds operator $[f]_P : g \mapsto [g, f]_P, g \in \mathcal{D}$, i.e.

\[
[f]_P := [\cdot, f]_P : \mathcal{D} \to \mathcal{D}. \quad (2.4)
\]

Putting $\xi := (q, p) = (q^1, \ldots, q^N, p_1, \ldots, p_N) \in \mathcal{F}$ and defining the map $\overline{h} : \mathcal{F} \to \mathcal{F}$ by $\overline{h} : (q, p) \mapsto ([h]_P (q^1), \ldots, [h]_P (q^N), [h]_P (p_1), \ldots, [h]_P (p_N))$, which map can be called *Hamiltonian operator*, we see that (2.2) is equivalent to

\[
\frac{d\xi}{dt} = \overline{h}(\xi). \quad (2.5)
\]

3. **Bundle description of dynamical functions in classical mechanics**

At first sight, it seems the solution of (2.5) might be written as $\xi(t) = \mathcal{U}(t, t_0)\xi(t_0)$ with $\mathcal{U}(t, t_0)$ being the Green's function for this equation. However, this is wrong as generally $h$ depends on $\xi$, $h = h(\xi; t)$, so $\mathcal{U}$ itself must depend on $\xi$. Consequently, we cannot apply to the Hamiltonian equation (2.5) the developed in [1] method for fibre bundle interpretation and reformulation of the Schrödinger equation. The basic reason for this is that the Hamilton equation is primary related to the (phase) space while the
Schrödinger one is closely related to the 'space of observables'. This suggests the idea of bundle description of dynamical functions which are the classical analogues of quantum observables. Below we briefly realize it for time-independent dynamical functions.

Let $g \in \mathcal{D}$ and $\partial g / \partial t = 0$. By (2.3) and (2.4), we have

$$\frac{dg}{dt} = [h]_P(g). \tag{3.1}$$

Writing for brevity $g(t)$ instead of $g(\xi(t); t) = g(\xi(t); t_0)$, we can put

$$g(t) = \mathcal{V}(t, t_0) (g(t_0)), \tag{3.2}$$

where $t_0$ is a fixed instant of time and the dynamical operator $\mathcal{V}$, the Green function of (3.1), is defined via the initial-value problem

$$\frac{\partial \mathcal{V}(t, t_0)}{\partial t} = [h]_P \circ \mathcal{V}(t, t_0), \quad \mathcal{V}(t_0, t_0) = 1, \tag{3.3}$$

where 1 is the corresponding unit operator and $\circ$ is the mappings' composition sign.

The explicit form of $\mathcal{V}(t, t_0)$ is

$$\mathcal{V}(t, t_0) = \left( \text{Texp} \int_{t_0}^{t} [h(\xi; \tau)]_P \, d\tau \right)_{\xi=t_0} \tag{3.4}$$

where $\text{Texp} \int_{t_0}^{t} \ldots$ denotes the so-called chronological (called also T-ordered, P-ordered, or path-ordered) exponent. One can easily check the linearity of $\mathcal{V}(t, t_0)$ and the equalities

$$\mathcal{V}(t_3, t_1) = \mathcal{V}(t_3, t_2) \circ \mathcal{V}(t_2, t_1), \tag{3.5}$$

$$\mathcal{V}(t_1, t_1) = 1, \tag{3.6}$$

$$\mathcal{V}^{-1}(t_1, t_2) = \mathcal{V}(t_2, t_1), \tag{3.7}$$

the last of which is a consequence of the preceding two. Here $t_1$, $t_2$ and $t_3$ are any three moments of time.

Let $M$ and $\mathbb{T}$ be respectively the Newtonian 3-dimensional space and 1-dimensional time of classical mechanics.\(^a\) Let $\gamma : J \to M$, $J \subseteq \mathbb{T}$, be the trajectory of some (point-like) observer (if the observer exists for all $t \in \mathbb{T}$, then $J = \mathbb{T}$.)

Now define a bundle $(\mathcal{R}, \pi_\mathcal{R}, M)$ with total space $\mathcal{R}$, base $M$, projection $\pi_\mathcal{R} : \mathcal{R} \to M$, and isomorphic fibres $\mathcal{R}_x := \pi_\mathcal{R}^{-1}(x) := d_x^{-1}(\mathbb{R})$ where $\mathbb{R}$ is

\(^a\) $M$ and $\mathbb{T}$ are isomorphic to $\mathbb{R}^3$ and $\mathbb{R}^1$ respectively. This is insignificant for the following.
regarded as a standard fibre of \((R, \pi_R, M)\) and \(d_x : R_x \to \mathbb{R}\) are (arbitrarily) fixed isomorphisms, which are free parameters in our theory. This means that \(R = \bigcup_{x \in M} R_x\) and \(\pi_R(r) = x, r \in R, \) if \(r \in R_x\) for some \(x \in M\).

To every function \(g : \mathcal{F} \times T \to \mathbb{R}\), we assign a lifting of paths\(^b\) \(g\) such that

\[
g : \gamma \mapsto g_\gamma : t \mapsto g_\gamma(\xi; t) := d_{\gamma(t)}^{-1}(g(\xi; t)) \in R_\gamma(t).
\]

In this way the dynamical algebra \(\mathcal{D}\) becomes isomorphic to a subalgebra of the algebra of liftings of paths (or sections along paths) of \((R, \pi_R, M)\).

For explicitly time-independent dynamical functions, substituting (3.8) into (3.2), we get

\[
g_\gamma(t) = V_\gamma(t, t_0)(g_\gamma(t_0)),
\]

where, for brevity, we write \(g_\gamma(t) := g_\gamma(\xi(t); t) = g_\gamma(\xi(t); t_0)\) and

\[
V_\gamma(t, t_0) := d_{\gamma(t)}^{-1} \circ V(t, t_0) \circ d_{\gamma(t_0)} : R_{\gamma(t_0)} \to R_{\gamma(t)}.
\]

The map \(V_\gamma(t, t_0)\) is linear and, due to (3.5) and (3.6), satisfies the equations

\[
V(t_3, t_1) = V(t_3, t_2) \circ V(t_2, t_1),
\]

\[
V(t_1, t_1) = 1.
\]

The last three equations show that \(V : \gamma \mapsto V_\gamma : (t, t_0) \mapsto V_\gamma(t, t_0)\) is a linear transport along paths in \((R, \pi_R, M)\) (cf. [1] or [12,13]). We call it the dynamical transport.

By [14, proposition 5.3] or [1, eq. (3.40)], equation (3.9) is equivalent to

\[
V^D(g) = 0.
\]

Here \(V^D\) is the derivation along paths corresponding to \(V\), viz. (see [12, definition 4.1], [13], or [1, definition 3.4])

\[
V^D : \text{PLift}^1(R, \pi_R, M) \to \text{PLift}^0(R, \pi_R, M),
\]

where \(\text{PLift}^k(R, \pi_R, M), k = 0, 1, \ldots\) is the set of \(C^k\) liftings of paths from \(M\) to \(R\), and its action on a lifting \(\lambda \in \text{PLift}^1(R, \pi_R, M)\) with \(\lambda : \gamma \mapsto \lambda_\gamma\) is given via

\[
V^D_t(\lambda) := \lim_{\varepsilon \to 0} \left\{ \frac{1}{\varepsilon} \left[ V_\gamma(t, t + \varepsilon)(\lambda_\gamma(t + \varepsilon)) - \lambda_\gamma(t) \right] \right\}
\]

\(^b\)Equivalently, the mapping \(g\) can be regarded as a (multiple-valued) section along paths; see [1, sect. 3 &4].
with \( \nabla D^\gamma \lambda (\lambda) := ((\nabla D \lambda)(\gamma))(t) = (\nabla D \lambda)_{\gamma}(t) \).

The equivalence of (3.13) and the conventional equation of motion (3.1) can easily be verified. Therefore (3.13) represents the bundle equation of motion for the dynamical functions.

To conclude, we emphasize on the fact that the application of the bundle approach, developed in [1,2], to classical mechanics results only in bundle description of dynamical functions.

4. Bundle description of the Liouville equation

In classical statistical mechanics [11], the evolution of a system is described via a distribution (function on the phase space) \( \mathcal{P} : \mathcal{F} \times \mathbb{T} \to \mathbb{R} \) satisfying the conditions \( \int_{\mathcal{F}} d\xi \mathcal{P}(\xi; t) = 1 \) and \( \mathcal{P}(\xi; t) \geq 0, \xi \in \mathcal{F}, t \in \mathbb{T}, \) and whose time evolution is governed by the Liouville equation

\[
\frac{\partial \mathcal{P}}{\partial t} = \mathcal{L}(\mathcal{P})
\]

with \( \mathcal{L} = \mathcal{L}(\xi; t) \) being the Liouville operator (the Liouvillian) of the investigated system [11, § 2.2]. If the system is Hamiltonian, i.e. if it can be described via a Hamiltonian \( h \), its Liouvillian is \( \mathcal{L} = -[h]_\mathcal{P} \).

Since equations (3.1) and (4.1) are similar, we can apply the already developed ideas and methods to the bundle reformulation of the basic equation of classical statistical mechanics.

We can write the solution of (4.1) as

\[
\mathcal{P}(\xi; t) = \mathcal{W}(\xi; t, t_0)(\mathcal{P}(\xi; t_0))
\]

where the distribution operator \( \mathcal{W} \) is defined via the initial-value problem

\[
\frac{\partial \mathcal{W}(\xi; t, t_0)}{\partial t} = \mathcal{L}(\xi; t) \circ \mathcal{W}(\xi; t, t_0), \quad \mathcal{W}(\xi; t_0, t_0) = 1,
\]

i.e. \( \mathcal{W}(\xi; t, t_0) = \text{Exp} \int_{t_0}^{t} \mathcal{L}(\xi; \tau) d\tau \).

Since \( \mathcal{W} \) satisfies (3.5) and (3.6) with \( \mathcal{V} \) for \( \mathcal{V} \), a fact that can easily be checked, the maps

\[
W(\xi; t, t_0) := d_{\gamma(t)}^{-1} \circ \mathcal{W}(\xi; t, t_0) \circ d_{\gamma(t_0)} : R_{\gamma(t_0)} \to R_{\gamma(t)}
\]

satisfy (3.11) and (3.12). Therefore these maps define a linear transport \( W \) along paths in \((R, \pi_R, M)\). It can be called the distribution transport.
Now to any distribution $\mathcal{P}: \mathcal{F} \times \mathbb{T} \rightarrow \mathbb{R}$, we assign a (distribution) lifting $P$ of paths in the fibre bundle $(R, \pi_R, M)$, introduced in Sect. 3, such that

$$P: \gamma \mapsto P_\gamma: t \mapsto P_\gamma(\xi; t) := d_{\gamma(t)}^{-1}(\mathcal{P}(\xi; t)) \in R_{\gamma(t)}. \quad (4.5)$$

The so-defined lifting $P: \gamma \rightarrow P_\gamma$ of paths in $(R, \pi_R, M)$ is linearly transported along arbitrary observer's trajectory $\gamma$ by means of $W$. In fact, combining (4.2) and (4.5), using (4.5) for $t = t_0$ and (4.4), we get

$$P_\gamma(\xi; t) = W_\gamma(\xi; t, t_0)(P_\gamma(\xi; t_0)) \quad (4.6)$$

which proves our assertion. We want to emphasize on the equivalence of (4.6) and the Liouville equation (4.1), a fact following from the derivation of (4.6) and the definitions of the quantities appearing in it. This result, combined with [14, proposition 5.3], shows the equivalence of (4.1) with the invariant equation

$$^WD(P) = 0 \quad (4.7)$$

where $^WD$ is the derivation along $\gamma$ corresponding to $W$ (see (3.14)). The last equation can naturally be called the bundle Liouville equation.

5. Conclusion

In this paper we tried to apply the methods developed in [1,2,3,4,5] for quantum mechanics to classical mechanics and classical statistical mechanics. Regardless that these methods are fruitful in quantum mechanics, they do not work with the same effectiveness in classical mechanics and statistics. The main cause for this is that these mechanics are more or less theories of space (and time), i.e. they directly depend on the accepted space (and time) model. So, since the fibre bundle formalism, we are attempting to transfer from quantum mechanics and statistical to classical ones, is suitable for describing quantities directly insensitive to the space(-time) model, we can realize the ideas of [1,2,3,4,5] in the classical region only partially.

In this work we represented dynamical and distribution functions as liftings of paths of a suitably chosen fibre bundle over space. These liftings, as it was demonstrated, appear to be linearly transported along any observer's trajectory with respect to corresponding (possibly parallel) transports along paths in the bundle mentioned. As a consequence of that fact, the equations of motion for distributions and time-independent dynamical functions have one and the same mathematical form: the derivations, generated by the corresponding transports, of these liftings vanish along observer's trajectory.
Thus, we have seen that (some) quantities arising over space admit natural bundle formulation which is equivalent to the conventional one. We demonstrated that for time-independent dynamical functions in classical Hamiltonian mechanics and distribution functions in classical statistical mechanics. Other classical quantities also admit bundle description.

The fibre bundle formalism is extremely suitable for describing all sorts of fields over the space(-time). Therefore it seems naturally applicable to quantum physics. In particular, this is true for nonrelativistic and relativistic quantum mechanics (and statistics) whose full self-consistent bundle (re)formulation is developed in the series of papers [1,2,3,4,5,6,7].

References


EXISTENCE OF INDEFINITE KÄHLER METRICS OF CONSTANT SCALAR CURVATURE ON COMPACT COMPLEX SURFACES

Dedicated to Professor Kouei Sekigawa on his sixtieth birthday

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In the present note, some recent results concerning the existence of indefinite Kähler metrics of constant scalar curvature (e.g., indefinite Kähler-Einstein metrics and scalar-flat Kähler metrics) on compact complex surfaces are reported. It turns out that such an existence problem is closely related to a certain obstruction, which is a generalized version of the Bando-Calabi-Futaki character. Related problems and questions are also proposed.

1. Introduction

Kähler-Einstein metrics and scalar-flat Kähler metrics (both including Ricci-flat Kähler ones) are two important subjects in complex differential geometry. There have been many studies for these Kähler metrics in the positive-definite case, concerning existence, constructions, uniqueness, obstructions, and relationships with other geometric structures. Furthermore, such metrics have their natural indefinite counterparts, which have also been studied in mathematical physics. Motivated by the work of Ooguri-Vafa [28] on string theory, Petean [29] studied the existence problem for indefinite Kähler-Einstein metrics, especially Ricci-flat ones, on compact complex surfaces. As another generalization of the Ricci-flat case, the existence problem for scalar-flat indefinite Kähler metrics on compact complex

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surfaces has been studied in [13,15]. On complex surfaces, scalar-flat Kähler metrics (possibly indefinite) are very important, since these yield self-dual metrics.

Next to Kähler-Einstein metrics and scalar-flat Kähler metrics, we are interested in the geometry of Kähler metrics of constant scalar curvature. In the positive-definite case, there are many extensive studies in this topic. However, compared with the positive-definite one, the indefinite case has not been well-explored. In this note, we will introduce several results for the existence of (mainly indefinite) Kähler-Einstein metrics, scalar-flat Kähler metrics and Kähler metrics of constant scalar curvature on compact complex surfaces, and will pose several problems and questions.

2. Preliminaries

Let $M = (M, J)$ be a complex manifold of complex dimension $n$ with complex structure $J$. A pseudo-Riemannian metric $g$ on $M$ is called a Kähler metric if $g$ is $J$-invariant (i.e., $g(J\cdot, J\cdot) = g$) and the fundamental form $\omega := g(J\cdot, \cdot)$ is $d$-closed, and hence $(M, g)$ is called a Kähler manifold with Kähler form $\omega$. Note that $g$ is not assumed to be positive-definite. A metric $g$ is said to be definite if $g$ is positive or negative definite. The positivity and the negativity of metrics are of no importance, when we discuss their existence; indeed, $g$ is positive-definite if and only if $-g$ is negative-definite. If $n = 2$, then $g$ is definite (resp. indefinite) if and only if $\omega^2 > 0$ (resp. $\omega^2 < 0$) with respect to the complex orientation of $M$.

Let $(M, g)$ be a compact Kähler manifold with Kähler form $\omega$. Then the Kähler class of $\omega$ is defined to be the Bott-Chern class $[\omega]_{BC}$ of $\omega$, which is an element in the Bott-Chern cohomology group

$$H^{1,1}_{BC}(M; \mathbb{R}) = \frac{\text{Ker}(d : \Omega^{1,1}(M; \mathbb{R}) \to \Omega^3(M; \mathbb{R}))}{\text{Im}(\sqrt{-1}\partial\bar{\partial} : \Omega^0(M; \mathbb{R}) \to \Omega^{1,1}(M; \mathbb{R}))}$$

(Barth et al. [2]), where $\Omega^{1,1}(M; \mathbb{R})$ and $\Omega^r(M; \mathbb{R})$ denote the spaces of real $(1,1)$-forms and $r$-forms on $M$ ($r = 0, 1, 2, 3, \ldots, 2n$), respectively. In the definite case, the de Rham cohomology class $[\omega]_{DR}$ of the Kähler form $\omega$ is called the Kähler class, since it completely determines $[\omega]_{BC}$, by the $\partial\bar{\partial}$-lemma. However, in the indefinite case, the $\partial\bar{\partial}$-lemma does not hold in general. In fact, Gauduchon [12] showed the following exact sequence:

$$0 \to H^1(M; \mathcal{O}_M)/H^1_{DR}(M; \mathbb{R}) \to H^{1,1}_{BC}(M; \mathbb{R}) \to H^{1,1}_{DR}(M; \mathbb{R}) \to 0,$$

(1)

where $\mathcal{O}_M$ denotes the structure sheaf of $M$, and $H^1_{DR}(M; \mathbb{R})$ and $H^{1,1}_{DR}(M; \mathbb{R})$ are the de Rham cohomology groups determined by real
closed 1-forms and (1,1)-forms on \(M\), respectively. Hence we see that 
\[ H^{1,1}_{BC}(M; \mathbb{R}) \cong H^{1,1}_{DR}(M; \mathbb{R}) \]  
(i.e., the \(\partial \overline{\partial}\)-lemma holds for real (1,1)-forms) if and only if the first Betti number \(b_1(M)\) of \(M\) equals the twice of its irregularity \(q_M = \dim_C H^1(M; \mathcal{O}_M)\) (i.e., \(b_1(M) = 2q_M\)). When \(M\) is a compact complex surface, the \(\partial \overline{\partial}\)-lemma holds for (1,1)-forms if and only if \(b_1(M)\) is even, that is, \(M\) admits a definite Kähler metric. Note that the compactness of \(M\) plays an essential role in showing (1).

Let \(L\) be a holomorphic line bundle over a compact complex manifold \(M\) and \(h_L\) a Hermitian fiber-metric on \(L\). For a (local) non-zero holomorphic section \(\sigma\), set 
\[ c_1(L) := \left[ (\sqrt{-1}/2\pi) \partial \overline{\partial} \log |h_L(\sigma, \sigma)| \right]_{BC}. \]
Then \(c_1(L)\) is independent of the choice of \(h_L\) as an element in \(H^{1,1}_{BC}(M; \mathbb{Z})\) (Bott-Chern [3]) and is called the first Chern class of \(L\). If \((M, g)\) is a Kähler manifold (possibly indefinite), then the de Rham class \([\gamma]_{DR}\) (resp. the Bott-Chern class \([\gamma]_{BC}\)) of the Ricci form \(\gamma := -\sqrt{-1} \partial \overline{\partial} \log |\det g|\) of \((M, g)\) determines the first Chern class \(c_1(K^{-1}_M)\) (resp. \(c_1(K^{-1}_M)\)) of the anti-canonical bundle \(K^{-1}_M\) of \(M\). Here \(|\det g|\) stands for a Hermitian fiber-metric on \(K^{-1}_M\). We write \(c_1(M)\) and \(c_1(M)\) for \(c_1(K^{-1}_M)\) and \(c_1(K^{-1}_M)\), respectively.

We next recall the equation characterizing the existence of a Kähler metric of constant scalar curvature in a given Kähler class. Let \((M, g)\) be a compact Kähler manifold with Kähler form \(\omega\) and denote \(\eta\) its Kähler class. Then, \(\eta\) contains a Kähler metric of constant scalar curvature if and only if there is a function \(\varphi\) on \(M\) satisfying
\[
(\gamma - \sqrt{-1} \partial \overline{\partial} f) \wedge (\omega + \sqrt{-1} \partial \overline{\partial} \varphi)^{n-1} = \mu (\omega + \sqrt{-1} \partial \overline{\partial} \varphi)^{n},
\]
where \((\omega + \sqrt{-1} \partial \overline{\partial} \varphi)^n/\omega^n =: e^f(> 0)\) and \(\mu := 2\pi c_1(M) \cdot \eta^{n-1}/\eta^n\). It seems to be difficult to solve (2) in general, even in the definite case.

3. Obstructions

There are two famous obstructions to the existence of definite Kähler metrics of constant scalar curvature on compact complex manifolds: the Matsushima-Lichnerowicz obstruction and the Bando-Calabi-Futaki character. In what follows, let \(M\) be a compact complex manifold of complex dimension \(n\), \(\mathfrak{h}(M)\) denote the Lie algebra of all holomorphic vector fields on \(M\) and \(\mathfrak{h}_0(M)\) the subset of \(\mathfrak{h}(M)\) consisting of vector fields with zeros.

The Matsushima-Lichnerowicz obstruction means that, if \(M\) admits a definite Kähler metric of constant scalar curvature, then \(\mathfrak{h}_0(M)\) becomes a reductive Lie subalgebra of \(\mathfrak{h}(M)\) (Matsushima [22], Lichnerowicz [21]). As noted in LeBrun-Simanca [20], if \(M\) admits no definite Kähler metric,
the set $\mathfrak{h}_0(M)$ is not necessarily a linear subspace of $\mathfrak{h}(M)$. The author has not known yet whether this obstruction can be generalized to the indefinite case.

In the definite case, it is well-known that the complexification of the Lie algebra $\mathfrak{u}(M,g)$ of Killing vector fields on a compact Kähler manifold $(M,g)$ is a subalgebra of $\mathfrak{h}(M)$. In the author’s paper [15], a similar result for indefinite Kähler surfaces is reported, indeed, every Killing vector field on a compact indefinite Kähler surface is a real holomorphic vector field. In [14,13], the same conclusion in the result above was stated without compactness; however, the compactness assumption is essential and must be assumed. In fact, anonymous referees to [15] pointed out that, without compactness, the same conclusion does not hold in general.

On the other hand, the Bando-Calabi-Futaki character can be generalized to the indefinite case, under certain additional assumptions. We recall the definition of this generalization (Futaki-Mabuchi [11], [15]). Let $(M,g,\omega)$ be a compact (indefinite) Kähler manifold with Kähler form $\omega$ and $\eta := [\omega]_{BC}$ denote its Kähler class. A holomorphic vector field $V$ on $M$ is said to be Hamiltonian holomorphic, if $[\nu_V \omega] = 0$ in $H^1(M; \mathcal{O}_M)$, that is, there exists a smooth function $\nu$ on $M$ such that $\nu_V \omega = \bar{\partial} \nu$ ($\nu$ is called the holomorphy potential of $V$ with respect to $\omega$). In particular, when $M$ is regular (i.e., $H^1(M; \mathcal{O}_M) = \{0\}$), every holomorphic vector field on $M$ is Hamiltonian holomorphic. Note that the notion of Hamiltonian holomorphic vector fields is defined for the Kähler class $\eta$ rather than for each Kähler form $\omega$ in $\eta$.

Let $\mathfrak{h}(M,\eta)$ be the set of all Hamiltonian holomorphic vector fields on $(M,\eta)$. Suppose that $(M,\eta)$ and $V$ in $\mathfrak{h}(M,\eta)$ satisfy the following conditions:

(i) $M$ admits a definite Kähler metric.
(ii) $\eta = 2\pi c_1(L)$ for a holomorphic line bundle $L$ over $M$.
(iii) $V$ can lift holomorphically onto $L$.

Then we define a generalized version of the Bando-Calabi-Futaki character $F^{\eta/2\pi}_M(V)$ by

$$F^{\eta/2\pi}_M(V) := \frac{n}{(2\pi)^n} \int_M \nu(\gamma \wedge \omega^{n-1} - \mu \omega^n),$$

where $\nu$ is the holomorphy potential of $V$ with respect to $\omega$ in $\eta$ and $\mu := 2\pi c_1(M) \cdot \eta^{n-1}/\eta^n$. It is easy to show that $F^{\eta/2\pi}_M(V)$ is independent of the choice of $\nu$. Apparently, the right-hand side of (3) depends on the choice of $\omega$ in $\eta$. Under the assumptions (i)–(iii), it has been shown in [15] that
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$F^{\eta/2\pi}_M(V)$ is also independent of the choice of $\omega$ in $\eta$, by using a formula in Tian [30]. By definition, if $(M, \eta, V)$ satisfies the conditions (i)–(iii) and if $\eta$ contains the Kähler form of a Kähler metric of constant scalar curvature, then $F^{\eta/2\pi}_M(V)$ vanishes. Namely, $F^{\eta/2\pi}_M$ is an obstruction to the existence of Kähler metrics of constant scalar curvature.

If $\omega$ comes from a definite Kähler metric, then $F^{\eta/2\pi}_M$ coincides with the usual Bando-Calabi-Futaki character, without the assumptions (ii) and (iii). Indeed, since there exists a smooth function $f$ on $M$ such that $\gamma \wedge \omega^{n-1} - \mu \omega^n = (\Box f/n) \omega^n$, we have

$$F^{\eta/2\pi}_M(V) = \frac{1}{\sqrt{-1}} \int_M V f \left( \frac{\omega}{2\pi} \right)^n,$$

which coincides with the usual Bando-Calabi-Futaki character (Futaki [9], Calabi [6]).

4. Kähler surfaces

In this section, we focus our attention on the existence of Kähler metrics of constant scalar curvature on compact complex surfaces $M$. As stated above, $M$ admits a definite Kähler metric if and only if $b_1(M)$ is even. However, there is no such a simple criterion for the existence of an indefinite Kähler metric. Note that an indefinite Kähler metric on $M$ is a neutral metric (i.e., a metric of neutral signature $(++--)$). For the existence of neutral metrics on compact four-manifolds, see Matsushita [23], Matsushita-Law [26]. Concerning restrictions on compact indefinite Kähler surfaces, see Petean [29].

4.1. Ricci-flat case

If $M$ admits a Ricci-flat Kähler metric $g$ (possibly indefinite), then the first Chern class $c_1(M) = 0$ in $H^2(M; \mathbb{R})$ and the Kodaira dimension $\kappa(M)$ equals zero. By the Enriques-Kodaira classification of compact complex surfaces (Barth et al. [2]), we see that $M$ must be biholomorphic to the following possibilities:

- a hyperelliptic surface;
- a complex torus;
- a $K3$ surface or an Enriques surface (if $g$ is definite);
- a primary Kodaira surface (if $g$ is indefinite).

Conversely, these candidates admit Ricci-flat Kähler metrics (possibly of indefinite signature). It is well-known that a $K3$ surface and an Enriques
surface admit definite Ricci-flat Kähler metrics, due to Yau's theorem [33]. It is known that these surfaces admit neutral metrics ([23]), but no indefinite Kähler metric ([29]). Note that a primary Kodaira surface admits no definite Kähler metric, since its first Betti number equals three. A hyperelliptic surface and a complex torus admit Ricci-flat definite Kähler metrics, which are in fact known to be flat via the Gauss-Bonnet formula. In the indefinite case, a primary Kodaira surface, a hyperelliptic surface and a complex torus also admit flat indefinite Kähler metrics; however, these may admit nonflat, Ricci-flat ones. In [29], explicit examples of nonflat, Ricci-flat indefinite Kähler metrics are constructed on every primary Kodaira surface and on a certain complex torus (cf. [13]). Recently, it is reported in Matsushita [24,25] that Petean’s example is a special case of a certain family of Walker metrics.

4.2. Einstein case

In the definite case, there are many studies for the existence of Kähler-Einstein metrics on compact complex surfaces. For example, it is known that a minimal surface $M$ of general type admits a definite Kähler-Einstein metric with $\mu < 0$ if and only if $M$ contains no $(-2)$-curve. Concerning the case $\mu > 0$, the complex projective plane $\mathbb{P}^2$ and the product $\mathbb{P}^1 \times \mathbb{P}^1$ are typical examples. It is also known that the blow-up of $\mathbb{P}^2$ at one or two points admits no definite Kähler-Einstein metric, but a blow-up of $\mathbb{P}^2$ at suitable $k$ points with $k = 3, 4, \ldots, 8$ admits such a metric. For these and more results, see, e.g., Barth et al. [2], Bourguignon [4], Futaki [10], Tian [31] etc., and references therein.

In the indefinite case, Petean [29] showed that the existence of an indefinite Kähler-Einstein metric on $M$ with $\mu \neq 0$ implies $c_2(M) < 0$ and $\kappa(M) = -\infty$, and moreover that, if $M$ admits an indefinite Kähler-Einstein metric with $\mu \neq 0$, then $M$ must be biholomorphic to a minimal irrational ruled surface; a certain surface of class VII$_0$. He also gave examples of indefinite Kähler-Einstein metrics on the former surfaces; however, no surface in the latter case has been known (cf. [29]).

4.3. Scalar-flat case

In the definite case, if $M$ admits a non-Ricci-flat scalar-flat Kähler metric, then $c_2(M) < 0$ and $\kappa(M) = -\infty$, and hence $M$ must be biholomorphic to a (possibly non-minimal) ruled surface. LeBrun [19] and Kim et al. [17]
showed the existence of scalar-flat Kähler metrics on ruled surfaces blown-up at suitably chosen points.

Compared with Einstein case, the candidates of compact (non-Ricci-flat) scalar-flat indefinite Kähler surfaces are not well-understood. The product of two Riemann surfaces whose Euler characteristic are the same sign is a typical example of compact scalar-flat indefinite Kähler surfaces. Note that a certain surface of class VII₀ as in the Einstein case and a minimal surface of general type with positive even signature (e.g., a surface constructed by Atiyah [1] and Kodaira [18]) have still been in the candidates. Under additional assumptions, the existence of scalar-flat Kähler metrics gives strong restriction on the underlying complex surface. For example, if \( b₁(M) \) is even and \( \kappa(M) = -\infty \), then a compact scalar-flat indefinite Kähler surface \( M \) must be biholomorphic to a Hirzebruch surface \( F_d \) (\( d = 0, 1, 2, \ldots \)). Here \( F₀ \) and \( F₁ \) are biholomorphic to the product \( \mathbb{P}¹ \times \mathbb{P}¹ \) and the blow-up \( \mathbb{P}² \# \mathbb{P}² \) of \( \mathbb{P}² \) at one point, respectively. If \( d \geq 1 \), \( F_d \) admits no scalar-flat indefinite Kähler metric. This result is verified by using Nakagawa’s combinatorial formula in Nakagawa [27] for the Bando-Calabi-Futaki character on compact toric manifolds. For details, see [15]. Contrary, \( F₀ = \mathbb{P}¹ \times \mathbb{P}¹ \) admits many scalar-flat indefinite Kähler metrics, constructed by using an indefinite analogue of LeBrun’s ansatz (Tod [32], [13,15]).

4.4. Constant scalar curvature case

In the definite case, the existence of Kähler metrics of constant scalar curvature is related to the stability of the underlying polarized Kähler manifolds and is also to the properness of a certain functional defined for each Kähler form in a fixed Kähler class. These relations are central motivations to study such metrics (Tian [31]). However, this point of view seems not to be appropriate in the indefinite case. As another motivation, such metrics yield examples of extremal Kähler metrics (Calabi [5,6]), which also make sense in the indefinite case. In both definite and indefinite cases, the product of two compact Riemann surfaces admits Kähler metrics of constant scalar curvature, given by the product metrics of constant curvature metrics. In particular, \( F₀ = \mathbb{P}¹ \times \mathbb{P}¹ \) admits many Kähler metrics of constant scalar curvature. Fine [7] has lately studied the existence problem of definite Kähler metrics of constant scalar curvature on certain surface bundles over a holomorphic curve, and has obtained the existence result on such complex surfaces.
Recently, by computing the Bando-Calabi-Futaki character for each possible Kähler class with the help of Nakagawa's formula, the author has shown the following fact:

\textit{No Hirzebruch surface }$F_d$ \textit{(}d \geq 1\text{) admits an indefinite Kähler metric (possibly definite) of constant scalar curvature.}

For details, see [16].

5. Problems and questions

1. Is every Killing vector field on a compact indefinite Kähler manifold is real holomorphic in higher dimensions?

2. Does there exist a generalization of the Matsushima-Lichnerowicz obstruction in the indefinite case?

3. Generalize the Bando-Calabi-Futaki character for a compact indefinite Kähler manifold $M$ with Kähler class $\eta$ to the following cases:

- $M$ admits no definite Kähler metric;
- $\eta$ is not proportional to an integral class.

4. Find a criterion, similar to Nakai's criterion, to determine whether a given Bott-Chern class contains the Kähler form of an indefinite Kähler metric.

5. On Ricci-flat indefinite Kähler surfaces:

- Describe the set of all isometry classes of indefinite Ricci-flat Kähler metrics on each of complex tori, hyperelliptic surfaces and primary Kodaira surfaces.

- Classify complex tori that admit nonflat, Ricci-flat indefinite Kähler metrics.

- Does there exist a hyperelliptic surface that admits nonflat, Ricci-flat indefinite Kähler metrics? If exists, classify such surfaces.

- Can every Ricci-flat indefinite Kähler metric be connected by a smooth path with a flat one?

6. On indefinite Kähler-Einstein surfaces:

- Classify compact indefinite Kähler-Einstein surfaces.

7. On scalar-flat indefinite Kähler surfaces:

- Classify compact scalar-flat indefinite Kähler surfaces.

- Does there exist a compact scalar-flat indefinite Kähler surface with nonzero signature?

- Can every scalar-flat indefinite Kähler metric on a compact complex
surface with zero signature be connected by a smooth path with a conformally-flat one?

(d) Does there exist a compact complex surface $M$ with $c_2^2(M) = 0$ that admits non-Ricci-flat, scalar-flat indefinite Kähler metrics?

8. On indefinite Kähler surfaces of constant scalar curvature:
(a) What kinds of topological or analytical restrictions are there on a compact indefinite Kähler surface of constant scalar curvature?

9. On compact indefinite Hermitian surfaces:
(a) What kinds of compact complex surfaces do admit (anti-)self-dual indefinite Hermitian metrics?
(b) Classify compact conformally-flat indefinite Hermitian surfaces.

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We give an announcement of our work on drawing Bézier curves on a real space form. By extending the notion of rational Bézier curves on a Euclidean plane, we define projective Bézier curves. We show we can draw circle-arcs by them.

Keywords: projective Bézier curves, rational Bézier curves, standard sphere, hyperbolic plane, circles

1. Introduction

It is well-known that Bézier curves play extremely important role in Computer Aided Geometric Design (CAGD). This area was originally developed by automotive engineers who were familiar with describing automotive parts by lines and circles. Industrial sense it is natural to consider CAGD on a flat space. But on several occasions we need to draw pictures...
on curved objects. For example, on vases, on promotional airships and so on. Though it is usual to make linear approximations when we produce industrial products, it should be useful to consider CAGD on surfaces. In this paper, by extending the notion of rational Bézier curves on a Euclidean plane, we introduce projective Bézier curves on a 2-dimensional real space form, which is one of a standard sphere, a Euclidean plane and a real hyperbolic plane.

2. Bézier curves

Since our construction of projective Bézier curves is based on Bézier curves, we shall start by recalling the definition of Bézier curves. Given $(n+1)$ control points $P_0, P_1, \ldots, P_n \in \mathbb{R}^3$, we define a Bézier curve $\sigma$ of order $n$ by

$$
\sigma(t) = (1-t)^n P_0 + \cdots + \binom{n}{k} t^k (1-t)^{n-k} P_k + \cdots + t^n P_n,
$$

where points in $\mathbb{R}^3$ are identified with vectors in a vector space $\mathbb{R}^3$. Usually we consider it on the interval $0 \leq t \leq 1$. A Bézier curve of order 1 is a line segment joining given two control points, and a Bézier curve of order 2, which is given by $\sigma(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$, is a parabola. It is known that Bézier curves are obtained by interpolating control points by line segments inductively. This method is called the de Casteljau's algorithm. For detail see for example [1,2]. Here we just draw a picture in the case $n = 2$.

![Figure 1. a Bézier curve of order 2 drawn by de Casteljau's algorithm](image)

Bézier curves have the following basic properties. Let $\sigma$ be a Bézier curve of control points $P_0, \ldots, P_n$ on $\mathbb{R}^3$.

1. The origin $\sigma(0)$ is $P_0$ and the terminus $\sigma(1)$ is $P_n$.  


(2) The initial vector $\sigma'(0)$ is parallel to the vector $\overrightarrow{P_0P_1}$ and the terminus vector $\sigma'(1)$ is parallel to the vector $\overrightarrow{P_{n-1}P_n}$.

(3) $\sigma$ lies in the inside of the convex hull of the control points.

(4) Affine maps preserve Bézier curves: That is, for a affine map $f$ of $\mathbb{R}^3$, the curve $f \circ \sigma$ coincides with the Bézier curve of control points $f(P_0), \ldots, f(P_n)$.

3. Rational Bézier curves

Bézier curves on $\mathbb{R}^3$ are quite useful but there is a fault that we can not draw circles by Bézier curves. In order to improve this point, rational Bézier curves are introduced. As a Bézier curve of order 2 is a parabola, we consider it as a kind of conic. If we use conic projection, all conics should be obtained by Bézier curves of order 2.

We take a plane $\mathbb{R}^2 = \{(x, y, 1) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$ in $\mathbb{R}^3$. We say a curve to be a rational Bézier curve if it is obtained by projecting a Bézier curve through a conic projection centered at the origin. For example, given points $Q_0, Q_1, Q_2 \in \mathbb{R}^2$ and positive numbers $\omega_0, \omega_1, \omega_2$, a rational Bézier curve $\gamma$ of order 2 with these control points is of the form

$$y(t) = \frac{(1-t)^2\omega_0Q_0 + 2t(1-t)\omega_1Q_1 + t^2\omega_2Q_2}{(1-t)^2\omega_0 + 2t(1-t)\omega_1 + t^2\omega_2}.$$

We call the numbers $\omega_0, \omega_1, \omega_2$ weights of $\gamma$. This curve is obtained by projecting a Bézier curve $\sigma$ in $\mathbb{R}^3$ with these control points $\omega_0Q_0, \omega_1Q_1, \omega_2Q_2$.

![Figure 2. A Bézier curve $\sigma$ in $\mathbb{R}^3$ and a rational Bézier curve $\gamma$ on $\mathbb{R}^2$](image)

Rational Bézier curves clearly satisfy the same properties as of Bézier curves: If $\gamma$ is a rational Bézier curve with control points $Q_0, Q_1, \ldots, Q_n$, 

1. $\gamma(0) = Q_0$ and $\gamma(1) = Q_n$, 

...
(2) \( \gamma'(0) \) is parallel to \( \overrightarrow{Q_0Q_1} \) and \( \gamma'(1) \) is parallel to \( \overrightarrow{Q_{n-1}Q_n} \).
(3) \( \gamma \) lies in the inside of the convex hull of the control points,
(4) Affine maps preserve rational Bézier curves.

Rational Bézier curves depend not only on their control points but also on their weights. In order to control rational Bézier curves of order 2 interactively on our computer systems, their shoulder tangents are quite useful. For a rational Bézier curve \( \gamma \), we call a point

\[
\gamma\left(\frac{1}{2}\right) = \frac{w_0 Q_0 + 2w_1 Q_1 + w_2 Q_2}{w_0 + 2w_1 + w_2}
\]

its shoulder point and call a line \( T \) its shoulder tangent if it is tangent to \( \gamma \) at \( \gamma(1/2) \). When \( \gamma \) is of order 2 with control points \( Q_0, Q_1, Q_2 \), we denote by \( R_0 \) the intersection of \( T \) and the segment \( Q_0Q_1 \), and by \( R_1 \) the intersection of \( T \) and the segment \( Q_1Q_2 \). These points are expressed as

\[
R_0 = \frac{w_0 Q_0 + w_1 Q_1}{w_0 + w_1}, \quad R_1 = \frac{w_1 Q_1 + w_2 Q_2}{w_1 + w_2}.
\]

![Figure 3. shoulder tangent segment T and its endpoints R₀, R₁](image)

**Proposition 3.1.** Rational Bézier curves of order 2 are determined by their control points and their shoulder tangents. More precisely, given control points \( Q_0, Q_1, Q_2 \) and a segment \( T \) determined by \( R_0 = (1-\lambda)Q_0 + \lambda Q_1 \) and \( R_1 = (1-\mu)Q_1 + \mu Q_2 \) with \( 0 < \lambda, \mu < 1 \), we find that a rational Bézier curve with these control points has \( T \) as its shoulder tangent if and only if its weights satisfy the relation

\[
\omega_0 = k\lambda\mu, \quad \omega_1 = k\mu(1-\lambda), \quad \omega_2 = k(1-\lambda)(1-\mu)
\]

with some positive \( k \).
Proposition 3.2. Rational Bézier curves of order 2 are determined by their control points and their shoulder points. More precisely, given control points $Q_0, Q_1, Q_2$ which do not lie on a line and a point $S = \lambda Q_0 + \mu Q_1 + \nu Q_2$ ($\lambda + \mu + \nu = 1$, $\lambda, \mu, \nu > 0$), a rational Bézier curve with these control points has $S$ as its shoulder point if and only if its weights satisfy the relation

$$\omega_0 = k\lambda, \omega_1 = 2k\mu, \omega_2 = k\nu$$

with some positive $k$.

4. Projective Bezier curves on a standard sphere

We are now in the position to define “nice curves” on a standard sphere $S^2$. We represent a standard sphere $S^2$ as a subset

$$\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

of $\mathbb{R}^3$. Given points $Q_0, Q_1, Q_2 \in S^2 \subset \mathbb{R}^3$ whose distances satisfy $d(Q_{i-1}, Q_i) < \pi$ ($i = 1, 2$) and positive numbers $\omega_0, \omega_1, \omega_2$, we define a projective Bézier curve of order 2 by

$$\gamma(t) = \frac{(1-t)^2\omega_0 Q_0 + 2t(1-t)\omega_1 Q_1 + t^2 \omega_2 Q_2}{\|(1-t)^2\omega_0 Q_0 + 2t(1-t)\omega_1 Q_1 + t^2 \omega_2 Q_2\|},$$

where $\| \cdot \|$ denotes the Euclidean norm. We obtain this curve by projecting the Bézier curve $\sigma$ of control points $P_0 = \omega_0 Q_0$, $P_1 = \omega_1 Q_1$, $P_2 = \omega_2 Q_2$ through the projection centered at the origin.

Figure 4. a projective Bézier curve $\gamma$ of order 2
Therefore, in the case that $Q_0, Q_1, Q_2$ lie on a geodesic segment, we have to take care in choosing its weights so that $\sigma$ does not pass through the origin. In other cases, as control points lie on some open hemi-sphere, the curve $\sigma$ clearly does not pass through the origin.

When control points lie on some hemi-sphere, we can similarly define projective Bézier curves on $S^2$. They clearly satisfy the same properties as of Bézier curves:

**Proposition 4.1.** Let $\gamma$ be a projective Bézier curve on a standard sphere $S^2$ whose control points $Q_0, \ldots, Q_n$ lies on a hemi-sphere. We then have

1. $\gamma(0) = Q_0$ and $\gamma(1) = Q_n$. 
2. At $\gamma(0)$ it is tangent to the geodesic segment joining $Q_0, Q_1$, and at $\gamma(1)$ it is tangent to the geodesic segment joining $Q_{n-1}, Q_n$. 
3. It lies in the inside of the convex hull of the control points. 
4. Isometries of $S^2$ (i.e. elements of $O(3)$) preserve projective Bézier curves.

Also for projective Bézier curves their shoulder tangents are useful in controlling them interactively. For a projective Bézier curve $\gamma$ of order 2 whose control points $Q_0, Q_1, Q_2 \in S^2$ do not lie on a geodesic, we call a point $\gamma(1/2)$ its shoulder point and call a geodesic segment $T$ its shoulder tangent if it is tangent to $\gamma$ at $\gamma(1/2)$ and its endpoints lie on geodesic segments joining $Q_0, Q_1$ and $Q_1, Q_2$. Its end points are given by

\[ R_0 = \frac{\omega_0 Q_0 + \omega_1 Q_1}{\sqrt{\omega_0^2 + \omega_1^2 + 2\omega_0\omega_1(Q_0, Q_1)}}, \quad R_1 = \frac{\omega_1 Q_1 + \omega_2 Q_2}{\sqrt{\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2(Q_1, Q_2)}}, \]

where $(\cdot, \cdot)$ denotes the canonical Euclidean inner product.

**Proposition 4.2.** Projective Bézier curves of order 2 on a standard sphere $S^2$ are determined by their control points and their shoulder tangents. More precisely, given control points $Q_0, Q_1, Q_2$, and a segment $T$ determined by

\[ R_0 = \frac{(1 - \lambda)Q_0 + \lambda Q_1}{\| (1 - \lambda)Q_0 + \lambda Q_1 \|} \quad \text{and} \quad R_1 = \frac{(1 - \mu)Q_0 + \mu Q_1}{\| (1 - \mu)Q_0 + \mu Q_1 \|} \quad (4.1) \]

with $0 < \lambda, \mu < 1$, we find that a projective Bézier curve with these control points has $T$ as its shoulder tangent if and only if its weights satisfy the relation (3.1).

We should note that our representation (4.1) does not reflect the ratios of distances $d(Q_0, R_0) : d(R_0, Q_1)$ and $d(Q_1, R_1) : d(R_1, Q_2)$. Here we give some examples which show that shoulder tangents control projective Bézier curves.
In Figure 5, the left-hand side is a figure of a projective Bézier curve of weights $\omega_0 = \omega_1 = \omega_2 = 1$, and the right-hand side is a figure of a projective Bézier curve of weights $\omega_0 = \omega_2 = 1$ and $\omega_1 = 2$.

**Proposition 4.3.** Projective Bézier curves of order 2 are determined by their control points and their shoulder points. More precisely, given control points $Q_0, Q_1, Q_2$ which do not lie on a geodesic and a point $S = \lambda Q_0 + \mu Q_1 + \nu Q_2$ for some $\lambda + \mu + \nu = 1$, $\lambda, \mu, \nu > 0$, a rational Bézier curve with these control points has $S$ as its shoulder point if and only if its weights satisfy the relation (3.2).

In order to show that projective Bézier curves of order 2 are useful, we show that we can draw circle-arcs by them.

**Theorem 4.1.** For given a circle-arc which is not closed on a standard sphere $S^2$, there is a projective Bézier curve of order 2 on $S^2$ whose image coincides with this circle-arc.

We briefly explain our idea to draw circle-arcs on a standard sphere. When the circle-arc $C$ is a part of a great circle, we take control points $Q_0, Q_2$ to be its endpoints and $Q_1$ to be the midpoint of $C$. If we choose weights as $\omega_0 = \omega_1 = \omega_2 = 1$, we find the image of this projective Bézier curve of order 2 coincides with $C$.

When the circle-arc $C$ is a part of a small circle, we take a circular cone $Z$ in $\mathbb{R}^3$ whose summit is the origin and whose intersection with the sphere coincides with this small circle (see Figure 6). Consider a plane $\alpha$ which contains the endpoints of $C$ and is parallel to a plane containing a generatrix of $Z$. Since the intersection of $Z$ and $\alpha$ is a parabola, we have a
Bézier curve \( \sigma \) on \( \mathbb{R}^3 \) whose image coincides with this parabola. Then the projective Bézier curve generated by \( \sigma \) is a desirable one.

Figure 6. circle-arc drawn by a projective Bézier curve

To be more precise, we give formulae to get the second control points \( Q_1 \) in drawing small circles. Note that the first and the third control points \( Q_0, Q_2 \) are endpoints of a given circle-arc. Let \( C \) be a circle-arc which is a part of an intersection of a standard sphere of radius 1 and a circular cone \( Z \) of angle \( \theta \). We denote its endpoints by \( Q_0 \) and \( Q_2 \). We take their weights to be \( \omega_0 = \omega_2 = 1 \). Since \( C \) lies on some plane \( \beta \) in \( \mathbb{R}^3 \) and forms an arc of a sector, we denote by \( \varphi \) the angle of this sector.

When \( \varphi = \pi \), which is the case that \( C \) is a half of a small circle, if we choose \( Q_1 \) and its weight \( \omega_1 \) as

\[
Q_1 = \frac{Q_0 \times Q_2}{\sin 2\theta}, \quad \omega_1 = \sin \theta,
\]

then the image of this projective Bézier curve of order 2 coincides with \( C \). Here \( Q_0 \times Q_2 \) denotes the exterior product of vectors \( Q_0, Q_2 \). Thus in this case the vector \( Q_1 \) of second control point is parallel to the plane \( \beta \).

When \( 0 < \varphi < \pi \), which is the case that the arc is shorter than a half of a small circle, if we choose

\[
Q_1 = \frac{\sqrt{(Q_0, Q_2) - \cos 2\theta}}{\sqrt{2} \omega_1 \sin \theta(1 + (Q_0, Q_2))} (Q_0 + Q_2) + \frac{\cos \theta \sqrt{1 - (Q_0, Q_2)}}{\sqrt{2} \omega_1 \sin \theta(1 + (Q_0, Q_2))} (Q_0 \times Q_2)
\]
and
\[ \omega_1 = \frac{\sqrt{2((Q_0, Q_2) - \cos 2\theta) + \cos \theta (1 - (Q_0, Q_2))^2}}{\sqrt{2(1 + (Q_0, Q_2)) \sin \theta}}, \]

then the image of this projective Bézier curve of order 2 coincides with C. When \( \pi < \varphi < 2\pi \), which is the case that the arc is longer than a half of a small circle, if we choose
\[ Q_1 = -\frac{\sqrt{(Q_0, Q_2) - \cos 2\theta}}{\sqrt{2\omega_1 \sin \theta (1 + (Q_0, Q_2))}} (Q_0 + Q_2) \]
\[ + \frac{\cos \theta \sqrt{1 - (Q_0, Q_2)}}{\sqrt{2\omega_1 \sin \theta (1 + (Q_0, Q_2))}} (Q_0 \times Q_2) \]

and the same \( \omega_1 \) as for the case \( 0 < \varphi < \pi \), then the image of this projective Bézier curve of order 2 coincides with C. As we can see in Figures 7, 8 and 9, the second control point goes further from the circle-arc C as it becomes longer.

Figure 7. A half of a small circle
Figure 8. Circle-arc of length shorter than half
Figure 9. Circle-arc of length longer than half

5. Projective Bezier curves on a real hyperbolic plane

In this section we define projective Bézier curves on a real hyperbolic plane. Our technique for a standard sphere goes through in this case. We represent a real hyperbolic plane \( H^2 \) as a subset
\[ \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1 \} \]
of \( \mathbb{R}^3 \). Since \( H^2 \) is a typical example of a Hadamard surface, we can define projective Bézier curves on \( H^2 \) just the same way as of rational
Bézier curves. Given control points $Q_0, Q_1, \ldots, Q_n \in H^2 \subset \mathbb{R}^3$ and positive numbers $\omega_0, \omega_1, \ldots, \omega_n$, we take a Bézier curve $\sigma$ on $\mathbb{R}^3$ of control points $P_i = \omega_i Q_i$ ($i = 0, 1, \ldots, n$). We define a projective Bézier curve $\gamma$ by $\gamma(t) = \sigma(t)/\langle\sigma(t), \sigma(t)\rangle^{1/2}$, where $\langle , \rangle$ denotes the indefinite metric on $\mathbb{R}^3$ which is given by $\langle(x_1, y_1, z_1), (x_2, y_2, z_2)\rangle = -x_1 x_2 - y_1 y_2 + z_1 z_2$. Here we note that a Bézier curve $\sigma$ in $\mathbb{R}^3$ lies in the convex hull defined by $P_0, \ldots, P_n$, we see it lies in the inside of the set $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < z^2, \, z > 0\}$ and projective Bézier curves are well-defined. For example, a projective Bézier curve of order 2 is represented as

$$
\gamma(t) = \frac{(1 - t)^2 \omega_0 Q_0 + 2t(1 - t)\omega_1 Q_1 + t^2 \omega_2 Q_2}{\| (1 - t)^2 \omega_0 Q_0 + 2t(1 - t)\omega_1 Q_1 + t^2 \omega_2 Q_2 \|_1},
$$

where $\|Q\|_1$ is given by $\sqrt{\langle Q, Q \rangle}$.

Our definition guarantees the same properties as of projective Bézier curves on $S^2$ hold.

**Proposition 5.1.** Let $\gamma$ be a projective Bézier curve on a hyperbolic plane $H^2$ with control points $Q_0, \ldots, Q_n$. We then have

1. $\gamma(0) = Q_0$ and $\gamma(1) = Q_n$.
2. At $\gamma(0)$ it is tangent to the geodesic segment joining $Q_0, Q_1$, and at $\gamma(1)$ it is tangent to the geodesic segment joining $Q_{n-1}, Q_n$.
3. It lies in the inside of the convex hull of the control points.
4. Isometries of $H^2$ (i.e. elements of $O(2, 1)$) preserves projective Bézier curves.

For projective Bézier curves of order 2 on $H^2$, we define their shoulder
tangents and shoulder points just the same way as for projective Bézier curves of order 2 on $S^2$.

**Proposition 5.2.**

1. Projective Bézier curves of order 2 on $H^2$ are determined by their control points and their shoulder tangents.
2. Projective Bézier curves of order 2 on $H^2$ are determined by their control points and their shoulder points.

For about drawing circle-arcs on $H^2$ by projective Bézier curves of order 2, we can do the same argument as for $S^2$. The only one thing we have to take care is that we need to take the second control point on $H^2$ which is included in \( \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < z^2, \, z > 0 \} \).

![Figure 11. circle-arc](image)

![Figure 12. side view of a circle-arc](image)

**Theorem 5.1.** For given a circle-arc of sufficiently small length on a real hyperbolic plane $H^2$, there is a projective Bézier curve of order 2 on $H^2$ whose image coincides with this circle-arc.

![Figure 13. drawing a circle-arc by projective Bézier curve of order 2](image)
If the length of circle-arc longer, then the second control point goes further. Thus to draw a picture on a real hyperbolic plane we need more control points than the case of drawing a picture on a sphere.

References
The algebraic notion of Gieseker stability is related to the existence of balanced metrics which are zeros of a certain moment map. We investigate some properties of balanced metrics relative to the Harder-Narasimhan filtration of a vector bundle and to blowups in the case of projective surfaces.

In Section 1 and 2, we give an overview of the relation between Gieseker stability and the existence of a sequence of canonical metrics which converge towards a (weakly) Hermite-Einstein metric if the vector bundle is Mumford stable. In Section 3, we give an approximation of the curvature of a vector bundle using natural information coming from its Harder-Narasimhan filtration. Eventually in Section 4 we look at the case of a Gieseker stable vector bundle which is not Mumford stable over a projective surface.

Let \((M, \omega)\) be a smooth projective manifold of complex dimension \(n\) with \(\omega\) a Kähler form and let \(L\) be a very ample line bundle over \(M\) equipped with a smooth hermitian metric \(h_L\).

1. **Background material for stability of vector bundles**

The purpose of this section is to introduce some classical notions about stability of vector bundles. Let's define for a holomorphic vector bundle \(E\) on \(M\) of rank \(r(E)\),

\[
\mu(E) = \mu_L(E) = \frac{\deg_L(E)}{r(E)}
\]

the normalized degree of \(E\) (relative to the degree) with respect to the polarization \(L\). Moreover, we introduce the normalized Hilbert polynomial
(relative to the Euler characteristic) for $E$ by

$$p_{E}(k) = p_{E,L}(k) = \frac{\chi(E \otimes L^{k})}{r(E)}.$$ 

For $n \mapsto p_{1}(n)$ and $n \mapsto p_{2}(n)$ two functions with integer values, we will denote $p_{1} < p_{2}$ (resp. $p_{1} \preceq p_{2}$), if for $n$ large enough, $p_{1}(n) < p_{2}(n)$ (resp. $p_{1}(n) \leq p_{2}(n)$).

**Definition 1.1.** A vector bundle $E$ is said to be Mumford $L$-stable (resp. semi-stable) if for all subsheaf $\mathcal{F}$ of $E$ with $0 < r(\mathcal{F}) < r(E)$, we have $\mu(\mathcal{F}) < \mu(E)$ (resp. $\leq$). A Mumford semi-stable vector bundle is called polystable if it is a direct sum of Mumford stable bundles (of same normalized degree).

**Definition 1.2.** A vector bundle $E$ is said to be Gieseker-Maruyama $L$-stable (resp. $L$ semi-stable) if for all subsheaf $\mathcal{F}$ of $E$ with $0 < r(\mathcal{F}) < r(E)$, we have $p_{\mathcal{F}} < p_{E}$ (resp. $\preceq$). A Gieseker semi-stable vector bundle is called polystable if it is a direct sum of Gieseker stable bundles (of same normalized Hilbert polynomial).

By Riemann-Roch Theorem, Gieseker stability and Mumford stability are equivalent if $M$ is a curve. For higher dimension, we only have the following implications:

$$E \text{ Mumford stable } \Rightarrow E \text{ Gieseker stable}$$

$$E \text{ Gieseker semi-stable } \Rightarrow E \text{ Mumford semi-stable}$$

Gieseker proved that there always exists a projective scheme $\mathcal{M}$ that parametrizes the equivalence classes of torsion free Gieseker semi-stable sheaves with fixed Chern classes. Moreover, the Gieseker stable sheaves are parametrized by the closed points of an open subscheme of $\mathcal{M}$. Gieseker and Maruyama’s approach gives a natural compactification of the moduli space $\mathcal{M}$, whereas, in general, there may not exist a canonical structure for the moduli space of equivalence classes of Mumford semi-stable sheaves. However, for projective surfaces these structures of moduli spaces do exist with different compactifications related by contractions and flips and admit the same Donaldson polynomials.

2. **Gieseker stability and canonical metrics**

Let $E$ be a hermitian holomorphic vector bundle of rank $r$ on the projective manifold $M$. By Kodaira’s theorem, for $k$ large enough we get an
embedding $i_k$ given by a basis of sections $(S_i)$ of the space $H^0(M, E \otimes L^k)$:

$$i_k : M \hookrightarrow \text{Gr}(r, N)$$

$$z_0 \mapsto \ker (ev_{z_0} : H^0(M, E \otimes L^k) \to E \otimes L^k|_{z_0})^\vee$$

where we have set $N = N(k) = \dim H^0(M, E \otimes L^k) = \chi(E \otimes L^k)$. Moreover, our situation is fully described by

$$E \otimes L^k \to U_{r, N}$$

where $U_{r, N}$ is the dual of the universal bundle over the Grassmannian of quotients of dimension $r$ of $\mathbb{C}^N$. Over $Gr(r, N)$ equipped with the natural Fubini-Study metric, acts the group

$$SU(N) = \{ R \in U(N) : \det(R) = 1 \}$$

and the associated moment map for the standard Fubini-Study metric is:

$$\mu_{SU(N), Gr(r, N)} : [Q] \mapsto Q^\dagger \overline{Q} - \frac{r}{N} Id \in \text{su}(N) = \text{Lie}(SU(N)).$$

where we have identified $\text{su}(N)$ with its dual. We consider an element $[Q] \in Gr(r, N)$ as a matrix $Q \in M_{N \times r}(\mathbb{C})$ that represents $r$ vectors of $\mathbb{C}^N$ that form an orthonormal basis, by the natural identification

$$Gr(r, N) = \{ R \in M_{N \times r}(\mathbb{C}) : \mathbb{R}^T R = Id \}/U(r)$$

Moreover, one notices that the map

$$\tilde{\mu}_{r, N} : i_k \mapsto \int_M \mu_{SU(N), Gr(r, N(k))}(i_k(x))dV(x)$$

is a moment map for the action of $SU(N)$ acting on $C^\infty(M, Gr(r, N))$, which is an infinite dimensional Kähler manifold by [Hi].

We also need the following definition:

**Definition 2.1.** Let $h$ be a hermitian metric on a globally generated holomorphic vector bundle $E$. We define $B_h \in \text{End}(E)$ as the restriction to the diagonal of the Bergman kernel associated to the $L^2$ metric induced by $h$ on $H^0(M, E)$ (also called distorsion function). If we set $(s_i)_{i=1..m}$ an orthonormal basis of $H^0(M, E)$ for the metric $\int_M \langle \cdot, \cdot \rangle_h$ and $m = \dim(H^0(M, E))$, then for all $z \in M$,

$$B_h(z) = \sum_{i=1}^{m} s_i(z) \langle \cdot, s_i(z) \rangle_h$$

and this definition does not depend on the choice of the basis $(s_i)_{i=1..m}$. 

Note $V$ the volume of $(M, \omega)$. Inspired by the ideas of [Do3], we have

**Theorem 2.1. (Wang)** The holomorphic vector bundle $E$ is Gieseker stable if and only if its automorphism group is finite and there exists $k_0 \geq 0$ such that for all $k > k_0$, the embedding $i_k$ can be balanced, in the sense that there exists a unique $g \in SL(N)$ (up to action of $SU(N)$) such that

$$\tilde{\mu}_{r,N}(g \cdot i_k) = \int_M \mu_{SU(N), Gr(r, N)}(g \cdot i_k(x)) dV(x) = 0$$

This is equivalent to the existence of a sequence of hermitian metrics $h_k$ on $E$, called balanced metrics, such that pointwise

$$B_{h_k \otimes h_L} = \frac{\chi(E \otimes L^k)}{rV} Id_{E \otimes L^k}.$$

**Sketch of the proof**

**Gieseker stability and GIT**

We refer to [H-L] for the underlying construction of Quot scheme of Gieseker semi-stable sheaves and [M-F-K] for notions of Geometric Invariant Theory (GIT). We shall use the following stability criterion developed by Gieseker and Maruyama in [Gi] and [Ma] that relates the condition of stability for a vector bundle with a condition of GIT-stability :

**Theorem 2.2. (Gieseker-Maruyama)** Let $E$ be a globally generated vector bundle of rank $r$. Let $S_i$ be a basis of sections of $H^0(M, E)$ and let $T(E) \in \text{Hom}(\wedge^r H^0(M, E), H^0(M, \det(E)))$ defined by

$$T(E)(S_{i_1}, \ldots, S_{i_r}) = S_{i_1} \wedge \cdots \wedge S_{i_r}.$$

We can view $T(E)$ as a point in the space

$$Z_E := \mathbb{P} \text{Hom}(\wedge^r H^0(M, E), H^0(M, \det(E))).$$

The vector bundle $E$ is Gieseker stable (resp. semi-stable) if and only if for $k$ large enough, $T(E \otimes L^k) \in Z_{E \otimes L^k}$ is GIT-stable with respect to the action of $SL(H^0(M, E \otimes L^k))$ and the linearisation $O_{Z_{E \otimes L^k}}(1)$.

Suppose we have fixed a reference metric $h$ on the hermitian holomorphic vector bundle $E$ and that $E \otimes L^k$ is globally generated. We get a $L^2$-metric $H = \text{Hilb}(h) = \int_M \langle \cdot, \cdot \rangle_{h \otimes h_L} dV$ on the space $H^0(M, E \otimes L^k)$. From the embedding $i_k$ in the Grassmannian given by an $H$-orthonormal basis $S_i$, we get a metric on the bundle $U_{r,N}$. Since, $i_k^* U_{r,N} \simeq E \otimes L^k$, we get a natural metric on $E$ that we will call $FS(H)$, and therefore a metric on
that we will simply denote $\| \cdot \|$. Eventually, this gives us a metric on $Z := \mathcal{Z}_{E \otimes L^k}$, that we can evaluate at a point $z \in Z$:

$$
\|z\|_Z^2 = \sup_{\text{(S}_i\text{) } H\text{-orthonormal basis of } H^0(M, E \otimes L^k)} \int_M \sum_{1 \leq i_1 < \cdots < i_r \leq N} \|S_{i_1}(p) \wedge \cdots \wedge S_{i_r}(p)\|^2 \ dV
$$

where the sup is independent of the choice of the basis. The classical Kempf-Ness's result [K-N, Theorem 0.2], gives us an analytical criterion to check the GIT-stability of a point $z \in Z$, with respect to the linearization $O_z(1)$ and the $SL(N)$ action: $z$ is GIT-stable if and only if the application

$$
\mathcal{L}(g) : g \mapsto \log \int_M \sum_{1 \leq i_1 < \cdots < i_r \leq N} \|g \cdot S_{i_1}(p) \wedge \cdots \wedge g \cdot S_{i_r}\|^2 \ dV
$$

is bounded from below by a strictly positive constant and is proper (for all $t > 0$ there exists a compact set $K \subset SL(N)$ such that $\mathcal{L}(g) > t$ if $g \notin K$).

**Interlude about the notion of integral of a moment map**

Consider $\Xi$ a smooth symplectic manifold, $\omega$ its symplectic form and $\Gamma$ a compact Lie group acting symplectically on $\Xi$. If $\mu$ is a moment map associated to this action, then one can define the functional

$$
\Psi : \Xi \times \Gamma^C \to \mathbb{R}
$$

that we will call the "integral of the moment map $\mu$" and that satisfies the following properties:

- for all $p \in \Xi$, the critical points of the restriction $\Psi_p$ of $\Psi$ to $\{p\} \times \Gamma^C$ coincide with the points of the orbit $\text{Orb}_{\Gamma^C}(p)$ on which the moment map vanishes;
- the restriction $\Psi_p$ to the lines $\{e^{\lambda u} : u \in \mathbb{R}\}$ where $\lambda \in \text{Lie}(\Gamma^C)$ is convex.

**Theorem 2.3.** (Mundet i Riera) *There exists a unique application $\Psi : \Xi \times \Gamma^C \to \mathbb{R}$ that satisfies:

1. $\Psi(p, e^\lambda) = 0$ for all $p \in \Xi$;
2. $\frac{1}{\mu} \Psi(p, e^{i\lambda u})|_{u=0} = \langle \mu(p), \lambda \rangle$ for all $\lambda \in \text{Lie}(\Gamma)$;

Moreover, this functional enjoys the following properties: $\Psi$ is $\Gamma$-invariant and satisfies the cocycle relation $\Psi(p, \gamma) = \Psi(\gamma p, \gamma') = \Psi(p, \gamma' \gamma)$ and also the relation $\Psi(\gamma p, \gamma') = \Psi(p, \gamma^{-1} \gamma' \gamma)$ for all $p \in \Xi$, $\gamma, \gamma' \in \Gamma^C$. Eventually, $\frac{d^2}{du^2} \Psi(p, e^{i\lambda u}) \geq 0$ for all $\lambda \in \text{Lie}(\Gamma)$ with equality if and only if the vector field $\overline{X}_\lambda (e^{i\lambda u} p) = 0$. 
Remember that we have a diffeomorphism
\[
\Gamma \times \text{Lie}(\Gamma) \to \Gamma^C \\
(\gamma, u) \mapsto \gamma e^{iu}.
\] (1)

Let \( \rho : \Gamma^C \to GL(W) \) be a faithful representation on a finite dimensional complex vector space equipped with a hermitian metric such that \( \rho(\Gamma) \subset U(W) \). We still denote \( \rho \) the representation induced on \( \text{Lie}(\Gamma) \) and on \( \Gamma \). This leads to define the following metric on \( \text{Lie}(\Gamma) \) by
\[
\langle a, b \rangle_\Gamma = Tr(\rho(a) \rho(b)^*) .
\]

By the diffeomorphism (1), we can associate to each element \( \gamma e^{iu} \in \Gamma^C \) its logarithm \( \log_{\Gamma^C}(\gamma e^{iu}) = u \).

**Definition 2.2.** We will say that \( \Psi \) is linearly log-proper respectively to the metric \( \langle . , . \rangle_\Gamma \) on \( \Gamma^C \) if there exists two constants \( c_1 > 0 \) and \( c_2 > 0 \) such that for all \( g \in \Gamma^C \) and for all \( p \in \Xi \),
\[
|| \log_{\Gamma^C}(g) ||_\Gamma \leq c_1 \Psi_p(g) + c_2 .
\]

**Balanced condition and Kempf-Ness functional**

We want to measure the action of \( SL(N) \) on a point \( z \in Z \). For that reason, with the same notations as before, we introduce the following functional for \( g \in SL(N) \) which depends only on the choice of \( H \),
\[
\overline{KN}_{k,E} : g \mapsto \frac{1}{2} \int_M \log \frac{\sum_{1 \leq i_1 < \cdots < i_r \leq N} \| g \cdot S_{i_1}(p) \wedge \cdots \wedge g \cdot S_{i_r}(p) \|^2}{\sum_{1 \leq i_1 < \cdots < i_r \leq N} \| S_{i_1}(p) \wedge \cdots \wedge S_{i_r}(p) \|^2} dV(p) .
\]

where \( (S_i)_{i=1, \ldots, N} \) is an \( H \)-orthonormal basis of holomorphic sections of \( E \otimes L^k \). Let \( [Q(p)] \) be the point of \( Gr(r,N) \) given by the embedding \( i_k \) at \( p \in M \) and the metric \( H \) on \( H^0(M,E \otimes L^k) \). Our functional is related to a functional that plays a key role in Donaldson's theory [Do1,Do2,P-S]:

**Definition 2.3.** Let \( E \) be a hermitian holomorphic vector bundle on \( M \) and \( h_1, h_2 \) two hermitian metrics on \( E \). We define the Kempf-Ness functional as the integral of the first Chern-Weil form:
\[
KN_E(h_1, h_2) = \int_M \log \det (h_2^{-1} h_1) \frac{\omega^n}{n!}
\]
Lemma 2.0.1. For all \( g \in SL(N) \),
\[
\overline{K_{N_k,E}}(g) = \frac{1}{2} K_{N_E \otimes L^k} \left( FS(H \circ g), FS(H) \right)
= \frac{1}{2} \int_M \log \frac{\det (i\overline{Q'} g g Q)}{\det (i\overline{Q} Q)} dV,
\]
and \( \overline{K_{N_k,E}}(g) \) is the integral of the moment map \( \tilde{\mu}_{r,N} \).

Proof. One can represent \([Q(p)]\) for all \( p \in M \) as a Stiefel point
\[
Q' = \begin{pmatrix} Z \\ \text{Id}_{r \times r} \end{pmatrix}
\]
with \( Z(p) = [z_1, \ldots, z_r] \in M_{(N-r) \times r}(\mathbb{C}) \). There exists an antiholomorphic application \( \Phi : Gr(r, N) \to Gr(N-r, N) \) such that \( \Phi ([Q']) = [Q'^\perp] \), which implies \( \Phi \left( \begin{pmatrix} Z \\ \text{Id}_{r \times r} \end{pmatrix} \right) = \begin{pmatrix} \text{Id}_{(N-r) \times (N-r)} \\ -i\overline{Z} \end{pmatrix} \).

Set \( [\beta_1, \ldots, \beta_{N-r}] := -i\overline{Z} \) and fix a basis \( (e_i)_{i=1,\ldots,N} \) of \( \mathbb{C}^N \). As it is mentioned in [Mo, Chapter 7], the potential of the Fubini-Study metric on the Grassmannian is given explicitly by
\[
\log |(e_{N-r+1} + z_1) \wedge \cdots \wedge (e_N + z_r)|^2
= \log |(e_1 + \beta_1) \wedge \cdots \wedge (e_{N-r} + \beta_{N-r})|^2
= \log |(e_1 + \beta_1) \wedge \cdots \wedge (e_{N-r} + \beta_{N-r}) \wedge (e_{N-r+1} + z_1) \wedge \cdots \wedge (e_N + z_r)|
= \log \det \begin{pmatrix} \text{Id}_{(N-r) \times (N-r)} & Z \\ -i\overline{Z} & \text{Id}_{r \times r} \end{pmatrix}
= \log \det (\text{Id}_{(N-r) \times (N-r)} + Z^t \overline{Z})
= \log \det (\text{Id}_{r \times r} + i\overline{Z} Z)
= \log \det (i\overline{Q'} Q')
= \log \det (i\overline{Q} Q).
\]

A simple computation shows that at the point \([g \cdot Q] \), this potential is also given by
\[
\log \det (i\overline{Q} g g Q).
\]

For the second assertion, it is sufficient to consider the induced action by the 1-parameter subgroup of the form \( \{ u \to e^{Su} \in SL(N) \} \) (here \( S \neq 0 \) is a hermitian trace free matrix). Since we know that for all \( A_1, A_2 \in GL(N, \mathbb{C}) \),
one has \(D \det_{A_1}(A_2) = \det(A_1) \text{tr}(A_1^{-1}A_2)\), we obtain:

\[
\frac{d}{du} \left( \overline{K_{N,k,E}}(e^{S_u}) \right)
\]

\[
= \frac{1}{2} \int_M \frac{d}{du} \log \left( \det \left( tQ e^{S_u} e^{S_u'} Q \right) \right) dV,
\]

\[
= \frac{1}{2} \int_M \text{tr} \left( \left( tQ e^{S_u} e^{S_u'} Q \right)^{-1} \left( tQ e^{S_u} \left( S + tS \right) e^{S_u'} Q \right) \right) dV.
\]

Since we have chosen an orthonormal basis, \(tQ Q = Id\) and consequently, for \(u = 0\) we get,

\[
\frac{d}{du} \left( \overline{K_{N,k,E}}(e^{S_u}) \right) \bigg|_{u=0} = \int_M \text{tr}(tQ SQ) - \frac{r}{N} \int_M \text{tr}(S)
\]

\[
= \langle \bar{\mu}_{r,N}([Q]), S \rangle
\]

and we conclude by Theorem 2.3. \(\square\)

**Lemma 2.0.2.** The functional \(\overline{K_{N,k,E}}(\cdot)\) is linearly log-proper with respect to the action of \(SL(N)\).

**Proof.** Define the Kähler cone respectively to \(\omega\):

\[
Ka(M,\omega) = \left\{ \varphi \in C^\infty(M,\mathbb{R}) : \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi > 0 \right\}
\]

We set

\[
\varphi = \log \sum_{1 \leq i_1 < \cdots < i_r \leq N} ||S_{i_1}(p) \wedge \cdots \wedge S_{i_r}||^2.
\]

Since \(\varphi \in Ka(M,\omega)\), a theorem of Kähler geometry of G. Tian [Ti] asserts that there exists two constants \(\alpha = \alpha(M,\omega) > 0\) and \(C = C(M,\omega) > 1\) such that

\[
\int_M e^{-\alpha(\varphi - \sup_M \varphi)} \frac{\omega^n}{n!} < C,
\]

which implies

\[
\log \left( \int_M e^{-\alpha(\varphi - \sup_M \varphi)} \frac{\omega^n}{n!} \right) < C',
\]
and by concavity of log, that

\[ \int_M \varphi \frac{\omega^n}{n!} \geq \int_M \left( \sup_{\varphi} \frac{\omega^n}{n!} - \frac{1}{\beta} \right) \geq \log \left( \sup_{p \in M} \sum_{1 \leq i_1 < \cdots < i_r \leq N} \| S_{i_1}(p) \wedge \cdots \wedge S_{i_r}(p) \|^2 \right) - \frac{1}{\beta}. \]

Therefore, for all \( g \in SL(N) \),

\[ \overline{K}\mathcal{N}_{k,E}(g \cdot h) \geq \log ||| g \cdot z |||_Z^2 - \frac{1}{\beta}, \]

where \( \beta(M, \omega) > 0 \) depends only on \( M \) and \( \omega \). In fact, we also get,

\[ \log ||| g \cdot z |||_Z^2 \geq \overline{K}\mathcal{N}_{k,E}(g \cdot h) \geq \log ||| g \cdot z |||_Z^2 - \frac{1}{\beta(M)}. \]

By Lemma 2.0.2, the functional is \( \overline{K}\mathcal{N}_{k,E} \) is proper and bounded from below if and only if the functional \( \mathcal{L} \) is bounded from below and proper, which means that the point \( T(\mathcal{E} \otimes L^k) \in \mathcal{Z} \) is GIT-stable. Moreover we obtain that the embedding \( i_k \) is balanced if and only if the Fubini-Study metric induced by \( i_k \) is a scalar multiple of the original metric on \( E \). This means that there exists a hermitian metric \( h_k \) on \( E \) which is a fixed point for \( \mathcal{F}\mathcal{S} \circ \text{Hilb} \) (resp. \( \text{Hilb}(h_k) \) is a fixed point for \( \text{Hilb} \circ \mathcal{F}\mathcal{S} \)). Now, we remark that \( Q\overline{Q} = \lambda \text{Id} \) if and only if \( t\overline{Q}Q \) is the matrix of the orthogonal projection to \( \text{ker}(Q) \). But \( t\overline{Q}Q \) is a bundle morphism corresponding to the Bergman kernel of \( E \) for the metric \( h_k \). Therefore, we obtain the second part of Theorem 2.1.

An interesting consequence of Theorem 2.1 is an analogue of the work [Do4] of S. Donaldson in the case of vector bundles. It uses essentially two facts. First of all, one knows an asymptotic expansion in \( k \) of the Bergman kernel over a compact manifold:

\[ B_{h \otimes h_{L,k}} = k^n \text{Id} + k^{n-1} \left( \frac{1}{2} \text{Scal}(g_{i\bar{j}}) \text{Id} + \sqrt{1} \Lambda \omega F_h \right) + \cdots \]

if one has assumed that \( \omega = \frac{\sqrt{1}}{2} \sum g_{i\bar{j}}dz_i \wedge d\overline{z_j} \) represents \( c_1(L) \). Secondly, once one has fixed a holomorphic structure \( \mathcal{E} \) on \( E \), the Bergman kernel can be seen as a moment map for the action of the Gauge group \( \mathcal{G} \) of \( E \) on the infinite dimensional Kähler space

\[ \mathcal{H} = \left\{ (s_1, \ldots, s_N) \in C^\infty(M, \mathcal{E} \otimes L^k)^N : \frac{\partial}{\partial s_i} = 0 \right\}, \]

and the points in \( \mathcal{H}//(\mathcal{G} \times SU(N)) \) correspond to balanced metrics.
Theorem 2.4.

- Suppose $E$ is Gieseker stable. If the sequence $(h_k)$ of balanced metrics on $E$ converges, then its limit is weakly Hermite-Einstein.
- If $E$ is Mumford stable then the sequence of balanced metrics $h_k$ converges and the convergence is $C^m$ for all $m \geq 0$.

Theorems 2.1 and 2.4 were found independently by X. Wang [Wal, Wa2] and J. Keller. A similar problem had already been studied in [Dr] in the case of curves. See also [Ke] for a generalization of this theorem to the case of Vortex equations and stability of pairs.

Remark. A hermitian metric $h$ on vector bundle $E$ is weakly Hermite-Einstein if the curvature $F_h$ of the Chern connection relative to $h$ satisfies the equation

$$\sqrt{-1} \Lambda_\omega F_h = \lambda_h \text{Id}_E,$$

where $\lambda_h$ is a continuous function with real values. Since $M$ is compact, there exists a unique function $f \in C^\infty (M, \mathbb{R})$ (up to a constant), such that the new metric $e^f \cdot h$ is Hermite-Einstein (i.e. $\lambda_{e^f \cdot h} = \mu(E)$ is constant). A good reference on this subject is [L-T].

3. Harder-Narasimhan filtration

In this section, we give an application of Theorem 2.4. We will need to introduce the following classical notion:

Definition 3.1. If $\mathcal{F}$ is a torsion free sheaf, a Härder-Narasimhan filtration for $\mathcal{F}$ is an increasing filtration

$$0 = HN_0 (\mathcal{F}) \subset \cdots \subset HN_l (\mathcal{F}) = \mathcal{F},$$

such that the factors $gr_i^{HN} (\mathcal{F}) = HN_i (\mathcal{F}) / HN_{i-1} (\mathcal{F})$ for $i = 1, \ldots, l$ are torsion free Mumford semi-stable with normalized degree $\mu_i$ satisfying

$$\mu_{\max} (\mathcal{F}) := \mu_1 > \cdots > \mu_l =: \mu_{\min} (\mathcal{F}),$$

Such a filtration exists and is unique. The graduated object

$$gr^{HN} (\mathcal{F}) = \bigoplus gr_i^{HN} (\mathcal{F})$$

is uniquely determined by the isomorphism class of $\mathcal{F}$. Moreover, there exists a unique Mumford semi-stable saturated subsheaf $\mathcal{F}_1 \subset \mathcal{F}$, called maximal destabilizing subsheaf of $\mathcal{F}$, such that:

- If $\mathcal{F}_2 \subset \mathcal{F}$ is a proper subsheaf of $\mathcal{F}$, then $\mu (\mathcal{F}_2) \leq \mu (\mathcal{F}_1)$;
• If $\mu (\mathcal{F}_2) = \mu (\mathcal{F}_1)$, then $r (\mathcal{F}_2) \leq r (\mathcal{F}_1)$.

Notice that $HN_1 (\mathcal{F})$ is the maximal destabilizing subsheaf of $\mathcal{F}$.

We know that if $E$ is a Mumford polystable vector bundle, $E$ splits holomorphically as $E = \bigoplus_{i=1}^{l} E_i$, the induced metric on $E_i$ is Hermite-Einstein and the induced filtration is given by $HN_i (E) = \bigoplus_{j<i} E_j$. Now, for an unspecified holomorphic structure, the Harder-Narasimhan filtration may not split holomorphically nor be given by vector subbundles.

**Lemma 3.0.3.** Let $\mathcal{F}$ be a torsion free sheaf on $M$ and let $\mathcal{F}' = \mathcal{F}/HN_1 (\mathcal{F})$. Then

$$HN_{i+1} (\mathcal{F}) = \ker (\mathcal{F} \rightarrow \mathcal{F}' / HN_i (\mathcal{F}'))$$

and $HN_{i+1} (\mathcal{F}) / HN_1 (\mathcal{F}) = HN_i (\mathcal{F}')$.

**Proof.** The proof is outlined in [H-L]. It uses the fact that for a sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, one has $\ker (HN_1 (B) \rightarrow C) = HN_1 (A)$.

**Proposition 3.1.** Consider the exact sequence of torsion free sheaves

$$0 \rightarrow \mathcal{E}_1 \rightarrow E \rightarrow \mathcal{E}_2 \rightarrow 0,$$

with $l_1 := \mu_{\min} (\mathcal{E}_1) > \mu_{\max} (\mathcal{E}_2)$. Then the Harder-Narasimhan filtration of $E$ is given by

$$0 = HN_0 (E) \subset HN_1 (\mathcal{E}_1) \subset \cdots \subset HN_{l_1} (\mathcal{E}_1)$$

$$= \mathcal{E}_1 \subset HN_{l_1+1} (E) \subset \cdots \subset HN_l (E) = E,$$

with:

$$HN_i (E) = \ker (E \rightarrow \mathcal{E}_2 / HN_{i-1} (\mathcal{E}_2)) \quad \text{for} \quad i = l_1, \ldots, l,$$

$$HN_i (E) = HN_i (\mathcal{E}_1) \quad \text{for} \quad i = 0, \ldots, l_1.$$

Moreover, $gr^{HN}_E = gr^{HN}_{\mathcal{E}_1} \oplus gr^{HN}_{\mathcal{E}_2}$.

**Proof.** Let $\mathcal{F} \subset E$ be the maximal destabilizing sheaf. We get $\mu (\mathcal{F}) \geq \mu_{\max} (\mathcal{E}_1) \geq \mu_{\min} (\mathcal{E}_1) > \mu_{\max} (\mathcal{E}_2)$ by hypothesis. The application $\phi : \mathcal{F} \rightarrow \mathcal{E}_2$ is non trivial, because otherwise we would have $\mu (\text{Im} (\phi)) \geq \mu (\mathcal{F}) > \mu_{\max} (\mathcal{E}_2)$ which would contradict the semi-stability of $\mathcal{F}$. Therefore we have $\mathcal{F} \subset \mathcal{E}_1$ and $\mathcal{F} = \mathcal{E}_1$ or $\mathcal{F}$ is the maximal destabilizing subsheaf of $\mathcal{E}_1$. If $\mathcal{E}_1$ is Mumford semi-stable, then clearly $\mathcal{E}_1$ is the maximal destabilizing subsheaf of $E$ as shows the previous lemma.
For general $E_1$, we use the same kind of arguments by induction over the length of the Harder-Narasimhan filtration of $E_1$. The sequence

$$0 \to E_1/F \to E/F \to E_2 \to 0$$

still satisfies the inequality $\mu_{\min}(E_1/F) = \mu_{\min}(E_1) > \mu_{\max}(E_2)$. By induction hypothesis, we get

$$0 \subset HN_0(E_1/F) \subset \cdots \subset HN_{l_1-1}(E_1/F) = E_1/F$$

$$E_1/F \subset HN_{l_1}(E/F) \subset \cdots \subset HN_{l-1}(E/F) = E/F$$

with $HN_i(E/F) = \ker(E/F \to E_2/HN_{i-l_1+1}(E_2))$. From another side, by the lemma,

$$HN_i(E) = \ker(E \to (E/F)/HN_{i-1}(E/F))$$

and therefore,

$$HN_i(E/F) = \ker(E \to E_2/HN_{i-l_1}(E_2)).$$

By induction hypothesis, we know that for $i \leq l_1$,

$$HN_i(E) /F = HN_{i-1}(E/F) = HN_{i-1}(E_1/F)$$

and we also obtain by the lemma,

$$HN_{i+1}(E) = \ker(E \to (E/F)/HN_{i-1}(E_1/F))$$

$$= \ker(E \to (E/F)/HN_{i-1}(E_1/F))$$

$$= \ker(E \to (E/F)/(HN_i(E_1)/F))$$

$$= HN_{i+1}(E_1)$$

Finally, the last assertion is obvious.

**Definition 3.2.** To each factor $HN_i(E)$ of a vector bundle $E$ equipped with a hermitian metric $h_E$, corresponds a projection (orthogonal for $h_E$) $\pi_i^{E,h_E}$ on $HN_i(E)$. Define the hermitian endomorphism $\Pi_{HN(E)}^{\omega,h_E}$:

$$\Pi_{HN(E)}^{\omega,h_E} = \sum \mu \left( g_{\pi_i}^{HN}(E) \right) \left( \pi_i^{E,h_E} - \pi_{i-1}^{E,h_E} \right).$$

**Remark.** As we have seen, if $E$ is a holomorphic vector bundle equipped with a Hermite-Einstein metric $h_E$, then by Uhlenbeck-Yau’s Theorem $E$ is Mumford polystable and we have the decomposition

$$(E, h_E) = (E_1, h_1) \oplus \cdots \oplus (E_k, h_k)$$
by Mumford stable vector bundles with the same normalized degree $\mu(E)$. In particular, $\sqrt{-1}\Lambda F_{h_i} = \mu(E) Id_{E_i}$ and

$$\Pi^{\omega, h_E}_{HN(E)} = \mu(E) \begin{pmatrix} Id_{E_1} & & \\ & \ddots & \\ & & Id_{E_l} \end{pmatrix}.$$ 

**Definition 3.3.** Let $\mathcal{F}$ be a torsion free sheaf which is Mumford semi-stable. A Jordan-Hölder filtration of $\mathcal{F}$ is a filtration

$$0 = JH_0(\mathcal{F}) \subset \cdots \subset JH_l(\mathcal{F}) = \mathcal{F}$$

such that the factors $gr^{JH}_i(\mathcal{F}) = JH_i(\mathcal{F})/JH_{i-1}(\mathcal{F})$ are all Mumford stable with same normalized Hilbert polynomial. The graduated object

$$gr^{JH}(\mathcal{F}) = \bigoplus gr^{JH}_i(\mathcal{F})$$

does not depend on the choice of the filtration.

**Theorem 1.** Let $E$ be a holomorphic vector bundle on $(M, \omega)$. If the Harder-Narasimhan filtration $HN(E)$ of $E$ is given by subbundles, then for all $s > 0$, for all $r \geq 0$, there exists a smooth hermitian metric $h$ on $E$ compatible with the holomorphic structure such that

$$\left\| \sqrt{-1}\Lambda_\omega F_h - \Pi^{\omega, h}_{HN(E)} \right\|_{C^r} < \varepsilon.$$ 

**Proof.** We give a proof by induction on the length of the Harder-Narasimhan filtration of $E$.

If the rank of $E$ is 1, this comes from the fact that we can use the Jordan-Hölder filtration since $E$ is in particular Mumford stable, and we can apply Theorem 2.4 to get a sequence of metrics weakly Hermite-Einstein $h_k$ which are, up to a conformal change, Hermite-Einstein metrics $h_k'$. Therefore, for $k$ large enough,

$$\left\| \sqrt{-1}\Lambda_\omega F_{E,h_k'} - \mu(E) Id_E \right\|_{C^r} = O\left(\frac{1}{k}\right).$$

Now, if the length of the Harder-Narasimhan filtration of $E$ is bigger than 2, then

$$0 \to E_1 \to E \to E_2 \to 0,$$ 

where $E_1$ is the maximal destabilizing sheaf of $E$ which is, as $E_2$, a vector bundle. The filtrations $HN(E_1)$ and $HN(E_2)$ are given by vector bundles
by Proposition 3.1, and for the metrics $h_1$ et $h_2$ (and respectively their
curvatures $F_1$, $F_2$ of $E_1$ and $E_2$), we get
\[ \left\| -i\Lambda F_1 - \Pi_{HN(E_1)}^{\omega,h_1} \right\|_{C^r} < \varepsilon/3, \quad \left\| -i\Lambda F_2 - \Pi_{HN(E_2)}^{\omega,h_2} \right\|_{C^r} < \varepsilon/3. \]
From (2) we have $\Pi_{HN(E)}^{\omega,h_1 \oplus h_2} = \Pi_{HN(E_1)}^{\omega,h_1} \oplus \Pi_{HN(E_2)}^{\omega,h_2}$ and the holomorphic
structure on $E$ has the following form:
\[ \overline{\partial}_E = \left( \begin{array}{cc} \partial_{E_1} & \alpha \\ 0 & \partial_{E_2} \end{array} \right), \]
with $\alpha$ a smooth section of $\Omega^{0,1}(\text{Hom}(E_1, E_2))$ (see [Ko, \$1.6]). Then,
\[ \left\| -i\Lambda F_E - \Pi_{HN(E)}^{\omega,h_1 \oplus h_2} \right\|_{C^r} \leq \left\| -i\Lambda F_1 - \Pi_{HN(E_1)}^{\omega,h_1} \right\|_{C^r} + \left\| \Lambda F_2 - \Pi_{HN(E_2)}^{\omega,h_2} \right\|_{C^r} + 2 \sup |\alpha|^2 + 2 \sup |\overline{\partial}^* \alpha|^2, \]
Up to a Gauge change of the form $g = \left( \begin{array}{cc} \delta & 0 \\ 0 & \delta^{-1} \end{array} \right)$, we can assume that
\[ 2 \left( \sup |\alpha|^2 + \sup |\overline{\partial}^* \alpha|^2 \right) < \varepsilon/3. \]
This allows us to conclude, considering the new structure $g(\overline{\partial}_E)$. \hfill \Box

**Remark.** In the case of a curve, the terms of the filtration of $E$ are
locally free, and therefore subbundles of $E$. If we attach to each vector
bundle of this filtration a Jordan-Hölder filtration, we immediately get an
improvement of [Br, Theorem 5] which was our original motivation.

### 4. The case of surfaces

For complex surfaces, a Gieseker stable vector bundle may not be Mumford
stable. By Theorem 2.4, we know that for Gieseker stable vector bundles
which are not Mumford stable, the sequence of balanced metrics will not
converge.

From another side, we know that in the case of surfaces, the singularities
of torsion free sheaves are just points, and the reflexive sheaves are locally
free [Ko, \$5]:

**Proposition 4.1.** Let $M$ be a complex manifold. The singular set $S(F)$
of the analytic coherent sheaf $F$ is defined as the closed subvariety upon
which $F$ is not locally free. If $F \to M$ is a torsion free sheaf, $S(F)$ has
codimension at least 2. If $F$ is reflexive, then $S(F)$ has codimension at
least 3.
This remark is the key point of the "gluing construction" technique introduced by N. Buchdahl in [Bu2]. To a Mumford semi-stable sheaf $F$ on $M$, one can associate a semi-stable vector bundle $\Sigma (F)$ that admits a Hermite-Einstein metric in the following way: $\Sigma (F) = \Sigma (F^{**})$ and if $F' \subset F$ satisfies $\mu (F') = \mu (F)$ then $\Sigma (F) = \Sigma (F') \oplus \Sigma (F/F')$. One checks by induction on the rank that we get a unique vector bundle $\Sigma (F)$. This vector bundle and $F$ has same rank and determinant and $\Sigma (F)$ is a direct sum of Mumford stable vector bundles with the same normalized degree, i.e Mumford polystable. Moreover, we have non trivial homomorphisms $F \rightarrow \Sigma (F)$ and $\Sigma (F) \rightarrow F^{**}$. For a semi-stable vector bundle $E$, we shall denote by $\mathbb{B} (E)$ the set of points $x \in M$ for which there exists a Mumford semi-stable vector bundle $E'$ with $\mu (E') = \mu (E)$ and an inclusion $E' \rightarrow E$ such that $E'_x \nrightarrow E_x$ is not of maximal rank. By an induction on the rank, one proves that $\mathbb{B} (E)$ is finite. In the case of a Kähler compact surface $M$, the following result holds [Bu2, Proposition 4.3]:

**Lemma 4.0.4.** Let $E$ be a Mumford semi-stable vector bundle such that $\Sigma (E) = \bigoplus \gamma_i \otimes E_i$ where $\gamma_i$ is vector space of dimension $d_i$ and $E_i$ is a Mumford stable vector bundle with $\mu (E_i) = \mu (E)$ and $E_i \ncong E_j$ for $i \neq j$. Let choose $e > r(E) \max_i (d_i / r(E_i))$. Then for all choice of $e$ points $(x_i)_{i=1..e} \in M \setminus \mathbb{B} (E)$, there exists a vector bundle $\tilde{E}$ on the resolution $\tilde{M} \xrightarrow{\pi} M$ of these points such that:

- $\tilde{E}$ restricts to $\mathcal{O} (1) \oplus \sum_{i=1}^{r-1} \mathcal{O}$ on each component of the exceptional divisor,
- $(\pi_* \tilde{E})^{**} = E$,
- $\tilde{E}$ is Mumford stable with respect to the polarization $\omega_e = \pi^* \omega + e \sum_{i=1}^p s_i$ for $e$ small enough and where $s_i$ is the non trivial holomorphic section that represents exactly the divisor $-\pi^{-1} (x_i)$.

Now the fact that Mumford stability implies Gieseker stability gives us directly the following result.

**Theorem 2.** Let $M$ be a projective surface and $E$ be a Gieseker stable vector bundle on $M$ which is not Mumford stable. There exists a resolution $\tilde{M} \xrightarrow{\pi} M$ consisting of a blowup of a finite number of points and a vector bundle $\tilde{E}$ on $\tilde{M}$ such that the balanced metrics $(h_k)$ associated to $\tilde{E}$ converge towards a weakly Hermite-Einstein metric and $(\pi_* \tilde{E})^{**} = E$.

We now give a consequence of [Bu2, Proposition 2.4] (see also for details [Bu1]):
Definition 4.1. Let $\mathcal{F}$ be a reflexive sheaf and $\mathcal{F}_1 \subset \mathcal{F}$ with $\mathcal{F}/\mathcal{F}_1 = \mathcal{F}_2$. We note $\text{Tor}_2 (\mathcal{F}_2)$ the torsion of $\mathcal{F}_2$. If $\mathcal{F}_2$ is torsion free, $\mathcal{F}_1$ is said to be saturated; otherwise its saturation is $\text{Sat} (\mathcal{F}_1, \mathcal{F}) = \ker (\mathcal{F} \to \mathcal{F}_2/\text{Tor} (\mathcal{F}_2))$.

Lemma 4.0.5. Let $E$ be a holomorphic vector bundle on a smooth projective surface. Consider the following filtration of $E$

$$0 = E_0 \subset \cdots \subset E_l = E$$

by saturated sheaves. Then, there exists a resolution $\widetilde{M} \to M$ consisting of a blowup of a finite number of points and a filtration

$$0 = \widetilde{E}_0 \subset \cdots \subset \widetilde{E}_l = \widetilde{E}$$

with $(\pi_* \widetilde{E})^{**} = E$ and $\widetilde{E}_l = \text{Sat}(\pi^* E_l, \widetilde{E})$ is a subbundle of $\widetilde{E}$.

For a blowup $\widetilde{M} \to M$ with $L_\pi := \pi^{-1} (x_0)$ as exceptional divisor associated, the metric $\pi^* \omega$ is positive and degenerates only along the tangent directions to $L_\pi$. Let $F_{L_\pi}$ be the curvature form of any smooth hermitian metric on the associated line bundle $O (-L_\pi)$. For $\delta > 0$ sufficiently small, $\omega_\delta = \pi^* \omega + \delta F_{L_\pi}$ is a smooth closed definite positive $(1, 1)$-form. It is with respect to this polarization $\omega_\delta$ that we will speak of stability on the manifold $\widetilde{M}$. We obtain under this setting a generalization of Theorem 2.

Theorem 3. Let $E$ be a holomorphic vector bundle on a smooth projective surface. Then, there exists a resolution $\widetilde{M} \to M$ consisting of a blowup of a finite number of points and a vector bundle $\widetilde{E}$ on $\widetilde{M}$ such that for $\delta > 0$ sufficiently small, and for all $\varepsilon > 0$, $r \geq 0$, there exists a smooth hermitian metric $\widetilde{h}$ on $\widetilde{E}$ with

$$\left\| \sqrt{-1} \Lambda F_{\widetilde{h}} - \Pi^{\omega_\delta, \widetilde{h}}_{HN (\widetilde{E})} \right\|_{C^r} < \varepsilon$$

and $(\pi_* \widetilde{E})^{**} = E$.

Proof. If the Harder-Narasimhan filtration is given by vector bundles, we apply Theorem 1. Otherwise, we prove by induction on the rank. The result is clear for rank 1. For $r (E) > 1$, we apply the previous lemma to get a filtration $\widetilde{E}_i$ of $\widetilde{E}$ and for all $\varepsilon > 0$, Theorem 1 and the hypothesis of induction allow us to find a hermitian metric $h_i (\varepsilon)$ for $\widetilde{E}_i/\widetilde{E}_{i-1}$ such that

$$\left\| \sqrt{-1} \Lambda F_{h_i (\varepsilon)} - \Pi^{\omega_\delta, h_i (\varepsilon)}_{HN (\widetilde{E}_i/\widetilde{E}_{i-1})} \right\|_{C^r} < \varepsilon.$$ 

By considering the smooth metric $\widetilde{h} = \bigoplus_{i=1}^l h_i (\varepsilon)$, and by using the same kind of argument that for Theorem 1, we can conclude. \qed
References


The main purpose of this paper is to survey characterizations of totally umbilic hypersurfaces and isoparametric hypersurfaces related to the results in [1] and [3].

1. Introduction

In differential geometry it is interesting to know the shape of a Riemannian submanifold by the extrinsic shape of geodesics and circles of positive curvature of the submanifold in an ambient Riemannian manifold. For example, it is known that a hypersurface $M^n$ in a Euclidean space $\mathbb{R}^{n+1}$ is locally a standard sphere if and only if every geodesic of $M$ is a circle of positive curvature in $\mathbb{R}^{n+1}$. Such a characterization is quite natural. But, it requires a very large quantity of information because of its condition "every geodesic of $M$". Therefore we shall give a practical criterion for a hypersurface to be totally umbilic, which is better than this characterization (cf. [4]).

In this paper we first study hypersurfaces $M^n$'s isometrically immersed into a space form $\widetilde{M}^{n+1}(c)$ of constant curvature $c$ (that is, $\widetilde{M}^{n+1}(c)$ is con-
gruent to $\mathbb{R}^{n+1}$, a standard sphere $S^{n+1}(c)$ or a hyperbolic space $H^{n+1}(c)$ according as the curvature $c$ is zero, positive or negative) from this point of view.

We here recall the definition of circles in Riemannian geometry. A smooth regular curve $\gamma = \gamma(s)$ parametrized by its arclength $s$ is called a circle of curvature $\kappa(\geq 0)$ if there exists a field of unit vectors $Y = Y_s$ along the curve which satisfies the differential equations: $\nabla_\gamma \dot{\gamma} = \kappa Y$ and $\nabla_\gamma Y = -\kappa \gamma$, where $\kappa(\geq 0)$ is a constant and $\nabla_\gamma$ denotes the covariant differentiation along $\gamma$ with respect to the Riemannian connection $\nabla$ of $M$. Needless to say, a circle of null curvature is nothing but a geodesic.

In Section 2 we give a characterization of totally umbilic but not totally geodesic hypersurfaces by observing the extrinsic shape of geodesics on hypersurfaces (Proposition 2.2). Totally umbilic hypersurfaces in a space form $\tilde{M}^{n+1}(c)$ are typical examples of isoparametric hypersurfaces $M^n$'s (that is, all principal curvatures of $M^n$ in $\tilde{M}^{n+1}(c)$ are constant). In this context, in Section 3 we characterize isoparametric hypersurfaces in a space form (Theorems 3.1, 3.2 and 3.3). In these sections we pay particular attention to the extrinsic shape of geodesics on hypersurfaces. In the last section, observing the extrinsic shape of circles of positive curvature on hypersurfaces, we again characterize totally umbilic hypersurfaces in a space form (Theorem 4.1). We note that Theorem 4.1 is closely related to Theorems 3.1, 3.2 and 3.3.

2. Characterization of totally umbilic hypersurfaces I

The following is well-known:

**Proposition 2.1.** Let $M^n$ be a hypersurface isometrically immersed into a space form $\tilde{M}^{n+1}(c)$. Then the following three conditions are equivalent:

1. $M^n$ is totally umbilic in $\tilde{M}^{n+1}(c)$.
2. Every geodesic on $M^n$ is a circle in $\tilde{M}^{n+1}(c)$.
3. Every circle on $M^n$ is a circle in $\tilde{M}^{n+1}(c)$.

We shall prove the following which is an improvement of Proposition 2.1 related to the condition (2):

**Proposition 2.2.** Let $M^n$ be a hypersurface isometrically immersed into a space form $\tilde{M}^{n+1}(c)$. Then the following two conditions are equivalent:

1. $M^n$ is totally umbilic but not totally geodesic in $\tilde{M}^{n+1}(c)$.
2. For each point $p$ in $M$ there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of $T_p(M)$ such that all geodesics on $M$ through $p$ in the direction $v_i + v_j$
(1 ≤ i ≤ j ≤ n) are circles of positive curvature in the ambient space \( \overline{M}^{n+1}(c) \).

**Proof.** We have only to show that a hypersurface \( M^n \) satisfying the condition (2) is umbilic at a fixed point \( p \in M \). As a matter of course our hypersurface \( M \) is not totally geodesic in \( \overline{M}^{n+1}(c) \) (with Riemannian metric \( \langle , \rangle \)). We denote by \( \nabla \) and \( \overline{\nabla} \) the Riemannian connections of \( M \) and \( \overline{M}^{n+1}(c) \), respectively. Let \( \gamma_i = \gamma_i(s) \) (1 ≤ i ≤ n) be geodesics of \( M \) with \( \gamma_i(0) = p \) and \( \dot{\gamma}_i(0) = v_i \). By the Gauss formula we have

\[
\overline{\nabla}_{\dot{\gamma}_i} \dot{\gamma}_i = \nabla_{\dot{\gamma}_i} \dot{\gamma}_i + (A\dot{\gamma}_i, \dot{\gamma}_i)N = (A\dot{\gamma}_i, \dot{\gamma}_i)N,
\]

where \( N \) is a unit normal vector field of \( M \) in \( \overline{M}^{n+1}(c) \) and \( A \) is the shape operator with respect to \( N \). Hence the Weingarten formula \( \tilde{\nabla}_X N = -AX \) implies

\[
\overline{\nabla}_{\dot{\gamma}_i}(\overline{\nabla}_{\dot{\gamma}_i} \dot{\gamma}_i) = \langle (\nabla_{\dot{\gamma}_i} A)\dot{\gamma}_i, \dot{\gamma}_i \rangle N - \langle A\dot{\gamma}_i, \dot{\gamma}_i \rangle A\dot{\gamma}_i. \tag{1}
\]

On the other hand, since the curve \( \gamma_i \) is a circle in the ambient space \( \overline{M}^{n+1}(c) \) by hypothesis, there exists a positive constant \( \kappa_i \) satisfying that

\[
\overline{\nabla}_{\dot{\gamma}_i}(\overline{\nabla}_{\dot{\gamma}_i} \dot{\gamma}_i) = -\kappa_i^2 \dot{\gamma}_i. \tag{2}
\]

Comparing the tangential components of (1) and (2), we find the following:

\[
\langle A\dot{\gamma}_i, \dot{\gamma}_i \rangle A\dot{\gamma}_i = \kappa_i^2 \dot{\gamma}_i
\]

so that at \( s = 0 \) we have

\[
\langle Av_i, v_i \rangle Av_i = \kappa_i^2 v_i. \tag{3}
\]

Hence we get

\[
\langle Av_i, v_j \rangle = 0 \quad \text{for } 1 \leq i < j \leq n. \tag{4}
\]

Let \( \gamma_{ij} = \gamma_{ij}(s)(1 \leq i < j \leq n) \) be geodesics of \( M \) with \( \gamma_{ij}(0) = p \) and \( \dot{\gamma}_{ij}(0) = (v_i + v_j)/\sqrt{2} \). Then we similarly see that

\[
\langle A(v_i + v_j), v_i + v_j \rangle A(v_i + v_j) = 2\kappa_{ij}^2 (v_i + v_j)
\]

for some positive constant \( \kappa_{ij} \). So we obtain

\[
\langle A(v_i + v_j), v_i - v_j \rangle = 0 \quad \text{for } 1 \leq i < j \leq n.
\]

Thus we have

\[
\langle Av_i, v_i \rangle = \langle Av_j, v_j \rangle \quad \text{for } 1 \leq i, j \leq n. \tag{5}
\]
It follows from (4) and (5) that $AX = \kappa X$ for each $X \in T_p(M)$ and for some $\kappa$. Therefore we get the conclusion. 

Remark 2.1. If we replace "circles of positive curvature" by "circles" in the condition (2) of Proposition 2.2, this proposition does not hold. The following non-totally umbilic hypersurface is worth mentioning:

Example 2.1. We consider a Clifford hypersurface $M^n = S^k(2c) \times S^{n-k}(2c)$ in $S^{n+1}(c)$ for $k \in \{1, \ldots, n-1\}$. Then this hypersurface has parallel second fundamental form, besides it has two constant principal curvatures $\lambda_1 = \sqrt{c}$ with multiplicity $k$ and $\lambda_2 = -\sqrt{c}$ with multiplicity $n-k$. Take an orthonormal basis $\{v_1, \ldots, v_n\}$ at an arbitrary point $p$ of $M$ in such a way that $v_1, \ldots, v_k$ (resp. $v_{k+1}, \ldots, v_n$) are principal curvature vectors with principal curvature $\lambda_1$ (resp. $\lambda_2$). Then straightforward computation yields the following:

(1) All geodesics on $M$ through $p$ in the direction $v_i + v_j$ $(1 \leq i \leq j \leq k$ or $k + 1 \leq i \leq j \leq n)$ are circles of positive curvature $\sqrt{c}$ in $S^{n+1}(c)$.

(2) All geodesics on $M$ through $p$ in the direction $v_i + v_j$ $(1 \leq i \leq k, k + 1 \leq j \leq n)$ are also geodesics in $S^{n+1}(c)$.

3. Characterizations of isoparametric hypersurfaces

The following proposition clarifies a fundamental property of all isoparametric hypersurfaces in a space form.

Proposition 3.1. Let $M^n$ be an isoparametric hypersurface in a space form $\tilde{M}^{n+1}(c)$. Then for each point $p$ of $M^n$, there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of $T_p(M)$ such that all geodesics on $M^n$ through $p$ in the direction $v_i$ $(1 \leq i \leq n)$ are circles in the ambient space $\tilde{M}^{n+1}(c)$.

Proof. Let $M$ be an isoparametric hypersurface of a space form $\tilde{M}(c)$ with constant distinct principal curvatures $\kappa_1, \ldots, \kappa_g$. Then the tangent bundle $TM$ is decomposed as: $TM = T_{\kappa_1} \oplus \ldots T_{\kappa_g}$, where $T_{\kappa_i} = \{X \in TM : AX = \kappa_i X\} \ (i = 1, \ldots, g)$. We here recall the fact that each distribution $T_{\kappa_i}$ is integrable and moreover, every leaf of $T_{\kappa_i}$ is totally geodesic in the hypersurface $M$ and totally umbilic in the ambient space $\tilde{M}(c)$ (see [2]), which implies that every geodesic of such leaves is a circle in $\tilde{M}(c)$ as well as a geodesic in $M$. Hence, for each point $p$ of $M$, taking an orthonormal basis $\{v_1, \ldots, v_n\}$ of $T_p(M)$ in such a way that each $v_i$ $(1 \leq i \leq n)$ is a principal curvature vector, we find that the vectors $v_1, \ldots, v_n$ satisfy the statement of our Proposition. 

It is natural to consider the converse of Proposition 3.1. We pose the following problem:

**Problem.** Let $M^n$ be a connected hypersurface isometrically immersed into a space form $\tilde{M}^{n+1}(c)$. If for each point $p$ of $M^n$ there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of $T_p(M)$ such that all geodesics on $M^n$ through $p$ in the direction $v_i$ ($1 \leq i \leq n$) are circles in $\tilde{M}^{n+1}(c)$, is $M$ isoparametric in the ambient space $\tilde{M}^{n+1}(c)$?

This problem is still open until now. Our aim here is to survey partial positive answers to the above problem (see [3]):

The following gives a characterization of isoparametric hypersurfaces with nonzero constant principal curvatures.

**Theorem 3.1.** Let $M^n$ be a connected hypersurface isometrically immersed into a space form $\tilde{M}^{n+1}(c)$. Then $M^n$ is an isoparametric hypersurface with nonzero constant principal curvatures if and only if for each point $p$ of $M^n$ there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of $T_p(M)$ such that all geodesics on $M^n$ through $p$ in the direction $v_i$ ($1 \leq i \leq n$) are circles of positive curvature in $\tilde{M}^{n+1}(c)$.

**Proof.** The “only if” part is clear from the proof of Proposition 3.1. We shall prove the “if” part.

Let $M^n$ be a connected hypersurface isometrically immersed into a space form $\tilde{M}^{n+1}(c)$ satisfying that for each point $p$ of $M^n$ there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of $T_p(M)$ such that all geodesics on $M^n$ through $p$ in the direction $v_i$ ($1 \leq i \leq n$) are circles of positive curvature in $\tilde{M}^{n+1}(c)$. We consider the open dense subset $\mathcal{U} = \{ p \in M : \text{the multiplicity of each principal curvature of } M \text{ in } \tilde{M}(c) \text{ is constant on some neighborhood } \mathcal{V}_p(\subseteq \mathcal{U}) \text{ of } p \}$ of $M$. Note that all principal curvatures are differentiable on $\mathcal{U}$ and in a neighborhood of any point $p$ in $\mathcal{U}$ the principal curvature vectors can be chosen to be smooth. In the following, we shall study on a fixed neighborhood $\mathcal{V}_p$.

By the same argument as in the proof of Proposition 2.2 we get equation (3) again, so that we have

$$Av_i = \kappa_i v_i \quad \text{or} \quad Av_i = -\kappa_i v_i \quad (1 \leq i \leq n),$$

which means that the tangent space $T_p(M)$ is decomposed as:

$$T_p(M) = \{ v \in T_p(M) : Av = -\kappa_i v \} \oplus \{ v \in T_p(M) : Av = \kappa_i v \}$$

$$\oplus \ldots \oplus \{ v \in T_p(M) : Av = -\kappa_i v \} \oplus \{ v \in T_p(M) : Av = \kappa_i v \},$$
where $0 < \kappa_{i_1} < \kappa_{i_2} < \ldots < \kappa_{i_g}$ and $g$ is the number of positive distinct principal curvatures at the point $p$. Hence our discussion yields that every $\kappa_{i_j}$ is differentiable on $\mathcal{V}_p$. Next, we shall show the constancy of $\kappa_{i_j}$. It suffices to check the case that $Av_{i_j} = \kappa_{i_j}v_{i_j}$. By hypothesis $v_{i_j} \kappa_{i_j} = 0$. For any $v_\ell$ ($1 \leq \ell \neq i_j \leq n$), since $A$ is symmetric, we see

$$\langle (\nabla v_{i_j} A) v_\ell, v_{i_j} \rangle = \langle v_\ell, (\nabla v_{i_j} A) v_{i_j} \rangle.$$  

In order to compute equation (6) easily, we extend an orthonormal basis \{\(v_1, \ldots, v_n\)\} of $T_p(M)$ to principal curvature unit vector fields on some neighborhood $\mathcal{W}_p(\subset \mathcal{V}_p)$, say \{\(V_1, \ldots, V_n\)\}. Moreover we can choose $\nabla V_{i_j} V_{i_j} = 0$ at the point $p$, where $(V_{i_j})_p = v_{i_j}$. Such a principal curvature vector field $V_{i_j}$ can be obtained as follows:

First we define a smooth vector field $W_{i_j}$ on some sufficiently small neighborhood $\mathcal{W}_p(\subset \mathcal{V}_p)$ by using parallel displacement for the vector $v_{i_j}$ along each geodesic with origin $p$. We remark that in general $W_{i_j}$ is not principal on $\mathcal{W}_p$, but $AW_{i_j} = \kappa_{i_j} W_{i_j}$ on the geodesic $\gamma = \gamma(s)$ with $\gamma(0) = p$ and $\gamma'(0) = v_{i_j}$. Here we define the vector field $U_{i_j}$ on $\mathcal{W}_p$ as: $U_{i_j} = \prod_{\alpha \neq \kappa_{i_j}} (A - \alpha I) W_{i_j}$, where $\alpha$ runs over the set of all distinct principal curvatures of $M$ except for the principal curvature $\kappa_{i_j}$. Then

$$AU_{i_j} = A(\prod_{\kappa \neq \kappa_{i_j}} (A - \kappa I)) W_{i_j} = (\prod_{\kappa \neq \kappa_{i_j}} (A - \kappa I)) A V_{\kappa_{i_j}} \text{ component of } W_{i_j}$$

$$= \kappa_{i_j} U_{i_j} \neq 0$$

on $\mathcal{W}_p$. We put $V_{i_j} = U_{i_j}/||U_{i_j}||$. Our construction guarantees that $AV_{i_j} = \kappa_{i_j} V_{i_j}, (V_{i_j})_p = v_{i_j}$ and the integral curve of $V_{i_j}$ through the point $p$ is a geodesic on $M$. In particular, we obtain $\langle \nabla V_{i_j}, V_{i_j} \rangle_p = 0$.

Thanks to the Codazzi equation $\langle (\nabla_X A) Y, Z \rangle = \langle (\nabla_Y A) X, Z \rangle$, at the point $p$ we find

$$(\text{the left-hand side of (6)}) = \langle (\nabla v_{i_j} A) v_{i_j}, v_{i_j} \rangle$$

$$= \langle (\nabla V_{i_j} A) V_{i_j}, V_{i_j} \rangle$$

$$= \langle \nabla V_{i_j} (\kappa_{i_j} v_{i_j}) - A \nabla v_{i_j} V_{i_j}, V_{i_j} \rangle$$

$$= \langle (V_{i_j} \kappa_{i_j}) V_{i_j} + (\kappa_{i_j} I - A) \nabla v_{i_j} V_{i_j}, V_{i_j} \rangle$$

$$= v_{i_j} \kappa_{i_j}.$$
Similarly we get
\[
\langle V_t, (\nabla_{V_i} A)V_{ij} \rangle = \langle V_t, \nabla_{V_i} (\kappa_{ij} V_j) - A \nabla_{V_i} V_j \rangle = \langle v_t, (v_i \kappa_{ij}) v_j \rangle = 0.
\]
Therefore we can see that the differential \(d\kappa_{ij}\) of \(\kappa_{ij}\) vanishes at the point \(p\), which shows that every \(\kappa_{ij}(>0)\) is constant on \(\mathcal{W}_p\), since \(p\) is an arbitrary point of \(\mathcal{W}_p\). This, together with the continuity of each principal curvature function on \(M\) and the connectivity of \(M\), shows that our hypersurface \(M\) is isoparametric (with nonzero constant principal curvatures) in the ambient space \(\tilde{M}^{n+1}(c)\).

The proof of Theorem 3.1 gives us the following which is a characterization of all isoparametric hypersurfaces in a space form \(\tilde{M}^{n+1}(c)\):

**Theorem 3.2.** Let \(M^n\) be a connected hypersurface of a space form \(\tilde{M}^{n+1}(c)\) of constant curvature \(c\). Then \(M^n\) is isoparametric in \(\tilde{M}^{n+1}(c)\) if and only if for each point \(p\) in \(M\) there exists an orthonormal basis \(\{v_1, \ldots, v_m\}\) of the orthogonal complement of \(\ker A\) in \(T_p(M)\) (\(m = \text{rank } A\)) such that all geodesics of \(M\) through \(p\) in the direction \(v_i\) (\(1 \leq i \leq m\)) are circles of positive curvature in the ambient space \(\tilde{M}^{n+1}(c)\).

At the end of this section we write Theorem 3.2 as follows:

**Theorem 3.3.** Let \(M^n\) be a connected hypersurface of a space form \(\tilde{M}^{n+1}(c)\). Then \(M^n\) is isoparametric in \(\tilde{M}^{n+1}(c)\) if and only if for each point \(p\) of \(M^n\) there exists an orthonormal basis \(\{v_1, \ldots, v_n\}\) of \(T_p(M)\) of principal curvature vectors such that all geodesics on \(M^n\) through \(p\) in the direction \(v_i\) (\(1 \leq i \leq n\)) are circles in the ambient space \(\tilde{M}^{n+1}(c)\).

**Proof.** If \(\langle Av_i, v_i \rangle = 0\), then \(Av_i = 0\), because \(v_i\) is a principal curvature vector. Hence the proof of Theorem 3.1 yields that all principal curvatures of \(M\) are constant.

4. Characterization of totally umbilic hypersurfaces II

Paying particular attention to the extrinsic shape of circles of positive curvature on hypersurfaces, we give a characterization of all totally umbilic hypersurfaces in a space form \(\tilde{M}^{n+1}(c)\) [1]:
Theorem 4.1. A hypersurface $M^n$ isometrically immersed into a space form $\tilde{M}^{n+1}(c)$ of constant curvature $c$ is totally umbilic in $\tilde{M}^{n+1}(c)$ if and only if there exists $\kappa > 0$ with the following condition:

At each point $p \in M$, there is an orthonormal basis $\{v_i, \ldots, v_n\}$ of $T_pM$ such that for each distinct $i, j$ circles $\gamma_{i,j}$, $\gamma_{i,-j}$ of curvature $\kappa$ on $M$ with initial condition that

$$
\gamma_{i,j}(0) = \gamma_{i,-j}(0) = p, \quad \gamma_{i,j}'(0) = \gamma_{i,-j}'(0) = v_i,
$$

$$
\nabla_{\gamma_{i,j}}\gamma_{i,j}'(0) = \kappa v_j, \quad \nabla_{\gamma_{i,-j}}\gamma_{i,-j}'(0) = -\kappa v_j
$$

are circles in the ambient space $\tilde{M}^{n+1}(c)$.

Proof. The "only if" part of our Theorem follows from Proposition 2.1 (3). We shall prove the "if" part. Let $\gamma_{a,b} = \gamma_{a,b}(s)$ be a circle of curvature $\kappa$ satisfying the hypothesis at an arbitrary point $p = \gamma_{a,b}(0)$ on the hypersurface $M$. By use of the formulae of Gauss and Weingarten, we find by regarding $\gamma_{a,b}$ as a curve on $\tilde{M}$ that

$$
\tilde{\nabla}_{\gamma_{a,b}}\gamma_{a,b} = \nabla_{\gamma_{a,b}}\gamma_{a,b} + \langle A\gamma_{a,b}, \gamma_{a,b}\rangle N, \quad (7)
$$

$$
\tilde{\nabla}_{\gamma_{a,b}}\tilde{\nabla}_{\gamma_{a,b}}\gamma_{a,b} = -\kappa^2\gamma_{a,b} - \langle A\gamma_{a,b}, \gamma_{a,b}\rangle A\gamma_{a,b}
$$

$$
+ \{3\langle A\gamma_{a,b}, \nabla_{\gamma_{a,b}}\gamma_{a,b}\rangle + \langle (\nabla_{\gamma_{a,b}}A)\gamma_{a,b}, \gamma_{a,b}\rangle\}N. \quad (8)
$$

Thus we have $\|	ilde{\nabla}_{\gamma_{a,b}}\gamma_{a,b}\|^2 = \kappa^2 + \langle A\gamma_{a,b}, \gamma_{a,b}\rangle^2$. Since $\gamma_{a,b}$ is also a circle as a curve in $\tilde{M}$, we find $\langle A\gamma_{a,b}, \gamma_{a,b}\rangle$ is constant along this curve and obtain

$$
\langle A\gamma_{a,b}, \gamma_{a,b}\rangle\{\langle A\gamma_{a,b}, \gamma_{a,b}\rangle\gamma_{a,b} - A\gamma_{a,b}\}
$$

$$
+ \{3\langle A\gamma_{a,b}, \nabla_{\gamma_{a,b}}\gamma_{a,b}\rangle + \langle (\nabla_{\gamma_{a,b}}A)\gamma_{a,b}, \gamma_{a,b}\rangle\}N = 0 \quad (9)
$$

by comparing equation (8) with

$$
\tilde{\nabla}_{\gamma_{a,b}}\tilde{\nabla}_{\gamma_{a,b}}\gamma_{a,b} + \|	ilde{\nabla}_{\gamma_{a,b}}\gamma_{a,b}\|^2\gamma_{a,b} = 0.
$$

Taking the normal component of equation (9) for the hypersurface we get

$$
3\langle A\gamma_{a,b}, \nabla_{\gamma_{a,b}}\gamma_{a,b}\rangle + \langle (\nabla_{\gamma_{a,b}}A)\gamma_{a,b}, \gamma_{a,b}\rangle = 0.
$$

Evaluating this equation at $s = 0$, we have

$$
\pm 3\kappa\langle Av_i, v_j\rangle + \langle (\nabla_{v_i}A)v_i, v_i\rangle = 0,
$$

where the double sign takes plus if $(a, b) = (i, j)$ and takes minus if $(a, b) = (i, -j)$. From these equations for $(i, j)$ and $(i, -j)$ we obtain

$$
\langle Av_i, v_j\rangle = 0 \quad \text{and} \quad \langle (\nabla_{v_i}A)v_i, v_i\rangle = 0 \quad \text{for every distinct } i, j. \quad (10)
$$
On the other hand, taking the tangential component of equation (9), we have

\[ \langle A\gamma_a, b, \gamma_a, b \rangle \{ \langle A\gamma_a, b, \gamma_a, b \rangle \gamma_a, b - A\gamma_a, b \} = 0. \]

When \( \langle Av_i, v_i \rangle \neq 0 \), the constant \( \langle A\gamma_i, b, \gamma_i, b \rangle \) is not 0 for every \( b \). Therefore \( A\gamma_i, b = \langle A\gamma_i, b, \gamma_i, b \rangle \gamma_i, b \) holds for every \( b \). Differentiating both sides of this equation along \( \gamma_i, b \), we get

\[ (\nabla_{\gamma_i, b} A)\gamma_i, b + AV_i, b \gamma_i, b \]

\[ = \left\{ (\langle A\gamma_i, b, A \rangle)\gamma_i, b, \nabla_{\gamma_i, b} \gamma_i, b \right\} \gamma_i, b + \langle A\gamma_i, b, \gamma_i, b \rangle \nabla_{\gamma_i, b} \gamma_i, b. \]

Evaluating this equation at \( s = 0 \), from equation (10) we have

\[ (\nabla_{vi} A)v_i \pm \kappa Av_j = \pm \kappa \langle Av_i, v_i \rangle v_j \]

for every \( j \neq i \), where the double signs take plus if \( b = j \) and take minus if \( b = -j \). Thus we obtain \( (\nabla_{vi} A)v_i = 0 \) and \( Av_j = \langle Av_i, v_i \rangle v_j \) for every \( j \neq i \) in this case.

When \( \langle Av_i, v_i \rangle = 0 \), we have \( \langle A\gamma_i, b, \gamma_i, b \rangle = 0 \) for every \( b \). Differentiating this equation we get

\[ (\langle A\gamma_i, b, A \rangle)\gamma_i, b, \nabla_{\gamma_i, b} \gamma_i, b \] \[ + 2\langle A\gamma_i, b, \gamma_i, b \rangle \nabla_{\gamma_i, b} \gamma_i, b = 0. \]

Evaluating this equation at \( s = 0 \), we have \( (\langle A\gamma_i, b, A \rangle)v_i, v_i \rangle \pm 2\kappa \langle Av_i, v_j \rangle = 0 \) for every \( j \neq i \), where the rule of double sign is the same as above. Thus we obtain \( \langle Av_i, v_j \rangle = 0 \) for every \( j \). As \( \{v_1, \ldots, v_n\} \) is a basis of \( T_pM \), we get \( Av_i = 0 \).

We now show that \( M \) is umbilic at \( p \). Since we already see that \( \langle Av_i, v_j \rangle = 0 \) for every distinct \( i, j \), it is enough to verify \( \langle Av_i, v_i \rangle = \langle Av_j, v_j \rangle \). When \( Av_j = \langle Av_i, v_i \rangle v_j \) holds, it is trivial. When \( Av_i = 0 \), we have \( \langle Av_j, v_j \rangle = 0 \), because either \( Av_j = 0 \) or \( Av_i = \langle Av_j, v_j \rangle v_i \) holds. Thus in this case we have \( \langle Av_i, v_i \rangle = \langle Av_j, v_j \rangle = 0 \). As \( p \in M \) is an arbitrary point we get our conclusion.

\[ \square \]

References

In a recent paper two of us (J.L. and L.M.T.S., 2001) have given a complete list (modulo the (8,8)-periodicity) of 145 basic type-changing transformations of Hurwitz pairs. By the Complementarity Theorem (Thm. 3 of that paper) it is enough to study Hermitian Hurwitz pairs (responsible for 49 such transformations) which, by the Atomization Theorem due to J.L. and O.Suzuki (2001) connects quasiregular functions in the sense of Clifford analysis with hyperkahlerian holomorphic chains (P. Dolbeault, J. Kalina, and J.L., 1999). Explicitly, using fractal representation of Clifford algebras corresponding to Hermitian Hurwitz pairs we are able to prove a counterpart of the Atomization Theorem for fractal gemmae, distinguishing 11 basic type-changing transformations corresponding to 11 atoms appearing in the original Atomization Theorem.

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1. Introduction

In Part 1 of the paper [22] we are dealing, in particular, with Hermitian Hurwitz pairs (cf. also [10]). If their bidimension \((p, n)\) has \(p\) odd or, better to say, the bidimension is \((2p - 1, n)\), \(p = 2, 3, \ldots\), and \(A_1^1, \ldots, A_{2p-1}^1\) are generators of the corresponding Clifford algebra \(Cl_{2p-1}(\mathbb{C})\), we consider a sequence of Clifford algebras

\[ Cl_{2p+2q-1}(\mathbb{C}), \quad q = 1, 2, \ldots \] 

with generators \(A_1^{q+1}, \ldots, A_{2p+2q-1}^{q+1}\). Of course this induces the corresponding sequence of Hermitian Hurwitz pairs of bidimension \((p_q, n(p_q))\), \(p_q = 2p + 2q - 1\).

Consider further the sequence of corresponding systems of closed squares \(Q_{q+1}^\alpha\) of diameter 1, centered at the origin of \(\mathbb{C}\). Then we decompose each \(Q_{q+1}^\alpha\) into the corresponding \(2^{2p+2q-2}\) equal squares parallel to the sides of the original square. In the case of \(pq = 2p + 2q - 1\), \(A = (a_{ij}); j, k = 1, 2, \ldots, 2p+q-2\), we include to the object constructed all closed squares corresponding to the matrix elements equal \(a_{ij}^{qk}\) whenever it is different from zero. We may say that we consider the bundle \((\Sigma_{\alpha})\) of \(a_{ij}^{qk}\)-graded fractals of the flower type:

\[ \Sigma_{\alpha} = (Q_{q}^{\alpha}), \quad q = 1, 2, \ldots ; \] 

we endow them with functions

\[ g_{\alpha}^q(a_{ij}^{qk}; z) = a_{ij}^{qk} \text{ if } g_{\alpha}^q(z) = a_{ij}^{qk}; \quad g_{\alpha}^q(a_{ij}^{qk}; z) = 0 \text{ if } g_{\alpha}^q(z) \neq a_{ij}^{qk}, \]

where \(g_{\alpha}^q\) is the grading function: \(g_{\alpha}^q(z) = a_{ij}^{qk}\) inside the square corresponding to the pair \((j, k)\). By (2), for \(\alpha \geq 2p\) each \(Q_{q}^{\alpha}\) is decomposed into \(4^{q+1}\) equal squares.

It is convenient to consider the sequence (1) with generators

\[ A_{\alpha}^{q+1} = \sigma_3 \otimes A_{\alpha}^q \equiv \begin{pmatrix} A_{\alpha}^q & 0 \\ 0 & -A_{\alpha}^q \end{pmatrix}, \quad \alpha = 1, 2, \ldots, 2p + 2q - 3; \]

\[ A_{2p+2q-2}^{q+1} = \sigma_1 \otimes I_{p,q} \equiv \begin{pmatrix} 0 & I_{p,q} \\ I_{p,q} & 0 \end{pmatrix}, \]

\[ A_{2p+2q-1}^{q+1} = -\sigma_2 \otimes I_{p,q} \equiv \begin{pmatrix} 0 & iI_{p,q} \\ -iI_{p,q} & 0 \end{pmatrix}, \]

where

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
and $I_{p,q} = I_{2p+q-2}$, the unit matrix of order $2^{p+q-2}$.

When considering the sequences $g_1^p(z)$, $g_2^p(z), \ldots$ at the points (called stigmas):

$$z = \frac{1}{2\sqrt{2}} \frac{m}{2^n} (1 - i), \quad m = 0, \pm 1, \ldots, \pm (2^n - 1); \quad n = 0, 1, \ldots, \quad (4)$$

of the diagonal or at the related points $z_-, z_+, z^-, z^+$ defined in [12, 21] we establish (in the quoted papers) their periodicity with two kinds of periods only, corresponding to

$$1, 2, 4, 7, 8, 11, 13, 14, 16, 19, 21, 22, 25, 26, 28, 31, 32, 35, 37, 38, \ldots \quad (5)$$

and

$$3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24, 27, 29, 30, 33, 34, 36, 39, 40, \ldots \quad (6)$$

respectively, independently of $p$ and $\alpha$ in $\Sigma_\alpha$.

This result seems to be the key point to look for a fractal counterpart of the Atomization Theorem [20] connecting quasiregular functions in the sense of Clifford analysis with hyperkählerian holomorphic chains [5] and, consequently, to distinguish the corresponding basic type-changing transformations.

2. Branch-type fractal representation and Atomization Theorem for Hermitian Hurwitz pairs

As noticed in [13], in order to investigate the sequences (5) and (6) it seems natural to construct the corresponding bundle $\Xi^\alpha$ of graded fractals of the branch type which is possible by the duality theorem proved in that paper; cf. also [3]. Since the theorem quoted refers to inoculated fractals, we recall the corresponding definition; for an introductory staff we can refer, e.g., to [8].

Let $K_0$ be a compact metric space. Choosing a reference point $p_0$ in $K_0$, the fundamental branches $L_j^0, j = 1, 2, \ldots, Q$, and a system of contractible mappings $\gamma_\ell, \ell = 1, 2, \ldots, Q$, where we assume the usual separation condition (where $K_0^0$ is the open kernel of $K_0$):

$$\gamma_j(K_0^0) \cap \gamma_\ell(K_0^0) = \emptyset \quad \text{for} \quad j \neq \ell; \quad j, \ell = 1, 2, \ldots, Q,$$
the fractal set of the branch type is the lattice

\[ L = \bigcup_{n=0}^{\infty} L^n, \text{ where } L^n = L^{n-1} \cup \hat{L}^n, \hat{L}^n = \bigcup_{\ell=1}^{Q} \gamma_{\ell}(L^{n-1}), \text{ for } n = 1, 2, \ldots, \]

\[ L^0 = \bigcup_{j=1}^{Q} L^0_j, \text{ where } L^0_j \text{ is the closed segment in } K_0 \text{ joining } p_0 \text{ to } \gamma_j(p_0). \]

We complete this definition as follows:

(i) If, for some \( n \) and \( j \); \( \gamma_{\ell}(L^{n-1}), \ell = 1, 2, \ldots, Q, \) in (7), depending on a gradating function \( g^1 \) related to a fractal \( \Xi_1 \) within the graded fractal bundle in question is replaced by another term

\[ \bigcup_{\ell=1}^{Q'} \gamma'_{\ell}(L^{n-1}), \]

depending on other contractible mappings \( \gamma'_{\ell}, \ell = 1, 2, \ldots, Q', \) and a gradation function \( g^2 \) related to a fractal \( \Xi_2 \) within the graded fractal bundle in question, we say that \( \Xi_1 \) is inoculated of the first kind at its \( n \)-th embranchment by a branch of \( \Xi_2 \).

(ii) If, for some \( n, \) to the sum \( \hat{L}^n \) in (7), depending on a gradation function \( g^1 \) related to a fractal \( \Xi_1 \) within the graded fractal bundle in question, an extra sum \( \hat{L}'^n \) is added, depending on other contractible mappings \( \gamma'_{\ell}, \ell = 1, 2, \ldots, Q', \) and a gradation function \( g^2 \) related to a fractal \( \Xi_2 \) within the graded fractal bundle in question, we say that \( \Xi_1 \) is inoculated of the second kind at its \( n \)-th embranchment by a branch of \( \Xi_2 \).

(iii) If, for some \( n, \) the \( n \)-th branch of a fractal \( \Xi_1 \) within the graded fractal bundle in question is considered in the bundle together with the 1-st embranchment of a fractal \( \Xi_2 \) within the fractal bundle in question, we say that \( \Xi_1 \) is inoculated of the third kind at its \( n \)-th embranchment by \( \Xi_2 \).

Effectively, with the meaning of \( n \) as in the formula (4), for \( p = 2, \alpha = 5, \) and \( n = 11, \) construction of the fractal dual to \( \Sigma_5 \) is visualized on Fig. 1, where we clearly see two kinds of periods only, being of real dimension 16, which we call bipetals [9, 21]. Their appearance is related to the Hurwitz formula [7] corresponding to \( (2p - 1) + 6 = 9: \)

\[ (x_1^2 + \cdots + x_9^2)(y_1^2 + \cdots + y_{16}^2) \equiv (x \circ_9 y)_1^2 + \cdots + (x \circ_9 y)_{16}^2 \]

with

\[ (x \circ_9 y)_j = \sum_{\alpha=1}^{9} \sum_{k=1}^{16} C_j^{\alpha k} x_\alpha y_k, \quad (x_1, \ldots, x_9) \in R^9, \quad (y_1, \ldots, y_{16}) \in R^{16}. \]
stands for the bipetal

\[ s = 9 \]

\[ s = 3 \]

\[ s = 1 \]

Figure 1. Construction of the fractal dual to \( \Sigma_5 \) for \( p = 2 \)
Small squares resp. circles in Fig. 1 correspond to the sequences (5) resp. (6). The symbols \( m_q \) inside those squares resp. circles indicate the position of a number \( m \) belonging to the sequence (5) resp. (6) in the \( q \)-th step of iteration in (3) which corresponds now to the \( q \)-th embranchment; also it is represented by the \( q \)-th column in the figure. The rows correspond to the points (4), namely to the ratios \( m/2^n \), but for \( q + 1 \) the scale of the column is divided by two comparing with the scale for \( q \), e.g. \( 8 \cdot 2^{-4} \) for \( q = 2 \) corresponds to \( 8 \cdot 2^{-3} \) for \( q = 1 \). Blank areas stay for the situations where the periodicity has not yet started; areas marked with \( ||| \) indicate that no entry exists because \( q \) is too small; finally, areas \( \equiv \) indicate that no entry exists because the further dividing of the scale by two keeping \( m \) an integer in \( m/2^n \) is impossible.

By the Periodicity Theorems 1-6 in [21] summarized, in quaternionic formulation, as the Periodicity Theorem in [12], when passing to an analogue of Fig. 1 for \( (p, \alpha, n) \neq (2, 5, 11) \), the position of symbols \( m_q \) will remain unchanged. The bipetals will in general change and for \( n \) odd the upper parallelograms resp. ellipses will remain their upper position, whereas for \( n \) even these upper objects have to be drawn as the lower ones. In such a way, in addition to the inoculated graded fractal bundle \( (\Xi^\alpha) \) of the branch type, of the third kind, dual to \( (\Sigma_\alpha) \) as in (2), depending on the petals in question, we can consider an additional inoculated graded fractal \( \Xi \) of the branch type, of the first kind, being a kind of skeleton of all fractals \( \Xi^\alpha \) in \( (\Xi^\alpha) \), with gradation two, depending on two sequences (5) and (6) only.

Our purpose now is to find a counterpart of the following

**Atomization Theorem** (for Hermitian Hurwitz pairs) [20]. Suppose that the pseudometric corresponding to the \( p \)-dimensional real vector space appearing in the definition of an Hermitian Hurwitz pair of bidimension \( (p, n) \) has the form

\[
\langle dx, dx \rangle = dx_1^2 - dx_2^2 - \cdots - dx_6^2 \text{ or } dx_1^2 + \cdots + dx_3^2 - dx_4^2 - dx_5^2 \text{ for } p = 5, \tag{8}
\]

resp.

\[
\begin{align*}
\langle dx, dx \rangle &= dx_1^2 - dx_2^2 - \cdots - dx_6^2 \\
&\text{or } dx_1^2 + \cdots + dx_3^2 - dx_4^2 - \cdots - dx_6^2, \\
&\text{or } dx_1^2 + \cdots + dx_7^2 - dx_8^2 - dx_9^2 \text{ for } p = 9. \tag{9}
\end{align*}
\]

Then in each of these cases there exists a complex structure on the corre-
sponding isometric embedding

\[ \iota : \mathbb{C}^2 \cong \mathbb{R}^4 \to G(2, 4) \text{ of the form } \mathbb{R}^4 \ni x \mapsto \sum_{\alpha=1}^{3} x_\alpha S_\alpha + x_4 S_4 \text{ for } p = 5, \]

resp.

\[ \iota : \mathbb{C}^4 \cong \mathbb{R}^8 \to G(8, 16) \text{ of the form } \mathbb{R}^8 \ni x \mapsto \sum_{\alpha=1}^{7} x_\alpha S_\alpha + x_8 S_8 \text{ for } p = 9, \]

with the property that the embedding is the real part of a holomorphic mapping.

Here \( G(m, 2n) \) denotes the \( m \)-dimensional Grassmanian submanifold of \( \mathbb{C}^n \) and \( S_\alpha, I_4 \) or \( I_8 \) are generators of a central Clifford algebra over \( \mathbb{C} \). It appears that the totality of the embeddings \( \iota \) depends on eleven \( 2 \times 2 \) complex matrices called atoms. Both the matrices and the dependence of the embeddings on them are explicitly given in [20].

3. Fractal gemmae, their type-changing transformations and the corresponding Atomization Theorem

The fractal \( \Xi \) (shown on Fig. 1 in [13]) has six types of embranchments:

Here \( \square \) resp. \( \circ \) represent members of \( (5) \) resp. \( (6) \). Only two latter structures correspond to inoculation at the embranchment. They will be called gemmae of \( \Xi \). We are going to prove

**Atomization Theorem for Fractals.** Suppose that the pseudometric corresponding to the \( p \)-dimensional real vector space appearing in the definition of an Hermitian Hurwitz pair of bidimension \( (p, n) \) has the form (8) resp. (9) resp.

\[
\begin{align*}
\langle dx, dx \rangle &= dx_1^2 - dx_2^2 - \cdots - dx_{13}^2 \\
\text{or } dx_1^2 + \cdots + dx_7^2 - dx_8^2 - \cdots - dx_{13}^2, \\
\text{or } dx_1^2 + \cdots + dx_3^2 - dx_4^2 - \cdots - dx_{13}^2, \\
\text{or } dx_1^2 + \cdots + dx_{11}^2 - dx_{12}^2 - dx_{13}^2, \\
\text{or } dx_1^2 + \cdots + dx_9^2 - dx_{10}^2 - \cdots - dx_{13}^2 \quad \text{for } p = 13.
\end{align*}
\]
Then in each of these cases there exists a finite subfractal $\Xi_0 \subset \Xi$, where all gemmae can be obtained with the help of type-changing transformations listed in Table 1.

In the above table $\begin{pmatrix} 11 \\ 3 \end{pmatrix}$ etc. stays for the pair of gemmae

$\begin{pmatrix} 21 \\ 11 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$

with $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$ taken away. The other symbols and numbers can correspondingly be deduced from Figs. 2-4 illustrating the (constructive) proof. For the full transformations listed in the table, the gemmae indicated have to be extended to finite sequences of petals corresponding to these gemmae and related to the generators of the Clifford algebra concerned. Under the type of gemmae we mean the class of abstraction of all the structures $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ resp. $\begin{pmatrix} \circ \end{pmatrix}$. A transformation of gemmae is type-changing if it sends $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \circ \end{pmatrix}$ or vice versa. In analogy to the fact that $0 \in \mathbb{C}$ lies in the real and imaginary axis as well, we complete the both sequences (5) and (6) by 0 at the beginning, so that we have objects $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \circ \end{pmatrix}$. This is caused by the fact that we have no pairing for $\begin{pmatrix} 3 \\ [13] \end{pmatrix}$, so we need this extension.
The table is illustrated by Fig. 4. The choice of coordinates is motivated by the notions of pistil and stamens introduced and discussed in [21]. The choice of symbols $r$ and $s$ for particular basic type-changing transformations, called fractal atoms, as well as the corresponding lower indices seem natural. Except for $r^1$, the choice of upper indices refers to the length of the corresponding vectors. The notation $\#_5$, $\#_7$, etc. informs on the end of the sequence (4) for $p = 5, 7$, etc.

Proof of the Theorem. From [13] we know that numbers preceding the inoculations are of the form

$$(2m - 1)2^{2p-1}; \quad m, p = 1, 2, \ldots, \quad (11)$$

whereas numbers following the inoculations are of the form

$$(2m - 1)2^{2p} - 1; \quad m, p = 1, 2, \ldots. \quad (12)$$
Moreover, if we order the set of numbers (11) to make the sequence $(a_n)$, $a_n < a_{n+1}; n = 1, 2, \ldots$, then $a_n$ with $n$ odd correspond to (5), and $a_n$ with $n$ even correspond to (6). If we order the set of numbers (12) to make the sequence $(b_n)$, $b_n < b_{n+1}; n = 1, 2, \ldots$, then $b_n$ with $n$ odd correspond to (6), and $b_n$ with $n$ even correspond to (5). It is then natural to look for pairings of the type

Figure 3. Type-changing transformation of the inoculated graded fractal $\Xi$ to the bundle $(\Xi_1, \Xi_2, \Xi_3, \ldots)$ of inoculated fractals without gradation, gemma by gemma.
Figure 4. Type-changing transformations (□) ↔ (○): (a) decomposed as stigmas of the pistil, (b) compared with the stamens

This is done in [13], where the Proposition says that \( \Xi \) can be decomposed to the bundle of inoculated fractals \( \Xi_1 \) and \( \Xi_2 \) without gradation (containing only copies of □ and ○, respectively), where \( \Xi_2 \) is repeated infinitely many times. Precisely, the embranchments of \( \Xi_1 \) are renumbered according to the scheme \( n \mapsto n + 1 \). At the new first embranchment of \( \Xi_1 \) this fractal is inoculated of the second kind by the first copy \( \Xi_2^1 \) of \( \Xi_2 \). At the first embranchment of \( \Xi_2^1 \) this fractal is inoculated of the second kind by the second copy \( \Xi_2^2 \) of \( \Xi_2 \). At the first embranchment of \( \Xi_2^2 \) this fractal is inoculated of the second kind by the third copy \( \Xi_2^3 \) of \( \Xi_2 \), etc. (Fig. 2 in [13]). Clearly, this formulation includes the statement that we have no pairing for \( \Xi \) and then infinitely many copies of \( \Xi \) have to start infinitely many copies of \( \Xi_2 \).

Let us analyze the procedure more closely: it determines a type-changing transformation of gemmae related to \( \Xi \) (Fig. 2) and, consequently, a type-changing transformation of \( \Xi \) to the bundle \( (\Xi_1, \Xi_2^1, \Xi_2^2, \ldots) \) of frac-
tals without gradation, gemma by gemma (Fig. 3). The global transformation can be decomposed into the basic transformations, fractal atoms, listed in Table 1. By the (8,8)-periodicity of the Clifford structure or, in the context of dual fractal bundles \((\Sigma_\alpha), (\Xi^{\alpha})\), the related fractal \(\Xi\), and the bundle \((\Xi_1, (\Xi^{2}_2))\), by self-similarity of the properly chosen subfractals, we have no need of continuing the table. This completes the proof.

4. Relationship with hyperkählerian holomorphic chains

It is clear that the both Atomization Theorems give preference, among odd dimensions, to those congruent to one modulo four. In a series of papers [4-6], Dolbeault, Kalina and Lawrynowicz extended the Dolbeault theory of holomorphic chains to hyperkählerian manifolds of real dimension \(4n, n = 1, 2, \ldots\), and applied it to a study of almost hyperbolic pseudodistances. Let \(K = K(\mathbb{R}) = (K, g)\) be such a manifold. This means that there exist on \(K\) two complex structures \(\vec{t}\) and \(\vec{j}\) such that \(g\) is a Kähler metric for each of them and \(\vec{t}\vec{j} = -\vec{j}\vec{t}\) (cf. e.g. [1], Proposition 14.10). Then \(g\) is still a Kähler metric for a complex structure \(\vec{k} = \vec{t}\vec{j}\). Moreover, if \(z = (z_1, z_2, z_3) \in \partial B_1(0)\) (the unit two-dimensional sphere in \(\mathbb{R}^3\)), then \(\vec{j}z = \vec{z} \cdot (\vec{i}, \vec{j}, \vec{k}) = z_1\vec{i} + z_2\vec{j} + z_3\vec{k}\) is again a complex structure of \(K\) and \(g\) is a metric of it.

The quoted results on holomorphic chains and almost hyperbolic pseudodistances (cf. also [2]) depend on \(\vec{j}z\), but in the case where \(K \subset M\) or \(M \supset K\) and \(M\) is a manifold equipped with one of the pseudometrics (8)-(10), we can relate \(\vec{j}z\) to the corresponding isometric embedding \(\iota\), determined in the cases of (8) and (9) in the Atomization Theorem. This enables us, in particular, to consider as dual the almost hyperbolic pseudodistances related to two dual structures on \(M\).

Let us analyze the situation more closely. In [22] we can find such dualities, expressed as type-changing transformations, relating manifolds \(M\) with the following pseudometrics.

Case I. \(dx_1^2 + \cdots + dx_3^2 - dx_4^2 - dx_5^2\) and \(dx_1^2 - dx_2^2 - \cdots - dx_6^2\) for \(p = 5\).
In analogy to Penrose twistors [24] we arrive at the structure of Hurwitz twistors (H for short) [16], determined by a system of \((5) = 5\) algebraic equations, and at their anti-objects (aH), corresponding to the 5-dimensional Kaluža-Klein theory [14].

Case II. \(dx_1^2 + \cdots + dx_3^2\) and \(dx_1^2 + \cdots + dx_5^2 - dx_6^2 + dx_7^2\) for \(p = 9\).
Here we arrive at the structure of pseudotwistors (p) [20, 21], determined by a system of \((9) = 126\) algebraic equations, and at their anti-objects (ap).
Case III. $dx_1^2 + \cdots + dx_7^2 - dx_8^2 - \cdots - dx_{13}^2$ and $dx_1^2 + \cdots + dx_5^2 - dx_6^2 - \cdots - dx_{13}^2$

for $p = 13$.

Here we arrive at the structure of bitwistors (b) [11, 17, 18], determined by a system of $(13^2) = 715$ algebraic equations, and at their anti-objects (ab).

Case IV. $dx_1^2 + \cdots + dx_5^2 - dx_6^2 - \cdots - dx_9^2$ and $dx_1^2 + \cdots + dx_3^2 - dx_4^2 - \cdots - dx_9^2$

for $p = 9$.

Here we arrive at the structure of pseudobitwistors (pb) [20, 21], determined by a system of 126 algebraic equations, and at their anti-objects (apb).

The other cases appearing in (10) were not discussed in [22] because of the (8,8)-periodicity of the Clifford structure.

The system of four-type preserving transformations, listed above as dualities, is not satisfactory for us from the point of view of relationship with the generalized Dolbeault theory. For instance, the Hurwitz-twistor counterpart of Penrose's fundamental theorem [24] in the local version states a one-to-one correspondence between the real-analytic solutions of the corresponding spinor equations of spin $\frac{1}{2}n$ in an open set in $\mathbb{C}^2$ and harmonic forms with respect to the $(1,1)$-metric $ds^2 = dz^1 dz^1 - dz^2 d\bar{z}^2$. The construction depends on the indefinite fibre $(2,0)$-metric $d\rho^2 = d\xi^1 d\bar{\xi}^1 + d\xi^2 d\bar{\xi}^2$.

On the other hand the analogous pseudotwistor counterpart of the Penrose theorem states a one-to-one correspondence between the respective real-analytic solutions of an open set in $\mathbb{C}^4$ and harmonic forms with respect to the $(0,4)$-metric $d\sigma^2 = -dz^1 d\bar{z}^1 - \cdots - dz^4 d\bar{z}^4$. The construction depends on the indefinite fibre $(8,0)$-metric $d\sigma^2 = d\xi^1 d\bar{\xi}^1 + \cdots + d\xi^8 d\bar{\xi}^8$. Therefore the both structures have to be linked by a proper type-changing transformation.

Another interesting field of interference between hyperkaherian structures and harmonicity is provided by the problem of characterizing hyperholomorphic functions in the space $\mathcal{Q}_p$ in terms of harmonic majorants associated with the Dirichlet problem [15].

The Hurwitz-twistor counterpart of the Penrose theorem in the semiglobal version states a one-to-one correspondence of the space of holomorphic solutions of the above mentioned spinor equations of spin $\frac{1}{2}n$ with the one-dimensional Dolbeault cohomology group $H^1$ depending on $\mathcal{O}(n - 2) = \mathcal{O}([e]^{n-2})$, where $[e]$ is the canonical effective divisor of $\mathbb{P}^3(\mathbb{C})$. On the other side the analogous pseudotwistor counterpart of the Penrose theorem states a one-to-one correspondence of the respective space of holomorphic solutions with the group $H^1$ depending on $\mathcal{O}(-\alpha n - \beta)$, where
\( \alpha \) and \( \beta, \beta \geq 2, \) are some positive integers. Therefore, again, the both structures have to be linked by a proper type-changing transformation.

The above demands can be fulfilled with the help of Atomization Theorem for fractals (Fig. 5). Namely, we have, as a corollary to that theorem, the following

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Type-changing transformations (\( \square \) \( \Longleftrightarrow \) \( \bigcirc \)) related to points of the diagonal of generators of \( \text{Cl}_5(\mathbb{C}) \), \( \text{Cl}_9(\mathbb{C}) \) and \( \text{Cl}_{13}(\mathbb{C}) \) (i.e. of the pistil of the corresponding fractal of the flower type) and the resulting scheme for type-changing transformations between the objects and anti-objects listed in Table 2.}
\end{figure}

**Atomization Theorem for Hurwitz Twistor-like structures.** Suppose that the pseudometric corresponding to the \( p \)-dimensional real vector space appearing in the definition of an Hermitian Hurwitz pair of bidimension \( (p,n) \) has the form given in Cases I-IV. Then in each of these cases there exists a type-changing transformation of the form listed in Table 2.

Of course in the case of \( V \Longleftrightarrow W \) with \( \dim W - \dim V = q > 0, \) the transformation concerns the points of \( V \) and \( q \)-dimensional subspaces of \( W \) with the proper (e.g. Grassmanian) structure.

The theorem may be regarded as a further contribution to the so-called double Cartan-like triality of Hermitian Hurwitz pairs \([11,23]\).
<table>
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<tr>
<th>No.</th>
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<td>1</td>
<td>$H \leftrightarrow ap$</td>
<td>$p \leftrightarrow ab$</td>
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<td>$p \leftrightarrow ab$</td>
<td>$H \leftrightarrow ap$</td>
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<td>$ap \leftrightarrow pb$</td>
<td>$ap \leftrightarrow ab$</td>
<td>$b \leftrightarrow p$</td>
<td>$H \leftrightarrow aH$</td>
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<td>2</td>
<td>$aH \leftrightarrow p$</td>
<td>$p \leftrightarrow ab$</td>
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<td>$p \leftrightarrow ab$</td>
<td>$H \leftrightarrow ap$</td>
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<td>$ap \leftrightarrow pb$</td>
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<td>$b \leftrightarrow p$</td>
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SOME FOUR-DIMENSIONAL ALMOST HYPERCOMPLEX PSEUDO-HERMITIAN MANIFOLDS

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In this paper, a lot of examples of four-dimensional manifolds with an almost hypercomplex pseudo-Hermitian structure are constructed in several explicit ways. The received 4-manifolds are characterized by their linear invariants in the known aspects.

Introduction

In the study of almost hypercomplex manifolds the Hermitian metrics are well known. The parallel study of almost hypercomplex manifolds with skew-Hermitian metrics is in progress of development [6], [7].

Let \((M, H)\) be an almost hypercomplex manifold, i.e. \(M\) is a 4n-dimensional differentiable manifold and \(H\) is a triple \((J_1, J_2, J_3)\) of anticommuting almost complex structures, where \(J_3 = J_1 \circ J_2 \) [8], [2].

A standard hypercomplex structure for all \(x(y^i, x^i, u^i, v^i) \in T_pM, p \in M\) is defined in [8] as follows

\[ J_1x(y^i, x^i, v^i, -u^i), \quad J_2x(-u^i, -v^i, x^i, y^i), \quad J_3x(v^i, -u^i, y^i, -x^i). \] (1)

Let us equip \((M, H)\) with a pseudo-Riemannian metric \(g\) of signature \((2n, 2n)\) so that

\[ g(\cdot, \cdot) = g(J_1 \cdot, J_1 \cdot) = -g(J_2 \cdot, J_2 \cdot) = -g(J_3 \cdot, J_3 \cdot). \] (2)

We called such metric a pseudo-Hermitian metric on an almost hypercomplex manifold [6]. It generates a Kähler 2-form \(\Phi\) and two pseudo-Hermitian
metrics $g_2$ and $g_3$ by the following way

$$\Phi := g(J_1, \cdot), \quad g_2 := g(J_2, \cdot), \quad g_3 := g(J_3, \cdot). \quad (3)$$

The metric $g$ ($g_2$, $g_3$, respectively) has an Hermitian compatibility with respect to $J_1$ ($J_3$, $J_2$, respectively) and a skew-Hermitian compatibility with respect to $J_2$ and $J_3$ ($J_1$ and $J_2$, $J_1$ and $J_3$, respectively).

On the other hand, a quaternionic inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{H}$ generates in a natural way the bilinear forms $g$, $\Phi$, $g_2$ and $g_3$ by the following decomposition: $\langle \cdot, \cdot \rangle = -g + i\Phi + jg_2 + kg_3$.

The structure $(H, G) := (J_1, J_2, J_3; g, \Phi, g_2, g_3)$ is called a hypercomplex pseudo-Hermitian structure on $M^{4n}$ or shortly a $(H, G)$-structure on $M^{4n}$. The manifold $(M, H, G)$ is called an almost hypercomplex pseudo-Hermitian manifold or shortly an almost $(H, G)$-manifold [6].

The basic purpose of the recent paper is to construct explicit examples of the $(H, G)$-manifolds of the lowest dimension at $n = 1$ and to characterize them.

The following structural $(0,3)$-tensors play basic role for the characterization of the almost $(H, G)$-manifold

$$F_\alpha(x, y, z) = g((\nabla_x J_\alpha) y, z) = (\nabla_x g_\alpha)(y, z), \quad \alpha = 1, 2, 3,$$

where $\nabla$ is the Levi-Civita connection generated by $g$.

It is well known [2], that the almost hypercomplex structure $H = (J_\alpha)$ is a hypercomplex structure if the Nijenhuis tensors


vanish for each $\alpha = 1, 2, 3$. Moreover, one $H$ is hypercomplex iff two of $N_\alpha$ are zero.

Since $g$ is a Hermitian metric with respect to $J_1$, we use the classification of the almost Hermitian manifolds given in [5]. According to it the basic class of these manifolds of dimension 4 are the class of almost Kähler manifolds $\mathcal{AK} = \mathcal{W}_2$ and the class of Hermitian manifolds $\mathcal{H} = \mathcal{W}_4$. The class of the $\mathcal{AK}$-manifolds are defined by condition $d\Phi = 0$ or equivalently $\sigma_{x,y,z} F_1(x, y, z) = 0$. The class of the Hermitian 4-manifolds is determined by $N_1 = 0$ or

$$F_1(x, y, z) = \frac{1}{2} \left[ g(x, y) \theta_1(z) - g(x, z) \theta_1(y) 
- g(x, J_1 y) \theta_1(J_1 z) + g(x, J_1 z) \theta_1(J_1 y) \right]$$

where $\theta_1(\cdot) = g^{ij} F_1(e_i, e_j, \cdot) = \delta \Phi(\cdot)$ for any basis $\{e_i\}_{i=1}^4$, and $\delta$ — the coderivative.
On other side, the metric $g$ is a skew-Hermitian one with respect to $J_2$ and $J_3$. A classification of all almost complex manifolds with skew-Hermitian metric (Norden metric or B-metric) is given in [3]. The basic classes are:

$$\mathcal{W}_1 : F_{\alpha}(x, y, z) = \frac{1}{4} \left[ g(x, y)\theta_{\alpha}(z) + g(x, z)\theta_{\alpha}(y) \\
+ g(x, J_{\alpha}y)\theta_{\alpha}(J_{\alpha}z) + g(x, J_{\alpha}z)\theta_{\alpha}(J_{\alpha}y) \right],$$

$$\mathcal{W}_2 : \sigma_{x, y, z} F_{\alpha}(x, y, J_{\alpha}z) = 0, \quad \mathcal{W}_3 : \sigma_{x, y, z} F_{\alpha}(x, y, z) = 0,$$

where $\theta_{\alpha}(\cdot) = g^{ij}F_{\alpha}(e_i, e_j, \cdot)$, $\alpha = 2, 3$, for an arbitrary basis $\{e_i\}_{i=1}^4$.

We denote the main subclasses of the respective complex manifolds by $\mathcal{W}(J_{\alpha})$, where $\mathcal{W}(J_1) := \mathcal{W}_4(J_1)$ [5], and $\mathcal{W}(J_{\alpha}) := \mathcal{W}_1(J_{\alpha})$ for $\alpha = 2, 3$ [3].

In the end of this section we recall some known facts from [6] and [7].

A sufficient condition an almost $(H, G)$-manifold to be an integrable one is following

**Theorem 0.1.** Let $(M, H, G)$ belongs to $\mathcal{W}(J_{\alpha}) \cap \mathcal{W}(J_{\beta})$. Then $(M, H, G)$ is of class $\mathcal{W}(J_{\gamma})$ for all cyclic permutations $(\alpha, \beta, \gamma)$ of $(1, 2, 3)$.

A pseudo-Hermitian manifold is called a pseudo-hyper-Kähler manifold (denotation $(M, H, G) \in \mathcal{K}$), if $F_{\alpha} = 0$ for every $\alpha = 1, 2, 3$, i.e. the manifold is Kählerian with respect to each $J_{\alpha}$ (denotation $(M, H, G) \in \mathcal{K}(J_{\alpha})$).

**Theorem 0.2.** If $(M, H, G) \in \mathcal{K}(J_{\alpha}) \cap \mathcal{W}(J_{\beta}) (\alpha \neq \beta \in \{1, 2, 3\})$ then $(M, H, G) \in \mathcal{K}$.

As $g$ is an indefinite metric, there exists isotropic vector fields $X$ on $M$. Following [4] we consider the invariants

$$\|\nabla J_{\alpha}\|^2 = g^{ij}g^{kl}((\nabla_{e_i}J_{\alpha})e_k, (\nabla_{e_j}J_{\alpha})e_l), \quad \alpha = 1, 2, 3,$$

where $\{e_i\}_{i=1}^4$ is an arbitrary basis of $T_pM$, $p \in M$.

**Definition 0.1.** An $(H, G)$-manifold is called: (i) isotropic Kählerian with respect to $J_{\alpha}$ if $\|\nabla J_{\alpha}\|^2 = 0$ for some $\alpha \in \{1, 2, 3\}$; (ii) isotropic hyper-Kählerian if it is isotropic Kählerian with respect to every $J_{\alpha}$ of $H$.

**Theorem 0.3.** Let $M$ be an $(H, G)$-manifold of class $\mathcal{W} = \bigcap_{\alpha} \mathcal{W}(J_{\alpha})$ ($\alpha = 1, 2, 3$) and $\|\nabla J_{\alpha}\|^2$ vanishes for some $\alpha = 1, 2, 3$. Then $(M, H, G)$ is an isotropic hyper-Kähler manifold, but it is not pseudo-hyper-Kählerian in general.
A geometric characteristic of the pseudo-hyper-Kähler manifolds according to the curvature tensor \( R = [\nabla, \nabla] - \nabla[ , ] \) induced by the Levi-Civita connection is given in [7].

**Theorem 0.4.** Each pseudo-hyper-Kähler manifold is a flat pseudo-Riemannian manifold with signature \((2n, 2n)\).

1. The two known examples of almost \((H, G)\)-manifolds

1.1. **A pseudo-Riemannian spherical manifold with \((H, G)\)-structure**

Following [10] we have considered in [6] and [7] a pseudo-Riemannian spherical manifold \( S^4_2 \) in pseudo-Euclidean vector space \( \mathbb{R}^5_2 \) of type \((-++++)\). The structure \( H \) is introduced on \( \hat{S}^4_2 = S^4_2 \setminus \{(0,0,0,0,\pm 1)\} \) as in (1) and the pseudo-Riemannian metric \( g \) is the restriction of the inner product of \( \mathbb{R}^5_2 \) on \( \hat{S}^4_2 \). Therefore \( \hat{S}^4_2 \) admits an almost hypercomplex pseudo-Hermitian structure. The corresponding manifold is of the class \( W(J_1) \) but it does not belong to \( W \) and it has a constant sectional curvature \( k = 1 \). Moreover, we established that the considered manifold is conformally equivalent to a flat \( K(J_1) \)-manifold, which is not a \( K \)-manifold and \( (\hat{S}^4_2, H, G) \) is an Einstein manifold.

1.2. **The Thurston manifold with \((H, G)\)-structure**

In [6] we have followed the interpretation of Abbena [1] of the Thurston manifold. We have considered a 4-dimensional compact homogeneous space \( L/\Gamma \), where \( L \) is a connected Lie group and \( \Gamma \) is the discrete subgroup of \( L \) consisting of all matrices whose entries are integers. We have introduced the almost hypercomplex structure \( H = (J_\alpha) \) on \( TE L \) as in (1) and we translate it on \( T_A L, A \in L \), by the action of the left invariant vector fields. The \( J_\alpha \) are invariant under the action of \( \Gamma \), too. By analogy we have defined a left invariant pseudo-Riemannian inner product in \( TE L \). It generates a pseudo-Riemannian metric \( g \) on \( M^4 = L \). Then the generated 4-manifold \( M \) is equipped with a suitable \((H, G)\)-structure and \((M, H, G)\) is a \( W(J_1) \)-manifold but it does not belong to the class \( W \).

2. **Engel manifolds with almost \((H, G)\)-structure**

In the next two examples we consider \( M = \mathbb{R}^4 = \{(x^1, x^2, x^3, x^4)\} \) with a basis \( \{e_1 = \frac{\partial}{\partial x^1}, e_2 = \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4}, e_3 = -\frac{\partial}{\partial x^4}, e_4 = -\frac{\partial}{\partial x^4}\} \) and an
Engel structure $\mathcal{D} = \text{span}\{e_1, e_2\}$, i.e. an absolutely non-integrable regular two-dimensional distribution on $TM$ [4].

2.1. **Double isotropic hyper-Kählerian structures but neither hypercomplex nor symplectic**

At first we use the introduced there a pseudo-Riemannian metric and almost complex structures given by

$$
\begin{align*}
g &= (dx^1)^2 + \{1 - (x^1)^2 - (x^3)^2\}(dx^2)^2 - (dx^3)^2 \\
&\quad - (dx^4)^2 - 2x^1 dx^2 dx^3 + 2x^3 dx^2 dx^4, \\
J : &\quad Je_1 = e_2, Je_2 = -e_1, Je_3 = e_4, Je_4 = -e_3, \\
J' : &\quad J'e_1 = e_2, J'e_2 = -e_1, J'e_3 = -e_4, J'e_4 = e_3.
\end{align*}
\tag{4}
$$

It is given in [4] that $(J, g)$ and $(J', g)$ are a pair of indefinite almost Hermitian structures which are isotropic Kähler but neither complex nor symplectic.

It is clear that $\{e_i\}_{i=1}^4$ is an orthonormal $(++--)$-basis. We accomplish the introduction of an $(H, G)$-structure on $M$ by

$$
J_1 := J'; J_2 : J_2 e_1 = e_3, J_2 e_2 = e_4, J_2 e_3 = -e_1, J_2 e_4 = -e_2; J_3 := J_1 J_2.
$$

By direct computations we verify that the constructed manifold is an $(H, G)$-manifold and it is isotropic hyper-Kählerian but not Kählerian and not integrable with non-vanishing Lie forms with respect to any $J_\alpha (\alpha = 1, 2, 3)$.

**Remark.** If we define $J_1$ as $J$ instead of $J'$ then the kind of example is not changed. So we receive a pair of almost $(H, G)$-structures corresponding to the given almost complex structures.

The non-zero components of the curvature tensor $R$ and the basic linear invariant of the almost Hermitian manifold $(M, J_1, g)$ are given in [4] by

$$
R_{1221} = \frac{3}{4}, R_{1331} = -R_{2142} = -R_{2442} = -R_{3143} = R_{3443} = \frac{1}{4}, R_{2332} = 1; \\
\|F_1\|^2 = 0, \quad \|N_1\|^2 = 8, \quad \tau = 0, \quad \tau_1^* = -2,
$$
where the following denotations are used for $\varepsilon_a = \|e_a\|^2$

$$\|F_1\|^2 = \|\nabla \Phi\|^2 = \sum_{a,b,c=1}^4 \varepsilon_a \varepsilon_b \varepsilon_c F_1(e_a, e_b, e_c)^2,$$

$$\|N_1\|^2 = \sum_{a,b=1}^4 \varepsilon_a \varepsilon_b \|N_1(e_a, e_b)\|^2,$$

$$\tau = \sum_{a,b=1}^4 \varepsilon_a \varepsilon_b R(e_a, e_b, e_b, e_a), \quad \tau^*_1 = \frac{1}{2} \sum_{a,b=1}^4 \varepsilon_a \varepsilon_b R(e_a, J_1 e_a, e_b, J_1 e_b).$$

We get the corresponding linear invariants with respect to $J_2$ and $J_3$:

$$\|F_2\|^2 = 0, \quad \|N_2\|^2 = 0, \quad \tau^*_2 = 0;$$

$$\|F_3\|^2 = 0, \quad \|N_3\|^2 = -8, \quad \tau^*_3 = 0,$$

where $\tau^*_\alpha = \sum_{a,b=1}^4 \varepsilon_a \varepsilon_b R(e_a, e_b, J_\alpha e_b, e_a); \alpha = 2, 3$.

### 2.2. Double isotropic hyper-Kählerian structures which are non-integrable but symplectic

Now we consider the same Engel manifold $(M = \mathbb{R}^4, D)$ but let the pseudo-Riemannian metric and the pair of almost complex structures be defined by other way: [4]

$$g = (dx^1)^2 - \{1 - (x^1)^2 + (x^3)^2\}(dx^2)^2 + (dx^3)^2$$

$$- (dx^4)^2 - 2x^1 dx^2 dx^3 + 2x^3 dx^2 dx^4,$$

$$J : J e_1 = e_3, \quad J e_2 = e_4, \quad J e_3 = -e_1, \quad J e_4 = -e_2,$$

$$J' : J' e_1 = e_3, \quad J' e_2 = -e_4, \quad J' e_3 = -e_1, \quad J' e_4 = e_2.$$

In this case $\{e_i\}_{i=1}^4$ is an orthonormal basis of type $(-+++)$.

It is shown that $(M, J, g)$ and $(M, J', g)$ are a pair of isotropic Kähler almost Kähler manifolds with vanishing linear invariants.

We accomplish the introduced almost complex structures to almost hypercomplex structures on $M$ by using the following way: we set the given $J$ (resp. $J'$) as $J_1$ (resp. $J'_1$), then we introduce $J_2$ (resp. $J'_2$) by

$$J_2 : J_2 e_1 = e_2, \quad J_2 e_2 = -e_1, \quad J_2 e_3 = -e_4, \quad J_2 e_4 = e_3,$$

$$J'_2 : J'_2 e_1 = e_2, \quad J'_2 e_2 = -e_1, \quad J'_2 e_3 = e_4, \quad J'_2 e_4 = -e_3,$$  \hspace{1cm} (5)

and finally we set $J_3 := J_1 J_2$ (resp. $J'_3 := J'_1 J'_2$).

It is easy to check that $H = (J_\alpha)$ and $H' = (J'_\alpha)$ together with $g$ generate a pair of almost hypercomplex pseudo-Hermitian structures on $M$. 

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We characterize the both received \((H,G)\)-manifolds as isotropic hyper-Kähler but not Kähler manifolds and not integrable manifolds with non-vanishing Lie forms with respect to any \(J_\alpha\). Moreover, they have the following linear invariants:

\[
\|N_1\|^2 = 0, \quad \|N_2\|^2 = -\|N_3\|^2 = 8, \quad \|F_\alpha\|^2 = 0, \quad \tau = \tau_\alpha^* = 0 (\alpha = 1, 2, 3).
\]

3. Real spaces with almost \((H,G)\)-structure

3.1. Real semi-space with almost \((H,G)\)-structure

Let us consider the real semi-space \(\mathbb{R}^4_+ = \{(x^1, x^2, x^3, x^4), \ x^i \in \mathbb{R}, \ x^1 > 0\}\) with the basis given by \(\{e_1 = x^1 \frac{\partial}{\partial x^1}, \ e_2 = x^1 \frac{\partial}{\partial x^2}, \ e_3 = x^1 \frac{\partial}{\partial x^3}, \ e_4 = x^1 \frac{\partial}{\partial x^4}\}\). It is clear that this basis is orthonormal of type \((++--)\) with respect to the pseudo-Riemannian metric \(g = \{(dx^1)^2 + (dx^2)^2 - (dx^3)^2 - (dx^4)^2\}/(x^1)^2\).

We introduce an almost hypercomplex structure \(H = (J_\alpha)\) as follows

\[
J_1: \quad J_1 e_1 = e_2, \quad J_1 e_2 = -e_1, \quad J_1 e_3 = e_4, \quad J_1 e_4 = -e_3;
\]

\[
J_2: \quad J_2 e_1 = e_3, \quad J_2 e_2 = -e_4, \quad J_2 e_3 = -e_1, \quad J_2 e_4 = e_2; \quad J_3 = J_1 J_2
\]

and we check that \(H\) and \(g\) generates an almost \((H,G)\)-structure on \(\mathbb{R}^4_+\).

We verify immediately that \(H\) is integrable and the obtained hypercomplex pseudo-Hermitian manifold \((\mathbb{R}^4_+, H, G)\) belongs to the class \(\mathcal{W} = \bigcap_\alpha \mathcal{W}(J_\alpha)\) but it is not isotropic Kählerian with respect to \(J_\alpha\) \((\alpha = 1, 2, 3)\).

By direct computations we obtain for the curvature tensor that \(R = -\tau_1\), i.e. the manifold has constant sectional curvatures \(k = -1\) and it is an Einstein manifold. Moreover, the linear invariants are

\[
\|N_3\|^2 = 0, \quad 2\|F_1\|^2 = 4\|\theta_1\|^2 = -\|F_\beta\|^2 = -\|\theta_\beta\|^2 = 16,
\]

\[
\tau = -3, \quad \tau_1^* = -12, \quad \tau_\beta^* = 0,
\]

where \(\alpha = 1, 2, 3; \beta = 2, 3;\) and \((\mathbb{R}^4_+, H, G)\) is conformally equivalent to a pseudo-hyper-Kähler manifold by the change \(\tilde{g} = (x^1)^2 g\).

3.2. Real quarter-space with almost \((H,G)\)-structure

Let the real quarter-space

\[
M = \mathbb{R}^2_+ \times \mathbb{R}^2_+ = \{(x^1, x^2, x^3, x^4), \ x^i \in \mathbb{R}, \ x^1 > 0, \ x^3 > 0\}
\]

be equipped with a pseudo-Riemannian metric

\[
g = \frac{1}{(x^1)^2} \left\{(dx^1)^2 + (dx^2)^2\right\} - \frac{1}{(x^3)^2} \left\{(dx^3)^2 + (dx^4)^2\right\}.
\]
Then the basis \( \{ e_1 = x^1 \frac{\partial}{\partial x^1}, e_2 = x^1 \frac{\partial}{\partial x^2}, e_3 = x^3 \frac{\partial}{\partial x^3}, e_4 = x^3 \frac{\partial}{\partial x^4} \} \) is an orthonormal one of type \((+++--)\). We introduce an almost hypercomplex structure \( H = (J_\alpha) \) \((\alpha = 1, 2, 3)\) as in the previous example by (6).

The received almost \((H,G)\)-manifold is a \( \mathcal{K}(J_1) \)-manifold and an isotropic hyper-Kähler manifold. As a corollary, \( N_1 = 0, F_1 = 0, \theta_1 = 0 \) and hence \( \| N_1 \|^2 = \| F_1 \|^2 = \| \theta_1 \|^2 = 0 \). For the \( J_\alpha \) \((\alpha = 2, 3)\) the Nijenhuis tensors \( N_\alpha \), the tensors \( F_\alpha \), and the Lie forms \( \theta_\alpha \) are non-zero (therefore \( H \) is not integrable), but the linear invariants \( \| N_\alpha \|^2, \| F_\alpha \|^2 \) and \( \| \theta_\alpha \|^2 \) vanish.

The non-zero components of the curvature tensor are given by \( R_{1221} = -R_{3443} = -1 \). For the Ricci tensor we have \( \rho_{ii} = -1 \) \((i = 1, \ldots, 4)\). Therefore the basic non-zero sectional curvatures are \( k(e_1, e_2) = -k(e_3, e_4) = -1 \) and the scalar curvatures \( \tau, \tau^*_\alpha \) \((\alpha = 1, 2, 3)\) are zero.

4. **Real pseudo-hyper-cylinder with almost \((H,G)\)-structure**

Let \( \mathbb{R}^5 \) be a pseudo-Euclidean real space with an inner product \( \langle \cdot, \cdot \rangle \) of signature \((+++-+)\). Let us consider a pseudo-hyper-cylinder defined by

\[
S : (z^2)^2 + (z^3)^2 - (z^4)^2 - (z^5)^2 = 1,
\]

where \( Z(z^1, z^2, z^3, z^4, z^5) \) is the positional vector at \( p \in S \). We use the following parametrization of \( S \) in the local coordinates \((u^1, u^2, u^3, u^4)\) of \( p \):

\[
Z = Z(u^1, \cosh u^4 \cos u^2, \cosh u^4 \sin u^2, \sinh u^4 \cos u^3, \sinh u^4 \sin u^3).
\]

We consider a manifold on the surface \( \tilde{S} = S \setminus \{ u^4 = 0 \} \). Then the basis \( \{ e_1 = \partial_1, e_2 = \frac{1}{\cosh u^4} \partial_2, e_3 = \frac{1}{\sinh u^4} \partial_3, e_4 = \partial_4 \} \) of \( T_p \tilde{S} \) at \( p \in \tilde{S} \) is an orthonormal basis of type \((+++-)\) with respect to the restriction \( g \) of \( \langle \cdot, \cdot \rangle \) on \( \tilde{S} \). Here and further \( \partial_i \) denotes \( \frac{\partial Z}{\partial u^i} \) for \( i = 1, \ldots, 4 \).

We introduce an almost hypercomplex structure by the following way

\[
\begin{align*}
J_1 & : J_1 e_1 = e_2, J_1 e_2 = -e_1, J_1 e_3 = -e_4, J_1 e_4 = e_3; \\
J_2 & : J_2 e_1 = e_3, J_2 e_2 = e_4, J_2 e_3 = -e_1, J_2 e_4 = -e_2; \quad J_3 = J_1 J_2
\end{align*}
\]

and check that \( H = (J_\alpha) \) and the pseudo-Riemannian metric \( g \) generate an almost \((H,G)\)-structure on \( \tilde{S} \).

By straightforward calculations with respect to \( \{ e_i \} \) \((i = 1, \ldots, 4)\) we receive that the almost \((H,G)\)-manifold \( \tilde{S} \) is not integrable with non-zero
Lie forms regarding any $J_\alpha$ of $H$ and it has the following linear invariants:

$$
\|N_1\|^2 = 2\|F_1\|^2 = 2\|\nabla J_1\|^2 = 8\|\theta_1\|^2 = -8\tanh^2 u^4;
$$

$$
\|N_2\|^2 = -8\coth^2 u^4, \quad \|\theta_2\|^2 = (2\tanh u^4 + \coth u^4)^2;
$$

$$
\|F_2\|^2 = \|\nabla J_2\|^2 = 4(2\tanh^2 u^4 + \coth^2 u^4);
$$

$$
\|N_3\|^2 = -8(\tanh u^4 - \coth u^4)^2, \quad \|\theta_3\|^2 = (\tanh u^4 + \coth u^4)^2;
$$

$$
\|F_3\|^2 = \|\nabla J_3\|^2 = 4(\tanh^2 u^4 + \coth^2 u^4).
$$

The non-zero components of the curvature tensor and the corresponding Ricci tensor and scalar curvatures are given by

$$
R_{2332} = -1, \quad R_{2442} = -\tanh^2 u^4, \quad R_{3443} = \coth^2 u^4
$$

$$
\rho_{22} = 1 + \tanh^2 u^4, \quad \rho_{33} = -1 - \coth^2 u^4, \quad \rho_{44} = -\tanh^2 u^4 - \coth^2 u^4
$$

$$
\tau = 2(1 + \tanh^2 u^4 + \coth^2 u^4), \quad \tau^*_\alpha = 0, \quad \alpha = 1, 2, 3.
$$

Hence $(\tilde{S}, H, G)$ has zero associated scalar curvatures and $H$ is a non-integrable structure on it.

5. Complex surfaces with almost $(H, G)$-structure

The following three examples concern several surfaces $S^2_\mathcal{C}$ in a 3-dimensional complex Euclidean space $(\mathbb{C}^3, \langle \cdot, \cdot \rangle)$. It is well known that the decomplexification of $\mathbb{C}^3$ to $\mathbb{R}^6$ using the $i$-splitting, i.e. $(Z^1, Z^2, Z^3) \in \mathbb{C}^3$, where $Z^k = x^k + iy^k$ ($x^k, y^k \in \mathbb{R}$), is identified with $(x^1, x^2, x^3, y^1, y^2, y^3) \in \mathbb{R}^6$. Then the multiplying by $i$ in $\mathbb{C}^3$ induces the standard complex structure $J_0$ in $\mathbb{R}^6$. The real and the opposite imaginary parts of the complex Euclidean inner product $\Re\langle \cdot, \cdot \rangle$ and $-\Im\langle \cdot, \cdot \rangle$ are the standard skew-Hermitian metrics $g_0$ and $\bar{g}_0 = g_0(\cdot, J_0 \cdot)$ in $(\mathbb{R}^6, J_0, g_0, \bar{g}_0)$, respectively. So, the natural decomplexification of an $n$-dimensional complex Euclidean space is the $2n$-dimensional real space with a complex skew-Hermitian structure $(J_0, g_0, \bar{g}_0)$.

5.1. Complex cylinder with almost $(H, G)$-structure

Let $S^2_\mathcal{C}$ be the cylinder in $(\mathbb{C}^3, \langle \cdot, \cdot \rangle)$ defined by $(Z^1)^2 + (Z^2)^2 = 1$. Let the corresponding surface $S^4$ in $(\mathbb{R}^6, J_0, g_0, \bar{g}_0)$ be parameterized as follows

$$
S^4: Z = Z(\cos u^1 \cosh u^3, \sin u^1 \cosh u^3, u^2, \sin u^1 \sinh u^3, -\cos u^1 \sinh u^3, u^4).
$$
Then the local basis \( \{ \partial_1, \ldots, \partial_4 \} \) is orthonormal of type \((++--)\) and it generates the metric \( g = (du^1)^2 + (du^2)^2 - (du^3)^2 - (du^4)^2 \) on \( S^4 \). The almost hypercomplex structure \( H \) is determined as usually by (1). It is easy to verify that the received \((H,G)\)-manifold is a flat pseudo-hyper-Kähler manifold.

5.2. Complex cone with almost \((H,G)\)-structure

Now let \( S'_C \) be the complex cone in \((\mathbb{C}^3, \langle \cdot, \cdot \rangle)\) determined by the equation \( (Z^1)^2 + (Z^2)^2 - (Z^3)^2 = 0 \). Then we consider the corresponding 4-dimensional surface \( S \) in \((\mathbb{R}^6, J_0, g_0, \tilde{g}_0)\) by the following parametrization of \( Z \):

\[
(u^1 \cos u^2 \cosh u^4 - u^3 \sin u^2 \sinh u^4, u^1 \sin u^2 \cosh u^4 + u^3 \cos u^2 \sinh u^4, u^1, u^1 \sin u^2 \sinh u^4 + u^3 \cos u^2 \cosh u^4, -u^1 \cos u^2 \sinh u^4 + u^3 \sin u^2 \cosh u^4, u^3). 
\]

Further we consider a manifold on \( \tilde{S} = S \setminus \{0,0,0,0,0,0\} \), i.e. we exclude the plane \( u^1 = u^3 = 0 \) from the domain of \( S \) which maps the origin. Then the derived metric \( g \) on \( \tilde{S} \) has the following non-zero components regarding \( \{ \partial_k \} \):

\[
g_{11} = -g_{33} = 2, \quad g_{22} = -g_{44} = (u^1)^2 - (u^3)^2, \quad g_{24} = g_{42} = 2u^1u^3.
\]

We receive the following orthonormal basis of signature \((++--)\):

\[
\left\{ e_1 = \frac{1}{\sqrt{2}} \partial_1, \quad e_2 = \lambda \partial_2 + \mu \partial_4, \quad e_3 = \frac{1}{\sqrt{2}} \partial_3, \quad e_4 = -\mu \partial_2 + \lambda \partial_4 \right\},
\]

where \( \lambda = u^1/(u^1)^2 + (u^3)^2 \), \( \mu = u^3/(u^1)^2 + (u^3)^2 \). We introduce a structure \( H \) as in (1). It is easy to check that \( H \) and \( g \) generate an almost \((H,G)\)-structure on \( \tilde{S} \). By direct computations we get that the received \((H,G)\)-manifold is a flat hypercomplex manifold which is Kählerian with respect to \( J_1 \) but it does not belong to \( \mathcal{W}(J_2) \) or \( \mathcal{W}(J_3) \) and the Lie forms \( \theta_2 \) and \( \theta_3 \) are non-zero. The corresponding linear invariants are given by

\[
\|F_2\|^2 = \|\nabla J_2\|^2 = 2\|\theta_2\|^2 = 16 \{\mu^2 - \lambda^2\}, \quad \|F_3\|^2 = \|\nabla J_3\|^2 = 2\|\theta_3\|^2 = 4 \{\mu^2 - \lambda^2\}.
\]

5.3. Complex sphere with almost \((H,G)\)-structure

In this case let \( S'_C \) be the unit sphere in \((\mathbb{C}^3, \langle \cdot, \cdot \rangle)\) defined by \( (Z^1)^2 + (Z^2)^2 + (Z^3)^2 = 1 \). After that we consider the corresponding 4-surface \( S \) in
\((\mathbb{R}^6, J_0, g_0, \tilde{g}_0)\) with the following parametrization of \(Z(x_1, x_2, x_3, y_1, y_2, y_3)\):

\[
\begin{align*}
    x_1 & = \cos u^1 \cos u^2 \cosh u^3 \sinh u^4 - \sin u^1 \sin u^2 \sinh u^3 \sinh u^4, \\
    x_2 & = \cos u^1 \sin u^2 \cosh u^3 \cos u^4 + \sin u^1 \cos u^2 \sinh u^3 \sinh u^4, \\
    x_3 & = \sin u^1 \cosh u^3, \\
    y_1 & = \cos u^1 \sin u^2 \cosh u^3 \sinh u^4 + \sin u^1 \cos u^2 \sinh u^3 \cosh u^4, \\
    y_2 & = -\cos u^1 \cos u^2 \cosh u^3 \sinh u^4 + \sin u^1 \sin u^2 \sinh u^3 \cosh u^4, \\
    y_3 & = -\cos u^1 \sinh u^3.
\end{align*}
\]

Further we consider a manifold on \(\tilde{S} = S \setminus \{0, 0, \pm 1, 0, 0, 0\}\), i.e. we exclude the set \(u^1 = \pm \pi/2, u^3 = 0\) from the domain \((-\pi, \pi)^2 \times \mathbb{R}^2\) of \(S\) which maps the pair of "poles".

The induced metric on \(\tilde{S}\) has the following non-zero local components:

\[
\begin{align*}
   g_{11} = -g_{33} & = 1, \\
   g_{22} = -g_{44} & = \cos^2 u^1 \cosh^2 u^3 - \sin^2 u^1 \sinh^2 u^3, \\
   g_{24} = g_{42} & = 2 \sin u^1 \cos u^1 \sinh u^3 \cosh u^3.
\end{align*}
\]

Further we use the following orthonormal basis of signature \((+++--)\):

\[
\{ e_1 = \partial_1, \ e_2 = \lambda \partial_2 + \mu \partial_4, \ e_3 = \partial_3, \ e_4 = -\mu \partial_2 + \lambda \partial_4 \},
\]

where \(\lambda = \frac{\cos u^1 \cosh u^3}{\cos^2 u^1 + \sinh^2 u^3}, \mu = \frac{\sin u^1 \sinh u^3}{\cos^2 u^1 + \sinh^2 u^3}\). We introduce a structure \(H\) as in (1) and we verify that \(H\) and \(g\) generate an almost \((H, G)\)-structure on \(\tilde{S}\). By direct computations we get that \((\tilde{S}, H, G)\) is a \(\mathcal{K}(J_2)\)-manifold of pointwise constant totally real sectional curvatures

\[
\nu = \frac{\sinh^2 u^3 - \sin^2 u^1}{4(\cos^2 u^1 + \sinh^2 u^3)^4}, \quad \nu_2^* = \frac{\sin 2u^1 \sinh 2u^3}{2(\cos^2 u^1 + \sinh^2 u^3)^4},
\]

where \(\nu := \frac{R(x,y,y,x)}{\pi_1(x,y,y,x)} \), \(\nu_2^* := \frac{R(x,y,y,J_2 x)}{\pi_1(x,y,y,x)}\) for a basis \(\{x, y\}\) of any non-degenerate totally real section \(\sigma\) (i.e. \(\sigma \perp J_2 \sigma\)). \((\tilde{S}, J_2, g)\) is an almost Einstein manifold since its Ricci tensor is \(\rho = 2(\nu g - \nu_2^* g_2)\). But, the Nijenhuis tensors and the Lie forms corresponding to other two almost complex structures \(J_1\) and \(J_3\) are non-zero. Beside that, we receive the following linear invariants:

\[
\tau = 8\nu, \quad \tau_1^* = 0, \quad \tau_2^* = 8\nu_2^*, \quad \tau_3^* = 0, \\
\|N_1\|^2 = 2\|\nabla J_1\|^2 = 8\|\theta_1\|^2 = -32\nu, \ -\|N_3\|^2 = 2\|\nabla J_3\|^2 = 8\|\theta_3\|^2 = 32\nu.
\]
6. Lie groups with almost \((H, G)\)-structure

The next two examples are inspired from an example of a locally flat almost Hermitian surface constructed in [9]. Let \(\mathcal{L}\) be a connected Lie subgroup of \(GL(4, \mathbb{R})\) consisting of matrices with the following non-zero entries

\[
\begin{align*}
a_{11} &= a_{22} = \cos u_1, & a_{12} &= -a_{21} = \sin u_1, \\
a_{13} &= u_2, & a_{23} &= u_3, & a_{33} &= 1, & a_{44} &= \exp u_4
\end{align*}
\]

for arbitrary \(u^1, u^2, u^3, u^4 \in \mathbb{R}\).

The Lie algebra of \(\mathcal{L}\) is isomorphic to the Lie subalgebra of \(gl(4; \mathbb{R})\) generated by the matrices \(X_1, X_2, X_3, X_4\) with the following non-zero entries:

\[
(X_1)_{13} = (X_2)_{12} = -(X_2)_{21} = (X_3)_{23} = (X_4)_{44} = 1.
\]

6.1. A Lie group as a complex manifold but non-hypercomplex one

For the first recent example let us substitute the following pseudo-Riemannian \(g\) for the metric on \(\mathcal{L}\) used there: \(g(X_i, X_j) = \varepsilon_i \delta_{ij}\), where \(1 \leq i, j \leq 4; \varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -\varepsilon_4 = 1\). Further we introduce an \(\mathcal{L}\)-invariant almost hypercomplex structure \(H\) on \(\mathcal{L}\) as in (1). Then, there is generated an almost \((H, G)\)-structure on \(\mathcal{L}\) and the received manifold is complex with respect to \(J_2\) but non-hypercomplex and the Lie forms do not vanish. The non-zero components of the curvature tensor \(R\) is determined by \(R_{1221} = R_{1331} = -R_{2332} = 1\) and the linear invariants are the following:

\[
\begin{align*}
\|N_1\|^2 &= 2\|\nabla J_1\|^2 = 8\|\theta_1\|^2 = -\|\nabla J_2\|^2 = -2\|\theta_2\|^2 = \|N_3\|^2 = -8, \\
\|\nabla J_3\|^2 &= 12\|\theta_3\|^2 = 12, & \tau &= -\tau_1^* = 2, & \tau_2^* = \tau_3^* = 0.
\end{align*}
\]

6.2. A Lie group as a flat Kähler manifold but non-hypercomplex one

For the second example we use the following pseudo-Riemannian \(g\) on \(\mathcal{L}\):

\[
\begin{align*}
g(X_i, X_j) = \varepsilon_i \delta_{ij}, & \quad 1 \leq i, j \leq 4; \varepsilon_1 = -\varepsilon_2 = \varepsilon_3 = -\varepsilon_4 = 1. \quad \text{We actually substitute only the type of the signature:} \quad \text{(+-+-) for} \quad \text{(++--);} \\
& \quad \text{of the basis} \quad \{X_1, X_2, X_3, X_4\}. \quad \text{Then we introduce} \quad H \quad \text{by the following different way}:
\end{align*}
\]

\[
\begin{align*}
J_1 X_1 &= X_3, & J_1 X_2 &= X_4, & J_1 X_3 &= -X_1, & J_1 X_4 &= -X_2, \\
J_2 X_1 &= -X_4, & J_2 X_2 &= X_3, & J_2 X_3 &= -X_2, & J_2 X_4 &= X_1, & J_3 &= J_1 J_2.
\end{align*}
\]
Therefore we obtain that the constructed \((H,G)\)-manifold is flat and Kählerian with respect to \(J_1\) but regarding \(J_2\) and \(J_3\) it is not complex and the structural tensors have the form
\[
F_2(X, Y, Z) = -\theta_2(J_3X)g(Y, J_3Z),
\]
\[
F_3(X, Y, Z) = -\theta_3(J_2)g(Y, J_2Z).
\]
The non-zero linear invariants for \(\beta = 2, 3\) are the following:
\[-\|N_\beta\|^2 = 2\|\nabla J_\beta\|^2 = 2\|F_\beta\|^2 = 8\|\theta_\beta\|^2 = 8.
\]

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References

10. J. Wolf, *Spaces of constant curvature*, University of California, Berkley, California, 1972.
The Navier-Stokes equations are considered by the use of the method of Lagrangians with covariant derivatives (MLCD) over spaces with affine connections and metrics. It is shown that the Euler-Lagrange equations appear as sufficient conditions for the existence of solutions of the Navier-Stokes equations over (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion. By means of the corresponding \((n-1)+1\) projective formalism the Navier-Stokes equations for radial and tangential accelerations are found.

1. Introduction

By the use of the method of Lagrangians with covariant derivatives (MLCD) [1], [2] the different energy-momentum tensors and the covariant Noether's identities for a field theory as well as for a theory of continuous media can be found. On the basis of the \((n-1)+1\) projective formalism and by the use of the notion of covariant divergency of a tensor of second rank the corresponding covariant divergencies of the energy-momentum tensors could be found. They lead to Navier-Stokes' identity and to the corresponding generalized Navier-Stokes' equations.

Let the following structure

\[(M, V, g, \Gamma, P)\]

be given, where

(i) \(M\) is a differentiable manifold with \(\text{dim } M = n\),
(ii) \(V = V^A_B \cdot e_A \otimes e^B \in \otimes^k l(M)\) are tensor fields with contravariant rank \(k\) and covariant rank \(l\) over \(M\), \(A\) and \(B\) are collective indices,
(iii) \(g \in \otimes^{sym2}(M)\) is a covariant symmetric metric tensor field over \(M\),
(iv) \( \Gamma \) is a contravariant affine connection, \( P \) is a covariant affine connection related to the covariant differential operator along a basis vector field \( \partial_i \) or \( e_i \) in a co-ordinate or non-co-ordinate basis respectively

\[
\nabla_{\partial_i} V = V^{A}_{B:i} \cdot \partial_A \otimes dx^B,
\]

\[
V^{A}_{B:i} = V^{A}_{B,i} + \Gamma^A_{Ci} \cdot V^{C}_{B} + P^{D}_{Bi} \cdot V^{A}_{D},
\]

\[
V^{A}_{B,i} = \frac{\partial V^{A}_{B}}{\partial x^i}.
\]

A Lagrangian density \( L \) can be considered in two different ways as a tensor density of rank 0 with the weight \( q = 1/2 \), depending on tensor field's components and their first and second covariant derivatives

(i) As a tensor density \( L \) of type 1, depending on tensor field's components, their first (and second) partial derivatives, (and the components of contravariant and covariant affine connections), i.e.

\[
L = \sqrt{-d_g} \cdot L(g_{ij}, g_{ij,k}, g_{ij,k,l}, V^A_B, V^A_B,i, V^A_B,i,j, \Gamma^i_{jk}, \ldots, P^i_{jk}, \ldots),
\]

where \( L \) is a Lagrangian invariant,

\[
d_g = \det(g_{ij}) < 0, \quad g = g_{ij} \cdot dx^i \cdot dx^j,
\]

\[
dx^i \cdot dx^j = \frac{1}{2} \cdot (dx^i \otimes dx^j + dx^j \otimes dx^i),
\]

\[
V^A_B,i,j = \frac{\partial V^A_B}{\partial x^j} \frac{\partial}{\partial x^i}.
\]

The method using a Lagrangian density of type 1 is called Method of Lagrangians with partial derivatives (MLPD).

(ii) As a tensor density \( L \) of type 2, depending on tensor field's components and their first (and second) covariant derivatives, i.e.

\[
L = \sqrt{-d_g} \cdot L(g_{ij}, g_{ij,k}, g_{ij,k,l}, V^A_B, V^A_B,i, V^A_B,B;i,j).
\]

By the use of the variation operator \( \delta \), commuting with the covariant differential operator

\[
\delta \circ \nabla_\xi = \nabla_\xi \circ \delta + \nabla_\delta \xi, \quad \xi \in T(M), \ T(M) = \cup_{x \in M} T_x(M),
\]

we could find the Euler-Lagrange equations.

By the use of the Lie variation operator (identical with the Lie differential operator) \( \mathcal{L}_\xi \), we could find the corresponding energy-momentum tensors.

The method using a Lagrangian density of type 2 is called Method of Lagrangians with covariant derivatives (MLCD) \[3\].

The way of obtaining the Navier-Stokes equations could be given in the following rough scheme
2. Navier-Stokes' identities and Navier-Stokes' equations

If we consider the projections of the first Noether identity along a non-null (non-isotropic) vector field \( u \) and its corresponding contravariant and covariant projective metrics \( h^u \) and \( h_u \) we will find the first and second Navier-Stokes identities.

From the Noether identities in the form

\[
\bar{g}(F) + \bar{g}(\delta \theta) \equiv 0, \quad \text{(first covariant Noether's identity),}
\]
\[
(\theta)\bar{g} - (\theta^T)\bar{g} \equiv (Q)\bar{g}, \quad \text{(second covariant Noether's identity),}
\]

we can find the projections of the first Noether identity along a contravariant non-null vector field \( u = u^i \cdot \partial_i \) and orthogonal to \( u \).

Since

\[
g(\bar{g}(F), u) = g_{ik} \cdot \bar{g}_{kl} \cdot F_l \cdot u^i = g_{i}^l \cdot F_l \cdot u^i = F_i \cdot u^i = F(u), \quad \text{(1)}
\]
\[
g(\bar{g}(\delta \theta), u) = (\delta \theta)(u), \quad F = F_k \cdot dx^k, \quad \text{(2)}
\]
we obtain the first Navier-Stokes identity in the form
\[ F(u) + (\delta \theta)(u) = 0. \]  

(3)

By the use of the relation
\[ \bar{g}[h_u(\bar{g})(F)] = \bar{g}(h_u[\bar{g}(F)]) = h^u(F), \quad \bar{g}(h_u)\bar{g} = h^u, \]  
\[ \bar{g}[h_u(\bar{g})(\delta \theta)] = \bar{g}(h_u[\bar{g}(\delta \theta)]) = h^u(\delta \theta), \]  
the first Noether identity could be written in the forms
\[ h_u[\bar{g}(F)] + h_u[\bar{g}(\delta \theta)] = 0, \]  
\[ h^u(F) + h^u(\delta \theta) = 0. \]  

(4) (5) (6) (7)

The last two forms of the first Noether identity represent the second Navier-Stokes identity.

If the projection \( h^u(F), \) orthogonal to \( u, \) of the volume force \( F \) is equal to zero, we obtain the generalized Navier-Stokes equation in the form
\[ h^u(\delta \theta) = 0, \]  

(8)
or in the form
\[ h_u[\bar{g}(\delta \theta)] = 0. \]  

(9)

Let us now find the explicit form of the first and second Navier-Stokes identities and the explicit form of the generalized Navier-Stokes equation. For this purpose we can use the explicit form of the covariant divergency \( \delta \theta \) of the generalized canonical energy-momentum tensor \( \theta. \)

(a) The first Navier-Stokes identity follows in the form
\[
F(u) + (\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot g(a, u) \\
+ e \cdot [u(\rho_\theta + \frac{1}{e} \cdot L \cdot k) + (\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot \delta u + \delta \theta s] \\
- (KrL)(u) - L \cdot (\delta K r)(u) + g(\nabla_u \theta \pi, u) + g(\nabla_\sigma u, u) \\
+ (\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot (\nabla_u g)(u, u) + (\nabla_u g)(\theta \pi, u) + (\nabla_\sigma g)(u, u) \\
+ [\delta((\theta \xi) g)](u) \equiv 0.
\]  

(10)

Since
\[
g(u, a) = \pm l_u \cdot \frac{dl_u}{d\tau} - \frac{1}{2} \cdot (\nabla_u g)(u, u) \\
= \frac{1}{2} \cdot [\frac{d}{d\tau}(\pm l_u^2) - (\nabla_u g)(u, u)],
\]
the first Navier-Stokes identity could be interpreted as a definition for the change of $I_u^2$ along the world line of the observer. The length of the non-isotropic contravariant vector $u$ is interpreted as the velocity of a signal emitted or received by the observer \[6\]. On this basis, the first Navier-Stokes identity is related to the change of the velocity of signals emitted or received by an observer moving in a continuous media or in a fluid.

(b) The second Navier-Stokes identity can be found in the form

$$h_u[\bar{g}(F)] + h_u[\bar{g}(\delta \theta)]$$

$$\equiv (\rho \theta + \frac{1}{e} \cdot L \cdot k) \cdot h_u(a)$$

$$- h_u[\bar{g}(KrL) - L \cdot h_u[\bar{g}(\delta Kr)] + \delta u \cdot h_u(\theta \pi)$$

$$+ h_u(\nabla_u \theta \pi) + h_u(\nabla_{\bar{g}}u)$$

$$+ (\rho \theta + \frac{1}{e} \cdot L \cdot k) \cdot h_u[\bar{g}(\nabla_u g)(u)] + h_u[\bar{g}(\nabla_u g)(\theta \pi)]$$

$$+ h_u[\bar{g}(\nabla_{\bar{g}}g)(u)] + h_u[\bar{g}(\delta (\bar{S} g))] + h_u[\bar{g}(F)] = 0. \quad (11)$$

(c) The generalized Navier-Stokes equation $h_u[\bar{g}(\delta \theta)] = 0$ follows from the second Navier-Stokes identity under the condition $h_u[\bar{g}(F)] = 0$ or under the condition $F = 0$

$$(\rho \theta + \frac{1}{e} \cdot L \cdot k) \cdot h_u(a)$$

$$- h_u[\bar{g}(KrL) - L \cdot h_u[\bar{g}(\delta Kr)] + \delta u \cdot h_u(\theta \pi)$$

$$+ h_u(\nabla_u \theta \pi) + h_u(\nabla_{\bar{g}}u)$$

$$+ (\rho \theta + \frac{1}{e} \cdot L \cdot k) \cdot h_u[\bar{g}(\nabla_u g)(u)] + h_u[\bar{g}(\nabla_u g)(\theta \pi)]$$

$$+ h_u[\bar{g}(\nabla_{\bar{g}}g)(u)] + h_u[\bar{g}(\delta (\bar{S} g))] = 0, \quad (12)$$

$$h_u(a) = g(a) - \frac{1}{e} \cdot g(u, a) \cdot g(u). \quad (13)$$

The second Navier-Stokes identity could be considered as a definition for the density of the inner force. If the density of the inner force is equal to zero, i.e. if $\bar{F} = \bar{g}(F) = 0$, then the covariant divergency, $\delta \bar{\theta} = \bar{g}(\delta \theta)$ of the generalized canonical energy-momentum tensor $\theta$ is also equal to zero, i.e. $\delta \bar{\theta} = \bar{g}(\delta \theta) = 0$. Then the orthogonal to the contravariant vector field $u$ projection of the second Navier-Stokes identity leads to the equations

$$\bar{g}[h_u(\bar{F})] = 0 \quad \Leftrightarrow \quad \bar{g}[h_u(\delta \bar{\theta})] = 0.$$

The last equation is the Navier-Stokes equation in spaces with affine connections and metrics. Now, we can prove the following proposition:
Proposition 1. The necessary and sufficient condition for the existence of the Navier-Stokes equation in a space with affine connections and metrics is the condition for the vanishing of the density of the inner force in a dynamic system described by the use of a Lagrangian invariant \( L \), interpreted as the pressure \( p \) of the system, i.e. the necessary and sufficient condition for

\[
\bar{g}[h_u(\delta \theta)] = 0
\]

is the condition

\[
\bar{g}[h_u(F)] = 0.
\]

The proof follows directly from the projective second Navier-Stokes identity \( \bar{g}[h_u(F)] + \bar{g}[h_u(\delta \theta)] \equiv 0 \).

3. Invariant projections of Navier-Stokes' equations

3.1. Navier-Stokes' equations and Euler-Lagrange's equations

Let us now consider the second Navier-Stokes identity in the form [3]

\[
\bar{g}[h_u(\bar{g}(F))] + \bar{g}[h_u(\bar{g}(\delta \theta))] \equiv 0
\]

or in the form

\[
F_\perp + \delta \theta_\perp \equiv 0, \\
F_\perp = \bar{g}[h_u(\bar{g}(F))], \quad \delta \theta_\perp = \bar{g}[h_u(\bar{g}(\delta \theta))], \\
g(u, F_\perp) = 0, \quad g(u, \delta \theta_\perp) = 0.
\]

The explicit form of the density \( F \) of the inner force could be given as [3]

\[
F = \overline{F}_i \cdot dx^i, \\
\overline{F}_i = \frac{\delta L}{\delta V^A_B} \cdot V^A_{B;i} + W_i, \\
W_i = W_i(T_{ki}^j, g_{jk;l}),
\]

where \( T_{ki}^j \) are the components of the torsion tensor (in a co-ordinate basis \( T_{ki}^j = \Gamma^j_{lk} - \Gamma^j_{kl} \)).

For (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion \( (T_{ki}^j = 0) \) the quantity \( W \) is equal to zero \( (W_i = 0) \) and the density of the inner force \( F \) has the form

\[
\overline{F}_i = \frac{\delta L}{\delta V^A_B} \cdot V^A_{B;i}
\]
If the Euler-Lagrange equations are fulfilled in (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion, i.e. if

$$\frac{\delta L}{\delta V^A{}_B} = 0,$$

then \(F = 0\) and

$$\bar{F}_i = \frac{\delta L}{\delta V^A{}_B} \cdot V^A{}_{B; i} = 0,$$

and the following propositions could be proved:

**Proposition 2.** Sufficient conditions for the existence of the Navier-Stokes equation in (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion are the Euler-Lagrange equations.

**Proposition 3.** Every contravariant vector field \(u \in T(M)\) in (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion is a solution of the Navier-Stokes equation if the Euler-Lagrange equations are fulfilled for the dynamic system, described by a given Lagrangian invariant \(L = p\) interpreted as the pressure of the system.

**Corollary.** If \(L = p = p(u^i, u^i; j; k, g_{ij}, g_{ij; k}, g_{ij; k};, V^A{}_{B}, V^A{}_{B; i}, V^A{}_{B; i; j})\) is a Lagrangian density fulfilling the Euler-Lagrange equations for \(u^i\) and \(V^A{}_{B}\) in (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion, then the contravariant non-isotropic vector field \(u\) is also a solution of the Navier-Stokes equation.

### 3.2. Representation of \(F_\perp\) and \(\delta \theta_\perp\)

Now, we can use the corresponding to a vector field \(\xi_\perp, g(u, \xi_\perp) = 0\) (orthogonal to the vector field \(u\)) projective metrics \(h_{\xi_\perp}\) and \(h^{\xi_\perp}\).

$$h_{\xi_\perp} = g - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot g(\xi_\perp) \otimes g(\xi_\perp),$$

$$h^{\xi_\perp} = \bar{g} - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp \otimes \xi_\perp.$$

The vector field \(F_\perp\) could be written in the form \([4], [5]\)

$$F_\perp = \frac{g(F_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \bar{g}[h_{\xi_\perp}(F_\perp)] = \mp g(F_\perp, n_\perp) \cdot n_\perp + \bar{g}[h_{\xi_\perp}(F_\perp)]$$

$$= F_{\perp z} + F_{\perp c}, \quad \xi_\perp = l_{\xi_\perp} \cdot n_\perp, \quad g(n_\perp, n_\perp) = \mp 1.$$
$F_{lZ}$ is the radial inner force density and $F_{lC}$ is the tangential (Coriolis) inner force density

$$F_{lZ} = \mp g(F_{l\perp}, n_{\perp}) \cdot n_{\perp}, \quad F_{lC} = \mp g[h_{\xi_{\perp}}(F_{l\perp})],$$

$$g(F_{lZ}, u) = 0, \quad g(F_{lC}, \xi_{\perp}) = 0, \quad g(F_{lC}, u) = 0.$$

The Navier-Stokes equation could now be written in the form

$$\delta \theta_{\perp} = \mp g(\delta \theta_{\perp}, n_{\perp}) \cdot n_{\perp} + \mp h_{\xi_{\perp}}(\delta \theta_{\perp}) = 0,$$

or in the forms

$$\delta \theta_{\perp} := \mp g(\delta \theta_{\perp}, n_{\perp}) \cdot n_{\perp} = 0,$$

Navier-Stokes' equation for radial accelerations,

$$\delta \theta_{\perp} := \mp h_{\xi_{\perp}}(\delta \theta_{\perp}) = 0,$$

Navier-Stokes' equation for tangential accelerations.

### 3.3. Radial projections of Navier-Stokes' equation.

**Navier-Stokes' equation for radial accelerations**

If we use the explicit form of the Navier-Stokes equation

$$\left(\rho \theta + \frac{1}{e} \cdot L \cdot k\right) \cdot a_{\perp}$$

$$- \left[\mp g(KrL)_{\perp} - L \cdot [\mp g(\delta K)_{\perp}] + \delta u \cdot \theta \pi_{\perp}\right]$$

$$+ (\nabla u \theta \pi_{\perp})_{\perp} + (\nabla \sigma g)_{\perp}$$

$$+ (\rho \theta + \frac{1}{e} \cdot L \cdot k) \cdot [\mp g(\nabla u g)(u)]_{\perp} + [\mp g(\nabla u g)(\theta \pi_{\perp})]_{\perp}$$

$$+ [\mp g(\nabla \sigma g)(u)]_{\perp} + [\mp g(\delta(\theta \sigma g))]_{\perp} = 0,$$

(14)

$$L = p,$$

(15)

and apply the projection of the Navier-Stokes equation along and orthogonal to the vector field $\xi_{\perp}$, by the use of the representation of the acceleration $a_{\perp}$ in the form

$$a_{\perp} = g(a_{\perp}, n_{\perp}) \cdot n_{\perp} + \mp h_{\xi_{\perp}}(a_{\perp}) = a_{z} + a_{c},$$

$$a_{z} = g(a_{\perp}, n_{\perp}) \cdot n_{\perp}, \quad a_{c} = \mp h_{\xi_{\perp}}(a_{\perp}),$$

where $a_{z} = g(a_{\perp}, n_{\perp}) \cdot n_{\perp} = \mp l_{a_{z}} \cdot n_{\perp}$ is the radial (centrifugal, centripetal) acceleration and $a_{c} = \mp h_{\xi_{\perp}}(a_{\perp}) = \mp l_{a_{c}} \cdot m_{\perp}$, $g(n_{\perp}, m_{\perp}) = 0$, is the tangential (Coriolis) acceleration, we could fine the explicit form of the
Navier-Stokes equation for radial (centrifugal, centripetal) accelerations in the form

\[
\begin{align*}
(\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot a_z \\
- [\bar{g}(KrL)]_{\perp z} - L \cdot [\bar{g}(\delta Kr)]_{\perp z} + \delta u \cdot \theta \pi_{\perp z} \\
+ (\nabla_u \theta \pi)_{\perp z} + (\nabla_{\theta \bar{g}} u)_{\perp z} \\
+ (\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot [\bar{g}(\nabla_u g)(u)]_{\perp z} + [\bar{g}(\nabla_u g)(\theta \pi)]_{\perp z} \\
+ [\bar{g}(\nabla_{\theta \bar{g}} g)(u)]_{\perp z} + [\bar{g}(\delta((\theta \bar{g}) g))]_{\perp z} = 0,
\end{align*}
\]

(16)

\[ L = p. \]

Special case: Perfect fluids: \( \theta \pi = 0, \theta \bar{g} = 0, \theta \bar{S} = 0, L = p. \)

\[
\begin{align*}
(\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot a_z - [\bar{g}(KrL)]_{\perp z} - L \cdot [\bar{g}(\delta Kr)]_{\perp z} \\
+ (\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot [\bar{g}(\nabla_u g)(u)]_{\perp z} = 0.
\end{align*}
\]

(18)

Special case: Perfect fluids in (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion: \( \theta \pi = 0, \theta \bar{g} = 0, \theta \bar{S} = 0, L = p, \nabla_u g = 0, \delta Kr = 0. \)

\[
(\rho_\theta + \frac{1}{e} \cdot p) \cdot a_z = [\bar{g}(Kr p)]_{\perp z}, \quad a_z = \frac{1}{\rho_\theta + \frac{1}{e} \cdot L} \cdot [\bar{g}(Kr p)]_{\perp z}.
\]

3.4. Tangential projections of Navier-Stokes' equation.

Navier-Stokes' equation for tangential accelerations

For tangential (Coriolis') accelerations the Navier-Stokes equation takes the form [4]

\[
\begin{align*}
(\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot a_c \\
- [\bar{g}(KrL)]_{\perp c} - L \cdot [\bar{g}(\delta Kr)]_{\perp c} + \delta u \cdot \theta \pi_{\perp c} \\
+ (\nabla_u \theta \pi)_{\perp c} + (\nabla_{\theta \bar{g}} u)_{\perp c} + \\
+ (\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot [\bar{g}(\nabla_u g)(u)]_{\perp c} + [\bar{g}(\nabla_u g)(\theta \pi)]_{\perp c} \\
+ [\bar{g}(\nabla_{\theta \bar{g}} g)(u)]_{\perp c} + [\bar{g}(\delta((\theta \bar{g}) g))]_{\perp c} = 0,
\end{align*}
\]

(19)

\[ L = p. \]

Special case: Perfect fluids: \( \theta \pi = 0, \theta \bar{g} = 0, \theta \bar{S} = 0, L = p. \)

\[
\begin{align*}
(\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot a_c - [\bar{g}(KrL)]_{\perp c} - L \cdot [\bar{g}(\delta Kr)]_{\perp c} \\
+ (\rho_\theta + \frac{1}{e} \cdot L \cdot k) \cdot [\bar{g}(\nabla_u g)(u)]_{\perp c} = 0.
\end{align*}
\]

(21)
Special case: Perfect fluids in (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion: \(\theta \pi = 0, \theta \delta = 0, \delta \mathcal{L} = 0, L = p, \nabla_u g = 0, \delta K r = 0\).

\[
\left(\rho \frac{1}{e} \cdot p\right) \cdot a_c = [\mathcal{G}(Kr p)]_{\perp c},
\]

\[a_c = \frac{1}{\left(\rho \frac{1}{e} \cdot p\right)} \cdot [\mathcal{G}(Kr p)]_{\perp c}.
\]

4. Conclusions

The representations of the Navier-Stokes equation in its forms for radial (centrifugal, centripetal) and tangential (Coriolis') accelerations could be used for description of different motions of fluids and continuous media in continuous media mechanics, in hydrodynamics and in astrophysics. The method of Lagrangians with covariant derivatives (MLCD) appears to be a fruitful tool for working out the theory of continuous media mechanics and the theory of fluids in spaces with affine connections and metrics, considered as mathematical models of space-time.

References


A QUATERNION APPROACH IN PHYSICS AND ENGINEERING CALCULATION

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Our the purpose of this paper is to show how the quaternion formalism can be applied with great success not only to the interpolation between coordinate frames, but also to a remarkably elegant description of the evolving coordinate-frame geometry of curves. Specific applications of these techniques include the generation of optimal kinematics solution corresponding to smooth mathematical curves appearing in computer graphics or scientific molecular kinematics. The correspondence between the orientation of a 3D object represented by a $3 \times 3$ orthogonal matrix in the group $SO(3)$ and unit quaternions has long been known to physicists and mathematicians, and was brought to attention of the computer graphics community by Shoemake, 1985. Unit quaternions are isomorphic to the topological 3-sphere $S^3$, which is also the topological space of the Lie group $SU(2)$, the simply connected twofold cover of the group $SO(3)$ describing rotations in ordinary 3D Euclidean space.

1. Introduction

The discovery of quaternions is attributed to the Irish mathematician William Rowan Hamilton (1805–1865) whose persistent struggle to extend the notion of complex numbers represented as algebraic pairs to triplets, finally gave birth to a very interesting non-commutative algebra [1, 2]. Mathematicians hailed the discovery as one of the three highly significant developments in the nineteenth century, the other two being the developments of the non-Euclidean geometry and the theoretical foundation for calculus. Even though quaternions remained a solution in search of a problem for many years after their discovery, applications in the fields of clas-
chical mechanics and the theory of relativity were identified in the early twentieth century. The capability of quaternions to succinctly represent three-dimensional rotations about an arbitrary axis motivated researchers to employ quaternion algebra in rotational kinematics equations. As a result, several new application areas involving quaternion based algorithms emerged [3]. These include diverse fields such as robotics, orbital mechanics, aerospace technologies, and inertial navigation systems.

The motivation for this paper is derived from the recent developments in the fields the physics and the engineering, that's means inverse kinematics, molecule kinematics, calculation, computer graphics and games programming where quaternions have been very effectively used. Graphics programmers have now realized the potential of quaternions as a very general and powerful rotation operator. Recent graphics APIs provide functions for quaternion operations. For example, the Java-3D API has classes (javax.vecmath.Quat4d) for creating quaternion objects, Povray (Persistence of Vision Ray Tracer) supports quaternions, Quickdraw 3D provides routines for quaternion operations, and Mathematica has an add-on package Algebra 'Quaternions'. Quaternions are also used in advanced algorithms in games programming and animation such as keyframe interpolations and the simulation of camera motion in a three-dimensional space [2]. Some of the virtual world interaction devices employ quaternions to parameterize rotations.

This article is organized as follow: The second part deals with quaternionic solution of the inverse kinematics problem for redundant planar robot. The third part deals with following problem. We obtaining a constrained conformation of a known kinematic structure (the ligand). The 3D positions of certain parts of the structure are predetermined for the pharmacophore model. The fourth part of the manuscript describes an algorithm for construction of spherical line interpolation of two quaternions. This is a general solution for solving a motion in $\mathbb{R}^3$.

2. Quaternionic solution of the inverse kinematics problem for planar robot

The present part discusses the following inverse kinematics problem: planar robot with open chain is given. A conditions and disadvantages for solving these problem are to be found. There are several mathematics ways for presentation the problem but our interest will be connected with quaternionic polynomial solutions. In this paper we call the attention to the basic
algebraic features of the space of the hypercomplex and derive consistent algebraic representation for robot kinematics. The robot link geometry is represented as hypercomplex geometry.

2.1. Quaternionic solution of the inverse kinematics problem for a two link planar robot

Consider the planar robot with the link lengths $l_1, l_2$ in the $z$-plane. the hypercomplex representation of the two links relative to their joint points $O_1, O_2$ and the axis $e_x, e_y$ is

\[
 l(v_1) = l_1 \cos(v_1)e_x + l_1 \sin(v_1)e_y \\
 l(v_1, v_2) = l_2 \cos(v_1 + v_2)e_x + l_2 \sin(v_1 + v_2)e_y.
\]

By exploiting relation $e_y = ze_x, e_x = ze + y$ between quaternionic units $e_x, e_y z$ one obtains $l_1(v_1) = l_1(\cos(v_1) + z \sin(v_1)) e_x = l_1 e^{v_1 z}$ is the $z$ complex representation ($z^2 = -1$) of the first robot link. Similarly, one obtains for the second link $L_2(z) = L_2 e^{(v_1 + v_2)z}$. The end point position relative to the first link origin can be given directly as $r_e(z) = l_1 e^{v_1 z} + l_2 (v_1 + v_2) z$. The distance $r_e$ of the end point from the first link origin is given by the relation $r_e^2 = (l_1 + l_2) + 2 l_1 l_2 \cos(v_2)$. By defining the following relations: $l_1 - l_2 = r_{\text{min}}$ and $l_1 + l_2 = r_{\text{max}}$ one obtains

\[
 r_{\text{max}} \cos \frac{\theta_2^2}{2} + r_{\text{min}} \sin \frac{\theta_2^2}{2} = r_e^2.
\]

The above equation is the well known "Euler"-equation of surface curvature (1).

\[
 \chi_{\text{min}} = \frac{r_{\text{min}}}{r_{\text{max}}} = \frac{l_2 - l_1}{l_2 + l_1}, \\
 \chi_e = \frac{r_e}{r_{\text{max}}}, \\
 \sin \frac{\theta_2}{2} = \pm \sqrt{\frac{r_{\text{max}} - r_e^2}{r_{\text{max}}^2 + r_{\text{min}}^2}},
\]

and the link $O O'$ is fixed (i.e. nothing rotation or translation) by introducing the parameters:

\[
 k_{1,2} = \frac{1}{2} (1 \pm \chi_{\text{min}}) \quad \text{and} \quad \hat{k}_{1,2} = \frac{1}{2} (1 \pm \chi_e).
\]
One obtains the relation \( \frac{\sin \Delta_1}{\sin \Delta_2} = \frac{k_2}{k_1} \) where

\[
\theta_2 = \Delta_1 + \Delta_2 \quad \text{and} \quad \theta_1 = \chi_{\epsilon} \pm \Delta_1 \quad \text{and}
\]

\[
\sin \frac{\theta_2}{2} = \pm \sqrt{\frac{k_1 k_2}{k_1 k_2}} = \pm \sqrt{\frac{1 - \chi_{\epsilon}^2}{1 - \chi_{\min}^2}}.
\]

Concerning to solving of the inverse kinematics problem is related with the characteristic polynomial of a corresponding eigenvalue problem:

\[
P(\lambda) = (1 - \chi_{\min}^2) \lambda^2 - (1 - \chi_{\epsilon}^2).
\]

The roots of this characteristic polynomial are: \( \lambda_{1,2}^2 = \pm \sqrt{\frac{1 - \chi_{\epsilon}^2}{1 - \chi_{\min}^2}} \) Now, we propose a new recursive algorithm for inverse kinematics problem:

**Proposition 1.** The algebraic forms of the solutions of inverse kinematics problem of planar robot with \( i, i + 1 \) links is the roots of polynomial \( P_i \) are obtained as:

\[
P(\lambda_i) = (1 - \chi_{i,\min}^2) \lambda_i^2 - (1 - \chi_{i,\epsilon}^2).
\]

Our purpose will be to provide consideration a pair of two rotations chains with base joints \( O_i, O_{i+1} \), lengths \( l_i, l_{i+1} \) and separated by distance \( l_{i-2} \). If we regally join the end link, then planar robot becomes a chain of \( i + 1 \) links of lengths \( l_i, l_{i+1}, l_{i-2} \), where \( l_{i+1} \) is the distance between the joints of the combined end links. The constraint surface for both chain \( i \) and chain \( i + 1 \) have two-link robot. The algebraic forms of the constraint surfaces \( P_i \) are obtained as:

\[
P(\lambda_i) = (1 - \chi_{i,\min}^2) \lambda_i^2 - (1 - \chi_{i,\epsilon}^2).
\]

The solutions are required in computer parts (in order to animate objects) and in visualization of moving objects as well as in path planning of robot manipulators.

The solution of the inverse kinematics problem is analytically similar to the known reflection conditions for optic rays with \( k_1, k_2 \) as reflection coefficients. Based on such similarity, one can conclude that the solutions of the inverse kinematics problem can formulated as an optimisation problem for the path geometry. With the same reasoning, the solution of the inverse kinematics problem is related to the certain eigenvalue problem.
3. The molecule structure and discrete curves

A molecule structure can be represented as a collection of atoms together with their coordinates in three-dimensional space $\mathbb{R}^3$. This may be a list of all atoms in the protein, or a list of the ones that can be observed. Thus, if $N$ atoms are listed with coordinates we have a vector in $(\mathbb{R}^3)^N$. Two structures are the same if one can be transformed into the other by a sense-preserving Euclidean motion. If $E$ is the group generated by rotations and translations, it can be thought of as acting on all the coordinates listed, and a structure is an element of $(\mathbb{R}^3)^N/E$. As example: The protein molecule is a sequence of amino acids of 20 different kinds. The peptide bond links into a polymeric backbone individual amino acids with 20 types of side-chains. It is convenient to think of a protein as a collection of discrete curves. This is useful both in understanding the torsion angle description of molecule (protein) structures and the method of using orientational restraints and dynamics to determine molecule structures. A discrete curve is a sequence of points $p_0, \ldots, p_n$ in three-dimensional space. These points can be thought of as atoms and the line segments joining atoms in the sequence can be thought of as covalent bonds. As example: The backbone of a protein is a discrete curve consisting of points representing the atoms $-C'N-C-C'$ proceeding from the $N$-terminus to the $C$-terminus. By putting the atoms in sequential order, side-chains can also be made into a discrete curve. Thinking of a protein as a curve allows us to abstract some ideas from differential geometry to study the structure.

3.1. Frenet frames in $H$

The differential geometry of curves traditionally begins with a vector $\tilde{x}(s)$ that describes the curve parametrically as a function of $s$ that is least theist-differentiable. Then the tangent vector $\tilde{t}(s)$ is well-defined at every point $\tilde{x}(s)$ and we may choose two additional orthogonal vectors in the plane perpendicular to $\tilde{t}(s)$ to form a complete local orientation frame. Provided the curvature of $\tilde{x}(s)$ vanishes nowhere, we can choose this local coordinate system to be the Frenet frame consisting of the tangent $\tilde{t}(s)$, the binormal $\tilde{b}(s)$, and principal normal $\tilde{n}(s)$, which given in terms of the curve itself by the expressions in [2].

A Frenet formalism for discrete curves will be described briefly. The idea of a Frenet frame, or moving frame, for differentiable space curves can be modified for use with discrete space curves.

Next we sketch the correspondence between the unit quaternion and
orthonormal coordinate frames; this will take us to our main result, which is a reformulation of the Frenet and parallel-transport frames in terms of quaternions only. A quaternion frame is a unit-length for-vector \( q = (q_0, q_1, q_2, q_3) \) that corresponds to exactly one 3D coordinate and is characterized by the following mapping to 3D rotation: Every possible 3D rotation \( R \) can be constructed from either of two related quaternions \( q \) using the transformation law:

\[
[q \cdot \hat{V} \hat{q}] = \sum_{j=1}^{3} R_{ij} V_j
\]

where, \( v = (0, \hat{V}) \) a pure 3-vector, we can compute \( R_{ij} \) directly to be the quadratic formula

\[
R = \begin{pmatrix}
q_0^2 + q_1^2 - q_2^2 - q_3^2 \\
2q_1q_2 + 2q_0q_3 \\
2q_1q_3 - 2q_0q_2
\end{pmatrix}
\]

This quadratic form for a general orthonormal \( SO(3) \) frame suggests that the Frenet and parallel transport frames and their evolution equations might be expressible directly in terms of a linear equation in the quaternion variables. If we identify the columns of this rotational matrix as \((\vec{t}, \vec{n}, \vec{b})\), respectively, we find that differentiation yields

\[
d\vec{t} = 2A.dq, \quad d\vec{n} = 2B.dq, \quad d\vec{b} = 2C.dq,
\]

where

\[
A = \begin{pmatrix}
q_0 & q_1 - q_2 - q_3 \\
q_3 & q_2 & q_1 & q_0 \\
-q_2 & q_3 - q_0 & q_1
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
-q_3 & q_2 & -q_1 - q_0 \\
q_0 & -q_1 & q_2 - q_3 \\
q_1 & q_0 & -q_3 & q_2
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
q_2 & q_3 & -q_0 - q_1 \\
-q_1 - q_0 & q_3 & q_2 \\
q_0 & -q_1 & -q_2 & q_3
\end{pmatrix}.
\]

The Frenet equation themselves must take the form

\[
\vec{t}' = v\sigma \vec{n}, \quad \vec{n}' = -v\sigma \vec{t} + v\sigma \vec{b} \quad \vec{b}' = -v\sigma \vec{n}.
\]

By simply writing out the right-hand sides of these equations and grouping terms, we derive the following fundamental expression, the quaternion
Frenet frame equation:

\[ q'(s) = \begin{pmatrix} q'_0 \\ q'_1 \\ q'_2 \\ q'_3 \end{pmatrix} = \frac{1}{2} v \begin{pmatrix} 0 & -\sigma & 0 & -k \\ \sigma & 0 & k & 0 \\ 0 & -k & 0 & \sigma \\ k & 0 & -\sigma & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \]  

(2)

where the \( v \) is the scalar magnitude of the curve derivative and the geometry of the curve is embodied in the curvature \( k \) and torsion \( \sigma \).

Orthogonal frames correspond to orthogonal matrices and right-handed orthogonal frames to rotation matrices. The Frenet frames can be thought of as molecular frames along the molecule. The plane formed by the tangent and the normal vector at a point contains the point together with the previous and the subsequent points. The Frenet frames are also related to the torsion angles used in the study of molecular structure.

The relationship of one Frenet frame to the next is given by

\[ F_{j+1} = F_j R(\sigma_j) R(\theta_{j+1}) \]

where \( \theta_j = \arccos(t_{j-1} \cdot t_j) \) is the exterior bond angle at \( p_j \), and \( \sigma_j \) is the angle of torsion about the bond direction \( t_j \).

\[
R(\sigma_j) = \begin{pmatrix} i \\ \hat{n} \\ \hat{b} \end{pmatrix} 
\]

\[
= \left( l^2(1 - \cos(\sigma)) + \cos(\sigma) \right) \begin{pmatrix} l m(1 - \cos(\sigma)) - c \sin(\sigma) \\ d(1 - \cos(\sigma)) + m \sin(\sigma) \end{pmatrix} 
\]

\[
= \begin{pmatrix} l m(1 - \cos(\sigma)) + \sin(\sigma) \\ m^2(1 - \cos(\sigma)) + \cos(\sigma) \end{pmatrix} \begin{pmatrix} \cos(\sigma) \\ - \sin(\sigma) \end{pmatrix} + \begin{pmatrix} \sin(\sigma) \\ \cos(\sigma) \end{pmatrix} + \begin{pmatrix} \cos(\sigma) \\ \sin(\sigma) \end{pmatrix}
\]

\[
\cdot \begin{pmatrix} t \\ n \\ b \end{pmatrix}
\]

is rotation of vector \( v = (0, t, n, b) \) and quaternion

\[ q = (\cos(\frac{\sigma}{2}), \sin(\frac{\sigma}{2}), m \sin(\frac{\sigma}{2}), n \sin(\frac{\sigma}{2})) \).

Obviously about rotation \( R(\theta_j + 1) \) we must work analogically. Thus the discrete curve can be reconstructed up to a Euclidean motion from the sequences \( s_j = p_{j+1} - p_j \) of bond lengths, \( t_j - 1 \cdot t_j \) of bond angle cosines, and \( \sigma_j \) of torsion angles.

### 3.2. Standart protein geometry

For finding and describing protein structures with limited structural data it is often assumed that bond lengths and angles have standard (or ideal)
values depending on the type of bond. With bond angles and bond lengths
given, the above discussion shows how the structure of the protein backbone
can be determined given a torsion angle for each bond.

4. The molecule kinematics model

A molecule is characterized by a pair $\langle A, B \rangle$, in which $A$ represents a collection of atoms, and $B$ represents a collection of bonds between pairs of atoms. An underlying graph can be considered for the molecule, in which vertices represent atoms and edges represent bonds. Thus, usual graph-theoretic concepts such as connectedness, paths, trees, and cycles can be applied to molecules. It will be convenient to choose one atom, $a_{\text{anch}} \in A$, as the anchor for the molecule (or the root of the corresponding graph). We assume that the underlying graph structure is a tree (i.e., no flexible rings). We represent rigid rings by considering the entire ring as a special atom that is attached by a bond to the rest of the molecule. Our assumption about rings remains valid for a large set of molecules that are of interest in drug design. Our general approach could be extended to cyclic molecules by exploiting computational algebra techniques that obtain kinematics solutions to cyclic chains, by using techniques for large cyclic chains, or by breaking bonds and adding dummy constraints. Information used for kinematics and energy computation is associated with each of the atoms and bonds. Each atom carries standard information, such as its van der Waals radius [4]. Three pieces of information are associated with each bond, $b_i \in B$: (i) the bond length, $l_i$; (ii) the bond angle $\theta_i$, is the angle between $b_i$ and the previous bond, in the direction toward $a_{\text{anch}}$; (iii) the set of possible torsion angles, $\sigma_i$, which represents the ability of the bond to rotate about its own axis. The part of the molecule that is attached to $b$ in the direction away from the anchor will also undergo rotation about this axis. If $i$ must remain constant, the bond is fixed; otherwise, it is considered rotatable. In most molecular studies, bond lengths and bond angles are considered fixed while torsions are allowed to vary. We follow this assumption in our work. From now on we represent the conformation of the ligand as an $m$-dimensional vector of torsion angles, in which each component corresponds to a rotatable bond. We model the kinematics of the ligand using techniques common in robotics. Although powerful analytical techniques exist for searching the solution spaces of similar structures in robotics, [3] such techniques are impractical for handling high degrees of freedom. Our ligands have many torsion degrees of freedom; thus, we focus on randomized
solutions to the conformational search problem. The need for efficiency has also motivated randomized search techniques in robotics. We use our experience with these methods to develop a randomized conformational search technique, which simultaneously reduces energy and maintains the pharmacophore constraints. Molecular kinematics gives the positions of all of the atoms of the ligands in terms of the torsion angles. The bond lengths, bond angles, and torsion angles can be conveniently used as parameters in the quaternionical representation for spatial kinematics chains. This representation is useful for the molecule, suppose that a local coordinate frame is attached at the beginning of each link (or atom center). If a bond \( b_i \) follows a bond \( b_{i+1} \) in the chain, then the coordinate frame of \( b_i \) is related to \( b_{i+1} \) and the fictitious bond is used to define \( 0 \).

The fictitious bond is used to define \( 0 \). If \( b_i \) is not rotatable, then \( \sigma_i \) is a constant; otherwise, \( i \) is a conformation parameter, included in \( \sigma \). The position of any atom in the molecule can be determined by chaining matrices of the form (3). For example, suppose \( b_i, b_{i-1}, \ldots, b_1 \) represents the sequence of bonds in the path from a particular atom, \( a \in A \), to \( a_{\text{anch}} \). Taking into account the anchor orientation the position of a particular atom, \( a \in A \), at the end of a path, \( b_i, b_{i-1}, \ldots, b_1 \) to \( a_{\text{anch}} \) is composed by (3).

5. Spherical linear interpolation

In 1985 were described the SLERP (Spherical Line intERPolation algorithm) [5]. This article made Slerp widely popular among the engineers in area of computer graphics. Most of modern realisation of 3D rotational algorithms are based or derived from SLERP. In [5] was given only final formulae of SLERP, in this part we describe two derivations of this method and discuss the case of “small angles collision”, because these topics are beyond the scope of widely used papers [5], [6].

5.1. Interpolation using quaternions

We had a construction for spherical linear interpolation of two quaternions \( v_0 \) and \( v_1 \) treated as unit length vectors in \( R^4 \) space, the angle between them acute. The unit quaternion \( w \) is a of \( \bar{v}_0 \times \bar{v}_1 \). The idea is that \( \bar{v}_t = q_t \bar{v}_0 q^* \)

Rotate \( \bar{v}_0 \) by angle \( t\theta \) around \( w \)

\[
w = \frac{\bar{v}_0 \times \bar{v}_1}{|\bar{v}_0 \times \bar{v}_1|} = \frac{\bar{v}_0 \times \bar{v}_1}{\sin \theta}\]
using quaternions
\[ q(t) = \cos \left( \frac{t\theta}{2} \right) + \sin \left( \frac{t\theta}{2} \right)w \]
\[ \vec{v}_t = q_t \vec{v}_0 q^* \]
\[ \vec{v}_t = \cos \left( \frac{t\theta}{2} \right) + \sin \left( \frac{t\theta}{2} \right)w \vec{v}_0 \cos \left( \frac{t\theta}{2} \right) - \sin \left( \frac{t\theta}{2} \right)w \]
\[ \vec{v}_t = (\cos \left( \frac{t\theta}{2} \right) + \sin \left( \frac{t\theta}{2} \right)w \cdot \vec{v}_0 + \sin \left( \frac{t\theta}{2} \right)w \times \vec{v}_0) \]
\[ (\cos \left( \frac{t\theta}{2} \right) - \sin \left( \frac{t\theta}{2} \right)w) \]
\[ \vec{v}_t = \cos^2 \left( \frac{\theta}{2} \right) \vec{v}_0 + \sin (\theta/2) \cos (\theta/2)(w \times \vec{v}_0) \]
\[ - \sin (\theta/2) \cos (\theta/2)(\vec{v}_0 \times w) + \sin^2 (\theta)((w \times \vec{v}_0) \times w) \]

Here we easily can obtain:
\[ \vec{v}_t = \cos^2 \left( \frac{\theta}{2} \right) \vec{v}_0 + i2 \sin (\theta/2) \cos (\theta/2)(w \times \vec{v}_0) \]
\[ - \sin^2 \left( \frac{\theta}{2} \right)((w \times \vec{v}_0) \times w) \] (4)

We can use following properties of defined basis:
\[ (\vec{v}_0 \times \vec{v}_1) \times \vec{v}_0 = (\vec{v}_0 \cdot \vec{v}_0)\vec{v}_1 - (\vec{v}_0 \cdot \vec{v}_1)\vec{v}_0 = \vec{v}_1 - \cos (t\theta/2)\vec{v}_0 \] (5)

and also:
\[ (w \times \vec{v}_0) \times w = (w \cdot w)\vec{v}_0 - (w \cdot \vec{v}_0)w = \vec{v}_0 \] (6)

Playing (5) (6) on (4) we can write \( v_t \) in following form:
\[ \vec{v} = \cos^2 \left( \frac{t\theta}{2} \right) + \frac{\sin (t\theta)}{\sin (\theta)}(\vec{v}_1 - \cos (t\theta/2)\vec{v}_0) - \sin^2 \left( \frac{t\theta}{2} \right)\vec{v}_0 \]
\[ = \left( \frac{\sin \theta (\cos^2 \left( \frac{t\theta}{2} \right) - \sin^2 \left( \frac{t\theta}{2} \right)) - \sin \left( t\theta \right) \cos \left( \theta \right) \sin \left( \theta \right) }{\sin \left( \theta \right) } \right) \vec{v}_0 + \left( \frac{\sin \left( t\theta \right) }{\sin \left( \theta \right) } \right) \vec{v}_1 \]
\[ = \left( \frac{\sin \left( \theta - t\theta \right) }{\sin \left( \theta \right) } \right) \vec{v}_0 + \left( \frac{\sin \left( t\theta \right) }{\sin \left( \theta \right) } \right) \vec{v}_1 \] (7)

5.2. Spherical linear interpolation

Alternative derivation [6]. We have a construction for spherical linear interpolation of two quaternions \( q_0 \) and \( q_1 \) treated as unit length vectors in \( R^4 \) space, the angle between them acute. The idea was that
\[ q(t) = c_0(t)q_0 + c_1(t)q_1 \]
where \( c_0(t) \) and \( c_1(t) \) are real valued functions for \( 0 \leq t \leq 1 \) with \( c_0(0) = 1, c_1(0) = 0, c_0(1) = 0, \) and \( c_1(1) = 1. \) The quantity \( q(t) \) is required to be a unit vector, so
\[ 1 = q(t) \cdot q(t) = c_0(t)^2 + 2 \cos \left( \theta \right) c_0(t)c_1(t) + c_1(t)^2. \]
This is the equation of an ellipse that is factored using methods of analytic geometry to obtain formulas for \( c_0(t) \) and \( c_1(t) \).

A simpler construction uses only trigonometry and solving two equations in two unknowns. As \( t \) uniformly varies between 0 and 1, the values \( q(t) \) are required to uniformly vary along the circular arc from \( q_0 \) to \( q_1 \). That is, the angle between \( q(t) \) and \( q_0 \) is \( \cos(t\theta) \) and the angle between \( q(t) \) and \( q_1 \) is \( \cos((1-t)\theta) \). Dotting the equation for \( q(t) \) with \( q_0 \) yields

\[
\cos t = c_0(t) + \cos(\theta)c_1(t)
\]

and dotting the equation with \( q_1 \) yields

\[
\cos((1-t)) = \cos(\theta)c_0(t) + c_1(t).
\]

These are two equations in the two unknowns \( c_0 \) and \( c_1 \). The solution for \( c_0 \) is

\[
c_0(t) = \frac{\cos(t\theta) - \cos(\theta)\cos((1-t)\theta)}{1 - \cos^2(\theta)} = \frac{\sin((1-t)\theta)}{\sin(\theta)}
\]

The last equality is obtained by applying double-angle formulas for sine and cosine. By symmetry, \( c_1(t) = c_0(1-t) \). Or solve the equations for

\[
c_1(t) = \frac{\cos((1-t)\theta) - \cos(\theta)\cos(t\theta)}{1 - \cos^2(\theta)} = \frac{\sin(t\theta)}{\sin(\theta)}.
\]

The spherical linear interpolation, abbreviated as Slerp, is defined by

\[
q_t = \frac{q_0 \sin((1-t)\theta) + q_1 \sin(t\theta)}{\sin(\theta)}
\]

for \( 0 \leq t \leq 1 \). Although \( q_1 \) and \(-q_1\) represent the same rotation, the values of \( \text{Slerp}(t; q_0, q_1) \) and \( \text{Slerp}(t; q_0, -q_1) \) are not the same. It is customary to choose the sign \( \sigma \) on \( q_1 \) so that \( q_0 \cdot (\sigma q_1) \geq 0 \) (the angle between \( q_0 \) and \( q_1 \) is acute). This choice avoids extra spinning caused by the interpolated rotations.

5.3. The numerical problem in occasion of smal values of \( \theta \)

In numerical application we must avoid the collision of so-called machine null division. This occurs when we divide on values smaller than machine \( \varepsilon \).

Definition 1. Machine \( \varepsilon \) is the smallest real value representing in float point notation different from 0. In numerical system with floating point representation we can define \( \varepsilon \) as follow: if

\[
\varepsilon_0 = \frac{1}{2}
\]
and $\varepsilon_i = \varepsilon_{i-1}/2$ then $\varepsilon_{n-1}$ which satisfied

$$\varepsilon_{n+1} + 1 = 1$$

then $\varepsilon_n$ is the machine epsilon and we will denote it with $\varepsilon$.

When $\sin(\theta) \leq \varepsilon$ we will get as result division on 0 in (8), to avoid this obstacle we must use LERP (Line interpolation) instead of SLERP

$$q_t = q_0(1-t) + q_1t$$

5.4. Algorithm for using slerp in numerical engineering calculation

1. In first step we must denote preliminary and starting conditions.
   We have a two quaternion $q_0$ and $q_1$. We have a quaternion $w$ so $w = q_0xq_1$, and $w \in H$. $w$ is the axis of rotation. We must define $t \in [0, 1]$. As $t$ uniformly varies between 0 and 1, the values $q(t)$ are required to uniformly vary along the circular arc from $q_0$ to $q_1$. We must calculate $\varepsilon$.

2. Normalisation of $q_1$ and $q_2$: $|q_1|^2 = 1$ and $|q_2|^2 = 1$.

3. Solving the the small angles collision: if $\sin(\theta) \leq \varepsilon$ we continue with step 4 if $\sin(\theta) > \varepsilon$ we continue with step 5.

4. In this step we use Lerp instead of using Slerp $q_t = q_0(1-t) + q_1t$.

5. In this step we use Slerp

$$q_t = \frac{q_0 \sin((1-t)\theta) + q_1 \sin(t\theta)}{\sin(\theta)}.$$

References

THE EXISTENCE OF INDEFINITE METRICS OF SIGNATURE (++--) AND TWO KINDS OF ALMOST COMPLEX STRUCTURES IN DIMENSION FOUR

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In the present note, we mainly review existence theorems of the indefinite metrics of signature (+ + --), i.e., neutral metrics in four dimension, and the existence of two kinds of almost complex structures which are associated with such neutral metrics. Recent results on Walker 4-manifolds, which exhibit a large variety of neutral geometries, are also included.

Keywords: indefinite metrics, neutral metrics, almost complex structures, opposite almost complex structures, Chern numbers, fields of 2-planes, symplectic structures, almost Kähler metrics, Kähler metrics, Walker metrics

Indefinite metrics have their root in relativity theory, and go back to just a hundred years ago. In classical relativity theory, metrics must be indefinite and solely of Lorentz signature. There were, of course, some works on indefinite metrics from a purely mathematical point of view, but in most cases they were not major themes in the development of geometry. The author recognized Steenrod’s theorem [30, §40] as inception of the study of pseudo-Riemannian geometry for itself.

Theorem 1. A compact smooth manifold of dimension n admits an indefinite metric of signature $(p,n-p)$ $(p \geq 1)$ if and only if the manifold admits a nonsingular field of tangent $p$-planes.

Nowadays, this assertion is well known, but it is not trivial. Its proof involves the problem of the extension of sections of fibre bundles. Steenrod

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recognized that the existence of an indefinite metric on a manifold is equivalent to the existence of a global section of a Grassmann manifold bundle over the manifold. There were a lot of differential geometers who had been working on distributions and foliations, but it was very rare for them to have paid attention to the relation to indefinite metrics. In fact, research on metrics of indefinite signature, other than of Lorentz signature, is still not a major activity in the field of differential geometry. See the following table:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Metrics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-dim</td>
<td>no indefinite metric</td>
</tr>
<tr>
<td>2-dim</td>
<td>(++)</td>
</tr>
<tr>
<td>3-dim</td>
<td>(++) (++)</td>
</tr>
<tr>
<td>4-dim</td>
<td>(++++) (+++-) (++--)</td>
</tr>
<tr>
<td>5-dim</td>
<td>(+++++) (+++-) (+++-) (+---) ...</td>
</tr>
</tbody>
</table>

From this table, we immediately see that the lowest dimensional example of indefinite signature that is not Lorentz is the four-dimensional neutral signature. The author began studying these neutral metrics in the early 80s and Law, independently, in the early 90s (see [13], [14], [15]).

From Steenrod's theorem, we know that the existence of a (++--)-metric on a 4-manifold is equivalent to the existence of a nonsingular field of tangent 2-planes. In the terminology of G-structures, the existence of a (++--)-metric on a 4-manifold is nothing but the existence of an O(2, 2)-structure, while a field of 2-planes is a GL(2, \mathbb{R}) \times GL(2, \mathbb{R})-structure. Both these structures reduce to O(2) \times O(2) structures, as O(2) \times O(2) is in each case the maximal compact subgroup.

We then focus on the completely orientable case, i.e., the structure group G is the identity component SO_0(2, 2) of O(2, 2). For this case, Steenrod's theorem implies that a 4-manifold can admit a (++--)-metric, with G = SO_0(2, 2), if and only if it admits a field of orientable tangent 2-planes. In 1958, Hirzebruch and Hopf obtained necessary and sufficient conditions for the existence of such a field of 2-planes. Let M be a compact oriented 4-manifold, with \chi[M] the Euler characteristic and \tau[M] the Hirzebruch index (signature) of M. We denote by \mu_M the intersection form of M, which is a bilinear form on the free part of the second cohomology group as follows:

$$\mu_M : H^2(M, \mathbb{Z})/\text{Tor} \times H^2(M, \mathbb{Z})/\text{Tor} \to H^4(M, \mathbb{Z}) \cong \mathbb{Z}. \quad (1)$$
A cohomology class $w \in H^2(M, \mathbb{Z})/\text{Tor}$ is called characteristic if and only if it satisfies the following

$$\mu_M(w, x) \equiv \mu_M(x, x) \mod 2 \quad \forall x \in H^2(M, \mathbb{Z})/\text{Tor}. \quad (2)$$

**Theorem 2.** (Hirzebruch-Hopf [9]) A compact oriented 4-manifold $M$ admits a field of oriented tangent 2-planes if and only if there exist two characteristic elements $w, w'$ such that

$$\mu_M(w, w) = 3\tau[M] + 2\chi[M] \quad (3)$$
$$\mu_M(w', w') = 3\tau[M] - 2\chi[M]. \quad (4)$$

To establish this condition, they considered the conditions for $M$ to admit a reduction of the structure group $G$ from $SO(4)$ to $SO(2) \times SO(2)$, which is the maximal compact subgroup of $SO_0(2,2)$. The intersection form $\mu_M$ can be considered as a symmetric bilinear form with integral entries. It has been known that not every form can be realized as an intersection form on a compact simply connected 4-manifold (e.g., Rohlin’s Theorem [28]). For the classification of $\mu_M$, see [29], [24], [16]. Not until S.K. Donaldson applied gauge theory to four-dimensional topology in the early 80s, however, was it not known which definite bilinear forms could be realized as intersection forms on a 4-manifold. In fact, he proved [6] that the intersection form on a compact orientable simply-connected 4-manifold, if definite, must be the standard form. Therefore, the classification of $\mu_M$ on a compact simply connected 4-manifold $M$ is completed as follows:

- **Indefinite case**: $\mu_M$ is isomorphic to one of the following forms [29]:
  
  **Type I**  
  $$\mu_M = m\langle 1 \rangle \oplus n\langle -1 \rangle \quad (m, n \geq 1) \quad (5)$$
  
  **Type II**  
  $$\mu_M = m \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus nE_8 \quad (m \geq 1, n \in \mathbb{Z}) \quad (6)$$

- **Definite case**: $\mu_M$ is isomorphic to one of the following forms [6]:
  
  $$\mu_M = m\langle 1 \rangle \quad \text{(positive definite)}, \quad (7)$$
  $$\mu_M = m\langle -1 \rangle \quad \text{(negative definite)}. \quad (8)$$

After the ICM at Berkeley, where Donaldson was celebrated as a Fields Laureate, the author applied the classification of $\mu_M$ given in (5) - (8) to the conditions (3), (4), and arrived at the following

**Theorem 3.** (Matsushita [17]) Let $M$ be a compact simply connected 4-manifold.
(a) If $\mu_M$ is indefinite of Type I, then $M$ admits a nonsingular field of oriented tangent 2-planes, and equivalently a $(++--)$-metric with $G = SO_0(2,2)$, if and only if both $m$ and $n$ are odd, or equivalently if and only if
\[ \chi[M] + \tau[M] \equiv 0 \mod 4, \quad \chi[M] - \tau[M] \equiv 0 \mod 4. \] (9)

(b) If $\mu_M$ is indefinite of Type II, then $M$ admits a nonsingular field of oriented tangent 2-planes, and equivalently a $(++--)$-metric with $G = SO_0(2,2)$, if and only if $m$ is odd, or equivalently if and only if
\[ \chi[M] - \tau[M] \equiv 0 \mod 4. \] (10)

(c) If $\mu_M$ is definite, then $M$ cannot admit a nonsingular field of oriented tangent 2-planes, and equivalently a $(++--)$-metric with $G = SO_0(2,2)$.

We must recall important results on fields of tangent planes due to Atiyah. In 1970, he showed from the index theorems [1] that if a $4k$-dimensional compact oriented manifold $M^{4k}$ admits a field of oriented tangent $q$-planes, with $q \equiv 2 \mod 4$, then the $4k$-manifold $M^{4k}$ must satisfy the following topological constraints:
\[ \chi[M^{4k}] + \tau[M^{4k}] \equiv 0 \mod 4, \quad \chi[M^{4k}] - \tau[M^{4k}] \equiv 0 \mod 4. \] (11)

It should be noted that Theorem 2 asserts that this necessary condition is in fact a sufficient condition for a simply connected 4-manifold.

In this line of thought, Donaldson proved further in 1986 [7] that the classification of $\mu_M$ given in (5)-(8) can be available for all compact orientable 4-manifolds, without the restriction of simply connectedness. In 1991, Saeki generalized Theorem 2 to all compact 4-manifolds as follows (see [18]):

**Theorem 4.** (a) Let $M$ be a compact 4-manifold whose intersection form $\mu_M$ is indefinite. Then $M$ admits a field of oriented tangent 2-planes if and only if $\tau[M]$ and $\chi[M]$ satisfy the pair of conditions
\[ \tau[M] + \chi[M] \equiv 0 \mod 4, \quad \tau[M] - \chi[M] \equiv 0 \mod 4. \] (12)

(b) Let $M$ be a 4-manifold whose intersection form $\mu$ is definite, then $M$ admits a field of oriented tangent 2-planes if and only if $\tau[M]$ and $\chi[M]$ satisfy the pair of conditions (12), and
\[ |\tau[M]| + \chi[M] \geq 0. \] (13)
At this stage, we will step into the interpretation of the two conditions (3), (4) of Hirzebruch and Hopf. It is well-known that the first condition (3) is nothing but Wu’s condition for a 4-manifold $M$ to admit an almost complex structure [34]. If there exists a characteristic element $w$, then there exists on $M$ an almost complex structure $J$ so that $w$ is the first Chern class $c_1(J)$ of $J$. Therefore, (3) can be written as
\[
\mu_M(c_1(J), c_1(J)) = c_1^2(J) = 3\tau[M] + 2\chi[M].
\] (14)

It is known that the second Chern number coincides with the Euler characteristic $c_2(J) = \chi[M]$. Since $c_1^2(J)$ and $c_2(J)$ are topological invariants, they do not depend on the choice of $J$, and can be written as $c_1^2(M)$ and $c_2(M)$, or more simply as $c_1^2$ and $c_2$. In terms of these Chern numbers, we have
\[
\chi[M] = c_2, \quad \tau[M] = \frac{1}{3}(c_1^2 - 2c_2).
\] (15)

We now turn our attention to the second condition (4). If we denote by $-M$ the 4-manifold $M$ with orientation reversed, then $\chi[-M] = \chi[M]$, $\tau[-M] = -\tau[M]$, and $\mu_{-M} = -\mu_M$. We can rewrite (4) in terms of the topological invariants $\chi[-M]$, $\tau[-M]$ as follows:
\[
\mu_{-M}(w', w') = 3\tau[-M] + 2\chi[-M].
\] (16)

This is nothing but Wu’s condition for $-M$ to admit an almost complex structure. An almost complex structure on $-M$ is called an opposite almost complex structure on $M$ [18]. Similarly to the first condition (3), the characteristic element $w'$ can be considered as the first Chern class $c_1(J')$ determined by an opposite almost complex structure $J'$. The equation (16) can be written as
\[
\mu_{-M}(c_1(J'), c_1(J')) = c_1(J') \cup c_1(J')[-M]
= -c_1(J') \cup c_1(J')[M]
= 3\tau[-M] + 2\chi[-M]
= -3\tau[M] + 2\chi[M].
\] (17)

Denoting $c_1(J') \cup c_1(J')[-M]$ by $c_1^2(-M)$, we have a formula for a 4-manifold which admits an opposite almost complex structure as follows:
\[
c_1^2(-M) = -3\tau[M] + 2\chi[M], \quad c_2(-M) = \chi[M].
\] (18)

Similarly to (15), we have the opposite version:
\[
\chi[M] = c_2(-M), \quad \tau[M] = -\frac{1}{3}(c_1^2(-M) - 2c_2(-M)).
\] (19)
Now let $M$ be a 4-manifold admitting a $(++--)$-metric. Then $M$ has both Chern numbers $(c_1^2, c_2)$ and opposite Chern numbers $(c_1^2(-M), c_2(-M))$, which are related to each other as follows:

$$c_1^2(-M) = 4c_2 - c_1^2, \quad c_2(-M) = c_2$$

Then Atiyah's condition (9) and (12) are written in terms of the Chern numbers as follows:

$$\chi + \tau \equiv 0 \mod 4 \iff c_1^2 + c_2 \equiv 0 \mod 12,$$
$$\chi - \tau \equiv 0 \mod 4 \iff c_1^2(-M) + c_2(-M) = 5c_2 - c_1^2 \equiv 0 \mod 12.$$  

The former is the Noether theorem, and the latter is its opposite version. These conditions are equivalent to the following

$$c_1^2 + c_2 \equiv 0 \mod 12, \quad c_2 \equiv 0 \mod 2,$$  \hspace{1cm} (20)

which are the only restrictions on the Chern numbers of an indefinite 4-manifold of signature $(++--)$, with indefinite intersection form.

If $M$ has positive definite intersection form $\mu_M$ ($\tau[M] > 0$), then in addition to (20), the inequality (13) restricts the Chern numbers as

$$c_1^2 > 2c_2, \quad c_1^2 \geq -c_2.$$  \hspace{1cm} (21)

Similarly, if $M$ has the negative definite intersection form $\mu_M$ ($\tau[M] < 0$), then in addition to (20), the Chern numbers must satisfy the inequalities

$$c_1^2 < 2c_2, \quad c_1^2 \leq 5c_2.$$  \hspace{1cm} (22)

We have thus obtained the necessary and sufficient conditions for three equivalent objects as follows (cf. [23]):

(i) a $(++--)$-metric with $G = SO_0(2, 2)$

(ii) a pair $(J, J')$ of an almost complex structure $J$ and an opposite almost complex structure $J'$

(iii) a field of orientable tangent 2-planes.

It is also important to understand such equivalent objects from a group theoretical point of view. We may suppose, as Hirzebruch and Hopf did [9], that the structure group $G$ of $M$ is primarily $G = SO(4)$, i.e., $M$ can be assumed to have implicitly a Riemannian metric. It is known that $SU(2)$ ($\cong_{\text{topologically}} S^3$) is the spin group of $SO(3)$. It is well-known that the spin group $Spin(4)$ of $SO(4)$ is isomorphic to $SU(2) \times SU(2) \cong S^3 \times S^3$. It is also well-known that the space of almost complex structures at a point
of \( M \) that are orthogonal with respect to some implicit Riemannian metric, is a quotient space of \( SO(4) \) as follows:

\[
\{ \text{space of } J \} = SO(4)/U(2), \tag{23}
\]

where the unitary group \( U(2) \) is the stabilizer of a standard almost complex structure. Similarly, the space of (orthogonal) opposite almost complex structures \( J' \) at a point is written as

\[
\{ \text{space of } J' \} = SO(4)/U'(2), \tag{24}
\]

where \( U'(2) \) is the stabilizer of a standard opposite almost complex structure. There is an exact sequence

\[
0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(4) \cong S^3 \times S^3 \xrightarrow{\alpha} SO(4) \longrightarrow 1. \tag{25}
\]

We now attention to the three subgroups of \( Spin(4) \) (\( \cong S^3 \times S^3 \)) as follows:

\[
\begin{align*}
S^1 \times S^3, & \quad S^3 \times S^1 \subset S^3 \times S^3, \tag{26} \\
S^1 \times S^1 = (S^1 \times S^3) \cap (S^3 \times S^1) \subset S^3 \times S^3, & \tag{27}
\end{align*}
\]

which have as their images in \( SO(4) \) by the covering map \( \alpha \) respectively as follows:

\[
\begin{align*}
\alpha(S^1 \times S^3) &= U(2), \quad \alpha(S^3 \times S^1) = U'(2) \tag{28} \\
\alpha(S^1 \times S^1) &= U(2) \cap U'(2) = U(1) \times U(1) = SO(2) \times SO(2). \tag{29}
\end{align*}
\]

It is important to recognize that the two subgroups \( U(2) \) and \( U'(2) \) have a common subgroup \( SO(2) \times SO(2) \), which is again evidence of the equivalence between fields of 2-planes and pairs \( (J, J') \) of an almost complex structures and an opposite ones. Now recall that the space of oriented 2-planes at a point of \( M \) is the Grassmann manifold \( \widetilde{G}_{T2}(\mathbb{R}^4) \). In terms of the spin group and its subgroups:

\[
\begin{align*}
\widetilde{G}_{T2}(\mathbb{R}^4) &= \frac{SO(4)}{SO(2) \times SO(2)} = \frac{\alpha(S^3 \times S^3)}{\alpha(S^1 \times S^1)} \times \frac{\alpha(S^3 \times S^3)}{\alpha(S^1 \times S^1)} = \frac{SO(4)}{U(2)} \times \frac{SO(4)}{U'(2)} \\
&= \{ \text{space of } J \} \times \{ \text{space of } J' \}. \tag{30}
\end{align*}
\]

This correspondence globalizes and we see that fields of 2-planes are in one to one correspondence with the pairs \( (J, J') \) (cf. [20]).

We must consider an indefinite 4-manifold \( (M, g) \) of signature \((++--)\). A survey on the structure group \( SO_0(2,2) \), associated with a \((++--)\)-metric, cannot be overlooked. Consider the exact sequence (cf. [23])

\[
0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin_0(2,2) \xrightarrow{\alpha} SO_0(2,2) \longrightarrow 1. \tag{31}
\]
It is known that

\[ \text{Spin}_0(2,2) \cong SU(1,1) \times SU(1,1) \cong H^3 \times H^3, \]  

(32)

where \( SU(1,1) \cong \text{topologically} \ H^3 \) is the 3-dimensional pseudo-sphere of \( \mathbb{R}^{2,2} \). Since \( S^1 \) is a subgroup of \( H^3 \), the spin group \( \text{Spin}_0(2,2) \) has \( S^1 \times H^3, H^3 \times S^1 \) and \( S^1 \times S^1 \), as subgroups.

The existence of \( J \) on \((M, g)\) implies that the structure group of \( TM \) reduces to \( \alpha(S^1 \times SU(1,1)) = U(1,1) \subset SO_0(2,2) \). Similarly, the existence of \( J' \) on \((M, g)\) reduces the structure group to \( \alpha(SU(1,1) \times S^1) = U'(1,1) \subset SO_0(2,2) \). Here, \( U(1,1) \) (resp. \( U'(1,1) \)) is the stabilizer of a standard (resp. opposite) almost complex structure, which is orthogonal with respect to the \((++--)-\)metric \( g \). Therefore, the existence of a pair \((J, J')\) corresponds to the reduction to

\[ U(1,1) \cap U'(1,1) = U(1) \times U(1) = SO(2) \times SO(2), \]  

(33)

the maximal torus in \( SO_0(2,2) \). It is important to note that the existence of a field of 2-planes on \( M \) implies the reduction to \( G = SO(2) \times SO(2) \), which is nothing but \( U(1) \times U(1) \). See the following diagrams.

**Indefinite \((++--)\):**

<table>
<thead>
<tr>
<th>Spin group</th>
<th>Structure group</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Spin}_0(2,2) )</td>
<td>( SO_0(2,2) )</td>
</tr>
<tr>
<td>( S^1 \times SU(1,1) )</td>
<td>( SU(1,1) \times S^1 )</td>
</tr>
<tr>
<td>( U(1,1) )</td>
<td>( U'(1,1) )</td>
</tr>
<tr>
<td>( S^1 \times S^1 )</td>
<td></td>
</tr>
</tbody>
</table>

**Positive definite \((++++)\):**

<table>
<thead>
<tr>
<th>Spin group</th>
<th>Structure group</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Spin}(4) )</td>
<td>( SO(4) )</td>
</tr>
<tr>
<td>( S^1 \times SU(2) )</td>
<td>( SU(2) \times S^1 )</td>
</tr>
<tr>
<td>( U(2) )</td>
<td>( U'(2) )</td>
</tr>
<tr>
<td>( S^1 \times S^1 )</td>
<td></td>
</tr>
</tbody>
</table>
This is a good position to consider examples of compact 4-manifolds which admit a \((++--)\)-metric, or a pair \((J, J')\) of two kinds of almost complex structures, and also a field of 2-planes. Since Theorems 3 and 4 are clear and simple, we can easily find a lot of examples. In fact, a 4-manifold of the form \(W_{m,n,k} = m \mathbb{CP}^2 \# n \overline{\mathbb{CP}}^2 \# k (\mathbb{CP}^1 \times R) \) (\(R\): a curve of genus 2) [31], with indefinite form \((m + k)(1) \oplus (n + k)(-1)\), admits a \((++--)\)-metric if and only if \(n \equiv m \mod 2\) and \(k \equiv m + 1 \mod 2\) [18]. The 4-manifolds \(V_{m,n} = m (S^1 \times S^3) \# n \mathbb{CP}^2 \) (\(m, n \geq 0\), with positive definite form \(\mu_M = n(1)\), admits a \((++--)\)-metric if and only if \(m\) is odd, \(n\) even, and \(m \leq n + 1\) [18]. It is worthwhile to show that the simply connected 4-manifolds

\[
W_{m,n,0} = m \mathbb{CP}^2 \# n \overline{\mathbb{CP}}^2,
\]

with indefinite form \(m(1) \oplus n(-1)\), admits a \((++--)\)-metric, equivalently a pair \((J, J')\) of two kinds of almost complex structures, or also equivalently a field of 2-planes, if and only if \(n\) and \(m\) are both odd (see [17]).

If we restrict our attention to compact complex surfaces as listed in the Enriques-Kodaira classification [2, VI, Table 10], we see from Theorem 4 the following [4, Proposition 5]:

**Theorem 5.** A surface \(X\) admits a \((++--)\)-metric if and only if the second Chern number \(c_2(X) = \chi[X]\) is even.

**Proof.** Since a surface \(X\) has an integrable almost complex structure \(J\), \(X\) admits a \((++--)\)-metric if and only if \(X\) admits an opposite almost complex structure \(J'\). The first condition (Noether's theorem) in (20) already holds. For the surfaces with definite intersection forms, we must verify the additional conditions (21) and (22). These conditions follow from results in [18, Theorems 14, 15].

This theorem is a powerful tool for finding examples of surfaces which admit a \((++--)\)-metric, equivalently an opposite almost complex structure, or also equivalently a field of 2-planes. We can see that the following surfaces are such examples:

- (a) minimal rational surfaces with \((c_1^2, c_2) = (8, 4)\)
- (b) Hopf surfaces and Inoue surfaces of minimal surfaces of Class VII
- (c) Ruled surfaces of genus \(g \geq 1\)
- (d) Enriques surfaces
- (e) Hyper elliptic surfaces
(f) Kodaira surfaces
(g) K3 surfaces
(h) Tori
(i) minimal properly elliptic surfaces
(j) surfaces of general type with \( c_2 \) even

We now take a K3 surface in order to illustrate our analysis for the \( (++--) \)-metrics. By a classical K3 surface, denoted by \( X \), we mean the underlying real 4-manifold of a K3 surface. We explain why \( X \) admits a \( (++--) \)-metric. The intersection form of \( X \) is indefinite of Type \( I_{1} \) of the form:

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix} \oplus -2E_8.
\]

\( X \) is a simply connected 4-manifold, and has Euler characteristic \( \chi[X] = 24 \) \((= 2 + \text{rank of } \mu_X)\) and Hirzebruch index \( \tau[X] = -16 \). From Theorem 3 (b), we can recognize whether or not \( X \) can admit a \( (++--) \)-metric. The condition (10) there becomes

\[
\chi[X] - \tau[X] = 24 - (-16) = 40 \equiv 0 \mod 4,
\]

which by Theorem 3 (b) is equivalent to the coefficient of the hyperbolic part being odd, as it is: 3. Thus, \( X \) satisfies (10), and therefore it admits a \( (++--) \)-metric. As a real 4-manifold, \( X \) admits infinitely many almost complex structure \( J \), with the Chern numbers

\[
c_1^2 = 3 \tau[X] + 2 \chi[X] = 3 \cdot (-16) + 2 \cdot 24 = 0, \quad c_2 = \chi[X] = 24.
\]

Moreover, \( X \) admits an opposite almost complex structure \( J' \), with the opposite Chern numbers

\[
c_1^2(-X) = -3 \tau[X] + 2 \chi[X] = -3 \cdot (-16) + 2 \cdot 24 = 96,
\]

\[
c_2(-X) = \chi[X] = 24.
\]

Now, recognize that a K3 surface is a classical K3 surface \( X \) with an integrable almost complex structure \( J \) whose first Chern class is \( c_1 = 0 \). \( X \) can admits infinitely many non-integrable almost complex structures, whose first Chern classes \( c_1 \) are in general not zero, but with zero squared norm \( c_1^2 = 0 \). We can easily see that the Miyaoka-Yau inequality holds [25]:

\[
c_1^2 \leq 3c_2 \Rightarrow 0 \leq 3 \cdot 24 = 72.
\]

However, the opposite version of the Miyaoka-Yau inequality becomes

\[
c_1^2(-X) \leq 3c_2(-X) \iff 4c_2 - c_1^2 \leq 3c_2 \iff c_2 \leq c_1^2,
\]
which clearly fails for $X$, i.e., $24 \not\leq 0$. Thus, we can conclude that $X$ cannot admit an integrable opposite almost complex structure (see [3], [12]).

One of the interesting facets of neutral geometry is that despite the indefinite signature, there are strong parallels with positive definite signature $(+++)$ and the neutral signature $(++--)$. There are similar isomorphisms for the Lie algebras $so(2, 2)$ and $so(4)$ as follows:

$$so(2, 2) = so(1, 2) + so(1, 2)$$
$$so(4) = so(3) + so(3).$$

These decompositions are closely related with the decompositions of the bundle $\Lambda^2(M)$ of two-forms on a 4-manifold $M$ into a sum of ± eigen 2-forms of the Hodge star operator.

It is highly desirable to have explicit examples of pairs $(J, J')$ on both Riemannian and neutral 4-folds. We give explicit orthogonal complex structures on a 4-dimensional vector spaces $(\mathbb{R}^4, (++--))$ of neutral signature, and $(\mathbb{R}^4, (++++)$) of Euclidean norm.

i) $(\mathbb{R}^4, (++--))$: An orthogonal complex structure $J$ on $(\mathbb{R}^4, (++--))$ has the form

$$J = \begin{bmatrix} 0 & -a & b & c \\ a & 0 & c & -b \\ b & c & 0 & -a \\ c & -b & a & 0 \end{bmatrix}, \quad a^2 - b^2 - c^2 = 1,$$

where parameters $a$, $b$ and $c$ represent the quotient space $SO_0(2, 2)/U(1, 1) = H^2$.

An orthogonal opposite complex structure $J'$ on $(\mathbb{R}^4, (++--))$ has the form

$$J' = \begin{bmatrix} 0 & -a' & b' & c' \\ a' & 0 & -c' & b' \\ b' & -c' & 0 & a' \\ c' & b' & -a' & 0 \end{bmatrix}, \quad a'^2 - b'^2 - c'^2 = 1,$$

where parameters $a'$, $b'$ and $c'$ represent the quotient space $SO_0(2, 2)/U'(1, 1) = H^2$. Note that $J$ and $J'$ commute with each other.

ii) $(\mathbb{R}^4, (++++)$: An orthogonal complex structure $J$ on $(\mathbb{R}^4, (++++))$ has the form

$$J = \begin{bmatrix} 0 & -a & b & c \\ a & 0 & c & -b \\ -b & -c & 0 & -a \\ -c & b & a & 0 \end{bmatrix}, \quad a^2 + b^2 + c^2 = 1.$$
where parameters $a$, $b$ and $c$ represent the quotient space $SO(4)/U(2) = S^2$.

An orthogonal opposite complex structure $J$ on $(\mathbb{R}^4, (++++)$) has the form

$$J' = \begin{bmatrix} 0 & -a' & b' & -c' \\ a' & 0 & c' & b' \\ -b' & -c' & 0 & a' \\ c' & -b' & -a' & 0 \end{bmatrix}, \quad a'^2 + b'^2 + c'^2 = 1, \quad (45)$$

where parameters $a'$, $b'$ and $c'$ represent the quotient space $SO(4)/U'(2) = S^2$. Note that $J$ and $J'$ commute with each other: $JJ' = J'J$.

We end this note with a brief review of significant families of $(++--)$-metrics of 4-dimensional Walker geometry [32], [33], which have been intensively studied by the author and colleagues in recent years. See [21], [22], [5].

A Walker 4-manifold is a triple $(M, g, D)$ consisting of a 4-manifold $M$, together with an indefinite metric $g$ and a nonsingular field of 2-dimensional planes $D$ (or distribution) such that $D$ is parallel and null with respect to $g$. From Walker's theorem [32, Theorem 1 and §6 Case 1], there is a system of coordinates $(x^1, x^2, x^3, x^4)$ with respect to which $g$ takes the canonical form

$$g = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{bmatrix}, \quad (46)$$

where $a$, $b$ and $c$ are functions of the coordinates $(x^1, x^2, x^3, x^4)$. We see that $g$ is of signature $(++--)$ (or neutral). The parallel null 2-plane $D$ is spanned locally by $\{ \partial_1, \partial_2 \}$, where $\partial_i$ are the abbreviated forms of $\frac{\partial}{\partial x^i}$, $(i = 1, \ldots, 4)$.

We call a $g$-orthogonal almost complex structure $J$ on a Walker 4-manifold $M$ proper if $J$ defines a standard generator of a positive $\frac{\pi}{2}$-rotation on $D$, i.e., explicitly

$$J\partial_1 = \partial_2 \quad J\partial_2 = -\partial_1. \quad (47)$$

The following is a fundamental fact for the present issue.

**Theorem 6.** The canonical form (46) of $g$ defines a unique proper almost complex structure $J$ on a Walker 4-manifold $M$; namely, the $J$ defined by
the following action on the coordinate basis:

\[
\begin{align*}
J \partial_1 &= \partial_2, \\
J \partial_3 &= -c \partial_1 + \frac{1}{2} (a - b) \partial_2 + \partial_4 \\
J \partial_4 &= \frac{1}{2} (a - b) \partial_1 + c \partial_2 - \partial_3.
\end{align*}
\] (48)

In terms of the metric $g$ and the proper almost complex structure $J$, we can define a Kähler form $\Omega(X, Y) = g(JX, Y)$, whose explicit form is given by

\[
\Omega = dx^1 \wedge dx^4 - dx^2 \wedge dx^3 + \frac{1}{2} (a + b) dx^3 \wedge dx^4.
\] (49)

We are interested in when $\Omega$ is symplectic, i.e., $\Omega$ is closed. (In what follows, we shall use the abbreviation $\partial p(x^1, x^2, x^3, x^4)/\partial x^i = \partial p/\partial x^i = p_i$, for any function $p$ and $i = 1, \ldots, 4$.)

**Theorem 7.** $\Omega$ is symplectic if and only if the sum $a + b$ is independent of $x^1$ and $x^2$. In fact, $a$ and $b$ satisfy the following PDE’s:

\[
a_1 + b_1 = 0, \quad a_2 + b_2 = 0.
\] (50)

Then the Kähler form (49) becomes symplectic.

The proper almost complex structure $J$ in (48) is integrable if and only if the torsion of $J$ (Nijenhuis tensor) vanishes.

**Theorem 8.** The proper almost complex structure $J$ is integrable if and only if the following PDE’s hold:

\[
a_1 - b_1 - 2c_2 = 0, \quad a_2 - b_2 + 2c_1 = 0.
\] (51)

As an important result, we have the Kähler condition as follows.

**Theorem 9.** The triple $(g, J, \Omega)$ is Kähler if and only if the following PDE’s hold

\[
a_1 = -b_1 = c_2, \quad a_2 = -b_2 = -c_1.
\] (52)

Moreover, if the triple $(g, J, \Omega)$ is Kähler, then the functions $a$, $b$ and $c$ are all harmonic with respect to the first two arguments $(x^1, x^2)$. That is,

\[
a_{11} + a_{22} = 0, \quad b_{11} + b_{22} = 0, \quad c_{11} + c_{22} = 0.
\] (53)
Theorem 9 provides a useful method of producing examples of indefinite Kähler 4-manifolds. We begin with a harmonic function $h(x, y)$ of two variables $(x, y)$, which satisfies the following Laplace equation

$$ (\partial_{xx} + \partial_{yy})h(x, y) = 0. \quad (54) $$

Many harmonic functions $h(x, y)$ of two variables are known, e.g., as follows:

\[
x^2 - y^2, \quad 2x(1 - y), \quad (x - y)(x^2 + 4xy + y^2), \quad x^3 - 3xy^2, \\
\cos x \sinh y, \quad e^x \cos y, \quad e^x(x \cos y - y \sin y), \quad \log(x^2 + y^2), \quad \text{etc.} \quad (55)
\]

We shall construct an indefinite Kähler 4-manifold, starting from a harmonic function, for example $h(x, y) = \cos x \sinh y$. First put $a = h(x^1, x^2) + \psi(x^3, x^4)$, i.e., as follows:

$$ a = a(x^1, x^2, x^3, x^4) = \cos x^1 \sinh x^2 + \psi(x^3, x^4), \quad (56) $$

where $\psi$ is an arbitrary smooth function of $(x^3, x^4)$. Then, $a$ is also harmonic with respect to $(x^1, x^2)$, and we have

$$ a_1 = -\sin x^1 \sinh x^2, \quad a_2 = \cos x^1 \cosh x^2. \quad (57) $$

From (52), we have PDE’s for $b$ to satisfy as

$$ b_1 = -a_1 = \sin x^1 \sinh x^2, \quad b_2 = -a_2 = -\cos x^1 \cosh x^2, \quad (58) $$

and similarly PDE’s for $c$ to satisfy as

$$ c_1 = -a_2 = -\cos x^1 \cosh x^2, \quad c_2 = a_1 = -\sin x^1 \sinh x^2. \quad (59) $$

These PDE’s are easily solved, and we have solutions

$$ b = b(x^1, x^2, x^3, x^4) = -\cos x^1 \sinh x^2 + \lambda(x^3, x^4), \quad (60) $$

$$ c = c(x^1, x^2, x^3, x^4) = \sin x^1 \cosh x^2 + \mu(x^3, x^4), \quad (61) $$

where $\lambda(x^3, x^4), \mu(x^3, x^4)$ are arbitrary smooth functions of $(x^3, x^4)$. Thus the indefinite Kähler metric takes the form

$$ g = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & \cos x^1 \sinh x^2 + \psi(x^3, x^4) & \sin x^1 \cosh x^2 + \mu(x^3, x^4) \\
0 & 1 & \sin x^1 \cosh x^2 + \mu(x^3, x^4) - \cos x^1 \sinh x^2 + \lambda(x^3, x^4)
\end{bmatrix}. \quad (62) $$

**Remark.** Petean’s nonflat indefinite Kähler-Einstein metric [27] can be constructed in this way as a very special case. Assume first that $h = 0$. Then, we have that $a = \psi(x^3, x^4)$, $b = \lambda(x^3, x^4)$, and $c = \mu(x^3, x^4)$. If we
further assume that \( c = \mu(x^3, x^4) = 0 \), and that \( \psi(x^3, x^4) = \lambda(x^3, x^4) \), i.e., \( a = b = \psi(x^3, x^4) \), then the metric becomes

\[
g = \begin{bmatrix} g_{ij} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \psi(x^3, x^4) & 0 \\ 0 & 1 & 0 & \psi(x^3, x^4) \end{bmatrix},
\]

which is precisely Petean's example. Concerning indefinite Kähler metrics, see [10].

Einstein-Walker metrics can also be determined [5].

**Theorem 10.** A Walker metric (3), with \( c = 0 \), is Einstein if and only if the defining functions \( a(x^1, x^2, x^3, x^4) \) and \( b(x^1, x^2, x^3, x^4) \) are either of Type I: \( a \) is a quadratic function with respect to \( x^1 \) and is independent of \( x^2 \), and similarly \( b \) is a quadratic function with respect to \( x^2 \) and is independent of \( x^1 \), i.e.,

\[
\begin{align*}
a &= a(x^1, x^3, x^4) = k(x^1)^2 + x^1 R(x^3, x^4) + \xi(x^3, x^4) \\
b &= b(x^2, x^3, x^4) = k(x^2)^2 + x^2 P(x^3, x^4) + \eta(x^3, x^4),
\end{align*}
\]

where \( R \) and \( P \) are subject to the following PDE:

\[
R_{x^4} + P_{x^3} = 0,
\]

or Type II: \( a \) and \( b \) are linear functions with respect to \( x^1 \) and \( x^2 \) as follows:

\[
\begin{align*}
a &= a(x^1, x^2, x^3, x^4) = x^1 R(x^3, x^4) + x^2 S(x^3, x^4) + \xi(x^3, x^4) \\
b &= b(x^1, x^2, x^3, x^4) = x^2 P(x^3, x^4) + x^1 Q(x^3, x^4) + \eta(x^3, x^4),
\end{align*}
\]

where \( P(x^3, x^4), Q(x^3, x^4), R(x^3, x^4) \) and \( S(x^3, x^4) \) are subject to the following PDE's:

\[
S_{x^4} = \frac{1}{2} SP, \quad Q_{x^3} = \frac{1}{2} QR, \quad R_{x^4} + P_{x^3} = SQ.
\]

Finally we show an example, due to Haze, of noncompact indefinite Ricci flat almost-Kähler non-Kähler 4-manifolds [22]. This is an indefinite version of the example given by Nurowski and Przanowski [26]. The metric

\[
g = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \frac{2x^1}{x^3} & 0 \\ 0 & 1 & 0 & \frac{2x^1}{x^3} \end{bmatrix}
\]
is a Ricci flat metric on the coordinate patch $x^3 > 0$ (or $x^3 < 0$), which admits an almost Kähler structure, but not Kähler structure. Note that the metric (68) belongs to Type II Einstein-Walker metrics as shown in (66), with (67). This is a counter example of the Goldberg conjecture [8] for the case of indefinite and noncompact 4-manifolds.

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ON THE COMPLEX WKB ANALYSIS FOR A 2ND ORDER LINEAR O.D.E. WITH THE MOST GENERAL CHARACTERISTIC POLYGON

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Asymptotics of O.D.E. appearing in the turning point problems can be characterized literally by its characteristic polygon. The simplest case is the Airy equation which has a one-segment characteristic polygon. Nakano [15], [16], [20]~[22], Nakano et al. [23], and Roos [28], [29] study O.D.E.'s with a several-segment one. Here, we study an O.D.E. with an arbitrarily finitely many-segment one.

1. Introduction

1.1. We study the following one-dimensional Schrödinger equation

\[ \varepsilon^2 h \frac{d^2 y}{dx^2} = Q(x, \varepsilon)y, \quad Q(x, \varepsilon) := \sum_{k=0}^{s} \sum_{l=0}^{l_k-1} a_{jk+l} e^{j_k+l} x^{m_{jk}-l} \alpha_{jk}^{-1}; \]

\[ h, s \in \mathbb{N}; \quad x, y, a_{jk+l} \in \mathbb{C}, \ \forall a_{jk} \neq 0, \ a_{j_k+l} = 0 \ (l \geq 1); \]

\[ 0 < \varepsilon < 1; \ D := \{ x : 0 \leq |x| \leq x_0 \}, \]

where \( x_0 \) is a small constant, and

\[
\begin{align*}
\alpha_{jk}^{-1} := & \frac{m_{jk} - m_{jk+1}}{j_{k+1} - j_k} > 0, \quad \alpha_{j_k}^{-1} := \frac{m_{j_k} + 2}{h}, \\
\alpha_{jk}^{-1} > & \alpha_{j_k+1}^{-1} > 0 \ (k = 0, 1, 2, \ldots, s - 1), \\
l_k := & j_{k+1} - j_k \ (> 0) \ (k = 0, 1, 2, \ldots, s - 1), \\
l_0 := & 0, \ j_s := h, \ \sum_{k=0}^{s-1} l_k = h \ (l_k \in \mathbb{N}), \\
h > & \frac{j_k + \alpha_{jk}(m_{jk} + 2)}{2} \quad (k = 0, 1, 2, \ldots, s).
\end{align*}
\]
The zeros of \( Q(x,0) (= a_{j_0} x^{m_{j_0}}) \) are called turning points of (1.1), and so (1.1) has a turning point at \( x = 0 \) of order \( m_{j_0} \). The inequality (1.4) is called the singular perturbation condition.

Our aim is to analyze the asymptotics of solutions of (1.1) in \( D \) by using the concept of the characteristic polygon (defined below) and by applying what we call the stretching-matching method (Nakano [15], [16], [18], [20]~[22], Nakano et al. [23], Nishimoto [25], Wasow [33]).

1.2. We set another \((X,Y)\)-plane, on which we put points \( P_k \)'s according to the indexes of \( \varepsilon \) and \( x \) of terms in \( Q(x,\varepsilon) \) and a point \( R \) according to the index of \( \varepsilon \) on the left hand side of (1.1) defined respectively by

\[
\begin{align*}
    P_k^{(l)} &= \left( \frac{j_k + l \cdot m_{j_k} - l \cdot \alpha_{j_k}^{-1}}{2}, \frac{j_k + l \cdot m_{j_k} - l \cdot \alpha_{j_k}^{-1}}{2} \right), \\
    R &= (h, -1).
\end{align*}
\]

Thus the point \( P_k^{(l)} \) corresponds to the term \( a_{j_k+l} \varepsilon^{j_k+l} x^{m_{j_k}+l \cdot \alpha_{j_k}^{-1}} \) and vice versa.

Let \( L_k \) \((k = 1, 2, \ldots, s)\) be a segment on which lie the points

\[
P_{k-1}^{(0)}, P_{k-1}^{(1)}, P_{k-1}^{(2)}, \ldots, P_{k-1}^{(l_k-1)}, P_k^{(0)},
\]

and let \( L_{s+1} \) be a segment connecting two points \( P_s^{(0)} \) and \( R \). The relations in (1.2) represent inclinations of the segments and a relation between them. The characteristic polygon for (1.1) is, by definition, a polygon connecting all the segments \( L_k \)'s in order (Iwano-Sibuya [12]). This characteristic polygon is convex downward due to the third relation of (1.2) and the singular perturbation condition (1.4), and it snaps at \( P_k^{(0)} \)'s so that it consists of \( s+1 \) segments where \( s \) is any positive integer.

The differential equations with a two-, a three- or a much more-segment characteristic polygon are studied in Fedoryuk [6], Nakano [15], [21], [22], Nakano et al. [23], Roos [28], [29] and they are a particular case of (1.1). Nakano [18], [20] study the \( n \)-th order O.D.E. Third order differential equations are studied from a different point of view from them (Aoki et al. [1], Berk et al. [2], Matsubara et al. [14], Nakano [17], [19], Nakano et al. [24]).

Remark. An asymptotic contribution of a point over the characteristic polygon corresponding to a term \( x^m \) \((m > m_{j_k} - l \cdot \alpha_{j_k}^{-1})\) of \( \varepsilon^{j_k+l} \) is known to be smaller than the terms corresponding to the points on the characteristic polygon (Iwano-Sibuya [12]). Therefore, in this sense, (1.1) is the most general Schrödinger equation of this type.
1.3. The contents of this paper are as follows. In §2, we reduce (1.1) asymptotically to the simpler differential equations in appropriate subdomains of $D$. In §3, the WKB approximations for the reduced differential equations are got. The WKB approximations are the truncated formal solutions. In §4, we give a brief review of a canonical domain and a formal computation of matching matrices for a differential equation which is rather concrete but rather general. Two sets of the solutions of the reduced differential equations are matched, i.e., they are connected linearly by the matching matrix by using their WKB approximations. In the last section (§ 5) we obtain matching matrices for (1.1). Thus we can know any asymptotic information about solutions in $D$.

2. The asymptotic reductions of (1.1)

2.1. Each term of $Q(x, \varepsilon)$ can be considered to be "the asymptotically dominant term" in some subdomain of $D$. For example, we show the term $a_{j_k} \varepsilon^{i_k} x^{m_{j_k}}$ is dominant. In order to do so, we pick up the term to the head and separate $Q(x, \varepsilon)$ into three parts as follows:

$$Q(x, \varepsilon) = a_{j_k} \varepsilon^{i_k} x^{m_{j_k}} \left\{ \sum_{k=0}^{\hat{k}-1} \sum_{l=0}^{l_k-1} + \sum_{l=0}^{i_k-1} + \sum_{k=k+1}^{x} \sum_{l=0}^{l_k-1} \frac{a_{j_k+l}}{a_{j_k}} \varepsilon^{j_k+k+l} x^{m_{j_k}-m_{j_k}-l\cdot\varepsilon^{-1}} \right\}. \quad (2.1)$$

The first $\Sigma\Sigma$ represents

$$\sum_{k=0}^{\hat{k}-1} \sum_{l=0}^{l_k-1} \frac{a_{j_k+l}}{a_{j_k}} \left( \varepsilon\varepsilon^{-1} x^{\alpha_{j_k-1}} \right)^{l_k-l} \prod_{n=0}^{k-1} \left( \varepsilon\varepsilon^{-1} x^{\alpha_{j_k+n}} \right)^{l_k+n}, \quad (2.2)$$

which tends to zero when $\varepsilon^{-1} x^{\alpha_{j_k-1}} \rightarrow 0$, namely, which is small as $\varepsilon \rightarrow 0$ for $x \in \{ x : |x| \leq \tilde{k} \varepsilon^{\alpha_{j_k-1}} \}$ ($\tilde{k}$ a sufficiently small constant) because $\varepsilon^{-1} x^{\alpha_{j_k-1}} \rightarrow 0$ as $\varepsilon \rightarrow 0$ for $x$ in the smallest region $\{ x : |x| \leq \tilde{k} \varepsilon^{\alpha_{j_k-1}} \}$.

The second $\Sigma$ in (2.1) represents

$$\sum_{l=0}^{i_k-1} \frac{a_{j_k+l}}{a_{j_k}} \left( \varepsilon x^{\alpha_{j_k}} \right)^l, \quad (2.3)$$

which is tends to one when $\varepsilon x^{\alpha_{j_k}} \rightarrow 0$, namely, when $\varepsilon \rightarrow 0$ for $x \in \{ x : |x| \geq \tilde{K} \varepsilon^{\alpha_{j_k}} \}$ ($\tilde{K}$ a sufficiently large constant).
The last $\Sigma\Sigma$ represents

\[(2.4) \quad \sum_{k=\bar{k}+1}^{s} \sum_{l=0}^{l_k-1} \frac{a_{jk+l}}{a_{jk}} (\varepsilon x^{-\alpha_{jk}^{-1}})^{l} \prod_{n=0}^{k-\bar{k}-1} (\varepsilon x^{-\alpha_{jk+n}^{-1}})^{l_k+n}, \]

which tends to zero when $\varepsilon x^{-\alpha_{jk}^{-1}} \to 0$, namely, which is small as $\varepsilon \to 0$ for $x \in \{ x : |x| \geq \bar{K} \varepsilon^{\alpha_{jk}} \}$ ($\bar{K}$ a sufficiently large constant) because $\varepsilon x^{-\alpha_{jk}^{-1}} \to 0$ as $\varepsilon \to 0$ for $x$ in the largest region $\{ x : |x| \geq \bar{K} \varepsilon^{\alpha_{jk}} \}$.

Thus, we can conclude that the term $a_{jk} \varepsilon^{j_k} x^{m_{jk}}$ is dominant to obtain the reduced differential equation

\[(2.5) \quad \varepsilon^{2h-j_k} \frac{d^2y}{dx^2} = a_{jk} x^{m_{jk}} y \]

as $\varepsilon \to 0$ for $x$ in the subdomain $D_{out,j_k} (\subset D)$ defined by

\[(2.5)' \quad D_{out,j_k} := \{ (x, \varepsilon) : \bar{K} \varepsilon^{\alpha_{jk}} \leq |x| \leq \bar{K} \varepsilon^{\alpha_{jk-1}} \}. \]

2.2. In the intermediate domain, designated by $D_{j_{k+1}}$, between $D_{out,j_k}$ and $D_{out,j_{k+1}}$, (1.1) can be asymptotically reduced as follows. Applying the stretching transformation, which is the first step of the stretching-matching method,

\[(2.6) \quad x := t \varepsilon^{\alpha_{jk}} \]

to (1.1), we obtain the differential equation

\[(2.7) \begin{cases} 
\varepsilon^{2}_{j_{k+1}} \frac{d^2y}{dt^2} = Q_{j_{k+1}} (t)y \quad (\varepsilon^{2}_{j_{k+1}} := \varepsilon^{2h-(j_{k}+\alpha_{jk}(m_{jk}+2))}) \\
Q_{j_{k+1}} (t) := \sum_{l=0}^{l_k} a_{jk+l} t^{m_{jk}-l-\alpha_{jk}^{-1}} \end{cases} \]

in the domain

\[(2.8) \quad D_{j_{k+1}} := \{ t : \bar{k} \leq |t| \leq \bar{K} \}. \]

Since the coefficient $Q_{j_{k+1}} (t)$ is a polynomial, the differential equation (2.7) possesses its own turning points at zeros of $Q_{j_{k+1}} (t)$ which are called secondary turning points of (1.1). (Thus, analyzing differential equations with secondary turning points is called a secondary turning point problem. The first secondary turning problem is studied in Nakano et al. [23].)
We show how to get (2.7) in the following. After substituting (2.6) for \( x \) in (1.1) we get the equation

\[
\varepsilon^{2h-2\alpha_{jk}} \frac{d^2y}{dt^2} = \tilde{Q} y
\]

(2.9)

\[
\tilde{Q} = \varepsilon^{j_k + \alpha_{jk}} m_k \sum_{k=0}^{s} \sum_{l=0}^{l_k-1} a_{jk+l} \varepsilon^{j_k+l+\alpha_{jk}(m_{jk}-l^{-1})-j_k-\alpha_{jk} m_{jk}}
\]

where \( \sum_1 \) contains all the terms corresponding to the points

\[
P_{00}^{(0)}, P_{01}^{(1)}, P_{02}^{(2)}, \ldots, P_{0l_k-1}^{(l_k-1)}, P_{10}^{(0)}, P_{11}^{(1)}, \ldots, P_{1l_k-1}^{(l_k-1)}
\]

on the segments \( L_1, L_2, \ldots, L_k \); \( \sum_2 \) contains terms corresponding to the points

\[
P_{k0}^{(0)}, P_{k1}^{(1)}, P_{k2}^{(2)}, \ldots, P_{kl_k-1}^{(l_k-1)}, P_{k+10}^{(0)}
\]

on \( L_{k+1} \); and \( \sum_3 \) contains terms corresponding to the points

\[
P_{k+10}^{(1)}, P_{k+11}^{(2)}, \ldots, P_{s0}^{(0)}
\]

on \( L_{k+2}, \ldots, L_s \). Then we can see that

\[
\sum_1 = \sum_{k=0}^{k-1} \sum_{l=0}^{l_k-1} a_{jk+l} \varepsilon^{g t^{m_{jk}-l^{-1}}} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0 \quad \text{for} \quad t \in D_{j_{k+1}}
\]

because the \( \varepsilon \)'s exponent

\[
g := \sum_{n=k+1}^{k-1} l_n \cdot \alpha_j^{-1}(\alpha_{jk} - \alpha_{j_n}) + (l_k - l) \cdot \alpha_j^{-1}(\alpha_{j_{k-1}} - \alpha_{j_k})
\]

is positive. The second sum is a polynomial of \( t \) such as

\[
\sum_2 = \sum_{l=0}^{l_k} a_{jk+l} t^{m_{jk}-l^{-1}}
\]

because the exponent of \( \varepsilon \) vanishes. In the third sum, the \( \varepsilon \)'s exponent

\[
g' := \sum_{n=1}^{k-k-1} l_{k+n} \cdot \alpha_{j_{k+n}}^{-1}(\alpha_{jk} + n - \alpha_{j_k}) + l \cdot \alpha_j^{-1}(\alpha_{j_{k-1}} - \alpha_{j_k})
\]

is also positive so that

\[
\sum_3 \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0 \quad \text{for} \quad t \in D_{j_{k+1}}.
\]
Thus we can get the equation (2.7).

We call (2.5) is an outer equation for (1.1) and (2.7) an inner equation for (1.1) according to Nakano et al. [23], and their solutions are called an outer and an inner solution, respectively.

2.3. Summing up the above consideration, we get the following result.

**Theorem 2.1.** The differential equation (1.1) is asymptotically reduced in (2.5)' to (2.5) which corresponds to the point $P_k^{(0)}$ on the characteristic polygon.

The differential equation (1.1) is also asymptotically reduced in (2.8) to (2.7) which corresponds to the points $P_k^{(0)}, P_k^{(1)}, P_k^{(2)}, \ldots, P_k^{(d_k-1)}, P_k^{(0)}$ on the segment $L_{k+1}$ of the characteristic polygon.

We should notice that the domain (2.8) is bounded but it must be extended to an unbounded domain

$$(2.8)' \quad D_{in} := \{ t : 0 < |t| < \infty \}$$

in order to match, i.e., to connect linearly two sets of inner and outer solutions. It is the second step of the stretching-matching method (§4.4, 4.5 and §5). We call (2.5)' (resp. (2.8)') the outer (resp. inner) domain of (1.1).

3. The WKB approximations

3.1. Both the outer equation (2.5) and the inner equation (2.7) have a common form to a singular perturbation. Then, we here study

$$\varepsilon^{2h} \frac{d^2 y}{dx^2} = Q(x) y \quad (h \in \mathbb{N}; x, y \in \mathbb{C}; 0 \leq |x| < \infty; 0 < \varepsilon < 1),$$

where $Q(x)$ is a polynomial of $x$. $x = \infty$ is an irregular singular point of (3.1). A point $x = a$ is called a turning point of (3.1) if $Q(a) = 0$. There exist many turning points when $Q(x)$ is of high degree.

WKB approximations $\tilde{y}^\pm(x, \varepsilon)$ for (3.1) are, by definition, given by

$$\tilde{y}^\pm(x, \varepsilon) := \frac{1}{\sqrt{Q(x)}} \exp \left( \pm \frac{1}{\varepsilon h} \xi(a, x) \right),$$

where

$$\xi(a, x) := \int_a^x \frac{1}{\sqrt{Q(x)}} \, dx.$$
(3.2) is sometimes called a formal WKB solutions. A curve on the $x$-plane defined by the equation
\[ \Re \xi(a, x) = C \quad (Q(a) = 0) \]
is called a level curve of level $C$ and it is called a Stokes curve for (3.1) when $C = 0$, and a curve defined by the equation
\[ \Im \xi(a, x) = C \quad (Q(a) = 0) \]
is called a level curve of level $C$ and it is called an anti-Stokes curve for (3.1) when $C = 0$. Near $x = \infty$, Stokes and anti-Stokes curves are defined by $\xi(\infty, x) = 0$.

3.2. The main property of the Stokes and the anti-Stokes curves is as follows:

(i) Stokes curves and anti-Stokes curves from a turning point tend to other turning points or to the irregular singular point $x = \infty$.

(ii) A Stokes curve can not cross itself. An anti-Stokes curve can not cross itself, too.

(iii) There are no (sums of several) Stokes curves or anti-Stokes curves homotopic to a circle.

The map $\xi := \xi(a, x) \ (Q(a) = 0)$ defined by (3.3) is a conformal mapping from the $x$-plane to the $\xi$-plane except for the turning points because $d\xi/dx \neq 0$ at $x \neq a$. Then the level curves defined by (3.4) and ones defined by (3.5) are mapped perpendicularly on the $\xi$-plane by $\xi := \xi(a, x)$.

When an unbounded simply connected domain $D^{\text{can}}$ on the $x$-plane is mapped onto the whole $\xi$-plane except for one or several slits, it is called a canonical domain for (3.1). Several canonical domains exist generally for one differential equation and it is known that at least one canonical domain certainly exists for (3.1). But we cannot construct any canonical domain for an abstract $Q(x)$ because we cannot get any turning points and then any Stokes curve. In the case of a concrete polynomial, we can construct them as given in Fedoryuk [6], Nakano [22], Wasow [33]. In every canonical domain two independent solutions of (3.1) exist and it is the maximal domain in which true solutions have the WKB approximations as their asymptotic expansions as stated in the theorem below. The detail information is given in several references (Fedoryuk [6], [7], Wasow [33]).

When $Q(x)$ is a rational function, the Stokes curve configuration is very different from the case of a polynomial. Several examples can be seen in Fedoryuk [6] and Nakano [16], [18]. Another brief review of canonical domains will be given in §4.2.
The WKB approximations $\tilde{y}^\pm(x, \varepsilon)$ possess the double asymptotic property stated as follows (Evgrafov et al. [4], Fedoryuk [6]).

**Theorem 3.1.** Let $\tilde{y}^\pm(x, \varepsilon)$ and $\mathcal{D}_{\text{can}}$ be the WKB approximations and the canonical domain for (3.1) respectively. Then, there exist the true solutions $y^\pm(x, \varepsilon)$ of (3.1) having $\tilde{y}^\pm(x, \varepsilon)$ as their asymptotic expansions:

\[
y^\pm(x, \varepsilon) \sim \tilde{y}^\pm(x, \varepsilon) \quad \text{as} \quad \begin{cases} x \to \infty, & \varepsilon \to 0, \\ \varepsilon \to 0, & x \in \mathcal{D}_{\text{can}} \end{cases} \quad (0 < \varepsilon < 1) .
\]

The WKB approximation is valid near both the irregular singular point ($x = \infty$) and the turning points. If $\mathcal{D}_{\text{can}}$ is bounded from $x = \infty$, as in a case where it is regarded as a bounded subset of some canonical domain, the first relation is naturally vacant.

### 3.3 According to §3.1, the outer and inner WKB approximations $\tilde{y}_{out,k}^\pm(x, \varepsilon)$ for (2.5) and $\tilde{y}_{in,k+1}^\pm(t, \varepsilon)$ for (2.7) respectively are given by

\[
\tilde{y}_{out,k}^\pm(x, \varepsilon) := \frac{1}{\sqrt{a_{j_k} x^{j_k}}} \exp\left( \pm \frac{2 \sqrt{a_{j_k}} x^{(m_{j_k} + 2)/2}}{\varepsilon^{j_k/2} m_{j_k} + 2} \right),
\]

\[
\tilde{y}_{in,k+1}^\pm(t, \varepsilon) := \frac{1}{\sqrt{Q_{j_{k+1}}(t)}} \exp\left( \pm \frac{1}{\varepsilon_{j_{k+1}}} \int_{t}^{\varepsilon_{j_{k+1}}} \sqrt{Q_{j_{k+1}}(t)} \, dt \right),
\]

where $Q_{j_{k+1}}(a) = 0$. The asymptotic property of the inner WKB approximations is as follows:

\[
\tilde{y}_{in,k+1}^\pm(t, \varepsilon) \sim \begin{cases} \frac{1}{\sqrt{a_{j_{k+1}} t^{j_{k+1}}}} \exp\left( \pm \frac{2 \sqrt{a_{j_{k+1}}} t^{(m_{j_{k+1}} + 2)/2}}{\varepsilon_{j_{k+1}} m_{j_{k+1}} + 2} \right) & (t \to \infty) \\ \frac{1}{\sqrt{a_{j_{k+1}} t^{m_{j_{k+1}}}}} \exp\left( \pm \frac{2 \sqrt{a_{j_{k+1}}} t^{(m_{j_{k+1}} + 2)/2}}{\varepsilon_{j_{k+1}} m_{j_{k+1}} + 2} \right) & (t \to 0) \end{cases},
\]

where we notice that $j_k + l_k = j_{k+1}$, $m_{j_k} - l_k \cdot \alpha_{j_k}^{-1} = m_{j_{k+1}}$. Then, from Theorem 3.1, we obtain

**Theorem 3.2.** Assume the necessary canonical domains. Then there exist true solutions $y_{out,k}^\pm(x, \varepsilon)$ (resp. $y_{in,k+1}^\pm(t, \varepsilon)$) of (2.5) (resp. (2.7)) such that the following asymptotic properties are valid:

\[
y_{out,k}^\pm(x, \varepsilon) \sim \tilde{y}_{out,k}^\pm(x, \varepsilon) \quad \text{as} \quad \varepsilon \to 0, \quad x \in \mathcal{D}_{out} \cap \mathcal{D}_{\text{can}}.
\]
where $\mathcal{D}^\text{can}_{\text{out}}$ is a canonical domain for (2.5), and

$$\begin{align*}
y_{\text{in},j+1}(t, \varepsilon) \sim \tilde{y}_{\text{in},j+1}(t, \varepsilon) \quad \text{as} \quad & \begin{cases} 
\varepsilon \to 0, & t \in \mathcal{D}^\text{can}_{\text{in}}, \\
\varepsilon \to \infty, & 0 < \varepsilon < 1,
\end{cases} 
\end{align*}$$

where $\mathcal{D}^\text{can}_{\text{in}}$ is a canonical domain for (2.7).

We should notice that the outer WKB approximations $\tilde{y}_{\text{out},j}(x, \varepsilon)$ have a single asymptotic property as shown below Theorem 3.1, because the $x$-domain $\mathcal{D}^\text{can}_{\text{out}}$ is bounded.

4. Formal computation of matching matrices

4.1. Two differential equations (2.5) and (2.7) seem to be rather complicated on the surface. In order to understand their essence, then, we consider tentatively the following rather concrete differential equation and try to formally compute matching matrices between an inner and an outer solution:

$$\begin{align*}
(((a \neq 0, b \neq 0; h, \alpha, M, M', m', m \in \mathbb{N}; M > M' > m' > m))
\end{align*}$$

where we suppose that all the points corresponding to terms of $Q$ lie on one segment of the characteristic polygon. Then, by the similar reduction to §2, we get the following dominant equations in each specified region:

\begin{align*}
(4.2)_{\text{out,1}} & \quad \varepsilon^2 h^2 y'' = ax^M y, \quad \varepsilon \to 0 \quad \text{for} \quad x : \tilde{K} \varepsilon^\alpha \leq |x| \leq \tilde{K} \varepsilon^\alpha', \\
(4.2)_{\text{out,2}} & \quad \varepsilon^2 h - \alpha(M-m) y'' = b x^m y, \quad \varepsilon \to 0 \quad \text{for} \quad x : (\tilde{K} \varepsilon^\alpha'' \leq |x| \leq \tilde{K} \varepsilon^\alpha,
\end{align*}

where $\alpha'$ and $\alpha''$ are constants such as $\alpha' < \alpha < \alpha''$, and $\tilde{K}$ and $\tilde{K}$ are sufficiently large and small constants, respectively. $(4.2)_{\text{out,1,2}}$'s are outer equations and the specified regions are outer domains. Putting

$$x = t \varepsilon^\alpha \quad \text{(a stretching transformation),}$$

we can transform (4.1) to

$$\begin{align*}
(4.3)_{\text{in}} \quad & \begin{cases} 
\varepsilon^{2h-\alpha(M+2)} y'' = Q(t) y \quad (\tilde{k} \leq |t| \leq \tilde{K}), \\
Q(t) := at^M + a' t^{M'} + \cdots + b' t^{m'} + b t^m \sim \begin{cases} 
at^M \quad (t \to \infty), \\
b t^m \quad (t \to 0).
\end{cases}
\end{cases}
\end{align*}$$
which is an inner equation. In order to match an outer and an inner solutions (§4.4), we should extend the bounded region in (4.3) to

\[(4.3)_{in}'\]  \[D_{in} : 0 < |t| < \infty,\]

which is called an inner domain and in which we must construct a canonical domain \(D_{in}^{can}\) for \((4.3)_{in}\).

4.2. Since the coefficient of \((4.3)_{in}\) is a polynomial, a number of turning points of \((4.3)_{in}\) is finite and a number of the Stokes and anti-Stokes curves is also finite. Every canonical domain on the \(t\)-plane is an unbounded simply connected set and it is mapped by the mapping \(\xi = \xi(a, t) \; (:= \int_a^t \sqrt{Q(t)} \, dt, \; a = 0 \; \text{or} \; \infty)\) onto the whole \(\xi\)-plane with a finite number of slits each of which starts from the image of the turning point and goes to infinity \((+i \cdot \infty\; \text{or} \; -i \cdot \infty)\) along a direct line parallel to the imaginary axis. Here, we should notice that \(t = \infty\) is an irregular singular point for \((4.3)_{in}\), and so there exist \(M + 2\) Stokes curves and \(M + 2\) anti-Stokes curves near \(t = \infty\). The slit is an image of the Stokes (or level) curve which is a component of the boundary of the canonical domain \(D_{in}^{can}\).

There exists a direct line which starts from an image of some turning point parallel to the real axis on the \(\xi\)-plane and along which a value \(\Re \xi\) increases to infinity as \(t\) tends to infinity. The inverse image of this line is one of anti-Stokes (or level) curves, which is labeled by \(\gamma_{+\infty}\). Also, there exists another direct line which starts from an image of the same or other turning point parallel to the real axis and along which a value \(\Re \xi\) decreases to \(-\infty\) as \(t\) tends to infinity. The inverse image of this line is also one of anti-Stokes (or level) curves, which is labeled by \(\gamma_{-\infty}\).

Since \(t = 0\) is a turning point of \((4.3)_{in}\) (a secondary turning point of \((4.1)\)), there exist \(m + 2\) Stokes curves and there exist a same number of anti-Stokes curves emerging from the origin. Hence, there exists a line in the \(\xi\)-plane which is an image of an anti-Stokes curve and along which \(\Re \xi\) decreases to 0 as \(t\) tends to 0. We label this anti-Stokes curve by \(\gamma_{+0}\). Also, there exists another line which is an image of an anti-Stokes curve and along which \(\Re \xi\) increases to 0 as \(t\) tends to 0. We label this anti-Stokes curve by \(\gamma_{-0}\).

The inner domain \((4.3)_{in}'\) is unbounded and it contains anti-Stokes (or level) curves such as \(\gamma_{\pm \infty}\) and \(\gamma_{\pm 0}\), and we choose a canonical domain \(D_{in}^{can}\) including both \(\gamma_{\pm \infty}\) and \(\gamma_{\pm 0}\). We notice that those four curves are partly contained in the domain of \((4.3)_{in}'\) and they continue into the outer domains because \(\arg x = \arg t\) since the parameter \(\varepsilon\) is real \((x = t \varepsilon^\alpha, \; \varepsilon > 0)\).
4.3. The WKB approximations of \((4.2)_{out,1}\) and \((4.3)_{in}\) are, respectively, given by

\[
(4.2)_{out,1}^{WKB} \quad \tilde{y}^\pm_{out,1}(x, \varepsilon) := \frac{1}{\sqrt{a x}} \exp \left( \pm \frac{\sqrt{a}}{\varepsilon^h} \frac{2}{M + 2} x^{(M+2)/2} \right),
\]

\[
(4.2)_{out,2}^{WKB} \quad \tilde{y}^\pm_{out,2}(x, \varepsilon) := \frac{1}{\sqrt{b x}} \exp \left( \pm \frac{\sqrt{b}}{\varepsilon^h} \frac{2}{M + 2} x^{(m+2)/2} \right),
\]

\[
(4.3)_{in}^{WKB} \quad \tilde{y}^\pm_{in}(t, \varepsilon) := \frac{1}{\sqrt{Q(t)}} \exp \left( \pm \frac{1}{\varepsilon^h} \frac{1}{(M+2)/2} \int_0^t \sqrt{Q(t)} \, dt \right),
\]

and an asymptotic form of \((4.3)_{in}^{WKB}\) is as follows:

\[
\tilde{y}^\pm_{in}(t, \varepsilon) \sim \begin{cases} 
\frac{1}{\sqrt{a t}} \exp \left( \pm \frac{\sqrt{a}}{\varepsilon^h} \frac{2}{M + 2} t^{(M+2)/2} \right) & (t \to \infty) \\
\frac{1}{\sqrt{b t}} \exp \left( \pm \frac{\sqrt{b}}{\varepsilon^h} \frac{2}{M + 2} t^{(m+2)/2} \right) & (t \to 0)
\end{cases}
\]

4.4. We define solution vectors as follows:

\[
(4.5)_{out,1} \quad [\tilde{O}_j] := t \left[ \tilde{y}^+_{out,1}(x, \varepsilon), \tilde{y}^-_{out,1}(x, \varepsilon) \right] \quad (j = 1, 2),
\]

\[
(4.5)_{in} \quad [\tilde{I}] := t \left[ \tilde{y}^+_{in}(t, \varepsilon), \tilde{y}^-_{in}(t, \varepsilon) \right].
\]

A matching matrix

\[
(4.6)_{j} \quad \tilde{M}_j := M[\tilde{O}_j, \tilde{I}] \quad (j = 1, 2)
\]

connecting \([\tilde{O}_j]\) to \([\tilde{I}]\) is a \(2 \times 2\) matrix and it is, by definition,

\[
(4.6)'_{j} \quad \tilde{M}_j \cdot [\tilde{O}_j] \sim [\tilde{I}] \quad (\varepsilon \to 0).
\]

We now compute \(\tilde{M}_1\). Putting \(\tilde{M}_1 := \begin{bmatrix} a & b \\ c & d \end{bmatrix}\), we get from \((4.6)'_1\)

\[
\begin{align*}
\begin{cases}
\frac{\tilde{y}^+_{out,1}(x, \varepsilon)}{\tilde{y}^-_{out,1}(x, \varepsilon)} + \frac{b \tilde{y}^-_{out,1}(x, \varepsilon)}{\tilde{y}^+_{out,1}(x, \varepsilon)} & \sim 1 \\
\frac{\tilde{y}^+_{out,1}(x, \varepsilon)}{\tilde{y}^-_{out,1}(x, \varepsilon)} & \sim 1 \quad (\varepsilon \to 0).
\end{cases}
\end{align*}
\]

Since \(x\) and \(t\) are related such as \(x = t \varepsilon^\alpha\), we put

\[
x := \eta \varepsilon^{\alpha-\beta}, \quad t := \eta \varepsilon^{-\beta} \quad (0 < \beta < \alpha, \ |\eta| = 1).
\]
Then, $x$ belongs to the outer domain $\{ x : \tilde{K} \varepsilon^\alpha \leq |x| \}$, and $x \to 0$ and $t \to \infty$ as $\varepsilon$ tends to 0. Substituting (4.8) for $x$ and for $t$ in the WKB approximations, we get

$$
\begin{align*}
\tilde{y}_{\text{out},1}(x, \varepsilon) &= \frac{1}{\sqrt{a(\eta \varepsilon^{-\beta})^M}} \exp\left( \pm \frac{2 \sqrt{a}}{M + 2} \eta^{(M+2)/2} \varepsilon^{-\hat{g}} \right) \\
\tilde{y}_{\text{in}}(t, \varepsilon) &\sim \frac{1}{\sqrt{a(\eta \varepsilon^{-\beta})^M}} \exp\left( \pm \frac{2 \sqrt{a}}{M + 2} \eta^{(M+2)/2} \varepsilon^{-\hat{g}} \right) \quad (t \to \infty),
\end{align*}
$$

where $\hat{g} := h - (\alpha - \beta)(M + 2)/2 > 0$, and we can see that, as $\varepsilon$ tends to 0,

$$(4.10)_1 \quad \frac{\tilde{y}_{\text{out},1}(x, \varepsilon)}{\tilde{y}_{\text{in}}(t, \varepsilon)} = \varepsilon^{-\alpha M/4} \exp(0),$$

$$(4.10)_2 \quad \frac{\tilde{y}_{\text{out},1}(x, \varepsilon)}{\tilde{y}_{\text{in}}(t, \varepsilon)} = \varepsilon^{-\alpha M/4} \exp\left( -\frac{4 \sqrt{a}}{M + 2} \eta^{(M+2)/2} \varepsilon^{-\hat{g}} \right) \to \infty \quad (\Re \eta < 0)$$

if a new parameter $\eta$ is taken such that $\arg \eta = \arg \gamma_{-\infty}$ for a sufficiently large $|t|$ in the canonical domain $\mathcal{D}_{\text{in}}^{\text{can}}$ of (4.3)$_{\text{in}}$ (cf. §4.2.), or in other words, if $t$ goes to $\infty$ along $\gamma_{-\infty} \in \mathcal{D}_{\text{in}}^{\text{can}}$. Similarly, we can see that

$$(4.10)_3 \quad \frac{\tilde{y}_{\text{out},1}(x, \varepsilon)}{\tilde{y}_{\text{in}}(t, \varepsilon)} = \varepsilon^{-\alpha M/4} \exp\left( \frac{4 \sqrt{a}}{M + 2} \eta^{(M+2)/2} \varepsilon^{-\hat{g}} \right) \to \infty \quad (\Re \eta > 0)$$

if another new parameter $\eta$ is taken such that $\arg \eta = \arg \gamma_{+\infty}$ for a sufficiently large $|t|$ in the canonical domain $\mathcal{D}_{\text{in}}^{\text{can}}$ (or, if $t \to \infty$ along $\gamma_{+\infty} \in \mathcal{D}_{\text{in}}^{\text{can}}$), and

$$(4.10)_4 \quad \frac{\tilde{y}_{\text{out},1}(x, \varepsilon)}{\tilde{y}_{\text{in}}(t, \varepsilon)} = \varepsilon^{-\alpha M/4} \exp(0).$$

Therefore, (4.7) becomes

$$(4.11) \quad \begin{cases} a \cdot \varepsilon^{-\alpha M/4} + b \cdot \infty \sim 1 \\
c \cdot \infty + d \cdot \varepsilon^{-\alpha M/4} \sim 1, \end{cases}$$

then we get

$$(4.11)' \quad a \sim \varepsilon^{\alpha M/4}, \quad b \sim 0, \quad c \sim 0, \quad d \sim \varepsilon^{\alpha M/4} \quad (\varepsilon \to 0),$$

thus

$$(4.12) \quad \tilde{M}_1 = \varepsilon^{\alpha M/4} E,$$

where $E$ is the 2-dim. unit matrix.
4.5. We can obtain another matching matrix $\tilde{M}$ defined by

$$\tilde{M} := M[\tilde{I}, \tilde{O}_2]$$

connecting $[\tilde{I}]$ to $[\tilde{O}_2]$:

$$\tilde{M} \cdot [\tilde{I}] \sim [\tilde{O}_2] \quad (\varepsilon \to 0).$$

In fact, when we write (4.13) as

$$(4.13)' \quad \tilde{M}^{-1} \cdot [\tilde{O}_2] \sim [\tilde{I}] \quad (\varepsilon \to 0),$$

we can compute $\tilde{M}^{-1}$ in the very similar way to $\tilde{M}_1$, namely,

$$\tilde{M}^{-1} = \tilde{M}_2.$$ 

Putting $\tilde{M}^{-1} := \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]$ again, we get a similar relation to (4.7):

$$(4.14) \quad \begin{cases} a \frac{\tilde{y}_{\text{out},2}(x, \varepsilon)}{\tilde{y}_{\text{in}}(t, \varepsilon)} + b \frac{\tilde{y}_{\text{out},2}(x, \varepsilon)}{\tilde{y}_{\text{in}}(t, \varepsilon)} \sim 1 \\ c \frac{\tilde{y}_{\text{out},2}(x, \varepsilon)}{\tilde{y}_{\text{in}}(t, \varepsilon)} + d \frac{\tilde{y}_{\text{out},2}(x, \varepsilon)}{\tilde{y}_{\text{in}}(t, \varepsilon)} \sim 1 \end{cases} \quad (\varepsilon \to 0).$$

Since $x$ and $t$ are related such as $x = t\eta^\alpha$, we put, in this time,

$$(4.15) \quad x := \eta \varepsilon^{\alpha + \beta}, \quad t := \eta \varepsilon^\beta \quad (\alpha > \beta > 0, \ |\eta| = 1).$$

Then, we can easily see that $x$ belongs to the outer domain $\{ x : |x| \leq \tilde{k} \varepsilon^\alpha \}$, and $x \to 0$ and $t \to 0$ as $\varepsilon \to 0$. Substituting (4.15) for $x$ and for $t$ in the WKB approximations, we get

$$\begin{align*}
\frac{\tilde{y}_{\text{out},2}(x, \varepsilon)}{\tilde{y}_{\text{in}}(t, \varepsilon)} & \sim \frac{1}{\sqrt{b(\eta \varepsilon^{\alpha + \beta})^m}} \exp \left( \pm \frac{2\sqrt{b}}{m+2} \eta^{(m+2)/2} \varepsilon^{-\hat{g}} \right) \\
\tilde{y}_{\text{in}}(t, \varepsilon) & \sim \frac{1}{\sqrt{b(\eta \varepsilon^\beta)^m}} \exp \left( \pm \frac{2\sqrt{b}}{m+2} \eta^{(m+2)/2} \varepsilon^{-\hat{g}} \right) \quad (t \to 0)
\end{align*}$$

(4.16)

where $\hat{g} := h - \alpha(M + 2)/2 - \beta(m + 2)/2 > 0$. Therefore we get the following relations, as $\varepsilon$ tends to 0,

$$\begin{align*}
(4.17)_1 & \quad \frac{\tilde{y}_{\text{out},2}(x, \varepsilon)}{\tilde{y}_{\text{in}}(t, \varepsilon)} = \varepsilon^{-\alpha m/4} \exp(0), \\
(4.17)_2 & \quad \frac{\tilde{y}_{\text{out},2}(x, \varepsilon)}{\tilde{y}_{\text{in}}(t, \varepsilon)} = \varepsilon^{-\alpha m/4} \exp \left( -\frac{4\sqrt{b}}{m+2} \eta^{(m+2)/2} \varepsilon^{-\hat{g}} \right) \to \infty \quad (\Re \eta < 0)
\end{align*}$$
if a new parameter $\eta$ is taken such that $\arg \eta = \arg \gamma_{-0}$ for a sufficiently small $|t|$ in the canonical domain $D_{in}^{can}$, or in other words, if $t$ approaches to the origin along $\gamma_{-0} \in D_{in}^{can}$. Similarly, we can see that, as $\varepsilon$ tends to 0,

$$(4.17)_3 \quad \frac{\tilde{y}_{out,2}(x, \varepsilon)}{\tilde{y}_{in}(t, \varepsilon)} = \varepsilon^{-a m/4} \exp \left( \frac{4 \sqrt{b}}{m + 2} \eta^{(m+2)/2} \varepsilon \tilde{g} \right) \to \infty \quad (\Re \eta > 0)$$

if another new parameter $\eta$ is taken such that $\arg \eta = \arg \gamma_{+0}$ for a sufficiently small $|t|$ in the canonical domain $D_{in}^{can}$ (or, if $t \to 0$ along $\gamma_{+0} \in D_{in}^{can}$), and

$$(4.17)_4 \quad \frac{\tilde{y}_{out,2}(x, \varepsilon)}{\tilde{y}_{in}(t, \varepsilon)} = \varepsilon^{-a m/4} \exp(0).$$

Therefore, (4.14) becomes

$$(4.18) \quad \begin{cases} a \cdot \varepsilon^{-a m/4} + b \cdot \infty \sim 1 \\ c \cdot \infty + d \cdot \varepsilon^{-a m/4} \sim 1' \end{cases}$$

then we get

$$(4.18)' \quad a \sim \varepsilon^{a m/4}, \quad b \sim 0, \quad c \sim 0, \quad d \sim \varepsilon^{a m/4} \quad (\varepsilon \to 0),$$

thus

$$(4.19) \quad \tilde{M}^{-1} = \varepsilon^{a m/4} E,$$

and

$$(4.20) \quad \tilde{M} = \varepsilon^{-a m/4} E.$$

Therefore we can obtain the following

**Theorem 4.1.** Let $D_{in}^{can}$ be a canonical domain for $(4.3)_{in}$. Then three solution vectors $(4.5)_{out,j}$, $(4.5)_{in}$ are related by the matching matrices defined by $(4.6)_j$ (or $(4.6)_j'$) and $(4.13)$, which are given by

$$(4.21) \quad \tilde{M}_1 = \varepsilon^{a M/4} E, \quad \tilde{M}_2 = \varepsilon^{a m/4} E, \quad \tilde{M} = \varepsilon^{-a m/4} E.$$

**Remark.** We notice the relation $\tilde{M}_2 = \tilde{M}^{-1}$. These matching matrices depend on the $\varepsilon$'s exponent of the stretching transformation and the $x$'s exponent of the first or the last term of the polynomial coefficient $Q(x, \varepsilon)$. These terms correspond to the end points of a segment of the characteristic polygon (cf. §7 of Nakano [22]).
5. Matching matrices for (1.1)

5.1 Since the reduced differential equations (2.5) and (2.7) are asymptotically derived from (1.1), their solutions have linear relations, which can be represented by matching matrices as shown in §4.

Let \( y_{\text{out}, j_k}(x, \varepsilon) \) be the true solutions of (2.5) and \( y_{\text{in}, j_{k+1}}(t, \varepsilon) \) be the true solutions of (2.7), then the corresponding WKB approximations are (3.7) and (3.8), respectively. And we define solution vectors such as

\[
\begin{align*}
O_{j_k} & := \begin{bmatrix} y_{\text{out}, j_k}^+(x, \varepsilon), y_{\text{out}, j_k}^-(x, \varepsilon) \end{bmatrix}, \\
I_{j_{k+1}} & := \begin{bmatrix} y_{\text{in}, j_{k+1}}^+(t, \varepsilon), y_{\text{in}, j_{k+1}}^-(t, \varepsilon) \end{bmatrix}.
\end{align*}
\]

Then we can obtain matching matrices between them as follows:

**Theorem 5.1.** The matching matrix \( M[O_{j_k}, I_{j_{k+1}}] \) connecting a solution vector \( O_{j_k} \) to a solution vector \( I_{j_{k+1}} \) is a \( 2 \times 2 \) matrix and it is, by definition, given by the relation

\[
M[O_{j_k}, I_{j_{k+1}}] \circ \quad [0_{j_k}] = [I_{j_{k+1}}].
\]

Then the matching matrix is asymptotically given by

\[
M[O_{j_k}, I_{j_{k+1}}] \sim \varepsilon^{a_{j_k}m_{j_k}/4} E \quad (\varepsilon \to 0).
\]

Another matching matrix defined by \( M[I_{j_{k+1}}, O_{j_k}] \) is

\[
M[I_{j_{k+1}}, O_{j_k}] \sim \varepsilon^{-a_{j_k}m_{j_{k+1}}/4} E \quad (\varepsilon \to 0).
\]

**Proof.** By substituting the WKB approximations for the corresponding true solutions in (5.1), we get the asymptotic relation

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_{\text{out}, j_k}^+(x, \varepsilon) \\ y_{\text{out}, j_k}^-(x, \varepsilon) \end{bmatrix} \sim \begin{bmatrix} y_{\text{in}, j_{k+1}}^+(t, \varepsilon) \\ y_{\text{in}, j_{k+1}}^-(t, \varepsilon) \end{bmatrix} \quad (\varepsilon \to 0),
\]

where we put

\[
M[O_{j_k}, I_{j_{k+1}}] := \begin{bmatrix} a & b \\ c & d \end{bmatrix}.
\]

We write (5.3) simply as

\[
\tilde{M} \cdot \begin{bmatrix} \tilde{O}_{j_k} \\ \tilde{I}_{j_{k+1}} \end{bmatrix} \sim \begin{bmatrix} \tilde{O}_{j_k} \\ \tilde{I}_{j_{k+1}} \end{bmatrix} \quad (\varepsilon \to 0).
\]

The coefficient of (2.7) is

\[
Q_{j_{k+1}}(t) = a_{j_k} y_{j_{k+1}}^m + \ldots + a_{j_{k+1}} y_{j_{k+1}}^{m_{j_{k+1}}}
\]
and the relation between $x$ and $t$ is $x = t^\xi$ (in (2.6)). If we read $a_{jk}$, $m_{jk}$, $a_{j+1}$, and $m_{j+1}$ as $a$, $M$, $b$, and $m$ in (4.1), respectively, and we read also $\alpha_{jk}$ as $\alpha$ (of the stretching transformation in §4.1.) then we get

$$(5.6) \quad \tilde{M} = \varepsilon^{\alpha_{jk} m_{jk} / 4} E.$$  

By the similar way, we get the matching matrix $\tilde{M}$:

$$(5.7) \quad \tilde{M} \sim \varepsilon^{-\alpha_{jk} m_{j+1} / 4} E \quad (\varepsilon \to 0)$$

which is defined by

$$(5.7)' \quad \tilde{M} \cdot [I_{j+1}] = [O_{j+1}].$$  

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SPEED AND ACCELERATION OF PROJECTIVE BÉZIER CURVES OF ORDER 2 ON REAL SPACE FORMS

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We calculate derivatives of projective Bézier curves of order 2 and study the $C^2$ condition in joining them.

Keywords: Bézier curves of order 2, Bézier splines, smoothness, standard sphere, projective Bézier curves, real hyperbolic plane

1. Introduction

In the area of Computer Aided Geometric Design Bézier curves provide a powerful tool for designing curves. But when we need curves of complicated shape, we have to use Bézier curves of high degrees. In order to draw complicated curves, it is hence better to use Bézier spline curves which are curves obtained by being composited Bézier curves. As Bézier curves are defined by use of polynomials, it is needless to say that they are of $C^\infty$. But for Bézier spline curves, as they are piecewise polynomial curves, we need to study their smoothness at their junctions.

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ACCELERATION OF PROJECTIVE BÉZIER CURVES

For Bézier spline curves and rational Bézier spline curves some studies were already done (see for example [1, 2]). In this paper we restrict ourselves on Bézier spline curves of order 2 and investigate conditions to join two rational Bézier curves of order 2 smoothly. We also study speed and acceleration of projective Bézier curves on a standard sphere and a real hyperbolic space. Projective Bézier curves are introduced in [4] by Kawabata and the second author as generalizations of rational Bézier curves on a Euclidean space. They are obtained as images of Bézier curves through a conic projection centered at the origin. We extend our results on rational Bézier spline curves of order 2 to projective Bézier spline curves of order 2 on these real space forms.

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2. Bézier spline curves in a Euclidean space

We shall start with recalling Bézier curves in a Euclidean space. A Bézier curve $\sigma$ of order 2 in a Euclidean space $\mathbb{R}^3$ with control points $P_0, P_1$ and $P_2$ is expressed as

$$\sigma(t) = (1 - t)^2 P_0 + 2t(1 - t) P_1 + t^2 P_2 \quad (0 \leq t \leq 1),$$

where points in $\mathbb{R}^3$ are identified with vectors in a vector space $\mathbb{R}^3$. Its differentials are given as

$$\sigma'(t) = 2(1 - t)(P_1 - P_0) + 2t(P_2 - P_1),$$
$$\sigma''(t) = 2(P_2 - 2P_1 + P_0). \quad (2.1)$$

Hence the initial vector is tangent to the line joining $P_0$ and $P_1$, and the terminus vector is tangent to the line joining $P_1$ and $P_2$.

**Example 2.1.** Here we show an example. In order to see the speed of a Bézier curve $\sigma$, we plot points $\sigma(k/6), k = 0, 1, \ldots, 6$. We also attach tangent and acceleration vectors at these points.

![Figure 1. Speed and acceleration of a Bézier curve](image-url)
In order to get a smooth curve by joining two Bézier curves, we take 4 control points $P_0$, $P_1$, $P_3$ and $P_4$ and define a junction $P_2$ by $P_2 = (1 - \lambda)P_1 + \lambda P_3$ with some $0 < \lambda < 1$. We consider Bézier curves $\sigma_1, \sigma_2$ of control points $P_0, P_1, P_2$ and $P_2, P_3, P_4$, and define a curve $\sigma$ for positive numbers $\alpha_1, \alpha_2$ by

$$\sigma(s) = \begin{cases} \sigma_1(\alpha_1 s), & 0 \leq s \leq \frac{1}{\alpha_1}, \\ \sigma_2(\alpha_2(s - \frac{1}{\alpha_1})), & \frac{1}{\alpha_1} \leq s \leq \frac{1}{\alpha_1} + \frac{1}{\alpha_2}. \end{cases} \quad (2.2)$$

We call this a Bézier spline curve of order 2. By just the same way we can define a Bézier spline curve from given $(n + 2)$ control points $P_0, P_1, P_3, P_5, \ldots, P_{2n-1}, P_{2n}$, which joins $n$ Bézier curves of order 2. One can easily find the following well-known results by (2.1) (see for example [1]).

**Proposition 2.1.** A Bézier spline curve $\sigma$ of order 2 with 4 control points is of $C^1$ if and only if we choose $\alpha_i$ so that they satisfy $\alpha_1 \lambda = \alpha_2 (1 - \lambda)$.

**Proposition 2.2.** A Bézier spline curve $\sigma$ is of $C^2$ if and only if four control points lie on a plane and there is an auxiliary point $R$ such that

i) $P_1$ divides the segment $P_0R$ with the ratio $\lambda : (1 - \lambda)$,

ii) $P_3$ divides the segment $RP_4$ with the ratio $\lambda : (1 - \lambda)$.

In particular, in this case junction $P_2$ is uniquely determined.

![Figure 2. $C^2$ Bézier spline curve of order 2](image)

This proposition shows that if $P_4$ lies on the opposite side of $P_0$ on a plane with respect to the line $P_1P_3$ we can not take $C^2$ Bézier spline curve with these control points.

### 3. Rational Bézier spline curves on a Euclidean plane

In this section we study rational Bézier spline curves, which we also call projective Bézier spline curves on a Euclidean plane. We represent a Eu-
clidean plane \( \mathbb{R}^2 \) as a plane \( \{ (x, y, 1) \mid (x, y) \in \mathbb{R}^2 \} \) in a Euclidean space \( \mathbb{R}^3 \). Given control points \( Q_0, Q_1, Q_2 \in \mathbb{R}^2 \) and positive numbers \( \omega_0, \omega_1, \omega_2 \), we define a rational Bézier curve \( \gamma \) of order 2 by

\[
\gamma(t) = \frac{(1-t)^2 \omega_0 Q_0 + 2t(1-t)\omega_1 Q_1 + t^2 \omega_2 Q_2}{(1-t)^2 \omega_0 + 2t(1-t)\omega_1 + t^2 \omega_2} \quad (0 \leq t \leq 1).
\]

This is the image of a Bézier curve \( \sigma \) in \( \mathbb{R}^3 \) with control points \( \omega_0 Q_0, \omega_1 Q_1, \omega_2 Q_2 \) through a conic projection. The positive numbers \( \omega_i, i = 0,1,2 \) are called weights of \( \gamma \). If we denote \( \sigma(t) \) by \( (\sigma_x(t), \sigma_y(t), \sigma_z(t)) \), we see \( \gamma(t) = \sigma(t)/\sigma_z(t) \), hence its derivatives are given as

\[
\gamma'(t) = \frac{\sigma'(t)}{\sigma_z(t)} - \frac{\sigma_z'(t)}{\sigma_z(t)} \gamma(t),
\]

\[
\gamma''(t) = \frac{\sigma''(t)}{\sigma_z(t)} - \frac{2\sigma_z'(t)}{\sigma_z(t)^2} \sigma'(t) + \left( \frac{2\sigma_z'(t)^2}{\sigma_z(t)^2} - \frac{\sigma_z''(t)}{\sigma_z(t)} \right) \gamma(t).
\]

In particular, we find the initial and terminus vectors are

\[
\gamma'(0) = \frac{2\omega_1}{\omega_0} (Q_1 - Q_0), \quad \gamma'(1) = \frac{2\omega_1}{\omega_2} (Q_2 - Q_1),
\]

and the second derivatives at initial and terminus are

\[
\gamma''(0) = \frac{2\omega_2}{\omega_0} (Q_2 - Q_0) + \frac{4\omega_1}{\omega_0^2} (\omega_0 - 2\omega_1)(Q_1 - Q_0),
\]

\[
\gamma''(1) = -\frac{2\omega_0}{\omega_2} (Q_2 - Q_0) - \frac{4\omega_1}{\omega_0^2} (\omega_2 - 2\omega_1)(Q_2 - Q_1).
\]

**Example 3.1.** Here we give two examples of rational Bézier curves of order 2. We take 3 control points \( Q_0, Q_1 \) and \( Q_2 \) on a plane \( \mathbb{R}^2 \). In Figure 3, the weights of the rational Bézier curve of left hand-side are \( \omega_0 = 2, \omega_1 = 1, \omega_2 = 2 \), and the weights of the right hand-side are \( \omega_0 = 4, \omega_1 = 1, \omega_2 = 1 \). Comparing these examples we can see the influence of weights.

![Figure 3. Speed and acceleration of rational Bézier curves](image)
Rational Bézier spline curves of order 2 are defined just the same way as for Bézier spline curves. Given control points \( Q_0, Q_1, Q_3, Q_4 \in \mathbb{R}^2 \) we take a junction \( Q_2 = (1 - \lambda)Q_1 + \lambda Q_3 \) with some \( \lambda \) (\( 0 < \lambda < 1 \)). For two rational Bézier curves \( \gamma_1, \gamma_2 \) with control points \( Q_0, Q_1, Q_2 \) and \( Q_2, Q_3, Q_4 \) and with weights \( \omega_0, \omega_1, \omega_2, \omega_3, \omega_4 \), we define rational Bézier spline curve with positive numbers \( \alpha_1, \alpha_2 \) by the same way as in (2.2). We see by (3.1) the following well-known result.

**Proposition 3.1.** A rational Bézier spline curve \( \gamma \) is of \( C^1 \) if and only if we choose \( \alpha_1, \alpha_2 \) so that they satisfy \( \alpha_1 \omega_1 \lambda = \alpha_2 \omega_3 (1 - \lambda) \).

Next we consider the \( C^2 \) condition. Is it possible to adjust the weights so that the resulting Bézier spline curve is of \( C^2 \)? The answer is the following:

**Theorem 3.1.** We consider a rational Bézier spline curve which is obtained by joining two rational Bézier curves of order 2 on \( \mathbb{R}^2 \).

1. If three points \( Q_0, Q_1, Q_3 \) lie on some line \( \ell \), we can take \( C^2 \) rational Bézier spline curves if and only if \( Q_4 \) also lies on \( \ell \). In this case, its image is a segment.

2. If \( Q_0 \) and \( Q_4 \) lie on the same side with respect to the line \( Q_1Q_3 \), then there are infinitely many \( C^2 \) rational Bézier spline curves with these control points and given junction \( Q_2 \) on the segment \( Q_1Q_3 \).

3. If \( Q_4 \) lies on the opposite side of \( Q_0 \) with respect to the line \( Q_1Q_3 \), then we can not take \( C^2 \) rational Bézier spline curves with these control points.

As a matter of fact, when \( Q_0, Q_1, Q_2 \) do not lie on a line in \( \mathbb{R}^2 \), we linearly decompose the vector \( Q_4 - Q_3 \) as

\[
Q_4 - Q_3 = a(1 - \lambda)(Q_3 - Q_1) + b(Q_1 - Q_0).
\]

Combining the \( C^1 \) condition in Proposition 4.1, we obtain from (3.2) that the rational Bézier spline curve is of \( C^2 \) if and only if the following equalities on weights hold:

\[
\begin{aligned}
\omega_0 \omega_3^2 (1 - \lambda)^2 + b \omega_1^2 \omega_4 \lambda^2 &= 0, \\
\{ 2 \omega_1 (2 \omega_1 - \omega_2) - \omega_0 \omega_2 \} \omega_3^2 (1 - \lambda) &= \omega_1^2 \lambda \{ \omega_2 \omega_4 (a + 1) + 2 \omega_3 (\omega_2 - 2 \omega_3) \}.
\end{aligned}
\]

Since weights are positive, we find from the first equality of (3.4) that \( b < 0 \). In order to solve (3.4), we normalize these weighs as \( \omega_2 = 1 \). If we
set \( \omega_1 = \omega_3 = k > 0 \), these equalities turn to
\[
\begin{align*}
(1 - \lambda)^2 \omega_0 + b \lambda^2 \omega_4 &= 0 \\
(1 - \lambda) \omega_0 + (a + 1) \lambda \omega_4 &= 2k(2k - 1),
\end{align*}
\]

(3.5)

We can solve these equations on \( \omega_0, \omega_4 \) when \( \lambda \neq (a + 1)/(a + b + 1) \). As \( 0 < \lambda < 1 \), under the condition that \( b < 0 \), we see (3.5) has a solution
\[
\omega_0 = \frac{-2b\lambda k(2k - 1)}{(1 - \lambda)((a + 1) - (a + b + 1)\lambda)}, \quad \omega_4 = \frac{2(1 - \lambda)k(2k - 1)}{\lambda((a + 1) - (a + b + 1)\lambda)}
\]
for each \( k > 1/2 \) except the case \( a + 1 < 0 \) and \( \lambda = (a + 1)/(a + b + 1) \). When \( \lambda = (a + 1)/(a + b + 1) \), we find (3.5) for \( k = 1/2 \) has infinitely many solutions satisfying \( (1 - \lambda)^2 \omega_0 + b \lambda^2 \omega_4 = 0 \).

**Example 3.2.** We take 4 control points
\[
Q_0 = (0, 0), \; Q_1 = (1, 2), \; Q_3 = (3, 2), \; Q_4 = (2, -2)
\]
on a Euclidean plane. If we choose \( \lambda = 1/2 \), then we have \( a = 1, \; b = -2 \) in (3.3) and \( Q_2 = (2, 2) \). When \( \omega_1 = \omega_2 = \omega_3 = 1 \), by taking \( \omega_0 = 2, \omega_4 = 1 \), we obtain a \( C^2 \) Bézier spline curve of order 2 which is shown in Figure 4. When \( \omega_2 = 1 \) and \( \omega_1 = \omega_3 = 2 \), by taking \( \omega_0 = 12, \omega_4 = 6 \), we obtain another \( C^2 \) Bézier spline curve of order 2.

![Figure 4. C² Bézier spline curve of order 2](image)

Next we consider the situation that \( \omega_0, \omega_1, \omega_2 \) are given. When \( b < 0 \), by substituting the first equality in (3.4) to the second, we obtain the following equation on \( \omega_3 \):
\[
\begin{align*}
\{4b\omega_1^2 \lambda^2 + b(4\omega_1^2 - 2\omega_1 \omega_2 - \omega_0 \omega_2)\lambda(1 - \lambda)
&+ (a + 1)\omega_0 \omega_2 (1 - \lambda)^2\} \omega_3 = 2b\omega_1^2 \omega_2 \lambda^2.
\end{align*}
\]

(3.6)
Thus if we choose \( \lambda \) sufficiently near 1 so that the coefficient of \( \omega_3 \) in (3.6) is negative, we obtain positive solutions for \( \omega_3, \omega_4 \). We should note that the number \( a \) depends on \( \lambda \) but we can choose such \( \lambda \) in view of (3.3). Therefore if we do not stick on the position of junction we can take a \( C^2 \) rational Bézier spline curves of given weights \( \omega_0, \omega_1, \omega_2 \) in this case.

**Example 3.3.** We take 4 control points as in Example 3.2 and choose \( \lambda = 1/2 \). When \( \omega_0 = 1, \omega_1 = 2, \omega_2 = 3 \), by taking \( \omega_3 = 12/7, \omega_4 = 18/49 \), we obtain a \( C^2 \) Bézier spline curve of order 2. But when \( \omega_0 = 3, \omega_1 = \omega_2 = 1 \), we can not take \( C^2 \) Bézier spline curve of order 2 with these control points and the junction.

As a one point compactification of a Euclidean plane, we usually use a Riemann sphere. Here we briefly make mention of rational Bézier curves on this Riemann sphere. We take a subset

\[
RS = \{ (X, Y, Z) \in \mathbb{R}^3 \mid X^2 + Y^2 + (Z - 1)^2 = 1 \}
\]

of \( \mathbb{R}^3 \). We identify a Euclidean plane with \( RS \setminus \{(0,0,0)\} \) by the restriction of a canonical projection \( \varphi : \mathbb{R}^3 \setminus \{(0,0,0)\} \to RS \) given by

\[
(x, y, z) \mapsto \left( \frac{2xz}{x^2 + y^2 + z^2}, \frac{2yz}{x^2 + y^2 + z^2}, \frac{2z^2}{x^2 + y^2 + z^2} \right).
\]

Visually the distance between two points near the top \((0,0,2) \in RS\) and the distance between two points near the bottom \((0,0,0) \in RS\) are quite different by comparing their Euclidean distances. This property might be

![Figure 5. A rational Bézier curve on a Riemann sphere](image)
useful when we consider industrial applications such as controlling arms of robots. Given control points $Q_0$, $Q_1$, $Q_2 \in RS$ and weights $\omega_0$, $\omega_1$, $\omega_2$, we take a Bézier curve $\sigma$ in $\mathbb{R}^3$ with control points $\omega_0 Q_0$, $\omega_1 Q_1$, $\omega_2 Q_2$ and define a rational Bézier curve $\gamma$ on $RS$ by $\gamma(t) = 2\sigma_2(t)\sigma(t)/\|\sigma(t)\|^2$, where $\| \cdot \|$ denotes the Euclidean norm. Of course we can compute its derivatives directly, but it is better to use the map $d\varphi$ and the results on rational Bézier curves on a Euclidean plane. We here only give a figure of a rational Bézier curve on $RS$ and a Bézier curve in $\mathbb{R}^3$ (see Figure 5) and a figure of rational Bézier spline curve on $RS$.

**Example 3.4.** We take four control points and a junction

$$Q_0 = (0, 0, 2), \quad Q_1 = \left(\frac{8}{21}, \frac{16}{21}, \frac{32}{21}\right), \quad Q_2 = \left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\right),$$

$$Q_3 = \left(\frac{24}{29}, \frac{16}{29}, \frac{32}{29}\right), \quad Q_4 = \left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\right)$$

on $RS$. If we choose their weights as $\omega_0 = 1$, $\omega_1 = 21/32$, $\omega_2 = \omega_4 = 3/4$, $\omega_3 = 29/32$, we obtain a $C^2$ rational Bézier spline curve on $RS$. This is an image of a rational Bézier spline curve on a Euclidean plane through $\varphi$ whose control points are $\hat{Q}_0 = (0, 0)$, $\hat{Q}_1 = (1/4, 1/2)$, $\hat{Q}_3 = (3/4, 1/2)$, $\hat{Q}_4 = (1/2, -1/2)$, whose junction is $\hat{Q}_2 = (1/2, 1/2)$, and whose weights are $\hat{\omega}_0 = 2$, $\hat{\omega}_1 = \hat{\omega}_2 = \hat{\omega}_3 = \hat{\omega}_4 = 1$ (c.f. Example 3.2). Here we note that the rational Bézier curve of control points $Q_2$, $Q_3$, $Q_4$ in Figure 6 has interesting features.

![Figure 6. A rational Bézier spline curve on a Riemann sphere](image)
4. Projective Bézier spline curves on a standard sphere

In this section we consider the differential of projective Bézier curves of order 2 on a standard sphere $S^2$. We regard $S^2$ as a subset $\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}$ of $\mathbb{R}^3$, and identify points on $S^2$ with vectors in a vector space $\mathbb{R}^3$. Given points $Q_0, Q_1, Q_2 \in S^2$ whose distances satisfy $d(Q_0, Q_1) < \pi$, $d(Q_1, Q_2) < \pi$ and given positive numbers $\omega_0, \omega_1, \omega_2$, we define a projective Bézier curve $\gamma$ by

$$\gamma(t) = \frac{(1-t)^2\omega_0 Q_0 + 2t(1-t)\omega_1 Q_1 + t^2\omega_2 Q_2}{||(1-t)^2\omega_0 Q_0 + 2t(1-t)\omega_1 Q_1 + t^2\omega_2 Q_2||} \quad (0 \leq t \leq 1).$$

Here, when $Q_0, Q_1, Q_2$ lie on a geodesic, we omit the case of $(\omega_0, \omega_1, \omega_2)$ that $(1-t)^2\omega_0 Q_0 + 2t(1-t)\omega_1 Q_1 + t^2\omega_2 Q_2 = 0$ for some $0 < t < 1$.

In order to compute the differential of $\gamma$ we put $\sigma(t) = (1-t)^2\omega_0 Q_0 + 2t(1-t)\omega_1 Q_1 + t^2\omega_2 Q_2$, which is a Bézier curve of order 2 in $\mathbb{R}^3$ with control points $\omega_0 Q_0, \omega_1 Q_1, \omega_2 Q_2$. Projective Bézier curve $\gamma$ is an image of $\sigma$ through conic projection centered at the origin. By direct calculation we have

$$\gamma'(t) = \frac{\sigma'(t)}{||\sigma(t)||} - \frac{(||\sigma(t)||)^2}{||\sigma(t)||^3} \gamma(t) = \frac{\sigma'(t)}{||\sigma(t)||} - \frac{(||\sigma(t)||^2)'}{2||\sigma(t)||^2} \gamma(t),$$

$$||\sigma(t)||^2 = (1-t)^4\omega_0^2 + 4t^2(1-t)^2\omega_1^2 + t^4\omega_2^2 + 2t(1-t)[2(1-t)^2\omega_0\omega_1\langle Q_0, Q_1 \rangle + t(1-t)\omega_0\omega_2\langle Q_0, Q_2 \rangle + 2t^2\omega_1\omega_2\langle Q_1, Q_2 \rangle],$$

$$(||\sigma(t)||^2)' = 4(-1-t)^3\omega_0^2 + 2t(1-t)(1-2t)\omega_1^2 + t^3\omega_2^2 + (1-t)^2(1-4t)\omega_0\omega_1\langle Q_0, Q_1 \rangle + t(1-t)(1-2t)\omega_0\omega_2\langle Q_0, Q_2 \rangle + t^2(3-4t)\omega_1\omega_2\langle Q_1, Q_2 \rangle,$$

where $\langle , \rangle$ denotes the Euclidean inner product. In particular, we have

$$\left\{ \begin{array}{l}
\gamma'(0) = \frac{2\omega_1}{\omega_0} \{Q_1 - \langle Q_0, Q_1 \rangle Q_0 \}, \\
\gamma'(1) = -\frac{2\omega_1}{\omega_2} \{Q_1 - \langle Q_1, Q_2 \rangle Q_2 \}.
\end{array} \right. \quad (4.1)$$

Here we should note that if we denote the distance between two points $Q, R$ on $S^2$ by $d(Q, R)$ we see $\langle Q, R \rangle = \cos d(Q, R)$.

Next we compute the second derivative of a projective Bézier curve $\gamma$
where $\nabla_t$ denotes the covariant differentiation along $\gamma$. In particular, we obtain

$$\nabla_t \gamma'(0) = \frac{\omega_2}{\omega_0} (Q_2 - \langle Q_2, Q_0 \rangle Q_0)$$

$$+ \frac{4\omega_1}{\omega_0^2} (\omega_1 - 2\omega_1 \langle Q_0, Q_1 \rangle) (Q_1 - \langle Q_1, Q_0 \rangle Q_0),$$

$$\nabla_t \gamma'(1) = \frac{2\omega_0}{\omega_2} (Q_0 - \langle Q_0, Q_2 \rangle Q_2)$$

$$+ \frac{4\omega_1}{\omega_2^2} (\omega_2 - 2\omega_1 \langle Q_1, Q_2 \rangle) (Q_1 - \langle Q_1, Q_2 \rangle Q_2).$$

Here we show some figures of a projective Bézier curves on $S^2$. In order to see their speed, we also plot points which correspond to the parameter $t = k/6$, $k = 0, 1, \ldots, 6$ on each projective Bézier curve.

![Figure 7. Speed of projective Bézier curves on a standard sphere]

Projective Bézier spline curves are defined just the same way as for Bézier spline curves. Suppose control points $Q_0, Q_1, Q_3, Q_4 \in S^2$ satisfy that $Q_0, Q_1, Q_3$ lie on an open hemi-sphere and $Q_1, Q_3, Q_4$ also lie on an open hemi-sphere. We take a junction

$$Q_2 = \frac{(1 - \lambda)Q_1 + \lambda Q_3}{\|(1 - \lambda)Q_1 + \lambda Q_3\|}, \quad 0 < \lambda < 1$$
on a minimal geodesic joining $Q_1$ and $Q_3$. We consider projective Bézier curves $\gamma_1, \gamma_2$ with control points $Q_0, Q_1, Q_2$ and $Q_3, Q_4$ and with weights $\omega_0, \omega_1, \omega_2$ and $\omega_3, \omega_4$, and define projective Bézier spline curve $\gamma$ for positive numbers $\alpha_1, \alpha_2$ by the same way as in (2.2). We find the following by the definition of projective Bézier spline curves and by (4.1).

**Theorem 4.1.** A projective Bézier spline curve $\gamma$ of order 2 with four control points on $S^2$ is of $C^1$ if and only if we choose $\alpha_1, \alpha_2$ so that they satisfy $\alpha_1 \omega_1 \lambda = \alpha_2 \omega_3 (1 - \lambda)$.

Next we study the $C^2$ condition on projective Bézier spline curves. In order to apply the argument in the previous section, we suppose four control points $Q_0, Q_1, Q_3, Q_4$ lie on some open hemi-sphere. Since we have a canonical bijection

\[
\{(x, y, z) \in S^2 \mid z > 0\} \ni (x, y, z) \mapsto \left(\frac{x}{z}, \frac{y}{z}, 1\right) \in \{(X, Y, 1) \mid (X, Y) \in \mathbb{R}^2\}
\]

of an open hemi-sphere centered at the origin to a plane, we see the following:

**Theorem 4.2.** Suppose four control points $Q_0, Q_1, Q_3, Q_4 \in S^2$ lie in an open hemi-sphere.

1. If three points $Q_0, Q_1, Q_3$ lie on some great circle $\ell$, we can take $C^2$ projective Bézier spline curves of order 2 on $S^2$ if and only if $Q_4$ also lies on $\ell$. In this case, its image is a part of this great circle.
2. If $Q_0$ and $Q_4$ lie on the same side with respect to the great circle $Q_1Q_3$, then there are infinitely many $C^2$ projective Bézier spline curves of order 2 on $S^2$ with these control points and given junction $Q_2$ on the geodesic segment $Q_1Q_3$.
3. If $Q_4$ lies on the opposite side of $Q_0$ with respect to the great circle $Q_1Q_3$, then we can not take $C^2$ projective Bézier spline curves of order 2 with these control points.

5. Projective Bézier spline curves on a hyperbolic plane

For about projective Bézier curves on a real hyperbolic plane $H^2$ we have the same properties as those on a standard sphere. We regard $H^2$ as a subset \[\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1, z > 0\}\] of $\mathbb{R}^3$, and identify points on $H^2$ with vectors in a vector space $\mathbb{R}^3$. Given points $Q_0, Q_1, Q_2 \in H^2$ and positive numbers $\omega_0, \omega_1, \omega_2$, we define a projective Bézier
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curve $\gamma$ by

$$
\gamma(t) = \frac{(1 - t)^2 \omega_0 Q_0 + 2t(1 - t)\omega_1 Q_1 + t^2 \omega_2 Q_2}{\| (1 - t)^2 \omega_0 Q_0 + 2t(1 - t)\omega_1 Q_1 + t^2 \omega_2 Q_2 \|_1} \quad (0 \leq t \leq 1).
$$

Here for $P = (x, y, z) \in \mathbb{R}^3$ we set $\|P\|_1 = -x^2 - y^2 + z^2$. This projective Bézier curve $\gamma$ is an image of a Bézier curve $\sigma$ on $\mathbb{R}^3$ with control points $\omega_0 Q_0, \omega_1 Q_1, \omega_2 Q_2$ through a conic projection centered at the origin. We should note that $\omega_i Q_i, i = 0, 1, 2$ lie in the inside of the set $\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < z^2, z > 0 \}$ hence so does $\sigma$. Thus $\gamma$ is well-defined.

We can compute the derivatives of $\gamma$ just the same way as for the case of projective Bézier curves on $S^2$, if we change $(\cdot, \cdot)$ to $\langle \cdot, \cdot \rangle$, which is given by $\langle P_1, P_2 \rangle = -x_1 x_2 - y_1 y_2 + z_1 z_2$ for $P_i = (x_i, y_i, z_i), i = 1, 2$.

Since we have a canonical bijection

$$H^2 \ni (x, y, z) \mapsto \left( \frac{x}{z}, \frac{y}{z}, 1 \right) \in \{ (X, Y, 1) \mid (X, Y) \in \mathbb{R}^2 \},$$

we obtain the corresponding results on smoothness of projective Bézier spline curves of order 2 on $H^2$.

**Theorem 5.1.** A projective Bézier spline curve $\gamma$ of order 2 with four control points on $H^2$ is of $C^1$ if and only if we choose $\alpha_1, \alpha_2$ so that they satisfy $\alpha_1 \omega_1 \lambda = \alpha_2 \omega_3 (1 - \lambda)$.

**Theorem 5.2.** We consider projective Bézier spline curves of order 2 with four control points on $H^2$.

1. If three points $Q_0, Q_1, Q_3$ lie on a geodesic $\ell$, we can take $C^2$ projective Bézier spline curves on $H^2$ if and only if $Q_4$ also lies on $\ell$. In this case, its image is a part of this geodesic.

2. If $Q_0$ and $Q_4$ lie on the same side with respect to the geodesic through $Q_1$ and $Q_3$, then there are infinitely many $C^2$ projective Bézier spline curves $S^2$ with these control points and given junction $Q_2$ on the geodesic segment $Q_1 Q_3$.

3. If $Q_4$ lies on the opposite side of $Q_0$ with respect to the geodesic through $Q_1$ and $Q_3$, then we can not take $C^2$ projective Bézier spline curves with these control points.

**References**


4. S. Kawabata and T. Adachi, “Projective Bézier curves of order 2 on a real space form” *in this volume*. 
We search for the shape of a soap film stretched to a certain boundary condition by numerical computation. When the shape for one boundary condition is not only one, we consider which shape is a true soap film among these. We also consider about the phenomenon which a soap film cause when we change this boundary condition.

1. Introduction

Suppose $\Gamma_1$ is a Jordan curve in $\mathbb{R}^3$, bounding a minimal surface. If $\Gamma_t$ is a homotopy of $\Gamma_1$, it will often happen that there is a corresponding homotopy $S_t$ of minimal surfaces; but it may also fail to happen, or fail to happen in a unique way. There may, for instance, be a "bifurcation" at $\Gamma_1$, in which (say) three different minimal surfaces are bounded by $\Gamma_t$ for $t > 1$, but all three converge to $S_1$ as $t$ converges to 1. This is deeply related to the the question, how many minimal surfaces are bounded by $\Gamma_t$.
We consider the equations of Enneper’s surface given by

\[ u(r, \theta) = (r \cos \theta - \frac{r^3}{3} \cos 3\theta, -r \sin \theta - \frac{r^3}{3} \sin 3\theta, r^2 \cos 2\theta) \]

and define \( \Gamma_r(\theta) \) by \( u(r, \theta) \) for \( r \) which is a Jordan curve in \( \mathbb{R}^3 \) for \( r \leq \sqrt{3} \).

The number of the minimal surfaces, which \( \Gamma_r \) bounds, is known as follows.

If \( r \leq 1/\sqrt{3} \), the projection \( \Gamma_r \) on the \( xy \) plane is convex. A theorem of Radó guarantees that every minimal surface bounded by \( \Gamma_r \) can be written in the form \( z = f(x, y) \); hence there is only one such surfaces, Enneper’s surface (Nitsch [1]).

For \( 1/\sqrt{3} < r < 1 \), the projection on the \( xy \) plane is not convex, although it is not convex, although it is one-to-one. In this case, in a beautiful paper, Ruchert [2] has shown the uniqueness still holds.

For \( r > 1 \), the projection on the \( xy \) plane is no longer one to one. Nitsch has proved that for \( r \) slightly greater than 1, \( \Gamma_r \) is not a curve of uniqueness. He has also proved that \( r \) is near \( \sqrt{3} \), there are three minimal surfaces bounded by \( \Gamma_r \). One of them is Enneper’s surface passing through the origin. The other two are images of each other under the natural symmetry of \( \Gamma_r \).

In this situation, Beeson and Tromba [3] obtain a complete local description of the bifurcation process at \( r = 1 \) in terms of the “cusp catastrophe” of Thom’s morphogenesis. We do not know a bifurcation process of a minimal surface in terms of the “cusp catastrophe” of Thom’s morphogenesis except their result.

In this paper, we consider a Jordan curve in \( \mathbb{R}^3 \) as follows (Figure 1). Let

\[ P = (-0.5, -0.5, -0.5H), Q = (0.5, -0.5, -0.5H), R = (0.5, -0.5, 0.5H), \]
\[ S = (0.5, 0.5, 0.5H), T = (0.5, 0.5, -0.5H), U = (-0.5, 0.5, -0.5H), \]
\[ V = (-0.5, 0.5, 0.5H), W = (-0.5, -0.5, 0.5H) \]

be points in \( \mathbb{R}^3 \), and join \( P, Q, R, S, T, U, V, W \) by line segments and make a Jordan curve \( J \).

We consider the minimal surfaces bounded by \( J \). Using \( H = \infty \) in place of \( H = 2.4 \), we obtain the Scherk’s surface bounded by 4 lines. We perturb \( J \) by pulling line segments \( PQ \) and \( UT \), \( RS \) and \( WV \) with the same angle.

The obtained Jordan curves \( J_t \) is similar to \( \Gamma_r \). We conjecture the same behavior about minimal surfaces bounded by \( J_t \).
Since the minimal surfaces bounded by $J$ are graphs on $(-0.5, 0.5) \times (-0.5, 0.5)$, it is unique and stable. So the minimal surface is still unique for $J_t$ (small $t$). It is an interesting question, how many minimal surfaces are bounded by $J_t$, when a bifurcation occurs. Furthermore we perturb the Jordan curve $J_{t_0}$, where a bifurcation happen, by pulling line segments $PQ$ and $UT, RS$ and $WV$ with the different angle and obtain two parameter family of the Jordan curve. This two parameter family of Jordan curves is also similar to $\Gamma_{ab}$ [3]. We may get a bifurcation process in terms of the "cusp catastrophe" of Thom's morphogenesis.

There may be many Jordan curves admitting a bifurcation and it seems to be very difficult to give a mathematical proof [3] for each Jordan curve. We hope to have a computer simulation to find such a Jordan curve and a bifurcation to investigate the Plateau problem. We shall try a computer simulation on above question and the existence of a bifurcation and the estimate of $t_0$, a bifurcation process in terms of the "cusp catastrophe" of Thom's morphogenesis.

We use "Mathematica" for the numerical computation. Mathematica is a synthetic software system for mathematics and its application. We can

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*aThis software is the registered trademark of Wolfram Research, Inc.*
treat numerical processing by the unific method by using Mathematica.

2. Conclusion

According to the numerical computation, we obtain the following.

Observation
- The frame with three surfaces such that their surface areas become extreme values exists, and its frame has two equilibrium shapes in a nature. For example, the frame of Figure 1 is this.
- Relation between the height of center of surface stretched to the frame of Figure 1 and its SurfaceArea is the polynomial function of degree 4.
- In Figure 1, by changing boundary condition suitably, we can see Catastrophe which jumps from equilibrium shape (minimal surface) of one side to that of the other side.

3. A frame with two equilibrium shapes

Generally, the shape of a soap film is called 'Minimal surface'. Minimal surface means a surface such that its surface area becomes a extreme value.

In Figure 1, we consider the shape of the soap film stretched to this frame when the upper part inclines only $\theta$ [deg.] and the lower part inclines only $\phi$ [deg.] from a perpendicular state.

We search for the shape of a surface such that its SurfaceArea becomes a extreme value in the case of $H = 2.4$. At first, we search for the shape of soap film in perpendicular state (Figure 2).

![Figure 2. The shape of a soap film in perpendicular state ($H = 2.4$, ($\theta = \phi = 0$))](image)
Next, changing \((\theta, \phi)\) from \((0,0)\) to \((1,1)[\text{deg.}]\), we consider the following three cases (Figure 3).

- A \((d\theta = d\phi)\)
- B, C \((d\theta \neq d\phi)\)

![Figure 3. How to change of boundary condition](image)

In Figure 4, ShapeA, ShapeB and ShapeC are shapes of change A, B and C in state M \((\theta = \phi = 1[\text{deg.}])\), respectively.

We set \('Z_O'\) as the height of the center of surface (Figure 5). \(Z_O = 0\) line is the height of the center of frame. \(Z_{O2}(= 0)\) is the height of ShapeA. \(Z_{O3}(> 0)\) is ShapeB's, \(Z_{O1}(= -Z_{O3})\) is ShapeC's.

![Figure 4. Three shapes of solutions \((H = 2.4, (\theta = \phi = 1))\)](image)
We also calculate these SurfaceAreas. Table 1 is the result of calculation.

<table>
<thead>
<tr>
<th></th>
<th>ShapeA</th>
<th>ShapeB</th>
<th>ShapeC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Z_0)</td>
<td>0.00 ((Z_{O2}))</td>
<td>0.77 ((Z_{O3}))</td>
<td>-0.77 ((Z_{O1}))</td>
</tr>
<tr>
<td>(S(\text{SurfaceArea}))</td>
<td>5.10 ((S_2))</td>
<td>5.08 ((S_3))</td>
<td>5.08 ((S_1))</td>
</tr>
</tbody>
</table>

Figure 6 is the relation between SurfaceArea and \(Z_0\). In Figure 6, We can calculate only three coordinates \((Z_{O1}, S_1), (Z_{O2}, S_2), (Z_{O3}, S_3)\).

We want to find the relation between SurfaceArea and \(Z_0\) in state \(M(\theta = \phi = 1\, \text{[deg.]})\). But, if we search for the minimal value from ShapeA, we get either ShapeB and ShapeC. Then, we assume that SurfaceArea serves as a continuous function of \(Z_0\) which becomes the minimal value at ShapeB, ShapeC and the maximal value at ShapeA. Furthermore, we assume that relation between \(S(\text{SurfaceArea})\) and \(Z_0\) is approximated by the following polynomial function of degree 4.

\[
S(Z_0) = C_1 \int (Z_0 - Z_{O1})(Z_0 - Z_{O2})(Z_0 - Z_{O3})dZ_0 + C_2 \tag{1}
\]

\(C_1\) and \(C_2\) are unknown quantities.
There are three simultaneous equations to determine \( C_1 \) and \( C_2 \), though there are only two unknown quantities (\( C_1 \) and \( C_2 \))

\[
S_1 = S(Z_{O1}) = C_1 \alpha_1 + C_2 \tag{2a}
\]

\[
S_2 = S(Z_{O2}) = C_1 \alpha_2 + C_2 \tag{2b}
\]

\[
S_3 = S(Z_{O3}) = C_1 \alpha_3 + C_2 \tag{2c}
\]

\( \alpha_1, \alpha_2, \alpha_3 \) are constant. The case of \( \theta = \phi \),

\[
Z_{O3} = -Z_{O1}, \quad Z_{O2} = 0, \quad S_1 = S_3
\]

These draw following relation. \( \alpha_3 = \alpha_1 \)

Equation (2c) coincides with equation (2a). Therefore, \( C_1, C_2 \) are determined by equations (2a) and (2b).

In Figure 6, The shape of graph of expression (1) determined by equations (2a) and (2b) is DoubleWell.

But, This expression (1) is our assumption, not necessarily true expression.

SurfaceArea of a soap film is proportional to potential energy. Because of 'the principle of the minimum potential energy', in a nature, the soap film of ShapeA does not exist (Figure 6). A soap film becomes either ShapeB or ShapeC. We call ShapeB and ShapeC 'equilibrium shapes'. Equilibrium means stable.
A conclusion of 3rd topic is the following.

In one boundary condition, there are three surfaces such that their SurfaceAreas become extreme values, and two equilibrium shapes exist in a nature.

As development of this topic, we calculate the state of $(\theta, \phi) = (-0.1, -0.1)$ to $(1.5, 1.5)[\text{deg.}]$ in the case of $H = 2.4$, similarly. Figure 7 is this result.

![Figure 7. Bifurcation ($H = 2.4$)](image)

In Figure 7, the more we enlarge $\theta (= \phi)$, the more both maximal value and minimal value become small —— those differences become large.

The more we enlarge $\theta (= \phi)$, the more DoubleWell becomes deeper and the center of wells separates from each other. In the case of $H = 2.4$, if $\theta (= \phi) \leq 0.06[\text{deg.}]$, the shape of a surface such that its SurfaceArea becomes a extreme value is only one, otherwise, these are three. Therefore, $\theta (= \phi) = 0.06[\text{deg.}]$ is Turning point.

Then, we calculate Turning point in the case of $H =$ from 1.4 to 2.8, similarly (Figure 8).

In Figure 8, we can expect that Turning point is $\theta (= \phi) = 0[\text{deg.}]$ in the case of $H = \infty$. 
4. Analysis of the catastrophe phenomenon of a soap film

- In Figure 6, how can equilibrium ShapeB change into ShapeC?

This question is very humorous. Our hypothesis to this question is following.
The shape of DoubleWell change if we change either $\theta$ or $\phi$ little by little from state $M$ ($(\theta, \phi) = (1, 1)[\text{deg.}]$) —— minimal valueB denies maximal valueA and suits in certain $(\theta, \phi)$.

We do the simulation of the change of the shape of DoubleWell. We consider the following two cases.

(i) To fix $\theta(=1[\text{deg.}])$ and enlarge $\phi$ little by little
(ii) To fix $\phi(=1[\text{deg.}])$ and make $\theta$ small little by little

Figure 9 is the result of simulation in case (i).

At first, we want to find the relation between SurfaceArea and $Z_0$ in the case of $\theta \neq \phi$. Even when we change $\theta \neq \phi$, can we use this approximate expression (1)? The case of $\theta \neq \phi$, equations (2a), (2b), and (2c) are different from each other.

Because of this reason, there is no method except adding the graph of expression (1) determined by equation (2a), (2b) in the simulation (Fig-
Figure 9. Mechanism of disappearance of ShapeB ($H = 2.4, \phi = 1.01, 1.29, 1.57, 1.85$ in case (i)). In case (i) of Figure 9, we discover that this graph passes along coordinate $(Z_{O3}, S_3)$. Case (ii) is also the same result although we do not
show a figure. Due to the numerical computation, we discover that relation between SurfaceArea and $Z_O$ is always expression (1).

Next, we consider the change of the shape of DoubleWell by using this expression (1). Figure 10 is the locus of three point (maximal valueA, minimal valueB, C) in case (i) (and case (ii)).

![Figure 10. Locus of Maximal ValueA and Minimal ValueB and C ($H = 2.4$, in case(i),(ii))](image)

In Figure 9 and Figure 10, we see $Z_{O2}$ become large, and $Z_{O3}$ and $Z_{O1}$ become small. The SurfaceArea of ShapeC becomes smaller than that of ShapeB. When we change $\phi$ to $1.85[\text{deg}]$, The maximal valueA and the minimal valueB deny and they disappear mutually. Immediately, ShapeB becomes unstable shape, ShapeB jumps to ShapeC of bottom of well of the side of opposite. This phenomenon is called 'Catastrophe'.

In Figure 10, we can also see Catastrophe similarly in case (ii). Catastrophe in case (ii) differs from Catastrophe in case (i) a little. In case (ii), Catastrophe is caused by pulling up SurfaceArea of ShapeA and ShapeB,
when we change $\theta$ to 0.62[deg.]

Besides case (i) and (ii), There are many cases where we see Catastrophe. We searched for the angle $(\theta, \phi)$ where we see Catastrophe. This result is Figure 11.

In Figure 11, relation between $S$(Surface Area) and $Z_O$ when we see Catastrophe is either between the following two expression.

$$S(Z_O) = C_1 \int (Z_O - Z_{O1})(Z_O - Z_{O3})^2 dZ_O + C_2$$

$$S(Z_O) = C_1 \int (Z_O - Z_{O1})^2(Z_O - Z_{O3})dZ_O + C_2$$

In Figure 11, the curve which connected smoothly the point($\theta, \phi$) where we see Catastrophe is a "Wedge shape". In Figure 1, by changing boundary condition suitably, we can see the Catastrophe phenomenon.
References

The following note explains, how one may extend "kinematics in number Spaces", (especially from [B2], [Fr/Sp], [Sp7]) to kinematics in differentiable Spaces, i.e. in "manifolds with or without singularities", and with - in singularity-cases - relevant locally integrable vectorfields ([Sp1],[Sp2]). Singularities appear already in classical situations and hence should be included in the theory right from the beginning. This needs some new and more general start than in [Fr/Sp]. From the starting points in this note one may try to go on along similar lines as in [Fr/Sp]. Special cases are known from classical branches of kinematics: See the literature in [P] for more classical examples, see especially [B2], [Fr/Sp], [Sp7] for some later development. Early beginnings of this general frame in some special case are due especially to H.R. Müller [M] and O. Giering, H. Frank [F/G], [F]. As an extension of [Fr/Sp], this paper is written in german.

Einleitung


Um solche Inkonsistenzen, Brüche zu vermeiden, wird man von vorneherein Räume [Spl] einbeziehen und von Kinematiken auf Räumen ausgehen. Dabei werden zugleich allgemeinere Liegruppen, gar nur Gruppen (nämlich gewisse Transformationsgruppen auf Räumen) relevant.

Die im folgenden vorgeschlagenen Ansätze orientieren sich in Fragestellungen, Festlegungen und Resultaten an [Fr/Sp], über die im Spezialfall [Fr/Sp], auch [B2], bekannten Resultate hinaus treten im allgemeineren Kontext jedoch neue Phänomene auf.

1. erinnert (zur Bequemlichkeit des Lesers) an einige notwendige Definitionen und Resultate im Kontext von Räumen und macht deutlich, daß man im hier benötigten Rahmen - insbesondere bei Zugrundelegung ganz spezieller Felder - auf Räumen genauso arbeiten kann wie auf Zahlenräumen oder Mannigfaltigkeiten (dazu vgl. z.B. auch [Sp2], [Sp4], [Sp5], [Sp6]).


1. Vorbereitungen


Im Folgenden beschränken wir uns auf den Spezialfall der sog. reduzierten, lokal kompakten differenzierbaren Räume. Das Wort “reduziert”
Ein (eingebetteter) differenzierbarer Raum im $R^n$ ist eine beliebige lokal kompakte Teilmenge $A \subset R^n$ zusammen mit der Garbe $A$ von Keimen von jenen Funktionen als sog. "differenzierbaren" Funktionen auf $A$, die sich jeweils lokal zu differenzierbaren Funktionen in den umgebenden $R^n$ fortsetzen lassen. Zusammen mit diesem Differenzierbarkeitsbegriff $A$ heißt $A$, genauer $(A,A)$ differenzierbarer Raum im $R^n$. Statt $(A,A)$ schreiben wir kürzer nur $A$. Differenzierbare Abbildungen von offenen Teilmengen $U \subset A$ in einen $R^m$, dann in ein $B \subset R^m$ sind entsprechend erklärt. Das führt in kanonischer Weise zum Begriff der differenzierbaren Abbildung $f$ zwischen eingebetteten differenzierbaren Räumen $f : A \to B$. Formal ist $f$ gerade ein Morphismus lokal-geringter Räume [Spl].: $f : (A,A) \to (B,B)$ $f$ heißt Diffeomorphismus, wenn $f : A \to B$ topologisch und mit $f$ auch die Umkehrabbildung $f^{-1} : B \to A$ differenzierbar ist. Für $p \in A$ erhalten wir als "lineare Approximation in p an A" den "Tangentialvektorraum" von $A$ in $p$:

$$T_pA := \{ v \in R^n \mid \forall U(p) \subset R^n \text{ (offene Umgebung von } p), \forall f : U \to R \text{ differenzierbar mit } f[U \cap A = 0 \text{ ist } Df(p)(v) = 0 \}.$$  
Hier ist $Df(p)$ das Differential von $f$ an der Stelle $p$. Für jede Umgebung $V(p) \subset A$, jedes differenzierbare $h : V \to R^l$ ist das Differential $Dh(p) : T_pA \to T_pB$ wohl definiert, indem man zu einer differenzierbaren Fortsetzung in den umgebenden $R^n$ geht, dort das Differential bildet und dann auf $T_pA$ einschränkt. Jede differenzierbare Abbildung $f : A \to B$ liefert so für jedes $p \in A$ eine lineare Abbildung (das "Differential" von $f$ in $p$) $Df(p) : T_pA \to T_{f(p)}B$. Ist $f$ ein Diffeomorphismus, so sind alle Differentiale $Df(p)$ Isomorphismen. Das Analogon der klassischen Kettenregel gilt auch hier, ebenso der Satz: Ist $Df(p)$ injektiv, so liefert $f$ in einer Umgebung $V(p) \subset A$ von $p$ einen Diffeomorphismus $f : V \to f(V)$. Indes: Ist $Df(p)$ bijektiv, so braucht $f(V) \subset B$ keine Umgebung von $f(p)$ auf $B$ zu sein. Die einzelnen Tangentialräume $T_pA$ fügen sich zusammen zum Tangentialraum $TA$ von $A$:

$$TA := \{ (p,v) \mid p \in A, v \in T_pA \},$$

und eine differenzierbare Abbildung $f : A \to B$ liefert das differentierbare Differential $Df : TA \to TB$, $Df(p,v) := (f(p), Df(p)(v)) =: Df(v)$ in Kurzschreibweise.

Klassischer Spezialfall: Alle $A$, $B$ sind nur differenzierbare Untermannigfaltigkeiten.
Wie schon gesagt: Da wir die Differenzierbarkeitsklasse fixiert haben, identifizieren wir im Folgenden $A$ mit $(A,A)$.


Am bequemsten beschreibt man einen abstrakten differenzierbaren Raum $X$ als einen lokal-geringten Raum $(X,X)$, der in einer Umgebung $U(p)$ eines jeden $p \in X$ wie ein eingebetteter differenzierbarer Raum $(A,A)$ aussieht (d.h.: $(U(p),X|U(p))$ ist bimorph äquivalent zu $(A,A)$ in der Kategorie der lokal-geringten Räume).

Differenzierbare Räume bilden zusammen mit ihren differenzierbaren Abbildungen eine Kategorie mit Produkten (die eine volle Unterkategorie der Kategorie aller reduzierten lokal-geringten Räume ist).

Zu jedem differenzierbaren Raum $X$ existiert die Gruppe $\text{Aut}(X)$ der Automorphismen (= Diffeomorphismen $X \to X$) von $X$ mit vielen interessanten Untergruppen. Genau die lokal kompakten Untergruppen $G$ von $\text{Aut}(X)$ sind Liegruppen. Diese operieren dann auch schon differenzierbar auf $X$: die natürliche Abbildung $G \times X \to X$, $(g,x) \to g(x)$, ist differenzierbar (Spezialfall von [Sp5]). Interessante Untergruppen erhält man etwa so:

Metriken auf eingebetteten Räumen $A$ sind u.a. solche Abbildungen, die lokal zu Metriken in den einbettenden, umgebenden $\mathbb{R}^n$ fortsetzbar sind. Man schränke z.B. die euklidische Standardmetrik des $\mathbb{R}^n$ auf $A$, genauer $TA$ ein. Damit ist auch klar, was eine Metrik $d$ auf einem abstrakten Raum $X$ ist. „Metriken“ in einem allgemeineren Sinne können hier auch „entartet“ sein, brauchen nicht positiv definit (Riemannsch) zu sein, können alternierend sein, usw. Interessant können nun jene Untergruppen $G$ von $\text{Aut}(X)$ sein, deren Elemente Isometrien bezgl. einer „Metrik“ $d$ auf $X$ sind, oder: die Geodätische (bzgl. eines $d$ auf $X$) auf Geodätische abbilden.

Etwas genauer: Eine Metrik $d$ auf $X$ ist eine differenzierbare Abbildung $d : TX \oplus_{\pi} TX \to R$, die in den Fasern $T_pX \oplus T_pX$ bilinear (und evtl. nicht entartet oder/und alternierend, oder/und symmetrisch oder/und positiv definit oder “hermitesch” usw. ist). Hier ist $\pi : TX \to X$ die „natürliche“ Projektion mit $\pi^{-1}(p) = T_pX \forall p \in X$ und $TX \oplus_{\pi} TX := \{(u,v) \mid u,v \in TX \text{ mit } \pi(u) = \pi(v)\}$.

$TX$, $TX \oplus_{\pi} TX$ sind in natürlicher Weise differenzierbare Räume, $d$ wird i.a. als differenzierbar vorausgesetzt.

$f : X \to X$ heißt Isometrie bzgl. $d$, falls gilt:
Einparametrische Scharen, speziell Gruppen von Diffeomorphismen hängen mit speziellen Vektorfeldern und speziellen Tangentenbegriffen zusammen. Dies erläutern wir jetzt. Dazu bemerken wir vorweg:

**Notiz.** Auf Räumen gibt es beliebig viele sinnvolle und relevante Tangentenbegriffe, die i.a. nicht miteinander übereinstimmen, wohl aber im Sonderfall von Mannigfaltigkeiten. Sie sind -je nach Kontext- unterschiedlich relevant. Der Metrikbegriff ist dann auf die jeweiligen Tangentenbegriff zu beziehen: $T^*X \subset TX$, $d : T^*X \oplus T^*X \to R$. Auch die Differenzierbarkeit von $d$ kann u.a. wie folgt abgeschwächt werden: $d$ braucht nur in dem Sinne "schwächer differenzierbar" zu sein, daß bei Einsetzen von differenzierbaren Feldern $V, W$ (siehe weiter unten), das Resultat $d(V, W)$ eine differenzierbare Funktion wird (vgl. [Sp9]).

Solche relevanten Verallgemeinerungen beziehen wir jedoch im vorliegenden Entwurf des Umfanges wegen nur bis auf folgenden, hier wichtigen Fall mit ein. Dieser jetzt zu erläuternde Tangenten-Begriff ist differentialgeometrisch auf Räumen besonders relevant und nur bei Mannigfaltigkeiten auch schon der übliche. Zunächst:

Ein *Vektorfeld* (kurz: *Feld*) $V$ auf $X$ ist eine differenzierbare Abbildung $V : X \to TX$ mit $\pi \circ V = id$. Wie auf Mannigfaltigkeiten hat man auch für Vektorfelder auf $X$ ein Lieprodukt, mit dem die Vektorfelder eine Liealgebra bilden (dieses gewinnt man auf eingebetteten Räume $A \subset R^n$ per Einschränkung vom $R^n$ her, so dann auch auf abstrakten Räumen), [Sp1], [Sp6]. Auf Mannigfaltigkeiten führen Felder lokal stets zu Flüssen (per Integration), auf Räumen i.a. nicht. Es sind genau die sog. local integrablen Felder, die auch hier zu Flüssen führen:

Ein Feld $V : U \to TU$ auf einer offenen Teilmenge $U \subset X$ heißt lokal *integrabel*, wenn durch jedes $q \in U$ eine Integralkurve von $V$ geht, d.h. $\exists \varphi : I \to U$ mit $t^0 \in I \setminus \partial I$, $\varphi(t^0) = q$, $\varphi' = V \circ \varphi$. Man setze für $p \in X$:

$$T^i_pX := \{ v \in T_pX \mid \text{in einer Umgebung } U \subset X \text{ von } p \text{ gibt es ein lokal integrablen Feld } V \text{ mit } V(p) = v \}.$$  

Jeder Vektor $v \in T^i_pX$ heiße lokal *integrabel*.

Man zeigt für lokal kompakte differenzierbare Räume $X$ ([Sp2], [Sp4], [Sp6], [Sp9]):
Satz 1.1.

i) \( T^1_p X \subset T_p X \) ist stets ein Untervektorraum

ii) Ein Feld \( V : U \rightarrow TU \) ist lokal integrabel. \( \iff \forall p \in U \quad (\Rightarrow \text{ ist trivial}) \)

iii) Die lokal integrablen Felder bilden eine Lie-Algebra.

iv) Es gibt eine Zerlegung \( Z \) von \( X \) in disjunkte 1-1-immersierte zusammenhängende Untermannigfaltigkeiten mit: \( \forall M \in Z, \forall p \in M \) gilt: 

\( T_p M = T^1_p X \)

v) \( T^1_p (X \times Y) = T^1_p X \times T^1_p Y \) gilt z.B. in folgenden Fällen: \( X \) oder \( Y \) ist eine differenzierbare Mannigfaltigkeit, oder die Menge der Punkte \( q \in X \times Y \), bei denen \( X \times Y \) eine (differenzierbare) Mannigfaltigkeit ist, liegt dicht in \( X \times Y \).

vi) Genau jedes lokal integrable Feld \( V \) liefert lokal um jeden Punkt \( p \in U \) einen Fluss: eine einparametrige differenzierbare Familie (sogar Pseudogruppe) von Diffeomorphismen.

Achtung: v) erfaßt sehr allgemeine Fälle, so alle subanalytischen Fälle. Ob schon alle Fälle, ist nicht bekannt. Wir vermuten: nicht alle. I.a. ist \( T^1_p X \neq T_p X \), Beispiel: \( X = \{ x \in R \mid x \geq 0 \} \). Dann ist \( T_0 X = R \), \( T^*_0 X = \{0\} \). Z.B. ist das Feld \( V : X \rightarrow TX \) mit \( V(p) = (p, 1) \forall p \in X \), nicht lokal integriabel. Im Falle analytischer Räume sind alle (analytischen, sogar differenzierbaren) Felder lokal integriabel. Auf semi-analytischen Räumen gilt das i.a. nicht, wohl aber in hinreichend allgemeinen Spezialfällen. Ist \( X \) nicht lokal kompakt, so ist die Situation komplizierter als in 1.1 (vgl. [Sp2], [Sp11]). Diffeomorphismen (jedoch i.a. nicht differenzierbare Abbildungen) transportieren die \( T^1 \)-Räume ineinander.

Wichtig ist später noch folgender Quotientensatz ([Rei1], [Rei2]).

Satz 1.2. Die Liegruppe \( G \) operiere eigentlich auf dem differenzierbaren Raum \( X \). Dann ist der Bahnen-Raum (Quotientenraum) \( X/G \) mit der Quotienten topologie in "natürlicher Weise" ein differenzierbarer Raum, so daß die Quotientenabbildung \( q : X \rightarrow X/G \) differenzierbar wird und so daß gilt: Ist \( f : X \rightarrow Y \) eine \( G \)-invariante differenzierbare Abbildung (: \( f(g(x)) = f(x) \forall q \in G, x \in X \)), so gibt es genau eine differenzierbare Abbildung \( f^* : X/G \rightarrow Y \) mit folgender Eigenschaft:

\[ f^* \circ q = f. \]
Spezialfall: $G$ ist endlich oder $G$ operiert eigentlich diskontinuierlich. Es gilt im Falle einer Mannigfaltigkeit $X$: $X/G$ hat Singularitäten $\iff \exists g \in G, g \neq \text{id}$, mit Fixpunkten $g(p) = p$. (Der fixpunktfreie Fall ist klassisch und trivial.)

Zur Erinnerung: "G operiert eigentlich auf X" heißt
1) im Spezialfall, daß $X$ lokal kompakt ist: $\varphi: G \times X \to X \times X$, $(g, x) \to (g(x), x)$ ist eigentlich, d.h. $\forall K \subset X \times X$ kompakt ist $\varphi^{-1}(K) \subset G \times X$ kompakt
2) im Allgemeinfall (der den Spezialfall natürlich enthält): $\varphi$ (wie oben) ist abgeschlossen und $G \times \{y\} \supset (x, y)$ ist kompakt $\forall x, y \in X$.

"$G$ operiert eigentlich diskontinuierlich", heißt z.B.: $G$ operiert eigentlich und ist diskret. Oder: Zu je zwei Punkten $p \neq q$ auf $X$ gibt es Umgebungen $U(p)$, $V(q)$ so, daß $\{ g \in G | g(U) \cap V \neq \emptyset \}$ endlich ist.

2. Parametrische Familien von “Bewegungungen”.

Allgemeine Bewegungsabläufe


Zu spezielle Ansätze erweisen sich oft und rasch als nicht ausreichend genug, weil zunächst noch verborgene Gründe, Notwendigkeiten für Allgemeineres früher oder später deutlich werden.

Im folgenden starte ich daher von sehr allgemeinen Ansätzen, gehe jedoch über reduzierte differenzierbare Räume hier nicht hinaus (was möglich wäre).

$\mathcal{I}$ sei im folgenden ein fixierter differenzierbarer Raum, speziell kann $\mathcal{I}$ ein eindimensionales (Zeit-) Intervall oder ein mehrdimensionales (Zeit-, Temperatur-,...-) Intervall sein.

Räume sind im folgenden also stets reduzierte differenzierbare Räume.

Definition 2.1. $\alpha$ Ein (verallgemeinerter) Bewegungsablauf (oder eine Dynamik oder eine Kinematik) eines Raumes $X$ in einem (gegen einen) Raum $Y$ über (im Laufe von) $\mathcal{I}$ ist eine differenzierbare Abbildung

$\varphi: \mathcal{I} \times X \to Y$, wobei $\varphi(t, -): X \to \varphi(t, X)(\subset Y)$

ein Diffeomorphismus ist für jedes fixe $t \in \mathcal{I}$. 
Kurz: $X$ bewegt sich unter $\varphi$ in $Y$ über (den "Zeitraum") $I$.

$\beta$ $\varphi$ heißt [eigentlicher] Bewegungsablauf, falls gilt:

$$\varphi(t, X) = Y \quad \forall t \in I.$$ 

**Bemerkung 2.2.**

$\alpha$ Liefert ein Bewegungsablauf $\varphi : I \times X \to Y$ für ein $t \in I$ einen Diffeomorphismus $\varphi(t, -) : X \to Y$, so darf man $Y$ durch $X$ ersetzen und also von $\varphi : I \times X \to X$ ausgehen. Es ist jedoch oft günstiger, $X$ und $Y$ in der Notation zu trennen.

$\beta$ Mit $(\text{id} \times \varphi)(t, x) := (t, \varphi(t, x))$ ist eine Abbildung $\text{id} \times \varphi : I \times X \to (\text{id})(I \times X)( \subset I \times Y)$ ein Diffeomorphismus genau dann, wenn sich $X$ unter $\varphi$ in $Y$ bewegt.

$\gamma$ Bewegt sich $X$ unter $\varphi$ in $Y$ und $Y$ unter $\psi$ in einem Raum $Z$, so bewegt sich $X$ unter $\psi \circ \varphi$ in $Z$ (alles über $I$). Dabei ist $\psi \circ \varphi$ definiert durch $(\text{id} \times \psi \circ \varphi) = (\text{id} \times \psi) \circ (\text{id} \times \varphi)$, also durch $\psi \circ \varphi(t, -) = \psi(t, \varphi(t, -)) \quad \forall t \in I$.

$\delta$ Ein Bewegungsablauf $\varphi$ von $X$ in $Y$ ist eigentlich genau dann, falls $\text{id} \times \varphi : I \times X \to I \times Y$ ein Diffeomorphismus ist. Im eigentlichen Fall existiert dann der inverse Bewegungsablauf (der sog. Gegenlauf) $\varphi^{-1} : I \times Y \to X$ von $Y$ in $X$, wobei gilt: $(\text{id} \times \varphi^{-1}) := (\text{id} \times \varphi)^{-1}$ also $\varphi^{-1}(t, -) = \varphi(t, -)^{-1} \quad \forall t \in I$.

$\varepsilon$ Im Falle $X = Y$ nennen wir eigentliche Bewegungsabläufe (über $I$) auch automorphe Abläufe auf $X$ (über $I$). Es folgt: Die automorphen Abläufe auf $X$ (über $I$) bilden eine Gruppe.

$\zeta$ Jeden (bzw. jeden eigentlichen) Bewegungsablauf $\varphi$ kann man auch als "differenzierbare" Abbildung $\varphi^* : I \to \text{Einb.}(X, Y) := \{ f \mid f : X \to Y \text{ Einbettung} \}$ (bzw. $\varphi^* : I \to \text{Diff.}(X, Y)$) auffassen, nämlich vermöge: $\varphi^*(t) := \varphi(t, -)$. Umgekehrt erhält man $\varphi$ bei gegebenem $\varphi^* : \varphi(t, -) := \varphi^*(t)(-).$ Die Zuordnung $\varphi \to \varphi^*$ ist bijektiv, wenn man die Differenzierbarkeit von $\varphi^*$ vermöge der Differenzierbarkeit von $\varphi(t, -) := \varphi^*(t)(-)$ erklärt.

Der Spezialfall automorpher Abläufe $\varphi$ führt auf "differenzierbare" Abbildungen

$$\varphi \to \varphi^* : I \to \text{Aut}(X) := \{ f \mid f : X \to X \text{ ist ein Diffeo} \}$$

vermöge $\varphi^*(t) := \varphi(t, -)$, und umgekehrt. Aut$X$ ist eine Gruppe, relevante Untergruppen sind sogar Liegruppen (vgl. 1. oder 4.).

**Anschauliche Deutung** $\varphi$ als "Bewegungsablauf": Für jedes $t \in I$
legt \( \varphi^*(t, -) \) den Raum \( X \) in \( Y \) hinein und bewegt ihn mit variablem \( t \in I \) gegen \( Y \).

**Definition 2.3.** Ein *Morphismus* \( \Phi \) (über \( I \)) von einem Bewegungsablauf \( \varphi : I \times X \to Y \) nach einem Bewegungsablauf \( \psi : I \times X^* \to Y^* \) ist ein Paar von Bewegungsabläufe \( \Phi_1 : I \times X \to X^*, \Phi_2 : I \times Y \to Y^* \), für die folgendes Diagramm kommutiert:

\[
\begin{array}{ccc}
I \times X & \xrightarrow{id \times \varphi} & I \times Y \\
\downarrow{id \times \Phi_1} & & \downarrow{id \times \Phi_2} \\
I \times X^* & \xrightarrow{id \times \psi} & I \times Y^*
\end{array}
\]

*)

\( d.h.: \Phi_2(t, \varphi(t, x)) = \psi(t, \Phi_1(t, x)) \). Wir sagen: \( \Phi = (\Phi_1, \Phi_2) \) *koppelt* \( \phi \) an \( \psi \) an.

Für einige Spezialfälle mit eigenen Namen vgl. [Fr/Sp], [Sp7].

**Bemerkung 2.4.** In natürlicher Weise ist die Komposition von Morphismen zwischen Bewegungsabläufen über \( I \) erklärt. Man erhält: Bewegungsabläufe (über \( I \)) und deren Morphismen bilden eine (sog. kinematische) Kategorie.

Interessant werden unterschiedliche (sog. kinematische) Unterkategorien sein: Nämlich solche mit speziellen Räumen \( I \) und \( X \) (z.B. riemannschen Räumen, Mannigfaltigkeiten, Zahlenräumen, Liegruppen) und speziellen Bewegungen (z.B. isometrischen oder winkeltreuen oder inhaltstreuen oder geodätischen oder gruppentreuen). So ersetze man z.B. die abelsche Liegruppe \( \mathbb{R}^n \) durch eine be-liebige Liegruppe \( G \) und die linearen, affinen oder isometrischen Abbildungen des \( \mathbb{R}^n \) durch deren Analoga auf \( G \). Speziell lasse man eine Teilmenge, gar Untergruppe \( I \subset G \) auf \( G \) operieren: \( I \times G \to X, (t, g) \to t \cdot g \). Man beachte: Bei unserer kategoriellen Sicht sind Bewegungsabläufe die entscheidenden Objekte. Man erhält andere Kategorien, wenn man die Räume \( I \times X \) als Objekte, die Bewegungsabläufe als Morphismen ansieht. Unsere Gewichtung ist anders, nämlich *kinematisch*. I.a. sind Bewegungsabläufe \( \varphi, \psi \) *nicht* durch Morphismen verbunden. Und Paare \( \Phi = (\Phi_1, \Phi_2) \) von Bewegungsabläufen führen einen gegebenen Bewegungsablauf \( \varphi \) i.a. nicht in einen weiteren Bewegungsablauf \( \Psi \) über. Jedoch:

**Bemerkung 2.5.** Sind \( \Phi_1 : I \times X \to X^*, \Phi_2 : I \times Y \to Y^* \) eigentliche Bewegungsabläufe, so liefert das obige Diagramm *) eine Bijektion der Bewegungsabläufe von \( X \) in \( Y \) auf die von \( X^* \) in \( Y^* \). Im Falle \( X = Y \),
\[ X^* = Y^* \] wird man oft \( \Phi_1 = \Phi_2 \) haben, darüber hinaus sogar \( X = X^* \), also \( X = X^* = Y^* = Y \).

**Definition 2.6.** \( \varphi \) heißt \( (\Phi_1, \Phi_2) \)-invariant, falls \( \psi = \varphi \).

Gegeben seien eine Menge \( G \) (oft eine Gruppe) und Abbildungen \( T_X : G \to \text{Aut}X, T_Y : G \to \text{Aut}Y \) (oft Homomorphismen). \( G \) "operiert" also auf \( X \) und auf \( Y \). Wir nennen \( X, Y \) dann \( G \)-Räume und schreiben abkürzend:

\[ g \cdot p := g(p) \quad \forall p \in X, \quad g(p) := T_Y(g)(p) \quad \forall p \in Y, \quad g \in G. \]

**Definition 2.7.** Ein Bewegungsablauf \( \varphi : I \times X \to Y \) heißt (bzgl. obiger \( G \)-Operationen)

a) \( G \)-(Bahnen)invariant, falls gilt: Zu jedem \( t \in I, \ x \in X, \ g \in G \) gibt es ein \( g^* \in G \) mit \( \varphi(t, g(x)) = g^*(\varphi(t, x)) \).

b) \( G \)-(Bahnen)treu, falls gilt: \( \varphi(t, G \cdot x) = G \cdot (\varphi(t, x)) \quad \forall t \in I, \ x \in X. \)

c) \( G \)-äquivariant, falls gilt: \( \ varphi(t, g(x)) = g(\varphi(t, x)) \quad \forall t \in I, \ x \in X, \ g \in G. \)

Jedes \( g \in G \) operiert vermöge \( id \times g : I \times X \to I \times X \), \( (id, g)(t, x) := (t, g(x)) \) auf \( I \times X \) als Automorphismus, entsprechend auch auf \( I \times Y \).

Mit den \( G \)-Bahnen \( (id \times G)(t, x) := (t, G \cdot x) \) in \( I \times X \) erhält man für \( \varphi : I \times X \to Y \):

**Bemerkung 2.8.**

\( \alpha \) \( \varphi \) ist \( G \)-invariant (bzgl. \( G \)-treu) genau dann, wenn \( id \times \varphi \) die \( G \)-Bahnen von \( I \times X \) in \( (id \times G) \) jene von \( I \times Y \) abbildet.

\( \beta \) \( \varphi \) ist \( G \)-äquivariant genau dann, wenn \( (id \times \varphi) \) als Abbildung \( (id \times G) \)-äquivariant ist, d.h. wenn gilt: \( (id \times \varphi) \circ (id \times g) = (id \times g) \circ (id \times \varphi) \) \( \forall g \in G. \) D.h.: \( \varphi \) ist \( (g, g) \)-invariant gemäß 2.6 \( \forall g \in G. \)

\( \gamma \) Äquivarianz \( \Rightarrow \) Bahnentreu \( \Rightarrow \) Invarianz (umgekehrt i.a. nicht).

Sei \( G \) jetzt eine Gruppe, \( T_X, T_Y \) seien Homomorphismen. Dann existiert \( (I \times X)/(id \times G) \), der Bahnenraum von \( I \times X \) nach \( (id \times G) \), versehen mit der Quotiententopologie. In natürlicher Weise ist

\[ (I \times X)/(id \times G) = I \times (X/G). \]

Ein \( G \)-invarianter Bewegungsablauf \( \varphi : I \times X \to Y \) läßt sich in die Bahnenräume zu einem "Quotientenablauf" \( \varphi^* : I \times X/G \to Y/G \) durchdrücken, so daß zusammen mit den Quotientenabbildungen \( q_x : X \to \)
$X/G, q_y : Y \rightarrow Y/G$ folgendes Diagramm kommutiert:

\[
\begin{array}{ccc}
I \times X & \xrightarrow{\varphi} & Y \\
\downarrow \text{id} \times q_x & & \downarrow q_y \\
I \times X/G & \xrightarrow{\varphi^*} & Y/G
\end{array}
\]

Ist $\varphi$ $G$-Bahnentreu, so folgt: $\varphi(t, X) \subset Y$ ist $G$-invariant für jedes $t \in I$, $\varphi^*(t, ) : X/G \rightarrow \varphi^*(t, X/G) \subset Y/G$ ist topologisch; $\varphi^*$ ist also auf der topologischen Ebene das Analogon eines Bewegungsablaufes von $X/G$ in $Y/G$.

1.a. sind $X/G$, $Y/G$ gerade nur topologische Räume. Es gibt wichtige Ausnahmen:

Ist $G$ eine Liegruppe, die eigentlich auf $X$ und $Y$ operiert (speziell: $G$ operiere eigentlich diskontinuierlich, noch spezieller: $G$ ist endlich), so sind $X/G$, $Y/G$ in "natürlicher Weise" differenzierbare Räume, und zwar so, daß $X/G$, $Y/G$ Quotienten in der Kategorie der differenzierbaren Räume mit differenzierbaren Quotientenabbildungen werden. Jede $G$-invariante differenzierbare Abbildung $f : X \rightarrow Z$ zwischen Räumen ($G$ operiere auf $Z$ als id) läßt sich eineindeutig in den Quotienten hinein durchdrücken (faktorisieren) zu genau einer differenzierbaren Abbildung $f^* : X/G \rightarrow Y$, so dass gilt:

\[ f = f^* \circ q^* \]

Diese Zuordnung $f \rightarrow f^*$ ist funktoriell.Damit erhält man:

**Satz 2.9.** $G$ sei eine auf $X$ und $Y$ operierende Liegruppe und $\varphi : I \times X \rightarrow Y$ ein $G$-Bahnentreuer Bewegungsablauf. Die Quotienten $X/G$, $Y/G$ mögen als differenzierbare Räume existieren, mit differenzierbaren Quotientenabbildungen $q_x : X \rightarrow X/G$, $q_y : Y \rightarrow Y/G$. $\Rightarrow \varphi$ läßt sich in den Quotienten durchdrücken zu $\varphi^* : I \times X/G \rightarrow Y/G$. $\varphi^*$ ist wieder ein (differenzierbarer) Bewegungsablauf, und man hat folgendes kommutatives Diagramm:

\[
\begin{array}{ccc}
I \times X & \xrightarrow{\varphi} & Y \\
\downarrow \text{id} \times q_x & & \downarrow q_y \\
I \times X/G & \xrightarrow{\varphi^*} & Y/G
\end{array}
\]

$X/G, Y/G$ haben im allgemeinen Singularitäten, selbst wenn $Y = X = \mathbb{R}^n$ ist und $G$ endlich ist (vgl. Satz 1.2).
Die Existenz der Quotienten als differenzierbare Räume ist in vielen Fällen durch Satz 1.2 gesichert. Gehen wir im Folgenden von einer Kategorie von G-Räumen (und deren G-treuen Morphismen) aus, für welche die Quotienten $X/G$ als differenzierbare Räume existieren, so liefert 2.9 unmittelbar:

**Korollar 2.10.** Die Zuordnung $\varphi \rightarrow \varphi^*$ unter 2.9 liefert einen Funktor der Kategorie der G-treuen Bewegungsabläufe und G-treuen Morphismen in die Kategorie der Bewegungsabläufe und ihrer Morphismen (über fixiertem $I$).


**Beispiele.**

α) Die Räume $X$, $Y$ sind Zahlenräume $R^n$, $I$ ist ein Zeitt intervall. Bei Bewegungsabläufen $\varphi : I \times R^n \rightarrow R^n$ mit $\varphi(1,0) = 0$ mag $\varphi(t,-)$ stets eine Isometrie bzgl. der euklidischen Metriken oder stets eine geodätische Abbildung sein (die also Geraden in Geraden abbildet). Im ersten Fall entspricht jedem $\varphi(t,-)$ eine orthonormale Matrix, im zweiten Fall eine beliebige invertierbare Matrix $M(t) : \varphi(t,x) = M(t) \cdot x$.

Die Gruppe $G := R_+$ oder $= R\setminus\{0\}$ operiert per Streckung auf dem $R^n$. Es ist $(R^n\setminus\{0\})/R_+ = S^n$ (Sphäre), $(R^n\setminus\{0\})/(R\setminus\{0\}) = P^n$ (Projektiver Raum).

Die obigen Bewegungsabläufe sind alle G-treu, drücken sich also in die Quotienten durch. Man erhält als Quotienten-Kinematiken:

1. die sog. sphärische Kinematik, im Falle $G = R_+$
2. die sog. projektive Kinematik, im Falle $G = R\setminus\{0\}$.

β) Jede Gruppenoperation einer Liegruppe $G$ auf einem Raum $X$ ist über $I := G$ ein eigentlicher Bewegungsablauf $\varphi : G \times X \rightarrow X$ von $X$ in $X$. Speziell darf $X = G$ sein. Im letzteren Fall wird $X/G$ einpunktit. Oder: Eine Untergruppe $I = U$ von $G$ operiert (z.B. per Linksmultiplikation) auf $G : U \times G \rightarrow G$, $(u,g) \rightarrow u \cdot g$.

γ) In naheliegender Weise erhält man n-dimensionale Tori als Quotienten des $R^n$ nach geeigneten Gruppen. Dies führt zu torischen Kinematiken.

δ) Ein einfaches Beispiel mit Singularitäten erhält man so:

δ1) $X$ sei ein Doppelkegel $K$ im $R_3$, allgemeiner im $R^n$. Bewegungsabläufe sind u.a.Drehungen des Kegels in sich.

δ2) $X = K \times R^k$. Bewegungen sind jetzt Drehungen des Kegels in sich, kombiniert mit Bewegungen (z.B. Verschiebungen) längs $R^k$. 
Beispiele für Quotienten mit Singularitaten gibt es zuhauf (vgl. z.B.: [Spl0]): $\mathbb{R}^2$ mit $G$ als Gruppe der 90°-Drehungen, oder der 45°-Drehungen usw. $\mathbb{R}^n \times \mathbb{R}^n$ mit $G = \{\text{id}, \varphi\}$, $\varphi(x, y) := (x, -y)$. $\varphi$ ist hier eine Spiegelung. Die zugehörigen Kinematiken sollen hier nicht beschrieben werden.

3. Bewegungsabläufe und lokal integrable Felder auf Räumen

Hier beschränken wir uns auf Parameter-Räume $I$, die offene Teilmengen von Zahlenräumen $\mathbb{R}^n$, speziell des $\mathbb{R}^1$ sind.

$b : I \times X \to Y$ sei ein eigentlicher Bewegungsablauf von $X$ gegen $Y$, $I \subset \mathbb{R}^n$ etwa ein "Intervall". Zunächst nehmen wir $n = 1$, also $I \subset \mathbb{R}$ an. Zu $b$ gehört das folgende kinematische Feld $W$ auf $I \times Y$:

$$I \times Y \ni (t, y) \to W(t, y) := b_t(t, b^*(t, y)) \in T_y Y \, (!)$$

Dabei schreiben wir von hier ab stets

$$b^*(t, y) := b^{-1}(t, -)(y) \, \quad \text{(vgl. 2).}$$

und führen einige Abkürzungen für jedes Produkt $I \times X$ von Räume $I$, $X$ und jede differenzierbare Abbildung $f : I \times X \to Y$ ein. Für $v = (v_t, v_x) \in T_t I \times T_x X =: T_{(t, x)}(I \times X)$ und $D(\ldots)$ als totales Differential setzen wir

$$D_t f(t, x)(v_t) := Df(t, x)(v_t, 0), \quad f_t(t, x) := D_t f(t, x)(1) \text{ im Falle } I \subset \mathbb{R}^1,$$

$$D_x f(t, x)(v_x) := Df(t, x)(0, v_x).$$

Das kinematische Feld $W$ ist nun lokal integrabel im folgenden Sinne:

**Satz 3.1.** $W(t, y) \in T_y^i Y \ \forall (t, y) \in I \times Y$.

**Beweis.** Wir haben den Diffeomorphismus: $\varphi := \text{id} \times b : I \times X \to I \times Y$, $(t, x) \to (t, b(t, x))$. Das lokal integrable Feld $(1, 0) = 1 \times 0$ auf $I \times X$ geht unter $\varphi$ auf folgendes lokal integrable Feld auf $I \times Y$ über: $\varphi^*(1 \times 0)(t, y) := d\varphi(1 \times 0) \circ \varphi^{-1}(t, y) = \varphi_t \circ \varphi^{-1}(t, y) = (1 \times W)(t, y) \in T_{(t, y)}^i (I \times Y)$. Wegen $T_{(t, y)}^i(I \times Y) = T_t^i I \times T_y^i Y$ (Satz 1.1, v) folgt also die Behauptung.

Umgekehrt gilt für $I \subset R$, wenn wir $Y$ mit $X$ identifizieren (O.E.):

**Satz 3.2.** Jedes Feld

$W : I \times X \to T_x^i X, \quad W(t, x) \in T^i_x X \ \forall t \in I, \ x \in X$
lernt zu jedem \((t^0, x^0) \in I \times X\) in einer Umgebung \(U(t^0) \times U^*(x^0) \subset I \times X\) einen Diffeomorphismus
\[
\text{id} \times b : U \times U^* \rightarrow (\text{id} \times b)(U \times U^*) \subset \text{offen} \ I \times X
\]
mit der Eigenschaft: \(W(t, x) = b(t, b^*(t, x))\) in \((\text{id} \times b)(U \times U^*)\). Insbesondere hat \(W\) "viele" Integralkurven \(b(-, x) : W(t, b(t, x)) = (t, x)\) und
\[
* \quad \psi_s(s, t, x) = (1 \times W)(\psi(s, t, x)) = 1 \times W(\psi(s, t, x))
\]
füllt eine "kleine" Umgebung \(J(s^0) \times U(t^0) \times U^*(x^0) \rightarrow I \times X\) mit \(s^0 = 0\) sowie \(\psi(0, t, x) = (t, x)\) und
\[
*)\psi_s(s, t, x) = (1 \times W)(\psi(s, t, x)) = 1 \times W(\psi(s, t, x))
\]
für eine "kleine" Umgebung \(J(s^0) \times U(t^0) \times U^*(x^0)\) von \((t^0, s^0, x^0)\). Die erste Komponente unter *) liefert \(\psi_1(s, t, x) = s + \psi_1^1(t, x)\), und wegen \(\psi_1^1(t, x) = \psi_1(0, t, x) = t\) sogar \(\psi_1(s, t, x) = s + 1\). Ist O.E. \(J \subset I\), so liefert
\[
\varphi(s, x) := \psi(s, t^0, x) = \psi(s, 0, x)
\]
eine Abbildung \(J \times U^* \rightarrow I \times X\), die bei hinreichend kleinem \(J\) und \(U^*\) ein Diffeomorphismus auf das Bild \(\varphi(J \times U^*)\) wird, wobei \(\varphi(J \times U^*)\) eine Umgebung von \((t^0, x^0) \in I \times X\) ist (Satz 1.1, vi). Mit \(b(t, x) := \varphi_2(t, x)\) folgt die Behauptung.

Satz 3.2 verallgemeinert Satz 1.1. Der Begriff "lokal integrabel" ist also auch für die hier auftretenden einparametrigen Felder begründet. Zur Frage der "globalen" Integrabilität vgl. [Mei].

3.1 und 3.2 haben mehrparametrige Verallgemeinerungen. Für diese ist \(I \subset R^k = \{ t = (t_1, \ldots, t_k) \mid t_i \in R\}\), und jeder Diffeomorphismus
\[
\varphi = \text{id} \times b : I \times X \rightarrow I \times X
\]
bildet die Lie-Algebra-Garbe \(V_1\) der Vektorfelder auf \(I \times X\), die nur in \(I\)-Richtung Komponenten haben (und damit schon lokal integrabel sind), ab auf die (damit ebenfalls lokal integrable) Lie-Algebra-Garbe \(\varphi^*(V_1)\).

Die Projektion \(\pi_1 : T^i I \times T^i X \rightarrow T^i I = TI\) ist auf \(\varphi^*(V_1)\) bijektiv, d.h. für jedes \((t, x) \in I \times X\) ist die eingeschränkte Projektion bijektiv:
\[
**) \quad T^i t I \times T^i x X \supset \varphi^*(V_1)(t, x) \rightarrow T^i t I.
\]

Umgekehrt führt eine Lieverteilung \(V^*\) von (Keimen von) lokal integrablen Feldern auf \(I \times X\), für die
\[
\pi_1 : V^*(t, x) \rightarrow T^i t I \quad \forall (t, x) \in l \times X
\]
bijektiv ist, lokal jeweils zu einer l-parametrischen Schar von Diffeomorphismen. Zum Beweis hierzu verallgemeinere man obige Argumente. Man beachte: $V^*(t, x)$ bedeutet den Vektorraum in $T_{t}I \times T_{x}X$, der entsteht, wenn man die Felder aus $V^*$ in $(t, x)$ auswertet.

Zu einem eigentlichen Bewegungsablauf $b : I \times X \to Y$ und dessen sog. Gegenlauf $b^* := b^{-1}$ gehören insgesamt die folgenden sog. kinematischen Felder (mit $W = V^*$):

**Definition 3.3.** Für $y := b(t, x)$, damit auch $x = b^*(t, y)$ definieren wir: $I \times X \to V(t, x) := \partial_y b^*(t, y)(b_t(t, x)) \in T_{y}^iX$, $I \times Y \to V^*(t, y) := \partial_x b(t, x)(b^*_t(t, y)) \in T_{y}^iY$ mit deren "transportierten" Feldern $I \times X \to V'(t, x) := \partial_y b^*(t, y)(V^*(t, b(t, x))) = b^*_t(t, b(t, x)) \in T_{y}^iX$, $I \times Y \to V'^*(t, y) := \partial_x b^*(t, x)(V(t, b^*(t, y))) = b_t(t, b^*(t, y)) \in T_{y}^iY$.

Diese hängen in folgender Weise miteinander zusammen:

**Satz 3.4.** $V = -V'$ auf $I \times X$, $V^* = -V'^*$ auf $I \times Y$.

**Beweis.** Mit $D$ als totalem Differential und $D_y$ bzw. $D_x$ als totalem Differential nur bzgl. der Teilvariablen $y$ bzw. $x$ erhalten wir: Aus $x = b^*(t, b(t, x))$ und $y = b(t, b^*(t, y))$ folgt

$$0 = b^*_t(t, b(U)) + D_y b^*(t, b(t, x))(b_t(t, x)) = V'(t, x) + V(t, x)$$

und

$$0 = b_t(t, b^*(t, y)) + D_x b(t, b^*(t, y))(b^*_t(t, y)) = V'^*(t, y) + V^*(t, y) \quad \square$$


Isometrien, also "wirkliche" Bewegungsabläufe sollten von besonderer Relevanz sein. Dazu seien $X$, $Y$ Räume mit Riemannschen Metriken $d^x$ auf $X$, $d^y$ auf $Y$: $d^x : T^iX \times_{\pi} T^iX \to R$, $d^y : T^iY \times_{\pi} T^iY \to R$.

**Definition 3.5.** Ein Bewegungsablauf $b : I \times X \to Y$ heißt isometrisch, falls gilt:

$$d^y(D_x b(t, x)(V), D_x b(t, x)(W)) = d^x(V, W) \quad \forall V, W \in T^i_x X$$
(hier ist $D_x b(t, x)(V), D_x b(t, x)(W) \in T^i_y Y$ mit $y = b(t, x)$).
Isometrische eigentliche Bewegungsabläufe kann man durch $V$ und die gemäß 3.3 induzierten Vektorfelder $V = -V'$, $V^* = -V'^*$ charakterisieren; z.B. für $V$:

**Satz 3.6** Ein eigentlicher Bewegungsablauf $b : I \times X \to Y$ ist genau dann isometrisch, wenn für $V(t, x) := D_y b^*(t, b(t, x))(b_t(t, x))$ gilt: $b(t^0, -) : X \to Y$ ist für ein $t^0 \in I$ isometrisch, und es ist darüberhinaus

$$0 = d^x([V, v], w) + d^x(v, [V, w])$$

für alle Keime von lokal integrablen Feldern $v(x)$, $w(x)$ auf $X$ mit $d^x(v, w) = \text{konstant}$.

**Beweis.** Da es nur um lokale Aussagen geht, sei weiterhin O.E. $X, Y \subset \mathbb{R}^n$, also auch $T^iX, T^iY \subset \mathbb{R}^{2n}$ usw. Abbildungen wie $b, V$ usw. sind dann auch Abbildungen in Zahlenräume hinein. Deren Differentiale $D(\cdots)$ oder "partiellen" Differentiale $D_x(\cdots), D_y(\cdots), D_t(\cdots)$, nur bzgl. von Teilvariablen $x, y, t$ genommen, können dann ebenfalls als Abbildungen in Zahlenräume gelesen werden. Wir haben so für $b : I \times X \to Y$:

$$D_t b : (TI) \times X \to T^iY, \quad (t, v, x) \to b_t(t, x) \cdot v$$

$$D_x b : I \times T^iX \to T^iY, \quad (t, x, w) \to D_x b(t, x) \cdot w$$

Wir fixieren $t^0 \in I, x^0 \in X, y^0 := b(t^0, x^0)$ und nehmen zunächst folgenden Spezialfall an: $x \to \dim T^i_x X = k = \text{konstant für alle } x \text{ aus einer Umgebung von } x^0 \text{ (o.E. bald zunächst für alle } x \in X)$. Dies trifft für die Punkte $x^0$ aus einer offenen und dichten Teilmenge von $X$ zu, weil die Abbildung $x \to \dim T^i_x X$ aufgrund der Definition von $T^i_x X$ halbstetig nach oben ist. Dann sieht $X$ in einer Umgebung $U(x^0)$, zur Schreibvereinfachung zunächst also in ganz $X$ wie folgt aus:

$$X = X^* \times \mathbb{R}^k, \quad T^i_x X = 0 \times \mathbb{R}^k \quad \forall x \in X. \quad ([Sp2])$$

Entsprechendes gilt für $Y$:

$$Y = Y^* \times \mathbb{R}^k, \quad T^i_y Y = 0 \times \mathbb{R}^k \quad \forall y(= b(t, x)) \in Y,$$

da $b(t, -) : X \to Y$ ein Diffeomorphismus ist ($\forall t \in I$).

Alle lokal integrablen Felder haben jetzt also nur Komponenten in Richtung von $R$. Ferner seien o.E. $X^*, Y^* \in \mathbb{R}^m$. Richtungsableitungen von Abbildungen und Funktionen in $T^i$-Richtungen sind also jetzt auf $R^k$ die üblichen Ableitungen in $R^k$-Richtungen, wobei die auftretenden Objekte jetzt nur zusätzliche Parameter in $X^*$ bzw. $Y^*$ haben. Die Metrik $d^x$ auf $T^iX$ sieht jetzt so aus: Z.B. für $x = (x^*, x') \in X, x^* \in X^*, x' \in \mathbb{R}^k$ ist
$d(x, -) R^k$ eine Riemannsche Metrik auf $R^k$ der Gestalt: $d^x(x^*, x', v, w) = \sum a_{ij}(x^*, x') \cdot v_i \cdot w_j \forall v, w \in R^k$. Entsprechendes gilt für $Y$ mit analogen Bezeichnungen. Jeder Diffeomorphismus $\sigma: X = X^* \times R^k \rightarrow Y = Y^* \times R^k$ induziert für jedes $x^* \in X^*$ per Einschränkung einen Diffeomorphismus $\sigma|\{x^*\} \times R^k : \{x^*\} \times R^k \rightarrow \{y^*\} \times R^k$ mit $y^* = \pi_1 \circ \sigma(x^* \times R^k)$, $\pi_1 : Y^* \times R^k \rightarrow Y^*$ Projektion (vgl. [Sp9], [Sp6] p. 84) Die Ableitung der $a_{ij}$ in Richtung eines Vektors $u \in R^k = 0 \times R^k = T_{x^0}X$ beschreiben wir durch $D_x d^x(\cdot)(v)$. Zu jedem $v^0, w^0 \in T_{x^0}X$ gibt es (lokal um $(t^0, x^0)$) lokal integrable Felder $v(t, x), w(t, x)$ mit $v(t^0, x^0) = v^0, w(t^0, x^0) = w^0$, deren transportierte Felder $D_x b(v)(t, y) := v^*(t, y) := D_x b(t, x)(v(t, x)) - y := b(t, x)$ - und $w^* := D_x b(v)$ nur von $y$ abhängen, und für welche $d^v(id, v^*, w^*) = konst$ ist (zum Beweis: man transportiere erst $v^0, w^0$ nach $Y$, setze dort zu lokal integrierbaren Feldern fort, normiere und transportiere zurück). Es sei nun weiter $b$ isometrisch. Dann folgt:

konst = $d^y(y, v^*(y), w^*(y)) = d^x(b^*(t, y), v^*(t, y), w^*(t, y))$,
mit $v^*(t, y) := v(t, b^*(t, y)), w^*(t, y) := w(t, b^*(t, y)), x = b^*(t, y), y = b(t, x)$. Mit $V^*(t, x) := b^*_t(t, b(t, x))$ und den bisherigen Festlegungen ergibt sich jetzt

$(D_v V^*)(t, x) := (D_x V^*)(v)(t, x)$, mit der Kettenregel also

$D_y b^*_t(t, b(t, x))(D_x b(t, x)(v(t, x))) (\in T_{x^0}X)$
$= D_y(D_{\tau} b^*(t, y))(1)(v^*(y))$ für $y = b(t, x)$
$= D_{\tau} b^*(t, y)(v^*(y))(1)$
$= D_{\tau} v(t, y)(1) = v^*_t(t, y)$
mit $y = b(t, x)$. Entsprechend: $(D_w V^*)(t, x) = w^*_t(t, y)$. Jetzt folgt:

$0 = D_{\tau}(d^x(b^*, v, w))(1)(t, y)$
$= (D_x d^x)(b^*, v, w)(b^*_t)(t, y) + (D_v d^x)(id, id, w^*)(v^*_t)(t, x)$
$+ (D_w d^x)(id, v^*, id)(w^*_t)(t, y)$
$= (D_x d^x)(id, v, w)(V^*)(t, x) + d^x(id, D_{\tau} V^*, w)(t, x)$
$+ d^x(id, v, D_w V^*)(t, x)$

(weil $d^x$ in den $v, w$-Variablen bilinear ist).

Insbesondere gilt dies in $(t^0, x^0)$, also:

$0 = (D_X d^x)(x^0, v^0, w^0)(V^*)(t^0, x^0)$
$+ d^x(x^0, D_v, v^0, w^0) + d^x(x^0, v^0, D_w V^*)$.

**)
Also gilt dies auch, wenn wir für \( v^0, w^0 \) beliebige Keime lokal integrierbarer Felder \( v = v(x), w = w(x) \) und \( x \) für \( x_0 \) einsetzen. Ist speziell noch \( d^x(\text{id}, v, w) = \text{konst} \), so folgt auch

\[
0 = D_V \cdot (d^x(\text{id}, v, w)) = (D_x d^x)(\text{id}, v, w)(V^*) + d^x(\text{id}, D_V \cdot v, w) + d^x(\text{id}, v, D_V \cdot w).
\]

Durch Subtraktion dieser von obiger Gleichung und ersetzen von \( V^* \) durch \( V = -V^* \) folgt schließlich: \( 0 = d^x ([V, v], w) + d^x (v, [V, w]) \), damit also \( * \). Das liefert die eine Richtung von 3.6 in einer dichten Menge von Punkten \( x \in X \), damit auch in ganz \( X \). Wir gehen nun von \( * \) aus. Verfolgt man alle Schlüsse rückwärts, so ergibt sich, daß die Lösung \( b \) von \( b_t(t, x) = V(t, b(t, x)) \) notwendig Isometrien liefert. \( \square \)

**Notiz.** Auf Mannigfaltigkeiten, aber auch auf Räumen gehört zu einer Riemannschen Metrik \( d \) eine kovariante Ableitung \( D \), mit welcher obiger Beweis hätte geführt werden können ([Sp9]). Der obige Beweis ist unabhängig davon, \( d \) braucht allgemeiner sogar nur eine differenzierbare Bilinearform zu sein (vgl. 5. für einen Spezialfall).

### 4. Gleitgleitkinematik, kinematische Unterräume

Die bisher eingeführten Bewegungsabläufe erweisen sich für einige Zwecke noch als etwas zu speziell. Wir gehen jetzt von Unterräumen (Teilmengen)

\[
X^I \subset I \times X, \quad Y^I \subset I \times Y
\]

mit der Eigenschaft \( \pi_1(X^I) = I \), \( \pi_1(Y^I) = I \) aus. Dabei sei \( \pi_i \), die Projektion von \( I \times X \) (entsprechend von \( I \times Y \)) auf die \( i \)-te Komponente. Für \( d\pi_i \), schreiben wir kürzer auch nur \( \pi_i \). Es ist nun z.B.

\[
\pi_2(T_{(t, x)}(I \times X)) = T_x X = 0 \times T_x X \subset T_{(t, x)}(I \times X).
\]

Wir setzen \( X^I_t := \pi^{-1}_1(t) \) im ersten Fall; \( \pi_1 : I \times X \to I \), und \( Y^I_t \) analog im zweiten Fall. Im Folgenden entspricht \( \varphi \) dem bisherigen \( \text{id} \times \varphi \).

Ein **Bewegungsablauf** \( \varphi \) von \( X^I \) gegen \( Y^I \) ist ein Diffeomorphismus \( \varphi : X^I \to \varphi(X^I) \subset Y^I \) mit \( \varphi(X^I_t) \subset Y^I_t \) für alle \( t \in I \). \( \varphi \) heiße wieder *eigentlich*, falls zusätzlich gilt \( \varphi(X^I_t) = Y^I_t \). In dem Fall ist auch die existierende Umkehrabbildung \( \varphi^{-1} : Y^I \to X^I \) wieder ein eigentlicher Bewegungsablauf. Die Komposition von Bewegungsabläufen ist wie bisher erklärt und wieder ein Bewegungsablauf. Wie bisher erhält man die (jetzt erweiterte) Kategorie der Bewegungsabläufe und ihrer Morphismen über \( I \).

Auf diese allgemeinere Situation lassen sich jetzt auch die Begriffe der Invarianz, der Treue und Äquivarianz erweitern. Auch G-Quotienten lassen
sich wieder konstruieren. Wir wollen die Allgemeinheit jedoch nicht zu weit treiben und die Unterräume $X^I, Y^I$ nur für folgenden Zweck verwenden. Die folgende Definition verallgemeinert [Fr/Sp], 1.9 p. 44 mit der geometrischen Bedeutung dort:

**Definition 4.1.** Unter einem Bewegungsablauf $(\text{id} \times b) =: b^* : I \times X \rightarrow I \times Y$ gleitgleitet ein Unterraum $X^I \subset I \times X$ (mit $\pi_1(X^I) = I$) auf einem Unterraum $Y^I \subset I \times Y$ ($\pi_1(Y) = I$) ab, wenn folgendes gilt:

- **a)** $b^*(X^I) = Y^I$, d.h. $b^*|X^I : X^I \rightarrow Y^I$ ist eigentlich, ausführlicher:
  
  $$ b^*(X^I_t) = Y^I_t \forall t \in I $$

  (Anlegebedingung)

- **b)** $\pi_2(T(t,y)Y^I) \cdot D_x b(\pi_2(T(t,x)X^I))$ mit einem $\bullet \in \{\subset, \supset\}$ für jedes einzelne $(t,x) \in X^I$, $y = b(t,x)$. (Tangentialbedingung für "tangentes Anlegen")

bzw. wenn für alle $(t,x) \in X^I$ sogar gilt:

$$ \beta^* \pi_2(T(t,y)Y^I) \subset D_x b(\pi_2(T(t,x)X^I)) $$

(spezifische Tangentialbedingungen für sog. strenges Gleitgleiten).

Oft werden $X^I, Y^I$ von eigentlichem Bewegungsabläufen $\varphi^* : I \times X^* \rightarrow X^I, \psi^* : I \times Y^* \rightarrow Y^I$ herrühren, wobei $\varphi^* = \text{id} \times \varphi, \psi^* = \text{id} \times \psi$ und $\varphi, \psi$ jeweils (lokal wenigstens) eine festgelegte Umgebung von $(t,x)$ auf $X^I$. Oft wird die obige zweite Inklusion $\subset$ eine Gleichheit sein (z.B. stets bei lokalem Diffemorphismus $\varphi$ oben). Vgl. auch Definition 4.5. Man erhält Varianten zu obigem Gleitgleiten, wenn man unter Definition 4.1, $\beta$ bzw. $\beta^*$) die Räume $\pi_2(T(t,x)X^I)$ (alle oder nicht alle) durch $T_x \pi_2(...)_I$ oder $T \ldots$ durch $T^i \ldots$ ersetzt. Die "schwächesten" Varianten zu $\beta^*$ ist dann durch $\pi_2(T^i(t,x)Y^I) \subset D_x b(T_x(\pi_2X^I(t,x)))$ gegeben (schwaches... Gleitgleiten), die "stärksten" Variante durch $T^i(\pi_2Y^I(t,y)) \subset D_x b(\pi_2T(t,y)X^I))$ (starkes... Gleitgleiten)). Stets sprechen wir aber kurz nur vom Gleitgleiten. Was hier jeweils das "Richtigere" ist, wird vom Kontext abhängen.

In den in [Fr/Sp], [Sp7] beschriebenen Fällen stimmen alle diese Varianten überein, sonst i.a. nicht. "Rollt" z.B. ein Rad auf der Neilschen Parabel auch um die Spitze herum ab, so liegt an der Stelle der Singularität ein Gleitgleiten vor, das nur ein schwaches strenges Gleitgleiten
aber kein strenges Gleitgleiten ist.

Die Symmetrie unter \( \beta \) bedeutet folgende Relativität der Standpunkte (dagegen ist \( \beta^* \) i.a. unsymmetrisch):

Satz 4.2. Gleitgleitet unter dem eigentlichen Bewegungsablauf \( b : I \times X \to Y \) der Unterraum \( X^I \subset I \times X \) auf \( Y^I \subset I \times Y \) ab, so gleitgleitet unter dem inversen Ablauf \( b^{-1} : I \times Y \to X \) der Unterraum \( Y^I \) auf \( X^I \) ab (wobei unter \( \beta \) die Rollen von • vertauscht werden).

Im Falle der speziellen Tangentenbedingung lassen sich wie in [Fr/Sp] spezielle Gleitgleit-Typen einführen (für einen eigentlichen Bewegungsablauf \( b : I \times X \to Y \) und für \( X^I \subset I \times X \), \((\text{id} \times b)(X^I) = Y^I \subset I \times Y \).

Zunächst gilt:

Satz 4.3. Die spezielle Tangentenbedingung \( \beta^* \) ist äquivalent zu

\[
(D_y b(t, x) \circ \pi_1)(T_{(t, x)} X^I) \subset (D_x b(t, x) \circ \pi_2)(T_{(t, x)} X^I),
\]

also zu

\[
D_y b^*(t, y)(D_t b(t, x) \circ \pi_1)(T_{(t, x)} X^I) \subset \pi_2(T_{(t, x)} X^I)
\]

\( \forall (t, x) \in X^I, y = b(t, x). \) (mit \( \pi_i \) als Projektion von \( TI \times TX \) auf den \( i \)-ten Faktor).

Beweis. Wegen \( Y^I = b^*(X^I) \) ist \( T_{(t, y)} Y^I = D b^*(T_{(t, x)} X^I) \), also folgt wegen \( b = \pi_2 \circ b^* \):

\[
\pi_2(T_{(t, y)} Y^I) = Db(T_{(t, x)} X^I).
\]

Damit ist \( \beta^* \), d.h. \( \pi_2(T_{(t, y)} Y^I) \subset D_x b(\pi_2(T_{(t, x)} X^I)) \) äquivalent zu

\[
Db(T_{(t, x)} X^I) \subset D_x b(\pi_2(T_{(t, x)} X^I)).
\]

Daraus folgt die Behauptung. \[\square\]

Notiz. Änderungen gemäß obiger Notiz ändern entsprechend 4.3.

Die folgenden, von formalen und inhaltlichen Standpunkten aus sinngewollten Unterscheidungen verallgemeinern [Fr/Sp], Definition 5.17, p. 88. Vgl. die Motivationen dort. Wir kürzen mit \( y = b(t, x) \) ab: \( G(t, x) := D_y b^*(t, x) \circ D_t b(t, x) : T_I I \to T_x X \) und gehen für die folgende Definition des Gleitgleitrollens von einer fixierten linearen Kontraktion \( \pi = \pi_{(t, x)} : \pi_2(T_{(t, x)} X^I) \to T_x X^I \) aus (beachte: \( T_x X^I \subset \pi_2(T_{(t, x)} X^I) \)). Wir schreiben suggestiv:

\[
\pi_2(T_{(t, x)} X^I) \Theta T_x X^I \ := \text{Kern} \pi_{(t, x)}.
\]
In vielen Fällen wird auf TX eine Metrik $d$ gegeben sein mit $\pi_{(t,x)}$ als zugehöriger orthogonaler Projektion (und $b$ wird winkeltreu, speziell sogar isometrisch bezüglich $d$ sein). Jedenfalls hängt das folgende "Gleitrollen" von der Wahl von $\pi$ ab.

**Definition 4.4.** Unter einem eigentlichen Bewegungsablauf $b : I \times X \rightarrow Y$ gleitgleite $X'(\subset I \times X)$ streng auf $Y'(\subset I \times Y)$. Also: $b^*(X^I) = Y^I$ und $G_{(t,x)}(\pi_1(T(t,x)X^I)) \subset \pi_2(T(t,x)X^I)$.

a) Das Gleitgleiten ist ein (reines) Rollen in $(t, x) \in X^I$, falls gilt:

$$G_{(t,x)} = 0 \text{ auf } \pi_1(T(t,x)X^I).$$

β) Das Gleitgleiten ist ein Rollgleiten in $(t, x) \in X^I$, falls gilt:

$$G_{(t,x)}(\pi_1(T(t,x)X^I)) \subset T_xX^I.$$  

γ) Das Gleitgleiten ist ein Gleitrollen in $(t, x) \in X^I$, falls gilt:

$$G_{(t,x)}(\pi_1(T(t,x)X^I)) \subset \pi_2(T(t,X)X^I) \Theta T_xX^I.$$  

δ) Das Gleitgleiten ist ein erweitertes Schrotten in $(t, x)$, falls ein Rollgleiten vorliegt in $(t, x')$ für alle $x' \in X'_t$ aus einer Umgebung von $x$, und falls $x' \rightarrow G_{(t,x)}(w(x'))$ konstant ist für jedes Feld $x' \rightarrow w(x') \in \pi_1(T(t,x')X^I)$, das in einer Umgebung von $x$ auf $X'_t$ gegeben und konstant ist ($t$ ist hier fixiert).

Es sei darauf hingewiesen, daß Definition 4.4 zwar [Fr/Sp], Definition 5.17, p. 88 verallgemeinert, aber für den Allgemeinfall δ vielleicht nicht immer das Richtige trifft und daher als vorläufig anzusehen ist. Das ist aber nicht besonders tragisch, weil das "Rollgleiten" die zentralere Eigenschaft ist (wie schon in [Fr/Sp] bemerkt).

**Definition 4.5.** Wir sagen, es liegt ein (reines) Rollen bzw. ein Rollgleiten bzw. . . . bei $(t^0, x)$ (oder auf $X^I$) vor, wenn Entsprechendes für alle $(t, x)$ aus einer Umgebung von $(t^0, x^0)$ auf $X^I$ (oder auf ganz $X^I$) gilt.

[Fr/Sp], Definition 5.17 betrifft in der Tat folgenden Spezialfall des obigen Gleitgleitens: $b^* = \text{id} \times b : I \times X \rightarrow I \times Y$, und per Einschränkung

$$b^* : X^I \quad \longrightarrow \quad Y^I$$

$$\text{Diffeo } \text{id} \times \varphi \uparrow \quad \quad \quad \quad \quad \quad \uparrow \text{id} \times \psi \text{ Diffeo}$$

$$\text{id} \times \tau : I \times X^* \quad \longrightarrow \quad I \times Y^*$$

Die Anlegebedingung 4.1, α), also $(\text{id} \times b)(X^I) = Y^I$ liefert einen eigentlichen Bewegungsablauf $\tau$, so daß obiges Diagramm kommutiert und
zu einer Anlegebedingung für "φ unter b an ψ* := ψ o (id × τ)" wird:

\[ ψ^*(t, x^*) = b(t, (φ(t, x^*))), \]

insbesondere folgt \( Dψ^* = Db o (id × Dφ) \).

Die tangentielle Anlegebedingung 4.1, \( β^* \) (bzw. \( β^* \)) wird zur tangentiellen Anlegebedingung für "φ unter b an ψ*":

\[ Dψ^*(T(t,x^*)(I × X^*)) \subseteq \text{oder}\ D_x b(t, x)(Dφ(T(t,x^*)(I × X^*))) \]

bzw. im speziellen Fall \( β^* \) nur mit Inklusionen \( ⊆ \)

\[ \forall (t, x^*) ∈ I × X^*, x = φ(t, x^*). \text{Man darf nun o.E.}\ X^* = Y^*, \tau = \text{id}, ψ = ψ* \text{annehmen. Gilt nun 4.1, α), speziell also}\ Dψ(v, w) = D_tb(v) + D_x b(D_t φ(v) + D_x φ(w)) \text{für alle (v, w) ∈ T(t,x^*)(I × X^*) = TtI × T_x X^*,} \]

so ist die Bedingung \( β^* \) äquivalent zu

\[ ψ_t = Dψ(I × 0) = D_x b o (D_t φ(ρ) + D_x φ(σ)) \]

für geeignete \( ρ ∈ T_t I = R, σ ∈ T_x X^*, \) bzw. (man beachte: 1 ∈ TtI, \( dψ(1 × 0) = D_t b(1) + D_x b(Dφ(1)) \) auch äquivalent zu

\[ D_t b(1) = D_x b o D_t φ(ρ - 1) + D_x b o D_x φ(σ). \]

Das ist aber 4.3. Das Rollen entspricht nach 4.4 also \( ρ = 1, σ = 0 \), das Rollgleiten entspricht \( ρ = 1 \), das Gleitrollen entspricht \( σ = 0 \), das erweiterte Schroten entspricht \( ρ = 1 \) und \( σ(t, x) = σ(t) \), falls \( D_x φ \) nur von \( t \) abhängt.

Im Falle \( X = Y = \mathbb{R}^n, X^* = Y^* = \mathbb{R}^k \) und \( φ, b \) affin isometrisch (bzgl. der euklidischen Metriken), ist das gerade \([Fr/Sp]\), Definition 5.17. \([Fr/Sp]\) entwickelt hierzu eine umfangreiche Theorie. Wir können diese im hier vorliegenden allgemeinen Rahmen nicht nachzeichnen. Schon beim folgenden ersten Schritt, nämlich zur Frage der Existenz von Unterräumen \( X^1, Y^I \), die unter einem eigentlichen Bewegungsablauf \( b : I × X → Y \) auferstander rollgleiten oder erweitert schroten, zeichnen sich über Spezialfälle wie \([Fr/Sp]\) oder \([Sp7]\) hinaus neue Phänomene ab (§5). Zunächst sondern wir unterschiedliche, zu b gehörende sog. kinematische Unterräume in \( X \) bzw. \( Y \) aus. Wir erweitern und ergänzen damit klassische Definitionen und auch Definitionen aus \([S]\), \([Fr/Sp]\), \([Sp7]\). In Anlehnung an klassische Redeweisen nennen wir diese kinematischen Unterräume wieder (wenn auch etwas irreführend in unserem allgemeinen Kontext) Achsenmannigfaltigkeiten, besser: Achsenräume (da Singularitäten auftreten können). Punkte dieser Räume erfassen unterschiedliches stationäres Verhalten der Dynamik \( b \) an diesen Punkten. 4.6 beschreibt einige der Möglichkeiten.

Zur Erinnerung: \( D_b^*(t, y)(b_t(t, x)) = V(t, x) \) für \( y = b(t, x) \). Weiter sei \( [V] := d(V, V) \) und \( d \) eine fixierte differenzierbare Bilinearform auf \( T^i X \),
$D[V]$ sei auf $T^1X$ das totale Differential der Funktion $[V]$; weiter sei - angewandt auf Felder $D$ eine fixierte kovariante Ableitung auf (den Feldern aus) $T^1X$ (die z.B. Riemannschn ist bzgl. einer fixierten Riemannschen Semi-Metrik $d$ auf $T^iX$).

**Definition 4.6. Kinematische Unterräume (Achsenräume) $A^i(t^0), P(t^0)$ auf $X$ zur Dynamik $b$ im Zeitpunkt $t^0$ (mit $A^i, P$ in abkürzender Schreibweise)

α) $A^0(t^0) := \{ x \in X \mid 0 = V(t^0, x) \}$

Polraum (mit Polpunkten) zu $b$ in $X$ im Zeitpunkt $t^0$.

β) $A^i := \cup A^i(t^0) := \{ (t^0, x) \mid t^0 \in I, x \in A^i(t^0) \}$.

γ) $G A^i(t^0) := b(t^0, A^i(t^0)$ (besser $G A^i_{b}(t^0) \subset Y$. Die $G A^i := \cup G A^i(t^0)$ heissen kinematische Gegenräume(-achsen) von $b$.

**Notiz.** Man erhält unter 4.6 weitere kinematische Unterräume, wenn man dort die "0" jeweils durch "relevante" Funktionen bzw. Felder ersetzt. Der Leser interpretiere die Räume $A^i(t^0)$ geometrisch, u.a. mit Hilfe der Integralkurven zu $V(t^0, -)$. Mit Hilfe des Feldes $V^*(t, y) := b_1(t, b^*(t, y))$ auf $I \times Y$ erhält man entsprechende dynamische Unterräume (Achsen) $A^{*i}(t^0)$, $A^{*i}$ zu $b^*$ auf $Y$ bzw. $I \times Y$. Es ist $A^{*i}(t^0) = b(t, A^i(t^0)) = G A^i(t^0)$ für $i = 0$. In den übrigen Fällen $i > 0$. müssen hierfür die ausgewählten kovarianten Ableitungen und Bilinearformen auf $X$ und $Y$ über $b$ "differentialgeometrisch zusammen passen", $b$ wird insbesondere "isometrisch" sein (man verwende 3.4).

**Bemerkung 4.7.** Für alle $t^0 \in I$ gilt:

α) $A^1(t^0) \subset A^2(t^0) \subset A^7(t^0), A^0(t^0) \subset A^7(t^0), A^4(t^0) \subset A^5(t^0)$,

β) $A^1(t^0) \subset A^4(t^0)$ und $A^2(t^0) \subset A^5(t^0) \cap A^6(t^0)$, falls $D$ mit $d$ verträglich ist.
\( A^0(t^0) \subset A^3(t^0) \subset A^4(t^0) \), falls \([V](t^0, -) \geq 0\) ist.

\( A^6(t^0) \supset A^7(t^0) \), falls \( D \) mit \( d \) verträglich ist. \( A^6(t^0) = A^7(t^0) \), falls zusätzlich \([V](t^0, x) > 0 \ \forall V(t^0, x) \neq 0\) ist. \( A^5(t^0) \cap A^7(t^0) = A^2(t^0) \), falls zusätzlich \([V](t^0, x) > 0 \ \forall V(t^0, x) \neq 0\) ist.

\textbf{Beweis.} Die Behauptungen sind sofort klar oder folgen aus den Identitaten (wobei \( D \) mit \( d \) verträglich ist):

\[ D_w d(V, V) = 2 \cdot d(D_w V, V), \]

\[ D_w (V \cdot [V]^{-1/2}) = -1/2[V]^{-3/2} \cdot D_w[V] \cdot V + [V]^{-1/2} \cdot D_w V. \]

Die mit den Achsen \( A^i \) zusammenhängenden Probleme stellen wir gleich in einen allgemeineren Rahmen. \( b : I \times X \rightarrow Y \) sei ein eigentlicher Bewegungsablauf. \( D \) sei eine fixierte kovariante Ableitung auf \( T^i X \).

\textbf{Definition 4.8.}

\( \alpha \) Ein Unterraum \( X^I \subset I \rightarrow X \) heiße \emph{invariant} unter \( b \), falls \( X^I \) unter \( b \) auf dem Gegenraum \( Y^I := (id \times b)(X^I) \) gleitgleitet.

\( \beta \) \( X^I \) heiße unter \( b \) \\emph{stark invariant} zum Zeitpunkt \( t^0 \), falls \( V|X^I_{t^0} \) tangentiell an \( X^I_{t^0} := \{ x \mid (t^0, x) \in X^I \} \) ist.

\( \gamma \) \( X^I \) heiße unter \( b \) \\emph{fix} (bzw. \emph{starr} bzw. \emph{bahnstarr} bzgl. \( D \)) zum Zeitpunkt \( t^0 \), falls gilt: \( V|X^I_{t^0} = 0 \) (bzw. \( V|X^I_{t^0} \) ist tangentiell an \( X^I_{t^0} \) und parallel längs \( X^I_{t^0} \) bzw. parallel nur längs allen Bahnkurven von \( V(t^0, -) \) in \( X^I_{t^0} := D_V V(t^0, -) = 0 \).

\textbf{Bemerkung 4.9.} \( \gamma \Rightarrow \beta \Rightarrow \alpha \)

\textbf{Frage 4.10.} Man finde Unterräume \( X \) mit obigen Eigenschaften. Diese würden \( b \) "geometrisch deuten": Ein Gleitgleiten, gar Rollgleiten oder erweitertes Schroten von \( X^I \) unter \( b \) auf dem Gegenraum \( Y^I \) "erzeugt" gewissermaßen \( b \). Was gilt in einem Fall wie \( X^I = A^i \)? Wie liegen die \( A^i \) zueinander?

\textbf{Satz 4.11.} \( \pi_1(P_b) = I. \) Dann rollt unter dem eigentlichen Bewegungsablauf \( b \) der Polraum \( P_b \) auf dem Gegenpolraum \( G P_b \) ab. \( P_b \) ist unter \( b \) invariant, stark invariant und fix.

\textbf{Beweis.} Aus \( y = \varphi(t, x), b_t(t, x) = 0 \) folgt \( V(t, x) = D_y b^*(t, y) \circ D_t b(t, x)(1) = 0 \), insbesondere \( G_{t, x} = 0 \) auf ganz \( P_b \). Der Rest ist klar.


\textbf{Satz 4.12} \( \pi_1(T_{t, x}X^I) = T_I(I = R) \) für alle \((t, x) \in X^I \). Dann ist \( X^I \) im Polraum \( P_b \), \( Y^I \) im Gegenpolraum \( G P_b \) von \( b \) enthalten.
Beweis. Aus $G(t,x) = 0$ folgt wegen der Voraussetzung $\pi_1(T(t,x)X^I) = R$ auch $b_t(t,x) = 0$ (da $D_yb^*$ überall ein Isomorphismus ist).

Mehr als die Inklusion $X^I \subset P_b$ kann man nicht erwarten: Z.B. sei $X = Y$, $b(t, -) = \text{id}$ $\forall t$, $X = I \times p$ für ein $p \in X$. Dann ist $I \times p \subset P_b = I \times X$. Die Voraussetzung $\pi_1(T(t,x)X^I) = R$ dürfte aber oft nicht erfüllt sein.


Für die zu einem eigentlichen isometrischen Bewegungsablauf $b : I \times X \to Y$ gehörenden Achsenräume $A^i$ gilt nun:

Satz 4.13.  
\begin{enumerate}
  \item $A^1 \subset A^2$, $A^0 \subset A^3 \subset A^2 = A^4 = A^5 = A^7$, $A^5 = I \times X$
  \item $A^2$ ist unter $b$ bahnenstarr, also auch stark invariant und damit invariant; insbesondere rollgleitet $A^2$ unter $b$ auf $GA^2$.
\end{enumerate}

Beweis. Da $b$ isometrisch ist, erfüllt mit $y := b(t,x)$ das Feld $V(t,x) := D_yb^*(t,y)(b_t(t,x))$ gerade Satz 3.6, *(1). Für jedes fixierte $t^0$ erfüllt also $V(t^0,-)$ gerade Satz 3.6, *(1). Jeder (zu jedem $x^0$) jeweils zugehörige (um $x^0$ lokale) Fluß $\psi(s,x) = V(t(s,x),x)$, und $\psi(0,x) = x$ liefert also bei fixiertem $s$ (lokal um $x^0$) einen isometrischen Diffeomorphismus in $X$ (mit $X = Y$ gedacht). Wegen
\begin{align*}
V(t^0,\psi(s,x)) &= \psi_t(s,x) = \psi_t(s+t,x)|_{t=0} = d_t(\psi(s,y(t,x)))|_{t=0} \\
&= d_x(\psi(s,x)(\psi_t(0,x))) = d_x(\psi(s,x)(V(t^0,x)))
\end{align*}
folgt: Mit $x^0 \in A^2(t^0)$ ist auch stets $\psi(s,x^0) \in A^2(t^0)$ und insbesondere $V(t^0,x^0) \in T_{x^0}A^2(t^0)$, sogar in $T_{x^0}^iA^2(t^0)$. Die in $A^2(t^0)$ startenden Bahnen von $\psi(s,x)$ bleiben also in $A^2(t^0)$. Nach Definition von $A^2(t^0)$ ist dort $D_VV = 0$. Damit folgt $\beta)$. Zum Nachweis von $\alpha):$ Wegen **) im Beweis zu 3.6 ist
\begin{align*}
*) \quad d(\nabla V, w) &= -d(V, \nabla w V) = 2D_w(\nabla V, V) \quad \forall w \in T^iX,
\end{align*}
also folgt: $(\nabla V)(t^0,x) = 0 \iff D(d(V,V))(t^0,x) = 0$, also $A^2(t^0) = A^4(t^0) \forall t^0$, also $A^2 = A^4$, $A^5(t^0) = X \forall t^0$ folgt aus *) oben, wenn man dort
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\[ w = V(t^0, -) \] setzt und \( d(D_V V, V) = 0 \) erhält. Also ist \( A^5 = I \times X \). Wir zeigen jetzt \( A^2 = A^7 \): Wegen \( \ast \) ist \( d(D_V V, V) = 0 \). Also folgt: \( D_V V, V \) l.a. \( \iff D_V V = 0 \). Wir zeigen weiter: \( A^2 = A^6 \) Für \( V(t^0, x) \neq 0 \) erhält man wegen \( A^5 = I \times X \: D_V (V/[V]^{1/2}) = [V]^{-3/2}D_V[V] \cdot V + [V]^{1/2}D_V V = [V]^{-1/2}D_V V \) und daraus die Behauptung. Die verbliebenen Inklusionen sind schließlich klar.

Im Sonderfall [Fr/Sp] ist das Rollgleiten unter 4.13, \( \beta \) sogar ein erweitertes Schrötens. Wie weit das auch "im allgemeinen" zutrifft, ist noch offen. Vielleicht auch trifft Definition 4.4, \( \delta \) mit ihrer Orientierung am Spezialfall [Fr/Sp] für den Allgemeinfall nicht immer das Erwünschte.

5. Geodätische Kinematik in Zahlenräumen \( R^n \)

Die folgende Spezialisierung erlaubt genauere Aussagen und gibt zugleich bestätigende Beispiele zum Bisherigen. Die "Bewegungen" hier sind sämtlich "geodätisch", d.h. sie bilden Geodätische im euklidischen \( R^n \) auf Geodätische ab und sind also affin. Als Spezialfall hiervon erhält man die affin isometrische Kinematik von [Fr/Sp], die affine konforme Kinematik von [Sp7]. Beide ordnen sich hier in einen größeren Kontext ein.

\[ b : I \times R^n \to R^n \] sei ein geodätischer Bewegungsablauf: Für jedes \( t \in I \) ist \( b(t, -) : R^n \to R^n \) geodatisch, bildet also "lokal kürzeste" Kurven auf "lokal kürzeste" Kurven ab (bzgl. der euklidischen Metrik im \( R^n \)). \( b(t, -) \) ist daher gerade eine affine Bijektion, also:

\[ b(t, x) \equiv k(t) + H(t) \cdot x = k + Hx \quad \text{(in Kurzscheibweise)} \]

Hier ist \( H(t) \) eine invertierbare \( n \times n \)-Matrix, die - wie auch \( k(t) \in R^n \) differenzierbar von \( t \) abhängt. Es ist

\[ x = b^*(t, y) = -H^{-1}(t) \cdot k(t) + H^{-1}(t) \cdot y = -H^{-1} \cdot k + H^{-1} \cdot y \]

\[ b^*_y(t, y) = H(t)^{-1} \]

wenn wir hier wie weiterhin Matrizen mit den zugehörigen Abbildungen identifizieren. Also ist

\[ V(t, x) = b^*_y(t, y)(b_t(t, x)) \quad \text{für} \quad y = b(t, x) \quad (3.3) \]

\[ = H^{-1} \cdot k' + H^{-1} \cdot H' \cdot x =: K + M \cdot x \]

\( D \) bezeichne den totalen Differentialoperator bzgl. der \( x \) - (oder auch der \( y \)-) Variablen, \( D_V (\ldots) \) die Richtungsableitung von (\ldots) in Richtung \( V \). Dann ist

\[ DV = M, \quad D_V V = M \cdot V \]
$C$ sei eine weitere $n \times n$-Matrix. Wir kürzen ab: $[V] := V^+ \cdot C \cdot V$ und $V^{(k)} := V/[V]^k$ an Stellen $[V] > 0$, $k \in \mathbb{R}$. $(...)^+$ bedeutet dabei "gestürzt".

Mit $V$ hängen nun nach 4.6 folgende Arten von "Achsen" zusammen, die unterschiedliches "stationäres Verhalten" von $b$ erfassen:

\[
A^0(t^0) = \{ x \mid 0 = K(t^0) + M(t^0) \cdot x(= V(t^0, x)) \}
\]

\[
A^1(t^0) = \{ x \mid 0 = DV(t^0, x) = M(t^0) \}
\]

\[
A^2(t^0) = \{ x \mid 0 = D_V V(t^0, x) = M \cdot V(t^0, x) \}
\]

\[
A^3(t^0) = \{ x \mid V^+ CV(t^0, -) \text{ hat in } x \text{ ein Extremum} \}
\]

\[
A^4(t^0) = \{ x \mid 0 = D[V](t^0, x) = V^+(t^0, x) \cdot (C + C^+) \cdot M(t^0) \}
\]

\[
A^5(t^0) = \{ x \mid 0 = D_V[V](t^0, x) = V^+(C + C^+) \cdot M \cdot V(t^0, x) \}
\]

\[
A^6(t^0) = \{ x \mid 0 \geq [V](t^0, x), \text{sonst } D_V(V/[V]^{1/2})(t^0, x) = 0 \}
\]

\[
A'^6(t^0) = \{ x \mid 0 \geq [V](t^0, x), \text{sonst } M \cdot V(t^0, x), V(t^0, x) \text{ l.a.} \}
\]

\[
A^7(t^0) = \{ x \mid D_V V(t^0, x), V(t^0, x) \text{ l.a.} \}
\]

\[
= \{ x \mid M \cdot V(t^0, x), V(t^0, x) \text{ l.a.} \}
\]

\[
A^{(k)}(t^0) = \{ x \mid [V](t^0, x) < 0, \text{sonst } D_g(V/[V]^k)(t^0, x) = 0 \}.
\]

Die letzte Festlegung ist neu gegenüber §4, aber i.w. überflüssig, wie wir gleich sehen werden. Ferner ist $A^0(t^0) = A'^6(t)$, wir wir auch gleich sehen werden.

**Bemerkung 5.1.**

\begin{itemize}
  \item[α)] $A^{(1/2)}(t^0) = A^6(t^0)$.
  \item[β)] $V(t^0, x) = 0 \Rightarrow [V](t^0, x) = 0$
  \item[γ)] $[V] = 0$, falls $C^+ = -C$ ist. Denn: $V^+ C \cdot V = (V^+ \cdot C \cdot V)^+ = V^+ \cdot C^+ \cdot V = -V^+ \cdot C \cdot V$.
\end{itemize}

Wir halten noch folgende Bedingungen fest:

\begin{itemize}
  \item[i)] $\forall x : V(t^0, x) = 0 \iff [V](t^0, x) = 0$,
  \item[ii)] $[V](t^0, x) \geq 0 \forall x$ (C ist z.B. positiv definit)
\end{itemize}

**Satz 5.2.**

\begin{itemize}
  \item[α)] Sei $[V](t^0, x) > 0$. Dann gilt:
    \[
    D_V(V/[V]^{1/2})(t^0, x) = 0 \iff M \cdot V(t^0, x), V(t^0, x) \text{ l.a. Also:}
    \]
  \item[β)] $A^7(t^0) \subset A^6(t^0) = A'^6(t^0) \forall t^0 \in I.$
  \item[γ)] Gelten i), ii), so folgt: $A^2(t^0) = A^{(k)}(t^0) \forall k \neq 1/2$.
\end{itemize}
δ) Gelten i), ii), so folgt

\[ 0 = D[V]^k(t^0, x) \iff 0 = D[V](t^0, x) \quad \forall k \neq 0, \]

\[ 0 = D_V[V]^k(t^0, x) \iff 0 = D_V[V](t^0, x) \quad \forall k \neq 0. \]

**Beweis.** Im Falle \([V](t^0, x) > 0\) existiert \(V = [V]^k \cdot V^{(k)}\) in \((t^0, x)\), und zwar für jedes \(k \in \mathbb{R}\). Dann ist in \((t^0, x): [V] = V^+ CV = V^+ C^+ V, \]

\[ D_V[V] = (D_V[V]^k) \cdot V^{(k)} + [V]^k \cdot D_V[V^{(k)}], \text{ mit } * = (k) \text{ also: } D_V \cdot V^* = 0 \]

\[ \iff D_V^* = 0 \iff D_V[V] = D_V[V]^k \cdot V^{(k)} \iff M \cdot V = D_V[V] = k \cdot [V]^k \cdot D_V^* \iff \]

1) \(MV, V\) sind l.a. (: \(M \cdot V = a \cdot V, \text{ sogar } a = V^+ CMV/[V]\)) und
2) \(V^+ CMV = k \cdot [V]^{-1} \cdot D_V[V] \cdot [V] = k \cdot D_V[V] = k \cdot (V^+ (C + C^+) \cdot MV)\)
3) Nun ist ebenso \(V^+ C^+ MV = k \cdot D_V[V]\), also ist 2) äquivalent zu

\[ V^+(C + C^+)MV = 2k \cdot (V^+(C + C^+)MV) \]

α) folgt, da 2) im Falle \(k = 1/2\) keine Bedingung ist. β) ist jetzt klar. γ)
Im Falle \(k \neq 1/2\) muß \(V^+(C + C^+MV = 2 \cdot V^+ CMV\) Null sein, also auch obiger Proportionalitätsfaktor \(a\), also folgt \(MV = 0\). Für \([V](t^0, x) > 0, \]

\(k \neq 1/2\) folgt also in \((t^0, x): D_V[V/[V]^k] = 0 \iff MV = 0\). Unter den Bedingungen i), ii) gilt damit: \(A^2(t^0) = A^{(k)}(t^0) \quad \forall k \neq 1/2. \) δ) schließlich ist klar.

Man hat weitere Beziehungen zwischen den verschiedenen Achsen, manche gelten nur unter Zusatzbedingungen, z.B.:

**Bemerkung 5.3.**
α) \(A^0(t^0) \subset A^1(t^0) \subset A^5(t^0)\) für \(i = 2, 4, A^1(t^0) = R^n \) oder \(= 0, A^0(t^0) \subset A^2(t^0) \subset A^7(t^0)\) \(\forall t^0 \in I.\)

β) Sei \(C + C^+ = p \cdot E\) mit \(p \neq 0, M(t^0)^+ = A \cdot M(t^0)\) mit det \(A \neq 0\) (speziell also \(M(t^0)^+ = \pm M(t^0)\) oder det \(M(t^0) \neq 0\) ⇒ \(A^2(t^0) = A^4(t^0)\)) Sei dann zusätzlich \(M(t^0)^+ = -M(t^0) \Rightarrow A^2(t^0) = A^7(t^0)\)
γ) Aus \([V](t^0, x) > 0\) im Falle \(V(t^0, x) \neq 0 \ \forall x\), folgt: \(A^5(t^0) \cap A^7(t^0) = A^2(t^0) \subset A^8(t^0) = A^{m6}(t^0) = A^7(t^0).\)

Die folgenden Sonderfälle unter 5.4, α), γ) betreffen speziell die euklidische Kinematik [Fr/Sp] mit höchstens nur noch zwei i.a. verschiedenen, nicht trivialen Achsentypen \(A^0 \subset A^3, \text{ unter } \beta\) und \(\gamma_2\) die \(W\)-Kinematik [Sp7] mit höchstens nur noch 3 i.a. nicht trivialen verschiedenen Achsentypen \(A^0 \subset A^5, A^7).\)
Satz 5.4.

a) Im Falle $C + C^+ = p \cdot E$, $p > 0$, $M(t^0)^+ = -M(t^0)$ gilt:

\begin{align*}
A^1(t^0) &= 0 \quad \text{oder} \quad R^n; \\
A^5(t^0) &= R^n; \\
A^0(t^0) &\subset A^3(t^0), \\
A^0(t^0) &\subset A^2(t^0) = A^4(t^0) = A^6(t^0) = A'^6(t^0) = A^7(t^0)
\end{align*}

\begin{itemize}
  \item[\beta_1] Im Falle $\det M(t^0) \neq 0$ gilt:
    \begin{align*}
    A^0(t^0) &= A^2(t^0), \\
    #A^0(t^0) &= 1, \\
    A^1(t^0) &= \emptyset, R^n.
    \end{align*}
  \item[\beta_2] Gilt zusätzlich $C + C^+ = p \cdot E$ mit $p \neq 0$ (bzw. sogar mit $p > 0$) so folgt weiter
    \begin{align*}
    A^2(t^0) &= A^4(t^0) \quad \text{(bzw. dazu noch $A^0(t^0) = A^7(t^0)$)}.
    \end{align*}
\end{itemize}

\begin{itemize}
  \item[\gamma] Im winkeltreuen Fall ($H \cdot H^+ = q \cdot E$) gilt zusätzlich:
    \item[\gamma_1] zu \alpha): $A^3(t^0) = A^2(t^0)$ im Falle $q \equiv 1$ (euklidischer Fall)
    \item[\gamma_2] zu \beta): $A^3(t^0) = A^0(t^0)$ im Falle $q(t^0) \neq 0$ ("echt" winkeltreuer Fall)
\end{itemize}

\textbf{Beweis.} Zu \gamma) siehe [Sp7], Satz 1.8. \hfill \Box

Einige der Achsen $A'(t^0)$ tragen eine natürliche, mit $V$ zusammenhängende Blätterung. Wir kürzen im folgenden ab: $M = M(t^0)$, $K = K(t^0)$, fixieren also $t^0$.

Weiter sei $L := \ker M$ (als Abbildung $x \to Mx$), und $A^2(t^0, x^0) := x^0 + L$ für jedes fixierte $x^0$.

\textbf{Bemerkung 5.5.}

\begin{align*}
A^2(t^0) &= \{ x \mid 0 = MV(t^0, x) = M(K + Mx) \} \\
&= \{ x^0 + x^* \mid M \cdot Mx^* = 0 \}, \quad \text{mit } x^0 \in A^2(t^0) \\
&\supset \{ x^0 + x^* \mid Mx^* = 0 \} = A^2(t^0, x^0).
\end{align*}

Wir haben eine Blätterung von $A^2(t^0)$ in gewisse Räume $A^2(t^0, x^0)$, die auch disjunkt gemacht werden kann ($\forall x, y$ ist $A^2(t^0, x) \cap A^2(t^0, y) = \emptyset$ oder $= A^2(t^0, x))$: $A^2(t^0) = \cup A^2(t^0, x)$, wobei $x$ ganz $A^2(t^0)$ durchläuft.

\textbf{Satz 5.6+}. 

a) Es gibt für jedes $j = 0, \ldots, 5$ je einen (im Fall $j = 0, 1, 2, 4$ affinen) Unterraum $U^j = U^j(t^0) \subset A^j(t^0)$ mit folgender disjunkten "Blätterung":

\begin{align*}
A^j(t^0) &= U^j + L = \cup A^2(t^0, x^j), \quad \text{wobei } x^j \text{ ganz } U^j \text{ durchläuft.}
\end{align*}

\begin{itemize}
  \item[\beta_1] Falls $M$ auf Bild $M$ injektiv ist (z.B. falls $M^+ = -M$, speziell $H^+ H = E$ ist, (vgl. [Sp7]) ist $U^2(t^0) = \{ x^0 \}$, also
    \begin{align*}
    A^2(t^0) &= A^2(t^0, x^0)
    \end{align*}
\end{itemize}
\( \beta_2 \) Falls \( M^+(C+C^+) \) injektiv auf Bild \( M \) ist (z.B. falls \( M^+ \) injektiv auf Bild \( M \), \( C+C^+ = p \cdot E \) mit \( p \neq 0 \) ist), folgt

\[
U^4(t^0) = \{ x^0 \}, \quad A^4(t^0) = x^0 + L = A^2(t^0, x^0).
\]

**Beweis.** klar. \( \square \)

Für die bisher ausgeschlossenen Räume \( A^6(= A'^6) \), \( A^7 \) gibt es etwas andere Zerlegungen.

Sei \( E(t^0) \) die Menge der Eigenvektoren von \( M(t^0) \), \( C \in \mathbb{R}^n \) und

\[
A^7_C(t^0) := \{ x \mid \exists s \neq 0 \text{ mit } K(t^0) + M(t^0) \cdot x = Cs \}.
\]

**Satz 5.7.** Für \( j = 6, 7 \) hat man folgende Blätterungen:

\[
A^j(t^0) = \bigcup A^7_C(t^0) \cup A^7(t^0),
\]

wobei im Falle \( j = 7 \) die Vereinigung über alle \( C \in E(t^0) \) zu nehmen ist, im Falle \( j = 6 \) über alle \( C \in E(t^0) \) mit \( |C| > 0 \). Speziell folgt: \( T_xA^j(t^0) \supseteq T_xA^7_C(t^0) \forall x \in A^7_C(t^0), C \) wie oben.

**Beweis.** Der Fall \( j = 7 \) ist klar, wegen \( A^6 = A'^6 \) ist dann auch der Fall \( j = 6 \) klar. \( \square \)

Für jedes \( C \in \mathbb{R}^n \setminus \{0\} \) gilt übrigens \( A^7_C(t) = \bigcup A^2(t, x) \), wobei die Vereinigung über alle \( x \) zu nehmen ist, für die \( V(t, x), C \text{ l.a. sind. Die Bedeutung dieser Zelegungen liegt in folgenden Aussagen zum kinematischen Verhalten. Für jedes fixierte } t \in I \text{ gilt nämlich:} \)

**Satz 5.8.**

a) \( V(t, x) = 0 \in T^0_{(t,x)}A^0(t) \forall x \in A^0(t) \). \( A^0(t) \) ist der größte (affine, evtl. leere) Unterraum zu \( t \), auf dem \( V(t, -) = 0 \) ist.

\( \beta \) \( V(t, x) \in T^0_{(t,x)}A^2(t, y) \forall x \in A^2(t, y), y \in A^2(t) \). \( V|A^2(t, y) = K(t) + M(t)y^0 = \text{konst} \forall y \in A^2(t) \). \( V|A^j(t) \) ist tangentiell an \( A^j(t) \), aber i.a. nicht konstant (\( \forall j = 1, 2, 3, 4, 5 \)).

\( \gamma \) \( V(t, x) \in T^0_{(t,x)}A^7(t) \), falls \( C := V(t, x) \) Eigenvektor von \( M(t) \) ist,

\( T^0_{(t,x)}A^7(t) \forall x \in A^7(t) \). \( V|A^7_C(t) \) ist oben also tangentiell an \( A^7_C(t) \), aber i.a. nicht konstant. Also ist auch \( V|A^6(t), V|A^7(t) \) jeweils tangentiell, aber i.a. nicht konstant.

**Beweis.** a), \( \beta \) sind aufgrund der Definitionen von \( A^0, A^2, A^2 \) klar (zur Nicht-konstanz gibt es Beispiele), \( \gamma \) Es ist \( V = K + M \cdot x; (t, x) \) sei fixiert. Für \( C := V(t, x), MC = \lambda \cdot C \) und alle \( r \in R \) folgt: \( x + r \cdot V(t, x) \in A_C(t) \). Es ist nämlich \( K + M(x + rV(t, x)) = K + Mx + rMV(t, x) = C + r\lambda C = (1 + r\lambda)C. \) Damit wird \( V(t, x)eT_xA^7_C(t) \in T^0_xA(t) \). \( \square \)
KINEMATICS AND VECTORFIELDS

Kinematisch gewendet bedeutet das:

**Korollar 5.9.** Unter einem geodätischen Bewegungsablauf gleitgleiten die zugehörigen Achsenräume $A^d$ auf ihren jeweils zugehörigen Gegenachsenräumen ab.

Die $A^d(t)$ sind nämlich nach obigem stark invariant, also invariant, wegen $\beta$) oben im Falle 5.6, $\beta$) sogar starr. Das enthält speziell [Fr/Sp]. Im Falle einer winkeltreuen Kinematik ([Sp7]) $b = k + l \cdot f \cdot x$ mit $f(t)^+ \cdot f(t) = E$, $l(t) \neq 0 \forall t$ ist $M(\cdot V = K + M \cdot x)$ sogar invertierbar. Dann sind $MV(t, x)$, $V(t, x)$ genau dann l.a., wenn $M \cdot (x - x^0), (x - x^0)$ l.a. sind, wobei $V(t, x^0) = 0$ sei. (denn: $MV, V$ l.a. $\iff V, M^{-1}V$ l.a., und $V(t, x) = K(t) + M(t) \cdot x = M(t) \cdot (x - x^0)$). Dies ist die von Somer [S] beschriebene Situation.

**Literatur**


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A COMPUTER-ASSISTED STUDY OF
THE DELAYED LOGISTIC MAP

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1. Introduction

Since ancient Greece, any phenomenon, however complicated it looks, is regarded as a union of simple phenomena. Thus by analyzing these simple phenomena, we can analyze a complicated phenomenon. That had been the guiding principle throughout science world. However, in late 1960's new phenomena were discovered. A new science, called the theory of chaos, has emerged. They are described by very simple equations, but they behave in very strange erratic ways. The most famous example of such a chaotic system is the logistic map in mathematical biology. The map, studied by Robert May in 1976 [5], is given by

\[ N_\mu(x) = \mu x(1 - x). \]

If \( x_n \) represents the population of some species in the \( n \)-th generation, then the population in the \((n + 1)\)-st generation is given by the logistic model:

\[ x_{n+1} = N_\mu(x_n) \quad (n = 0, 1, 2, \ldots). \quad (1.1) \]

May observed that the behavior of orbits depends in a dramatic way on the parameter \( \mu > 0 \).

In May's model given by (1.1), the effect of increase or decrease of population appears only in the following generation. This may not always be the case for species that have a long maturation time or that migrate to nesting or breeding areas. The amount of food available to the \( n \)-th generation in that area might depend on how much was eaten in the previous year. In this case the population of the \((n + 1)\)-st generation depends on the
population in the $n$-th and the $(n - 1)$-st generations. This motivates the following model of population growth called the delayed logistic model \cite{1},

$$x_{n+1} = \mu x_n (1 - x_{n-1}).$$ \hspace{1cm} (1.2)

In the present paper, we study the differences of properties of the dynamical systems (1.2) for different values of $\mu$. The paper is organized as follows. In the section 2 we recall what May observed. In the section 3 we study the delayed logistic model by numerical method as well as rigorous mathematical arguments. In the section 4 we study in detail what happens when the parameter $\mu$ hits certain values.

This is an announcement of a joint research with Professors Y. Nakamura and T. Natsume.

2. Orbit diagrams

A best way to describe a change in the orbit $\{x_n\}$ when value of $\mu$ is changed, is the orbit diagram, which assembles all the information into a single picture. The diagram shows how changes in value of $\mu$ would change the ultimate behavior of the system.

The diagram is obtained in the following way. Since we are interested in the fate of orbits, we omit early stage of orbits. To be precise, for a given value of $\mu$ and for a given value of $x_0$, we compute first 10,000 points on the orbit, and then we plot the following 100 points. These 100 points are considered as the final population. Values of $\mu$ are represented horizontally, and the final population is plotted on the vertical direction. The orbit diagram of the logistic map with $x_0 = 0.01$ is shown in the Figure 2.1.

One important aspect of the logistic model is that the ultimate fate of orbits is independent of the choice of the initial value $x_0$. In other words one gets the same Figure 2.1 for different values of $x_0$.

When the value of parameter is small ($0 < \mu \leq 1$), the population becomes extinct. As the parameter rises ($1 < \mu \leq 3$), so does the equilibrium levels. As the parameter rises further ($3 < \mu$), the equilibrium splits in two. This splitting is called a bifurcation. The population begins to alternate between two different levels. The bifurcations come faster and faster. Then the system turns chaotic ($\mu > 3.57$). The population visits infinitely many different values.

We now turn our attention to the delayed logistic model. The orbit $\{x_n\}$ of the delayed logistic model (1.2) depends on two values of $x_0$ and
Figure 2.1. The orbit diagram for $x_{n+1} = \mu x_n (1 - x_n)$ with $x_0 = 0.01$.

$x_1$. We choose $x_0 = 0.46$ and $x_1 = 0.37$, and draw the orbit diagram (Figure 2.2) following the same procedure as the logistic model. The reason why we choose those initial values will be explained in the section 4.

The orbit diagram of the delayed logistic map with $x_0 = 0.46$ and $x_1 = 0.37$ is shown in the Figure 2.2.

Figure 2.2. The orbit diagram for $x_{n+1} = \mu x_n (1 - x_{n-1})$ with $x_0 = 0.46, x_1 = 0.37$. 
When $\mu \leq 2$, the diagram is similar to that of the logistic map. But for $\mu > 2$, a quite different picture emerges. A blow-up of the portion $\mu \geq 2$ of Figure 2.2 is in the Figure 2.3.

Figure 2.3. Blow-up of the portion $\mu \geq 2$ of Figure 2.2.

Figure 2.3 looks like the chaotic region $(3.57 < \mu < 4.00)$ in the Figure 2.1.

3. Delayed logistic map

The delayed logistic model is in a sense very similar to order two difference equations, or to order two ordinary differential equations. In the study of ordinary differential equations, instead of dealing directly with order two equations, one converts them to systems of order one equations. We employ a similar method in order to study the delayed logistic model.

Set $y_n = x_{n-1}$ ($n = 1, 2, 3, \ldots$), then the delayed logistic map is expressed by the following pair of equations:

$$
\begin{align*}
x_{n+1} &= \mu x_n (1 - y_n) \\
y_{n+1} &= x_n.
\end{align*}
$$

A map $F_\mu : \mathbb{R}^2 \to \mathbb{R}^2$, which we call the delayed logistic map, is defined by

$$F_\mu(x, y) = (\mu x (1 - y), x).
$$

Then the recurrence formula (3.1) is written in the following form:

$$(x_{n+1}, y_{n+1}) = F_\mu(x_n, y_n).$$
The origin $O = (0, 0)$ is always a fixed point of $F_\mu$, and there exists another nontrivial fixed point $P_\mu = (\frac{\mu-1}{\mu}, \frac{\mu-1}{\mu})$, if $\mu > 1$.

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be $C^1$-class. Suppose that $P$ is a fixed point of $F$. Denote by $DF(P)$ the Jacobian matrix of $F$ at $P$. The eigen-values of $DF(P)$ classify the nature of $P$. A fixed point $P$ is hyperbolic if $|\lambda| \neq 1$ for all eigen-values $\lambda$ of $DF(P)$. A hyperbolic fixed point $P$ is a sink if there exists a neighbourhood $U$ of $P$ such that $F^n(Q) \to P$ as $n \to \infty$ for any $Q \in U$ [2]. The fixed point $P$ is a source if there exists a neighbourhood $V$ of $P$ such that for any $Q \in V$, different from $P$, there exists an $N$ such that $F^n(Q) \notin V$ for all $n \geq N$. If $P$ is neither a sink, nor source, $P$ is called a saddle point. We have the following well-known fact (see, e.g. [3]).

**Proposition 3.1.** Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be of $C^1$-class, and let $P$ be a fixed point of $F$. Denote by $\lambda_+, \lambda_-$ the eigen-values of the Jacobian matrix of $F$ at $P$ with $|\lambda_-| \leq |\lambda_+|$. We have the following:

1. If $|\lambda_+| < 1$, then $P$ is a sink,
2. If $|\lambda_+| > 1$, then $P$ is a source,
3. If $|\lambda_+| > 1$ and $|\lambda_-| < 1$, then $P$ is a saddle point.

We apply Proposition 3.1 to the delayed logistic map. As

$$DF_\mu(O) = \begin{bmatrix} \mu & 0 \\ 1 & 0 \end{bmatrix}$$

we know that

$$\lambda_+ = \mu \text{ and } \lambda_- = 0.$$ 

Hence, when $\mu = 1$, Proposition 3.1 cannot be applied. Yet, the restriction of $F_1$ on $\Omega = [0, 1] \times [0, 1]$ can be analyzed in a straightforward way. Take $(x_0, y_0) \in \Omega$, and the let $\{(x_n, y_n)\}$ be the orbit of $(x_0, y_0)$. We have

$$0 \leq x_{n+1} = x_n(1 - y_n) < x_n.$$ 

Thus $\{x_n\}$ is a monotone-decreasing sequence, and we have its limit $\lim_{n \to \infty} x_n = \alpha$. By formula (3.1), $\lim_{n \to \infty} y_n = \alpha$. Again by formula (3.1), $F_1(\alpha, \alpha) = (\alpha, \alpha)$. Consequently $\alpha = 0$.

Let us go back to the case of the logistic map $N_\mu : \mathbb{R} \to \mathbb{R}$. If $0 < \mu \leq 4$, the map $N_\mu$ maps $I = [0, 1]$ into itself, but it does not do so any more if $\mu > 4$. There exists a Cantor set $K \subset I$ such that $N_\mu(K) \subset K$. For detail, see [3]. A similar phenomenon occurs for the delayed logistic map. When
\(\mu > 1\), the set \(\Omega\) is not invariant under \(F_{\mu}\). For instance, \(F_{\mu}(1,0) \notin \Omega\). Define sets \(\Omega_n(\mu)\) and \(\Omega(\mu)\) by:

\[
\Omega_n(\mu) = \{ (x,y) \in \Omega : F_{\mu}^k(x,y) \in \Omega, 1 \leq k \leq 2 \},
\]

\[
\Omega(\mu) = \bigcap_{k=1}^{\infty} \Omega_k(\mu).
\]

Numerical analysis indicates that for large \(\mu\)'s \(\Omega(\mu)\) are “thin” sets like Cantor sets. We will investigate \(\Omega(\mu)\) further in the section 4. For now, in order to apply Proposition 3.1 to the delayed logistic map, we need only that the set

\[
\Omega_2(\mu) = \{ (x,y) \in \Omega_1(\mu) : \mu^2(x(1-x)(1-y) \leq 1 \}
\]

is invariant under \(F_{\mu}\).

**Lemma 3.1.** There exists a \(\mu_2 > 2\) such that \(\Omega_2(\mu)\) is invariant under \(F_{\mu}\) for \(0 < \mu \leq \mu_2\).

The proof will be given in the section 4. As we will see there \(\mu_2\) is a root of an algebraic equation of degree 6. Its approximate value is \(\mu_2 = 2.24628\). The origin \(O\) belongs to the interior of \(\Omega_1(\mu)\). Therefore (a modified version of) Proposition 3.1 can be applied, if \(1 < \mu \leq \mu_2\).

Summarizing the argument above, we have:

**Theorem 3.1.** As for the restriction of \(F_{\mu}\) on \(\Omega\) we have

1. If \(0 < \mu \leq 1\), then the origin \(O\) is a sink of \(F_{\mu}\).
2. If \(1 < \mu \leq \mu_2\), then the origin \(O\) is a saddle point of \(F_{\mu}\).

Let us turn to the other fixed point \(P_{\mu}\). We see that

\[
DF_{\mu}(P_{\mu}) = \begin{bmatrix} 1 & 1 - \mu \\ 1 & 0 \end{bmatrix}.
\]

Hence the eigen-values of \(DF_{\mu}(P_{\mu})\) are

\[
\lambda_+ = \frac{1 + \sqrt{5 - 4\mu}}{2} \quad \text{and} \quad \lambda_- = \frac{1 - \sqrt{5 - 4\mu}}{2}.
\]

Therefore,

1. If \(1 < \mu \leq 2\), then \(|\lambda_{\pm}| < 1\),
2. If \(\mu = 2\), then \(|\lambda_{\pm}| = 1\),
3. If \(\mu > 2\), then \(|\lambda_{\pm}| > 1\).
Notice that the fixed point $P_\mu$ belongs to the interior of $\Omega_2(\mu)$. Therefore, again by a modified version of Proposition 3.1 the following theorem is obtained.

**Theorem 3.2.** As for the restriction of $F_\mu$ on $\Omega_2(\mu)$ we have

1. If $1 < \mu < 2$, then $P_\mu$ is a sink.
2. If $2 < \mu < \mu_2$, then $P_\mu$ is a source.

We will investigate in detail the situation where $\mu \geq 2$ in the section 4.

### 4. Sudden break-up of limit cycles

In this section we see by means of numerical experiments what happens when $\mu$ surpasses 2. As observed above, $P_\mu$ is a source when $2 < \mu < \mu_2$. Since $\Omega_2(\mu)$ is compact and is invariant under $F_\mu$ by Lemma 3.1, any orbit $\{F_\mu(x_0,y_0)\}$ for arbitrary $(x_0,y_0) \neq P_\mu$ close to $P_\mu$ has an accumulation point. Those accumulation points form closed subsets, and attractors appear as shown in the Figure 4.1.

![Figure 4.1](image)

Figure 4.1. (a) Limit cycles when $\mu = 2.050, 2.100, 2.150$. (b) Periodic points when $\mu = 2.06653, 2.11757$.

As one can see in Figure 2.3, there exists a $\mu_p$ such that if $\mu > \mu_p$, then attractors disappear and periodic orbits appear. We do not know the exact value of $\mu_p$, though we have that to 3 decimal places $\mu_p = 2.177$. Numerical experiments indicate that there exists a $\mu_c$ ($\mu_p < \mu_c < \mu_2$) such that if $\mu > \mu_c$, then a limit cycles reappears. To 3 decimal places, $\mu_c = 2.200$. 
Numerical experiments indicate also that the portion $2.000 \leq \mu < 2.177$ of Figure 2.3 has structures similar to the portion $2.177 \leq \mu < 2.200$ and periodic orbits appear as shown in the Figure 4.1.(b).

When $\mu > 2.271$, limit cycles vanish. When the value of $\mu$ increases, any orbit $\{F_{\mu}(x_0, y_0)\}$ seems to tend to go eventually outside $\Omega$. This is experimentally checked by using different initial values. The gap in orbit diagram (Figure 2.2) shows this phenomenon. After this stage, depending on initial values, closed limit cycles, or pieces of arcs appear. These phenomena of appearance and disappearances of limit cycles do not seem to come from glitches in programing, or computational margin.

The orbit diagram when $\mu > 2.271$ is shown in the Figure 4.2, 4.3.

Figure 4.2. Blow-up of the portion $2.275 \leq \mu \leq 2.400$ of Figure 2.2.

Notice that Figure 4.3 looks like the mirror image of the portion $\mu > 3$ of the orbit diagram Figure 2.1.

Let us now prove the existence of $\mu_2 > 2$ such that, for any $2 < \mu < \mu_2$, there exists a closed subset $\Omega(\mu)$ of $\Omega$ invariant under $F_{\mu}$.

By a close look at $F_{\mu}$ we know that

$$F_{\mu}(I \times \{0\}) = \{(\mu y, y) : y \in I\}, \quad F_{\mu}(I \times \{1\}) = \{0\} \times I,$$

$$F_{\mu}(\{0\} \times I) = \{(0, 0)\}, \quad \text{and that } F_{\mu}(\{1\} \times I) = \{(x, 1) : 0 \leq x \leq \mu\}.$$

$F_{\mu}$ is one-to-one on the interior of $\Omega$, and that

$$F_{\mu}(\Omega) = \{(x, y) : 0 \leq x \leq \mu y, \ 0 \leq y \leq 1\}.$$  (4.1)
For any \( n \) set
\[
\mathcal{A}_n(\mu) = \{(x, y) \in \Omega : F^k_{\mu}(x, y) \in \Omega \text{ for } 1 \leq k < n, F^n_{\mu}(x, y) \notin \Omega \}.
\]
Then
\[
\Omega(\mu) = \Omega \setminus \left( \bigcup_{k=1}^{\infty} \mathcal{A}_n(\mu) \right).
\]
Notice that if \( \mathcal{A}_n(\mu) = \emptyset \), then \( \mathcal{A}_n \neq \emptyset (k \geq n+1) \) because \( F(\mathcal{A}_{n+1}(\mu)) \subset \mathcal{A}_n(\mu) \). For \((x, y) \in \Omega\), the condition that \( F_{\mu}(x, y) \notin \Omega \) is equivalent to the condition that \( \mu(1-y) > 1 \). From this it follows that
\[
\mathcal{A}_1(\mu) = \left\{ (x, y) \in \Omega : y < 1 - \frac{1}{\mu x} \right\}.
\]
Thus, \( \mathcal{A}_1(\mu) = \emptyset \) and \( \Omega(\mu) = \Omega \) for all \( 0 \leq \mu \leq 1 \).

For \( \mu > 1 \), by formula (4.1)
\[
F_{\mu}(\Omega \setminus \mathcal{A}_1(\mu)) = \{(x, y) \in \Omega : x \leq \mu y \}.
\]
Therefore, \( \mathcal{A}_2(\mu) = \emptyset \) if and only if
\[
1 - \frac{1}{\mu x} \leq \frac{x}{\mu} \quad \text{for } \quad 0 \leq x \leq 1.
\]
This holds for all \( 1 < \mu \leq 2 \). That is,
\[
\Omega(\mu) = \left\{ (x, y) \in \Omega : y \geq 1 - \frac{1}{\mu x} \right\}.
\]
for all $1 < \mu \leq 2$.

Define a curve $C_n$ by $C_n = F^n(I \times \{0\})$. Then

$$C_1 = \{(\mu y, y) : y \in I\}, \quad C_2 = \{(y(\mu - y), y) : 0 \leq y \leq \mu\}.$$  

From this, it follows that

$$F^2_\mu(\Omega \setminus (A_1(\mu) \cup A_2(\mu))) = \{(x, y) \in \Omega : x \leq y(\mu - y)\}.$$  

Therefore $A_3(\mu) = \emptyset$ if and only if $C_2$ is not contained in $A_1(\mu)$. The largest value of $\mu$ such that $A_3(\mu) = \emptyset$ is the value of $\mu$ when the curve $C_2$ is tangential to the curve

$$\left\{(x, y) \in \Omega : y = 1 - \frac{1}{\mu x}\right\} = \left\{(x, y) \in \Omega : x = \frac{1}{\mu(1 - y)}\right\}.$$  

This holds if and only if the algebraic equation in $y$:

$$\mu y(1 - y)(\mu - y) = 1$$  

has a root with multiplicity 2 in $I$. It is an easy exercise in freshman calculus to show that there exists a unique $\mu_2 > 2$ satisfying the condition (4.2). Hence for $2 < \mu \leq \mu_2$, we have

$$\Omega(\mu) = \Omega_2(\mu)$$  

$$= \left\{(x, y) \in \Omega : y \geq 1 - \frac{1}{\mu x} \text{ and } x \geq 1 - \frac{1}{\mu^2(1 - x)}\right\},$$  

and $\Omega(\mu)$ is invariant for any $\mu \leq \mu_2$.

We do not know the exact value of $\mu$, but we can describe it as a root of an algebraic equation and compute its approximate value.

Set $p(y) = \mu y(1 - y)(\mu - y) - 1$, $q(y) = p'(y)/\mu$. Then the equation (4.2) has a root of multiplicity 2 in $I$ if and only if $p(y)$ and $q(y)$ have a common zero. If you let

$$p_1(y) = 2\mu(\mu^2 - \mu + 1)y - (\mu^3 + \mu^2 - 9),$$  

$$q_1(y) = (4\mu^4 - 3\mu^3 - 3\mu^2 + 4\mu + 27)y - 2\mu^2(\mu^2 - \mu + 1),$$

then,

$$9p(y) = (3y - \mu - 1)q(y) - p_1(y),$$

$$2\mu(\mu^2 - \mu + 1)q(y) = 3yp_1(y) - q_1(y).$$

Therefore $p(y)$ and $q(y)$ have a common zero if and only if

$$\begin{vmatrix}
2\mu(\mu^2 - \mu + 1) & \mu^3 + \mu^2 - 9 \\
4\mu^4 - 3\mu^3 - 3\mu^2 + 4\mu + 27 & 2\mu^2(\mu^2 - \mu + 1)
\end{vmatrix}
= 9(\mu^6 - 2\mu^5 - 3\mu^4 + 6\mu^3 + 6\mu^2 - 4\mu - 27) = 0. \quad (4.3)$$
With a help of computer we know that the last equation (4.3) has a real root in \((2, \infty)\) to 5 decimal places \(\mu_2 = 2.24628\).

Repeating the arguments above for every \(n\) one can determine the values of \(\mu\) such that \(A_n(\mu) = \emptyset\). Figure 4.4 show that if \(\mu > \mu'\) then \(\Omega(\mu)\) is "smaller" than \(\Omega(\mu')\).

![Figure 4.4](image)

Figure 4.4. Pictures of \(A_n\)'s for: (a) \(\mu = 2.271\), (b) \(\mu = 2.375323\). The set \(\bigcup_{k=1}^{14} A_k\) is shown in the gray, the set \(\bigcup_{k=15}^{n} A_k\) is shown in the black, and the set \(\Omega(\mu)\) is shown in the white.

Computational experiment indicates that the initial value \((0.46, 0.37)\) used in the section 2 to compute the orbit diagram seems to belong to the set \(\Omega(\mu_0)\). We chose these initial values so that we can compute the orbit \(F_\mu^n(x_0, y_0) \in \Omega\) for sufficiently large \(n\)’s.

In order to have a better understanding of the delayed logistic map, a rigorous mathematical reasoning is required, assisted by numerical analysis, which we plan to continue to work on.

References

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CROFTON FORMULAE BY REFLECTIVE SUBMANIFOLDS IN RIEMANNIAN SYMMETRIC SPACES

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1. Introduction

A plane of any dimension in a Euclidean space is a typical example of reflective submanifolds. As an introduction we first explain about Crofton formulae by planes in a Euclidean space. We denote by $G$ the identity component of the group of isometries of $\mathbb{R}^n$ and take a plane $B$ of dimension $l$ in $\mathbb{R}^n$. We consider the set $\mathcal{R}(B)$ of all planes which are conjugate to $B$ by the action of $G$:

$$\mathcal{R}(B) = \{ gB \mid g \in G \}.$$  

Then $\mathcal{R}(B)$ has a structure of symmetric space and a $G$-invariant measure. For an integer $k$ with $k + l \geq n$ there exists a constant $\sigma$ such that

$$\int_{\mathcal{R}(B)} \text{vol}(N \cap C)d\mu(C) = \sigma \text{vol}(N)$$

holds for any submanifold $N$ of dimension $k$ in $\mathbb{R}^n$. For almost all $C$ in $\mathcal{R}(B)$ the intersection $N \cap C$ is empty or a submanifold of dimension $k + l - n$. The function $C \mapsto \text{vol}(N \cap C)$ is a measurable function on $\mathcal{R}(B)$ with respect to the $G$-invariant measure on $\mathcal{R}(B)$ and we can consider its integration. This integral formula is a classical Crofton formula. We shall show its explicit expression in Corollary 4.2. You can find various versions of Crofton formulae in a famous Santaló's textbook [7] on integral geometry.

The purpose of the present note is to show extended Crofton formulae by reflective submanifolds in Riemannian symmetric spaces. We shall recall the definitions of symmetric spaces and reflective submanifolds in the next section. We show that the set $\mathcal{R}(B)$ of all reflective submanifolds which are conjugate to one reflective submanifold $B$ in a Riemannian symmetric
space $M$ has a structure of symmetric space in Theorem 2.3 (T.[12]). If $M$ is a Riemannian symmetric space of compact type, $\mathcal{R}(B)$ is a Riemannian symmetric space of compact type. On the other hand, if $M$ is a Riemannian symmetric space of noncompact type, $\mathcal{R}(B)$ is a semi-Riemannian symmetric space. In order to formulate Crofton formulae with respect to $\mathcal{R}(B)$ we have to consider integrations on a semi-Riemannian manifold. So we define a canonical measure on a semi-Riemannian manifold. Using these preliminaries we can formulate Crofton formulae by reflective submanifolds in Riemannian symmetric spaces in Theorem 4.1 (T.[12]):

$$\int_{\mathcal{R}(B)} \text{vol}(N \cap C) d\mu(C) = \int_N \sigma_B(T_x N) d\mu(x),$$

where $N$ is a submanifold in $M$. We can formulate more detailed versions of Crofton formulae in complex space forms.

Several years ago the author defined the multiple Kähler angle in order to formulate Poincaré formulae of submanifolds in complex projective spaces in [8]. We can formulate Crofton formulae in complex space forms using the multiple Kähler angles more explicitly than general cases.

2. Reflective submanifolds in Riemannian symmetric spaces

We recall the definitions of symmetric spaces and reflective submanifolds. The following definition of symmetric space is due to Loos [6].

**Definition 2.1.** A differential manifold $M$ is called a *symmetric space* if for each $x$ in $M$ there exists a diffeomorphism $s_x$ which satisfies the following conditions.

1. Each $x$ is an isolated fixed point of $s_x$,
2. $s_x$ is involutive, that is, $s_x^2 = 1$,
3. $s_x(s_y(z)) = s_{s_x(y)}(s_x(z))$.

If $M$ is a (semi-)Riemannian manifold and if each $s_x$ is isometric, we call $M$ a (semi-)Riemannian symmetric space.

The following definition of reflective submanifold is due to Leung [4].

**Definition 2.2.** Let $M$ be a complete Riemannian manifold. A connected component of the fixed point set of an involutive isometry of $M$ is called a *reflective submanifold*.

**Theorem 2.3 (T.[12]).** Let $M$ be a Riemannian symmetric space and $B$ be a reflective submanifold of $M$. We denote by $G$ the identity component
of the group of all isometries on $M$ and by $\mathcal{R}(B)$ the set of all reflective submanifolds in $M$ which are conjugate to $B$ by the action of $G$:

$$\mathcal{R}(B) = \{ gB \mid g \in G \}.$$ 

Then $\mathcal{R}(B)$ has a structure of symmetric space. If $M$ is a Riemannian symmetric space of compact type, $\mathcal{R}(B)$ is a Riemannian symmetric space of compact type. If $M$ is a Riemannian symmetric space of noncompact type, $\mathcal{R}(B)$ is a semi-Riemannian symmetric space of semisimple type.

3. Integration on semi-Riemannian manifolds

In the case where $M$ is a Riemannian symmetric space of noncompact type, for a reflective submanifold $B$ in $M$ the manifold $\mathcal{R}(B)$ is a semi-Riemannian symmetric space of semisimple type as is shown in the previous section. We would like to consider integration on $\mathcal{R}(B)$, so we prepare for measures and integrations on semi-Riemannian manifolds in this section.

Let $E$ be a real vector space of finite dimension with inner product $\langle \ , \rangle$. We do not assume that $\langle \ , \rangle$ is positive definite. Then we can construct an inner product on $\wedge^p E$. From now on we always consider the inner product on the exterior algebra $\wedge^p E$ of a real vector space $E$ of finite dimension with inner product $\langle \ , \rangle$ mentioned above and also denote it by the same symbol $\langle \ , \rangle$. We denote the length of $u$ by $|u| = |\langle u, u \rangle|^{1/2}$.

**Definition 3.1.** Let $(M, g)$ be a semi-Riemannian manifold. We denote by $\mathcal{K}(M)$ the set of all real valued continuous functions on $M$ with compact support. We take a local coordinate system $(U; x_1, \ldots, x_n)$ of $M$. For each $x \in U$ we consider the inner product on $\wedge^n T_x(M)$ induced from the semi-Riemannian metric $g$. For $f \in \mathcal{K}(M)$ with $\text{supp } f \subset U$ we define $L(f)$ by

$$L(f) = \int_U f(x_1, \ldots, x_n) \left| \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} \right| dx_1 \cdots dx_n.$$ 

Here the integral on the right hand side is the Lebesgue integral on the Euclidean space. The integrand is a continuous function with compact support, so the integral is equal to the usual Riemann integral. We can see that $L(f)$ is independent of the choice of the local coordinate system by the formula of integral of variable change. In the case where an element whose support is not included in a single coordinate neighborhood, we can define the value of $L : \mathcal{K}(M) \to \mathbb{R}$ by the use of a partition of unity. This is also
independent of the choice of the partition of unity. By Riesz representation theorem there exists a Radon measure $\mu_M$ on $M$ which satisfies

$$L(f) = \int_M f \, d\mu_M \quad (f \in \mathcal{C}(M)).$$

We call this measure $\mu_M$ the canonical measure. We will consider only the canonical measures on semi-Riemannian manifolds. We denote $\text{vol}(M) = \mu_M(M)$ and call $\text{vol}(M)$ the volume of $M$.

4. Crofton formulae by reflective submanifolds

We use the notation in Section 2 and consider a submanifold $N$ of $M$ which satisfies $\dim N + \dim B \geq \dim M$ and integral formulae of the following type:

$$\int_{\mathcal{R}(B)} \text{vol}(N \cap C) \, d\mu(C) = \text{geometric amount of } N,$$

which is called Crofton formula. In order to formulate this we define

$$\mathcal{R}_0(B) = \{ kT_oB \mid k \in K \},$$

which coincides with the set of tangent spaces $T_oC$ of all reflective submanifolds $C$ belonging to $\mathcal{R}(B)$ through $o$. We denote

$$S(B) = \{ g \in G \mid g(B) = B \}.$$

Since $\mathcal{R}_0(B) \cong K/(K \cap S(B))$, we can consider a $K$-invariant Riemannian metric on $\mathcal{R}_0(B)$ induced from the biinvariant Riemannian metric on $K$. If a vector subspace $V \subset T_oM$ satisfies $\dim V + \dim B \geq \dim M$, we define $\sigma_B(V)$ by

$$\sigma_B(V) = \int_{\mathcal{R}_0(B)} |\tilde{V} \wedge \tilde{c}| \, d\mu(c).$$

Here $\tilde{V}$ is the wedge product of an orthonormal basis of $V$ and $\tilde{c}$ is similar. We consider only the norm of their wedge product, so there is no ambiguity. For a general vector subspace $V \subset T_xM$ of the same condition of its dimension, we take $g \in G$ which satisfies $go = x$ and define $\sigma_B(V) = \sigma_B(dg^{-1}V)$. Since $K$ acts isometrically on $\mathcal{R}_0(B)$, the definition of $\sigma_B(V)$ is independent of the choice of $g \in G$. Using $\sigma_B$ we can formulate Crofton formulae by reflective submanifolds as follows.

**Theorem 4.1 (T.[12]).** Let $M$ be a Riemannian symmetric space of compact type or noncompact type and $B$ be a reflective submanifold of $M$. For a
submanifold $N$ of $M$ which satisfies $\dim N + \dim B \geq \dim M$, the following equation holds.

$$\int_{\mathcal{R}(B)} \text{vol}(N \cap C)d\mu(C) = \int_{N} \sigma_B(T_Z N)d\mu(x).$$

Howard [1] touched on Crofton formulae of submanifolds in Riemannian homogeneous spaces. Although the class of submanifolds we treat is restricted to reflective submanifolds, the invariant measure on the class of submanifolds and the integrand of the right side are explicitly given later in Theorem 4.1 (T.[12]).

In the case where $M = G/K$ is a real space form any totally geodesic submanifold $B$ in $M$ is a reflective submanifold and $\sigma_B$ is constant, because the linear isotropy action of $K$ on the Grassmann manifold consisting of real vector subspaces of dimension $\dim B$ in $T_0 M$ is transitive. We obtain the following corollary.

**Corollary 4.2.** Let $B$ be a totally geodesic submanifold of dimension $l$ in a real space form $M = G/SO(n)$ of dimension $n$. For a submanifold $N$ of dimension $k$ in $M$ such that $k + l \geq n$, the following equation holds.

$$\int_{\mathcal{R}(B)} \text{vol}(N \cap C)d\mu(C) = \frac{\text{vol}(S^{k+l-n})}{\text{vol}(S^l)} \text{vol}(\mathcal{R}(S^l))\text{vol}(N).$$

Leung [5] gave a classification of all the reflective submanifolds in simply connected Riemannian symmetric spaces. Among them We can formulate Crofton formulae not only in the real space forms but also in the complex space forms more explicitly than Theorem 4.1 (T.[12]). We show this in the next section.

5. Complex space forms

In order to formulate Crofton formulae of submanifolds in complex space forms we use the notion of multiple Kähler angle which the author introduced in [8]. We denote by $\omega$ the standard Kähler form of $\mathbb{C}^n$.

**Definition 5.1 (T.[8]).** Let $1 < k \leq n$. For a real vector subspace $V$ of dimension $k$ in $\mathbb{C}^n$ we consider a canonical form of the restriction $\omega|_V$, that is, we take an orthonormal basis $\alpha^1, \ldots, \alpha^k$ of the dual space of $V$ which satisfies

$$\omega|_V = \sum_{i=1}^{[k/2]} \cos \theta_i \alpha^{2i-1} \wedge \alpha^{2i}, \quad 0 \leq \theta_1 \leq \cdots \leq \theta_{[k/2]} \leq \pi/2.$$
Then we put \( \theta(V) = (\theta_1, \ldots, \theta_{[k/2]}) \) and call it the *multiple Kähler angle* of \( V \). In the case where \( n < k \leq 2n - 1 \), for a real vector subspace \( V \) of dimension \( k \) in \( \mathbb{C}^n \) we define the multiple Kähler angle of \( V \) by \( \theta(V) = \theta(V^\perp) \).

The definition of the multiple Kähler angle depends only on the Hermitian structure of \( \mathbb{C}^n \), so we can consider the multiple Kähler angle of any real submanifold in an almost Hermitian manifold. Using the multiple Kähler angle the author formulated Poincaré formulae of any real submanifolds in the complex space forms in [8]. In the present paper we formulate Crofton formulae of any real submanifolds in the complex space forms.

Before we treat general cases, we review some classical Crofton formulae in complex space forms. We recall a result of Leung on reflective submanifolds in complex space forms which is stated in Theorem 7 in [5].

**Theorem 5.2 (Leung).** The reflective submanifolds of the complex projective space \( \mathbb{C}P^n \) are \( \mathbb{C}P^k \) (\( 1 \leq k < n \)) and the real projective space \( \mathbb{R}P^n \) which is naturally embedded in \( \mathbb{C}P^n \). The reflective submanifolds of the complex hyperbolic space \( \mathbb{C}H^n \) are the geodesic submanifolds which correspond to the reflective submanifolds of \( \mathbb{C}P^n \) under duality.

We can obtain the following corollaries.

**Corollary 5.3.** Let \( B \) be a totally geodesic complex submanifold of complex dimension \( l \) in a complex space form \( M = G/S(U(n) \times U(1)) \) of complex dimension \( n \). For a complex submanifold \( N \) of complex dimension \( k \) in \( M \) such that \( k + l \geq n \), the following equation holds.

\[
\int_{\mathcal{R}(B)} \text{vol}(N \cap C) d\mu(C) = \frac{\text{vol}(\mathbb{C}P^{k+l-n})}{\text{vol}(\mathbb{C}P^k)} \frac{\text{vol}(\mathcal{R}(\mathbb{C}P^l))}{\text{vol}(\mathbb{C}P^k)} \text{vol}(N).
\]

**Corollary 5.4.** Let \( B \) be a totally geodesic Lagrangian submanifold in a complex space form \( M = G/S(U(n) \times U(1)) \) of complex dimension \( n \). For a Lagrangian submanifold \( N \) in \( M \), the following equation holds.

\[
\int_{\mathcal{R}(B)} \#(N \cap C) d\mu(C) = \frac{n + 1}{\text{vol}(\mathbb{R}P^n)} \text{vol}(\mathcal{R}(\mathbb{R}P^n)) \text{vol}(N),
\]

where \( \#X \) denote the number of the points in \( X \).

Now we consider the case where \( \sigma_B \) is not constant. Poincaré formulae of real submanifolds in complex space forms stated in Theorem 8 in [8] implies the following corollary.
Corollary 5.5. For any positive integers $k$, $l$ and $n$ which satisfy $k, 2l < 2n \leq k + 2l$, there exists a function $\sigma_{k,l}^n(\theta^{(k)})$ of variable $\theta^{(k)} \in \mathbb{R}[\min\{k, 2n-k\}/2]$ such that the following Crofton formula holds. Let $H^l$ be a totally geodesic complex submanifold of complex dimension $l$ in a complex space form $M = G/S(U(n) \times U(1))$ of complex dimension $n$. For a real submanifold $N$ of dimension $k$ in $M$, the following equation holds.

$$\int_{\mathcal{R}(H^l)} \text{vol}(N \cap C)d\mu(C) = \int_N \sigma_{k,l}^n(\theta(T_x N))d\mu(x),$$

where $\theta(T_x N)$ is the multiple Kähler angle of $T_x N$.

For any positive integers $k$ and $n$ which satisfy $k < 2n \leq k + n$, there exists a function $\tau_{k}^n(\theta^{(k)})$ of variable $\theta^{(k)} \in \mathbb{R}[\min\{k, 2n-k\}/2]$ such that the following Crofton formula holds. Let $L$ be a totally geodesic Lagrangian submanifold in a complex space form $M$ of complex dimension $n$. For a real submanifold $N$ of dimension $k$ in $M$, the following equation holds.

$$\int_{\mathcal{R}(L)} \text{vol}(N \cap C)d\mu(C) = \int_N \tau_{k}^n(\theta(T_x N))d\mu(x).$$

In some cases we can express $\sigma_{k,l}^n$ and $\tau_{k}^n$ mentioned above more explicitly.

Corollary 5.6. Let $B$ be a totally geodesic complex submanifold of complex dimension $n-1$ in a complex space form $M = G/S(U(n) \times U(1))$ of complex dimension $n$. For a real submanifold $N$ of dimension 2 in $M$, the following equation holds.

$$\int_{\mathcal{R}(B)} \#(N \cap C)d\mu(C) = \frac{\text{vol}(\mathcal{R}(\mathbb{C}P^{n-1}))}{2\text{vol}(\mathbb{C}P^1)} \int_N (1 + \cos^2 \theta_x)d\mu(x),$$

where $\theta_x$ is the Kähler angle of $N$ at $x$.

Corollary 5.7. Let $B$ be a totally geodesic complex submanifold of complex dimension 1 in a complex space form $M = G/S(U(n) \times U(1))$ of complex dimension $n$. For a real submanifold $N$ of dimension $2n - 2$ in $M$, the following equation holds.

$$\int_{\mathcal{R}(B)} \#(N \cap C)d\mu(C) = \frac{\text{vol}(\mathcal{R}(\mathbb{C}P^1))}{2\text{vol}(\mathbb{C}P^{n-1})} \int_N (1 + \cos^2 \theta_x)d\mu(x),$$

where $\theta_x$ is the Kähler angle of $N$ at $x$. 
Corollary 5.8. Let $B$ be a totally geodesic Lagrangian submanifold in a complex space form $M = G/S(U(2) \times U(1))$ of complex dimension 2. For a real submanifold $N$ of dimension 2 in $M$, the following equation holds.

$$\int_{\mathcal{R}(B)} (N \cap C) d\mu(C) = \frac{\text{vol}(\mathcal{R}(\mathbb{R}P^2))}{\text{vol}(\mathbb{R}P^2)} \int_N (3 - \cos^2 \theta_x) d\mu(x).$$

Corollary 5.9. Let $B$ be a totally geodesic Lagrangian submanifold in a complex space form $M = G/S(U(3) \times U(1))$ of complex dimension 3. For a real submanifold $N$ of dimension 3 in $M$, the following equation holds.

$$\int_{\mathcal{R}(B)} (N \cap C) d\mu(C) = \frac{4\text{vol}(\mathcal{R}(\mathbb{R}P^3))}{3\text{vol}(\mathbb{R}P^3)} \int_N (3 - \cos^2 \theta_x) d\mu(x).$$

Corollary 5.10. Let $B$ be a totally geodesic complex submanifold of complex dimension 2 in a complex space form $M = G/S(U(4) \times U(1))$ of complex dimension 4. For a real submanifold $N$ of dimension 4 in $M$, the following equation holds.

$$\int_{\mathcal{R}(B)} (N \cap C) d\mu(C) = \frac{\text{vol}(\mathcal{R}(\mathbb{C}P^2))}{8\text{vol}(\mathbb{C}P^2)} \int_N (3 \cos^2 \theta_1 + \cos^2 \theta_2 + 3 \cos^2 \theta_1 \cos^2 \theta_2) d\mu,$$

where $(\theta_1, \theta_2)$ is the multiple Kähler angle of $N$.

6. Hypersurfaces

In this section we consider Crofton formulae of hypersurfaces in Riemannian symmetric spaces of rank one.

Let $G/K$ be a Riemannian symmetric space of rank one and $n = \dim(G/K)$. By a result of Howard [1], for a submanifold $M$ of dimension $p$ and hypersurface $N$ in $G/K$, the following equation holds.

$$\int_G \text{vol}(M \cap gN) d\mu(g) = C \text{vol}(M) \text{vol}(N),$$

where

$$C = \frac{\text{vol}(K) \text{vol}(S^{p-1}) \text{vol}(S^n)}{\text{vol}(S^p) \text{vol}(S^{n-1})}.$$ 

On the other hand

$$C = \int_K |\tilde{T}^\perp M \wedge k^{-1} \tilde{T}^\perp N| d\mu(k) = \int_K |k \tilde{T}^\perp M \wedge \tilde{T}^\perp N| d\mu(k).$$
If $M$ is equal to a reflective submanifold $B$, we have $\mathcal{R}_0(B) \cong K/K \cap S(B)$ and

$$C = \text{vol}(K \cap S(B)) \int_{\mathcal{R}_0(B)} |\tilde{T}_\perp \wedge T_\perp N|d\mu(c) = \text{vol}(K \cap S(B))\sigma_B(TN).$$

Therefore

$$\sigma_B(TN) = \frac{C}{\text{vol}(K \cap S(B))} = \frac{\text{vol}(K)\text{vol}(S^{p-1})\text{vol}(S^n)}{\text{vol}(K \cap S(B))\text{vol}(S^p)\text{vol}(S^{n-1})} = \frac{\text{vol}(\mathcal{R}_0(B))\text{vol}(S^{p-1})\text{vol}(S^n)}{\text{vol}(S^p)\text{vol}(S^{n-1})}.$$

Thus we obtain the following corollary from Theorem 4.1 (T. [12]).

**Corollary 6.1.** Let $B$ be a reflective submanifold of dimension $p$ in a Riemannian symmetric space $G/K$ of dimension $n$. For a hypersurface $N$ in $G/K$, the following equation holds.

$$\int_{\mathcal{R}(B)} \text{vol}(N \cap C)d\mu(C) = \frac{\text{vol}(\mathcal{R}_0(B))\text{vol}(S^{p-1})\text{vol}(S^n)}{\text{vol}(S^p)\text{vol}(S^{n-1})} \text{vol}(N).$$

**References**


The class of the complex manifolds with Norden metric is considered. The Yano connection is introduced. The properties of the curvature tensor and the Bochner tensor of the Yano connection are studied.

1. Introduction

Let \((M, J)\) be an almost complex manifold with an almost complex structure \(J\). It is well known that on such manifolds there exists Hermitian metric \(g\) with the property \(g(JX, JY) = g(X, Y)\) for any \(X, Y \in \mathfrak{X}(M)\). In this case the manifold \((M, J, g)\) is called an almost Hermitian manifold. On the other hand, according to A.P. Norden [11], on an almost complex manifold there exists also an indefinite metric \(g\) such that \(g(JX, JY) = -g(X, Y)\), \(X, Y \in \mathfrak{X}(M)\). Almost complex manifolds with such a metric are originally introduced in [9] under the name generalized \(B\)-manifolds and the metric \(g\) is called \(B\)-metric. Later, a classification of almost complex manifolds with \(B\)-metric is given in [5] and equivalent characteristic conditions for each of these classes are obtained in [6]. From another point of view, these manifolds are studied in [8] where they are called almost complex Riemannian manifolds. Further, in this paper such manifolds are called almost complex manifolds with Norden metric.

Furthermore, these manifolds are considered by many authors, for example [1,2,3,4]. Let us note that examples of three of the main classes of almost complex manifolds with Norden metric are given in [3].

An important problem in the geometry of almost complex manifolds with Norden metric is the existing of linear connections with respect to which the almost complex structure is parallel. In [6] there is introduced a \(B\)-connection with non-zero torsion tensor field on a complex manifold.
with Norden metric \((M, J, g)\) with respect to which \(g\) and \(J\) are parallel. In
the same paper there is proved that the Bochner curvature tensor of the \(B\)-
connection is an invariant with respect to special conformal transformations
of the metric \(g\).

In this paper we introduce and study the Yano connection on an almost
complex manifold with Norden metric.

2. Preliminaries

Let \((M, J)\) be a \(2n\)-dimensional almost complex manifold, \(J^2 = -id\). A
metric \(g\) on \(M\) is called Norden if the almost complex structure \(J\) is an
antiisometry of the tangent space at any point of \(M\), i.e.

\[
g(JX, JY) = -g(X, Y), \quad X, Y \in \mathfrak{X}(M).
\]

Then, the manifold \((M, J, g)\) is called an almost complex manifold with
Norden metric.

The associated metric \(\tilde{g}\) of the manifold is defined by

\[
\tilde{g}(X, Y) = g(JX, Y) = g(X, JY).
\]

Obviously, the metric \(\tilde{g}\) is also a Norden metric. Both metrics are neces-
sarily of signature \((n, n)\).

Let \(\nabla\) be the Levi-Civita connection of the metric \(g\). The tensor field
\(F\) of type \((0, 3)\) on the manifold is defined by

\[
F(X, Y, Z) = g(\langle \nabla_X J \rangle Y, Z).
\]  

(1)

This tensor has the following symmetries:

\[
F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ).
\]  

(2)

The associated Lee 1-forms \(\theta\) and \(\tilde{\theta}\) on \(M\) are given by \(\theta(x) = g^{ij}F(e_i, e_j, x)\), \(\tilde{\theta} = \theta \circ J\), where \(x\) is a tangent vector at an arbitrary point
\(p \in M\), \(\{e_i\}_{i=1,...,2n}\) is a basis of the tangent space \(T_p M\) and \(g^{ij}\) is the
inverse of the matrix associated to \(g\).

The Nijenhuis tensor field \(N\) of the manifold is defined as follows

\[
N(X, Y) = (\nabla_X J) JY - (\nabla_Y J) JX + (\nabla_JX J) Y - (\nabla_JY J) X.
\]  

(3)

In [5] the eight classes of almost complex manifolds with Norden metric
are characterized by conditions for \(F\) as follows

1. The class \(W_0\) of the Kähler manifolds with Norden metric:

\[
F = 0 \iff \nabla J = 0
\]
2. The class $W_1$:

$$F(X, Y, Z) = \frac{1}{2n} [g(X, Y)\theta(Z) + g(X, Z)\theta(Y) + g(X, JY)\theta(JZ) + g(X, JZ)\theta(JY)]$$

3. The class $W_2$ of the special complex manifolds with Norden metric:

$$F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0, \quad \theta = 0 \iff N = 0, \quad \theta = 0$$

4. The class $W_3$ of quasi-Kähler manifolds with Norden metric:

$$F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0$$

5. The class $W_1 \oplus W_2$ of the complex manifolds with Norden metric:

$$F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0 \iff N = 0$$

6. The class $W_2 \oplus W_3$ of semi-Kähler manifolds with Norden metric:

$$\theta = 0$$

7. The class $W_1 \oplus W_3$:

$$F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) =$$

$$\frac{1}{n} [g(X, Y)\theta(Z) + g(X, Z)\theta(Y) + g(Y, Z)\theta(X) + g(X, JY)\theta(JZ) + g(X, JZ)\theta(JY) + g(Y, JZ)\theta(JX)]$$

8. The class of almost complex manifolds with Norden metric: no conditions.

In the paper [6] there are considered two types of conformal transformations of the Norden metric $g$ on an almost complex manifold $(M, J, g)$:

1. Conformal transformations of type I

$$\bar{g} = e^{2u}g,$$

where $u$ is a pluriharmonic function on $M$.

2. Conformal transformations of type II

$$\bar{g} = e^{2u}(\cos 2vg + \sin 2\tilde{v}g),$$

where $u + iv$ is a holomorphic function on $M$, i.e. $dv = -du \circ J$.

It is proved that the subclass $W_1^0$ of $W_1$ with closed Lee 1-forms $\theta$ and $\tilde{\theta}$ is conformally equivalent to a Kähler manifold with Norden metric by a transformation of type I. The manifolds belonging to $W_1^0$ are called
conformal Kähler manifolds with Norden metric. The class $W^0_1$ is closed with respect to the conformal transformations of type I and type II.

Let us note that examples of some of the classes are given as follows: in [1] for $W_2$ and $W_2 \oplus W_3$; in [3] for $W_0$, $W_1$ and $W_2$; in [4] for $W_0$ and $W^0_1$; in [6] for $W^0_1$; in [7] for $W_0$.

A tensor $L$ of type $(0,4)$ is called a curvature-like tensor if it satisfies the following conditions for any $X, Y, Z, W \in \mathfrak{X}(M)$:

$$L(X, Y, Z, W) = -L(Y, X, Z, W);$$
$$L(X, Y, Z, W) + L(Y, Z, X, W) + L(Z, X, Y, W) = 0;$$
$$L(X, Y, Z, W) = -L(X, Y, W, Z).$$

A curvature-like tensor $L$ is called a Kähler tensor if it satisfies the condition

$$L(X, Y, JZ, JW) = -L(X, Y, Z, W), \quad X, Y, Z, W \in \mathfrak{X}(M).$$

Then, the associated tensor $\tilde{L}$ defined by $\tilde{L}(X, Y, Z, W) = L(X, Y, Z, JW)$ is also a Kähler tensor.

Let us consider the following tensors of type $(0,4)$, where $S$ is a tensor of type $(0,2)$:

$$\psi_1(S)(X, Y, Z, W) = g(Y, Z)S(X, W) - g(X, Z)S(Y, W)$$
$$+ g(X, W)S(Y, Z) - g(Y, W)S(X, Z);$$
$$\psi_2(S)(X, Y, Z, W) = g(Y, JZ)S(X, JW) - g(X, JZ)S(Y, JW)$$
$$+ g(X, JW)S(Y, JZ) - g(Y, JW)S(X, JZ);$$
$$\pi_1(X, Y, Z, W) = \frac{1}{2} \psi_1(g)(X, Y, Z, W)$$
$$= g(Y, Z)g(X, W) - g(X, Z)g(Y, W);$$
$$\pi_2(X, Y, Z, W) = \frac{1}{2} \psi_2(g)(X, Y, Z, W)$$
$$= g(Y, JZ)g(X, JW) - g(X, JZ)g(Y, JW);$$
$$\pi_3(X, Y, Z, W) = -\psi_1(\bar{g})(X, Y, Z, W) = \psi_2(\bar{g})(X, Y, Z, W)$$
$$= -g(Y, Z)g(X, JW) + g(X, Z)g(Y, JW)$$
$$- g(X, W)g(Y, JZ) + g(Y, W)g(X, JZ).$$

It is known [6] that the tensor $\psi_1(S)$ is a curvature-like tensor iff $S$ is symmetric and the tensor $\psi_2(S)$ is a curvature-like tensor iff $S$ is symmetric and hybrid with respect to $J$, i.e. $S(JX, Y) = S(JY, X)$. In this case the tensors $\pi_1 - \pi_2, \pi_3$ and $\psi_1(S) - \psi_2(S)$ are Kähler tensors.
Let $L$ be a Kähler tensor over $T_pM$, $p \in M$ and $\{e_i\}_{i=1,\ldots,2n}$ be a basis of $T_pM$. Then the Ricci tensor $\rho(L)$ and the scalar curvatures $\tau(L)$ and $\tilde{\tau}(L)$ are given by

$$\rho(L)(Y,Z) = g^{is}L(e_i,Y,Z,e_s);$$
$$\tau(L) = g^{ik}\rho(L)(e_j,e_k);$$
$$\tilde{\tau}(L) = \tilde{\tau}(L) = g^{ik}\rho(L)(e_j,Je_k).$$

The associated Bochner curvature tensor $B(L)$ is defined by

$$B(L) = L - \frac{1}{2(n-2)} \left\{ \psi_1(\rho) - \psi_2(\rho) \right\} + \frac{1}{4(n-1)(n-2)} \left\{ \tau(\pi_1 - \pi_2) + \tilde{\tau}\pi_3 \right\}, \quad n \geq 3.$$

In [6] there is introduced the $B$-connection $D$ on $(M,J,g) \in W_1$. It is proved that if $K$ is the Kähler curvature tensor for $D$ then the Bochner tensor $B(K)$ is a conformal invariant of type I and type II.

3. Curvature properties of $W_1$-manifolds

Let $(M,J,g)$ be a $W_1$-manifold. Then, having in mind (6), the Nijenhuis tensor vanishes on $M$. The Lee 1-forms $\theta$ and $\tilde{\theta}$ are said to be closed iff $d\theta = d\tilde{\theta} = 0$ or the following equivalent conditions hold:

$$(\nabla_X \theta)Y = (\nabla_Y \theta)X, \quad (\nabla_X \tilde{\theta})Y = (\nabla_Y \tilde{\theta})X. \quad (10)$$

Taking into account (1), (4) and (10) we obtain the following

**Lemma 1.** If $(M,J,g) \in W_1^0$ then the following conditions are valid:

$$(\nabla_X \theta)Y = (\nabla_Y \theta)X, \quad (\nabla_X \theta)Y = (\nabla_Y \theta)JX,$$

$$(\nabla_X \theta)Y = \frac{1}{2n} \left[ g(X,Y)\Omega + g(X,JY)J\Omega + \theta(Y)X + \theta(JY)JX \right], \quad (11)$$

where $\Omega$ is the Lee vector corresponding to $\theta$, i.e. $g(X,\Omega) = \theta(X)$.

Let $R$ be the curvature tensor of $\nabla$, i.e. $R(X,Y)Z = \nabla_X\nabla_YZ - \nabla_Y\nabla_XZ - \nabla_{[X,Y]}Z$. The corresponding tensor of type $(0,4)$ is denoted by the same letter and is given by $R(X,Y,Z,W) = g(R(X,Y)Z,W)$. Then Lemma 1, (1) and (4) imply the following conditions

$$(\nabla_X F)(Y,Z,W) = g((\nabla_X K)(Y,Z),W), \quad (12)$$

$$(\nabla_X F)(Y,Z,W) = g((\nabla_X K)(Y,Z),W), \quad (13)$$

where \( K(X, Y) = \frac{1}{2n} [g(X, Y)\Omega + g(X, JY)J\Omega + \theta(Y)X + \theta(JY)JX] \).

Now let us consider the following tensors of type \((0,2)\):

\[
S(X, Y) = (\nabla_X \theta) JY + \frac{1}{4n} [\theta(X)\theta(Y) - \theta(JX)\theta(JY)],
\]
\[
M(X, Y) = \theta(X)\theta(Y) + \theta(JX)\theta(JY).
\] (14)

They have the following symmetries

\[
S(JX, JY) = -S(X, Y), \quad M(JX, JY) = M(X, Y).
\]

**Theorem 2.** Let \((M, J, g)\) be a conformal Kähler manifold with Norden metric. Then the curvature tensor \(R\) of \(\nabla\) has the following property

\[
R(X, Y, JZ, JW) = -R(X, Y, Z, W)
\]

\[
+ \frac{1}{2n} \left\{ [\psi_1 + \psi_2](S) + \frac{1}{4n} [\psi_1 + \psi_2](M) + \frac{1}{2n} \theta(\Omega) [\pi_1 + \pi_2] \right\}(X, Y, Z, W).
\] (15)

**Proof.** Having in mind (12), the condition (13) implies

\[
\] (16)

Then, taking into account Lemma 1, (4), (7), (14) from (16) we receive (15). \(\Box\)

Next, we define the tensor field \(R^*\) of type \((0,4)\) by

\[
R^* = R - \frac{1}{2n} \psi_1(L),
\] (17)

where

\[
L = S + \frac{1}{4n} M + \frac{\theta(\Omega)}{4n} g.
\] (18)

Since the tensor \(L\) is symmetric then \(R^*\) is a curvature-like tensor on any \(W^0\)-manifold. Moreover, taking into account Theorem 1, (7), (17) and (18) we obtain \(R^*(X, Y, JZ, JW) = -R^*(X, Y, Z, W)\), i.e. \(R^*\) is a Kähler tensor.

Then, according to (7), (8) and (17) we get the following interconnections between the corresponding Ricci tensors and the scalar curvatures of \(R\) and \(R^*\):

\[
\rho^* = \rho - \frac{1}{2n} [g \text{tr} L + 2(n-1)L];
\]
\[
\tau^* = \tau - \frac{2n-1}{n} \text{tr} L, \quad \text{tr} L = \frac{n}{2n-1} (\tau - \tau^*).
\]
Hence we obtain
\[
L(Y, Z) = \frac{n}{n-1} \left\{ \rho(Y, Z) - \rho^*(Y, Z) - \frac{\tau - \tau^*}{2(2n-1)} g(Y, Z) \right\}.
\]
The last equality and (17) imply
\[
R^* - \frac{1}{2(n-1)} \left\{ \psi_1(\rho^*) - \frac{\tau^*}{2n-1} \pi_1 \right\} = R - \frac{1}{2(n-1)} \left\{ \psi_1(\rho) - \frac{\tau}{2n-1} \pi_1 \right\}.
\]
(19)
The Weyl tensor \(W(R)\) of \(R\) is defined as follows
\[
W(R) = R - \frac{1}{n-2} \left\{ \psi_1(\rho) - \frac{\tau}{n-1} \pi_1 \right\}.
\]
(20)
It is well known that the Weyl tensor of type \((0,4)\) of \(R\) is an invariant of the conformal transformation of type I, i.e. \(W(R) = e^{2\Psi}W(R)\).
Then, using (19) and (20) we obtain the following

**Theorem 3.** Let \((M, J, g)\) be a conformal Kähler manifold with Norden metric. Then the Weyl tensors of \(R\) and \(R^*\) coincide, i.e. \(W(R) = W(R^*)\).

4. The Yano connection on almost complex manifolds with Norden metric

Let \((M, J, g)\) be an almost complex manifold with Norden metric. Following [12] and [13] we consider the Yano connection defined by
\[
\nabla'_X Y = \nabla_X Y + T(X, Y),
\]
(21)
where
\[
T(X, Y) = \frac{1}{4} \left[ (\nabla_X J) JY + 2 (\nabla_Y J) JX - (\nabla_{JX} J) Y \right].
\]
(22)
The torsion tensor field \(Q\) of \(\nabla'\) is given by
\[
Q(X, Y) = \nabla'_X Y - \nabla'_Y X - [X, Y] = T(X, Y) - T(Y, X).
\]
(23)
Taking into account (3), (22) and (23) we receive the following

**Lemma 4.** Let \((M, J, g)\) be an almost complex manifold with Norden metric. Then the Yano connection is symmetric iff the Nijenhuis tensor field vanishes on \(M\).

Let us note that the Yano connection is symmetric on the classes \(W_1, W_2\) and \(W_1 \oplus W_2\) according to Lemma 4 and the conditions (4), (5) and (6).

**Theorem 5.** Let \((M, J, g)\) be an almost complex manifold with Norden metric and \(\nabla'\) be the Yano connection on \(M\). Then \(\nabla' J = 0\) iff \(N = 0\).
Proof. The well known equality \( (\nabla'_X J) Y = \nabla'_X JY - J\nabla'_X Y \) and (3), (21), (22) imply
\[
(\nabla'_X J) Y = -\frac{1}{2} N(X, JY).
\]
Thus, the vanishing of \( \nabla'_J \) is equivalent to the vanishing of \( N \).

Next, we consider the Yano connection on \( W_1 \)-manifolds. The conditions (11) and (22) imply
\[
T(X, Y) = \frac{1}{4n} \left[ g(X, JY)J\Omega - g(X, Y)J\Omega + \theta(JX)Y - \theta(X)JY + \theta(JY)X - \theta(Y)JX \right].
\]

**Theorem 6.** Let \( (M, J, g) \) be a \( W_1 \)-manifold. Then the covariant derivatives of \( g \) and \( \tilde{g} \) with respect to the Yano connection satisfy the following conditions:
\[
(\nabla'_X g)(Y, Z) = \frac{1}{2n} \left[ \theta(X)g(Y, JZ) - \theta(JX)g(Y, Z) \right];
\]
\[
(\nabla'_X \tilde{g})(Y, Z) = -\frac{1}{2n} \left[ \theta(X)g(Y, Z) + \theta(JX)g(Y, JZ) \right].
\]

Proof. From (1), (2), (6), (21) and (22) we obtain:
\[
(\nabla'_X g)(Y, Z) = \frac{1}{2} \left[ 2F(Y, X, JZ) + F(JX, Y, Z) - F(X, Y, JZ) \right];
\]
\[
(\nabla'_X \tilde{g})(Y, Z) = \frac{1}{2} \left[ F(JZ, X, JY) - F(Z, X, Y) \right].
\]
Then, taking into account (4), the equalities (27) and (28) imply (25) and (26), respectively.

Let \( R' \) be the curvature tensor of \( \nabla' \) of type \((1, 3)\). Then, according to (21) we have:
\[
+ T(X, T(Y, Z)) - T(Y, T(X, Z));
\]
\[
+ T(X, T(Y, Z, W)) - T(Y, T(X, Z, W)),
\]
where \( R'(X, Y, Z, W) = g(R'(X, Y)Z, W) \) and
\[
T(X, Y, Z) = \frac{1}{4n} \left[ g(X, JY)\theta(Z) - g(X, Y)\theta(JZ) + g(X, JZ)\theta(Y)
- g(X, JZ)\theta(Y) + g(Y, JZ)\theta(X) - g(Y, JZ)\theta(X) \right].
\]
Theorem 7. Let \((M, J, g)\) be a conformal Kähler manifold with Norden metric. Then the curvature tensors \(R\) and \(R'\) are related as follows

\[
R' = R - \frac{1}{4n} \left\{ [\psi_1 + \psi_2] (S) + \frac{1}{2n} \psi_1 (M) + \frac{1}{4n} \theta (\Omega) [3\pi_1 + \pi_2] - \frac{1}{4n} \theta (J\Omega) \pi_3 \right\}.
\]  

(31)

Proof. By the use of Lemma 1, (4), (7), (14), (30) and after straightforward calculations in the right side of (29) we receive (31).

The last theorem, (7) and (15) imply

Corollary 8. Let \((M, J, g)\) be a conformal Kähler manifold with Norden metric. Then the curvature tensor of the Yano connection is Kählerian.

Theorem 9. Let \((M, J, g)\) be a conformal Kähler manifold with Norden metric. Then the Bochner curvature tensors of the Kähler tensors \(R'\) and \(R^*\) coincide.

Proof. From (17), (18) and (31) we get

\[
R' = R^* + \frac{1}{4n} [\psi_1 - \psi_2] (A),
\]

(32)

where

\[
A = S + \frac{\theta (\Omega)}{8n} g - \frac{\theta (J\Omega)}{8n} \bar{g}.
\]

Then, for the corresponding Ricci tensors \(\rho' = \rho (R')\), \(\rho^* = \rho (R^*)\) and the scalar curvatures \(\tau' = \tau (R')\), \(\tau^* = \tau (R^*)\), \(\bar{\tau}' = \bar{\tau} (R')\), \(\bar{\tau}^* = \bar{\tau} (R^*)\) we obtain

\[
\rho' = \rho^* + \frac{\tau' - \tau^*}{4(n - 1)} g - \frac{\bar{\tau}' - \bar{\tau}^*}{4(n - 1)} \bar{g} + \frac{n - 2}{2n} A.
\]

From the last equality, (9) and (32) it follows \(B(R') = B(R^*)\).

Lemma 10. Let \((M, J, g)\) be a conformal Kähler manifold with Norden metric and let \((M, J, \bar{g})\) be its conformally equivalent complex manifold with Norden metric by a transformation of type I. Then the corresponding curvature tensors \(R\) and \(\bar{R}\) are related as follows

\[
\bar{R} = e^{2u} \left\{ R - \psi_1 (G) - \pi_1 \sigma (U) \right\},
\]

where \(G(X, Y) = (\nabla_X \sigma) Y - \sigma (X) \sigma (Y)\), \(\sigma (X) = Xu = du (X)\), \(U = \text{grad} u\).
Taking into account the last lemma and the definition of the tensor $R^*$, we obtain the following interconnection of $(1,3)$-tensors

$$\overline{R} = R^* + \frac{\theta(\Omega)}{4n^2} \pi_3.$$  \hspace{1cm} (33)

Having in mind (9), from (33) we receive the following

**Corollary 11.** The Bochner tensors of $\overline{R}$ and $R^*$ are coincident on a conformal Kähler manifold with Norden metric.

**References**


ON 4-DIMENSIONAL ALMOST HYPERHERMITIAN 
MANIFOLDS

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In this article, we introduce two topics of our recent works ([4], [6], [8]). One is about the integrability of almost hyperhermitian manifolds. The other is about 4-dimensional almost hyperhermitian manifolds, especially.

1. Classes of almost Hermitian manifolds
A smooth manifold $M$ admitting $(1,1)$-tensor field $J$ satisfying $J^2 = -id.$ is called an almost complex manifold with the almost complex structure $J$. The concept of almost complex manifold is a natural generalization of the one of complex manifold, which was introduced by C. Ehresmann in 1948. Almost complex structure $J$ of an almost complex manifold $(M, J)$ is said to be integrable if $M$ admits a complex structure and the derived almost complex structure coincides with the almost complex strucure $J$. An almost complex manifold $(M, J)$ equipped with a Riemannian metric $g$ satisfying $g(JX, JY) = g(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$ ($\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on $M$) is called an almost Hermitian manifold with the almost Hermitian structure $(J, g)$. Kähler manifolds ($\nabla J = 0$, where $\nabla$ denotes the Riemannian connection of $(M, g)$) are the most typical almost Hermitian manifolds and studied by many researchers.

A. Gray and L. M. Hervella [1] defined sixteen classes of almost Hermitian manifolds including the class of Kähler manifolds. Now, we shall recall several classes of almost Hermitian manifolds out of the sixteen classes. Let $M = (M, J, g)$ be an almost Hermitian manifold with al-

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most Hermitian structure \((J, g)\) and \(\Omega_J\) the Kähler form of \(M\) defined by \(\Omega_J(X, Y) = g(JX, Y)\), for \(X, Y \in \mathfrak{X}(M)\). We denote by \(\nabla\) the Riemannian connection of \(M\). The Nijenhuis tensor field \(N_J\) is defined by \(N_J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]\) for \(X, Y \in \mathfrak{X}(M)\).

Almost Hermitian manifold \(M\) is called a *Hermitian manifold* if the almost complex structure \(J\) is integrable. Almost Hermitian manifold \(M\) is Hermitian manifold if and only if the Nijenhuis tensor field \(N_J\) vanishes. It is known that a Hermitian manifold \(M\) satisfies

\[
(\nabla_X J)Y - (\nabla_{JX} J)JY = 0
\]

for \(X, Y \in \mathfrak{X}(M)\). By the definition, a Kähler manifold is a special Hermitian manifold such that the almost complex structure \(J\) is parallel with respect to the Riemannian connection \(\nabla\). An almost Hermitian manifold \(M = (M, J, g)\) is called an *almost Kähler manifold* if the Kähler form is closed \((d\Omega_J = 0)\) or equivalently \(M\) satisfies

\[
\mathcal{G}_{X,Y,Z} g((\nabla_X J)Y, Z) = 0
\]

for \(X, Y, Z \in \mathfrak{X}(M)\), where \(\mathcal{G}_{X,Y,Z}\) denotes the cyclic sum with respect to \(X, Y, Z\). It is well-known that an almost Kähler manifold with the integrable almost complex structure is a Kähler manifold. A symplectic manifold \((M, \Omega)\) \((d\Omega = 0)\) is an almost Kähler manifold with respect to an almost Hermitian structure \((J, g)\) compatible with the symplectic form \(\Omega\). The cotangent bundle \(T^*M\) of any smooth manifold \(M\) is the most typical example of symplectic manifold. The first example of compact symplectic manifold which is not a Kähler manifold was given by W.P. Thurston [12].

An almost Hermitian manifold \(M = (M, J, g)\) is called a *nearly Kähler manifold* if \(M\) satisfies the condition

\[
(\nabla_X J)Y + (\nabla_{JX} J)JY = 0
\]

for \(X, Y \in \mathfrak{X}(M)\). Further, an almost Hermitian manifold \(M = (M, J, g)\) is called a *quasi-Kähler manifold* if \(M\) satisfies the condition

\[
(\nabla_X J)Y + (\nabla_{JX} J)JY = 0
\]

for \(X, Y \in \mathfrak{X}(M)\). It is known that a quasi-Kähler manifold \(M = (M, J, g)\) is necessarily a *semi-Kähler manifold* \((d\Omega_J = 0, \text{ where } \delta\text{ denotes the coderivative})\). On one hand, an almost Hermitian manifold \(M = (M, J, g)\) is called a *generalized locally conformal almost Kähler manifold* if there exists a 1-form \(\omega_J\) on \(M\) satisfying

\[
d\Omega_J = \omega_J \wedge \Omega_J.
\]
Further, an almost Hermitian manifold $M = (M, J, g)$ is called a para-Kähler manifold if $M$ satisfies the condition

$$R(X, Y) \cdot J = 0$$

for $X, Y \in \mathfrak{X}(M)$, where $R$ denotes the curvature tensor field of $M$. We denote by $\mathcal{H}$, $\mathcal{K}$, $\mathcal{AK}$, $\mathcal{NK}$, $\mathcal{QK}$, $\mathcal{SK}$ and $\mathcal{GLCAK}$ the classes of Hermitian, Kähler, almost Kähler, nearly Kähler, quasi-Kähler, semi-Kähler and generalized locally conformal almost Kähler manifolds, respectively. Then we have the following inclusion relations among these classes ([1]):

$$\mathcal{K} \subset \mathcal{AK} \subset \mathcal{NK} \subset \mathcal{QK} \subset \mathcal{SK},$$

$$\mathcal{K} = \mathcal{AK} \cap \mathcal{H}, \quad \mathcal{K} = \mathcal{QK} \cap \mathcal{H},$$

$$\mathcal{K} = \mathcal{NK} \cap \mathcal{AK}, \quad \mathcal{AK} \subset \mathcal{GLCAK}.$$  

We note that the above inclusion relations are all strict. In particular, we have $\mathcal{K} = \mathcal{NK}$ and $\mathcal{AK} = \mathcal{QK} = \mathcal{SK}$ in dimension 4.

### 2. Integrability of almost hyperhermitian manifolds

An almost hypercomplex manifold is a quadruple $M = (M, I, J, K)$, where $I$, $J$ and $K$ are three almost complex structures on a smooth manifold $M$ satisfying $K = IJ = -JI$. It is well-known that an almost hypercomplex manifold is $4m$-dimensional and orientable. An almost hypercomplex manifold $M = (M, I, J, K)$ equipped with a Riemannian metric $g$ is called an almost hyperhermitian manifold if $(I, g)$, $(J, g)$ and $(K, g)$ are simultaneously almost Hermitian structures on $M$. Especially, if $(I, g)$, $(J, g)$ and $(K, g)$ are simultaneously Kähler structures on $M$, then $M = (M, I, J, K, g)$ is called a hyperkähler manifold. In the same manner, by the condition that $(I, g)$, $(J, g)$ and $(K, g)$ satisfy simultaneously, we may define the classes of almost hyperhermitian manifolds which correspond to the classes of almost Hermitian manifolds described in §1.

Concerning the integrability of almost hyperhermitian manifolds, the following result obtained by N.J. Hitchin [3, Lemma 6.8] is well-known.

**Theorem 2.1.** Any almost hyperkähler manifold is a hyperkähler manifold.

N. Murakoshi, K. Sekigawa and A. Yamada [4] studied a generalization of the above Theorem 2.1 and obtained the following

**Theorem 2.2.** Any quasi-hyperkähler manifold is a hyperkähler manifold.
And also, K. Sekigawa and A. Yamada [8] gave the following

**Theorem 2.3.** Any $4m$ ($m \geq 2$)-dimensional generalized locally conformal almost hyperkähler manifold is a locally conformal hyperkähler manifold.

**Remark 2.1.** We cannot remove the restriction on the dimension of the manifold in Theorem 2.3.

So, we shall study about 4-dimensional case in the next section §3.

**Remark 2.2.** Any 4-dimensional almost hyperhermitian manifold is necessarily a generalized locally conformal almost hyperkähler manifold.

We denote by $\mathcal{HH}$, $\mathcal{HK}$, $\mathcal{AHK}$, $\mathcal{NHK}$, $\mathcal{QHK}$, $\mathcal{SHK}$ and $\mathcal{GLCAHK}$ the classes of hyperhermitian, hyperkähler, almost hyperkähler, nearly hyperkähler, quasi-hyperkähler, semi-hyperkähler and generalized locally conformal almost hyperkähler manifolds, respectively. Joining the above theorems 2.1, 2.2 and remark 2.2 with the inclusion relation (7) in §1, we have the following inclusion relations among these classes:

$$\mathcal{HK} = \mathcal{AHK} = \mathcal{NHK} = \mathcal{QHK},$$

especially, in dimension 4,

$$\{ \text{almost hyperhermitian manifolds} \} = \mathcal{GLCAHK},$$

$$\mathcal{HK} = \mathcal{AHK} = \mathcal{NHK} = \mathcal{QHK} = \mathcal{SHK}. \quad (9)$$

### 3. 4-dimensional almost hyperhermitian manifolds

In this section, we shall investigate some fundamental structures of 4-dimensional almost hyperhermitian manifolds. By Remark 2.2, any 4-dimensional almost hyperhermitian manifold is regarded as a generalized locally conformal almost hyperkähler manifold.

T. Nihonyanagi, K. Sekigawa and A. Yamada [6] studied 4-dimensional almost hyperhermitian manifolds and obtained the following results.

**Theorem 3.1.** Let $(M, I, J, K, g)$ be a 4-dimensional compact almost hyperhermitian Einstein manifold. Then $(M, I, J, K, g)$ is a Ricci-flat and $\ast$-Ricci-flat para-hyperkähler manifold. Furthermore, $(M, g)$ is either flat or its universal covering is a Ricci-flat $K3$ surface with a hyperkähler structure.
Theorem 3.2. Let $M = (M, I, J, K, g)$ be a 4-dimensional compact hyperhermitian manifold with non-positive scalar curvature. Then $M$ is a hyperkähler manifold.

Remark 3.1. For any 4-dimensional almost hyperhermitian manifold $(M, I, J, K, g)$, we may easily observe that, if any two almost complex structures of $I$, $J$, $K$ are integrable, then the remaining one is also integrable.

Concerning Theorems 3.1 and 3.2, we shall introduce an example of 4-dimensional compact locally flat almost hyperhermitian manifold which is not hyperkähler manifold (cf. [13]).

Example 3.1. Let $G$ be a connected Lie subgroup of $GL(4, \mathbb{R})$ consisting of the matrices

\[
\begin{pmatrix}
x & y & z & 0 \\
y & x & w & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad x^2 + y^2 = 1.
\]

Let $g$ be the Lie algebra of $G$. Then $g = \text{span}_\mathbb{R}\{e_1, e_2, e_3, e_4\}$, where

\[
e_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

We define a $G$-invariant Riemannian metric $g$ on $G$ by $g(e_i, e_j) = \delta_{ij}$ ($1 \leq i, j \leq 4$), and almost hypercomplex structure $(I, J, K)$ by

\[
Ie_1 = e_2, \quad Ie_2 = -e_1, \quad Ie_3 = e_4, \quad Ie_4 = -e_3,
\]

\[
Je_1 = e_3, \quad Je_3 = -e_1, \quad Je_2 = -e_4, \quad Je_4 = e_2,
\]

\[
Ke_1 = e_4, \quad Ke_4 = -e_1, \quad Ke_2 = e_3, \quad Ke_3 = -e_2.
\]

In the above Example 3.1, we may observe that $(G, I, J, K, g)$ is a 4-dimensional locally flat almost hyperhermitian manifold which is not a hyperkähler manifold. We may check $N_J = 0$, $N_I \neq 0$, $N_K \neq 0$ on
Let $\Gamma$ be the discrete subgroup of $G$ consisting of matrices
\[
\begin{pmatrix}
1 & 0 & l & 0 \\
0 & 1 & m & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & e^{n}
\end{pmatrix}
l, m, n \in \mathbb{Z}.
\] (13)

Then, we may see that the $\Gamma$-orbit space $\Gamma \backslash G$ is compact.

Next, we shall state a way to construct an almost hyperhermitian manifold from a manifold equipped with almost contact metric 3-structure.

**Example 3.2.** Let $N = (N, \varphi, \xi, \eta, g_0)$ be a $(4m - 1)$-dimensional almost contact metric manifold and $M = N \times \mathbb{R}$ (resp. $N \times S^1$) be the direct product manifold of $N$ and a real line $\mathbb{R}$ (resp. a unit circle $S^1$). We define almost complex structure $J$ on $M = N \times \mathbb{R}$ (resp. $N \times S^1$) by
\[
J = \begin{pmatrix}
\varphi & \xi \\
-\eta & 0
\end{pmatrix},
\] (14)
that is, $JX = \varphi X$, for $X \in \mathfrak{X}(M)$, $J\xi = -\nu$ and $J\nu = \xi$, where $\nu$ is a unit tangent vector field on $\mathbb{R}$ (resp. $S^1$). Further, we denote by $g$ the canonical product Riemannian metric on $M$. Then, we may easily see that $(M, J, g)$ is an almost Hermitian manifold. A $(4m - 1)$-dimensional manifold $N$ equipped with three almost contact structures $(\phi_a, \xi_a, \eta_a)$ $(a = 1, 2, 3)$ satisfying the following for an even permutation $(i, j, k)$ of $(1, 2, 3)$,
\[
\phi_k = \phi_i \phi_j - \eta_j \otimes \xi_i = -\phi_j \phi_i + \eta_i \otimes \xi_j, \\
\xi_k = \phi_i \xi_j = -\phi_j \xi_i, \\
\eta_k = \eta_i \circ \phi_j = -\eta_j \circ \phi_i,
\] (15)
then $\{(\phi_a, \xi_a, \eta_a)\}$ $(a = 1, 2, 3)$ is called an almost contact 3-structure. Let $N$ be a $(4m - 1)$-dimensional manifold equipped with almost contact 3-structure $\{(\phi_a, \xi_a, \eta_a)\}$ $(a = 1, 2, 3)$, and $g_0$ be a Riemannian metric compatible with three almost contact structure $(\phi_a, \xi_a, \eta_a)$ $(a = 1, 2, 3)$. Then we see that $(M, I, J, K, g)$ $(M = N \times \mathbb{R}$ (or $N \times S^1$)) is a $4m$-dimensional almost hyperhermitian manifold with almost hyperhermitian structure $(I, J, K, g)$, where $I, J$ and $K$ are almost complex structures defined from $(\phi_1, \eta_1, \xi_1)$, $(\phi_2, \eta_2, \xi_2)$ and $(\phi_3, \eta_3, \xi_3)$, respectively, and $g$ is the canonical product metric on $M$.

Especially, it is well-known that 3-dimensional unit sphere $S^3$ admits a Sasakian 3-structure $(\phi_a, \xi_a, \eta_a, g_0)$ $(a = 1, 2, 3)$. Thus, we see also that
the corresponding product manifold \( M = S^3 \times \mathbb{R} \) or \( S^3 \times S^1 \) equipped with the canonical product metric \( g \) is a 4-dimensional hyperhermitian manifold \( (M, I, J, K, g) \). For example, we may easily see that a manifold \( (M, I, J, K, g) \) for \( M = S^3 \times S^1 \) is a compact 4-dimensional hyperhermitian manifold with constant positive scalar curvature 12. In stead of the product metric \( g \) on \( S^3 \times \mathbb{R} \), considering the metric \( \bar{g} = e^{-2t} g \) \((t \in \mathbb{R})\), we may also see that \( (S^3 \times \mathbb{R}, I, J, K, \bar{g}) \) is a locally flat hyperkahler manifold.

Now, we shall give another example concerning Theorems 3.1 and 3.2.

**Example 3.3.** Let \( M = \mathbb{R}^4_+ = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 > 0\} \) and \( e_i = \frac{\partial}{\partial x_i} \) \((1 \leq i \leq 4)\). We define a Riemannian metric \( g \) on \( M \) by
\[
g(e_i, e_j) = \delta_{ij},
\]
then we may observe that \( (M, g) \) can be regarded as a solvable Lie group space and \( \{e_i\}_{i=1,2,3,4} \) is an orthonormal basis of the corresponding Lie algebra. Further, we define the almost hypercomplex structure \((I, J, K)\) by
\[
Ie_1 = e_2, \quad Ie_2 = -e_1, \quad Ie_3 = e_4, \quad Ie_4 = -e_3,
\]
\[
Je_1 = e_3, \quad Je_3 = -e_1, \quad Je_2 = -e_4, \quad Je_4 = e_2,
\]
\[
Ke_1 = e_4, \quad Ke_4 = -e_1, \quad Ke_2 = e_3, \quad Ke_3 = -e_2.
\]

In the above Example 3.3, we may see that \( (M, I, J, K, g) \) is a 4-dimensional hyperhermitian, non-hyperkahler manifold of constant sectional curvature \(-1\). However, we may also observe that \( (M, I, J, K, g) \) does not have any compact quotient as a hyperhermitian manifold.

Recently, T. Nihonyanagi [5] show that a para-hyperkahler structure on a 4-dimensional Einstein manifold is locally \( SO(3) \)-deformable to a hyperkahler structure.

**Theorem 3.3.** Let \( M = (M, I, J, K, g) \) be a 4-dimensional para-hyperkahler Einstein manifold. Then, for each point \( p \in M \), there exists a local smooth \( SO(3) \)-valued function \( \Theta \) such that the almost hyperhermitian structure \((\tilde{I}, \tilde{J}, \tilde{K}, g)\) given by
\[
\begin{pmatrix}
\tilde{I} \\
\tilde{J} \\
\tilde{K}
\end{pmatrix}
= \Theta
\begin{pmatrix}
I \\
J \\
K
\end{pmatrix}
\]
is a hyperkahler structure near the point \( p \).

Taking account of the above Theorems 3.1~3.3, we have also the following

**Corollary 3.1.** Let \( M = (M, I, J, K, g) \) be a 4-dimensional compact simply-connected almost hyperhermitian Einstein manifold. Then, \( M \) is a hyperkahler manifold. Furthermore, \( M \) is isometric to a K3-surface with
Ricci-flat metric and \((I, J, K, g)\) is globally \(SO(3)\)-deformable to a hyperkähler structure on a K3-surface in the sense of Theorem 3.3.

References