Rigid geometry is one of the modern branches of algebraic and arithmetic geometry. It has its historical origin in J. Tate's rigid analytic geometry, which aimed at developing an analytic geometry over non-archimedean valued fields. Nowadays, rigid geometry is a discipline in its own right and has acquired vast and rich structures, based on discoveries of its relationship with birational and formal geometries.

In this research monograph, foundational aspects of rigid geometry are discussed, putting emphasis on birational and topological features of rigid spaces. Besides the rigid geometry itself, topics include the general theory of formal schemes and formal algebraic spaces, based on a theory of complete rings which are not necessarily Noetherian. Also included is a discussion on the relationship with Tate’s original rigid analytic geometry, V.D. Berkovich’s analytic geometry and R. Huber’s adic spaces. As a model example of applications, a proof of Nagata’s compactification theorem for schemes is given in the appendix. The book is encyclopedic and almost self-contained.
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Foundations of Rigid Geometry I
To the memory of Professor Masayoshi Nagata 
and Professor Masaki Maruyama
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Introduction

In the early stage of its history, rigid geometry has been first envisaged in an attempt to construct a non-Archimedean analytic geometry, an analogue over non-Archimedean valued fields, such as \( p \)-adic fields, of complex analytic geometry. Later, in the course of its development, rigid geometry has acquired several rich structures, considerably richer than being merely ‘copies’ of complex analytic geometry, which endowed the theory with a great potential of applications. This theory is nowadays recognized by many mathematicians in various research fields to be an important and independent discipline in arithmetic and algebraic geometry. This book is the first volume of our prospective book project, which aims to discuss the rich overall structures of rigid geometry, and to explore its applications.

Before explaining our general perspective on this book project, we first provide an overview of the past developments of the theory.

0. Background. After K. Hensel introduced \( p \)-adic numbers by the end of the 19th century, the idea arose of constructing \( p \)-adic analogues of already existing mathematical theories that were formerly considered only over the field of real or complex numbers. One such analogue was the theory of complex analytic functions, which had by then already matured into one of the most successful and rich branches of mathematics. Complex analysis saw further developments and innovations later on. Most importantly, from extensive works on complex analytic spaces and analytic sheaves by H. Cartan and J. P. Serre in the mid-20th century, after the pioneering work by K. Oka, arose the new idea that the theory of complex analytic functions should be regarded as part of complex analytic geometry. According to this view, it was only natural to expect the emergence of \( p \)-adic analytic geometry, or more generally, non-Archimedean analytic geometry.

However, all first attempts encountered essential difficulties, especially in establishing a reasonable link between the local and global notions of analytic functions. Such a naive approach is, generally speaking, characterized by its inclination to produce a faithful imitation of complex analytic geometry, which can be already seen at the level of point sets and topology of the putative analytic spaces. For example, for the ‘complex plane’ over \( \mathbb{C}_p \) (= the completion of the algebraic closure \( \overline{\mathbb{Q}}_p \) of \( \mathbb{Q}_p \)), one takes the naive point set, that is, \( \mathbb{C}_p \) itself, and the topology simply induced by the \( p \)-adic metric. Starting from a situation like the one described, one
goes on to construct a locally ringed space $X = (X, \mathcal{O}_X)$ by introducing the sheaf $\mathcal{O}_X$ of ‘holomorphic functions,’ a conventional definition of which is something like this: $\mathcal{O}_X(U)$ for any open subset $U$ is the set of all functions on $U$ that admit a convergent power series expansion at every point. But this leads to an extremely cumbersome situation. Indeed, since the topology of $X$ is totally disconnected, there are too many open subsets, and this causes the patching of functions to be extremely ‘wobbly,’ so much so that one fails to have good control of the global behavior of analytic functions. For example, if $X$ is the ‘$p$-adic Riemann sphere’ $\mathbb{C}_p \cup \{\infty\}$, one would expect that $\mathcal{O}_X(X)$ consists only of constant functions, which, however, is far from being true in this situation.

Let us call the problem described above the **Globalization Problem**.\(^1\) Although in its essence it may be seen, inasmuch as being concerned with patching of analytic functions, as a topological problem, as it will turn out, it deeply links with the issue of how to define the notion of points. In the prehistory of rigid geometry, this Globalization Problem has been one, and perhaps the most crucial one, of the obstacles in the quest for a good non-Archimedean analytic geometry.\(^2\)

1. **Tate’s rigid analytic geometry.** The Globalization Problem found its fundamental solution when J. Tate introduced his rigid analytic geometry\(^3\) in a seminar at Harvard University in 1961. Tate’s motivation was to justify his construction of the so-called Tate curves, a non-Archimedean analogue of 1-dimensional complex tori, build by means of an infinite quotient\(^4\). Tate’s solution to the problem consists of the following items:

   - a ‘reasonable’ and ‘sufficiently large’ class of analytic functions and
   - a ‘correct’ notion of analytic coverings.

Here, one can find behind this idea the influence of A. Grothendieck in at least two ways: first, Tate introduced spaces by means of local characterization in terms of their function rings, as typified by scheme theory; second, he used the machinery of Grothendieck topology to define analytic coverings.

Let us briefly review Tate’s theory. First of all, Tate introduced the category $\text{Aff}_K$ of so-called affinoid algebras over a complete non-Archimedean valuation field $K$. Each affinoid algebra $\mathfrak{A}$, which is a $K$-Banach algebra, is considered to be the ring of ‘reasonable’ analytic functions over the ‘space’ $\text{Sp} \mathfrak{A}$, called the affinoid, which is the corresponding object in the dual category $\text{Aff}_K^{\text{opp}}$ of $\text{Aff}_K$. Moreover, based on the notion of admissible coverings, he introduced a new ‘topology,’ in fact, a Grothendieck topology, on $\text{Sp} \mathfrak{A}$, which we call the admissible topology.

\(^1\)This problem is, in classical literature, usually referred to as the problem in analytic continuation.

\(^2\)In his pioneering works\(^5\) and\(^6\), M. Krasner studied in deep the problem and gave a first general recipe for a meaningful analytic continuation of non-Archimedean analytic functions.

\(^3\)Elliptic curves and elliptic functions over $p$-adic fields have already been studied by É. Lutz at the suggestion of A. Weil, who was inspired by classical works of Eisenstein (cf.\(^7\), p. 538).

\(^4\)In his pioneering works\(^5\) and\(^6\), M. Krasner studied in deep the problem and gave a first general recipe for a meaningful analytic continuation of non-Archimedean analytic functions.
The admissibility imposes, most importantly, a strong finiteness condition on analytic coverings, which establishes close ties between the local and global behaviors of analytic functions, as is well described by the famous Tate’s acyclicity theorem (II.B.2.3). An important consequence of this nice local-to-global connection is the good notion of ‘patching’ of affinoids, by means of which Tate was able to solve the Globalization Problem, and thus to construct global analytic spaces.

In summary, Tate overcame the difficulty by ‘rigidifying’ the topology itself by imposing the admissibility condition, a strong restriction on the patching of local analytic functions. It is for this reason that this theory is nowadays called rigid analytic geometry.

Aside from the fact that it gave a beautiful solution to the Globalization Problem, it is remarkable that Tate’s rigid analytic geometry proved that it is possible to apply Grothendieck’s way of constructing geometric objects in the setting of non-Archimedean analytic geometry. Thus, rather than complex analytic geometry, Tate’s rigid analytic geometry resembles scheme theory. There seemed to be, however, one technical difference between scheme theory and rigid analytic geometry, which was considered to be quite essential at the time when rigid analytic geometry appeared: rigid analytic geometry had to use Grothendieck topology, not classical point set topology.

There is yet another aspect of rigid analytic geometry reminiscent of algebraic geometry. In order to have a better grasp of the abstractly defined analytic spaces, Tate introduced a notion of points. He defined points of an affinoid $\text{Sp} A$ to be maximal ideals of the affinoid algebra $A$; viz., his affinoids are visualized by the maximal spectra, that is, the set of all maximal ideals of affinoid algebras, just like affine varieties in the classical algebraic geometry are visualized by the maximal spectra of finite type algebras over a field. Note that this choice of points is essentially the same as the naive one that we have mentioned before. This notion of points was, despite its naivety, considered to be natural, especially in view of his construction of Tate curves, and practically good enough as far as being concerned with rigid analytic geometry over a fixed non-Archimedean valued field.

2. Functoriality and topological visualization. Tate’s rigid analytic geometry has, since its first appearance, proven itself to be useful for many purposes, and been further developed by several authors. For example, H. Grauert and R. Remmert [49] laid the foundations of topological and ring theoretic aspects of affinoid algebras, and R. Kiehl [69] and [70] promoted the theory of coherent sheaves and their cohomologies on rigid analytic spaces.

However, it was widely perceived that rigid analytic geometry still has some essential difficulties, some of which are listed below.

\footnote{One might be apt to think that Tate’s choice of points is an ‘easygoing’ analogue of the spectra of complex commutative Banach algebras, for which the justification, Gelfand–Mazur theorem, is, however, only valid in complex analytic situation, and actually fails in $p$-adic situation (see below).}
• *Functoriality of points does not hold.* If $K'/K$ is an extension of complete non-Archimedean valuation fields, then one expects to have, for any rigid analytic space $X$ over $K$, a mapping from the points of the base change $X_{K'}$ to the points of $X$, which, however, does not exist in general in Tate’s framework.

Let us call this problem the *Functoriality Problem.* The problem is linked with the following more fundamental one.

• *The analogue of the Gelfand–Mazur theorem does not hold.* The Gelfand–Mazur theorem states that there exist no Banach field extension of $\mathbb{C}$ other than $\mathbb{C}$ itself. In the non-Archimedean case, in contrast, there exist many Banach $K$-fields other than finite extensions of $K$. This would imply that there should be plenty of ‘valued points’ of an affinoid algebra not factoring through the residue field of a maximal ideal; in other words, there should be much more points than those that Tate has introduced.

It is clear that in order to overcome the difficulties of this kind one has to change the notion of points. More precisely, the problem lies in what to choose as the spectrum of an affinoid algebra. To this, there are at least two solutions:

(I) Gromov–Berkovich style spectrum;

(II) Stone–Zariski style spectrum.

The spectrum of the first style, which turns out to be the ‘smallest’ spectrum allowing to solve the Functoriality Problem in the category of Banach algebras, consists of height-one valuations, that is, *seminorms* (of a certain type) on affinoid algebras. The resulting point sets carry a natural topology, the so-called Gelfand topology. This kind of spectra was adopted by V. G. Berkovich in his approach to non-Archimedean analytic geometry, so-called *Berkovich analytic geometry* [11]. A nice feature of this approach is that, in principle, it can deal with a wide class of Banach $K$-algebras, including affinoid algebras, and thus solve the Functoriality Problem (in the category of Banach algebras). Moreover, the spectra of affinoid algebras in this approach are Hausdorff, thereby providing intuitively familiar spaces as the underlying topological spaces of the analytic spaces.

However, the Gelfand topology differs from the admissible topology; it is even weaker, in the sense that, as we will see later, the former topology is a *quotient* of the latter. Therefore, this topology does not solve the Globalization Problem for affinoid algebras compatibly with Tate’s solution, and, in order to do analytic geometry, one still has to use the Grothendieck topology just imported from Tate’s theory.

It is thus necessary, in order to simultaneously solve the Globalization Problem (for affinoids) and the Functoriality Problem, to further improve the notion of points and the topology. In the second style, the Stone–Zariski style, which we will take up in this book, each spectrum has more points by valuations, not only of height one,
but of higher height.\textsuperscript{5} It turns out that the topology on the point set thus obtained coincides with the admissible topology on the corresponding affinoid, thus solving the Globalization Problem without using the Grothendieck topology. Moreover, the spectra have plenty of points to solve the Functoriality Problem as well.

As we have seen, to sum up, both the Globalization Problem and the Functoriality Problem are closely linked with the more fundamental issue concerned with the notions of points and topology, that is, the problem of the choice of spectra. What lies behind all this is the philosophical tenet that every notion of space in commutative geometry should be accompanied with ‘visualization’ by means of topological spaces, which we call the topological visualization (Figure 1). It can be stated, therefore, that the original difficulties in the early non-Archimedean analytic geometry in general, Globalization and Functoriality, are rooted in the lack of adequate topological visualizations. We will dwell on more on this topic later.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (comm) at (0,0) {Commutative geometry};
  \node (top) at (2,0) {Topological spaces};
  \draw[->] (comm) -- (top);
\end{tikzpicture}
\caption{Topological visualization.}
\end{figure}

3. **Raynaud’s approach to rigid analytic geometry.** To adopt the spectra as in the Stone–Zariski style, in which points are described in terms of valuation rings of arbitrary height, one more or less inevitably has to deal with finer structures, somewhat related to integral structures, of affinoid algebras.\textsuperscript{6} The approach is, then, further divided into the following two branches:

- (II-a) R. Huber’s adic spaces\textsuperscript{7} [59], [60], and [61];
- (II-b) M. Raynaud’s viewpoint via formal geometry\textsuperscript{8} as a model geometry [88].

The last approach, which we will adopt in this book, fits in the general framework in which a geometry as a whole is a package derived from a so-called model geometry. Here is a toy model that exemplifies the framework. Consider, for example, the category of finite-dimensional $\mathbb{Q}_p$-vector spaces. We observe that this

\textsuperscript{5}Note that this height tolerance is necessary even for rigid spaces defined over complete valuation fields of height one.

\textsuperscript{6}Such a structure, which we call a rigidification, will be discussed in detail in II, §A.2. (c). In the original Tate rigid analytic geometry, the rigidifications are canonically determined by classical affinoid algebras themselves, and this fact explains why Tate’s rigid analytic geometry, unlike the more general Huber’s adic geometry, could work without reference to integral models of affinoid algebras.

\textsuperscript{7}Note that Huber’s theory is based on the choice of integral structures of topological rings. We will give, mainly in II, §A, a reasonably detailed account of Huber’s theory.

\textsuperscript{8}By formal geometry, we mean in this book the geometry of formal schemes, developed by A. Grothendieck.
category is equivalent to the quotient category of the category of finitely generated \( \mathbb{Z}_p \)-modules modulo the Serre subcategory consisting of \( p \)-torsion \( \mathbb{Z}_p \)-modules, since any finite-dimensional \( \mathbb{Q}_p \)-vector space has a \( \mathbb{Z}_p \)-lattice, that is, a ‘model’ over \( \mathbb{Z}_p \). This suggests that the overall theory of finite-dimensional \( \mathbb{Q}_p \)-vector spaces is derived from the theory of models, in this case, the theory of finitely generated \( \mathbb{Z}_p \)-modules.

In our context, what Raynaud discovered on rigid analytic geometry consists of the following statements.

- **Formal geometry**, which has already been established by Grothendieck prior to Tate’s work, can be adopted as a model geometry for Tate’s rigid analytic geometry.

- **The overall theory of rigid analytic geometry arises from Grothendieck’s formal geometry** (Figure 2), which leads to the extremely useful idea that, between formal geometry and Tate’s rigid analytic geometry, one can use theorems in one setting to prove theorems in the other.

![Figure 2. Raynaud’s approach to rigid geometry.](image)

To make more precise the assertion that formal geometry can be a model geometry for rigid analytic geometry, consider, just as in the toy model as above, the category of rigid analytic spaces over \( K \). Raynaud showed that the category of Tate’s rigid analytic spaces (with some finiteness conditions) is equivalent to the quotient category of the category of finite type formal schemes over the valuation ring \( V \) of \( K \). Here the ‘quotient’ means inverting all ‘modifications’ (especially, blow-ups) that are ‘isomorphisms over \( K \),’ the so-called *admissible modifications* (blow-ups).

There are several important consequences of Raynaud’s discovery; let us mention a few of them. First, guided by the principle that rigid analytic geometry is derived by formal geometry, one can build the theory afresh, starting from defining the category of rigid analytic spaces as the quotient category of the category of formal schemes modulo all admissible modifications.\(^9\) Second, Raynaud’s theorem says that rigid analytic geometry can be seen as the birational geometry of formal schemes, a novel viewpoint, which motivates one to explore the link with traditional birational geometry. Third, as already mentioned above, the bridge between formal

\(^9\)The rigid spaces obtained in this way are, more precisely, what we call coherent (= quasi-compact and quasi-separated) rigid spaces, from which general rigid spaces are constructed by patching.
schemes and rigid analytic spaces, established by Raynaud’s viewpoint, gives rise to fruitful interactions between these theories. Especially useful is the fact that theorems in the rigid analytic side can be deduced, at least when one works over complete discrete valuation rings, from theorems in the formal geometry side, available in EGA and SGA works by Grothendieck et al., at least in the Noetherian case.

4. Rigid geometry of formal schemes. We can now describe, along the line of Raynaud’s discovery, the basic framework of our rigid geometry that we promote in this book project. For us rigid geometry is a geometry obtained from a birational geometry of model geometries. This being so, the main purpose of this book project is to develop such a theory for formal geometry, thus generalizing Tate’s rigid analytic geometry and building a more general analytic geometry. Thus to each formal scheme $X$ we associate an object of a resulting category, denoted by $X^{\text{rig}}$, which itself should already be regarded as a rigid space. Then we define general rigid spaces by patching these objects. Note that, here, the rigid spaces are introduced as an ‘absolute’ object, without reference to a base space.

Among several classes of formal schemes we start with, one of the most important is the class of what we call locally universally rigid-Noetherian formal schemes; see I.2.1.7. The rigid spaces obtained from this class of formal schemes are called locally universally Noetherian rigid spaces, see II.2.2.23, which cover most of the analytic spaces that appear in contemporary arithmetic geometry. Note that the formal schemes of the above kind are not themselves locally Noetherian. A technical point resulting from the demand of removing Noetherian hypothesis is that one has to treat non-Noetherian adic rings of fairly general kind, for which classical theories, including EGA, do not give us enough tools; for example, valuation rings of arbitrary height are necessary in order to describe points on rigid spaces, and we accordingly need to treat fairly wide class of adic rings over them for describing fibers of finite type morphisms.

Besides, we would like to propose another viewpoint, which classical theory does not offer. Among what Raynaud’s theory suggests, the most inspiring is, we think, the idea that rigid geometry should be a birational geometry of formal schemes. We would like to adopt this perspective as one of the core ideas of our theory. In fact, as we will see soon below, it tells us what should be the most natural notion of point of a rigid space, and thus leads to an extremely rich structure concerned with visualizations (that is, spectra), whereby to obtain a satisfactory solution to the above-mentioned Globalization and Functoriality problems. We explain this in the sequel.

5. Revival of Zariski’s approach. The birational geometric aspect of our rigid geometry is best explained by means of O. Zariski’s classical approach to birational geometry as a model example. Around 1940’s, in his attempt to attack the desin-
gularization problem for algebraic varieties, Zariski introduced abstract Riemann spaces for function fields, which we call Zariski–Riemann spaces, generalizing the classical valuation-theoretic construction of Riemann surfaces by Dedekind and Weber. This idea has been applied to several other problems in algebraic geometry, including, for example, Nagata’s compactification theorem for algebraic varieties.

Let us briefly overview Zariski’s idea. Let \( Y \hookrightarrow X \) be a closed immersion of schemes (with some finiteness conditions), and set \( U = X \setminus Y \). We consider \( U \)-admissible modifications of \( X \), which are by definition proper birational maps \( X' \to X \) that are isomorphisms over \( U \). This class of morphisms contains the subclass consisting of \( U \)-admissible blow-ups, that is, blow-ups along closed subschemes contained in \( Y \). In fact, \( U \)-admissible blow-ups are cofinal in the set of all \( U \)-admissible modifications (due to the flattening theorem; cf. II, §E.1. (b)). The Zariski–Riemann space, denoted by \( \langle X \rangle_U \), is the topological space defined as the projective limit taken along the ordered set of all \( U \)-admissible modifications, or equivalently, \( U \)-admissible blow-ups, of \( X \). Especially important is the fact that the Zariski–Riemann space \( \langle X \rangle_U \) is quasi-compact (essentially due to Zariski [107]; cf. II,E.2.5), a fact that is crucial in proving many theorems, for example, the above-mentioned Nagata’s theorem.10

As is classically known, points of the Zariski–Riemann space \( \langle X \rangle_U \) are described in terms of valuation rings. More precisely, these points are in one-to-one correspondence with the set of all morphisms, up to equivalence by ‘domination,’ of the form \( \text{Spec } V \to X \), where \( V \) is a valuation ring (possibly of height 0), that map the generic point to points in \( U \) (see II, §E.2. (e) for details). Since the spectra of valuation rings are viewed as ‘long paths’ (cf. Figure 1 in 0, §6), one can say intuitively that the space \( \langle X \rangle_U \) is like a ‘path space’ in algebraic geometry (Figure 3).

Now, what we have meant by adopting birational geometry as one of the core ingredients in our theory is that we apply Zariski’s approach to birational geometry to the main body of our rigid geometry. Our basic dictionary for doing this, e.g., for rigid geometry over the \( p \)-adic field, is as follows:

- \( X \leftrightarrow \) formal scheme of finite type over \( \text{Spf } \mathbb{Z}_p \);
- \( Y \leftrightarrow \) the closed fiber, that is, the closed subscheme defined by ‘\( p = 0 \).’

In this dictionary, the notion of \( U \)-admissible blow-ups corresponds precisely to the admissible blow-ups of formal schemes.

6. Birational approach to rigid geometry. As we have already mentioned above, our approach to rigid geometry, called the birational approach to rigid geometry, is, so to speak, the combination of Raynaud’s algebro-geometric interpretation of

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10Zariski–Riemann spaces are also used in O. Gabber’s unpublished works in 1980’s on algebraic geometry problems. Their first appearance in literature in the context of rigid geometry seems to be in [38].
rigid analytic geometry, which regards rigid geometry as a birational geometry of formal schemes, and Zariski’s classical birational geometry (Figure 4). Most notably, it will turn out that this approach naturally gives rise to the Stone–Zariski style spectrum, which we have already mentioned before.

Raynaud’s viewpoint of rigid geometry + Zariski’s viewpoint of birational geometry

A nice point in combining Raynaud’s viewpoint and Zariski’s viewpoint is that, while the former gives the fundamental recipe for defining rigid spaces, the latter endows them with a ‘visualization.’ Let us make this more precise, and alongside explain what kind of visualization we attach here to rigid spaces.

As already described earlier, from an adic formal scheme \( X \) (of finite ideal type; cf. I.1.1.16), we obtain the associated rigid space \( \mathcal{X} = X^{\text{rig}} \). Then, suggested by what we have seen in the previous section, we define the associated Zariski–Riemann space \( \langle \mathcal{X} \rangle \) as the projective limit

\[
\langle \mathcal{X} \rangle = \lim X',
\]

taken in the category of topological spaces, of all admissible blow-ups \( X' \to X \) (Definition II.3.2.11). We adopt this space \( \langle \mathcal{X} \rangle \) as the topological visualization of the rigid space \( \mathcal{X} \). In fact, this space is exactly what we have expected as the topological visualization in the case of Tate’s theory, since it can be shown that the canonical topology (the projective limit topology) of \( \langle \mathcal{X} \rangle \) actually coincides with the admissible topology.
To explain more about the visualization of rigid spaces, we would like to introduce three kinds of visualizations in a general context. One is the topological visualization, which we have already discussed. The second one, which we name standard visualization, is the one that appears in ordinary geometries, as typified by scheme theory; that is, visualization by locally ringed spaces. Recall that an affine scheme, first defined abstractly as an object of the dual category of the category of all commutative rings, can be visualized by a locally ringed space supported on the prime spectrum of the corresponding commutative ring. The third visualization, which we call the enriched visualization, or just visualization in this book, is given by what we call triples: these are objects of the form \((X, \mathcal{O}_X^+, \mathcal{O}_X^-)\) consisting of a topological space \(X\) and two sheaves of topological rings together with an injective ring homomorphism \(\mathcal{O}_X^+ \hookrightarrow \mathcal{O}_X\) that identifies \(\mathcal{O}_X^+\) with an open subsheaf of \(\mathcal{O}_X\) such that the pairs \(X = (X, \mathcal{O}_X)\) and \(X^+ = (X, \mathcal{O}_X^+)\) are locally ringed spaces; in this setting, \(\mathcal{O}_X\) is regarded as the structure sheaf of \(X\), while \(\mathcal{O}_X^+\) represents the enriched structure, such as an integral structure (whenever it makes sense) of \(\mathcal{O}_X\).

The enriched visualization is typified by rigid spaces. The Zariski–Riemann space \(h\) has two natural structure sheaves, the integral structure sheaf \(\mathcal{O}_X^{\text{int}}\), defined as the inductive limit of the structure sheaves of all admissible blow-ups of \(X\), and the rigid structure sheaf \(\mathcal{O}_X\), obtained from \(\mathcal{O}_X^{\text{int}}\) by ‘inverting the ideal of definition.’ What is intended here is that, while the rigid structure sheaf \(\mathcal{O}_X\) should, as in Tate’s rigid analytic geometry, normally come as the ‘genuine’ structure sheaf of the rigid space \(X\), the integral structure sheaf \(\mathcal{O}_X^{\text{int}}\) represents its integral structure. These data comprise the triple

\[\text{ZR}(X) = ((X), \mathcal{O}_X^{\text{int}}, \mathcal{O}_X),\]

called the associated Zariski–Riemann triple, which gives the enriched visualization of the rigid space \(X\). That the rigid structure sheaf should be the structure sheaf of \(X\) means that the locally ringed space \(((X), \mathcal{O}_X^-)\) visualizes the rigid space in an ordinary sense, that is, in the sense of standard visualization.

Note that the Zariski–Riemann triple \(\text{ZR}(X)\) for a rigid space \(X\) coincides with Huber’s adic space associated to \(X\); in fact, the notion of Zariski–Riemann triple gives not only an interpretation of adic spaces, but also a foundation for them via formal geometry, which we establish in this book; see II, §A.5 for more details.

Figure 5 illustrates the basic design of our birational approach to rigid geometry, summarizing all what we have discussed so far.

The figure shows a ‘commutative’ diagram, in which the arrow \((\ast 1)\) is Raynaud’s approach to rigid geometry (Figure 2), and the arrow \((\ast 2)\) is the enriched visualization by Zariski–Riemann triples, coming from Zariski’s viewpoint. The other visualizations are also indicated in the diagram, the standard visualization

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11See II, §A.1 for the generalities of triples.
by (⋆3), and the topological visualization by (⋆4); the right-hand vertical arrows represent the respective forgetful functors.

![Diagram](image)

Figure 5. Birational approach to rigid geometry.

All this is the outline of what we will discuss in this volume. Here, before finishing this overview, let us add a few words on the outgrowth of our theory. Our approach to rigid geometry, in fact, gives rise to a new perspective of rigid geometry itself: *rigid geometry in general is an analysis along a closed subspace in a ringed topos*. This idea, which tells us what the concept of rigid geometry in mathematics should ultimately be, is linked with the idea of *tubular neighborhoods* in algebraic geometry, already discussed in [38]. From this viewpoint, Raynaud’s choice, for example, of formal schemes as models of rigid spaces can be interpreted as capturing the ‘tubular neighborhoods’ along a closed subspace by means of the formal completion. Now that there are several other ways to capture such structures, e.g., Henselian schemes etc., there are several other choices for the model geometry of rigid geometry.  

This yields several variants, e.g., rigid Henselian geometry, rigid Zariskian geometry, etc., all of which are encompassed within our birational approach.

7. **Relation with other theories.** In the first three sections II, §A, II, §B, and II, §C of the appendices to Chapter II, we compare our theory with other theories related to rigid geometry. Here we give a digest of the contents of these sections for the reader’s convenience.

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[12] There is, in addition to formal geometry and Henselian geometry, the third possibility for the model geometry, by *Zariskian schemes*. We provide a general account of the theory of Zariskian schemes and the associated rigid spaces, the so-called *rigid Zariskian spaces*, in the appendices I, §B and II, §D.

[13] The reader might note that this idea is also related to the cdh-topology in the theory of motivic cohomology.

[14] A. Abbes has recently published another foundational book [1] on rigid geometry, in which, similarly to ours, he developed and generalized Raynaud’s approach to rigid geometry.
• **Relation with Tate’s rigid analytic geometry.** Let $V$ be an $a$-adically complete valuation ring of height one, and set $K = \text{Frac}(V)$ (the fraction field), which is a complete non-Archimedean valued field with a non-trivial valuation $\| \cdot \|: K \to \mathbb{R}_{\geq 0}$. In II, §8.2. (c) we will define the notion of *classical points* (in the sense of Tate) for rigid spaces of a certain kind, including locally of finite type rigid spaces over $S = (\text{Spf } V)^{\text{rig}}$. If $X$ is a rigid space of the latter kind, it will turn out that the classical points of $X$ are reduced zero-dimensional closed subvarieties in $X$ (cf. II.8.2.6).

We define $X_0$ to be the set of all classical points of $X$. The assignment $X \mapsto X_0$ has several nice properties, some of which are incorporated into the notion of *(continuous) spectral functor* (cf. II, §8.1). Among them is the important property that classical points detect quasi-compact open subspaces: for quasi-compact open subspaces $U, V \subseteq X$, $U_0 = V_0$ implies $U = V$. In view of all this, one can introduce on $X_0$ a Grothendieck topology $\tau_0$ and sheaf of rings $\mathcal{O}_{X_0}$, which are naturally constructed from the topology and the structure sheaf of $X$; for example, for a quasi-compact open subspace $U \subseteq X$, $U_0$ is an admissible open subset of $X_0$, and we have $\mathcal{O}_{X_0}(U_0) = \mathcal{O}_X((U))$. It will turn out that the resulting triple $X_0 = (X_0, \tau_0, \mathcal{O}_{X_0})$ is a Tate rigid analytic variety over $K$, and thus one has the functor $X \mapsto X_0$ from the category of locally of finite type rigid spaces over $S$ to the category of rigid analytic varieties over $K$.

**Theorem** (Theorem II.B.2.5, Corollary II.B.2.6). The functor $X \mapsto X_0$ is a categorical equivalence from the category of quasi-separated locally of finite type rigid spaces over $S = (\text{Spf } V)^{\text{rig}}$ to the category of quasi-separated Tate analytic varieties over $K$. Moreover, under this functor, affinoids (resp. coherent spaces) correspond to affinoid spaces (resp. coherent analytic spaces).

Note that Raynaud’s theorem (the existence of formal models) gives the canonical quasi-inverse functor to the above functor.\(^{15}\)

• **Relation with Huber’s adic geometry.** As we have already remarked above, the Zariski–Riemann triple $\text{ZR}(X)$, at least in the situation as before, is an adic space. This is true in much more general situation, for example, in case $X$ is *locally universally Noetherian* (II.2.2.23). In fact, by the enriched visualization, we have the functor $\text{ZR}: X \mapsto \text{ZR}(X)$.

\(^{15}\)To show the theorem, we need the Gerritzen–Grauert theorem [45], which we assume whenever discussing Tate’s rigid analytic geometry. Note that, when it comes to the rigid geometry over valuation rings, this volume is self-contained only with this exception. We will prove Gerritzen–Grauert theorem without a circular argument in the next volume.
from the category of locally universally Noetherian rigid spaces to the category of adic spaces (Theorem II.A.5.1), which gives rise to a categorical equivalence in the most important cases. In particular, we have the following theorem.

**Theorem** (Theorem II.A.5.2). Let $S$ be a locally universally Noetherian rigid space. Then $\text{ZR}$ establishes a categorical equivalence from the category of locally of finite type rigid spaces over $S$ to the category of adic spaces locally of finite type over $\text{ZR}(S)$.

- **Relation with Berkovich analytic geometry.** Let $V$ and $K$ be as before. We will construct a natural functor

$$\mathcal{X} \mapsto \mathcal{X}_B$$

from the category of locally quasi-compact and locally of finite type rigid spaces over $S = (\text{Spf } V)^{\text{rig}}$ to the category of strictly $K$-analytic spaces (in the sense of Berkovich).

**Theorem** (Theorem II.C.6.12). The functor $\mathcal{X} \mapsto \mathcal{X}_B$ establishes a categorical equivalence from the category of all locally quasi-compact locally of finite type rigid spaces over $(\text{Spf } V)^{\text{rig}}$ to the category of all strictly $K$-analytic spaces. Moreover, $\mathcal{X}_B$ is Hausdorff (resp. paracompact Hausdorff, resp. compact Hausdorff) if and only if $\mathcal{X}$ is quasi-separated (resp. paracompact and quasi-separated, resp. coherent).

The underlying topological space of $\mathcal{X}_B$ is what we call the *separated quotient* (II, §4.3. (a)) of $(\mathcal{X})$, denoted by $[\mathcal{X}]$, which comes with the quotient map

$$\text{sep}_\mathcal{X}: (\mathcal{X}) \to [\mathcal{X}]$$

(*separation map*). In particular, the topology of $\mathcal{X}_B$ is the *quotient* topology of the topology of $(\mathcal{X})$.

Figure 6 illustrates the interrelations among the theories we have discussed so far. In the diagram,

- the functors (⋆1) and (⋆2) are fully faithful; the functor (⋆3), defined on locally quasi-compact rigid analytic spaces, is fully faithful to the category of strictly $K$-analytic spaces;
- the functor (⋆4): $\mathcal{X} \to \mathcal{X}_0$, defined on locally of finite type rigid spaces over $(\text{Spf } V)^{\text{rig}}$, is quasi-inverse to (⋆1) restricted on quasi-separated spaces;

\footnote{Note that, if $\mathcal{X}$ is quasi-separated, then $\mathcal{X}$ is locally quasi-compact if and only if $(\mathcal{X})$ is taut in the sense of Huber, 5.1.2 in [61] (cf. 0.2.5.6).}
• the functor \((\ast 5)\) is given by the enriched visualization, defined on locally universally Noetherian rigid spaces; it is fully faithful in practical situations, including those of locally of finite type rigid spaces over a fixed locally universally Noetherian rigid space, and of rigid spaces of type (N) (II.A.5.3);

• the functor \((\ast 6): \mathcal{X} \mapsto \mathcal{X}_{\text{B}}, \) defined on locally quasi-compact locally of finite type rigid spaces over \((\text{Spf } V)^{\text{rig}}, \) establishes a categorical equivalence with the category of strictly \(K\)-analytic spaces.

Finally, we would like to mention that it has recently become known to experts in the field that it is possible that some of the non-Archimedean spaces that arise naturally in contemporary arithmetic geometry cannot be handled in Berkovich’s analytic geometry (see e.g. [57], 4.4). This state of affair makes it important to investigate in detail the relationship between Berkovich’s analytic geometry and rigid geometry (or adic geometry). In II, §C.5, we will study a spectral theory of filtered rings and introduce a new category of spaces, the so-called metrized analytic spaces. This new notion of spaces generalizes Berkovich’s \(K\)-analytic spaces, and gives a clear picture of the comparison; see II, §C.6. (d). Also, the newly introduced spaces turn out to be equivalent to Kedlaya’s reified adic spaces [67], to which our filtered ring approach in this book offers a new perspective.

8. Applications. We expect that our rigid geometry will have rich applications, not only in arithmetic geometry, but also in various other fields. A few of them have already been sketched in [42], which include

• arithmetic moduli spaces (e.g. Shimura varieties) and their compactifications;

• trace formula in characteristic \(p > 0\) (Deligne’s conjecture).
In addition to these, since our theory has set out from Zariski’s birational geometry, applications to problems in birational geometry, modern or classical, are also expected. For example, this volume already contains Nagata’s compactification theorem for schemes and a proof of it (II, §F), as an application of the general idea of our rigid geometry to algebraic geometry.

Some other prospective applications may be to $p$-adic Hodge theory (cf. [91] and [92]) and to rigid cohomology theory for algebraic varieties in positive characteristic. Here the visualization in our sense of rigid spaces will give concrete pictures for tubes and the dagger construction. As one application in this direction one can mention

- $p$-adic Hodge theory vs. rigid cohomology.

Finally, let us mention that the applications to

- moduli of Galois representations,
- mirror symmetry,

the second of which has been first envisaged by M. Kontsevich, should be among the future challenges.

9. Contents of this book. We followed two basic rules in designing the contents of this book, both of which may justify its length. First, in addition to being a front-line exposition presenting new theories and results, we hope that this book will serve as an encyclopedic source. It contains, consequently, as many notions and concepts, hopefully with only few omission, that should come about as basic and important ones for present and future use, as possible.

Second, we have aimed at making this book as self-contained as possible. All results that sit properly inside the main body of our arguments are always followed by proofs, except for some minor or not-too-difficult lemmas, some of which are placed at the end of each section as exercises; even in this case, if the result is used in the main text, we give a detailed hint in the end of the book, which, in many cases, almost proves the assertion. Note that, because of several laborious requirements on the groundwork, such as removing the Noetherian hypothesis, are also self-contained many of the preliminary parts.

This volume consists of the following three chapters:

- Chapter 0. Preliminaries
- Chapter I. Formal geometry
- Chapter II. Rigid spaces

Let us briefly sketch the contents of each chapter. More detailed summaries will be given at the beginning of each chapter.
Chapter 0 collects preliminaries, which, however, contain also new results. Sections 0, §1 to 0, §7 give necessary preliminaries on set theory, category theory, general topology, homological algebra, etc. In the general topology section, we put emphasis on Stone duality between topological spaces and lattices. In 0, §8 and 0, §9, we will conduct thorough study of topological and algebraic aspects of topological rings and modules. This part of the preliminaries will be the bases of the next chapter, the general theory of formal geometry.

Chapter I is devoted to formal geometry. The essential task here is to treat non-Noetherian formal schemes of a certain kind, e.g., finite type formal schemes over an $a$-adically complete valuation ring of arbitrary height, for reasons of functoriality (as stated in 4. above). Since this kind of generalities seem to be missing in the past literature, we provide a self-contained and systematic theory of formal geometry, generalizing many of the theorems in [54], III. To this end, we introduce several new notions of finiteness condition outside the ideal of definition and show that they allow one to build a versatile theory of formal schemes.

Chapter II is the main part of this volume, in which we develop rigid geometry, based on the foundational work done in the previous chapter. The geometrical theory of rigid spaces that we treat in this chapter includes

- cohomology theory of coherent sheaves (II, §5, §6); finiteness (II.7.5.19);
- local and global study of morphisms (II, §7);
- classification of points (II, §8, §11.1);
- GAGA (II, §9);
- relations with other theories (II, §A, §B, §C).

There are of course many other important topics that are not dealt with in this volume. Some of them, including several important applications, will be contained in the future volumes.

10. Use of algebraic spaces. In I, §6 we develop a full-fledged theory of formal algebraic spaces. It is, in fact, one of the characteristic features of our approach to rigid geometry that we allow formal algebraic spaces, not only formal schemes, to be formal models of rigid spaces. The motivation mainly comes from the applications to algebraic geometry.

In algebraic geometry, while it is often difficult to show that spaces, such as moduli spaces, are represented by schemes, the representability by algebraic spaces is relatively easy to establish, thanks to M. Artin’s formal algebraization theorem [6]. Therefore, taking algebraic spaces into the scope increases the flexibility of the theory. In order to incorporate algebraic spaces into our rigid geometry, one first needs to discuss formal algebraic spaces, some of which appear as the formal
completion of algebraic spaces, and then proceed to the rigid spaces associated to them. Now the important fact is that, although formal algebraic spaces seem to constitute, via Raynaud’s recipe, a new category of rigid spaces that enlarges the already existing category of rigid spaces derived from formal schemes, they actually do not; viz., we do not have to enlarge the category of rigid spaces by this generalization. This is explained by the following theorem, which we shall prove in the future volume.

**Theorem (equivalence theorem).** Let $X$ be a coherent adic formal algebraic space of finite ideal type. Then there exists an admissible blow-up $X' \to X$ from a formal scheme $X'$. Therefore, the canonical functor

$$
\begin{align*}
\left\{ \text{coherent adic formal schemes of finite ideal type} \right\} & \left/ \left\{ \text{admissible blow-ups} \right\} \right. \\
\left\{ \text{coherent adic formal algebraic spaces of finite ideal type} \right\} & \left/ \left\{ \text{admissible blow-ups} \right\} \right.
\end{align*}
$$

is a categorical equivalence.

The theorem shows that, up to admissible blow-ups, formal algebraic spaces simply fall into the class of formal schemes, and thus define the associated rigid space $X^\text{rig}$ just ‘as usual.’ As for GAGA, we can generalize the definition of GAGA functor for algebraic spaces (using a compactification theorem of Nagata type for algebraic spaces).\(^{17}\)

**11. Properness in rigid geometry.** In rigid geometry, we have the following three natural definitions of properness. A morphism $\varphi: \mathcal{X} \to \mathcal{Y}$ of coherent rigid spaces is proper if either one of the following conditions is satisfied.

1. $\varphi$ is universally closed (II.7.5.4), separated, and of finite type.

2. **Raynaud properness.** There exists a proper formal model $f: X \to Y$ of $\varphi$.

3. **Kiehl properness.** $\varphi$ is separated of finite type, and there exist an affinoid covering $\{U_i\}_{i \in I}$ and, for each $i \in I$, a pair of finite affinoid coverings $\{\mathcal{V}_{ij}\}_{j \in J_i}$ and $\{\mathcal{V}'_{ij}\}_{j \in J_i}$ of $\varphi^{-1}(U_i)$ indexed by a common set $J_i$ such that, for any $j \in J_i$, $\mathcal{V}_{ij} \subseteq \mathcal{V}'_{ij}$ and $\mathcal{V}_{ij}$ is relatively compact in $\mathcal{V}'_{ij}$ over $U_i$ (in the sense of Kiehl).

\(^{17}\)This ‘analytification of algebraic spaces’ was already considered in depth and developed by B. Conrad and M. Temkin [31] over complete non-Archimedean valued fields.
Historically, properness in Tate’s rigid geometry has been first defined by R. Kiehl by condition (3) in his work [69] on finiteness theorem. This condition, existence of affinoid enlargements, stems from the general idea by Cartan and Serre and by H. Grauert for proving finiteness of cohomologies of coherent sheaves. While the equivalence of (1) and (2) is an easy exercise, the equivalence of (2) and (3), especially the implication (2) \( \Rightarrow \) (3), is a very deep theorem. Lütkebohmert’s 1990 paper [78] proves this for rigid spaces of finite type over \((\text{Spf} \, V)\text{\textsuperscript{rig}}\), where \(V\) is a complete discrete valuation ring. In this book, we temporarily define properness by condition (1) (and hence equivalently by (2)), and postpone the proof of the equivalence of these three conditions, especially the implication (2) \( \Rightarrow \) (3), in the so-called adhesive case (II.2.2.23), to the next volume, in which we will show the Enlargement Theorem by expanding Lütkebohmert’s technique.

12. Contents of the future volumes. Our project will continue in future volumes. The next volume will contain the following chapters.

- **Chapter III. Formal flattening theorem**
  This chapter will also contain several applications of the formal flattening theorem, such as Gerritzen–Grauert theorem.

- **Chapter IV. Enlargement theorem**
  This chapter will contain the proof of the equivalence of the three ‘definitions’ of properness.

- **Chapter V. Equivalence theorem and analytification of algebraic spaces**
  This chapter will give the proof of the equivalence theorem stated above and the definition of the GAGA functor for algebraic spaces.

13. General conventions. Chapter numbers are bold-face Roman, while for sections and subsections we use Arabic numbers; subsubsections are numbered by letters in parentheses; for example, ‘I, §3.2. (b)’ refers to the second subsubsection of the second subsection in §3 of Chapter I. Cross-references will be given by sequences of numerals, like I.3.2.1, which specify the places of the statements in the text. The chapter numbers are omitted when referring to places in the same chapter. Almost all sections are equipped with some exercises at the end, which are selected in order to help the reader understand the content. We insert hints for some of the exercises at the end of this volume.
Let us list some mathematical conventions.

- We fix once for all a Grothendieck universe $U$ ([8], Exposé I, 0); cf. 0, §1.1. (a).

- By a Grothendieck topology (or simply by a topology) on a category $\mathcal{C}$ we always mean a functor $J : x \mapsto J(x)$, assigning to each $x \in \text{obj}(\mathcal{C})$ a collection of sieves, as in [80], III, §2, Definition 1. In many places, however, Grothendieck topologies are introduced by means of a base (covering families) as in [80], III, §2, Definition 2, (*prétopologie* in the terminology in [8], Exposé II, (1.3)); in this situation, we consider, without explicit mentioning, the Grothendieck topology generated by the base.

- A site will always mean a $U$-site (cf. [8], Exposé II, (3.0.2)), that is, a pair $(\mathcal{C}, J)$ consisting of a $U$-category $\mathcal{C}$ ([8], Exposé I, Definition 1.1) and a Grothendieck topology on $\mathcal{C}$.

- All compact topological spaces are assumed to be Hausdorff; that is, we adopt the Bourbaki convention

\[
\text{quasi-compact} + \text{Hausdorff} = \text{compact}.
\]

However, we sometimes use the term ‘compact Hausdorff’ just for emphasis. Other conventions, in which we do not follow Bourbaki, are the following ones.

- Locally compact spaces are only assumed to be *locally* Hausdorff;\(^{18}\) A topological space $X$ is said to be *locally compact* if every point of $X$ has a compact neighborhood contained in a Hausdorff neighborhood.

- Paracompact spaces are *not* assumed to be Hausdorff; see 0, §2.5. (c).

- Whenever we say $A$ is a ring, we always mean, unless otherwise stipulated that $A$ is a commutative ring having the multiplicative unit $1 = 1_A$. We also assume that any ring homomorphism $f : A \rightarrow B$ is unital, that is, maps $1_A$ to $1_B$. Moreover,
  - for a ring $A$ we denote by $\text{Frac}(A)$ the total ring of fractions of $A$;
  - for a ring $A$ the Krull dimension of $A$ is denoted by $\text{dim}(A)$;
  - when $A$ is a local ring, its unique maximal ideal is denoted by $m_A$.

- Let $A$ be a ring and $I \subseteq A$ an ideal. When we say $A$ is $I$-adically complete or complete with respect to the $I$-adic topology, we always mean, unless otherwise stipulated that $A$ is *Hausdorff complete* with respect to the $I$-adic topology.

\(^{18}\)Note that, in [24], Chapter I, §9.7, Definition 4, locally compact spaces are assumed to be Hausdorff.
By an exact functor between derived categories (of any sort) we always mean an exact functor of triangulated categories that preserves the canonical $t$-structures (hence also the canonical cohomology functors), which are clearly specified by the context.

We will often use, by abuse of notation, the equality symbol ‘$\cong$’ for ‘isomorphic by a canonical morphism.’

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Chapter 0

Preliminaries

This chapter collects basic notions and results from various fields, which are prepared not only for the rest of this volume, but also for later volumes. In spite of its preliminary nature, the chapter contains new notions, results and techniques. Sections 1–7 are devoted to the background concepts and results in set theory, category theory, general topology, homological algebras, ringed spaces, schemes and algebraic spaces, valuation rings, and topological rings and modules. One of the new items that come into play in these sections is the notion of valuative spaces, discussed in §2.3. It will be shown later, in Chapter II, that valuative spaces provide a nice topological visualization (cf. Introduction) of rigid spaces; viz., the Zariski–Riemann spaces associated to rigid spaces are all valuative. In this sense, valuative spaces should be regarded as an abstraction, in non-Archimedean geometry, of spectral spaces, leading to a spectral geometry that arises from non-Archimedean geometries. We will see that valuative spaces have many rich topological structures, such as separated quotients, overconvergent subsets, tube subsets, etc.

In Sections 8 and 9 we give a general treatment of adically topologized rings and modules. The main object of these sections is commutative rings equipped with an adic topology defined by a finitely generated ideal. At first, we develop the theory of these objects in the most general setting, and later, we will consider various kinds of finiteness conditions imposed outside the ideal of definition. This ‘finiteness conditions outside I’ will form the central part of our discussion. For example, \( I \)-adically complete and Noetherian-outside-\( I \) rings form a nice class of adic rings that enjoy many of the pleasant properties known to be satisfied by Noetherian topological rings (due to a theorem by Gabber, Theorem 8.2.19), such as ‘preservation of adicness,’ a property similar to the Artin–Rees property. Of particular importance among these topological rings are what we call topologically universally pseudo-adhesive and topologically universally adhesive (abbr. t.u. adhesive) rings. All these new notions and techniques will provide the basis for the next chapter, in which we develop the general theory of formal schemes.

In Section 9 we focus on topologically of finite type algebras over \( V \), a valuation ring that is \( a \)-adically complete by a non-zero \( a \in \mathfrak{m}_V \). It will be shown, using another theorem by Gabber, that such an algebra is always t.u. adhesive (Corollary 9.2.7). Note that here we do not assume that the valuation ring \( V \)
is of height one. We will moreover show several ring-theoretically important results, such as the Noether normalization theorem in the height-one situation (Theorem 9.2.10).

In the end of this chapter, we provide appendices on further concepts and techniques including Huber’s f-adic rings and basics on derived categories.

1 Basic Languages

This section gives a short glossary of set theory and category theory. Like in the modern approaches to algebraic geometry, we postulate Grothendieck’s axiom (UA) in [8], Exposé I, 0, on existence of Grothendieck universe and fix one universe once for all. Some of the related technical notions, such as U-small sets and U-categories, are briefly reviewed in §1.1 and §1.2. In §1.3 we discuss limits and colimits, especially filtered (cofiltered) and essentially small limits (colimits). The final subsection gives an overview of general categorical frameworks for several stabilities of properties of arrows, which are mainly taken from [72], I, and further developed.

1.1 Sets and ordered sets

1.1. (a) Sets. In this book, we entirely work in the ZFC set theory (cf. [36], Chapter 1, §3) with the language of classes (cf. [36], Chapter 1, §5.3), assuming Grothendieck’s axiom (UA) for existence of Grothendieck universes ([8], Exposé I, 0). We fix once for all a Grothendieck universe U containing at least one infinite ordinal. As in [8], Exposé I, 1.0, a set x is said to be U-small, if it is isomorphic to a member of U.

1.1. (b) Ordered sets and order types. Recall that an ordering on a (not necessarily U-small) set X is a relation ≤ on X satisfying the following conditions:

(O1) \( x \leq x \), for any \( x \in X \);

(O2) \( x \leq y \) and \( y \leq x \) imply \( x = y \), for any \( x, y \in X \);

(O3) \( x \leq y \) and \( y \leq z \) imply \( x \leq z \), for any \( x, y, z \in X \).

As usual, we write \( x < y \) if \( x \leq y \) and \( x \neq y \).

A (partially) ordered set (poset) is a pair \((X, \leq)\) consisting of a set and an ordering. It is clear that an ordered set \((X, \leq)\) is U-small (resp. a member of U) if and only if so is the set X.

For an ordered set \( X = (X, \leq) \), we denote by \( X^{\text{opp}} = (X, \leq^{\text{opp}}) \) the ordered set having the same underlying set X with all the inequalities reversed, that is, for \( x, y \in X \), \( x \leq^{\text{opp}} y \) if and only if \( y \leq x \).
Let \((X, \leq)\) and \((Y, \leq)\) be ordered sets. A map \(f : X \to Y\) is said to be **ordered** if, for any \(x, y \in X\) with \(x \leq y\), we have \(f(x) \leq f(y)\).

An ordered set \((X, \leq)\) is said to be **totally ordered** if, for any \(x, y \in X\), either one of \(x < y\), \(x = y\), or \(x > y\) holds. For example, any ordinal (cf. [36], Chapter 2, §2) is a totally ordered set by the membership relation \(\in\).

Consider the set \(\text{U-Ord}\) of all isomorphism classes of totally ordered sets belonging to \(\text{U}\). For a totally ordered set \((X, \leq)\) in \(\text{U}\), the unique element \(\rho \in \text{U-Ord}\) such that \((X, \leq) \in \rho\) is called the **order type** of \((X, \leq)\). Any finite ordinal \(n = \{0, 1, \ldots, n - 1\}\) defines a unique order type, denoted again by \(n\). Order types of this form are said to be **finite**.

### 1.2 Categories

#### 1.2. (a) Conventions

In this book, categories are always considered within set theory; our standard reference on category theory is [79]. For a category \(\mathcal{C}\) we denote by \(\text{obj}(\mathcal{C})\) the class of objects of \(\mathcal{C}\), and for each pair \((x, y)\) of objects of \(\mathcal{C}\), we denote by \(\text{Hom}_\mathcal{C}(x, y)\) the class of arrows from \(x\) to \(y\). A category \(\mathcal{C}\) is called a **U-category** if \(\text{Hom}_\mathcal{C}(x, y)\) is \(\text{U}\)-small for any \(x, y \in \text{obj}(\mathcal{C})\) ([8], Exposé I, Definition 1.1). Almost all categories in this book are \(\text{U}\)-categories; moreover, they most of the time satisfy the following conditions ([8], Exposé I, 1.1.2):

1. **(C1)** the class of objects \(\text{obj}(\mathcal{C})\) is a subset of \(\text{U}\);
2. **(C2)** for any \(x, y \in \text{obj}(\mathcal{C})\), the set \(\text{Hom}_\mathcal{C}(x, y)\) is a member of \(\text{U}\).

Let \(\mathcal{C}\) be a \(\text{U}\)-category. For each pair \((x, y)\) of objects, we denote by \(\text{Isom}_\mathcal{C}(x, y)\) the subset of \(\text{Hom}_\mathcal{C}(x, y)\) consisting of all isomorphisms. Also, for an object \(x\), we write \(\text{End}_\mathcal{C}(x) = \text{Hom}_\mathcal{C}(x, x)\) and \(\text{Aut}_\mathcal{C}(x) = \text{Isom}_\mathcal{C}(x, x)\).

For a category \(\mathcal{C}\), we denote by \(\mathcal{C}^{\text{opp}}\) the **opposite category** of \(\mathcal{C}\) ([79], Chapter II, §2), that is, the category such that

1. \(\text{obj}(\mathcal{C}^{\text{opp}}) = \text{obj}(\mathcal{C})\) and
2. \(\text{Hom}_{\mathcal{C}^{\text{opp}}}(x, y) = \text{Hom}_\mathcal{C}(y, x)\) for any \(x, y \in \text{obj}(\mathcal{C})\).

By a functor \(\mathcal{C} \to \mathcal{D}\) we always mean a **covariant** functor, unless otherwise clearly stated. Contravariant functors will be written as a covariant functor from the opposite category of the domain category.

#### 1.2. (b) Frequently used categories

The following categories are frequently used in this book:

- **Sets** = the category of all sets in \(\text{U}\);
- **Top** = the category of all topological spaces in \(\text{U}\);
- **Ab** = the category of all abelian groups in \(\text{U}\).
For a ring (commutative with unit) \( A \):

- \( \text{Mod}_A \) = the category of all \( A \)-modules in \( U \);
- \( \text{Alg}_A \) = the category of all \( A \)-algebras in \( U \).

These are \( U \)-categories satisfying \((C1)\) and \((C2)\) in 1.2. (a).

1.2. (c) **Functor category.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. We denote by \( \mathcal{C}^\mathcal{D} \) the category of functors from \( \mathcal{D} \) to \( \mathcal{C} \) and natural transformations. Note that (cf. [8], Exposé I, 1.1.1):

1. if both \( \mathcal{C} \) and \( \mathcal{D} \) are members of \( U \) (resp. \( U \)-small), then so is \( \mathcal{C}^\mathcal{D} \);
2. if \( \mathcal{D} \) is \( U \)-small and \( \mathcal{C} \) is a \( U \)-category, then \( \mathcal{C}^\mathcal{D} \) is a \( U \)-category.

1.2. (d) **Groupoids and discrete categories.** A category \( \mathcal{C} \) is a **groupoid** if all arrows are isomorphisms. \( \mathcal{C} \) is **discrete** if the set \( \text{Hom}_{\mathcal{C}}(x, y) \) is empty unless \( x = y \) and, for each object \( x \), we have \( \text{End}_{\mathcal{C}}(x) = \{ \text{id}_x \} \). Note that a discrete category \( \mathcal{C} \) is completely determined by its class of objects.

1.2. (e) **Category associated to an ordered set.** To an ordered set \( X = (X, \leq) \) one canonically associates a category, denoted again by \( X \), whose objects are the elements of \( X \), and for \( x, y \in X \),

\[
\text{Hom}_X(x, y) = \begin{cases} 
\{ (x, y) \} & \text{if } x \leq y, \\
\emptyset & \text{otherwise.}
\end{cases}
\]

If the set \( X \) is a member of \( U \) (resp. \( U \)-small), then so is the associated category. Note that the category associated to the inverse ordered set \( X^{\text{opp}} \) (§1.1. (b)) is the opposite category of the category associated to \( X \) (§1.2. (a)).

1.3 **Limits**

1.3. (a) **Definition and universal property.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. The **diagonal functor**

\[
\Delta: \mathcal{C} \longrightarrow \mathcal{C}^\mathcal{D}
\]

is the functor defined as follows: \( \Delta(x): \mathcal{D} \rightarrow \mathcal{C} \) for \( x \in \text{obj}(\mathcal{C}) \) is the constant functor given by \( \Delta(x)(y) = x \) for any object \( y \) of \( \mathcal{D} \), and \( \Delta(x)(f) = \text{id}_x \) for any arrow \( f \) of \( \mathcal{D} \). We denote the right (resp. left) adjoint to \( \Delta \), if it exists, by \( \text{lim} \) (resp. \( \text{lim}^\leftarrow \)). For a functor \( F: \mathcal{D} \rightarrow \mathcal{C} \), the object \( \text{lim} F \) (resp. \( \text{lim}^\leftarrow F \)) is called the **limit** (resp. **colimit**) of \( F \).
To describe the mapping universality of the limit \( \lim F \), consider a morphism \( \Delta(x) \rightarrow F \) of functors, which amounts to the same as a collection of arrows \( x \rightarrow F(y) \) in \( \mathcal{C} \) for all \( y \in \text{obj}(\mathcal{D}) \), such that for any arrow \( f : y \rightarrow z \) in \( \mathcal{D} \) the resulting triangle

\[
\begin{array}{ccc}
F(y) & \xrightarrow{F(f)} & F(z) \\
x & \downarrow & \\
F(f) & \xleftarrow{x} & \end{array}
\]

is commutative. Replacing \( x \) by \( \lim F \), and considering the adjunction morphism \( \Delta(\lim F) \rightarrow F \), one gets the compatible collection of arrows \( \lim F \rightarrow F(y) \) (\( y \in \text{obj}(\mathcal{D}) \)). The limit \( \lim F \) is then characterized up to isomorphism by the following universal mapping property: whenever a collection of arrows \( x \rightarrow F(y) \) as above is given, there exists a unique arrow \( x \rightarrow \lim F \) such that the diagram

\[
\begin{array}{ccc}
& \lim F & \\
x & \xleftarrow{} & F(y) \\
& \downarrow & \\
& \end{array}
\]

is commutative for any \( y \in \text{obj}(\mathcal{D}) \).

The mapping universality of the colimit \( \lim F \) can be described similarly; the details are left to the reader.

1.3. (b) Limits over ordered sets. We will most frequently deal with limits and colimits with the index category being an ordered set (§1.2.(e)). If \( I \) is an ordered set, then the functor \( F : I \rightarrow \mathcal{C} \) as above amounts to the same as what is usually called an inductive system (synonym: direct system) \( \{X_i, f_{ij}\} \) of objects and arrows in \( \mathcal{C} \). Similarly, a functor of the form \( G : I^\text{opp} \rightarrow \mathcal{C} \) corresponds to what is called a projective system (synonym: inverse system). The corresponding limits, inductive and projective, are respectively written in the usual way as

\[
\lim_{i \in I} X_i \quad \text{and} \quad \lim_{i \in I} X_i.
\]

1.3. (c) Final and cofinal functors. A category \( I \) is said to be filtered if it is non-empty and satisfies the following conditions:

(a) for any \( x, x' \in \text{obj}(I) \), there exist \( y \in \text{obj}(I) \) and arrows \( x \rightarrow y \) and \( x' \rightarrow y \);

(b) given two arrows \( f, f' : x \rightarrow y \), there exists \( g : y \rightarrow z \) such that \( g \circ f = g \circ f' \).
An ordered set $I$ is said to be directed if any finite subset has an upper bound. If $I$ is a directed set, then it is, viewed as a category as in §1.2.(e), a filtered category.

A functor $L: J \to I$ between filtered categories is said to be final if

(F) for any $i \in \text{obj}(I)$, there exist $j \in \text{obj}(J)$ and an arrow $i \to L(j)$.

**Proposition 1.3.1** ([79], Chapter IX, §3, Theorem 1). Let $L: J \to I$ be a final functor between filtered categories, and $F: I \to \mathcal{C}$ a functor. If $\lim F \circ L$ exists, then so does $\lim F$, and the canonical morphism

$$\lim(F \circ L) \longrightarrow \lim F$$

is an isomorphism.

Thus, when taking the colimit, one can replace the index category $I$ by a category $J$ that admits a final functor $J \to I$. If such a category $J$ can be chosen as a directed set, the index category $I$ is said to be essentially small. In this case, the limit ‘along category’ can be replaced by a limit ‘along set,’ which is, needless to say, easier to handle.

Similarly, by duality, one has the notions of cofiltered categories and cofinal functors $L: J \to I$. For example, if $I$ is a directed set, the category $I^{\text{opp}}$ is cofiltered. A cofiltered category $I$ is said to be essentially small if it admits a cofinal functor $J^{\text{opp}} \to I$, where $J$ is a directed set. The dual statement of 1.3.1 holds alike.

**Proposition 1.3.2.** Let $L: J \to I$ be a cofinal functor between cofiltered categories, and $F: I \to \mathcal{C}$ a functor. If $\lim F \circ L$ exists, then so does $\lim F$, and the canonical morphism

$$\lim F \longrightarrow \lim(F \circ L)$$

is an isomorphism.

### 1.4 Several stabilities for properties of arrows

#### 1.4. (a) Base-change stability

Let $\mathcal{C}$ be a category. We consider a subcategory $\mathcal{D}$ of $\mathcal{C}$ containing all isomorphisms of $\mathcal{C}$ (hence, in particular, $\text{obj}(\mathcal{D}) = \text{obj}(\mathcal{C})$). Typically, such a subcategory comes about in the following way. Let $P$ be a property of arrows in the category $\mathcal{C}$ such that

(I) any isomorphism satisfies $P$ and

(C) if $a: x \to y$ and $b: y \to z$ satisfy $P$, then the composite $b \circ a$ satisfies $P$.

Then the subcategory consisting of all arrows satisfying $P$, denoted by $\mathcal{D}_P$, is a subcategory of the above type.
Proposition 1.4.1. Let $\mathcal{D}$ and $\mathcal{E}$ be subcategories of $\mathcal{C}$ containing all isomorphisms. We assume that

- if $a: x \to y$ belongs to $\mathcal{E}$, then for any $b: z \to y$, the fiber product $x \times_y z$ is representable in $\mathcal{C}$, and the arrow $x \times_y z \to z$ belongs to $\mathcal{E}$.

Consider the following conditions.

(B_1) Suppose $x \xrightarrow{a} x'$ and $y \xrightarrow{b} y'$ are commutative diagrams in $\mathcal{C}$ such that (i) the arrows $a$ and $b$ are in $\mathcal{D}$, and (ii) either the arrows $x \to z$ and $x' \to z$ or the arrows $x \to z$ and $y' \to z$ belong to $\mathcal{E}$. Then the induced arrow

$$a \times_z b: x \times_z y \to x' \times_z y'$$

belongs to $\mathcal{D}$.

(B_2) Suppose a commutative diagram $x \xrightarrow{a} y$ in $\mathcal{C}$ and an arrow $z' \to z$ in $\mathcal{C}$ are given such that the arrow $a$ belongs to $\mathcal{D}$. Suppose, moreover, that either one of the following holds: (i) the arrows $x \to z$ and $y \to z$ belong to $\mathcal{E}$, or (ii) the arrow $z' \to z$ belongs to $\mathcal{E}$. Then the induced arrow

$$a_{z'}: x \times_z z' \to y \times_z z'$$

belongs to $\mathcal{D}$.

(B_3) Suppose a diagram $x \xrightarrow{a} y \leftarrow y'$ in $\mathcal{C}$ is given such that (i) the arrow $a$ belongs to $\mathcal{D}$, and (ii) either the arrow $x \to y$ or the arrow $y' \to y$ belongs to $\mathcal{E}$. Then the induced arrow

$$a_{y'}: x \times_y y' \to y'$$

belongs to $\mathcal{D}$.

Then we have the implications

$$(B_1) \iff (B_2) \implies (B_3).$$

If we assume, moreover, that

- for arrows $a: x \to y$ and $b: y \to z$ in $\mathcal{C}$, if $b \circ a$ and $b$ belong to $\mathcal{E}$, then $a$ belongs to $\mathcal{E}$,

then it also holds the implication $(B_3) \implies (B_2)$.  

Proof. First let us show \((B_1) \implies (B_2)\). In the situation as in \((B_2)\), if the arrows \(x \to z\) and \(y \to z\) belong to \(\mathcal{E}\), apply \((B_1)\) with \(y = y'\) replaced by \(z'\) and \(x'\) replaced by \(y\). If the arrow \(z' \to z\) belongs to \(\mathcal{E}\), apply \((B_1)\) with \(x = x'\) replaced by \(z'\) and with \(y\) and \(y'\) by \(x\) and \(y\), respectively. Conversely, to show \((B_2) \implies (B_1)\), we use the fact that the arrow \(a \times_z b\) coincides with the composites

\[
x \times_z y \longrightarrow x' \times_z y \longrightarrow x' \times_z y' \quad \text{and} \quad x \times_z y \longrightarrow x \times_z y' \longrightarrow x' \times_z y',
\]

cf. [53], (0, 1.3.9). In either case where the arrows \(x \to z\) and \(x' \to z\) or the arrows \(x \to z\) and \(y' \to z\) belong to \(\mathcal{E}\), one can apply \((B_2)\) to show that the arrow \(a \times_z b\) belongs to \(\mathcal{D}\).

Implication \((B_2) \implies (B_3)\) follows easily, since \((B_3)\) is the special case of \((B_2)\) with \(z = y\). To show \((B_3) \implies (B_2)\), we use the fact that the arrow \(a \times' \mathcal{Z}\) is isomorphic to

\[
x \times_y (y \times_z z') \longrightarrow y \times_z z'
\]

cf. [53], (0, 1.3.4)). If the arrows \(x \to z\) and \(y \to z\) belong to \(\mathcal{E}\), then one can apply \((B_3)\) with \(y'\) replaced by \(y \times_z z'\), since \(x \to y\) belongs to \(\mathcal{E}\). If the arrow \(z' \to z\) belongs to \(\mathcal{E}\), one can apply \((B_3)\) with \(y'\) replaced by \(y \times_z z'\), since \(y \times_z z' \to y\) belongs to \(\mathcal{E}\).

\[
\square
\]

**Definition 1.4.2.** Suppose \(\mathcal{C}\) has all fiber products. Then the subcategory \(\mathcal{D}\), or the property \(P\) with \(\mathcal{D} = \mathcal{D}_P\), is base-change stable in \(\mathcal{C}\) if the conditions in 1.4.1 with \(\mathcal{E} = \mathcal{C}\) are satisfied.

1.4.(b) **Topology associated to a base-change stable subcategory.** We insert here a brief review of the Grothendieck topology\(^1\) associated to a base-change stable subcategory (cf. [72], I.1). Let \(\mathcal{C}\) be a category with all fiber products.

**Definition 1.4.3.** (1) A family \(\{u_\alpha \to u\}_{\alpha \in L}\) of arrows in \(\mathcal{C}\) is said to be universally effectively epimorphic if, for any arrow \(w \to u\) in \(\mathcal{C}\) and for any object \(v \in \text{obj}(\mathcal{C})\), the induced sequence of maps

\[
\text{Hom}_\mathcal{C}(w, v) \longrightarrow \prod_{\alpha \in L} \text{Hom}_\mathcal{C}(w \times_u u_\alpha, v) \longrightarrow \prod_{\alpha, \beta \in L} \text{Hom}_\mathcal{C}(w \times_u u_\alpha \beta, v), \quad (*)
\]

where \(u_\alpha \beta = u_\alpha \times_u u_\beta\), is exact.

(2) An arrow \(a: v \to u\) is called a universally effectively epimorphism if \(\{a\}\) is a universally effectively epimorphic family.

**Definition 1.4.4.** Let \(\mathcal{D}\) be a base-change stable subcategory of \(\mathcal{C}\) (1.4.2). The topology associated to \(\mathcal{D}\) (or to \(P\) when \(\mathcal{D} = \mathcal{D}_P\)) is the Grothendieck topology on \(\mathcal{C}\), denoted by \(J_\mathcal{D}\) (or by \(J_P\)), such that the covering families are given by all universally effectively epimorphic families in \(\mathcal{C}\) consisting of arrows in \(\mathcal{D}\).

\(^1\)See General conventions in the introduction for our convention for Grothendieck topologies.
**Proposition 1.4.5.** The topology $J = J_\mathcal{D}$ satisfies the condition

$(A_0)$ any representable presheaf on the site $\mathcal{C}_J = (\mathcal{C}, J)$ is a sheaf.

In other words, the topology $J$ is coarser than the canonical topology on $\mathcal{C}$ (see [8] Exposé II, (2.5), and [80], p. 126).

**Convention.** Throughout this book, whenever a site $\mathcal{C}_J = (\mathcal{C}, J)$ satisfies $(A_0)$, the sheaf on $\mathcal{C}_J$ associated to an object $x \in \text{obj}(\mathcal{C})$ is simply denoted by the same symbol $x$.

There are several other useful conditions for the topology $J = J_\mathcal{D}$, some of which come from properties of the subcategory $\mathcal{D}$. It will be convenient for us to formulate some of them as axioms.

**Definition 1.4.6.** Let $\mathcal{C}$ be a category with all fiber products.

1. An object $\emptyset \in \text{obj}(\mathcal{C})$ is said to be strictly initial if it is an initial object and for any $x \in \text{obj}(\mathcal{C})$ the set $\text{Hom}_\mathcal{C}(x, \emptyset)$ is empty unless $x$ is isomorphic to $\emptyset$. (It can be shown that if a strictly initial object exists, then any initial object is strictly initial.)

2. Let $\{x_\alpha\}_{\alpha \in L}$ be a family of objects in $\mathcal{C}$. The coproduct $\bigsqcup_{\alpha \in L} x_\alpha$ is called the disjoint sum if each $x_\alpha \times_x x_\beta$ for $(\alpha, \beta) \in L \times L$ is a strictly initial object.

Now the axioms for the base-change stable subcategory $\mathcal{D}$ that we will use in the sequel are listed as follows.

**S**$_1$) For a family $\{x_\alpha \to y\}_{\alpha \in L}$ of arrows in $\mathcal{C}$ such that the disjoint sum $x = \bigsqcup_{\alpha \in L} x_\alpha$ exists, the induced arrow $x \to y$ belongs to $\mathcal{D}$ if and only if each $x_\alpha \to x$ belongs to $\mathcal{D}$.

**S**$_2$) An arrow in $\mathcal{D}$ is a universally effectively epimorphism in $\mathcal{C}$ if and only if it is an epimorphism in $\mathcal{C}$.

**S**$_3$) For a commutative diagram $x \xrightarrow{a} y$ in $\mathcal{C}$ such that $c$ belongs to $\mathcal{D}$,

$$
\begin{bmatrix}
\end{bmatrix}
$$

(S$_3$(a)) if $a$ is a covering of $J_\mathcal{D}$, then $b$ belongs to $\mathcal{D}$;

(S$_3$(b)) if $b$ belongs to $\mathcal{D}$, then so does $a$.

1.4. (c) **Stability and effective descent.** Let $\mathcal{C}$ be a category with all fiber products, and $J$ a Grothendieck topology on $\mathcal{C}$ satisfying $(A_0)$ in 1.4.5.

**Definition 1.4.7.** A subclass $\mathcal{S}$ of objects in $\mathcal{C}$ is said to be stable (under $J$) if, for any covering family $\{u_\alpha \to u\}_{\alpha \in L}$, the object $u$ belongs to $\mathcal{S}$ if and only if all $u_\alpha$ for $\alpha \in L$ belong to $\mathcal{S}$. 
Definition 1.4.8. Let $\mathcal{D} \subseteq \mathcal{C}$ be a subcategory of $\mathcal{C}$ containing all isomorphisms in $\mathcal{C}$.

1. The subcategory $\mathcal{D}$ (or $\mathcal{P}$ when $\mathcal{D} = \mathcal{D}_P$) is said to be local on the target (under $\mathcal{J}$) if, for any $a: x \to y$ in $\mathcal{C}$ and for any covering family $\{y_\alpha \to y\}_{\alpha \in L}$, the arrow $a$ belongs to $\mathcal{D}$ if the base change $a_\alpha: y_\alpha \times_y x \to y_\alpha$ belongs to $\mathcal{D}$ for each $\alpha \in L$. If it is, moreover, base-change stable, then $\mathcal{D}$ (or $\mathcal{P}$ when $\mathcal{D} = \mathcal{D}_P$) is said to be stable (under $\mathcal{J}$).

2. The subcategory $\mathcal{D}$ (or $\mathcal{P}$ when $\mathcal{D} = \mathcal{D}_P$) is said to be local on the domain (under $\mathcal{J}$) if it is stable under $\mathcal{J}$ and, for any $a: x \to y$ in $\mathcal{C}$ and for any covering family $\{x_\alpha \to x\}_{\alpha \in L}$, the arrow $a$ belongs to $\mathcal{D}$ if and only if the composite $x_\alpha \to x \to y$ belongs to $\mathcal{D}$ for each $\alpha \in L$.

3. The subcategory $\mathcal{D}$ is called an effective descent class (with respect to $\mathcal{J}$) if it is stable under $\mathcal{J}$ and the following condition is satisfied. Let $\mathcal{F}$ be a set-valued sheaf on $\mathcal{C}_{/\mathcal{J}}$ together with a map $\mathcal{F} \to u$ of sheaves; if each sheaf fiber product $u_\alpha \times_u \mathcal{F}$ is represented by an object $w_\alpha \in \text{obj}(\mathcal{C})$, and if all arrows $w_\alpha \to u_\alpha$ for $\alpha \in L$ belong to $\mathcal{D}$, then $\mathcal{F}$ is representable (consequently, if $w$ represents $\mathcal{F}$, the resulting arrow $w \to u$ belongs to $\mathcal{D}$).

Definition 1.4.9. Let $x \in \text{obj}(\mathcal{C})$.

1. A local construction $\Phi$ on $x$ consists of the following data:

   a) a cofinal set $\mathcal{V} = \{x_\lambda \to x\}_{\lambda \in \Lambda, \alpha \in L}$ of covering families of $x$;

   b) an arrow $\Phi_{\lambda \alpha} \to x_\alpha$ in $\mathcal{C}$ for each $x_\lambda \to x$ in a covering family in $\mathcal{V}$;

   c) an arrow $\Phi_{\lambda \alpha} \to \Phi_{\mu \beta}$ in $\mathcal{C}$ for any arrow $x_\lambda \to x_\mu$ over $x$ such that

   $\Phi_{\lambda \alpha} \to \Phi_{\mu \beta}$

   $x_\lambda \to x_\mu$

   is Cartesian in $\mathcal{C}$.

2. A local construction $\Phi$ on $x$ is said to be effective if there exist an arrow $y \to x$ and isomorphisms $x_\lambda \times_x y \cong \Phi_{\lambda \alpha}$ for all $\alpha \in L$ and $\lambda \in \Lambda$ that are compatible with the arrows $\Phi_{\lambda \alpha} \to \Phi_{\mu \beta}$ as in (c) above.

Proposition 1.4.10 ([72], (1.12)). Let $\Phi$ be a local construction on $x \in \text{obj}(\mathcal{C})$, and $\mathcal{D}$ a stable subcategory of $\mathcal{C}$ that is an effective descent class. Suppose that, for any $x_\lambda \to x$ in a covering family in the cofinal set $\mathcal{V}$ of $\Phi$ as above, the arrow $\Phi_{\lambda \alpha} \to x_\lambda$ belongs to $\mathcal{D}$. Then $\Phi$ is effective, and the arrow $y \to x$ as in 1.4.9 (2) belongs to $\mathcal{D}$. 
1. Basic Languages

1.4. (d) Categorical equivalence relations. Finally, let us include some generalities of categorical equivalence relations (cf. [72], I.5).

Let $\mathcal{C}_J = (\mathcal{C}, J)$ be a site with the underlying category $\mathcal{C}$ with a final object and all fiber products.

**Definition 1.4.11.** A $J$-equivalence relation in $\mathcal{C}$ is a diagram

$$
\begin{array}{ccc}
R & \xrightarrow{p_1} & Y \\
\downarrow p_2 & & \downarrow q \\
Y & & F
\end{array}
$$

consisting of arrows in $\mathcal{C}$ such that

(a) for any object $Z$ of $\mathcal{C}$, the induced map

$$
\text{Hom}_{\mathcal{C}}(Z, R) \to \text{Hom}_{\mathcal{C}}(Z, Y) \times \text{Hom}_{\mathcal{C}}(Z, Y)
$$

is injective and defines an equivalence relation on the set $\text{Hom}_{\mathcal{C}}(Z, Y)$;

(b) the maps $p_1, p_2$ are covering maps with respect to the topology $J$.

Let $\mathcal{C}$ be a category with a final object and all fiber products, and $\mathcal{B}$ a base-change stable subcategory of $\mathcal{C}$ enjoying $(S_1)$, $(S_3(a))$, and $(S_3(b))$ in §1.4.(b). We consider the associated topology $J = J_{\mathcal{B}}$ (1.4.4).

As usual, a map $\mathcal{F} \to \mathcal{G}$ of sheaves on the site $\mathcal{C}_J$ is said to be representable if, for any map $z \to \mathcal{G}$ of sheaves from a representable sheaf, the sheaf fiber product $z \times_{\mathcal{G}} \mathcal{F}$ is representable. Let $\mathcal{D}$ be a base-change stable subcategory of $\mathcal{C}$. We say that a map $\mathcal{F} \to \mathcal{G}$ of sheaves is represented by a map in $\mathcal{D}$ if it is representable and, for any map $z \to \mathcal{G}$ of sheaves from a representable sheaf, the morphism $w \to z$ in $\mathcal{C}$ representing the base-change $z \times_{\mathcal{G}} \mathcal{F} \to z$ belongs to $\mathcal{D}$. If, moreover, $\mathcal{D} = \mathcal{B}$ and all morphisms $w \to z$ as above are covering maps, then we say that $\mathcal{F} \to \mathcal{G}$ is represented by a covering map.

**Proposition 1.4.12 ([72], I.5.5).** Let $p_1, p_2: R \to Y$ be a $J$-equivalence relation in the site $\mathcal{C}_J = (\mathcal{C}, J)$, and $q: Y \to F$ the cokernel of $p_1, p_2$ in the category of sheaves on $\mathcal{C}_J$. Then for any map of sheaves $\alpha: T \to \mathcal{F}$ from a representable sheaf, there exists a covering map $\pi: U \to T$ in $\mathcal{C}_J$ sitting in a commutative diagram

$$
\begin{array}{ccc}
U \times_T U & \xrightarrow{pr_1} & U \\
\downarrow \gamma & & \downarrow \beta \\
R & \xrightarrow{p_1} & Y \\
\downarrow p_2 & & \downarrow q \\
Y & & F
\end{array}
$$

Here, by the commutativity of the left-hand square, we mean $\beta \circ pr_1 = p_1 \circ \gamma$ and $\beta \circ pr_2 = p_2 \circ \gamma$. 


Proposition 1.4.13. Let \( p_1, p_2 : R \to Y \) be a \( J \)-equivalence relation in the site \( C_J = (C, J) \), and suppose that the induced arrow \( R \to Y \times Y \) belongs to an effective descent class \( D \) (cf. 1.4.8 (3)). Let \( q : Y \to \mathcal{F} \) be the cokernel of \( (p_1, p_2) \) in the category of sheaves on \( C_J \). Then

1. the map \( q \) is represented by a covering map;
2. the canonical map \( R \to Y \times \mathcal{F} \) of sheaves is an isomorphism;
3. the diagonal map \( \Delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \times \mathcal{F} \) is representable by an arrow in \( D \).

Proof. Assertion (1) follows from [72], I.5.9, while (2) follows from [72], I.5.4. Finally, (3) follows easily from [72], I.5.10.

Exercises

Exercise 0.1.1. Let \( C \) be a category with all essentially small filtered colimits, and \( I \) a directed set. We consider an inductive system \( \{J_i, h_{ij}\}_{i \in I} \) of directed sets. Suppose we are given the following data:

(a) for any \( i \in I \), an inductive system \( \{X_{i, \alpha}\}_{\alpha \in J_i} \) consisting of objects and arrows in \( C \) indexed by \( J_i \);
(b) for any \( i \leq j \), an arrow \( X_{i, \alpha} \to X_{j, h_{ij}(\alpha)} \) in \( C \) such that, whenever \( \alpha \leq \beta \) in \( J_i \), the diagram

\[
\begin{array}{ccc}
X_{i, \alpha'} & \longrightarrow & X_{j, h_{ij}(\alpha')} \\
\uparrow & & \uparrow \\
X_{i, \alpha} & \longrightarrow & X_{j, h_{ij}(\alpha)}
\end{array}
\]

commutes.

1. For any \( i \leq j \), there exists a canonical arrow

\[
\lim_{\alpha \in J_i} X_{i, \alpha} \longrightarrow \lim_{\beta \in J_j} X_{j, \beta},
\]

by which one can consider the double inductive limit \( \lim_{i \in I} \lim_{\alpha \in J_i} X_{i, \alpha} \).

2. Set \( \Lambda = \{(i, \alpha) : i \in I, \ \alpha \in J_i\} \), and consider the order on \( \Lambda \) defined as follows: \( (i, \alpha) \leq (j, \beta) \) if \( i \leq j \) and \( h_{ij}(\alpha) \leq \beta \). Then \( \Lambda \) is a directed set, and \( \{X_{i, \alpha}\}_{(i, \alpha) \in \Lambda} \) is an inductive system indexed by \( \Lambda \).

3. There exists a canonical isomorphism

\[
\lim_{i \in I} \lim_{\alpha \in J_i} X_{i, \alpha} \sim \lim_{(i, \alpha) \in \Lambda} X_{i, \alpha}.
\]
2. General topology

**Exercise 0.1.2.** Let $I$ be a non-empty directed set that admits a final and at most countable subset. Show that there exists an ordered final map $L: \mathbb{N} \to I$.

**Exercise 0.1.3.** Let $(\mathcal{C}, J)$ be a site, where $\mathcal{C}$ has a final object and all fiber products, and

\[
R \xrightarrow{p_1} Y \xleftarrow{p_2} \]

a diagram of arrows in $\mathcal{C}$ such that for any $Z \in \text{obj}(\mathcal{C})$ the induced map

\[
\text{Hom}_\mathcal{C}(Z, R) \xrightarrow{\sim} \text{Hom}_\mathcal{C}(Z, Y) \times \text{Hom}_\mathcal{C}(Z, Y)
\]

is injective and that $p_1$ and $p_2$ are covering maps with respect to $J$. Define $T$ by the Cartesian diagram

\[
\begin{array}{ccc}
T & \xrightarrow{p_{12}} & R \\
p_{23} \downarrow & & \downarrow p_2 \\
R & \xrightarrow{p_1} & Y
\end{array}
\]

Show that diagram $(\ast)$ gives a $J$-equivalence relation if and only if the following conditions are satisfied.

1. (a) There exists an arrow $\iota: Y \to R$ such that $p_1 \circ \iota = p_2 \circ \iota = \text{id}_Y$.
2. (b) There exists an automorphism $\sigma: R \xrightarrow{\sim} R$ such that $p_1 \circ \sigma = p_2$ and $p_2 \circ \sigma = p_1$ (in particular, we have $\sigma^2 = \text{id}_R$).
3. (c) There exists an arrow $p_{13}: T \to R$ such that $p_1 \circ p_{13} = p_1 \circ p_{12}$ and $p_2 \circ p_{13} = p_2 \circ p_{23}$.

2 General topology

In this section, we discuss generality of the topological spaces that appear typically (if not generally) in algebraic geometry. One kind of such typical ‘algebraic topological spaces’ are the so-called (locally) coherent topological spaces, introduced in §2.2. For example, the underlying topological spaces of schemes are always locally coherent. Coherent and sober topological spaces can be regarded as an algebraic analogue of compact Hausdorff spaces, as indicated by the following important fact: the filtered projective limit of a family of coherent sober topological spaces by quasi-compact transition maps is again a coherent sober topological space. This fact, which we will prove in §2.2. (c) by means of Stone duality between topological spaces and lattices, contains the famous theorem by Zariski, proved in his 1944 paper [107], which asserts that his ‘generalized Riemann space’ is quasi-compact.

\footnote{\textsuperscript{2}‘Coherent’ means ‘quasi-compact and quasi-separated.’}
The topological spaces that appear in rigid geometry as a topological visualization (cf. Introduction) are locally coherent topological spaces of a special kind, called *valuative spaces*, which we will introduce in §2.3. Already at the general topology level, valuative spaces have several interesting special features, such as overconvergent subsets, separated quotients, etc., all of which will play significant roles in rigid geometry. Moreover, by means of a certain class of valuative spaces, the so-called reflexive (2.4.1) locally strongly compact (2.5.1) valuative spaces, one can develop an interesting variant of Stone duality, given in §2.6, which will be useful in understanding the relationship between our rigid geometry and Berkovich’s analytic geometry.

This section ends with a brief exposition of topos theory with special emphasis on the so-called *coherent topoi* and their limits.\(^3\)

In the sequel, we will follow the following conventions:

- compact spaces are always assumed to be Hausdorff; that is, being compact is, by definition, equivalent to being quasi-compact and Hausdorff;
- locally compact spaces are always assumed to be locally Hausdorff;
- paracompact spaces, however, are not assumed to be Hausdorff (cf. §2.5. (c)).

2.1 Some general prerequisites

2.1. (a) **Generalization and specialization.** Let \(X\) be a topological space. We say that \(y \in X\) is a *generalization* of \(x \in X\) or, equivalently, that \(x\) is a *specialization* of \(y\), if \(x\) belongs to the closure \(\{y\}\) of \(\{y\}\) in \(X\) or, equivalently, if \(y\) is contained in any open subset containing \(x\). Let us denote by \(G_x\) the set of all generalizations of \(x\), that is, the intersection of all open neighborhoods of \(x\):

\[
G_x = \bigcap_{x \in U} U.
\]

Clearly, if \(z\) is a generalization of \(y\) and \(y\) is a generalization of \(x\), then \(z\) is a generalization of \(x\). If \(X\) is a \(T_0\)-space,\(^4\) then the set \(G_x\) is an ordered set with respect to the following order: for \(y, z \in G_x, y \leq z\) if \(z\) is a generalization of \(y\). A point \(x\) is *maximal* (resp. *minimal*) if there exists no other generalization (resp. specialization) of \(x\) than \(x\) itself. For a point \(x\) to be minimal it is necessary and sufficient that the singleton set \(\{x\}\) is a closed subset of \(X\); for this reason, minimal points are also called *closed points*. Note that \(x\) is the unique closed point of the subspace \(G_x \subseteq X\) (that is, the topological subspace endowed with the subspace topology).

---

3 Due to lack of space, we limit ourselves to a brief overview of the theory of locales and localic topos. For these subjects, the reader is advised to consult [63] and [64].

4 A topological space \(X\) is said to be a \(T_0\)-space (or Kolmogorov space) if, for any pair of two distinct points \(x \neq y\) of \(X\), there exists an open neighborhood of one of them that does not contain the other.
2. General topology

2.1. (b) Sober spaces. Recall that a topological space $X$ is said to be *irreducible* if it is non-empty and cannot be written as the union of two closed subsets distinct from $X$. If $X$ is not irreducible, it is said to be *reducible*. Let us list some of the basic facts on irreducible spaces and irreducible subsets (cf. [27], Chapter II, §4.1):

- a non-empty topological space $X$ is irreducible if and only if any non-empty open subset is dense in $X$, and in this case, any non-empty open subset is again irreducible (and hence is connected);
- a subset $Y$ of a topological space is irreducible if and only if its closure $\overline{Y}$ is irreducible;
- the image of an irreducible subset under a continuous mapping is again irreducible.

Let $X$ be a topological space, $Z \subseteq X$ an irreducible closed subset, and $U \subseteq X$ an open subset such that $Z \cap U \neq \emptyset$. Then $Z \cap U$ is an irreducible closed subset of $U$, and the closure $\overline{Z \cap U}$ of $Z \cap U$ in $X$ coincides with $Z$. In particular, if $Z_1$ and $Z_2$ are irreducible closed subsets of $X$, and if $U$ is an open subset such that $Z_i \cap U \neq \emptyset$ for $i = 1, 2$, then $Z_1 \cap U = Z_2 \cap U$ implies $Z_1 = Z_2$.

For a topological space $X$ and a point $x \in X$, the subset $\{x\}$ consisting of all specializations of $x$ is an irreducible closed subset. For a closed subset $Z \subseteq X$, a point $x \in Z$ is called a *generic point* of $Z$ if $Z = \{x\}$. A topological space $X$ is said to be *sober* if it is a $T_0$-space and any irreducible closed subset has a (unique) generic point.\(^5\) Note that any $T_2$-space (= Hausdorff space) is sober, since any irreducible $T_2$-space is just a singleton set (with the unique topology). Thus we have

$$T_2 \implies \text{sober} \implies T_0,$$

while $T_1$-ness\(^6\) and soberness are not comparable (cf. Exercise 0.2.7).

The proofs of the following propositions are straightforward and left to the reader.

**Proposition 2.1.1.** Every locally closed\(^7\) subspace of a sober space is sober.

**Proposition 2.1.2.** If a topological space $X$ admits an open covering $X = \bigcup_{\alpha \in \Lambda} U_\alpha$ such that each $U_\alpha$ is sober, then $X$ is sober.

We denote by $\text{STop}$ the full subcategory of the category $\text{Top}$ of topological spaces consisting of the sober spaces.

---

\(^5\)The uniqueness of the generic point follows from $T_0$-ness.

\(^6\)A topological space $X$ is said to be a $T_1$-space (or *Kuratowski space*) if for any pair of distinct points $x \neq y$ of $X$ there exist an open neighborhood $U$ of $x$ and an open neighborhood $V$ of $y$ such that $x \notin V$ and $y \notin U$.

\(^7\)A subset of a topological space $X$ is said to be *locally closed* if it is the intersection of an open subset and a closed subset.
Proposition 2.1.3. The inclusion functor \( i : \text{STop} \hookrightarrow \text{Top} \) admits the left-adjoint functor

\[ \cdot_{\text{sob}} : \text{Top} \rightarrow \text{STop}, \quad X \mapsto X^{\text{sob}}. \]

Briefly, \( X^{\text{sob}} \) is the set of all irreducible closed subsets of \( X \) endowed with the topology with respect to which open subsets are of the form \( \widetilde{U} \), each associated to an open subset \( U \subseteq X \), consisting of the irreducible closed subsets of \( X \) that intersect \( U \). For more detail (and the proof of 2.1.3), see [53], 0, §2.9.

2.1. (c) Completely regular spaces. Recall that a completely regular space is a \( T_1 \)-space \( X \) that enjoys the following property: for any point \( x \in X \) and a closed subset \( Z \subseteq X \) not containing \( x \), there exists a continuous function \( f : X \rightarrow [0, 1] \subseteq \mathbb{R} \) such that \( f(x) = 0 \) and \( f(C) = \{1\} \). By Urysohn’s lemma, normal (that is, \( T_1 \) and \( T_4 \)) topological spaces and locally compact Hausdorff spaces are completely regular (consider the one-point compactification for the latter).

2.1. (d) Quasi-compact spaces and quasi-separated spaces

Definition 2.1.4. (1) A topological space \( X \) is said to be quasi-compact if for any open covering \( \{U_\alpha\}_{\alpha \in L} \) of \( X \) there exists a finite subset \( L' \subseteq L \) of indices such that \( \{U_\alpha\}_{\alpha \in L'} \) already covers \( X \).

(2) A continuous map \( f : X \rightarrow Y \) of topological spaces is said to be quasi-compact if for any quasi-compact open subset \( V \) of \( Y \) its pull-back \( f^{-1}(V) \) is a quasi-compact open subset of \( X \).

The set of all quasi-compact open subsets of a topological space is closed under finite union. The composition of two quasi-compact maps is again quasi-compact. If \( X \) is a quasi-compact space and \( Z \subseteq X \) is a closed subset, then \( Z \) is quasi-compact, and the inclusion map \( i : Z \hookrightarrow X \) is quasi-compact.

Proposition 2.1.5. Let \( X \) be a topological space, and \( Y \subseteq X \) a subspace. Suppose that \( X \) admits an open basis consisting of quasi-compact open subsets. Then for any quasi-compact open subset \( V \) of \( Y \), there exists a quasi-compact open subset \( U \) of \( X \) such that \( V = Y \cap U \).

Corollary 2.1.6. Under the assumptions of 2.1.5, let \( g : Z \rightarrow Y \) be a continuous map. If \( j \circ g : Z \rightarrow X \) is quasi-compact, then so is \( g : Z \rightarrow Y \).

Definition 2.1.7. Let \( X \) be a topological space. A subset \( Z \subseteq X \) is said to be retrocompact if for any quasi-compact open subset \( U \subseteq X \) the intersection \( Z \cap U \) is quasi-compact.

\( ^8 \)A topological space \( X \) is said to be a \( T_4 \)-space if for any disjoint pair \( F_1, F_2 \) of closed subsets of \( X \) there exists a disjoint pair \( U_1, U_2 \) of open neighborhoods of \( F_1, F_2 \), respectively.
That is, \( Z \) is retrocompact if and only if the inclusion map \( Z \hookrightarrow X \), where \( Z \) is endowed with the subspace topology, is quasi-compact (2.1.4 (2)). The union of finitely many retrocompact subsets is again retrocompact. The intersection of finitely many retrocompact open subsets is again a retrocompact open subset.

**Definition 2.1.8.** A topological space \( X \) is said to be *quasi-separated* if, for any two quasi-compact open subsets \( U, V \subseteq X \), the intersection \( U \cap V \) is again quasi-compact.

In other words, \( X \) is quasi-separated if and only if any quasi-compact open subset of \( X \) is retrocompact (2.1.7). The set of all quasi-compact open subsets in a quasi-separated space is closed under finite intersection. Any open subset of a quasi-separated space is again quasi-separated.

### 2.2 Coherent spaces

#### 2.2. (a) Definition and first properties

**Definition 2.2.1.** A topological space \( X \) is said to be *coherent* if the following conditions are satisfied:

(a) \( X \) has an open basis consisting of quasi-compact open subsets;

(b) \( X \) is quasi-compact and quasi-separated.

(See §2.7. (d) for the topos-theoretic interpretation.)

**Examples 2.2.2.** (1) The empty set and singleton sets (endowed with the unique topology) are coherent; more generally, any finite space is coherent. Any continuous mapping from a finite space to an arbitrary coherent space is quasi-compact.

(2) The underlying topological space of a scheme admits an open basis consisting of quasi-compact open subsets. Hence, the underlying topological space of a quasi-compact and quasi-separated scheme (e.g., a Noetherian schemes) is coherent (due to [54], IV, (1.2.7)); in particular, the underlying topological space of any affine scheme is coherent.

For the reason mentioned in (2), quasi-compact and quasi-separated schemes are often called *coherent schemes*.

**Proposition 2.2.3.** Let \( X \) be a coherent topological space. Then any quasi-compact locally closed subset \( Z \subseteq X \), endowed with the subspace topology, is coherent, and is retrocompact in \( X \).
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Proof. By 2.1.5, any quasi-compact locally closed subset of \( X \) is the intersection of a closed subset and a quasi-compact open subset. Hence, it suffices to check the assertion in the cases where \( Z \) is closed, and where \( Z \) is quasi-compact open. Both cases are easy to verify. \( \square \)

Remark 2.2.4. (1) Coherent and sober spaces are also called spectral spaces by some authors.

(2) Hochster [58] has shown that any coherent sober space is homeomorphic to the prime spectrum \( \text{Spec} \ A \) of a commutative ring \( A \).

One of the most remarkable features of coherent spaces is that, as we will see soon (2.2.10 (1)), the small filtered projective limit of a projective system consisting of coherent sober topological spaces with quasi-compact transition maps is again coherent and sober. In connection with this, the reader is invited, before proceeding to the next paragraph, to try out Exercise 0.2.3, which deals with an analogous topic on filtered projective limits of schemes.

2.2. (b) Stone’s representation theorem. For the reader’s convenience, we include basic facts on the relationship between distributive lattices and coherent sober spaces. Our basic reference to the theory of distributive lattices is [64].

Let \( A \) be a distributive lattice (we use the binary symbols \( \lor, \land, \) and \( \leq \) as in [64] and always assume that \( A \) has 0 and 1). We view \( A \) as a category with respect to the partial order of \( A \) in the manner mentioned in §1.2.(e). Then, as a category, \( A \) is stable under finite limits and finite colimits, and any finite disjoint sum is universally disjoint ([8], Exposé II, 4.5). A lattice homomorphism \( A \rightarrow B \) is, in the categorical language, a functor that commutes with finite limits and colimits. Let \( \text{DLat} \) be the category of distributive lattices with lattice homomorphisms.

Example 2.2.5. For a topological space \( X \) we denote by \( \text{Ouv}(X) \) the set of all open subsets of \( X \). \( \text{Ouv}(X) \) is a distributive lattice with the structure \( (0, 1, \lor, \land, \leq) = (\emptyset, X, \cup, \cap, \subseteq) \). If \( X \) is coherent, then the set \( \text{QCOuv}(X) \) of all quasi-compact open subsets of \( X \) forms a distributive sublattice of \( \text{Ouv}(X) \).

There are notions of ideals and filters, which are dual to each other.

Definition 2.2.6. Let \( A \) be a distributive lattice.

(1) A subset \( I \subseteq A \) is called an ideal if (a) \( 0 \in I \), (b) \( a, b \in I \) implies \( a \lor b \in I \), and (c) \( a \in I \) and \( b \leq a \) imply \( b \in I \).

(2) A subset \( F \subseteq A \) is called a filter if (a) \( 1 \in F \), (b) \( a, b \in F \) implies \( a \land b \in F \), and (c) \( a \in F \) and \( b \geq a \) imply \( b \in F \).
Accordingly, by an ideal (resp. a filter) of a topological space $X$, we mean an ideal (resp. a filter) of the complete distributive lattice $\text{Ouv}(X)$.

For $A \in \text{obj}(\text{DLat})$ the lattice $\text{Id}(A)$ of all ideals of $A$ forms a complete Heyting algebra, which admits the embedding $A \hookrightarrow \text{Id}(A)$ of lattices that maps $a$ to the principal ideal $\{x : x \leq a\}$ generated by $a$. The image of $A$ in $\text{Id}(A)$ consists exactly of the sets of finite elements ([64], Chapter II, §3.1).

**Definition 2.2.7.** An ideal $P$ is said to be prime if (a) $1 \notin P$, and (b) $a \land b \in P$ implies $a \in P$ or $b \in P$.

In other words, an ideal $P$ is prime if and only if $A \setminus P$ is a filter. A filter of the form $A \setminus P$ (where $P$ is a prime ideal) is called a prime filter. Here is another description of prime ideals. Let $2 = \{0, 1\}$ be the Boolean lattice consisting of two elements. Then any prime ideal is the kernel of a surjective lattice homomorphism $A \twoheadrightarrow 2$.

Any non-zero distributive lattice has at least one prime ideal (one needs, similarly to the case of commutative rings, the axiom of choice to prove this).

Let $\text{Spec} A$ denote the set of prime ideals of $A$. For any ideal $I \subset A$ we define $V(I)$ to be the set of prime ideals containing $I$. The Zariski topology on $\text{Spec} A$ is the topology for which $\{V(I)\}_{I \in \text{Id}(A)}$ is the system of closed sets. We set $D(I) = \text{Spec} A \setminus V(I)$.

**Theorem 2.2.8** (Stone’s representation theorem; cf. [64], Chapter II, §3.4). (1) For any distributive lattice $A$, the topological space $\text{Spec} A$ is coherent and sober, and we have the functor

$$\text{Spec}: \text{DLat}_{\text{opp}} \longrightarrow \text{CSTop},$$

where $\text{CSTop}$ denotes the category of coherent sober topological spaces and quasi-compact maps. The distributive lattice $\text{Ouv}(\text{Spec} A)$ consisting of all open subsets of $\text{Spec} A$ is isomorphic to $\text{Id}(A)$ by $I \mapsto D(I)$. Moreover, the lattice $A$ is identified, by means of the embedding $A \hookrightarrow \text{Id}(A)$ and the isomorphism $\text{Id}(A) \cong \text{Ouv}(\text{Spec} A)$, with the distributive lattice of all quasi-compact open subsets of $\text{Spec} A$.

(2) The functor $(*)$ is an equivalence of categories. The quasi-inverse functor is given by $X \mapsto \text{QCOuv}(X)$ (2.2.5).

**2.2. (c) Projective limit of coherent sober spaces.** The category $\text{Top}$ is closed under small projective limits. For a projective system $\{X_i, p_{ij}\}$ of topological spaces indexed by an ordered set $I$ (cf. [24], Chapter I, §4.4), the projective limit $X = \lim_{\leftarrow i \in I} X_i$ has, as its underlying set, the set-theoretic projective limit of the underlying sets of $X_i$’s, endowed with the coarsest topology for which all projection maps $p_i: X \to X_i$ are continuous, or equivalently, the topology generated by the subsets of the form $p_i^{-1}(U)$ for $i \in I$ and open subsets $U \subseteq X_i$. 
**Theorem 2.2.10.** Let \( \{ X_i, p_{ij} : X_j \to X_i \} \) be a projective system of topological spaces, indexed by a directed set \( (I, \leq) \). Suppose that the topology of each \( X_i \) \((i \in I)\) is generated by quasi-compact open subsets, and that each transition map \( p_{ij} \) \((i \leq j)\) is quasi-compact (2.1.4 (2)). Set \( X = \lim_{\leftarrow i \in I} X_i \), and let \( U \subseteq X \) be a quasi-compact open subset. Then there exist an \( i \in I \) and a quasi-compact open subset \( U_i \subseteq X_i \) such that \( p_{i1}^{-1}(U_i) = U \), where \( p_i : X \to X_i \) is the canonical projection map.

**Proof.** By the definition of the topology of \( X \) and by the quasi-compactness of \( U \), there exist finitely many \( i_1, \ldots, i_n \in I \) and, for each \( k = 1, \ldots, n \), a quasi-compact open subset \( U_{ik} \subseteq X_{ik} \), such that \( U = \bigcup_{k=1}^n p_{ik}^{-1}(U_{ik}) \) (cf. [24], Chapter I, §4.4, Proposition 9). Take \( i \in I \) such that \( i \geq i_1, \ldots, i_n \), and set \( U_i = \bigcup_{k=1}^n p_{ik}^{-1}(U_{ik}) \). Then \( U_i \) is quasi-compact and \( U = p_i^{-1}(U_i) \), as desired. \( \square \)

**Theorem 2.2.10.** Let \( (I, \leq) \) be a directed set, and \( \{ X_i, p_{ij} \}_{i \in I} \) a projective system of coherent sober spaces with quasi-compact transition maps \( p_{ij} \) for \( i \geq j \). Set \( X = \lim_{\leftarrow i \in I} X_i \).

1. The topological space \( X \) is coherent and sober, and the canonical projection maps \( p_i : X \to X_i \) for \( i \in I \) are quasi-compact.

2. If, moreover, each \( X_i \) \((i \in I)\) is non-empty, then \( X \) is non-empty.

By (1) one finds, in particular, that the category \( \text{CSTop} \) is closed under small filtered projective limits. Note that the first assertion implies that the projective limit space \( X \) is, in particular, quasi-compact.

**Proof.** Set \( A_i = \text{QCou}(X_i) \) for each \( i \in I \). The transition maps \( p_{ij} \), \( i \leq j \), induce lattice homomorphisms \( u_{ij} : A_j \to A_i \), which form an inductive system \( \{ A_i \}_{i \in I} \) of lattices. Let \( A = \lim_{\leftarrow i \in I} A_i \) be the inductive limit. By 2.2.8, the topological space \( X' = \text{Spec} A \) is the projective limit of \( \{ X_i, p_{ij} \} \) in the category \( \text{CSTop} \). If all \( X_i \)'s are non-empty, each distributive lattice \( A_i \) is non-zero, that is, \( 0 \neq 1 \) in \( A_i \). This implies that \( 0 \neq 1 \) also in \( A \), hence \( X' \neq \emptyset \).

Hence, to prove the assertions, it remains to show that \( X \) and \( X' \) are homeomorphic. By the mapping universality of projective limits, we have the canonical map \( g : X' = \text{Spec} A \to X \). We are first going to check that \( g \) is bijective. To see this, note that each point \( x \) of \( X \) is uniquely represented by a sequence \( \{ x_i \in X_i \}_{i \in I} \) of points such that \( p_{ij}(x_i) = x_j \) for \( i \leq j \), which is, furthermore, uniquely interpreted as a system \( \{ h_i : A_i \to 2 \}_{i \in I} \) of surjective lattice homomorphisms such that \( h_j \circ u_{ij} = h_i \) for \( i \leq j \) \((h_i \) is defined as follows: it maps \( U \in A_i = \text{QCou}(X_i) \) to \( 1 \) if \( x_i \in U \), and to \( 0 \) otherwise). Giving the last data is then equivalent to giving a surjective lattice homomorphism \( h : A \to 2 \), whose kernel corresponds to
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a point \( x' \in X' = \text{Spec } A \). It is easy to see that \( x \mapsto x' \) thus constructed gives the inverse mapping of \( g: X' \to X \), and hence \( g \) is a continuous bijection. Moreover, quasi-compact open subsets of \( X' \) are of the form \( p_i^{-1}(U) \) for a quasi-compact open subset \( U \) of some \( X_i \) (2.2.9). Since \( g \) is bijective, \( g(p_i^{-1}(U)) = p_i^{-1}(U) \), and hence \( g \) maps any quasi-compact open subset of \( X' \) to an open subset of \( X \). Since quasi-compact open subsets of \( X' \) form an open basis of the topology on \( X' \), we conclude that \( g: X' \to X \) is a homeomorphism.

Remark 2.2.11. The above theorem can be more properly formulated in topos theory, where the projective system \( \{ X_i \}_{i \in I} \) is translated into the projective system of corresponding coherent topoi \( \{ \text{top}(X_i) \}_{i \in I} \). Under this interpretation, the theorem follows from [9], Exposé VI, (8.3.13), and Deligne’s theorem [9], Exposé VI, (9.0); see 2.7.17 below.

Theorem 2.2.10 is not only important in its own right, but has many useful consequences.

Corollary 2.2.12. Let \( \{ X_i, p_{ij} \}_{i \in I} \) be a filtered projective system of coherent sober topological spaces and quasi-compact maps indexed by a directed set \( I \). Let \( i \in I \) be an index, and \( U, V \subseteq X_i \) open subsets, where \( U \) is quasi-compact. Then the following conditions are equivalent.

(a) \( p_i^{-1}(U) \subseteq p_i^{-1}(V) \), where \( p_i: X = \lim_{\leftarrow j \in I} X_j \to X_i \) is the canonical mapping.

(b) There exists an index \( j \) with \( i \leq j \) such that \( p_{ij}^{-1}(U) \subseteq p_{ij}^{-1}(V) \).

Proof. Implication (b) \( \Rightarrow \) (a) is clear. To show (a) \( \Rightarrow \) (b), set

\[
Z_j = p_{ij}^{-1}(U) \setminus p_{ij}^{-1}(V), \quad j \in I, i \leq j.
\]

We need to show that \( Z_j \) is empty for some \( j \). Suppose that all \( Z_j \) are non-empty. By 2.2.3 and 2.1.1, each \( Z_j \) is coherent and sober. Applying 2.2.10 (2), we deduce that the projective limit \( Z = \lim_{\leftarrow j \geq i} Z_j \) is non-empty. On the other hand, since

\[
p_i^{-1}(U) = \lim_{\leftarrow j \geq i} p_{ij}^{-1}(U) \quad \text{and} \quad p_i^{-1}(V) = \lim_{\leftarrow j \geq i} p_{ij}^{-1}(V),
\]

we have \( Z = p_i^{-1}(U) \setminus p_i^{-1}(V) \). But then \( Z \neq \emptyset \) contradicts the assumption \( p_i^{-1}(U) \subseteq p_i^{-1}(V) \), whence the result. \( \square \)
**Theorem 2.2.13.** Let \( \{X_i, p_{ij}\}_{i \in I} \) and \( \{Y_i, q_{ij}\}_{i \in I} \) be two filtered projective systems of coherent sober topological spaces with quasi-compact transition maps indexed by a common directed set \( I \), and \( \{f_i\}_{i \in I} \) a map of the systems, that is, a collection of continuous maps \( f_i : X_i \to Y_i \) such that the diagram

\[
\begin{array}{ccc}
X_j & \xrightarrow{f_j} & Y_j \\
\downarrow{p_{ij}} & & \downarrow{q_{ij}} \\
X_i & \xrightarrow{f_i} & Y_i \\
\end{array}
\]

commutes for any \( i \leq j \). Let

\[
f = \lim_{i \in I} f_i : X = \lim_{i \in I} X_i \longrightarrow Y = \lim_{i \in I} Y_i
\]

be the induced map.

1. If \( f_i \) is quasi-compact for all \( i \in I \), then \( f \) is quasi-compact.
2. If \( f_i \) is quasi-compact and surjective for all \( i \in I \), then \( f \) is surjective.
3. If \( f_i \) is quasi-compact and closed for all \( i \in I \), then \( f \) is closed.

Before proving the theorem, let us present a useful corollary of it.

**Corollary 2.2.14.** Let \( \{X_i, p_{ij}\}_{i \in I} \) be a projective system of coherent sober spaces with quasi-compact transition maps indexed by a directed set \( I \). Suppose that all transition maps \( p_{ij} \) are surjective (resp. closed). Then for any \( j \in I \) the canonical projection \( p_j : X = \lim_{i \in I} X_i \to X_j \) is surjective (resp. closed).

**Proof.** For each \( j \in I \), replace \( I \) by the cofinal subset \( \{i \in I : i \geq j\} \) and apply 2.2.13 (2) and (3) to the constant projective system \( \{Y_i, q_{ij}\} \) with \( Y_i = X_j \). \( \square \)

The rest of this subsection is devoted to the proof of 2.2.13. We denote by \( p_i : X \to X_i \) and \( q_i : Y \to Y_i \) the canonical projections for \( i \in I \).

**Proof of Theorem 2.2.13 (1).** We need to show that, for any quasi-compact open subset \( V \) of \( Y \), its pull-back \( f^{-1}(V) \) is quasi-compact. In view of 2.2.9 one can choose \( i \in I \) and a quasi-compact open subset \( V_i \) of \( Y_i \) such that \( V = q_i^{-1}(V_i) \). Then \( f^{-1}(V) = p_i^{-1}(f_i^{-1}(V_i)) \), which is quasi-compact by 2.2.10 (1), and this shows (1) of 2.2.13. \( \square \)

To show the other assertions of the theorem, we need a few lemmas.

**Lemma 2.2.15.** Let \( X \) be a coherent sober topological space, and \( F \) a filter of the distributive lattice \( \text{Ouv}(X) \) (2.2.6 (2)) generated by quasi-compact open subsets. Then the subspace \( \bigcap_{U \in F} U \) is coherent and sober, and the inclusion map \( \bigcap_{U \in F} U \hookrightarrow X \) for any \( U \in F \) (in particular, \( \bigcap_{U \in F} U \hookrightarrow X \)) is quasi-compact.
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**Proof.** Consider the subfilter $F' = F \cap \mathcal{QCOuv}(X)$ of $F$. Since the topology on $X$ is generated by quasi-compact open subsets, the inclusion map $F' \hookrightarrow F$ is cofinal. Since $\bigcap_{U \in F} U = \lim_{\leftarrow U \in F'} U$, the lemma follows from 2.2.10 (1).

**Corollary 2.2.16.** Let $X$ be a coherent sober space, and $x \in X$. Then the subset $G_x$ of all generizations of $x$ with the subspace topology is coherent and sober, and the inclusion map $G_x \hookrightarrow X$ is quasi-compact.

**Proof.** Since $X$ has an open basis consisting of quasi-compact open subsets, and any quasi-compact open subset is coherent (2.2.3) and sober, the subset $G_x$ coincides with the intersection of all coherent open neighborhoods of $x$ (cf. §2.1.(a)), and hence the assertion follows from 2.2.15.

**Lemma 2.2.17.** Let $f: X \to Y$ be a quasi-compact map between coherent sober topological spaces.

1. Let $C$ be a closed subset of $Y$. Then $C$ and $f^{-1}(C)$ are coherent and sober. The map $f^{-1}(C) \to C$ induced by $f$ is quasi-compact.

2. Let $F$ be a filter of $\mathcal{Ouv}(Y)$ generated by quasi-compact open sets. Then $\bigcap_{U \in F} U$ and $f^{-1}(\bigcap_{U \in F} U)$ are coherent and sober, and the map $f^{-1}(\bigcap_{U \in F} U) \to \bigcap_{U \in F} U$ induced by $f$ is quasi-compact.

In particular, it follows from (2) and 2.2.15 that the map $f^{-1}(\bigcap_{U \in F} U) \to X$ is quasi-compact.

**Proof.** The first assertion follows immediately from 2.2.3, while the second follows from 2.2.15.

**Lemma 2.2.18.** (1) Let $f: X \to Y$ be a quasi-compact map between coherent sober topological spaces. Then for any $y \in Y$ the set $f^{-1}(y)$ with the subspace topology is coherent and sober, and the inclusion $f^{-1}(y) \hookrightarrow X$ is quasi-compact.

(2) Consider a commutative diagram of coherent sober spaces and quasi-compact maps

$$
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y.
\end{array}
$$

Let $y' \in Y'$ and let $y \in Y$ be the image of $y'$ by the map $Y' \to Y$. Then the natural map $f'^{-1}(y') \to f^{-1}(y)$ is quasi-compact.
Proof. (1) By 2.2.16, the subset $G_y$ of all generizations of $y$ is coherent and sober, and the inclusion map $G_y \hookrightarrow Y$ is quasi-compact. By 2.2.17, $f^{-1}(G_y)$ is coherent and sober, and the inclusion map $f^{-1}(G_y) \hookrightarrow X$ is quasi-compact. Since $y$ is the closed point of $G_y$, $f^{-1}(y)$ is a closed subset of $f^{-1}(G_y)$. Hence, by 2.2.3, $f^{-1}(y)$ is coherent and sober, and the inclusion map $f^{-1}(y) \hookrightarrow f^{-1}(G_y)$ is quasi-compact. Then the inclusion $f^{-1}(y) \hookrightarrow X = f^{-1}(y) \hookrightarrow f^{-1}(G_y) \hookrightarrow X$ is quasi-compact.

(2) It suffices to show the following: if (i) $g: X' \to X$ is a quasi-compact map between coherent sober spaces, (ii) $Z_0 \subseteq X'$ and $Z \subseteq X$ are subspaces such that $g(Z_0) \subseteq Z$, and (iii) the inclusions $Z_0 \hookrightarrow X'$ and $Z \hookrightarrow X$ are quasi-compact, then the map $g: Z' \to Z$ is quasi-compact (indeed, $Z' = f^{-1}(y')$ and $Z = f^{-1}(y)$ give our original situation). This follows from 2.1.6. \hfill \Box

Now, we can prove the second part of 2.2.13.

Proof of Theorem 2.2.13 (2). Take $y \in Y$ and set $y_i = q_i(y)$ for each $i \in I$. By our assumptions and 2.2.18, the set $Z_i = f_i^{-1}(y_i)$ is coherent, sober, and non-empty, and the map $Z_j \to Z_i$ induced by $p_{ij}$ is quasi-compact. Then it follows from 2.2.10 (2) that

$$f^{-1}(y) = \lim_{i \in I} Z_i \neq \emptyset,$$

which verifies the claim. \hfill \Box

For the proof of the third part, we need the following lemma; the proof is straightforward and left to the reader.

Lemma 2.2.19. Let $\{X_i, p_{ij}\}_{i \in I}$ be a projective system of topological spaces indexed by a directed set $I$. We set $X = \lim_{\leftarrow i \in I} X_i$ and denote the projection $X \to X_i$ by $p_i$.

(1) Suppose we are given a projective system $\{C_i, p_{ij}|_{C_i}\}_{i \in I}$ consisting of subspaces $C_i \subseteq X_i$ for $i \in I$. Then we have

$$\lim_{\leftarrow i \in I} C_i = \bigcap_{i \in I} p_i^{-1}(C_i).$$

(2) For any subset $C \subseteq X$ we have

$$\bar{C} = \lim_{\leftarrow i \in I} p_i(C) = \bigcap_{i \in I} p_i^{-1}(p_i(C)).$$
Proof of Theorem 2.2.13 (3). We need to show that for any closed subset $C \subseteq X$ the image $f(C)$ is closed. Set $C_i = p_i(C)$ and $D_i = f_i(C_i)$. These are closed subsets of $X_i$ and $Y_i$, respectively. By 2.2.19 we have $C = \text{lim} \leftarrow i \in I C_i$. On the other hand, $\text{lim} \leftarrow i \in I D_i = \bigcap_{i \in I} q_i^{-1}(D_i)$ is a closed subset of $Y$. Hence it suffices to show the equality

$$f(C) = \text{lim} \leftarrow i \in I D_i.$$ 

It is clear that the left-hand side is contained in the right-hand side. In order to show the converse, take $y \in \text{lim} \leftarrow i \in I D_i = \bigcap_{i \in I} q_i^{-1}(D_i)$. For each $i$, the set $Z_i = f_i^{-1}(q_i(y)) \cap C_i$ is non-empty, and $\{Z_i\}_{i \in I}$ forms a projective system of coherent sober spaces with quasi-compact transitions maps by 2.2.18 and 2.2.17 (1). Then it follows that $\text{lim} \leftarrow i \in I Z_i \neq \emptyset$, and any point $x$ in this projective limit, regarded as a point of $C$, is mapped to $y$ by construction.

Here we include a theorem, known as Steenrod’s theorem (cf. [102]), communicated to the authors by O. Gabber, as well as the proof suggested by him, on quasi-compactness of the projective limit spaces, although we will not use it in our ensuing discussion. Later in §2.7. (g), we will indicate a topos-theoretic proof.

**Theorem 2.2.20.** Let $(I, \leq)$ be a directed set, and $\{X_i, p_{ij}\}$ a projective system indexed by $I$ consisting of quasi-compact sober topological spaces. (Here we do not assume that the transition maps $p_{ij}$ are quasi-compact.)

1. The projective limit $X = \text{lim} \leftarrow i \in I X_i$ is quasi-compact and sober.

2. If each $X_i$ ($i \in I$) is non-empty, then so is the limit $X$.

**Proof.** First we show (2). Let $\Phi$ be the set of all projective systems consisting of non-empty closed subsets of $X_i$’s; that is, each element of $\Phi$ is a collection $\{Y_i\}_{i \in I}$ of non-empty closed subsets $Y_i \subseteq X_i$ for $i \in I$ such that $p_{ij}(Y_j) \subseteq Y_i$ for any $i \leq j$. We define a partial order $\leq$ on the set $\Phi$ as follows: for $y = \{Y_i\}$ and $y' = \{Y'_i\}$,

$$y \leq y' \iff Y'_i \subseteq Y_i, \quad i \in I.$$ 

One deduces from quasi-compactness of each $X_i$ that any totally ordered subset of $\Phi$ has an upper bound, and thus, by Zorn’s lemma, that there exists a maximal element $z = \{Z_i\}$.

**Claim 1.** Let $i \in I$, and consider a non-empty closed subset $W \subseteq Z_i$. Suppose there exists a cofinal subset $J$ of $I_{\geq i} = \{ j \in I : j \geq i \}$ such that $p_{ij}^{-1}(W) \cap Z_j \neq \emptyset$ for any $j \in J$. Then $W = Z_i$. 

To show this, set

\[ Z_k = \bigcap_{j \in J \cap I \geq k} p_{kj}(p_{ij}^{-1}(W) \cap Z_j), \]

for any \( k \in I \), and consider the collection \( C_{i,W,J} = \{Z'_k\}_{k \in I} \). Note that for any cofinal subset \( J' \) of \( J \) we have \( C_{i,W,J'} = C_{i,W,J} \); in particular, for \( k \leq l \), \( Z'_l \) coincides with the \( l \)-th component of \( C_{i,W,J} \), and thus we have \( p_{kl}(Z'_l) \subseteq Z'_k \).

Moreover, by our assumption and the quasi-compactness of each \( X_k \), the closed subsets \( Z'_k \) are non-empty for any \( k \in I \). Hence we have \( C_{i,W,J} = \emptyset \), and thus \( C_{i,W,J} = \emptyset \) by the minimality of \( z \). Since the \( i \)-th component of \( C_{i,W,J} \) is a closed subset of \( Y \), we conclude that \( Y = Z_i \), as desired.

**Claim 2.** Each \( Z_i \) for \( i \in I \) is irreducible.

Indeed, if \( Z_i = W_1 \cup W_2 \), where \( W_1, W_2 \) are closed subsets, then at least one of the subsets \( J_t = \{ j \in I_{\geq i} : p_{ij}^{-1}(W) \cap Z_j \neq \emptyset \} \) is cofinal in \( I_{\geq i} \); by Claim 1 we have \( Z_i = W_1 \) or \( Z_i = W_2 \).

One can similarly show that \( p_{ij}(Z_j) = Z_i \) for \( i \leq j \). Since \( X_i \) is sober, \( Z_i \) admits a unique generic point \( \eta_i \), and then the collection of points \( \{ \eta_i \}_{i \in I} \) determines a point in the limit \( X = \lim_{\leftarrow i \in I} X_i \). Hence \( X \) is non-empty, as desired.

Next, we show (1). Let \( \{ Z_\alpha \}_{\alpha \in L} \) be a collection of closed subsets of \( X \) such that \( \bigcap_{\alpha \in L} Z_\alpha = \emptyset \). We want to show that there exists a finite subset \( L' \subseteq L \) such that \( \bigcap_{\alpha \in L'} Z_\alpha = \emptyset \). We may assume that each \( Z_\alpha \) is of the form \( p_\alpha^{-1}(W) \) for a closed subset \( W \subseteq X_i \) for some \( i \) (where \( p_\alpha : X \to X_i \) is the canonical projection). We choose by the axiom of choice a function \( \alpha \mapsto (i(\alpha), W_{i(\alpha)}) \) such that \( Z_\alpha = p_{i(\alpha)}^{-1}(W_{i(\alpha)}) \). Now, for any \( i \in I \), we set

\[ L_i = \{ \alpha \in L : i(\alpha) \leq i \}, \quad Z_i = \bigcap_{\alpha \in L_i} p_{i(\alpha)}^{-1}(W_{i(\alpha)}). \]

Then we have \( p_{ij}(Z_j) \subseteq Z_i \) for any \( i \leq j \), and hence \( \{ Z_i \}_{i \in I} \) forms a projective system such that \( \lim_{\leftarrow i \in I} Z_i = \bigcap_{\alpha \in L} Z_\alpha = \emptyset \). Since each \( Z_i \) is quasi-compact and sober (2.1.1), there exists \( i \in I \) such that \( Z_i = \emptyset \) (due to (2) proved above). Since \( X_i \) is quasi-compact, there exists a finite subset \( L' \subseteq L_i \) such that \( \bigcap_{\alpha \in L'_i} p_{i(\alpha)}^{-1}(W_{i(\alpha)}) = \emptyset \), and hence \( \bigcap_{\alpha \in L'} Z_\alpha = \emptyset \), as desired.

\( \square \)

2.2. (d) Locally coherent spaces

**Definition 2.2.21.** A topological space \( X \) is locally coherent if \( X \) admits an open covering by coherent subspaces.
Since any coherent space admits an open basis of quasi-compact open subsets, and since quasi-compact open subsets of a coherent space are again coherent (2.2.3), a locally coherent topological space admits an open basis consisting of coherent open subsets. It follows that for a locally coherent space $X$ and a point $x \in X$ the set $G_x$ of all generizations of $x$ coincides with the intersection of all coherent open neighborhoods of $x$ (cf. §2.1. (a)); in particular, if $X$ is sober, then $G_x$ is coherent by 2.2.16.

Note that by 2.2.2 (2) the underlying topological space of any scheme is locally coherent.

**Proposition 2.2.22.** Let $X$ be a topological space. Then the following conditions are equivalent.

(a) $X$ is locally coherent.

(b) Every open subset of $X$ is locally coherent.

(c) $X$ admits an open covering $X = \bigcup_{\alpha \in L} U_{\alpha}$ by locally coherent spaces.

(d) The topology on $X$ is generated by coherent open subsets.

The proof is straightforward and left to the reader. The following proposition follows immediately from 2.2.3.

**Proposition 2.2.23.** Any locally closed subspace of a locally coherent space is again locally coherent.

In particular, any open subset of a coherent space is locally coherent.

For a locally coherent space $X$ to be coherent it is necessary and sufficient that $X$ is quasi-compact and quasi-separated. But note that a quasi-compact locally coherent space is not necessarily coherent. Indeed, if $Y$ is a coherent space, and $U \subseteq Y$ is an open subset that is not quasi-compact, then $X = Y \amalg_U Y$ (the gluing of two copies of $Y$ along $U$) is quasi-compact and locally coherent, but is not coherent.

As the Stone duality (§2.2. (b)) indicates, quasi-compact maps give a good notion for morphisms between coherent topological spaces. The appropriate notion of maps for locally coherent spaces is similarly defined as follows.

**Definition 2.2.24.** A continuous map $f: X \rightarrow Y$ between locally coherent spaces is said to be *locally quasi-compact* if, for any pair $(U, V)$ consisting of coherent open subsets $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, the map $f|_U: U \rightarrow V$ is quasi-compact (2.1.4 (2)).
The following facts are easy to check.

- A continuous map \( f : X \to Y \) between coherent spaces is locally quasi-compact if and only if it is quasi-compact.
- A continuous map \( f : X \to Y \) between locally coherent spaces is locally quasi-compact if and only if there exist an open covering \( \{V_\alpha\}_{\alpha \in L} \) of \( Y \) by coherent open subsets and, for each \( \alpha \in L \), an open covering \( \{U_{\alpha, \lambda}\}_{\lambda \in \Lambda_\alpha} \) of \( f^{-1}(V_\alpha) \) by coherent open subsets, such that each \( f|_{U_{\alpha, \lambda}} : U_{\alpha, \lambda} \to V_\alpha \) is quasi-compact.

**Proposition 2.2.25.** Let \( f : X \to Y \) be a locally quasi-compact map between locally coherent spaces. If \( V \subseteq Y \) is a retrocompact open subset of \( Y \), then \( f^{-1}(V) \) is retrocompact in \( X \).

**Proof.** It suffices to show that, for any coherent open subset \( U \subseteq X \), \( f^{-1}(V) \cap U \) is quasi-compact; here we use the fact that the topology of \( X \) is generated by coherent open subsets. Since \( f(U) \) is quasi-compact, there exists a coherent open subset \( W \subseteq Y \) such that \( f(U) \subseteq W \). Since \( f|_U : U \to W \) is quasi-compact, \( f^{-1}(V) \cap U = (f|_U)^{-1}(V \cap W) \) is quasi-compact. \( \square \)

**Theorem 2.2.26.** Let \( X \) be a locally coherent sober space, \( Y \) a locally coherent space, and \( f : X \to Y \) a quasi-compact continuous map. Then

\[
\overline{f(X)} = \bigcup_{y \in f(X)} \{y\}.
\]

In other words, the closure \( \overline{f(X)} \) is the set of all specializations of points of \( f(X) \).

**Proof.** The inclusion \( \overline{f(X)} \supseteq \bigcup_{y \in f(X)} \{y\} \) is clear. Let us show the converse. Considering an open covering of \( Y \) by quasi-compact open subsets, one reduces to the case where \( Y \) is quasi-compact. In this case, as the map \( f \) is quasi-compact, \( X \) is also quasi-compact. Hence, there exists a finite open covering \( X = \bigcup_{i=1}^n V_i \) by coherent open subsets. Replacing \( X \) by the disjoint union \( \bigsqcup_{i=1}^n V_i \), we may assume that \( X \) is coherent and sober.

For any \( y \in f(X) \), let \( \text{CN}_y \) be the set of coherent open neighborhoods of \( y \). We view \( \text{CN}_y \) as a directed set with respect to the reversed inclusion order. For any \( U \in \text{CN}_y \), since \( f \) is quasi-compact, the set \( f^{-1}(U) \) is non-empty and coherent. Moreover, for \( U \subseteq U' \), the inclusion \( f^{-1}(U) \subseteq f^{-1}(U') \) is quasi-compact. By 2.2.10 (2), we have

\[
\lim_{U \in \text{CN}_y} f^{-1}(U) = f^{-1}\left( \bigcap_{U \in \text{CN}_y} U \right) \neq \emptyset.
\]

On the other hand, \( \bigcap_{U \in \text{CN}_y} U \) is the set \( G_y \) of all generalizations of \( y \). This means that there is a generalization of \( y \) in \( f(X) \), and hence the claim follows. \( \square \)
Corollary 2.2.27. Let $X$ be a locally coherent sober space, and $U \subseteq X$ a retrocompact (2.1.7) open subset of $X$. Then

$$\bar{U} = \bigcup_{x \in U} \{x\}.$$ 

In other words, the closure $\bar{U}$ is the set of all specializations of points in $U$.

Corollary 2.2.28. In the situation as in (2.2.26), the following conditions are equivalent.

(a) $f$ is a closed map.
(b) $\overline{f(\{x\})} = \overline{\{f(x)\}}$ for any $x \in X$.

Proof. (a) $\implies$ (b) is clear. Suppose (b) holds, and let $C$ be a closed subset of $X$. By 2.2.26, $\overline{f(C)} = \bigcup_{x \in C} \overline{\{f(x)\}} = \bigcup_{x \in C} f(\{x\})$. Since $\{x\} \subseteq C$, one has $\overline{f(C)} \subseteq f(C)$. $\square$

2.3 Valuative spaces

2.3. (a) Valuative spaces

Definition 2.3.1. A topological space $X$ is said to be valuative if the following conditions are satisfied:

(a) $X$ is locally coherent (2.2.21) and sober (§2.1. (b));
(b) for any point $x \in X$ the ordered set $G_x$ of all generizations of $x$ (§2.1. (b)) is totally ordered;
(c) every point $x \in X$ has a maximal generization $\tilde{x} \in G_x$.

Note that, in view of (b), the maximal generization of $x$ as in (c) is uniquely determined. The order type (§1.1. (b)) of the totally ordered set $G_x$ is called the height of the point $x$.

Remark 2.3.2. (1) Note that, under the axiom of choice, (c) follows automatically from (a). Indeed, to show this, we may assume that $X$ is coherent and sober, hence is of the form $X = \text{Spec} \ A$ for a distributive lattice $A = \text{QCOuv}(X)$ by Stone duality (§2.2. (b)). In this situation, points of $G_x$ correspond to prime filters $F_y = \{U \in A : y \in U\}$ ($y \in G_x$). Any totally ordered subset $I$ of $G_x$ admits an upper bound $\bigcup_{F \in I} F$ in $G_x$; here, note that $y \in X$ is a generization of $z \in X$ if and only if $F_y \supseteq F_z$. Hence Zorn’s lemma implies that there exists a maximal element in $G_x$.

(2) It follows that any locally coherent and sober subspace of a valuative space is again valuative. In particular, by 2.2.23 and 2.1.1, any locally closed subspace of a valuative space is a valuative space.
The following proposition follows easily from 2.2.22.

**Proposition 2.3.3.** Let $X$ be a topological space. Then the following conditions are equivalent:

(a) $X$ is valuative;

(b) every open subset of $X$ is valuative;

(c) $X$ admits an open covering by valuative spaces.

An important example of valuative spaces, which motivates the terminology ‘valuative,’ is the underlying topological space of the spectrum Spec $V$ of a valuation ring $V$; cf. 6.3.1. In this case, the height of the closed point of Spec $V$ is nothing but the height of the valuation ring $V$ (6.2.6 (1)).

2.3. (b) Closures and tubes

**Definition 2.3.4.** Let $X$ be a valuative space. A tube closed subset of $X$ is a closed subset of the form $\bar{U}$ for a retrocompact (2.1.7) open subset $U$. The complement of a tube closed subset is called a tube open subset. Tube closed and tube open subsets are collectively called tube subsets.

The following proposition is an immediate consequence of 2.2.27.

**Proposition 2.3.5.** Let $X$ be a valuative space and $U \subseteq X$ a retrocompact open subset. Then for $x \in X$ to belong to the tube closed subset $\bar{U}$ it is necessary and sufficient that the maximal generization $\bar{x}$ of $x$ belongs to $U$.

**Corollary 2.3.6.** Let $X$ be a valuative space and $C = (X \setminus U)^{\circ}$ (where $U \subseteq X$ is retrocompact open) a tube open subset. Then for $x \in X$ to belong to $C$ it is necessary and sufficient that the maximal generization $\bar{x}$ of $x$ does not belong to $U$.

Recall the following general-topology fact: for two open subsets $U_1, U_2 \subseteq X$, we have

$$\overline{U_1 \cup U_2} = \overline{U_1} \cup \overline{U_2},$$

but not in general the intersections; in our situation, however, we have

**Proposition 2.3.7.** Let $X$ be a valuative space, and $\{U_\alpha\}_{\alpha \in L}$ a family of retrocompact open subsets of $X$. Then we have

$$\bigcap_{\alpha \in L} U_\alpha = \bigcap_{\alpha \in L} \overline{U_\alpha}.$$

**Proof.** The inclusion $\bigcap_{\alpha \in L} U_\alpha \subseteq \bigcap_{\alpha \in L} \overline{U_\alpha}$ is clear. To show the converse, take any point $y \in \bigcap_{\alpha \in L} U_\alpha$. Then by 2.3.5 the maximal generization $\bar{y}$ of $y$ belongs to $U_\alpha$ for any $\alpha \in L$. Consequently $y \in \{\bar{y}\} \subseteq \bigcap_{\alpha \in L} \overline{U_\alpha}$, as desired. \qed
Corollary 2.3.8. (1) Any finite union of tube closed (resp. tube open) subsets is a tube closed (resp. tube open) subset.

(2) Any finite intersection of tube closed (resp. tube open) subsets is a tube closed (resp. tube open) subset.

Proof. In view of 2.3.7, the closure operator $\overline{\cdot}$ commutes with finite intersections and finite unions of retrocompact open subsets. Hence, to show the corollary, it suffices to show that finite intersections and the finite unions of retrocompact open subsets are retrocompact, which is clear. 

2.3. (c) Separated quotients and separation maps. Let $X$ be a valuative space, and let $[X]$ denote the subset of $X$ consisting of all maximal points of $X$. We have the canonical retraction map

$$
\text{sep}_X: X \rightarrow [X], \quad x \mapsto \text{the maximal generization of } x,
$$

which we call the separation map. The separation map $\text{sep}_X$ is clearly surjective. We endow $[X]$ with the quotient topology induced from the topology on $X$. Then $[X]$ is a $T_1$-space (Exercise 0.2.10). The topological space $[X]$ thus obtained is called the separated quotient (or $T_1$-quotient) of $X$.

Note that, with this topology, the inclusion map $[X] \hookrightarrow X$ is not continuous in general.

Proposition 2.3.9 (universality of separated quotients). Let $X$ be a valuative space. Suppose we are given a continuous mapping $\varphi: X \rightarrow T$ to a $T_1$-space $T$. Then there exists a unique continuous map $\psi: [X] \rightarrow T$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & T \\
\downarrow{\text{sep}_X} & & \nearrow{\psi} \\
[X] & & \\
\end{array}
$$

commutes.

Proof. Let $x \in X$, and $y$ the maximal generization of $x$. We need to show that $\varphi(x) = \varphi(y)$. Since $T$ is $T_1$, the subset $\varphi^{-1}(\varphi(y))$ is closed and hence contains $\{y\}$. But this means $x \in \varphi^{-1}(\varphi(y))$, that is, $\varphi(x) = \varphi(y)$. 

Corollary 2.3.10 (functoriality). Any continuous map \( f : X \to Y \) between valuative spaces induces a unique continuous map

\[
[f] : [X] \longrightarrow [Y]
\]
such that diagram

\[
\begin{array}{c}
\xymatrix{ X \ar[r]^f \ar[d]_{\text{sep}_X} & Y \ar[d]_{\text{sep}_Y} \\
[X] \ar[r]^{[f]} & [Y] }
\end{array}
\]
commutes. (Hence, in particular, the mapping \( X \mapsto [X] \) is functorial.)

2.3. (d) Overconvergent sets

Definition 2.3.11. Let \( X \) be a valuative space. A closed (resp. an open) subset \( S \) of \( X \) is said to be overconvergent if for any \( x \in S \) any generization (resp. specialization) of \( x \) belongs to \( S \).

For example, if \( x \) is a maximal point of \( X \), then \( \{x\} \) is an overconvergent closed subset. Note that if \( S \subseteq X \) is overconvergent, then so is the complement \( X \setminus S \). Moreover, if \( S \) is an overconvergent closed or open subset and \( x \in S \), then both \( G_x \) (the set of all generizations of \( x \)) and \( \{x\} \) (the set of all specializations of \( x \)) are contained in \( S \).

The following propositions are easy to establish.

Proposition 2.3.12. Let \( X \) be a valuative space and \( X = \bigcup_{\alpha \in L} U_\alpha \) an open covering. Then \( S \subseteq X \) is an overconvergent closed (resp. open) subset if and only if \( S \cap U_\alpha \) is an overconvergent closed (resp. open) subset of \( U_\alpha \) for every \( \alpha \in L \).

Proposition 2.3.13. Let \( X \) be a valuative space, and \( S \) a closed or an open subset of \( X \). Then \( S \) is overconvergent if and only if \( S = \text{sep}_X^{-1}(\text{sep}_X(S)) \). Hence, in particular, \( T \mapsto \text{sep}_X^{-1}(T) \) gives a bijection between the set of all open (resp. closed) subsets of \( [X] \) and the set of all overconvergent open (resp. closed) subsets of \( X \).

Corollary 2.3.14. (1) Any finite intersection of overconvergent open subsets is an overconvergent open subset. Any union of open subsets is an overconvergent open subset.

(2) Any finite union of overconvergent closed subsets is an overconvergent closed subset. Any intersection of overconvergent closed subsets is an overconvergent closed subset.
By 2.3.5 and 2.3.6, we have the following result.

**Proposition 2.3.15.** Any tube closed (resp. tube open) subset of a valuative space $X$ is overconvergent closed (resp. overconvergent open) in $X$.

**Definition 2.3.16.** Let $X$ be a valuative space, and consider the separated quotient $[X]$ of $X$. A subset $T$ of $[X]$ is said to be a tube closed (resp. tube open) subset if $\text{sep}_X^{-1}(T)$ is a tube closed (resp. tube open) subset of $X$ (2.3.4).

Thus, by 2.3.13, $S \mapsto \text{sep}_X(S)$ gives a canonical order preserving bijection between the sets of all tube closed (resp. tube open) subsets of $X$ and of tube closed (resp. tube open) subsets of $[X]$.

**Proposition 2.3.17.** Let $X$ be a coherent valuative space.

1. For any overconvergent closed set $F$, the set of all tube open subsets containing $F$ is a fundamental system of neighborhoods of $F$.

2. For any overconvergent closed subsets $F_1, F_2$ with $F_1 \cap F_2 = \emptyset$, there exist tube open subsets $U_1$ and $U_2$ such that $F_i \subseteq U_i$ ($i = 1, 2$) and $U_1 \cap U_2 = \emptyset$.

**Proof.** (1) Take a quasi-compact open neighborhood $U$ of $F$. We want to find a tube open subset $C$ such that $F \subseteq C \subseteq U$. Take a quasi-compact open set $V$ such that $X \setminus U \subseteq V \subseteq X \setminus F$. Since $F$ is overconvergent, $\overline{V} \subseteq X \setminus F$ by 2.3.15, and $U$ and $\overline{V}$ cover $X$. This implies $F \subseteq X \setminus \overline{V} \subseteq U$.

(2) Since $X = (X \setminus F_1) \cup (X \setminus F_2)$, there exist quasi-compact open subsets $U'_i$ ($i = 1, 2$) such that $U'_1 \cup U'_2 = X$ and $U'_i \cap F_i = \emptyset$. As $F_i$ is overconvergent, we have $U'_i \cap F_i = \emptyset$ for $i = 1, 2$ by 2.2.27. Set $U_i = X \setminus U'_i$ for $i = 1, 2$, which are tube open subsets. We have $F_i \subseteq U_i$ ($i = 1, 2$) and $U_1 \cap U_2 = \emptyset$, as desired. □

**Corollary 2.3.18.** Let $X$ be a coherent valuative space.

1. The separated quotient $[X]$ is a normal topological space.

2. The space $[X]$ is compact (and hence is Hausdorff).

**Proof.** (1) follows from 2.3.17 (2), because $[X]$ is a $T_1$-space (Exercise 0.2.10).

(2) follows from (1) and the quasi-compactness of $X$. □

**Corollary 2.3.19.** Let $X$ be a coherent valuative space. Then the set of all tube open subsets of $[X]$ forms an open basis of the topological space $[X]$.

**Proof.** This follows from 2.3.17 (1) and the fact that, as $[X]$ is $T_1$, any singleton set $\{x\}$ for $x \in [X]$ is a closed subset. □
Recall here that a continuous map $f : X \to Y$ between topological spaces is said to be proper if for any topological space $Z$ the induced map

$$f \times \text{id}_Z : X \times Z \longrightarrow Y \times Z$$

is closed (cf. [24], Chapter I, §10.1, Definition 1).

**Corollary 2.3.20.** Let $X$ be a coherent valuative space. Then $\text{sep}_X$ is a proper map.

**Proof.** By [24], Chapter I, §10.2, Corollary 2, every continuous map from a quasi-compact space to a Hausdorff space is proper. The corollary follows from 2.3.18 (2).

**2.3. (e) Valuative maps**

**Definition 2.3.21.** A continuous map $f : X \to Y$ between valuative spaces is said to be valuative if $f([X]) \subset [Y]$, that is, $f$ maps maximal points to maximal points.

For example, open immersions between valuative spaces are valuative. Clearly, if $f : X \to Y$ is valuative, then $[f](x) = f(x)$ for any $x \in [X] (\subseteq X)$.

**Proposition 2.3.22.** Let $f : X \to Y$ be a valuative map between valuative spaces.

1. For any subset $C \subseteq Y$ that is stable under generization, $f^{-1}(C)$ is stable under generization in $X$, and the diagram

$$
\begin{array}{ccc}
  f^{-1}(C) \cap [X] & \longrightarrow & [X] \\
\downarrow & & \downarrow [f] \\
C \cap [Y] & \longrightarrow & [Y]
\end{array}
$$

is Cartesian.

2. We have $f^{-1}(\text{sep}_Y^{-1}(C \cap [Y])) = \text{sep}_X^{-1}(f^{-1}(C) \cap [X])$.

**Proof.** (1) is obvious by the definition of valuative maps. To show (2), first observe that the set $F = f^{-1}(\text{sep}_Y^{-1}(C \cap [Y]))$ is stable under specialization and generization in $X$ and that $F \cap [X] = [f]^{-1}(C \cap [Y])$. By (1), $F \cap [X] = f^{-1}(C) \cap [X]$, that is, $F$ coincides with the set of all specializations of points in $f^{-1}(C) \cap [X]$, whence the desired equality.

**Corollary 2.3.23.** Let $f : X \to Y$ be a valuative map between valuative spaces. Then, for any overconvergent closed (resp. overconvergent open) subset $C \subseteq Y$, $f^{-1}(C)$ is overconvergent closed (resp. overconvergent open) in $X$. 
Corollary 2.3.24. Let \( f: X \to Y \) be a locally quasi-compact (2.2.24) valuative map between valuative spaces.

(1) For any retrocompact open subset \( U \) of \( Y \), we have \( f^{-1}(\overline{U}) = \overline{f^{-1}(U)} \).

(2) For any tube closed (resp. tube open) (2.3.4) subset \( S \) of \( Y \), \( f^{-1}(S) \) is a tube closed (resp. tube open) subset of \( X \).

Proof. (1) Since \( U \) and \( f^{-1}(U) \) are retrocompact by 2.2.25, it follows from 2.2.27 that \( \overline{U} = \text{sep}_Y^{-1}([U]) \) and \( \overline{f^{-1}(U)} = \text{sep}_X^{-1}([f^{-1}(U)]) \). Then the desired equality follows from 2.3.22 (2).

(2) follows immediately from (1). \( \square \)

2.3. (f) Structure of separated quotients. Let \( X \) be a valuative space, and \( U \subseteq X \) an open subset. Then \( U \) is again a valuative space (2.3.3), and \( U \hookrightarrow X \) induces a continuous injection \([U] \hookrightarrow [X]\) with the image \( \text{sep}_X(U) \) (2.3.10).

The continuous bijection \([U] \to \text{sep}_X(U)\), where \( \text{sep}_X(U) \) is endowed with the subspace topology from \([X]\), may not be a homeomorphism; it is homeomorphic if, for example, \( U \) is overconvergent, or \( X \) and \( U \) are coherent; cf. Exercise 0.2.12.

If \( U \subseteq X \) is overconvergent open, then \([U] = \text{sep}_X(U)\) is open in \([X]\). But, in general, \( \text{sep}_X(U) \) for an open \( U \subseteq X \) is not necessarily in \([X]\). Note that if \( X \) is coherent and \( U \) is quasi-compact, then \([U]\) is compact in the Hausdorff space \([X]\) (2.3.18 (2)), and hence is closed.

Proposition 2.3.25 (continuity of \([\cdot]\)). Let \( \mathcal{C} \) be a \( \mathcal{U} \)-small category, and let

\[
F: \mathcal{C} \rightarrow \text{Top}
\]

be a functor. Suppose that \( F(X) \) is a valuative space for any object \( X \) in \( \mathcal{C} \) and that \( F(f) \) is an open immersion for any morphism \( f \) in \( \mathcal{C} \).

(1) The colimit \( X = \lim_{\to \mathcal{C}} F \) is representable by a valuative space.

(2) \( [X] \cong \lim_{\to \mathcal{C}} ([\cdot] \circ F) \).

Proof. (1) is clear. To show (2), observe first that the limit \( Q = \lim_{\to \mathcal{C}} [F(Z)] \) is a \( T_1 \)-space (easy to see). The maps \( F(x) \to X \) for \( x \in \text{obj}(\mathcal{C}) \) induce the maps \( [F(x)] \to [X] \), and hence the canonical map \( \alpha: Q \to [X] \). On the other hand, the composition \( F(x) \to [F(x)] \to Q \) for \( x \in \text{obj}(\mathcal{C}) \) gives rise to \( X \to Q \). As \( Q \) is a \( T_1 \)-space, we have \( \beta: [X] \to Q \) by the universality of separated quotients (2.3.9). It is easy to see that \( \alpha \) and \( \beta \) are inverses to each other. \( \square \)
Let $X$ be a valuative space, and $\{U_\alpha\}_{\alpha \in L}$ an open covering of $X$. Consider the coequalizer sequence

$$R \xrightarrow{\sim} \bigsqcup_{\alpha \in L} U_\alpha \longrightarrow X,$$

where $R = \bigsqcup_{\alpha, \beta \in L} U_\alpha \cap U_\beta$, and the induced sequence

$$[R] \xrightarrow{\sim} \bigsqcup_{\alpha \in L} [U_\alpha] \longrightarrow [X].$$

The functor $X \mapsto [X]$ commutes with disjoint unions and with finite intersection of open subsets. One sees easily that, in fact, $[R]$ defines an equivalence relation on $\bigsqcup_{\alpha \in L} [U_\alpha]$ (cf. Exercise 0.1.3). By 2.3.25 we immediately have the following corollary.

**Corollary 2.3.26.** The topological space $[X]$ is the quotient of $\bigsqcup_{\alpha \in L} [U_\alpha]$ by the equivalence relation $[R]$.

**Proposition 2.3.27.** Let $f : X \to Y$ be a quasi-compact valuative map of valuative spaces. Then the induced map $[f]$ is proper as a map of topological spaces.

**Proof.** First we assume $Y$ is coherent. Then $[X]$ is a quasi-compact space. Since $[Y]$ is Hausdorff (2.3.18 (2)), $[f]$ is proper by [24], Chapter I, §10.2, Corollary 2, and the claim is shown in this case.

In general, we take an open covering $\{U_\alpha\}_{\alpha \in L}$ of $Y$ by coherent spaces. We claim that $[f]$ is a closed map. Let $F \subseteq [X]$ be a closed subset. By 2.3.12, to show that $[f](F)$ is closed in $[Y]$, it suffices to show that $[f](F) \cap [U_\alpha]$ is closed in $[U_\alpha]$ for each $\alpha \in L$. First observe that, since $f$ is valuative, $[f]^{-1}([U_\alpha]) = [f^{-1}(U_\alpha)]$; similarly, $[f](F) \cap [U_\alpha] = [f](F \cap [f^{-1}(U_\alpha)])$. Since $[f]^{-1}([f^{-1}(U_\alpha)])$ is the separated quotient of the map $f|_{f^{-1}(U_\alpha)} : f^{-1}(U_\alpha) \to U_\alpha$, it follows from the coherent case that $[f](F) \cap [U_\alpha]$ is closed in $[U_\alpha]$ for any $\alpha \in L$, as desired.

To finish the proof, in view of [24], Chapter I, §10.2, Theorem 2, it suffices to show that $[f]^{-1}(y)$ is quasi-compact for any $y \in [Y]$. But this is reduced to showing the properness of $[f]|_{[f^{-1}(U_\alpha)]}$ with $y \in [U_\alpha]$, since $\{y\}$ is closed both in $[Y]$ and $[U_\alpha]$, and hence follows again from the coherent case.

## 2.3. (g) Overconvergent interior

**Definition 2.3.28.** Let $X$ be a valuative space, and $F$ a subset of $X$. The maximal overconvergent open subset in $X$ contained in $F$, of which the existence is guaranteed by 2.3.14 (1), is called the **overconvergent interior** of $F$ in $X$ and denoted by $\text{int}_X(F)$. 
**Proposition 2.3.29.** Let $X$ be a valuative space and $U \subseteq X$ a quasi-compact open subset. Suppose that

(*) there exists a coherent open subset $V \subseteq X$ such that $\overline{U} \subseteq V$.

Let $y \in X$ be a maximal point. Then $y \in \text{int}_X(U)$ if and only if $\{y\} \subseteq U$.

Note that condition (*) is automatic if $X$ is quasi-separated and locally strongly compact (2.5.1); cf. 2.5.5 below.

**Proof.** Suppose $\{y\} \subseteq U$. Since $\{y\}$ is overconvergent, there exists by 2.3.17 (1) an open subset $W \subseteq V$, overconvergent in $V$, such that $\{y\} \subseteq W \subseteq U$. Since $W \subseteq \overline{U} \subseteq V$, $W$ is also overconvergent in $X$. Hence $y \in \text{int}_X(U)$, as desired. The converse is clear. \hfill \Box

**Corollary 2.3.30.** In the situation as in 2.3.29,

$$\text{int}_X(U) = \text{sep}_X^{-1}([U] \setminus \text{sep}_X(\partial U)), $$

where $\partial U = \overline{U} \setminus U$.

**Proof.** A point $y \in X$ lies in $\text{sep}_X^{-1}([U] \setminus \text{sep}_X(\partial U))$ if and only if the maximal generization $\tilde{y}$ of $y$ satisfies $\tilde{y} \in U$ and $\{\tilde{y}\} \cap \partial U = \emptyset$. By 2.3.29, this is equivalent to $y \in \text{int}_X(U)$. \hfill \Box

**Corollary 2.3.31.** Let $X$ be a quasi-separated valuative space, $U$ a quasi-compact open subset of $X$, and $y \in X$ a maximal point. Assume $\overline{U}$ is quasi-compact. Then $y \in \text{int}_X(U)$ if and only if $\{y\} \subseteq U$.

**Proof.** Since $X$ is quasi-separated, any quasi-compact open subset of $X$ is coherent. Since $\overline{U}$ is quasi-compact, there exists a coherent open subset $V$ such that $\overline{U} \subseteq V$. Now the assertion follows from 2.3.29. \hfill \Box

### 2.4 Reflexive valuative spaces

#### 2.4. (a) Reflexive valuative spaces

**Definition 2.4.1.** A valuative space (2.3.1) $X$ is said to be reflexive if for any two coherent open subsets $U \subseteq V$ of $X$, $[U] = [V]$ implies $U = V$.

**Example 2.4.2.** Let $V$ be an $a$-adically complete valuation ring ($a \in \mathfrak{m}_V \setminus \{0\}$). Then the formal spectrum Spf $V$ (cf. I, §1.1.(b)) of $V$ is a valuative space with the unique maximal point $p_V = \sqrt{(a)}$, the associated height-one prime (cf. 6.7.4). It is reflexive if and only if $V$ is of height one (cf. 6.2.6 (1)).
Note that reflexiveness is a local property on $X$.

**Proposition 2.4.3.** A valuative space $X$ is reflexive if and only if, for any pair of open subsets $U \subseteq V$ of $X$ such that the inclusion $U \hookrightarrow V$ is quasi-compact, $[U] = [V]$ implies $U = V$.

**Proof.** The ‘if’ part is clear. Suppose $X$ is reflexive, and let $U \hookrightarrow V$ be as above. For any coherent open subset $W$ of $V$, $U \cap W$ is coherent. Since $[U \cap W] = [U] \cap [W] = [V] \cap [W] = [W]$, we have $U \cap W = W$. Since $V$ has an open basis consisting of coherent open subsets, this implies that $U = V$.

**Proposition 2.4.4.** Let $X$ be a reflexive valuative space. Then any retrocompact (2.1.7) open subset $U \subseteq X$ is regular, that is, $x_\U \overset{\text{ref}}{\longrightarrow} U$.

**Proof.** Considering an open covering of $X$ by coherent open subsets, one can easily reduce to the case where $X$ and $U$ are coherent. Since $U \subseteq (\bar{U})^\circ$, $U \cap V$ is coherent, and we have $[U \cap V] = [U] \cap [V] = [(\bar{U})^\circ] \cap [V] = [V]$. Hence $U \cap V = V$. Since this holds for any quasi-compact open subsets of $(\bar{U})^\circ$, $U = (\bar{U})^\circ$, as desired.

**Proposition 2.4.5.** Let $X, Y$ be coherent valuative spaces, and $f, g: X \rightarrow Y$ two valuative quasi-compact maps. Suppose $X$ is reflexive. Then $[f] = [g]$ implies $f = g$.

**Proof.** In view of the Stone duality (2.2.8), it suffices to show that for any quasi-compact open subset $V \subseteq Y$, we have $f^{-1}(V) = g^{-1}(V)$. By 2.2.27, $\text{sep}_X^{-1}([U]) = \bar{U}$. In particular, we have $[U] = ([\bar{U}]^\circ)\circ$. For any quasi-compact open subset $V \subseteq (\bar{U})^\circ$, $U \cap V$ is coherent, and we have $[U \cap V] = [U] \cap [V] = [(\bar{U})^\circ] \cap [V] = [V]$. Hence $U \cap V = V$. Since this holds for any quasi-compact open subsets of $(\bar{U})^\circ$, $U = (\bar{U})^\circ$, as desired.

**2.4. (b) Reflexivization.** Let us denote by $\text{Vsp}$ the category of valuative spaces and valuative and locally quasi-compact maps, and by $\text{RVsp}$ the full subcategory of $\text{Vsp}$ consisting of the reflexive valuative spaces.

**Theorem 2.4.6.** The canonical inclusion $I: \text{RVsp} \hookrightarrow \text{Vsp}$ admits a left adjoint functor $(\cdot)^\text{ref}: \text{Vsp} \rightarrow \text{RVsp}$, which has the following properties:

(a) $(\cdot)^\text{ref}$ preserves open immersions;

(b) the adjunction map $X^\text{ref} \rightarrow X$ induces a homeomorphism $[X^\text{ref}] \cong [X]$;

(c) the adjunction map $X^\text{ref} \rightarrow X$ is quasi-compact;

(d) if $X$ is quasi-separated, then so is $X^\text{ref}$.

The rest of this subsection will be devoted to showing this theorem.
2. General topology

2.4. (c) Coherent case. First, we construct \( X^{\text{ref}} \) for a coherent valuative space \( X \). Let \( A = \text{QCOuv}(X) \) be the distributive lattice of quasi-compact open subsets of \( X \). We have \( X \cong \text{Spec} A \) by Stone duality (2.2.8). Consider the map

\[
\varphi: A \rightarrow 2^{[X]}
\]

to the power-set lattice of \([X]\) given by \( U \mapsto \text{sep}_X(U) = [U] \). This is a lattice homomorphism, for we have \( \text{sep}_X(U) = U \cap [X] \) set-theoretically. Hence the image \( B \) is a distributive lattice, giving a coherent sober space \( \text{Spec} B \), which we denote by \( X^{\text{ref}} \). It is then easy to see that the surjective homomorphism \( \varphi: A \rightarrow B \) gives rise to a quasi-compact map \( i_X: X^{\text{ref}} \rightarrow X \), which induces a homeomorphism from \( X^{\text{ref}} \) to its image with the subspace topology. We thereby identify \( X^{\text{ref}} \) with its image in \( X \).

**Proposition 2.4.7.** \( X^{\text{ref}} \) is a valuative space, and the map \( i_X: X^{\text{ref}} \rightarrow X \) is valuative, inducing a homeomorphism \( [X^{\text{ref}}] \approx [X] \).

To show this, we need the following lemma.

**Lemma 2.4.8.** Let \( S \) be a compact space, and \( D \) a distributive sublattice of \( 2^S \) consisting of closed subspaces of \( S \). Suppose that \( D \) satisfies the following conditions:

(a) for any \( x \in S \), there exists \( C \in D \) such that \( x \in C \);

(b) for any \( x, y \in S \) with \( x \neq y \), there exists \( C \in D \) such that \( x \in C \) and \( y \notin C \).

For any \( x \in S \), set \( F_x = \{ C \in D: x \in C \} \) (non-empty due to (a)).

(1) Any prime filter (cf. 2.2. (b)) is contained in a filter of the form \( F_x \) for some \( x \in S \), and, for any \( x \in S \), \( F_x \) is a maximal filter (2.2.6 (2)) of \( D \).

(2) The map \( x \mapsto F_x \) gives a bijection between \( S \) and the set of all maximal filters of \( D \).

**Proof.** (1) It is easy to see that \( F_x \) for \( x \in S \) is a prime filter. Let \( F \subseteq D \) be a prime filter. For any finitely many elements \( C_1, \ldots, C_r \in F \), we have \( C_1 \cap \cdots \cap C_r \in F \), and since \( \emptyset \notin F \), we have \( C_1 \cap \cdots \cap C_r \neq \emptyset \). This implies, since \( S \) is compact, that the intersection of all \( C \)'s in \( F \) is non-empty, containing a point \( x \in S \). This shows \( F \subseteq F_x \).

To show that \( F_x \) is maximal, suppose \( F_x \) is contained in a prime filter \( F \). By what we have seen above, there exists \( y \in S \) such that \( F \subseteq F_y \), hence \( F_x \subseteq F_y \). But the assumption (b) implies \( \bigcap_{C \in F_x} C = \{x\} \), from which we deduce \( x = y \), and thus \( F = F_x \). Hence \( F_x \) is a maximal filter.

(2) By (1), the map \( x \mapsto F_x \) maps \( S \) surjectively onto the set of all maximal filters. Since \( \bigcap_{C \in F_x} C = \{x\} \), it is injective, too. \( \Box \)
Proof of Proposition 2.4.7. Let us first confirm that the distributive lattice $B$ satisfies the hypotheses of 2.4.8. Condition (a) clearly holds. For $x, y \in [X]$ ($x \neq y$), since $[X]$ is Hausdorff (2.3.18 (2)), $\text{sep}_X^{-1}(\{y\})$ is closed, and hence there exists a coherent open neighborhood $U$ of $x$ such that $y \not\in [U]$ and also (b) holds.

To show that $X_{\text{ref}}$ is valuative, since the topology of $X_{\text{ref}}$ is the subspace topology induced by $X$, which is valuative, we only need to show that any $x \in X_{\text{ref}}$ has a maximal generization (as in 2.3.2, this is automatic, but allows a direct proof as follows). By 2.4.8, $X_{\text{ref}} = \text{Spec} B$ contains all points in $[X]$, in which any point has a maximal generization. This means that $X_{\text{ref}}$ is valuative, that the map $i_X$ is valuative, and that $[i_X]$ gives a continuous bijection from $[X_{\text{ref}}]$ to $[X]$. Since both $[X_{\text{ref}}]$ and $[X]$ are compact (Hausdorff) (2.3.18 (2)), $[i_X]$ is a homeomorphism. □

Proposition 2.4.9. For any quasi-compact valuative map $f: Z \to X$ defined on a reflexive coherent valuative space $Z$, there exists uniquely a valuative map $h: Z \to X_{\text{ref}}$ such that $f = i_X \circ h$.

Proof. For any subset $S \subseteq [X_{\text{ref}}]$, set $H(S) = (\text{sep}_Z^{-1}([f]^{-1}(S)))^\circ$, where $(\cdot)^\circ$ denotes the interior kernel. For $S = [U]$ with $U \in A$, since $f$ is valuative, we have $H([U]) = (f^{-1}(\text{sep}_X^{-1}([U])))^\circ = (f^{-1}(\overline{U}))^\circ$ (due to 2.2.27), which is equal to $f^{-1}(U)$ by 2.3.24 (1) and 2.4.4. Since $f$ is quasi-compact, the map $H$ gives a lattice homomorphism from $B = \text{QCOuv}(X_{\text{ref}})$ to $\text{QCOuv}(Z)$, which defines a quasi-compact map $h: Z \to X_{\text{ref}}$. Since $f$ is valuative, so is $h$. Next, since the composition $H \circ \varphi: A = \text{QCOuv}(X) \to \text{QCOuv}(Z)$ clearly coincides with the homomorphism corresponding to $f: Z \to X$, we have $f = i_X \circ h$. Finally, the uniqueness of $h$ follows from 2.4.5 and the fact that $i_X$ is a homeomorphism. □

By the last proposition, $X \mapsto X_{\text{ref}}$ for $X$ coherent is functorial, giving the left adjoint functor of the inclusion functor from the category of coherent reflexive valuative spaces with quasi-compact maps to the category of coherent valuative spaces with quasi-compact maps.

2.4. (d) General case. Let $X$ be a valuative space, and consider the functor

$$F_X: \text{RVsp}^{\text{pp}} \longrightarrow \text{Sets}$$

defined as follows: for any reflexive valuative space $Z$, we set

$$F_X(Z) = \text{Hom}_\mathcal{E}(Z, X),$$

the set of all locally quasi-compact valuative maps from $Z$ to $X$. Let us consider the category $\mathcal{D}$ of quasi-separated reflexive valuative space with quasi-compact valuative maps, and the category $\text{CRVsp}$ of coherent reflexive valuative space with quasi-compact valuative maps. The discussion in the previous paragraph shows that the functor $F_X|_{\text{CRVsp}^{\text{pp}}}$ is representable for any coherent valuative space $X$. 


Lemma 2.4.10. Suppose that the functor $F_X|_{\text{CVsp}^{\text{opp}}}$ is representable by a coherent reflexive valuative space $X'$. Then $F_X$ itself is representable by $X'$.

Proof. Let us first show that $F_X|_{\mathcal{D}^{\text{opp}}}$ is representable by $X'$. For any quasi-separated reflexive valuative space $Z$, write $Z$ as a filtered inductive limit $\lim_{\rightarrow i \in I} Z_i$ of coherent open subspaces. Then giving $f: Z \to X$ is equivalent to giving the collection $\{f_i|_{Z_i}: Z_i \to X\}_{i \in I}$ of maps satisfying $f_j|_{Z_i} = f_i$ for $i \leq j$. By the assumption, each $f_i$ factors through $X'$ by $f'_i: Z_i \to X'$, and by the functoriality, we have $f'_j|_{Z_i} = f'_i$ for $i \leq j$. We thus obtain $f': Z \to X'$, which shows the representability of $F_X|_{\mathcal{D}^{\text{opp}}}$.

In general, given a reflexive valuative space $Z$, we take an open covering $\{Z_\alpha\}_{\alpha \in L}$ of $Z$ by coherent open subspaces. Then giving $f: Z \to X$ is equivalent to giving the collection of maps $\{f_\alpha: Z_\alpha \to X\}_{\alpha \in L}$ in such a way that $f_\alpha|_{Z_\alpha \cap Z_\beta} = f_\beta|_{Z_\alpha \cap Z_\beta}$. By the assumption, each $f_\alpha$ is canonically factored through $X'$ by $f'_\alpha: Z_\alpha \to X'$. Since $Z_\alpha \cap Z_\beta$ are quasi-separated, what we have already shown above implies $f'_\alpha|_{Z_\alpha \cap Z_\beta} = f'_\beta|_{Z_\alpha \cap Z_\beta}$. Hence we obtain $f': Z \to X'$ by patching $f'_\alpha$'s, which verifies the representability of $F_X$ by $X'$.

By the lemma, we have already shown that the functor $F_X$ is representable for any coherent valuative space $X$, and whence the existence of $(\cdot)^{\text{ref}}$ as a functor $\text{CVsp} \to \text{CRVsp}$ of coherent valuative spaces with quasi-compact maps.

The following lemma shows that the reflexivization functor $(\cdot)^{\text{ref}}$, if it exists, commutes with open immersions.

Lemma 2.4.11. Suppose that $F_X$ for a valuative space $X$ is representable by $(X^{\text{ref}}, i_X)$. Then for any open subset $U \subseteq X$, $(i_X^{-1}(U), i_X^{-1}(U))$ represents $F_U$.

Proof. It is enough to invoke that any open subspace of a reflexive valuative space is again reflexive, which is trivial. \qed

Now we are going to construct $X^{\text{ref}}$ for any valuative space. Take an open covering $\{X_\alpha\}_{\alpha \in L}$ of $X$ by coherent open subspaces. For each $\alpha \in L$, we have $(X^{\text{ref}}_\alpha, i_{X_\alpha})$. By 2.4.11, $i^{-1}_{X_\alpha}(X_\alpha \cap X_\beta)$ gives the reflexivization of $X_\alpha \cap X_\beta$, hence being equal to $i^{-1}_{X_\beta}(X_\alpha \cap X_\beta)$, which one can consistently denote by $(X_\alpha \cap X_\beta)^{\text{ref}}$. One can then patch $X^{\text{ref}}_\alpha$'s along $(X_\alpha \cap X_\beta)^{\text{ref}}$'s to obtain a valuative space $X'$, which is easily seen to be reflexive, and a map $i_X: X' \to X$.

In order to see that $X'$ gives the reflexivization of $X$, it remains to show that $X'$ enjoys the desired functoriality (which at the same time also confirms that the formation of $X'$ is independent, up to canonical isomorphism, of all choices we have made).
For a given \( f : Z \to X \), where \( Z \) is reflexive, define \( Z_\alpha = f^{-1}(X_\alpha) \) and \( f_\alpha = f|_{Z_\alpha} \) for any \( \alpha \in L \). For each \( \alpha \in L \), we have the canonical factorization

\[
Z_\alpha \xrightarrow{f_\alpha^{\text{ref}}} X^\text{ref}_\alpha \xrightarrow{i_X} X_\alpha
\]

of the map \( f_\alpha \). By the functoriality, \( f_\alpha^{\text{ref}}|_{Z_\alpha \cap Z_\beta} \) is equal to \((f_\beta|_{Z_\alpha \cap Z_\beta})^{\text{ref}}\), and hence is equal to \( f_\beta^{\text{ref}}|_{Z_\alpha \cap Z_\beta} \). Hence, by patching, we obtain a map \( f' : Z \to X' \) such that \( i_X \circ f' = f \). The map \( f \) is valuative and locally quasi-compact, since it is obtained by patching local maps having these properties. Since the representability holds locally and the above construction is functorial, it follows that \((X', i_X)\) represents the functor \( F_X \), and that \( X^\text{ref} = X' \) gives the reflexivization of \( X \).

To conclude the proof of Theorem 2.4.6, it remains to verify (a) \( \sim \) (d). Condition (a) follows immediately from 2.4.11. For (b), note that the functor \([\cdot]\) has the continuity (2.3.25), and, by our construction of \( X^\text{ref} \), the functor \((\cdot)^{\text{ref}}\) has the similar continuity. Hence checking (b) is reduced to that in the case where \( X \) is coherent, which has already been shown in 2.4.7. Condition (c) follows from (a) and the fact that \( X^\text{ref} \) is coherent if \( X \) is coherent. Finally, (d) follows from (a) and (c).

**Corollary 2.4.12.** Let \( X, Y \) be valuative spaces, and \( f, g : X \to Y \) two valuative locally quasi-compact maps. Suppose \( X \) is reflexive. Then \([f] = [g]\) implies \( f = g \).

### 2.5 Locally strongly compact valuative spaces

#### 2.5. (a) Locally strongly compact valuative spaces

**Definition 2.5.1.** A valuative space \( X \) is said to be **locally strongly compact** if for any \( x \in X \) there exists a pair \((W_x, V_x)\) consisting of an overconvergent open subset \( W_x \) and a coherent open subset \( V_x \) such that \( x \in W_x \subseteq V_x \).

If \( X \) is coherent, then one can take \( W_x = V_x = X \) for any \( x \in X \). Hence we have the following result.

**Proposition 2.5.2.** Any coherent valuative space is locally strongly compact.

**Proposition 2.5.3.** Let \( X \) be a locally strongly compact valuative space. Then for any \( x \in X \) the closure of \( \{x\} \) in \( X \) is quasi-compact.

**Proof.** Take \((W_x, V_x)\) as in 2.5.1. Since \( W_x \) is overconvergent, \( \overline{\{x\}} \) is contained in \( W_x \), and hence in \( V_x \). Thus \( \overline{\{x\}} \) is a closed subset of the quasi-compact space \( V_x \), and hence is quasi-compact. \( \square \)
Proposition 2.5.4. Let \( X \) be a quasi-separated valuative space. Then \( X \) is locally strongly compact if and only if the following condition is satisfied: for any \( x \in X \) there exists a pair \((U_x, V_x)\) of coherent open neighborhoods of \( \{x\} \) such that \( \overline{U_x} \subseteq V_x \).

Proof. Suppose \( X \) is locally strongly compact, and let \( x \in X \). Take \((W_x, V_x)\) as in 2.5.1. Since \( \{x\} \) is quasi-compact, one can take a quasi-compact open neighborhood \( U_x \) of \( \{x\} \) inside \( W_x \); since \( X \) is quasi-separated, \( U_x \) is coherent. By 2.2.27, we have \( \overline{U_x} \subseteq W_x \subseteq V_x \).

Conversely, suppose \( X \) satisfies the condition in the assertion. For \( x \in X \), take \((U_x, V_x)\) as above. We may assume without loss of generality that \( x \) is a maximal point. Then, by 2.3.31, the pair \((W_x, V_x)\) with \( W_x = \text{int}_X(U_x) \) gives a pair as in 2.5.1.

Proposition 2.5.5. Let \( X \) be a quasi-separated valuative space. Then the following conditions are equivalent:

(a) \( X \) is locally strongly compact;

(b) the closure \( \overline{U} \) of any quasi-compact open subset \( U \subseteq X \) is quasi-compact;

(c) there exists an open covering \( \{U_\alpha\}_{\alpha \in L} \) of \( X \) such that \( U_\alpha \) and \( \overline{U_\alpha} \) are quasi-compact for any \( \alpha \in L \).

Proof. First we prove (a) \( \implies \) (c). By 2.5.4, \( X \) has an open covering \( \{U_\alpha\}_{\alpha \in L} \) such that each \( U_\alpha \) is coherent and its closure \( \overline{U_\alpha} \) is contained in a coherent open subset \( V_\alpha \). Since \( V_\alpha \) is quasi-compact, each \( \overline{U_\alpha} \) is quasi-compact.

Next we show (c) \( \implies \) (b). Since \( U \) is quasi-compact, there exist finitely many \( \alpha_1, \ldots, \alpha_n \in L \) such that \( U = \bigcup_{j=1}^n U_\alpha \cap U_{\alpha_j} \). By 2.3.7,

\[
\overline{U} = \bigcup_{j=1}^n \overline{U_\alpha \cap U_{\alpha_j}}.
\]

Since each \( \overline{U_\alpha \cap U_{\alpha_j}} \) is quasi-compact, we deduce that \( \overline{U} \) is quasi-compact.

Finally, let us show (b) \( \implies \) (a). For any \( x \in X \), take a quasi-compact open neighborhood \( W \) of \( x \). We have \( \{x\} \subseteq \overline{W} \). Since \( \overline{W} \) is quasi-compact, so is \( \{x\} \). Then one has a quasi-compact (hence coherent) open subset \( U \) that contains \( \{x\} \). Since \( \overline{U} \) is again quasi-compact, it is further contained in a quasi-compact open subset \( V \). In view of 2.5.4, this shows (a).

Remark 2.5.6. Proposition 2.5.5 shows that, if \( X \) is a quasi-separated valuative space, then \( X \) is locally strongly compact if and only if it is taut in the sense of [61], 5.1.2.
2.5. (b) Characteristic properties

**Theorem 2.5.7.** Let $X$ be a locally strongly compact valuative space.

(a) The separated quotient $[X]$ is locally compact (hence locally Hausdorff).

(b) The separation map $\text{sep}_X: X \to [X]$ is proper.

(c) Each $x \in [X]$ admits an open neighborhood $Z_x \subseteq [X]$ such that $\text{sep}_X^{-1}(Z_x)$ is quasi-separated.

Conversely, if a valuative space $X$ satisfies (b) and (c), then $X$ is locally strongly compact.

**Proof.** For any $x \in [X]$, take $(W_x, V_x)$ as in 2.5.1. Since $W_x$ is overconvergent also in $V_x$, we have $\text{sep}_X(W_x) = \text{sep}_{V_x}(W_x)$, which is open both in $[V_x]$ and $[X]$. Now, by 2.3.18 (2), we know that $[V_x]$ is compact. Hence $W_x = \text{sep}_X(W_x) = \text{sep}_{V_x}(W_x)$ is a Hausdorff open neighborhood of $x$ both in $[V_x]$ and $[X]$, and $[V_x]$ gives a compact neighborhood of $x$ in $[X]$. This shows that $W_x$ is a locally compact Hausdorff space, whence (a). For (b), it suffices to show that $W_x = \text{sep}_X^{-1}(W_x) \to W_x$ is proper for any $x \in [X]$. Since $W_x = \text{sep}_X^{-1}(W_x)$, this follows from the properness of $V_x \to [V_x]$ (2.3.20). One verifies (c) with $Z_x = W_x$; indeed, $W_x = \text{sep}_X^{-1}(W_x)$ is an open subspace of the coherent space $V_x$, and hence is quasi-separated.

Now we show the converse. Suppose $X$ is a valuative space that satisfies (b) and (c). For any point $x \in X$, we need to find a pair $(W_x, V_x)$ as in 2.5.1. To this end, we may assume that $x$ is a maximal point, that is, $x \in [X]$. Take an open neighborhood $Z_x$ of $x$ in $[X]$ such that $Z_x = \text{sep}_X^{-1}(Z_x)$ is quasi-separated. It follows from (b) that the closure $\overline{\{x\}}$ of $\{x\}$ in $X$, being equal to the pull-back $\text{sep}_X^{-1}(\{x\})$, is quasi-compact. Since $\overline{\{x\}}$ is contained in $Z_x$, we have a quasi-compact open subset $U_x \subseteq Z_x$ such that $\overline{\{x\}}$. Since $Z_x$ is quasi-separated, in view of 2.2.27, the closure $\overline{U}$ of $U$ in $Z_x$ is equal to $\text{sep}_X^{-1}(\text{sep}_X(U_x))$. This shows that $[U_x]$ is closed in $[X]$, and hence $\overline{U}_x$ is quasi-compact by the properness of $\text{sep}_X$. Thus there exists a quasi-compact open neighborhood $V_x$ of $\overline{U}_x$ in $Z_x$. By 2.3.31, we have $x \in \text{int}_{Z_x}(U_x)$. Set $W_x = \text{int}_{Z_x}(U_x)$. Since $W_x$ is overconvergent in $Z_x$ and $Z_x$ is overconvergent in $X$, $W_x$ is overconvergent in $X$. Hence the pair $(W_x, V_x)$ is a desired one. \hfill $\square$

**Corollary 2.5.8.** Let $X$ be a valuative space. The following conditions are equivalent:

(a) $X$ is locally strongly compact;

(b) any overconvergent open subset $W \subseteq X$ is locally strongly compact;

(c) there exists an open covering $\{W_\alpha\}_{\alpha \in L}$ of $X$ consisting of overconvergent open subsets such that each $W_\alpha$ ($\alpha \in L$) is locally strongly compact.
Corollary 2.5.9. The separated quotient $[X]$ of a locally strongly compact valuative space $X$ is locally compact (hence locally Hausdorff). Moreover $[X]$ is Hausdorff, whenever $X$ is quasi-separated.

Proof. The first assertion is already proven in 2.5.7. Suppose $X$ is quasi-separated. Take two points $x_1 \neq x_2$ in $[X]$ and, for each $i = 1, 2$, a pair $(W_{x_i}, V_{x_i})$ as in 2.5.1. Since $U = V_{x_1} \cup V_{x_2}$ is quasi-compact open and $X$ is quasi-separated, $U$ is coherent open. Using 2.3.17 (2), take for each $i = 1, 2$ an open neighbourhood $W_i' = y_i$ that is contained and overconvergent in $U$, such that $W_1' \cap W_2' = \emptyset$. Then $W_i \cap W_i'$ for $i = 1, 2$ are overconvergent in $X$ and separate $\{y_1\}$ and $\{y_2\}$. □

Corollary 2.5.10. Let $X$ be a quasi-separated locally strongly compact valuative space. Then $[X]$ is a completely regular topological space (cf. §2.1. (c)).

Proposition 2.5.11. Let $X$ be a reflexive valuative space (2.4.1). Suppose there exists a family \( \{U_\alpha\}_{\alpha \in L} \) of open subsets $X$ such that

(a) each $U_\alpha$ is locally strongly compact and retrocompact (2.1.7) in $X$ and

(b) \( [X] = \bigcup_{\alpha \in L} [U_\alpha]^\circ \), where $(\cdot)^\circ$ denotes the interior kernel.

Then $X$ is locally strongly compact.

Proof. Set $U_\alpha = [U_\alpha]^\circ$ and $W_\alpha = \text{sep}_X^{-1}(U_\alpha)$ for $\alpha \in L$. We have

\[
W_\alpha \subseteq (\text{sep}_X^{-1}([U_\alpha]))^\circ = (\overline{U_\alpha})^\circ;
\]

see 2.2.27. By 2.4.4, $W_\alpha \subseteq U_\alpha$ for any $\alpha \in L$. Thus we have the diagram

\[
\begin{array}{ccc}
W_\alpha & \xrightarrow{\text{sep}_X^{-1}} & U_\alpha & \xrightarrow{\text{sep}_X} & X \\
\downarrow_{\text{sep}_X | W_\alpha} & & \downarrow_{\text{sep}_X | U_\alpha} & & \downarrow_{\text{sep}_X} \\
U_\alpha & \rightarrow & [U_\alpha] & \rightarrow & [X].
\end{array}
\]

By Exercise 0.2.12, the topology of $[U_\alpha]$ coincides with the subspace topology induced by $[X]$. Since $W_\alpha$ is overconvergent in $U_\alpha$, it is locally strongly compact due to 2.5.8. Since $\{W_\alpha\}_{\alpha \in L}$ covers $X$, again by 2.5.8, we deduce that $X$ is locally strongly compact. □

Corollary 2.5.12. Let $X$ be a reflexive and quasi-separated valuative space having a family $\{U_\alpha\}_{\alpha \in L}$ of coherent open subsets such that $[X] = \bigcup_{\alpha \in L} [U_\alpha]^\circ$. Then $X$ is locally strongly compact.
2.5. (c) Paracompact spaces

**Definition 2.5.13.** Let \( X \) be a topological space.

1. An open covering \( \{U_\alpha\}_{\alpha \in L} \) of \( X \) is said to be *locally finite* if any \( x \in X \) has an open neighborhood \( V \) such that \( V \cap U_\alpha \neq \emptyset \) for at most finitely many indices \( \alpha \in L \).

2. The space \( X \) is said to be *paracompact* if any open covering admits an open locally finite refinement.

Note that our definition of ‘paracompact’ differs from that in [24], Chapter I, §9.10, Definition 6, in that we do not assume Hausdorffness.

The following lemma is easy to see, and the proof is left to the reader.

**Lemma 2.5.14.** Let \( X \) be a locally coherent space.

1. The space \( X \) is paracompact if there is a locally finite covering by quasi-compact open subsets.

2. Let \( \{U_\alpha\}_{\alpha \in L} \) be a locally finite covering of \( X \) consisting of quasi-compact open subsets. Then for any \( \alpha \in L \) the set of all indices \( \beta \in L \) such that \( U_\beta \) intersects \( U_\alpha \) is finite.

**Proposition 2.5.15.** Let \( X \) be a paracompact quasi-separated valuative space. Then \( X \) is locally strongly compact.

**Proof.** By 2.5.5, it suffices to show that for any quasi-compact open subset \( U \) its closure \( \bar{U} \) is again quasi-compact. Let \( \{V_\alpha\}_{\alpha \in L} \) be a covering of \( \bar{U} \), that is, \( \bar{U} \subseteq \bigcup_{\alpha \in L} V_\alpha \). Together with \( X \setminus \bar{U} \) this gives an open covering of \( X \), and hence, there exists a locally finite refinement \( \{W_\lambda\}_{\lambda \in \Lambda} \) of \( \{V_\alpha\}_{\alpha \in L} \) by quasi-compact open subsets. Let \( \Lambda' \) be the subset of \( \Lambda \) consisting of all \( \lambda \) such that \( \bar{U} \cap W_\lambda \neq \emptyset \). Since \( U \cap V_\lambda \neq \emptyset \) for any \( \lambda \in \Lambda' \) and \( U \) is quasi-compact, the set \( \Lambda' \) is actually finite, and thus \( \bigcup_{\lambda \in \Lambda'} V_\lambda \) is quasi-compact. Since \( \bar{U} \subseteq \bigcup_{\lambda \in \Lambda'} V_\lambda \), \( \bar{U} \) is quasi-compact, as desired. \( \square \)

The following proposition can be shown by an argument similar to that in [24], Chapter 1, §9.10, Theorem 5.

**Proposition 2.5.16.** Let \( X \) be a paracompact quasi-separated locally strongly compact valuative space. Then \( X \) can be written as a disjoint sum \( X = \bigsqcup_{\lambda \in \Lambda} X_\lambda \) such that for each \( \lambda \in \Lambda \) there exists an increasing sequence \( U_{\lambda,0} \subseteq U_{\lambda,1} \subseteq \cdots \) of quasi-compact open subsets satisfying...
(a) $\bar{U}_{\lambda,n} \subseteq U_{\lambda,n+1}$ for any $n \geq 0$ and
(b) $X_\lambda = \bigcup_{n \geq 0} U_{\lambda,n}$.

**Proposition 2.5.17.** Let $X$ be a quasi-separated locally strongly compact valuative space. Then $X$ is paracompact if and only if $[X]$ is paracompact (in the sense as in [24], Chapter I, §9.10, Definition 6); note that, due to 2.5.9, $[X]$ is locally compact and Hausdorff).

**Proof.** Suppose $X$ is paracompact. We may assume in view of 2.5.16 that there exists an increasing sequence $U_0 \subseteq U_1 \subseteq \ldots$ of quasi-compact open subsets satisfying conditions similar to (a) and (b) in 2.5.16. For each $n \geq 0$ let $V_n$ be the interior kernel of $[U_n]$ in $[X]$. Since $\overline{U}_n$ is overconvergent (2.2.27), we have $\text{sep}_{X}^{-1}(\overline{V}_n) \subseteq \overline{U}_n$ and hence $\text{sep}_{X}^{-1}(\overline{V}_n) \subseteq \text{int}_X(U_{n+1})$ (cf. 2.3.31). Thus we have $\overline{V}_n \subseteq V_{n+1}$ for any $n \geq 0$, and hence $[X]$ is paracompact, by [24], Chapter 1, §9.10, Theorem 5.

Conversely, suppose $[X]$ is paracompact, and take three open coverings

$$
X = \bigcup_{\alpha \in L} U_\alpha = \bigcup_{\alpha \in L} V_\alpha = \bigcup_{\alpha \in L} W_\alpha
$$

of $X$ by quasi-compact open subsets such that for any $\alpha \in L$ we have $\overline{U}_\alpha \subseteq V_\alpha$ and $\overline{V}_\alpha \subseteq W_\alpha$ (this is possible due to 2.5.5 (b)). By 2.3.31,

$$
[X] = \bigcup_{\alpha \in L} [V_\alpha]^o = \bigcup_{\alpha \in L} [W_\alpha]^o
$$

(where $(\cdot)^o$ denotes the interior kernel). Take locally finite open coverings $[X] = \bigcup_{i \in I} \overline{V}_i$ and $[X] = \bigcup_{j \in J} \overline{W}_j$ that refine $\{[V_\alpha]^o\}_{\alpha \in L}$ and $\{[W_\alpha]^o\}_{\alpha \in L}$, respectively. Since for each $i \in I$ there exists an $\alpha \in L$ such that $\text{sep}_{X}^{-1}(\overline{V}_i) \subseteq \overline{V}_\alpha$, we deduce that the closure of $\text{sep}_{X}^{-1}(\overline{V}_i)$ is quasi-compact. Moreover, since $[X] = \bigcup_{j \in J} \overline{W}_j$ is a locally finite covering, $\text{sep}_{X}^{-1}(\overline{V}_i) \cap \text{sep}_{X}^{-1}(\overline{W}_j)$ is non-empty for only finitely many $j \in J$, say, $j_1, \ldots, j_{i,m_i}$. One can take a finite collection $\{U_{i_1}, \ldots, U_{i_{n_i}}\}$ of quasi-compact open subsets contained in $\bigcup_{k=1}^{m_i} \text{sep}_{X}^{-1}(\overline{W}_{j_{i,k}})$ such that $\text{sep}_{X}^{-1}(\overline{V}_i) \subseteq \bigcup_{k=1}^{n_i} U_{i_k}$. Then $\{U_{i_k}\}_{i \in I, 1 \leq k \leq n_i}$ is a locally finite covering of $X$ by quasi-compact open subsets. □

### 2.6 Valuations of locally Hausdorff spaces

In this subsection, we give a description of the category of reflexive (2.4.1) locally strongly compact (2.5.1) valuative spaces with valuative (2.3.21) and locally quasi-compact (2.2.24) maps. We will show, in 2.6.19 below, that this category is com-
pletely described, via separated quotients (§2.3.(c)), by locally compact (and locally Hausdorff) spaces with an extra structure, which we call a valuation. The description we give here can be regarded as a variant of Stone duality (cf. §2.2.(b)).

2.6. (a) Nets and coverings

Definition 2.6.1 (cf. [12] and [13]). Let $X$ be a topological space.

1. A collection $\tau$ of subsets of $X$ is called a quasi-net on $X$ if for each $x \in X$ there exists finitely many $U_1, \ldots, U_n \in \tau$ such that $x \in U_1 \cap \cdots \cap U_n$ and that $U_1 \cup \cdots \cup U_n$ is a (possibly not open) neighborhood of $x$.

2. A quasi-net $\tau$ is called a net on $X$ if for any $U, U' \in \tau$ the subset $\{U'' \in \tau : U'' \subseteq U \cap U'\}$ of $\tau$ is a quasi-net on the subspace $U \cap U'$.

Note that if $X$ is Hausdorff, then one can omit the condition ‘$x \in U_1 \cap \cdots \cap U_n$’ in (1). Let us recall some of the basic properties of quasi-nets and nets (cf. [12], §1.1). Let $\tau$ be a quasi-net on a topological space $X$.

- A subset $W \subseteq X$ is open if and only if for any $U \in \tau$ the intersection $U \cap W$ is open in $U$.
- If any element of $\tau$ is compact, then $X$ is Hausdorff if and only if for any $U, V \in \tau$ the intersection $U \cap V$ is again compact.

Suppose, moreover, that $X$ is locally Hausdorff and $\tau$ is a net consisting of compact subsets. Then

- for any $U, V \in \tau$, the intersection $U \cap V$ is locally closed in $U$ and $V$.

Proposition 2.6.2. Let $X$ be a locally strongly compact (2.5.1) valuative space, see 2.3.1, and $X = \bigcup_{\alpha \in L} U_\alpha$ an open covering by quasi-compact open subsets. Then the collection $\tau = \{[U_\alpha]\}_{\alpha \in L}$ gives a quasi-net on the separated quotient $[X]$ (§2.3.(c)). It is a net if and only if for any $\alpha, \beta \in L$ the collection $\{U_\gamma \mid U_\gamma \subseteq U_\alpha \cap U_\beta\}$ is a covering of $U_\alpha \cap U_\beta$.

Note that, due to 2.5.9, the separated quotient $[X]$ is a locally compact, hence locally Hausdorff, space.

Proof. Let $x \in [X]$ be a maximal point of $X$, and consider the closure $\overline{\{x\}}$ in $X$ of the singleton set $\{x\}$. Since $\{x\}$ is quasi-compact due to 2.5.3, there exists an
overconvergent open neighborhood $W$ of $\{x\}$ such that $\overline{W}$ is quasi-compact. Replacing, if necessary, $W$ by a smaller overconvergent open neighborhood of $\{x\}$, one can take $\alpha_1, \ldots, \alpha_n \in L$ such that $\overline{W} \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ and $\{x\} \cap U_{\alpha_i} \neq \emptyset$ for each $i = 1, \ldots, n$. Then $[U_{\alpha_1}] \cup \cdots \cup [U_{\alpha_n}]$ is a neighborhood of $x$ in $[X]$, since it contains $[W]$, which is open in $[X]$. Moreover, since $\{x\} \cap U_{\alpha_i} \neq \emptyset$ for each $i = 1, \ldots, n$, we have $x \in [U_{\alpha_1}] \cap \cdots \cap [U_{\alpha_n}]$, showing that $\tau = \{[U_{\alpha}]\}_{\alpha \in L}$ is a quasi-net on $[X]$. The other assertion is clear.

2.6. (b) Valuations of compact spaces. If $X$ is a coherent reflexive valuative space, then $X \cong X^{\text{ref}}$ is completely determined by the distributive lattice $QCOuv(X)$ of coherent open subsets, which is isomorphic to the distributive lattice

$$v = \{[U]: U \in QCOuv(X)\}$$

consisting of compact subsets of $[X]$ (cf. §2.4 (c)). This motivates the following definition.

**Definition 2.6.3.** Let $S$ be a compact (in particular, Hausdorff) space.

1. A distributive sublattice $v$ of $2^S$ is called a valuation of $S$ if there exists a continuous map $\pi: X \to S$ from a coherent reflexive valuative space $X$ such that

   (a) the map $[X] \to S$ induced by $\pi$ (cf. 2.3.9) is a homeomorphism and

   (b) $v$ coincides, through the identification $[X] \cong S$, with the lattice $\{[U]: U \subseteq X \in QCOuv(X)\}$.

2. If $v$ is a valuation of $S$, then the pair $(S, v)$ is called a valued compact space.

Note that if $(S, v)$ is a valued compact space, then the coherent reflexive valuative space $X$ is uniquely determined up to canonical homeomorphisms and isomorphic to $\text{Spec} v$ by Stone duality (cf. 2.2.8). Note also that if $v$ is a valuation of $S$, then every member $T \in v$ of $v$ is a compact (and hence closed) subset of $S$.

**Example 2.6.4.** The distributive lattice $2^S$ for a singleton set $S = \{\ast\}$ with the obvious compact topology is a valuation of $S$, for one has the obvious continuous map $\text{Spf} V \to S$ from the formal spectrum of an $a$-adically complete valuation ring ($a \in m_V \setminus \{0\}$) of height one (cf. 2.4.2).
Proposition 2.6.5. Let \( S \) be a compact (hence Hausdorff) space, and \( v \) a distributive lattice consisting of compact subspaces of \( S \). Then \( v \) is a valuation of \( S \) if and only if

(a) \( \Spec v \) is a valuative space and

(b) for any \( x \in S \) and any neighborhood \( U \) of \( x \), there exists \( T \in v \) such that \( x \in T \subseteq U \).

Moreover, in this situation, there exists a canonical continuous map \( \pi : \Spec v \to S \) that induces \( [\Spec v] \cong S \).

Proof. The ‘only if’ part is clear. For any \( p \in \Spec v \), consider the prime filter \( F_p \) corresponding to \( p \), that is, the complement of \( p \) in \( v \). As in the proof of 2.4.8, points in \( \bigcap_{T \in F_p} T \) canonically correspond to maximal filters \( F' \) that contain \( F_p \). Condition (a) implies that such \( F' \) is unique, and hence \( \bigcap_{T \in F_p} T = \{\pi(p)\} \) for some \( \pi(p) \in S \). This gives a map

\[
\pi : \Spec v \to S, \quad p \mapsto \pi(p),
\]

which is surjective due to 2.4.8. For each \( T \in v \), set \( U_T = \{p \in \Spec v : T \in F_p\} \). Then \( U_T \) is open in \( \Spec v \), and its image under \( \pi \) is contained in \( T \). Since \( \{U_T : T \in v\} \) forms an open basis of \( \Spec v \), the map \( \pi \) is continuous by (b). Moreover, \( \pi \) restricted on \( [\Spec v] \) is the inverse of the map \( x \mapsto F_x \) as in 2.4.8 (2), and thus induces the continuous bijection \( [\Spec v] \to S \) between compact Hausdorff spaces. Hence \( v \) is gives a valuation of \( S \), and the proposition is proved. \( \square \)

Corollary 2.6.6. Let \( S \) be a compact space, and \( v \) and \( v' \) distributive lattices consisting of compact subsets of \( S \). Suppose that \( v \subseteq v' \), that \( v \) is a valuation of \( S \), and that \( \Spec v' \) is valuative. Then \( v' \) is a valuation of \( S \).

Let \( S \) be a compact space, \( v \) a valuation of \( S \), and \( T = [U] \in v \), where \( U \subseteq X = \Spec v \) is a quasi-compact open subset. Then the following statements are easy to verify.

- The collection

\[
v|_T = \{T' \in v : T' \subseteq T\}
\]

is a net on \( T \) (thanks to 2.6.2), and is a valuation of \( T \); in particular, \( v \) is a net on \( S \).

- The homomorphism

\[
v \to v|_T, \quad T' \mapsto T' \cap T
\]

gives rise to the open immersion \( \Spec v|_T \hookrightarrow \Spec v \) corresponding to the inclusion map \( U \hookrightarrow X \).
**Definition 2.6.7.** A morphism \( f : (S, v) \to (S', v') \) between valued compact spaces is a continuous mapping \( f : S \to S' \) such that

(a) \( f \) induces \( f^* : v' \to v \) by \( T \mapsto f^{-1}(T) \)

(b) \([\text{Spec } f^*] = f\).

Note that a morphism of valued compact spaces \( f : (S, v) \to (S', v') \) induces a valuative map \( \text{Spec } f^* : X = \text{Spec } v \to X' = \text{Spec } v' \) of coherent reflexive valuative spaces such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\text{Spec } f^*} & X' \\
\downarrow & & \downarrow \\
S & \xrightarrow{f} & S'
\end{array}
\]

commutes. Now the following proposition is easy to see, essentially due to Stone duality (§2.2. (b)).

**Proposition 2.6.8.** The functor

\[
(S, v) \mapsto \text{Spec } v
\]

gives a categorical equivalence from the category of valued compact spaces (with the above-defined morphisms) to the category of coherent reflexive valuative spaces with valuative quasi-compact maps. The quasi-inverse to this functor is given by

\[
X \mapsto ([X], v),
\]

where

\[
v = \{[U] : U \in \text{QCOuv}(X)\}.
\]

**2.6. (c) Valuations of locally Hausdorff spaces**

**Definition 2.6.9.** Let \( X \) be a locally Hausdorff space.

(1) A pre-valuation \( v = (\tau(v), \{v_S\}_{S \in \tau(v)}) \) of \( X \) consists of

(a) a net \( \tau(v) \) of compact subspaces of \( X \)

(b) a valuation \( \tau(v) \) of \( S \) for each \( S \in \tau(v) \),

satisfying

- for any \( S, S' \in \tau(v) \) with \( S \subseteq S' \), we have \( v_S = \{T \in v_{S'} : T \subseteq S\} \).
(2) A pre-valuation \( v = (\tau(v), \{v_S\}_{S \in \tau(v)}) \) of \( X \) is said to be saturated if \( \bigcup_{S \in \tau(v)} v_S = \tau(v) \).

(3) A pre-valuation \( v = (\tau(v), \{v_S\}_{S \in \tau(v)}) \) is called a valuation if it is saturated and, for any finitely many elements \( S_1, \ldots, S_n \in \tau(v) \), \( \bigcup_{i=1}^{n} S_i \) again belongs to \( \tau(v) \) whenever it is Hausdorff as a subspace of \( X \).

(4) If \( v \) is a valuation of \( X \), then the pair \( (X, v) \) is called a valued locally Hausdorff space.

Note that

- a locally Hausdorff space that admits a pre-valuation is locally compact;
- if a pre-valuation \( v \) of \( X \) is saturated, then \( v \) is determined by the net \( \tau(v) \); indeed, for any \( S \in \tau(v) \), one has \( v_S = \{T \in \tau(v): T \subseteq S\} \);
- if \( X \) is compact, then the notion of valuations defined here coincides with the previous one (2.6.3 (1)) (whence the abuse of terminology); indeed, if \( v \) is a valuation of \( X \) in the sense of 2.6.3 (1), then \( (v, \{v|_S\}_{S \in \tau(v)}) \) gives a valuation in the sense of 2.6.9; conversely, given a valuation \( v \) of \( X \) in the latter sense, since \( X \) is a finite union of elements of \( \tau(v) \), \( X \) itself belongs to the net \( \tau(v) \).

Remark 2.6.10. Note that if \( (X, v) \) is a valued locally Hausdorff space, the net \( \tau(v) \) does not, in general, give a distributive lattice, for it may fail to contain \( X \) (the unit element). In other words, \( \tau(v) \) is a distributive lattice if and only if \( X \) is compact, in which case the valued locally Hausdorff space \( (X, v) \) is a valued compact space.

Definition 2.6.11. A morphism \( f: (X, v) \to (X', v') \) between valued locally Hausdorff spaces is a continuous mapping \( f: X \to X' \) such that, for any \( S \in \tau(v) \) and \( S' \in \tau(v') \) such that \( f(S) \subseteq S' \), the map \( f|_S: S \to S' \) with the valuations \( v_S \) and \( v_{S'} \) of \( S \) and \( S' \), respectively, is a morphism of valued compact spaces (2.6.7).

2.6. (d) Saturation and associated valuations

Definition 2.6.12. Let \( X \) be a locally Hausdorff space \( X \), and \( v_1, v_2 \) pre-valuations of \( X \). We say that \( v_2 \) is an extension of \( v_1 \), or that \( v_1 \) is a restriction of \( v_2 \), and we write

\[ v_1 \leq v_2, \]

if \( \tau(v_1) \subseteq \tau(v_2) \) and \( (v_1)_S = (v_2)_S \) for any \( S \in \tau(v_1) \).
Proposition 2.6.13. Any pre-valuation of a compact space extends uniquely to a valuation.

Proof. Let $X$ be a compact space, and $v = (\tau(v), \{v_S\}_{S \in \tau(v)})$ a pre-valuation of $X$. Let $\tilde{v}$ be the distributive sublattice of $2^X$ generated by all $v_S$’s. More precisely, a subset $T \subseteq X$ belongs to $\tilde{v}$ if it is a finite union of finite intersections of elements in $\bigcup_{S \in \tau(v)} v_S$. Then $\tilde{v}$ is a valuation of $X$ such that $\tilde{v}|_S = v_S$ for any $S \in \tau(v)$; indeed, one can glue the valuative spaces $\text{Spec } v_S$ by open immersions (cf. §2.6. (b)). It is then clear that $\tilde{v} = (\tilde{v}, \{\tilde{v}|_T\}_{T \in \tilde{v}})$ extends $v$. The uniqueness is also clear. □

Proposition 2.6.14. Let $X$ be a locally Hausdorff space, and let

$$v = (\tau(v), \{v_S\}_{S \in \tau(v)})$$

be a pre-valuation of $X$. Set

$$\tau(v_{\text{sat}}) = \bigcup_{S \in \tau(v)} v_S$$

and for any $T \in \tau(v_{\text{sat}})$

$$v_{T_{\text{sat}}} = \{T' \in \tau(v_{\text{sat}}): T' \subseteq T\}.$$ 

Then $v_{\text{sat}} = (\tau(v_{\text{sat}}), \{v_{T_{\text{sat}}}\}_{S \in \tau(v_{\text{sat}})})$ gives the smallest saturated extension of $v$ (called the saturation of $v$).

Proof. It is clear that $\tau(v_{\text{sat}})$ is a quasi-net. To show that it is a net, take $S_1, S_2 \in \tau(v_{\text{sat}})$ and set $S = S_1 \cap S_2$. We want to show that $\tau(v_{\text{sat}})|_S$ is a quasi-net. Since $X$ is locally compact, $S$ is locally closed both in $S_1$ and $S_2$, and also in $X$. This allows one to discuss locally on $X$ and to assume that $S$ is compact. Similarly to the proof of 2.6.13, consider the distributive sublattice $v'_S$ of $2^S$ generated by all $T' \in v_T$ for $T \in \tau(v)$ and $T \subseteq S$. Then $v'_S$ gives a valuation of $S$, and hence is a net. Since each element of $v'_S$ is a finite union of finite intersections of $T$’s as above, one verifies the conditions in 2.6.1 for any $x \in S$, showing that $\tau(v_{\text{sat}})|_S$ is a quasi-net, as desired. The rest of the proof is obvious. □

Let $X$ be a locally Hausdorff space, $v = (\tau(v), \{v_S\}_{S \in \tau(v)})$ a pre-valuation of $X$, and $C \subseteq X$ a subset. Set

$$v_C = (\{S \in \tau(v): S \subseteq C\}, \{v_S\}_{S \in \tau(v), S \subseteq C}).$$

If $v_C$ defines a pre-valuation of $C$, we call it the restriction of $v$ to $C$. The following assertions are easy to verify; cf. Exercise 0.2.18.
Proposition 2.6.15. Let $X$ be a locally Hausdorff space, $v = (\tau(v), \{v_S\}_{S \in \tau(v)})$ a pre-valuation of $X$. Define $\sigma(v)$ to be the set of all compact subsets $S \subseteq X$ such that the restriction of $v_{\text{sat}}$ to $S$, denoted consistently by $v_{\text{sat}}^S$, exists. For any $S \in \tau(v)$, set $v_S^{\text{val}}$ to be the unique valuation of $S$ that extends $v_{\text{sat}}^S$ (2.6.13). Then the data

$$v^{\text{val}} = (\tau(v), \{v_S^{\text{val}}\}_{S \in \tau(v)})$$

give the minimal valuation that extends $v$ (called the valuation associated to $v$).

Theorem 2.6.16. Let $X$ be a locally Hausdorff space, $v = (\tau(v), \{v_S\}_{S \in \tau(v)})$ a pre-valuation of $X$. Define

$$\text{Spec } v = \lim_{\to} \text{Spec } v_S.$$ 

Then $\text{Spec } v$ is a reflexive valuative space with a canonical homeomorphism

$$[\text{Spec } v] \cong X.$$ 

Moreover, $\sigma(v)$ coincides with the set of all subsets of $X$ that are images under the separation map $\text{sep}$ of coherent open subsets of $\text{Spec } v$.

Corollary 2.6.17. Any pre-valuation $v$ of a locally Hausdorff space $X$ has an extension to a valuation. Moreover, such an extension is unique.

2.6. (e) Reflexive locally strongly compact valuative spaces. For a valued locally Hausdorff space $(X, v)$, we have constructed the reflexive valuative space $\mathcal{X} = \text{Spec } v$ together with the canonical map $\mathcal{X} \to X$ inducing a homeomorphism $[\mathcal{X}] \cong X$; see 2.6.16.

Proposition 2.6.18. The valuative space $\mathcal{X} = \text{Spec } v$ is locally strongly compact. Moreover, if $X$ is Hausdorff (resp. compact), then $\mathcal{X}$ is quasi-separated (resp. coherent).

Proof. Since $\tau(v) = \{[U] : U \subseteq X \text{ coherent}\}$, for any open subset $W \subseteq [X]$ the restriction $v|_W$ of $v$ to $W$ exists, giving $\text{Spec } v|_W \cong \text{sep}^{-1}(W)$ (cf. Exercise 0.2.18). Hence, to show that $\mathcal{X}$ is locally strongly compact, we may assume that $X$ is Hausdorff.

Since $\tau(v)$ is a net, there exists an open covering $\{U_\alpha\}_{\alpha \in L}$ consisting of coherent open subsets of $\mathcal{X}$ such that $[X] = \bigcup_{\alpha \in L} [U_\alpha]$, where $(\cdot)^\circ$ denotes the interior kernel. Moreover, since $v$ is a valuation, we have $[U_\alpha] \in \tau(v)$ for any $\alpha \in L$. For any $\alpha, \beta \in L$, the restriction $\tau(v)|_{[U_\alpha \cap U_\beta]}$ of $\tau(v)$ to $[U_\alpha \cap U_\beta] = [U_\alpha] \cap [U_\beta]$ is a quasi-net. Hence there exists a family $\{V_\lambda\}_{\lambda \in \Lambda}$ of coherent open subsets of $U_\alpha \cap U_\beta$ such that $[U_\alpha \cap U_\beta] = \bigcup_{\lambda \in \Lambda} [V_\lambda]$. Then $U_\alpha \cap U_\beta$ is, since it is reflexive and quasi-separated valuative space, locally strongly compact due to 2.5.12.
Now by Exercise 0.2.12, the topology on \([U_\alpha \cap U_\beta]\) coincides with the subspace topology induced by \([U_\alpha]\). Since \([U_\alpha]\) and \([U_\beta]\) are compact, and since \(X = [X]\) is Hausdorff, we deduce that \([U_\alpha \cap U_\beta] = [U_\alpha] \cap [U_\beta]\) is compact, and hence, by 2.5.7 (b), \(U_\alpha \cap U_\beta\) is quasi-compact. This shows that \(X\) is quasi-separated, and hence is locally strongly compact by 2.5.12.

Finally, if \([X]\) is compact, then by 2.5.7 (b), \(X\) is quasi-compact, hence coherent.

**Theorem 2.6.19.** The functor \((X, v) \mapsto \text{Spec } v\) gives a categorical equivalence from the category of valued locally Hausdorff spaces (with the morphisms defined as in 2.6.11) to the category of reflexive locally strongly compact valuative spaces with valuative locally quasi-compact maps. The quasi-inverse to this functor is given by

\[ X \mapsto ([X], v = (\tau(v), \{v_S\}_{S \in \tau(v)}), \]

where

\[ \tau(v) = \{[U] : U \subseteq X \text{ coherent}\} \]

and, for any \([U] \in \tau(v),\]

\[ v[U] = \{[V] \subseteq [U] : V \in \text{QCOuv}(U)\}. \]

Moreover, \(X\) in \((X, v)\) is Hausdorff (resp. compact) if and only if \(\text{Spec } v\) is quasi-separated (resp. coherent).

**Proof.** We first show that any morphism \(f : (X, v) \to (X', v')\) of valued locally Hausdorff spaces induces a valuative locally quasi-compact map

\[ \text{Spec } f : \text{Spec } v \longrightarrow \text{Spec } v'. \]

Since we already know from 2.6.16 that \(\tau(v)\) (resp. \(\tau(v')\)) is the set of all \([U]\)’s for coherent open subsets \(U\) of \(X = \text{Spec } v\) (resp. \(X' = \text{Spec } v'\), \(f\) induces, for any \([U] \in \tau(v)\) and \([U'] \in \tau(v')\) with \(f([U]) \subseteq [U']\), a valuative quasi-compact map

\[ \text{Spec } f|_{[U]} : U = \text{Spec } v[U] \longrightarrow U' = \text{Spec } v'[U'] \]

such that

\[ [\text{Spec } f|_{[U]}] = f|_{[U]}. \]

Hence, in view of 2.4.12, the desired map

\[ \text{Spec } f : \text{Spec } v \longrightarrow \text{Spec } v' \]

is provided by

\[ \text{Spec } f = \lim_{S \in \tau(v)} \text{Spec } f|_S, \]
which is valuative and locally quasi-compact. In view of 2.6.18,

\[(X, v) \mapsto \text{Spec } v\]

provides the desired functor between the indicated categories.

Now, for a reflexive locally strongly compact valuative space \(X\), consider the data \(\{[X], v = (\tau(v), \{v_S\}_{S \in \tau(v)})\}\) as above. By 2.5.7, \([X]\) is a locally Hausdorff space. Moreover, due to 2.6.8, each \(v[U]\) gives the valuation of \([U]\) that corresponds to \(U\). Hence \(v\) gives a pre-valuation of \([X]\), and one can easily verify that it is a valuation. It is clear, then, that

\[X \mapsto ([X], v = (\tau(v), \{v_S\}_{S \in \tau(v)})\)

gives a functor. By 2.6.8 and a straightforward patching argument, this gives a quasi-inverse functor to the above functor.

Finally, the last statement follows from 2.6.18, 2.5.9, and 2.3.18 (2).

\[\square\]

2.7 Some generalities on topoi

In this book, a site will always mean a U-site ([8], Exposé II, (3.0.2)).\(^9\) Topos in this book is always a Grothendieck topos; a Grothendieck topos, or more precisely, Grothendieck U-topos is the category of U-sheaves of sets ([8], Exposé II, (2.1)) on a site ([8], Exposé IV, (1.1)). Our general reference to topos theory is [8], Exposé IV, and [80]. We denote by \(\text{TOPOI}\) the 2-category of topoi, that is, the 2-category in which the objects are topoi, 1-morphisms are morphisms of topoi, and for two 1-morphisms \(f, g: X \to Y\) the 2-morphisms are the natural transformations from \(f^*\) to \(g^*\) (see [55], Chapter I, §1, for generalities on 2-categories).

2.7. (a) Spacial topos. For a topological space \(X\), we denote by \(\text{top}(X)\) the associated topos, that is, the category of U-sheaves of sets over \(X\);\(^10\) we call \(\text{top}(X)\) the sheaf topos of \(X\). The category \(\text{Sets}\) of sets in U (§1.2. (b)) is a topos, as it is identical with the sheaf topos of the topological space consisting of one point (endowed with the unique topology).

The sheaf topos construction gives rise to a 2-functor

\[\text{top}: \text{Top} \longrightarrow \text{TOPOI}.\]

\(^9\)We sometimes employ such commonly-used expressions as ‘large’ sites and ‘small’ sites, which, as usual, have nothing to do with the set-theoretic size, such as U-small.

\(^{10}\)Note that since we only deal with the topological spaces in U (cf. 1.2. (b)), the topos \(\text{top}(X)\) is actually a U-topos.
A topos that belongs to the essential image of $\text{top}$ is said to be spacial; that is, a topos $E$ is spacial if it is equivalent to the sheaf topos of a topological space. Note that the canonical map $X \to X^{\text{sob}}$ (§2.1. (b)) induces an equivalence of topoi

$$\text{top}(X) \xrightarrow{\sim} \text{top}(X^{\text{sob}}).$$

Thus a spacial topos is always equivalent to the sheaf topos of a sober topological space.

**Theorem 2.7.1** ([8], Exposé IV, (4.2.3)). The 2-functor

$$\text{top}: \text{STop} \longrightarrow \text{TOPOI}$$

is 2-faithful, that is, for any sober topological spaces $X$ and $Y$, the functor

$$\text{Hom}_{\text{STop}}(X, Y) \longrightarrow \text{Hom}_{\text{TOPOI}}(\text{top}(X), \text{top}(Y))$$

is an equivalence of categories, where the left-hand side is considered to be a discrete category (§1.2. (d)).

**2.7. (b) Points.** Let $E$ be a topos. A point of $E$ is a morphism of topoi

$$\xi: \text{Sets} \longrightarrow E$$

or, equivalently, a functor (so-called fiber functor)

$$\xi^*: E \longrightarrow \text{Sets}$$

that commutes with finite projective limits and arbitrary inductive limits. A topos $E$ has enough points if, for any arrow $u$ in $E$, $u$ is a monomorphism (resp. an epimorphism) if and only if $\xi^*(u)$ is injective (resp. surjective) for any point $\xi$.

For a $U$-small topos $E$ we denote by $\text{pts}(E)$ and $\text{ouv}(E)$ the set of all isomorphism classes of points of $E$ and, respectively, the set of all isomorphism classes of subobjects of a fixed final object of $E$. For any $U \in \text{ouv}(E)$ we set

$$|U| = \{\xi \in \text{pts}(E): \xi^*(U) \neq \emptyset\}.$$ 

Then the collection $\{|U|: U \in \text{ouv}(E)\}$ is stable under finite intersections and under arbitrary unions and hence gives a topology on the set $\text{pts}(E)$. We denote the resulting topological space by $\text{sp}(E)$. The topological space $\text{sp}(E)$ is sober; if $E = \text{top}(X)$ for a topological space $X$, then

$$X^{\text{sob}} \cong \text{sp}(\text{top}(X))$$

([8], Exposé IV, (7.1.6)).
2.7. (c) **Localic topoi.** A spacial topos $E$ satisfies (cf. [63], §5.3)

(SG) $E$ is generated by subobjects of a fixed final object.

A (Grothendieck) topos that satisfies (SG) is said to be *localic*. Note that localic topoi are not always spacial.

**Theorem 2.7.2** (cf. [63], Theorem 7.25). *A localic topos $E$ is spacial if and only if $E$ has enough points.*

**Proof.** Here we give a sketch of the proof of the ‘if’ part; see loc. cit. for the details. Since $E$ has enough points, any open set $U$ of $X = \text{sp}(E)$ corresponds bijectively to an element $U_E$ of $\text{ouv}(E)$. For any object $u$ of $E$, $U \mapsto \text{Hom}_E(U_E, u)$ gives a sheaf on $X$. The construction gives a functor $E \to \text{top}(X)$. Since $U_E$’s for any open $U \subseteq X$ generate $E$, it follows that the functor gives a categorical equivalence. □

Let us finally remark that the notion of localic topoi is closely related with the theory of *locales*; cf. [64], Chapter II.

2.7. (d) **Coherent topoi.** In this subsection all topoi are considered with the canonical topology ([8], Exposé II, (2.5)), whenever considered as a site.

**Definition 2.7.3.** (1) An object $X$ of a site (or a topos; cf. [9], Exposé VI, (1.2)) is said to be *quasi-compact* if any covering family $\{X_i \to X\}_{i \in I}$ has a finite covering subfamily.

(2) An object $X$ of a topos $E$ is said to be *quasi-separated* if, for any arrows $S \to X$ and $T \to X$ in $E$ with $S$ and $T$ quasi-compact, the fiber product $S \times_X T$ is quasi-compact.

(3) An object $X$ of a topos $E$ is said to be *coherent* if it is quasi-compact and quasi-separated.

**Definition 2.7.4.** Let $E$ be a topos.

(1) An arrow $f : X \to Y$ in $E$ is *quasi-compact* if, for any arrow $Y' \to Y$ in $E$ with $Y'$ quasi-compact, the fiber product $X \times_Y Y'$ is quasi-compact.

(2) An arrow $f : X \to Y$ in $E$ is *quasi-separated* if the diagonal arrow $X \to X \times_Y X$ is quasi-compact.

(3) An arrow $f : X \to Y$ in $E$ is *coherent* if it is quasi-compact and quasi-separated.

We next define basic notions for topoi based on finiteness conditions for objects and morphisms listed above.
Definition 2.7.5. A topos $E$ is \textit{quasi-separated} (resp. \textit{coherent}) if it satisfies the following conditions:

(a) there exists a generating full subcategory consisting of coherent objects;

(b) every object $X$ of $E$ is quasi-separated over the final object, that is, the diagonal morphism $X \to X \times X$ is quasi-compact;

(c) the final object of $E$ is quasi-separated (resp. coherent).

Let $E$ be a topos, and consider the conditions: $E$ admits a generating full subcategory $C$ consisting of quasi-compact objects such that $C$ is

(a) stable under fiber products,

(b) stable under fiber products and by products of two objects, and

(c) stable under any finite projective limits.

Then

(b) $\iff$ $E$ is quasi-separated;

(c) $\iff$ $E$ is coherent.

Definition 2.7.6. We say that the topos $E$ is \textit{locally coherent} if it admits a generating full subcategory $C$ consisting of quasi-compact objects satisfying (a).

Note that, in this situation, any object of $C$ is coherent ([9], Exposé VI, (2.1)).

Definition 2.7.7. Let $f : E' \to E$ be a morphism of topoi. We say that $f$ is \textit{quasi-compact} (resp. \textit{quasi-separated}) if, for any quasi-compact (resp. quasi-separated) object $X$ of $E$, $f^*(X)$ is quasi-compact (resp. quasi-separated). If $f$ is quasi-compact and quasi-separated, we say that $f$ is \textit{coherent}.

Proposition 2.7.8. Let $X$ be a topological space that admits an open basis consisting of quasi-compact open subsets, and consider the associated topos $\text{top}(X)$. Then $\text{top}(X)$ is quasi-separated (resp. locally coherent, resp. coherent) if and only if $X$ is quasi-separated (resp. locally coherent, resp. coherent).

We state an important result by Deligne.

Theorem 2.7.9 ([9], Exposé VI, (9.0)). A \textit{locally coherent} topos $E$ has \textit{enough points}.

By 2.7.2 and 2.7.9, we have the following corollary.

Corollary 2.7.10. Any \textit{coherent} localic topos is \textit{spacial}.
2.7. (e) Fibered topoi and projective limits

**Definition 2.7.11** ([9], Exposé VI, (7.1.1)). Let $I$ be a category. A fibered category (considered with a cleavage)

$$p: E \to I$$

over $I$ is said to be a *fibered topos* or an *$I$-topos* if every fiber $F_i$ ($i \in \text{obj}(I)$) is a topos and for any arrow $f: i \to j$ of $I$ there exists a morphism of topoi $E_f: E_i \to E_j$ such that the pull-back morphism $f^*: E_j \to E_i$ by $f$ coincides with $E_f^*$.

Hence, by the cleavage construction of fibered categories ([52], Exposé VI, §8), giving a fibered topos is equivalent to giving a functor

$$E_*: I \to \text{TOPOI}.$$  

We denote by $\text{TOPOI}/I$ the 2-category of $I$-topoi.

Recall that for a fibered category $p: E \to I$ a morphism $\varphi: x \to y$ over a map $f: i \to j$ is said to be *Cartesian* if, for any object $z$ with $p(z) = i$, the canonical map

$$\text{Hom}_f(z, x) \to \text{Hom}_f(z, y), \quad \alpha \mapsto \varphi \circ \alpha,$$

is a bijection, where the left-hand set consists of arrows $z \to x$ over $\text{id}_i$, and the right-hand set consists of arrows $z \to y$ over $f$. An $I$-functor $E \to F$ of fibered categories over $I$ is said to be *Cartesian* if it maps Cartesian arrows to Cartesian arrows. The category of Cartesian functors from $E$ to $F$ is denoted by

$$\text{Cart}/I(E, F).$$

**Definition 2.7.12** ([9] Exposé VI, (8.1.1)). Let $p: E \to I$ be a fibered topos. A couple $(F, m)$ consisting of a topos $F$ and a Cartesian morphism $m: F \times I \to E$ of topoi over $I$ a *projective limit of the fibered topos* $F$ if for any topos $D$ the functor

$$\text{Hom}_{\text{TOPOI}}(D, F) \to \text{Cart}_{\text{TOPOI}/I}(D \times I, E)$$

obtained by the composition of $\text{Hom}_{\text{TOPOI}}(D, F) \to \text{Hom}_{\text{TOPOI}/I}(D \times I, F \times I)$ followed by $m$ is an equivalence of categories.

The projective limit is determined up to natural equivalences. We denote it by

$$\text{Lim} E = \text{Lim}_{I} E.$$  

In case $I$ is cofiltered and is essentially small (cf. §1.3. (c)), we have the following more down-to-earth description of the projective limit. Let $p: E \to I$ be an $I$-topos, and $S$ the set of all Cartesian arrows in $E$. We denote by $\text{Lim}_{I, \text{opp}} E$ the
category obtained from $E$ by inverting all arrows in $S$ (see [9], Exposé VI, (6.2), for details of the construction). We endow $\text{Lim}_{I^{\text{op}}} E$ with the weakest topology so that the canonical functor $E_i \to \text{Lim}_{I^{\text{op}}} E$ for each $i \in \text{obj}(I)$ is continuous. Then $\text{Lim}_{I^{\text{op}}} E$ becomes a site, and we have the natural equivalence

$$\text{Lim}_I E \cong (\text{Lim}_{I^{\text{op}}} E)^\sim$$

([9], Exposé VI, (8.2.3)).

What we have described here can be further boiled down to the following description, which starts from a simple observation. Assuming $\text{Lim}_I E$ exists as a topos, there is, for each $i \in \text{obj}(I)$, a projection $p_i: \text{Lim}_I E \to E_i$ of topoi. Then, for any object $F$ of $\text{Lim}_I E$, $F_i = p_i_* F$ satisfies $(E_f)_* F_i = F_j$ for any arrow $f: i \to j$ of $I$. Thus it is natural to define $\text{Lim}_I E$ to be the category whose objects are collections $\{F_i\}_{i \in \text{obj}(I)}$ of objects $F_i$ of $E_i$ that satisfy the compatibility conditions $(E_f)_* F_i = F_j$ for any $f: i \to j$. The verification of the equivalence with the above construction is left to the reader.

**Theorem 2.7.13** ([9], Exposé VI, (8.3.13)).* Let $p: E \to I$ be an $I$-topos, where $I$ is cofiltered and essentially small. Suppose that

- each fiber $E_i$ ($i \in \text{obj}(I)$) is a coherent topos (2.7.5) and
- for any $f: i \to j$ the morphism $E_f: E_i \to E_j$ of topoi is coherent (2.7.7).

Then the projective limit $\text{Lim}_I E$ is a coherent topos, and for each $i \in \text{obj}(I)$ the canonical projection $\text{Lim}_I E \to E_i$ is coherent. Moreover, if we denote by $E_{\text{coh}}$ the full subcategory of $E$ consisting of objects that are coherent in their fibers, then $E_{\text{coh}}$ is a fibered category over $I$, and $\text{Lim}_{I^{\text{op}}} E_{\text{coh}}$ (defined similarly as above) is canonically equivalent to the category $(\text{Lim}_I E)_{\text{coh}}$, the full subcategory of $\text{Lim}_I E$ consisting of coherent objects.

**2.7. (f) Projective limit of spacial topoi.** Let $I$ be a cofiltered and essentially small category, and consider a functor $X_\bullet: I \to \text{STop}$, which we denote by $i \mapsto X_i$. Then one can consider the projective limit $X_\infty = \text{lim}_I X_i$ in the category $\text{STop}$ (in fact, it is easy to see that the projective limit of sober spaces taken in the category $\text{Top}$ is sober). We are interested in comparing the topos $\text{top}(X_\infty)$ and the topos-theoretic projective limit $\text{Lim}_I \text{top}(X_i)$; note that the functor $X_\bullet$ yields the $I$-topos $\text{top}(X_\bullet) \to I$ with the fiber over $i$ being $\text{top}(X_i)$. Note that these two topoi may not be equivalent in general, for $\text{Lim}_I \text{top}(X_i)$ may not be spacial in general.
Let us first discuss the topos theoretic limit.

**Theorem 2.7.14.** Let $I$ be a cofiltered and essentially small category, and consider a functor $X_*: I \to \text{STop}$.

1. The topos $\text{Lim}_I \text{top}(X_i)$ is localic.

2. The topological space $\text{sp}(\text{Lim}_I \text{top}(X_i))$ is homeomorphic to $X_{\infty} = \lim_I X_i$.

**Proof.** Set $E = \text{Lim}_I \text{top}(X_i)$.

1. By the definition of the Grothendieck topology of $E$, $E$ is generated by all $p_i^*(U_i)$ for all $i \in \text{obj}(I)$ and subobjects $U_i$ of a final object of $\text{top}(X_i)$, where $p_i: E \to \text{top}(X_i)$ is the canonical projection. Thus $E$ is localic.

2. Consider the canonical map

$$F: \text{sp}(E) \longrightarrow \lim_{i \in \text{obj}(I)} \text{sp}(\text{top}(X_i)),$$

where each $\text{sp}(\text{top}(X_i))$ is, since $X_i$ is sober, identified with $X_i$ and the right-hand side is endowed with the projective limit topology. We claim that $F$ is bijective. Indeed, since each category of the points of $\text{top}(X_i)$ is discrete, any elements in the right-hand side lifts to a projective system of points of topoi $\text{top}(X_i)$, and hence to a point of $E$. Thus $F$ is surjective, and the injectivity is shown in a similar way. With the same notations as above, subobjects of the form $p_i^*(U_i)$ of the final object of $E$ generate the topology of $\text{sp}(E)$, and so the map $F$ gives a homeomorphism with respect to this topology. 

**Remark 2.7.15.** Note that the above proof of (1) shows that the projective limit of localic topoi is again localic.

**Corollary 2.7.16.** For a projective system of sober spaces $X_*: I \to \text{STop}$, the limit $\text{Lim}_I \text{top}(X_i)$ is spacial if and only if it has enough points. If it is spacial, the limit is equivalent to $\text{top}(\lim_I X_i)$.

An important case is the following one. Suppose, in addiction, that

- each $X_i$ is coherent and sober, and each transition map $X_i \to X_j$ is quasi-compact.

Then, by 2.7.13, $\text{Lim}_I \text{top}(X_i)$ is coherent; applying 2.7.10, 2.7.13, and 2.7.16, we conclude that $\text{Lim}_I \text{top}(X_i)$ is canonically equivalent to $\text{top}(\lim_I X_i)$, and $\lim_I X_i$ is coherent. Thus we have given a topos-theoretic proof of the first half of the following statement, which recasts 2.2.10.
Corollary 2.7.17. Let \( \{X_i\}_{i \in I} \) be a projective system of topological spaces indexed by a directed set such that each \( X_i \) is coherent and sober and each transition map \( X_i \to X_j \) is quasi-compact. Set
\[
X_\infty = \lim_{\rightarrow i \in I} X_i.
\]
Then \( X_\infty \) is coherent and sober. If, moreover, each \( X_i \) is non-empty, then \( X_\infty \) is non-empty.

Proof. The first assertion has been already shown above. For the second, we use 3.1.10 shown later independently:
\[
\Gamma(X, \mathbb{Z}) = \lim_{\rightarrow I} \Gamma(X_i, \mathbb{Z}).
\]
Note that, if each \( X_i \) is non-empty, then the inductive limit is clearly non-zero, whence the non-emptiness of \( X \) follows.

Remark 2.7.18. There is the ‘ringed’ version of the above argument, that is, we have the notion of ringed \( I \)-topoi and their projective limits as a ringed topos. The argument is quite similar to as above. For the details, see [9], Exposé VI, (8.6).

2.7. (g) Quasi-compact topoi and projective limits. Finally, let us include a few facts on quasi-compact topoi and their projective limits.

Definition 2.7.19. A topos \( E \) is said to be quasi-compact if it admits a quasi-compact final object.

Remark 2.7.20. In the literature (e.g., [65]), quasi-compact topoi are called compact topoi. In this book, however, we prefer to use the terminology quasi-compact, which is consistent with the standard usage in algebraic geometry.

Consider the 2-category \( \text{LocTOPOI} \) of localic topoi (§2.7. (c)) and the inclusion functor \( \text{LocTOPOI} \hookrightarrow \text{TOPOI} \). By collecting objects generated by subobjects of a fixed final objects for each topos \( E \), we have a right 2-adjoint functor \( E \mapsto E^{\text{loc}} \) from \( \text{TOPOI} \) to \( \text{LocTOPOI} \). This implies that the process of forming projective limits in \( \text{TOPOI} \) and \( \text{LocTOPOI} \) are compatible, and that, in particular, projective limits of localic topoi are again localic. Note that a topos \( E \) is quasi-compact if and only if so is \( E^{\text{loc}} \).

Theorem 2.7.21. Let \( p: E \to I \) be an \( I \)-topos, where \( I \) is cofiltered and essentially small. Suppose that each fiber \( E_i \) (\( i \in \text{obj}(I) \)) is a quasi-compact topos.

(1) The projective limit \( \lim_{\rightarrow i} E \) is a quasi-compact topos.

(2) If each \( E_i \) is non-empty, \( \lim_{\rightarrow i} E \) is non-empty.
Proof. If each $E_i$ for $i \in \text{obj}(I)$ is localic, then the claim is proven in [102], Theorem 2.3 and Corollary 2.4. The general case, by what we have remarked above, can be reduced to this case.

It follows from the adjunction between the functors $\text{top}: \text{STop} \to \text{LocTOPOI}$ and $\text{sp}: \text{LocTOPOI} \to \text{STop}$ that $\text{sp}$ preserves projective limits. Hence, as a corollary of the theorem and the following easy lemma, we obtain a topos-theoretic proof of 2.2.20.

Lemma 2.7.22. Let $E$ be a quasi-compact localic topos.

1. If $E$ is non-empty, then $\text{sp}(E)$ is non-empty.
2. The topological space $\text{sp}(E)$ is quasi-compact.

Exercises

Exercise 0.2.1. Let $X$ be an infinite set, and topologize it in such a way that a subset $Y \subseteq X$ is closed if and only if either it is $X$ itself, or is a finite subset. Show that the topological space $X$ is $T_1$, but is not sober.

Exercise 0.2.2. Show that a topological space $X$ is quasi-separated if and only if the diagonal mapping $\Delta: X \to X \times X$ is quasi-compact.

Exercise 0.2.3. Let $\{X_i, p_{ij}\}_{i \in I}$ be a projective system of topological spaces indexed by a directed set $I$. We assume that

(a) each $X_i$ is the underlying topological space of a coherent (= quasi-compact and quasi-separated) scheme and
(b) each transition map $p_{ij}: X_j \to X_i$ for $i \leq j$ is the underlying continuous mapping of an affine morphism of schemes.

Show that the projective limit $X = \lim_{\leftarrow i \in I} X_i$ (taken in $\text{Top}$) is a coherent sober topological space.

Exercise 0.2.4. Let $A$ be a distributive lattice.

1. A covering of $\alpha \in A$ is a finite subset $C$ of $A$ such that $\bigvee_{\beta \in C} \beta = \alpha$. Show that $A$ with this notion of coverings is a site. The associated topos is denoted by $\text{Spec } A$.

2. Show that the topos $\text{Spec } A$ is coherent.

3. Show that the topos associated to $\text{Spec } A$ is canonically equivalent to $\text{Spec } A$. 
Exercise 0.2.5. Show that any coherent sober space is homeomorphic to a projec-
tive limit of finite sober spaces.

Exercise 0.2.6. (1) Let $A$ be a Boolean distributive lattice. Show that Spec $A$ is a
profinite set and is homeomorphic to the spectrum of $A$ regarded as a commutative
ring.

(2) Show that $A \leftrightarrow$ Spec $A$ gives a categorical equivalence between the cat-
egory of Boolean distributive lattices and the opposite category of totally discon-
nected Hausdorff spaces.

Exercise 0.2.7. Let $X$ be a coherent sober space.

(1) Let $\{x_i\}_{i \in I}$ be a system of points of $X$ indexed by a directed set $I$ such that,
if $i \leq j$, $x_j$ is a generization of $x_i$. Show that there exists a point $x \in X$ that
is a generization of all $x_i$’s.

(2) Show that for any $x \in X$ the set $G_x$ of all generizations of $x$ contains a
maximal element.

Exercise 0.2.8. Let $\{X_i, p_{ij}\}_{i \in I}$ be a filtered projective system of coherent sober
topological spaces and quasi-compact maps, indexed by a directed set $I$, $X = \lim_{\leftarrow i \in I} X_i$, and $i \in I$ an index. Let $U_1, \ldots, U_n$ be a finite collection of quasi-
compact open subsets of $X_i$ such that $X = \bigcup_{k=1}^n p_i^{-1}(U_k)$, where $p_i: X \to X_i$
is the projection map. Show that there exists $j \in I$ with $i \leq j$ such that $X_j = \bigcup_{k=1}^n p_{ij}^{-1}(U_k)$.

Exercise 0.2.9. Let $\{X_i, p_{ij}\}_{i \in I}$ be a projective system of coherent sober spaces
with quasi-compact surjective transition maps indexed by a directed set $I$. Show
that $X = \lim_{\leftarrow i \in I} X_i$ is connected if and only if $X_i$ is connected for all $i \in I$.

Exercise 0.2.10. Show that for any valuative space $X$ the separated quotient $[X]$ is
a $T_1$-space.

Exercise 0.2.11. Let $X$ be a locally strongly compact valuative space.

(1) Show that any overconvergent open subset $U \subseteq X$ is locally strongly com-
pact.

(2) Show that for any quasi-compact open immersion $U \hookrightarrow X$ and any locally
strongly compact open subset $V \subseteq X$, $U \cap V$ is locally strongly compact. In
particular, an open subset $U \subseteq X$ is locally strongly compact if the inclusion
$U \hookrightarrow X$ is quasi-compact.
Exercise 0.2.12. (1) Let $X$ be a locally compact (hence locally Hausdorff) space, $Y$ a locally Hausdorff space, and $f: X \to Y$ a continuous injective map. Show that $f$ induces a homeomorphism onto its image endowed with the subspace topology induced from the topology of $Y$.

(2) Suppose, moreover, that $Y$ is locally compact. Show that $f(X)$ is locally closed in $Y$.

(3) Let $X$ be a locally strongly compact valuative space, and $U \subseteq X$ a locally strongly compact open subset. The injective map $[U] \to [X]$ maps $[U]$ homeomorphically onto a locally closed subspace of $[X]$.

Exercise 0.2.13 (structure theorem). Let $X$ be a locally strongly compact valuative space, and $U \subseteq X$ a locally strongly compact open subset.

(1) The inclusion map $j: U \to X$ is quasi-compact if and only if

$$[j]: [U] \hookrightarrow [X]$$

is a closed immersion.

(2) There exists an overconvergent open subset $Z \subseteq X$ containing $U$ such that the inclusion map $U \hookrightarrow Z$ is quasi-compact.

Exercise 0.2.14. Show that any finite intersection of locally strongly compact open subsets of a locally strongly compact valuative space is again locally strongly compact.

Exercise 0.2.15. Let $X$ be a locally strongly compact valuative space such that $[X]$ is Hausdorff. Show that $X$ is quasi-separated. In particular, $X$ is coherent if and only if $[X]$ is compact.

Exercise 0.2.16. Show that a distributive sublattice $v$ of $2^S$ for a compact space $S$ gives a valuation of $S$ if and only if the following conditions are satisfied:

(a) $\emptyset, S \in v$ and all element of $v$ are compact;

(b) for any $x \in S$ the family $\{T \in v: x \in T^o\}$ forms a fundamental system of neighborhoods of $x$;

(c) for any prime filter (cf. §2.2.(b)) $F \subseteq v$, there exists a unique maximal prime filter containing $F$;

(d) for two distinct maximal prime filters $F, F' \subseteq v$, we have $\bigcap_{T \in F} T \neq \bigcap_{T \in F'} T$;

(e) for any prime filter $F \subseteq v$ and a maximal filter $\widehat{F}$ containing $F$, the set of all prime filters between $F$ and $\widehat{F}$ is totally ordered with respect to the inclusion order.
**Exercise 0.2.17.** Let \( v = (\tau(v), \{v_S\}_{S \in \tau(v)}) \) be a pre-valuation of a locally Hausdorff space \( X \), and \( S_1, \ldots, S_n \in \tau(v) \). Show that if \( \bigcap_{i=1}^n S_i \) is Hausdorff, then \( \bigcap_{i=1}^n S_i \in \tau(v) \).

**Exercise 0.2.18.** Let \( X \) be a locally Hausdorff space, \( v = (\tau(v), \{v_S\}_{S \in \tau(v)}) \) a pre-valuation of \( X \).

1. Show that for any finite collection \( S_1, \ldots, S_n \in \tau(v) \), the restriction of \( v \) to the intersection \( C = \bigcap_{i=1}^n S_i \) exists.

2. Show that, for any finitely many \( S_1, \ldots, S_n \in \tau(v) \), the union \( C = \bigcup_{i=1}^n S_i \) is locally compact, and the restriction of \( v \) to \( C \) exists.

3. Suppose \( v \) is saturated. Show that, for any open subset \( U \subseteq X \), the restriction of \( v \) to \( U \) exists.

### 3 Homological algebra

In this section, we discuss two topics on homological algebra. The first topic concerns inductive and projective limits of sheaves and their cohomologies (§3.1 and §3.2). Given a projective system of topological spaces and an inductive system of sheaves on them, one has the inductive limit sheaf on the projective limit space. In this situation, we will give a general recipe to calculate the cohomologies of the inductive limit sheaf (§3.1.(d) and §3.1.(g)). This subsection also discusses Noetheriness of inductive limit rings (§3.1.(b)). As for projective limit sheaves, we discuss the so-called *Mittag–Leffler condition* and some of its consequences.

The second topic, discussed in §3.3, concerns coherent rings and modules. Here, a ring \( A \) is said to be coherent if every finitely generated ideal is finitely presented, or equivalently, the full subcategory of the category of \( A \)-modules consisting of finitely presented modules is an abelian subcategory (3.3.3).

**Convention.** Throughout this book, whenever we say \( A \) is a ring, we always mean that \( A \) is a commutative ring with multiplicative unit \( 1 = 1_A \), unless otherwise clearly stated; we also assume that any ring homomorphism \( f : A \to B \) is unitary, that is, maps 1 to 1. Here are other conventions:

- for a ring \( A \) we denote by \( \text{Frac}(A) \) the total ring of fractions of \( A \);
- for a ring \( A \) the Krull dimension of \( A \) is denoted by \( \dim(A) \);
- when \( A \) is a local ring, we denote by \( m_A \) the unique maximal ideal of \( A \);
- a ring homomorphism \( f : A \to B \) between local rings is said to be *local* if \( f(m_A) \subseteq m_B \).
3.1 Inductive limits

3.1. (a) Preliminaries. First, we collect basic known facts on inductive limits of rings and modules, which we quote, without proofs, mainly from [28].

**Proposition 3.1.1** ([28], Chapter I, §10.3, Proposition 3). Let \( \{ A_i, \phi_{ij} \}_{i \in I} \) be an inductive system of rings indexed by a directed set \( I \), and set \( A = \lim_{\longrightarrow i \in I} A_i \).

1. If \( A_i \neq 0 \) for each \( i \in I \), then \( A \neq 0 \).
2. If each \( A_i \) is an integral domain, then so is \( A \).
3. If each \( A_i \) is a field, then so is \( A \).

Note that (1) is a direct consequence of our convention: since each transition map \( \phi_{ij} \) maps \( 1_{A_i} \) to \( 1_{A_j} \), we immediately conclude that the inductive limit \( A \) has the element \( 1_A \) not equal to 0. Note also that (1) is the basis for the following well-known fact (somewhat similar to 2.2.10): if \( \{ X_i, f_{ij} \}_{i \in I} \) is a projective system of non-empty affine schemes indexed by a directed set, then the projective limit \( X = \lim_{\leftarrow i \in I} X_i \) in the category of schemes exists and is affine and non-empty (cf. Exercise 0.2.3).

**Proposition 3.1.2** ([27], Chapter II, §3, Exercise 16). Let \( \{ A_i, \phi_{ij} \}_{i \in I} \) be an inductive system of local rings and local homomorphisms indexed by a directed set \( I \).

1. The inductive limit \( A = \lim_{\longrightarrow i \in I} A_i \) is a local ring with the maximal ideal \( m_A = \lim_{\longrightarrow i \in I} m_{A_i} \).
2. Let \( k_i = A_i/m_{A_i} \) be the residue field of \( A_i, i \in I \). Then \( k = \lim_{\longrightarrow i \in I} k_i \) is the residue field of \( A \).
3. If, moreover, \( m_{A_j} = m_{A_i} A_j \) for \( j \geq i \), then \( m_A = m_{A_i} A \) for any \( i \in I \).

Let \( A = \{ A_i, \phi_{ij} \}_{i \in I} \) be an inductive system of rings. By an inductive system of \( A \)-modules we mean an inductive system \( M = \{ M_i, f_{ij} \}_{i \in I} \) such that each \( M_i \) is an \( A_i \)-module and that each \( f_{ij}: M_i \rightarrow M_j \) is compatible with \( \phi_{ij}: A_i \rightarrow A_j \), that is, for \( x \in M_i \) and \( a \in A_i \) we have \( f_{ij}(ax) = \phi_{ij}(a)f_{ij}(x) \). For another inductive system \( M' \) of \( A \)-modules, we can define the notion of homomorphism \( M \rightarrow M' \) in an obvious manner.

**Proposition 3.1.3** ([28], Chapter II, §6.2, Proposition 3). Let \( M = \{ M_i, f_{ij} \}_{i \in I} \), \( M' = \{ M'_i, f'_{ij} \}_{i \in I} \), and \( M'' = \{ M''_i, f''_{ij} \}_{i \in I} \) be filtered inductive systems of \( A \)-modules indexed by a directed set \( I \). Consider an exact sequence

\[
M' \longrightarrow M \longrightarrow M''
\]
that is, \( M'_i \to M_i \to M''_i \) is exact for every \( i \)). Then the induced sequence

\[
\lim_{i \in I} M'_i \longrightarrow \lim_{i \in I} M_i \longrightarrow \lim_{i \in I} M''_i
\]

is exact.

The following proposition is a consequence of the fact that the inductive limit functor has a right adjoint (cf. §1.3. (a)).

**Proposition 3.1.4** (cf. [28], Chapter II, §6.4, Proposition 7). Let \( M = \{ M_i, f_{ij} \}_{i \in I} \), \( M' = \{ M'_i, f'_{ij} \}_{i \in I} \), and \( M'' = \{ M''_i, f''_{ij} \}_{i \in I} \) be inductive systems of \( A \)-modules, and suppose we are given homomorphisms

\[
M' \leftarrow M \rightarrow M''
\]

of \( A \)-modules. Then the canonical map

\[
\lim_{i \in I} M'_i \otimes_{M_i} M''_i \longrightarrow (\lim_{i \in I} M'_i) \otimes_{\lim_{i \in I} M_i} (\lim_{i \in I} M''_i)
\]

is an isomorphism.

### 3.1. (b) Inductive limits and Noetherness.

Let us include here a useful technique, invented by M. Nagata (cf. [84], (43.10)), to show that an inductive limit ring in a certain situation is Noetherian.

**Proposition 3.1.5.** Let \( \{ R_\alpha \}_{\alpha \in L} \) be a filtered inductive system of rings, and \( R = \lim_{\alpha \in L} R_\alpha \). Suppose that the following conditions are satisfied:

(a) \( R_\alpha \) is a Noetherian local ring for any \( \alpha \in L \);

(b) for any \( \alpha \leq \beta \) the transition map \( R_\alpha \to R_\beta \) is local and flat;

(c) for any \( \alpha \leq \beta \) we have \( m_\alpha R_\beta = m_\beta \) (where \( m_\alpha \) denotes the maximal ideal of \( R_\alpha \)).

Then \( R \) is Noetherian.

**Proof.** By (c), the maximal ideal \( m \) of \( R \) coincides with \( m_\alpha R \) for any \( \alpha \in L \); in particular, \( m \) is a finitely generated ideal of \( R \). Hence the \( m \)-adic completion \( \hat{R} \) of \( R \) exists (cf. 7.2.16 below); moreover, since \( m \hat{R} \) is finitely generated, \( \hat{R} \) is Noetherian (cf. [81], (29.4)). Since \( R_\alpha/m_\alpha^{k+1} \to R/m^{k+1} \) is faithfully flat for any \( \alpha \in L \) and \( k \geq 0 \), by the local criterion of flatness (cf. §8.3. (b) below), we deduce that \( \hat{R}_\alpha \to \hat{R} \) is faithfully flat for any \( \alpha \in L \).
Now to prove the proposition it suffices to show that any increasing sequence

$$J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots$$

of finitely generated ideals of $R$ is stationary. Since $\hat{R}$ is Noetherian, the induced sequence

$$J_0 \hat{R} \subseteq J_1 \hat{R} \subseteq J_2 \hat{R} \subseteq \cdots$$

is stationary, that is, there exists $N \geq 0$ such that $J_n \hat{R} = J_m \hat{R}$ whenever $n, m \geq N$. We want to show the equality $J_n D J_m$ for $n, m \geq N$. It suffices to establish the following fact: if $J; J_0$ are finitely generated ideals of $R$ such that $J J_0 \subseteq R$ and $J_\alpha \hat{R} = J \hat{R} = J_\beta \hat{R}$, then $J_\alpha = J_\beta$ (since $\hat{R}$ is faithfully flat over $R_\alpha$), and thus $J = J'$, as desired.

**Corollary 3.1.6.** Let $\{R_\alpha\}_{\alpha \in L}$ be a filtered inductive system of rings, and $R = \lim \longrightarrow_{\alpha \in L} R_\alpha$. Suppose that the following conditions are satisfied:

(a) $R_\alpha$ is a Noetherian local ring for any $\alpha \in L$;

(b) for any $\alpha \leq \beta$ the transition map $R_\alpha \to R_\beta$ is local and formally smooth (that is, $R_\beta$ is $m_{R_\beta}$-smooth over $R_\alpha$, in the terminology of [81], §28);

(c) the set of integers $\{\dim(R_\alpha)\}_{\alpha \in L}$ is bounded.

Then $R$ is Noetherian.

**Proof.** Replacing $\{R_\alpha\}_{\alpha \in L}$ by a cofinal subsystem if necessary, we may assume that the numbers $\dim(R_\alpha)$ are all equal. Then for any $\alpha \leq \beta$ the closed fiber of $\text{Spec } R_\beta \to \text{Spec } R_\alpha$ is of dimension 0. Since $\text{Spec } R_\beta \to \text{Spec } R_\alpha$ is formally smooth, it is flat [54], IV, (19.7.1), and hence we deduce that $R_\beta/m_\alpha R_\beta$ are fields, that is, $m_\alpha R_\beta = m_\beta$. Now the assertion follows from 3.1.5.

**3.1. (c) Inductive limit of sheaves.** Let $X$ be a topological space, and consider a filtered inductive system $\{F_i, \phi_{ij}\}_{i \in I}$ of sheaves (of sets, abelian groups, rings, etc.) on $X$ indexed by a directed set $I$. The inductive limit sheaf $\lim \longrightarrow_{i \in I} F_i$ in the category of sheaves (of sets, abelian groups, rings, etc.) is described as follows (cf. e.g., [47], II, 1.11). Define a presheaf $F$ by

$$F(U) = \lim \longrightarrow_{i \in I} F_i(U)$$

for any open subset $U \subseteq X$; then the desired sheaf is the sheafification of the presheaf $F$. By construction,

$$\lim F_{i,x} = (\lim F_i)_x$$

for any point $x \in X$. From this and 3.1.2 (1) we deduce the following proposition.
Proposition 3.1.7. Let $X$ be a topological space, and $\{F_i, \phi_{ij}\}_{i \in I}$ a filtered inductive system consisting of sheaves of local rings (that is, every stalk $F_{i,x}$ is a local ring) and local homomorphisms (that is, $\phi_{ij,x}$ is a local homomorphism). Then the inductive limit $F = \lim_{\longrightarrow} F_i$ is a sheaf of local rings.

Proposition 3.1.8. Let $X$ be a topological space, and $\{F_i, \varphi_{ij}\}_{i \in I}$ a filtered inductive system of sheaves on $X$ indexed by a directed set $I$. Consider the canonical map

$$\Phi: \lim_{\longrightarrow} \Gamma(X, F_i) \longrightarrow \Gamma(X, \lim_{\longrightarrow} F_i).$$

1. If $X$ is quasi-compact (2.1.4 (1)), then $\Phi$ is injective.
2. If $X$ is coherent (2.2.1), then $\Phi$ is bijective.

This proposition can be seen as a special case of a topos-theoretic result [9], Exposé VI, Théorème 1.23. We give here a proof for the reader’s convenience. The ringed space version will be given in 4.1.6.

Proof. Set $F = \lim_{\longrightarrow} F_i$.

1. Let $\{s_i\}_{i \in I}$ and $\{t_i\}_{i \in I}$ be inductive systems of sections $s_i, t_i \in F_i(X)$ whose images under $\Phi$ in $F(X)$ coincide. (Here, if necessary, we replace $I$ by a cofinal subset.) We need to show that $s_k$ and $t_k$ coincide for $k$ sufficiently large. Since $F_x = \lim_{\longrightarrow} F_{i,x}$ for any $x \in X$, there exists $j$ (depending on $x$) such that $s_k|_x = t_k|_x$ for any $k \geq j$. There exists an open neighborhood $U$ of $x$ such that $s_j|_U = t_j|_U$. Since $X$ is quasi-compact, there exist a finite open covering $X = \bigcup_{\alpha=1}^n U_\alpha$ and indices $j_\alpha$ such that $s_k$ and $t_k$ coincide on $U_\alpha$ for $k \geq j_\alpha$. Taking $j$ to be the maximum of $\{j_1, \ldots, j_n\}$, we deduce that $s_k$ and $t_k$ coincide over $X$ for $k \geq j$, as desired.

2. We only need to show that $\Phi$ is surjective. Take $s \in F(X)$. The germ $s_x$ at $x \in X$ can be written as an inductive system $\{s_{x,i}\}_{i \in I}$ in $\{F_{i,x}\}_{i \in I}$. (Here, again, we replace $I$ by a cofinal subset if necessary.) Take an index $i \in I$ and a section $t_i \in F_i(U)$ over a quasi-compact open neighborhood $U$ of $x$ such that $s_{x,i} = t_i|_x$. Then for any $j \geq i$ one sets $t_j = \varphi_{ij}(U)(t_i)$ so that one gets an inductive system $\{t_j\}_{j \geq i}$ in $\{F_j(U)\}_{j \geq i}$. Replacing $U$ by a smaller quasi-compact open neighborhood if necessary, we may assume that the system $\{t_j\}_{j \geq i}$ is mapped by $\Phi(U)$ to $s|_U$.

Thus, replacing $I$ by a cofinal subset, we get a finite open covering $X = \bigcup_{\alpha=1}^n U_\alpha$ consisting of quasi-compact open subsets, and for each $\alpha$ an inductive system $\{t_{\alpha,i}\}_{i \in I}$ sitting in $\{F_i(U_\alpha)\}_{i \in I}$ that is mapped by $\Phi(U_\alpha)$ to $s|_{U_\alpha}$. Since $X$ is coherent, each $U_{\alpha\beta} = U_\alpha \cap U_\beta$ is quasi-compact, and hence by (1) there exists $j_{\alpha\beta} \in I$ such that $t_{\alpha,k}$ and $t_{\beta,k}$ coincide over $U_{\alpha\beta}$ for $k \geq j_{\alpha\beta}$. Taking $j$ to be the maximum of all $j_{\alpha\beta}$, the local sections $t_{\alpha,k}$ glue together to a section $t_k$ on $X$ for each $k \geq j$. Then the inductive system $\{t_k\}_{k \geq j}$ thus obtained is mapped by $\Phi$ to $s$. $\Box$
**Corollary 3.1.9.** Let \( f: X \to Y \) be a continuous mapping between topological spaces. Suppose that the following conditions are satisfied:

(a) \( X \) is quasi-separated (2.1.8) and has an open basis consisting of quasi-compact open subsets;

(b) \( Y \) has an open basis consisting of quasi-compact open subsets;

(c) \( f \) is quasi-compact (2.1.4 (2)).

Then for any filtered inductive system \( \{ \mathcal{F}_i, \varphi_{ij} \}_{i \in I} \) of sheaves on \( X \) indexed by a directed set \( I \), the canonical morphism

\[
\Phi: \lim_{i \in I} f_* \mathcal{F}_i \longrightarrow f_*(\lim_{i \in I} \mathcal{F}_i)
\]

is an isomorphism.

In other words, the direct image functor \( f_* \) commutes with arbitrary small filtered inductive limits.

**Proof.** The sheaf \( \lim_{i \in I} f_* \mathcal{F}_i \) is the sheafification of

\[
V \mapsto \lim_{i \in I} \Gamma(f^{-1}(V), \mathcal{F}_i);
\]

here, by the assumption, it is enough to consider only quasi-compact \( V \)'s. Since \( f \) is quasi-compact, and since \( X \) is quasi-separated, \( f^{-1}(V) \) is coherent (2.1.1). Hence we may apply 3.1.8 (2) to deduce that the map

\[
\lim_{i \in I} \Gamma(f^{-1}(V), \mathcal{F}_i) \longrightarrow \Gamma(f^{-1}(V), \lim_{i \in I} \mathcal{F}_i)
\]

is bijective. This means, in particular, that the map \( \Phi \) is stalkwise bijective, since taking stalks commutes with the inductive limit \( \lim_{i \in I} \), and hence that \( \Phi \) is an isomorphism. \( \square \)

### 3.1. (d) Sheaves on limit spaces.

Here we consider

- a filtered projective system of topological spaces \( \{ X_i, p_{ij}: X_j \to X_i \}_{i \in I} \) indexed by a directed set \( I \)

such that

(a) for any \( i \in I \) the topological space \( X_i \) is coherent (2.2.1) and sober (§2.1.(b)) and

(b) for any \( i \leq j \) the transition map \( p_{ij}: X_j \to X_i \) is quasi-compact (2.1.4 (2)).
Note that by 2.2.10 (1) the limit space $X = \lim_{\leftarrow i \in I} X_i$ is coherent and sober, and that the canonical projection maps $p_i: X \to \hat{X}_i$ for $i \in I$ are quasi-compact. Suppose, moreover, that we are given the following data:

- for each $i \in I$ a sheaf $\mathcal{F}_i$ (of sets, abelian groups, etc.) on $X_i$;
- for each pair $(i, j)$ of indices in $I$ with $i \leq j$, a morphism $\varphi_{ij}: p_{ij}^{-1}\mathcal{F}_i \to \mathcal{F}_j$

of sheaves such that $\varphi_{ik} = \varphi_{jk} \circ p^{-1}_{jk} \varphi_{ij}$ whenever $i \leq j \leq k$.

Then one has the inductive system $\{p_i^{-1}\mathcal{F}_i\}_{i \in I}$ of sheaves on $X$ indexed by $I$, and thus the sheaf

$$\mathcal{F} = \lim_{\leftarrow i \in I} p_i^{-1}\mathcal{F}_i$$

on $X$.

**Proposition 3.1.10.** The canonical map

$$\lim_{\leftarrow i \in I} \Gamma(X_i, \mathcal{F}_i) \longrightarrow \Gamma(X, \mathcal{F})$$

is an isomorphism.

**Proof.** We already know by 3.1.8 that $\Gamma(X, \mathcal{F}) \cong \lim_{\leftarrow i \in I} \Gamma(X, p_i^{-1}\mathcal{F}_i)$. We want to show that the map $\lim_{\leftarrow i \in I} \Gamma(X_i, \mathcal{F}_i) \to \lim_{\leftarrow i \in I} \Gamma(X, p_i^{-1}\mathcal{F}_i)$ is bijective.

Step 1. We first claim that the canonical map

$$\lim_{\leftarrow i \in I} \Gamma(X_i, \mathcal{F}_i) \longrightarrow \lim_{\leftarrow i \in I} \lim_{U} \Gamma(U, \mathcal{F}_i),$$

where $U$ in the right-hand side runs through the open subsets of $X_i$ such that $p_i(X) \subseteq U$, is an isomorphism. By Exercise 0.1.1, the double inductive limit in the right-hand side is canonically isomorphic to the inductive limit taken over the directed set

$$\Lambda = \{(i, U): i \in I, p_i(X) \subseteq U \subseteq X_i\},$$

where $(i, U) \leq (j, V)$ if and only if $i \leq j$ and $p_{ij}^{-1}(U) \supseteq V$. Hence the desired result follows if one shows that the subset of $\Lambda$ consisting of elements of the form $(i, X_i)$ is cofinal.

To see this, take any $(i, U) \in \Lambda$. Since $p_i(X)$ is quasi-compact and $X_i$ is coherent, we may assume that $U$ is quasi-compact. The condition $p_i(X) \subseteq U$ is equivalent to $p_i^{-1}(U) = X$. Then by 2.2.12 there exists $j \in I$ with $i \leq j$ such that $p^{-1}_{ij}(U) = X_j$. Hence, $(i, U) \leq (j, X_j)$, as desired.
Step 2. Take \( \{s_i\}_{i \in I} \in \lim_{i \in I} \Gamma(X, p_i^{-1}\mathcal{F}_i) \). By 2.2.9, there exist a finite open covering \( X = \bigcup_{\alpha \in \Lambda} U_\alpha \) by quasi-compact open subsets and an index \( i \in I \) such that

- each \( U_\alpha \) is of the form \( U_\alpha = p^{-1}_\alpha(U_{\alpha i}) \) for a quasi-compact open subset \( U_{\alpha i} \) of \( X_i \) and
- for each \( \alpha \), \( s_i|_{U_\alpha} \) lies in \( \lim_{V} \Gamma(V, \mathcal{F}_i) \), where \( V \) runs through all open subsets of \( X_i \) containing \( p_i(U_\alpha) \).

Then \( \{s_j|_{U_\alpha}\}_{j \geq i} \) defines a section in \( \lim_{j \geq i} \Gamma(V, \mathcal{F}_j) \), where \( U \) runs through all open subsets of \( U_{\alpha j} = p^{-1}_{ij}(U_{\alpha i}) \) containing \( p_j(U_\alpha) \). By Step 1, this defines a unique section in \( \lim_{j \geq i} \Gamma(U_{\alpha j}, \mathcal{F}_j) \). Since the spaces of the form \( U_{\alpha j} \cap U_{\beta j} \) are all coherent, these sections glue together to a unique section in \( \lim_{i \in I} \Gamma(X_i, \mathcal{F}_i) \), as desired.

**Corollary 3.1.11.** Let \( \{X_i, p_{ij}\}_{i \in I} \) and \( p_i: X \to X_i \) for \( i \in I \) be as above, and \( Z \) a topological space. Suppose we are given a system of continuous mappings \( \{g_i: X_i \to Z\}_{i \in I} \) such that \( g_j = g_i \circ p_{ij} \) whenever \( i \leq j \). Then for any sheaf \( \mathcal{G} \) on \( Z \) the canonical map

\[
\lim_{i \in I} \Gamma(X_i, g_i^{-1}\mathcal{G}) \longrightarrow \Gamma(X, g^{-1}\mathcal{G}),
\]

where \( g = \lim_{i \in I} g_i \), is an isomorphism.

**Proof.** Apply 3.1.10 to the situation where \( \mathcal{F}_i = g_i^{-1}\mathcal{G} \) for \( i \in I \).

**Corollary 3.1.12.** Let \( \{X_i, p_{ij}\}_{i \in I} \) and \( p_i: X \to X_i \) for \( i \in I \) be as above, and \( \mathcal{F} \) be a sheaf on \( X \). Then the canonical morphism

\[
\lim_{i \in I} p_i^{-1}p_{i*}\mathcal{F} \longrightarrow \mathcal{F}
\]

is an isomorphism.

**Proof.** By a similar reasoning as in the proof of 3.1.9, it suffices to show that the morphism in question induces an isomorphism between the sets of sections over each quasi-compact open subset \( U \subset X \). By 2.2.9, there exist an index \( i_0 \in I \) and quasi-compact open subset \( U_0 \) of \( X_{i_0} \) such that \( p_{i_0}^{-1}(U_0) = U \). Since each \( p_{i_0}^{-1}(U_0) \) with \( i \geq i_0 \) is a coherent sober space, and since \( U = \lim_{i \geq i_0} p_{i_0}^{-1}(U_0) \), we may assume, without loss of generality, that \( U = X \). (Here we replace the index set \( I \) by the cofinal subset \( \{i \in I : i \geq i_0\} \).) Then we have

\[
\Gamma(X, \lim_{i \in I} p_i^{-1}p_{i*}\mathcal{F}) \cong \lim_{i \in I} \Gamma(X_i, p_{i*}\mathcal{F}) = \lim_{i \in I} \Gamma(X, \mathcal{F}) = \Gamma(X, \mathcal{F}),
\]

where the first isomorphism is due to 3.1.10.
Next, in addition to the data fixed in the beginning of this subsection, we consider the following data:

- another filtered projective system of topological spaces \( \{ Y_i, q_{ij} \}_{i \in I} \), indexed by the same directed set \( I \), that satisfies the conditions similar to (a) and (b) in the beginning of this subsection;

- a map \( \{ f_i \}_{i \in I} \) of projective systems from \( \{ X_i, p_{ij} \}_{i \in I} \) to \( \{ Y_i, q_{ij} \}_{i \in I} \), that is, a collection of continuous maps \( f_i: X_i \to Y_i \) such that \( q_{ij} \circ f_j = f_i \circ p_{ij} \) for any \( i \leq j \).

We set \( Y = \varprojlim_{i \in I} Y_i \) and denote the canonical projection by \( q_i: Y \to Y_i \) for each \( i \in I \). Moreover, we have the continuous map

\[
f = \lim_{i \in I} f_i: X \to Y.
\]

We assume that

(c) for any \( i \in I \) the map \( f_i \) is quasi-compact.

Note that \( f \) is quasi-compact thanks to 2.2.13 (1).

**Corollary 3.1.13.** The canonical morphism of sheaves

\[
\varprojlim_{i \in I} q_i^{-1} f_i_* F_i \to f_* F
\]

is an isomorphism.

**Proof.** By a similar reasoning as in the proof of 3.1.9, we may restrict ourselves to showing that the map

\[
\varprojlim_{i \in I} \Gamma(V, q_i^{-1} f_i_* F_i) \to \Gamma(f^{-1}(V), F)
\]

is an isomorphism for any quasi-compact open subset \( V \) of \( Y \). Similarly to the proof of 3.1.12, we may assume, without loss of generality, that \( V = Y \). In this situation, one can replace the left-hand side by the double inductive limit

\[
\varprojlim_{i \in I} \varprojlim_{U} \Gamma(f_i^{-1}(U), F_i),
\]

where \( U \) runs through the open subsets of \( Y_i \) containing \( q_i(Y) \). By an argument similar to that in the proof of 3.1.10, one sees that this limit is isomorphic to \( \varprojlim_{i \in I} \Gamma(X_i, F_i) \). Then the desired result follows from 3.1.10. \( \square \)
Corollary 3.1.14. Let \(\{X_i, p_{ij}\}_{i \in I}\) and \(p_i: X \to X_i\) for \(i \in I\) be as above, and let \(Z\) be a coherent sober space. Suppose we are given a system of quasi-compact maps \(\{g_i: X_i \to Z\}_{i \in I}\) such that \(g_j = g_i \circ p_{ij}\) whenever \(i \leq j\). Then for any sheaf \(\mathcal{G}\) on \(Z\) the canonical morphism

\[
\lim_{i \in I} g_i * g_i^{-1} \mathcal{G} \longrightarrow g* g^{-1} \mathcal{G},
\]

where \(g = \lim_{i \in I} g_i\), is an isomorphism.

Proof. Apply 3.1.13 to the case when \(\{Y_i, q_{ij}\}_{i \in I}\) is the constant system \((Y_i = Z)\) and \(\mathcal{F}_i = g_i^{-1} \mathcal{G}\).

Corollary 3.1.15. Let \(\{X_i, p_{ij}\}_{i \in I}, \{Y_i, q_{ij}\}_{i \in I}\), and \(\{f_i\}_{i \in I}\) be as above, and \(\mathcal{F}\) a sheaf on \(X\). Then the canonical morphism

\[
\lim_{i \in I} q_i^{-1} f_i * p_i * \mathcal{F} \longrightarrow f* \mathcal{F}
\]

is an isomorphism.

Proof. By 3.1.12, \(\lim_{i \in I} q_i^{-1} f_i * p_i * \mathcal{F} = \lim_{i \in I} q_i^{-1} q_i * f* \mathcal{F} \cong f* \mathcal{F}\).

3.1. (e) Canonical flasque resolution. Let us recall the canonical flasque resolution for abelian sheaves; we only recall its basic properties, and refer to [47], II.4.3, for the construction. Let \(X\) be a topological space. We denote by \(\text{ASh}_X\) the abelian category of sheaves of abelian groups on \(X\) and by \(\mathbf{C}^+(\text{ASh}_X)\) the abelian category of complexes bounded below consisting of objects and arrows in \(\text{ASh}_X\) (cf. §C.2. (a)). The canonical flasque resolution is a functor

\[
\mathcal{C}^\bullet(X, \cdot): \text{ASh}_X \longrightarrow \mathbf{C}^+(\text{ASh}_X)
\]

such that for any abelian sheaf \(\mathcal{F}\) on \(X\)

(a) \(\mathcal{C}^q(X, \mathcal{F}) = 0\) for \(q < 0\);

(b) \(\mathcal{C}^q(X, \mathcal{F})\) for \(q \in \mathbb{Z}\) is flasque;

(c) \(\mathcal{C}^\bullet(X, \mathcal{F})\) is equipped with the augmentation \(\mathcal{F} \to \mathcal{C}^\bullet(X, \mathcal{F})\) such that the sequence

\[
0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(X, \mathcal{F}) \longrightarrow \mathcal{C}^1(X, \mathcal{F}) \longrightarrow \mathcal{C}^2(X, \mathcal{F}) \longrightarrow \cdots
\]

is exact.

Moreover, the functor \(\mathcal{C}^\bullet(X, \cdot)\) is exact.
The importance of the canonical flasque resolution lies in its canonicity. For instance, given an ordered set $I$, one can consider the category $\text{ASH}_X^I$ of inductive systems of abelian sheaves on $X$; the canonicity of the canonical flasque resolution allows one to construct the exact functor

$$\mathcal{C}^\bullet(X, \cdot): \text{ASH}_X^I \longrightarrow \mathcal{C}^+ (\text{ASH}_X)^I,$$

which maps each inductive system $\{F_i, \varphi_{ij}\}_{i \in I}$ of abelian sheaves to the inductive system of complexes $\{\mathcal{C}^\bullet(X, F_i), \varphi_{ij}^*\}$ consisting of canonical flasque resolutions of $F_i$’s.

3.1. (f) Inductive limit and cohomology

**Proposition 3.1.16.** Let $X$ be a coherent topological space (2.2.1), and $\{F_i, \varphi_{ij}\}_{i \in I}$ a filtered inductive system of sheaves of abelian groups on $X$ indexed by a directed set $I$. Then the canonical map

$$\Phi^q: \lim_{i \in I} H^q(X, F_i) \longrightarrow H^q(X, \lim_{i \in I} F_i)$$

is bijective for any $q \geq 0$.

**Proof.** First note that the case $q = 0$ has already been proved in 3.1.8 (2). In order to show the general case, let us take the filtered inductive system of complexes $\{\mathcal{C}^\bullet(X, F_i), \varphi_{ij}^*\}$ as above. By 3.1.3

$$0 \longrightarrow \lim_{i \in I} F_i \longrightarrow \lim_{i \in I} \mathcal{C}^\bullet(X, F_i)$$

is exact. This gives a quasi-flasque resolution of $\lim_{i \in I} F_i$ (cf. Exercise 0.3.1), and hence the desired result follows from 3.1.8 (2) and the exactness of inductive limits. 

**Corollary 3.1.17.** Let $f: X \rightarrow Y$ be a continuous mapping between topological spaces. Suppose that the following conditions are satisfied:

(a) $X$ is quasi-separated (2.1.8) and has an open basis consisting of quasi-compact open subsets;

(b) $Y$ has an open basis consisting of quasi-compact open subsets;

(c) $f$ is quasi-compact (2.1.4 (2)).

Then for any filtered inductive system $\{F_i, \varphi_{ij}\}_{i \in I}$ of sheaves of abelian groups on $X$ indexed by a directed set, the canonical morphism

$$\Phi^q: \lim_{i \in I} R^q f_* F_i \longrightarrow R^q (\lim_{i \in I} f_*) F_i$$

is an isomorphism.
The proof is similar to that of 3.1.9, where we now use 3.1.16 instead of 3.1.8.

3.1. (g) Cohomology of sheaves on limit spaces. Let us now return to the situation as in §3.1. (d).

**Lemma 3.1.18.** Suppose that each $\mathcal{F}_i$ is a flasque sheaf on $X_i$. Then the sheaf $\mathcal{F} = \lim_{\rightarrow i \in I} p_i^{-1} \mathcal{F}_i$ is quasi-flasque.

This follows easily from Exercise 0.3.1 and the fact that each $p_i^{-1} \mathcal{F}_i$ is flasque.

In the following statements, all sheaves are supposed to be sheaves of abelian groups, and morphisms of sheaves are morphisms of abelian sheaves.

**Proposition 3.1.19.** The canonical map

$$\lim_{\rightarrow i \in I} H^q(X_i, \mathcal{F}_i) \rightarrow H^q(X, \mathcal{F})$$

is an isomorphism for $q \geq 0$.

This follows from 3.1.10 and 3.1.18 as in the proof of 3.1.16. Combining this result and 3.1.12 we obtain the following corollary.

**Corollary 3.1.20.** Let $\{X_i, p_{ij}\}_{i \in I}$ and $p_i: X \rightarrow X_i$ for $i \in I$ be as in the beginning of §3.1. (d), and $\mathcal{F}$ a sheaf on $X$. Then the canonical morphism

$$\lim_{\rightarrow i \in I} H^q(X_i, p_{i*} \mathcal{F}) \rightarrow H^q(X, \mathcal{F})$$

is an isomorphism for $q \geq 0$.

**Corollary 3.1.21.** Let $\{X_i, p_{ij}\}_{i \in I}$ and $p_i: X \rightarrow X_i$ for $i \in I$ be as in the beginning of §3.1. (d), and $Z$ a topological space. Suppose we are given a system of continuous maps $\{g_i: X_i \rightarrow Z\}_{i \in I}$ such that, whenever $i \leq j$, we have $g_j = g_i \circ p_{ij}$. Then for any sheaf $\mathcal{G}$ on $Z$ the canonical map

$$\lim_{\rightarrow i \in I} H^q(X_i, g_i^{-1} \mathcal{G}) \rightarrow H^q(X, g^{-1} \mathcal{G}),$$

where $g = \lim_{\leftarrow i \in I} g_i$, is an isomorphism for $q \geq 0$.

This follows from 3.1.19 applied to the situation where $\mathcal{F}_i = g_i^{-1} \mathcal{G}$ for $i \in I$. Similarly to 3.1.13 and its subsequent results, one can show the following statements.

**Corollary 3.1.22.** Let $\{X_i, p_{ij}\}_{i \in I}$, $\{Y_i, q_{ij}\}_{i \in I}$, and $\{f_i\}_{i \in I}$ be as in §3.1. (d). Then the canonical morphism of sheaves

$$\lim_{\rightarrow i \in I} q_i^{-1} R^q f_{i*} \mathcal{F}_i \rightarrow R^q f_* \mathcal{F}$$

is an isomorphism for $q \geq 0$. 
Corollary 3.1.23. Let \( \{X_i, p_{ij}\}_{i \in I} \) be as in §3.1. (d), and \( Z \) a coherent sober space. Suppose we are given a system of quasi-compact maps \( \{g_i : X_i \to Z\}_{i \in I} \) such that \( g_j = g_i \circ p_{ij} \) whenever \( i \leq j \). Then for any sheaf \( \mathcal{G} \) on \( Z \) the canonical morphism

\[
\lim_{i \in I} R^q g_{i*}(g_i^{-1} \mathcal{G}) \to R^q g_* (g^{-1} \mathcal{G}),
\]

where \( g = \lim_{i \in I} g_i \), is an isomorphism for \( q \geq 0 \).

Corollary 3.1.24. Let \( \{X_i, p_{ij}\}_{i \in I}, \{Y_i, q_{ij}\}_{i \in I} \), and \( \{f_i\}_{i \in I} \) be as in §3.1. (d), and \( \mathcal{F} \) a sheaf on \( X \). Then the canonical morphism

\[
\lim_{i \in I} q_i^{-1} R^q f_{i*}(p_{i*} \mathcal{F}) \to R^q f_* \mathcal{F}
\]

is an isomorphism for \( q \geq 0 \).

3.2 Projective limits

3.2. (a) The Mittag-Leffler condition. Let \( \mathcal{C} \) be either the category of sets or an abelian category that has all small products (that is, for any family \( \{A_i\}_{i \in I} \) of objects indexed by a small set, the product \( \prod_{i \in I} A_i \) is representable). Then the category \( \mathcal{C} \) is small complete, that is, the limit \( \lim_{i \in I} F \) for a functor \( F : \mathcal{D} \to \mathcal{C} \) exists whenever \( \mathcal{D} \) is essentially small ([79], Chapter V, §2, Corollary 2).

Let \( A = \{A_i, f_{ij} : A_j \to A_i\} \) be a projective system of objects in \( \mathcal{C} \) indexed by a directed set \( I \), and set \( A = \lim_{i \in I} A_i \).

Definition 3.2.1. (1) The projective system \( A \) is said to be strict if all transition maps \( f_{ij} \) for \( i \leq j \) are epimorphic.

(2) Suppose \( I = \mathbb{N} \), the ordered set of all natural numbers. Then the projective system \( A \) is said to be essentially constant, if there exists \( N \) such that for \( N \leq i \leq j \) the transition maps \( f_{ij} \) are isomorphisms.

For each \( i \in I \) we set

\[
A'_i = \inf_{i \leq j} f_{ij}(A_j),
\]

and call it the universal image in \( A_i \); note that here the infimum is nothing but the projective limit of \( \{f_{ij}(A_j)\}_{j \geq i} \) and hence is a subobject of \( A_i \). Clearly, we have \( f_{ij}(A_j) \subset A'_i \) for any \( i \leq j \), and \( f_i(A) \subset A'_i \) for any \( i \in I \). Thus we have the projective system \( A' = \{A'_i, f_{ij}|A'_j\} \) such that \( A = \lim_{i \in I} A'_i \). Note that if \( A \) is strict, then \( A' = A \).
The **Mittag-Leffler condition** for a projective system \( A = \{ A_i, f_{ij} \} \) is:

\[
\text{(ML)} \quad \text{for any } i \in I \text{ there exists } j \geq i \text{ such that } f_{ik}(A_k) = f_{ij}(A_j) \text{ for any } k \geq j.
\]

Clearly, any strict projective system satisfies (ML). Conversely, if \( A \) satisfies (ML), then the induced projective system \( A' \) of universal images is strict. Condition (ML) is closely related to the non-emptiness of projective limits; by Exercise 0.1.2 and [26], Chapter III, §7.4, Proposition 5, we have the following result.

**Proposition 3.2.2.** Let \( X = \{ X_i, f_{ij} \} \) be a projective system of sets indexed by a directed set \( I \), and \( X = \lim_{\leftarrow \atop i \in I} X_i \). Suppose that \( I \) has a cofinal and at most countable subset and that the system \( X \) satisfies (ML). Then for any \( i \in I \) the canonical projection \( f_i: X \to X_i \) maps \( X \) surjectively onto the universal image \( X'_i \). (Hence, in particular, if \( X = \{ X_i, f_{ij} \} \) is strict, then all maps \( f_i \) for \( i \in I \) are surjective.)

**Proposition 3.2.3** ([54], 0 III, (13.2.1)). Let

\[
0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,
\]

where \( M \) etc., be an exact sequence in \( \text{Ab}^{I^\text{opp}} \) consisting of projective systems of abelian groups indexed by a common directed set \( I \).

1. If \( M \) satisfies (ML), then so does \( M'' \).
2. If \( M' \) and \( M'' \) satisfy (ML), then so does \( M \).

Since the functor \( \lim_{\leftarrow \atop i \in I} \) has a left-adjoint, we have the following result.

**Proposition 3.2.4.** The functor \( \lim_{\leftarrow \atop i \in I} : \text{Ab}^{I^\text{opp}} \to \text{Ab} \) is left-exact.

Hence one can consider the right derived functors \( \lim^{(q)}_{\leftarrow \atop i \in I} \) for \( q \geq 0 \).

**Lemma 3.2.5.** Let \( I \) be a directed set that contains a cofinal and at most countable subset. Consider an exact sequence

\[
\cdots \longrightarrow M^{q-1} \xrightarrow{d^{q-1}} M^q \xrightarrow{d^q} M^{q+1} \longrightarrow \cdots
\]

of projective system of abelian groups indexed by \( I \), where \( M^q = \{ M^q_i, f_{ij}^q \} \). Suppose that for any \( q \) the projective system \( M^q \) satisfies (ML). Then the induced sequence

\[
\cdots \longrightarrow \lim_{\leftarrow \atop i \in I} M^{q-1}_i \xrightarrow{d^{q-1}_\infty} \lim_{\leftarrow \atop i \in I} M^q_i \xrightarrow{d^q_\infty} \lim_{\leftarrow \atop i \in I} M^{q+1}_i \longrightarrow \cdots
\]

is exact.
Proof. It is easy to see that \( d^q_\infty \circ d^q_\infty^{-1} = 0 \). Let \( \{x_i\} \in \lim_{\leftarrow i \in I} M^q_i \) be such that \( d^q_\infty(\{x_i\}) = 0 \). We consider the following diagram with exact rows for \( i \leq j \):

\[
\begin{array}{cccc}
M^q_{j-2} & \overset{d^q_{j-2}}{\longrightarrow} & M^q_{j-1} & \overset{d^q_{j-1}}{\longrightarrow} & M^q_j & \overset{d^q_j}{\longrightarrow} & M^q_{j+1} \\
M^q_{i-2} & \overset{d^q_i}{\longrightarrow} & M^q_{i-1} & \overset{d^q_i}{\longrightarrow} & M^q_i & \overset{d^q_i}{\longrightarrow} & M^q_{i+1}.
\end{array}
\]

For each \( i \) we have \( d^q_i(x_i) = 0 \). Hence the subset \( S_i = (d^q_i)^{-1}(x_i) \) of \( M^q_i \) is non-empty, and \( \{S_i, f_{ij}^q | S_j\} \) is a projective system of sets.

Claim. The projective system \( \{S_i, f_{ij}^q | S_j\} \) satisfies (ML).

Let \( T_i \) be the image of \( d^q_i \). The projective system \( \{T_i, f_{ij}^q | T_j\} \) satisfies (ML) (3.2.3 (1)). Hence for any \( i \in I \) there exists \( j \geq i \) such that \( f_{ij}^q(T_i) = f_{ik}^q(T_k) \) for any \( j \leq k \). We want to show that \( f_{ij}^q(S_j) = f_{ik}^q(S_k) \) for any \( j \leq k \). Fix \( y_k \in S_k \), and set \( y_j = f_{jk}^q(y_k) \) and \( y_i = f_{ik}^q(y_k) \). Take any \( z_i \in f_{ij}^q(S_j) \), and put \( z_i = f_{ij}^q(z_j) \). We have \( z_j - y_j \in T_j \), and hence there exists \( w_k \in T_k \) such that \( z_i - y_i = f_{ik}^q(w_k) \). Hence \( z_i = f_{ik}^q(w_k) + y_i = f_{ik}^q(w_k + y_k) \in f_{ik}^q(S_k) \), which shows that \( f_{ij}^q(S_j) \subseteq f_{ik}^q(S_k) \). As the other inclusion is clear, we have \( f_{ij}^q(S_j) = f_{ik}^q(S_k) \), and the claim follows.

Now, by 3.2.2 and the fact that the universal images of the system \( \{S_i\} \) are non-empty, there exists \( \{y_i\} \in \lim_{\leftarrow i \in I} M^q_i \) such that \( d^q_\infty(\{y_i\}) = \{x_i\} \).

Corollary 3.2.6. The functor \( \lim_{\leftarrow i \in I} \) maps any acyclic complex in \( \mathbf{C}^+(\mathbf{Ab}^{\text{op}}) \) consisting of objects satisfying (ML) to an acyclic complex in \( \mathbf{C}^+(\mathbf{Ab}) \).

3.2. (b) Canonical strict resolution. Let \( M = \{M_i, f_{ij}\}_{i \in \mathbb{N}} \) be a projective system of abelian groups indexed by \( \mathbb{N} \). We are going to construct a short exact sequence

\[
0 \longrightarrow M \longrightarrow J^0 \longrightarrow J^1 \longrightarrow 0
\]

in \( \mathbf{Ab}^{\text{op}} \) such that \( J^0 \) and \( J^1 \) are strict.
The system \( J^0 = \{ J^0_i \}_{i \in \mathbb{N}} \) is constructed as follows. For \( i \in \mathbb{N} \) we set
\[
J^0_i = M_0 \oplus \cdots \oplus M_i
\]
and define the transition map \( J^0_j \to J^0_i \) for \( i \leq j \) by
\[
M_0 \oplus \cdots \oplus M_j \ni (x_0, \ldots, x_j) \mapsto \left( x_0, \ldots, x_{i-1}, \sum_{k=i}^j f_{ki}(x_k) \right) \in M_0 \oplus \cdots \oplus M_i.
\]
We have the obvious inclusion \( M \hookrightarrow J^0 \). The system \( J^1 = \{ J^1_i \}_{i \in \mathbb{N}} \) is defined to be the cokernel of this map. Explicitly, it is given by \( J^1_i = J^0_{i-1} \) for \( i > 0 \) and \( J^1_0 = 0 \); the transition maps are the canonical projections. Clearly, with the systems \( J^0 \) and \( J^1 \) thus constructed, we have the desired exact sequence of projective systems as above.

Note that, by the construction, the formation of the resolution \( 0 \to M \to J^* \) is functorial and hence defines a functor
\[
\text{Ab}^\mathbb{N}_{\text{opp}} \longrightarrow \mathcal{C}^{[0,1]}(\text{Ab}^\mathbb{N}_{\text{opp}}),
\]
which is also exact.

We call this resolution \( 0 \to M \to J^* \) the \textit{canonical strict resolution} of \( M \).

In view of 3.2.5, we can use the canonical strict resolution to compute the right derived functors \( \lim^{(q)} \) for \( q \geq 0 \) attached to \( \lim_{\leftarrow i \in I} \).

**Proposition 3.2.7.** Let \( M = \{ M_i, f_{ij} \}_{i \in \mathbb{N}} \) be a projective system of abelian groups indexed by a directed set \( I \) that has a cofinal and at most countable subset.

1. \( \lim_{\leftarrow i \in I}^{(q)} M_i = 0 \) for \( q \geq 2 \).
2. If \( M \) satisfies (ML), then \( \lim_{\leftarrow i \in I}^{(1)} M_i = 0 \).

**Proof.** By Exercise 0.1.2, we may assume that \( I = \mathbb{N} \). Since the canonical strict resolution is of length 1, (1) follows immediately. If \( M \) satisfies (ML), then by 3.2.5 the induced sequence of projective limits is exact, whence (2).

**Corollary 3.2.8** ([54], 0.40, (13.2.2)). Consider the exact sequence of projective systems of abelian groups \( (*) \) in 3.2.3 with \( I \) a directed set that has a cofinal and at most countable subset. If \( M' \) satisfies (ML), then the induced sequence
\[
0 \longrightarrow \lim_{\leftarrow i \in I} M'_i \longrightarrow \lim_{\leftarrow i \in I} M_i \longrightarrow \lim_{\leftarrow i \in I} M''_i \longrightarrow 0
\]
of abelian groups is exact.
3.2. (c) **Projective limit of sheaves.** Let $X$ be a topological space, and consider a projective system $\{F_i, p_{ij}\}_{i \in I}$ of sheaves (of sets, abelian groups, rings, etc.) on $X$ indexed by an ordered set $I$. The projective limit sheaf $\lim_{i \in I} F_i$ in the category of sheaves (of sets, abelian groups, rings, etc.) is described as follows. For any open subset $U \subseteq X$ set

$$F(U) = \lim_{i \in I} F_i(U).$$

This defines a presheaf $F$, which is easily seen to be a sheaf. The desired projective limit $\lim_{i \in I} F_i$ is given by the sheaf $F$ with the canonical maps $p_i: F \to F_i$ for $i \in I$. Note that, unlike the case of inductive limits, the canonical map

$$(\lim_{i \in I} F_i)_x \longrightarrow \lim_{i \in I} F_{i,x}$$

for $x \in X$ is in general neither surjective, nor injective.

**Proposition 3.2.9.** (1) For a topological space $X$ and an ordered set $I$, we have the canonical isomorphism

$$\lim_{i \in I} \Gamma_X \cong \Gamma_X \circ \lim_{i \in I}$$

of functors from $\text{ASh}_X^{\text{opp}}$ to $\text{Ab}$, where $\Gamma_X$ is the global section functor

$$\Gamma_X(F) = \Gamma(X, F).$$

(2) For a continuous map $f: X \to Y$ between topological spaces and an ordered set $I$, we have the canonical isomorphism

$$\lim_{i \in I} f_* \cong f_* \circ \lim_{i \in I}$$

of functors from $\text{ASh}_X^{\text{opp}}$ to $\text{ASh}_Y$.

Finally, note that the projective limit functor

$$\lim_{i \in I} : \text{ASh}_X^{\text{opp}} \longrightarrow \text{ASh}_X$$

is, as one can show similarly to 3.2.4, left-exact.
3.2. (d) **Canonical s-flasque resolution.** Let $X$ be a topological space.

**Definition 3.2.10.** A projective system $\{F_i, p_{ij}\}_{i \in I}$ of abelian sheaves on $X$ (that is, an object of $\text{ASH}_X^{\text{op}}$) is said to be s-flasque if

(a) each $F_i$ for $i \in I$ is flasque and

(b) for each $i < j$ the map $p_{ji} : F_j \to F_i$ is surjective and has a flasque kernel.

Let $\{F_i, p_{ij}\}_{i \in \mathbb{N}}$ be a projective system of abelian sheaves on $X$ indexed by $\mathbb{N}$. We are going to construct an s-flasque resolution

$$0 \to \{F_i\}_{i \in \mathbb{N}} \to \{G^\bullet_i\}_{i \in \mathbb{N}}.$$

First, we construct a resolution

$$0 \to \{F_i\}_{i \in \mathbb{N}} \to \{J^0_i\}_{i \in \mathbb{N}} \to \{J^1_i\}_{i \in \mathbb{N}} \to 0$$

where

$$J^0_i = F_0 \oplus \cdots \oplus F_i$$

and

$$J^1_i = \begin{cases} 0 & \text{if } i = 0, \\ F_0 \oplus \cdots \oplus F_{i-1} & \text{if } i > 0, \end{cases}$$

for each $i$ and define the maps similarly to that for canonical strict resolutions (cf. §3.2. (b)). Next we take the canonical flasque resolutions of sheaves in $\{J^0_i\}_{i \in \mathbb{N}}$ and in $\{J^1_i\}_{i \in \mathbb{N}}$. Thus we get the double complex $\{C^\bullet(X, F^\bullet_i)\}_{i \in \mathbb{N}}$ of projective systems. The desired resolution $\{G^\bullet_i\}_{i \in \mathbb{N}}$ is the single complex associated to this double complex; therefore,

$$\{G^q_i\}_{i \in \mathbb{N}} = \begin{cases} \{C^0(X, F^0_i)\}_{i \in \mathbb{N}} & \text{if } q = 0, \\ \{C^q(X, F^0_i) \oplus C^{q-1}(X, F^1_i)\}_{i \in \mathbb{N}} & \text{if } q > 0. \end{cases}$$

We want to show that this resolution is s-flasque. Condition (a) in 3.2.10 is clear. Condition (b) in 3.2.10 follows from the fact that the formation of the canonical flasque resolution of sheaves is an exact functor (§3.1. (e)). Indeed, if

$$0 \to \mathcal{K} \to J^k_j \to J^k_i \to 0$$

is exact for $k = 0, 1$ and $i < j$, then we have the induced exact sequence

$$0 \to \mathcal{C}^q(X, \mathcal{K}) \to \mathcal{C}^q(X, J^k_j) \to \mathcal{C}^q(X, J^k_i) \to 0.$$
By the construction, one sees that forming the above resolution defines a functor
\[ \text{ASH}_X^{\text{op}} \to \mathbf{C}^+(\text{ASH}_X^{\text{op}}), \]
which, moreover, is exact. We call the resolution \( 0 \to \{ \mathcal{F}_i \}_{i \in \mathbb{N}} \to \{ \mathcal{G}_i \}_{i \in \mathbb{N}} \) the canonical s-flasque resolution.

**Proposition 3.2.11.** Consider the exact sequence
\[ \cdots \to \mathcal{F}_{i+1} \to \mathcal{F}_i \to \cdots \]
of projective systems of abelian sheaves on \( X \) indexed by \( \mathbb{N} \). Suppose that for any \( p \) the projective system \( \{ \mathcal{F}_i \}_{i \in \mathbb{N}} \) is s-flasque. Then the induced sequence
\[ \cdots \to \mathcal{F}^{p-1}(X) \to \mathcal{F}^p(X) \to \mathcal{F}^{p+1}(X) \to \cdots \]
of abelian groups is exact, where \( \mathcal{F}^p = \varprojlim_{i \in \mathbb{N}} \mathcal{F}_i^p \).

**Proof.** The only non-trivial part of the proof is to show that any section \( s \in \mathcal{F}^p(X) \) that is mapped to 0 in \( \mathcal{F}^{p+1}(X) \) lies in the image of \( \mathcal{F}^{p-1}(X) \to \mathcal{F}^p(X) \). Let \( s_i \) be the image of \( s \) by the projection \( \mathcal{F}^p(X) \to \mathcal{F}_i^p(X) \) for each \( i \in \mathbb{N} \). Since \( \mathcal{F}^{p-1}(X) = \varprojlim_{i \in \mathbb{N}} \mathcal{F}_i^{p-1}(X) \), we want to construct the compatible system \( \{ t_i \}_{i \in \mathbb{N}} \) of liftings \( t_i \in \mathcal{F}_i^{p-1}(X) \) of \( s_i \). By [47], Chapter II, 3.1.3, \( s_0 \) has a lifting \( t_0 \in \mathcal{F}_0^{p-1}(X) \). Suppose one has already constructed compatible liftings up to \( t_{k-1} \in \mathcal{F}_{k-1}^{p-1}(X) \). Take any lifting \( t'_k \) of \( s_k \) (which exists according to [47], Chapter II, 3.1.3). The image \( t'_{k-1} \) of \( t'_k \) in \( \mathcal{F}_{k-1}^{p-1}(X) \) has the same image in \( \mathcal{F}_k^{p-1}(X) \) as \( t_{k-1} \), and hence there exists an element \( u_{k-1} \in \mathcal{F}_k^{p-2}(X) \) that is mapped to \( t_{k-1} - t'_{k-1} \). Since \( \{ \mathcal{F}_i^{p-2} \}_{i \in \mathbb{N}} \) is s-flasque, the projection \( \mathcal{F}_k^{p-2}(X) \to \mathcal{F}_{k-1}^{p-2}(X) \) is surjective, and hence we have an element \( u_k \in \mathcal{F}_k^{p-2}(X) \) that lifts \( u_{k-1} \). Let \( v_k \) be the image of \( u_k \) in \( \mathcal{F}_k^{p-1}(X) \), and set \( t_k = t'_{k} + v_k \). Then \( t_k \) is a lifting of both \( t_{k-1} \) and \( s_k \).

By induction, we get the desired system of liftings. \( \square \)

The proposition shows that the canonical s-flasque resolution can be used to compute the right derived functors of the left-exact functor
\[ \Gamma_X \circ \lim \left( \equiv \lim \circ \Gamma_X \right) \]
(cf. 3.2.9 (1)), that is, if
\[ 0 \to \{ \mathcal{F}_i \}_{i \in \mathbb{N}} \to \{ \mathcal{G}_i \}_{i \in \mathbb{N}} \]
is the canonical s-flasque resolution, then we have
\[ R^q(\Gamma_X \circ \lim)(\{ \mathcal{F}_i \}_{i \in \mathbb{N}}) = H^q(\Gamma(X, \mathcal{G}^\bullet)), \]
where \( \mathcal{G}^\bullet = \varprojlim_{i \in \mathbb{N}} \mathcal{G}_i^\bullet \).
Proposition 3.2.12. (1) If \( \{ \mathcal{G}_i \}_{i \in \mathbb{N}} \) is an s-flasque projective system of abelian sheaves, then \( \mathcal{G} = \lim_{\leftarrow i \in \mathbb{N}} \mathcal{G}_i \) is flasque.

(2) Suppose

\[
0 \rightarrow \{ \mathcal{F}_i \}_{i \in \mathbb{N}} \rightarrow \{ \mathcal{G}_i^* \}_{i \in \mathbb{N}}
\]

is an s-flasque resolution, and set \( \mathcal{F} = \lim_{\leftarrow i \in \mathbb{N}} \mathcal{F}_i \) and \( \mathcal{G}^* = \lim_{\leftarrow i \in \mathbb{N}} \mathcal{G}_i^* \). Then the induced sequence

\[
0 \rightarrow \mathcal{F} \rightarrow \mathcal{G}^*
\]

gives a flasque resolution of \( \mathcal{F} \).

Proof. (1) can be shown by an easy diagram similar to that in the proof of 3.2.11. By an argument similar to that in the proof of 3.2.11, one can see that for any open subset \( U \) the induced sequence

\[
0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}^*(U)
\]

is exact. Taking stalks at every point, one concludes the exactness of \( 0 \rightarrow \mathcal{F} \rightarrow \mathcal{G}^* \), as desired. \( \square \)

3.2. (e) Projective limits and cohomology. Let \( X \) be a topological space, and \( I \) a directed set containing a countable cofinal set. We consider the following conditions for a projective system \( \{ \mathcal{F}_i \}_{i \in I} \) of sheaves of abelian groups on \( X \) indexed by \( I \):

(E1) there exists an open basis \( \mathcal{B} \) of \( X \) such that for any \( U \in \mathcal{B} \) and \( q \geq 0 \) the projective system \( \{ H^q(U, \mathcal{F}_i) \}_{i \in I} \) satisfies (ML);

(E2) for any \( x \in X \) and any \( q > 0 \),

\[
\lim_{x \in U \in \mathcal{B}} \lim_{i \in I} H^q(U, \mathcal{F}_i) = 0,
\]

where \( U \) varies over the set of open neighborhoods of \( x \) in \( \mathcal{B} \).

Remark 3.2.13. (1) In practice, these conditions are considered in the situation where \( X \) has an open basis \( \mathcal{B} \) such that \( H^q(U, \mathcal{F}_i) = 0 \) for any \( U \in \mathcal{B}, i \in I, \) and \( q > 0 \). In this case, (E1) for \( q > 0 \) and (E2) are trivially satisfied, and hence one only need to check that the projective system \( \{ H^0(U, \mathcal{F}_i) \}_{i \in I} \) satisfies (ML).

(2) If \( X \) is a scheme, and if \( \{ \mathcal{F}_i \}_{i \in I} \) is a projective system consisting of quasi-coherent sheaves, then one can choose as \( \mathcal{B} \) the set of all affine open subsets of \( X \) (see [54], III, (1.3.1)). In this case, for example, if \( \{ \mathcal{F}_i \}_{i \in I} \) is strict, then \( \{ \mathcal{F}_i \}_{i \in I} \) satisfies (E1) and (E2); indeed, for \( i \leq j \) and \( U \in \mathcal{B} \), since the kernel of the surjective map \( \mathcal{F}_j \rightarrow \mathcal{F}_i \) is quasi-coherent, we deduce that \( H^0(U, \mathcal{F}_j) \rightarrow H^0(U, \mathcal{F}_i) \) is surjective, and hence \( \{ H^0(U, \mathcal{F}_i) \}_{i \in I} \) is strict.
Proposition 3.2.14. Let \( X \) be a topological space, and \( I \) a directed set containing a countable cofinal subset. Let \( \{ \mathcal{F}_i \}_{i \in I} \) be a projective system of sheaves of abelian groups on \( X \) indexed by \( I \), and set \( \mathcal{F} = \lim_{\leftarrow i \in I} \mathcal{F}_i \). Suppose \( \{ \mathcal{F}_i \}_{i \in I} \) satisfies (\( \text{E1} \)) and (\( \text{E2} \)).

1. For any \( q > 0 \) we have
\[
\lim_{i \in I}(q)\mathcal{F}_i = 0.
\]

2. For any \( q \geq 0 \) there exist canonical isomorphisms
\[
H^q(X, \mathcal{F}) \cong R^q(\Gamma_X \circ \lim_{i \in I}(\mathcal{F}_i)) \cong H^q(\lim_{i \in I}(\Gamma_X)(\{\mathcal{F}_i\}_{i \in I})).
\]

Remark 3.2.15. Combined with 3.2.11, the last statement (in the case \( I = \mathbb{N} \)) shows that one can use s-flasque resolutions (§3.2 (d)) to compute the cohomologies \( H^q(X, \mathcal{F}) \). Indeed, if
\[
0 \to \{\mathcal{F}_i\}_{i \in \mathbb{N}} \to \{\mathcal{G}^\bullet_i\}_{i \in \mathbb{N}}
\]
is the canonical s-flasque resolution, we have
\[
H^q(X, \mathcal{F}) \cong H^q(\Gamma(X, \mathcal{G}^\bullet)),
\]
where \( \mathcal{G}^\bullet = \lim_{i \in \mathbb{N}} \mathcal{G}^\bullet_i \).

Proof of Proposition 3.2.14. We assume without loss of generality that \( I = \mathbb{N} \) (Exercise 0.1.2 and Exercise 0.3.2). Let \( U \) be an open subset of \( X \). Since
\[
\Gamma_U \circ \lim_{i \in \mathbb{N}} \cong \lim_{i \in \mathbb{N}} \circ \Gamma_U,
\]
we have the following two spectral sequences that converge to the same infinity terms:
\[
\begin{aligned}
&\overset{I}{E}_2^{p,q}(U) = \lim_{i \in \mathbb{N}}(p)H^q(U, \mathcal{F}_{i}) \implies \overset{I}{E}_{\infty}^{p+q}(U) = R^{p+q}(\lim_{i \in \mathbb{N}}(\Gamma_U)(\{\mathcal{F}_i\}_{i \in \mathbb{N}}))
\end{aligned}
\]
and
\[
\begin{aligned}
&\overset{II}{E}_2^{p,q}(U) = H^p(U, \lim_{i \in \mathbb{N}}(q)\mathcal{F}_{i}) \implies \overset{II}{E}_{\infty}^{p+q}(U) = R^{p+q}(\Gamma_U \circ \lim_{i \in \mathbb{N}})(\{\mathcal{F}_i\}_{i \in \mathbb{N}}),
\end{aligned}
\]
with \( \overset{I}{E}_{\infty}^{p+q}(U) = \overset{II}{E}_{\infty}^{p+q}(U) \). For any point \( x \in X \) we set
\[
\overset{I}{E}_x = \lim_{U \in \mathcal{U}} \overset{I}{E}(U),
\]
where \( U \) runs through all open neighborhoods of \( x \) in \( \mathcal{U} \). Since the inductive limit functor is exact, this defines a spectral sequence that converges to the inductive limits of the corresponding \( \infty \)-terms. A similar notation will be used for \( \overset{II}{E}(U) \).
From (E1) and (E2) we have

(E1) \( I^{p,q}_{E_2,x} = 0 \) for \( p > 0 \);

(E2) \( I^{p,q}_{E_2,x} = 0 \) for \( p = 0 \) and \( q > 0 \).

Hence, the only non-zero \( E_2 \)-term of the spectral sequence \( I^{p,q}_{E_2,x} \) is \( I^{0,0}_{E_2,x} \). This shows that

\[
I^{p,q}_{E_2,x} = I^{0,q}_{E_2,x} = \begin{cases} 
\lim_{x \in U} \lim_{i \in \mathbb{N}} \Gamma(U, \mathcal{F}_i) & \text{if } p + q = 0, \\
0 & \text{otherwise.}
\end{cases}
\tag{\dagger}
\]

Now we claim that \( I^{p,0}_{E_2,x} = 0 \) for \( p > 0 \). Take the canonical s-flasque resolution \( 0 \to \{\mathcal{F}_i\}_{i \in \mathbb{N}} \to \{\mathcal{G}_i\}_{i \in \mathbb{N}} \). By 3.2.11

\[
0 = I^{p,0}_{E_2,x} = \lim_{x \in U} H^p(\Gamma(U, \mathcal{G}^\bullet)),
\]

where \( \mathcal{G}^\bullet = \lim_{i \in \mathbb{N}} \mathcal{G}_i^\bullet \). But then, since \( 0 \to \mathcal{F} \to \mathcal{G} \) gives a flasque resolution (3.2.12 (2)), we deduce that

\[
I^{p,0}_{E_2,x} = \lim_{x \in U} H^p(U, \mathcal{F}) = \lim_{x \in U} H^p(\Gamma(U, \mathcal{G}^\bullet)) = 0,
\]

as desired.

Now we start proving (1). We proceed by induction with respect to \( q \). Since

\[
0 = I^{p,-2,q}_{E_2,x} \xrightarrow{d} I^{p,0,q}_{E_2,x} \xrightarrow{d} I^{p,2,q}_{E_2,x} = 0,
\]

one has \( I^{0,1}_{E_2,x} = I^{0,1}_{E_{\infty,x}} \). But \( I^{0,1}_{E_{\infty,x}} \) is a subquotient of \( I^{1,1}_{E_{\infty,x}} \), which is zero by (\dagger), and hence we have \( \lim_{x \in X} I^{1,1}_{E_{\infty,x}} = 0 \). Since this is true for any point \( x \in X \), we get the desired vanishing (\text{*})$_1$. By induction, we assume that (\text{*})$_k$ is true for \( k = 1, \ldots, q - 1 \). Then for \( k \leq q - 1 \) we have \( I^{p,k}_{E_2,x} = 0 \). Since

\[
0 = I^{p,-2,q+1}_{E_2,x} \xrightarrow{d} I^{p,0,q}_{E_2,x} \xrightarrow{d} I^{p,2,q-1}_{E_2,x} = 0,
\]

one has \( I^{0,q}_{E_2,x} = I^{0,q}_{E_{\infty,x}} \). Arguing in much the same way as above, one sees the last term is zero for any \( x \in X \), and hence (\text{*})$_q$ holds as desired.

Now we look at the global spectral sequence \( I^p(E(X)) \). Since (1) holds, this spectral sequence degenerates at \( E_2 \)-terms. By this and (1), we get (2). \( \square \)
Corollary 3.2.16 (cf. [54], 0III. (13.3.1)). Let $X$ be a topological space, $I$ a directed set containing a countable cofinal subset, and $\{F_i\}_{i \in I}$ a projective system of abelian sheaves on $X$ indexed by $I$. Suppose $\{F_i\}_{i \in I}$ satisfies (E1) and (E2). Then for any $q > 0$ the canonical morphism

$$H^q(X, F) \longrightarrow \lim_{i \in I} H^q(X, F_i)$$

(\star\star)_q

is surjective. If, moreover, the projective system $\{H^{q-1}(X, F_i)\}_{i \in I}$ satisfies (ML), then $\star\star_q$ is bijective.

Remark 3.2.17. (1) Note that the fact that $\star\star_0$ is an isomorphism, not mentioned in the above statement, is always true due to 3.2.9 (1).

(2) In [54], 0III. (13.3.1), it is assumed in addition that the projective system $\{F_i\}_{i \in I}$ is strict. But this assumption is not necessary, since one can always replace $\{F_i\}_{i \in I}$ by a complex of strict systems.

Proof of Corollary 3.2.16. We use the spectral sequence $E = \mathcal{E}(X)$ as in the proof of 3.2.14. By 3.2.14 (2), this spectral sequence converges to $E_{\infty}^{p+q} = H^{p+q}(X, F)$. By 3.2.7 (1), the spectral sequence degenerates at $E_2$-terms. Hence we have the exact sequence

$$0 \longrightarrow E_2^{1,q-1} \longrightarrow H^q(X, F) \longrightarrow E_2^{0,q} \longrightarrow 0,$$

which shows, in particular, the surjectivity of (\star\star)_q. If $\{H^{q-1}(X, F_i)\}_{i \in \mathbb{N}}$ satisfies (ML), then by 3.2.7 (2) we have $E_2^{1,q-1} = 0$, whence the isomorphism $H^q(X, \lim_{k} F_k) \cong E_2^{0,q}$.

Proposition 3.2.18. Consider the situation as in 3.2.14 with $I = \mathbb{N}$, and let $q \geq 0$ be an integer. For each $i \in \mathbb{N}$ set $N_i = \ker(F \rightarrow F_i)$, and let $F^i$ be the image of the map $H^q(X, N_i) \rightarrow H^q(X, F)$. Then the following conditions are equivalent:

(a) the map $\star\star_q$ in 3.2.16 is injective;

(b) the map $\star\star_q$ in 3.2.16 is bijective;

(c) $\bigcap_{i \in \mathbb{N}} F^i = 0$. In other words, $H^q(X, F)$ is separated with respect to the filtration $F^\bullet = \{F^i\}_{i \in \mathbb{N}}$.

Proof. The equivalence of (a) and (b) follows from 3.2.16. Since the map

$$H^q(X, F)/F^i \rightarrow H^q(X, F_i)$$

is injective, we have the injective homomorphism

$$\lim_{i \in \mathbb{N}} H^q(X, F)/F^i \leftarrow \lim_{i \in \mathbb{N}} H^q(X, F_i).$$

(\dagger)
The canonical map \((**)_q\) clearly factors through \(\lim_{k \in \mathbb{N}} H^q(X, \mathcal{F})/F^i\) by using the canonical map
\[
H^q(X, \mathcal{F}) \longrightarrow \lim_{i \in \mathbb{N}} H^q(X, \mathcal{F})/F^i
\]
followed by the map \((\ddagger)\), and hence \((\ddagger)\) is bijective. Therefore, \((**)_q\) is injective if and only if \((\ddagger\ddagger)\) is injective, and the last condition is equivalent to (c).

**Proposition 3.2.19.** Let \(f : X \to Y\) be a continuous mapping between topological spaces, \(I\) a directed set containing a countable cofinal subset, and \(\{\mathcal{F}_i\}_{i \in I}\) a projective system of abelian sheaves on \(X\) indexed by \(I\). Set \(\mathcal{F} = \lim_{i \in I} \mathcal{F}_i\). Suppose that

- \(\{\mathcal{F}_i\}_{i \in I}\) satisfies (E1) and (E2)
- the projective systems \(\{R^q f_* \mathcal{F}_i\}_{i \in I}\) \((q \geq 0)\) of abelian sheaves on \(Y\) satisfies (E1) and (E2).

Then the canonical morphism
\[
R^q f_* \mathcal{F} \longrightarrow \lim_{i \in I} R^q f_* \mathcal{F}_i
\]
is an isomorphism.

By what we have seen in 3.2.13, the hypotheses in the proposition are satisfied in the following situation: \(f : X \to Y\) is a coherent (= quasi-compact and quasi-separated) morphism of schemes, and \(\{\mathcal{F}_i\}_{i \in I}\) is a strict projective system consisting of quasi-coherent sheaves on \(X\) such that the induced system \(\{R^q f_* \mathcal{F}_i\}_{i \in I}\) is also strict for \(q \geq 0\). Indeed, by a well-known fact in the theory of schemes (cf. 5.4.6 below), the sheaves \(R^q f_* \mathcal{F}_i\) for \(q \geq 0\) and \(i \in I\) are quasi-coherent.

**Proof of Proposition 3.2.19.** We consider the spectral sequences
\[
^I\mathcal{E}_{2}^{p,q} = \lim_{i \in \mathbb{N}} (p) R^q f_* \mathcal{F}_i \quad \Rightarrow \quad ^I\mathcal{E}_{\infty}^{p,q} = R^{p+q} (\lim_{i \in \mathbb{N}} f_*)(\{\mathcal{F}_i\}_{i \in \mathbb{N}}),
\]
and
\[
^II\mathcal{E}_{2}^{p,q} = R^p f_* \lim_{i \in \mathbb{N}} (q) \mathcal{F}_i \quad \Rightarrow \quad ^II\mathcal{E}_{\infty}^{p,q} = R^{p+q} (f_* \circ \lim_{i \in \mathbb{N}})(\{\mathcal{F}_i\}_{i \in \mathbb{N}}).
\]
By 3.2.9 (2), we have
\[
^I\mathcal{E}_{\infty}^{p+q} = ^II\mathcal{E}_{\infty}^{p+q}.
\]
By 3.2.14 (1), we have
\[
^II\mathcal{E}_{\infty}^{p+q} \cong R^{p+q} f_* \mathcal{F}
\]
and
\[
^I\mathcal{E}_{\infty}^{p+q} \cong \lim_{i \in I} R^{p+q} f_* \mathcal{F}_i.
\]
3.3 Coherent rings and modules

Definition 3.3.1 (cf. [27], Chapter I, §3, Exercise 11). Let $A$ be a ring.

(1) A finitely generated $A$-module $M$ is said to be coherent if every finitely generated $A$-submodule of $M$ is finitely presented. We denote by $\text{Coh}_A$ the full subcategory of $\text{Mod}_A$ consisting of coherent $A$-modules.

(2) $A$ is coherent if it is coherent as an $A$-module or, what amounts to the same, every finitely generated ideal of $A$ is finitely presented.

For example, any Noetherian ring is a coherent ring, and any finitely generated module over a Noetherian ring is a coherent module. It is easy to see that, if $A$ is a coherent ring and $S \subseteq A$ is a multiplicative subset, then $S^{-1}A$ is again a coherent ring. In Exercise 0.3.3 we will see that the integral closure $\hat{\mathcal{O}}$ of $\mathcal{O}$ in the ring of algebraic numbers $\hat{\mathbb{Q}}$ is a coherent ring, which, however, is not Noetherian (Exercise 0.3.4).

Let us first state here an easy proposition; the second part follows immediately from [27], Chapter I, §3.6, Proposition 11.

Proposition 3.3.2. (1) Let $A$ and $B$ be rings, $M$ a finitely generated $A$-module, and $N$ a finitely generated $B$-module. Then $M \times N$ is a coherent $A \times B$-module if and only if $M$ and $N$ are coherent over $A$ and $B$, respectively. In particular, $A \times B$ is coherent if and only if $A$ and $B$ are coherent.

(2) Let $A$ be a ring, and $B$ a faithfully flat $A$-algebra. Then a finitely generated $A$-module $M$ is coherent if and only if $M \otimes_A B$ is a coherent $B$-module. In particular, $A$ is coherent if $B$ is coherent.

Proposition 3.3.3. The following conditions for a ring $A$ are equivalent:

(a) $A$ is coherent;

(b) any finitely presented $A$-module is coherent;

(c) the category of finitely presented $A$-modules is an abelian subcategory of the abelian category $\text{Mod}_A$ of $A$-modules;

(d) if $f : M \to N$ is a homomorphism between finitely presented $A$-modules, then $\ker(f)$ is finitely presented;

(e) if the $A$-modules $M$ and $N$ sitting in the exact sequence

$$0 \to L \to M \to N \to 0$$

of $A$-modules are finitely presented, then so is $L$;

(f) if two of $L$, $M$, and $N$ in the exact sequence (*) are finitely presented, so is the rest.
(g) For any exact sequence of $A$-modules
\[ M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4 \rightarrow M_5, \]
if $M_1$ is finitely generated, and if $M_2$, $M_4$, and $M_5$ are finitely presented, then $M_3$ is finitely presented.

(h) For any complex
\[ M^\bullet = (\cdots \rightarrow M^{k-1} \rightarrow M^k \rightarrow M^{k+1} \rightarrow \cdots) \]
of $A$-modules, if every $M^k$ is finitely presented, then $H^q(M^\bullet)$ is finitely presented for any $q$.

The proof is easy with the aid of the following elementary lemma.

**Lemma 3.3.4.** Let $A$ be a ring, and
\[ 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \quad (\ast) \]
an exact sequence of $A$-modules.

1. If $L$ is a finitely generated and $M$ is a finitely presented, then $N$ is finitely presented.

2. If $L$ and $N$ are finitely presented, then $M$ is finitely presented.

**Corollary 3.3.5.** Let $A$ be a coherent ring. Then an $A$-module $M$ is coherent if and only if it is finitely presented.

If $A$ is a coherent ring, then any coherent $A$-module has a free resolution; hence we have the following corollary.

**Corollary 3.3.6.** Let $A$ be a coherent ring. Then for any coherent $A$-modules $M$ and $N$, $\text{Ext}_A^p(M, N)$ and $\text{Tor}_A^p(M, N)$ are coherent for $p \geq 0$.

The following notion will be particularly important later.

**Definition 3.3.7.** A ring $A$ is said to be **universally coherent** if every finitely presented $A$-algebra is coherent.

Clearly, if $A$ is universally coherent, then any finitely presented $A$-algebra is again universally coherent. Note also that if $A$ is universally coherent, then any localization of $A$ is again universally coherent.
4. Ringed spaces

Exercises

Exercise 0.3.1. Let $X$ be a coherent topological space. A sheaf $\mathcal{F}$ of sets on $X$ is said to be \textit{quasi-flasque} ([68]) if, for any quasi-compact open subset $U \subseteq X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

1. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of abelian sheaves on $X$, and suppose $\mathcal{F}'$ is quasi-flasque. Then show that

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \longrightarrow 0$$

is exact.

2. Show that, for a quasi-flasque sheaf of abelian groups $\mathcal{F}$, $\text{H}^q(X, \mathcal{F}) = 0$ for $q > 0$.

3. Let $\{\mathcal{G}_i, \varphi_{ij}\}_{i \in I}$ be a filtered inductive system of quasi-flasque sheaves of sets indexed by a directed set $I$. Show that the inductive limit $\mathcal{G} = \lim_{\longrightarrow i \in I} \mathcal{G}_i$ is quasi-flasque.

Exercise 0.3.2. Let $\{A_i, f_{ij}\}_{i \in I}$ be a projective system of sets indexed by a directed set $I$, and $J \to I$ a cofinal ordered map. Show that, if $\{A_i, f_{ij}\}_{i \in I}$ satisfies (ML), then so does $\{A_i, f_{ij}\}_{i \in J}$.

Exercise 0.3.3. Let $\{A_i, \phi_{ij}\}_{i \in I}$ be an inductive system of rings indexed by a directed set. Assume that each $A_i$ is coherent (resp. universally coherent) and that each transition map is flat. Show that $A = \lim_{\longrightarrow i \in I} A_i$ is coherent (resp. universally coherent).

Exercise 0.3.4. Show that the subring of all algebraic integers in $\mathbb{Q}$ is coherent, but not Noetherian.

Exercise 0.3.5. Let $k$ be a field, and consider the polynomial ring $k[x_1, x_2, \ldots]$ in countably many variables. Set $J = (x_1x_2, x_1x_3, \ldots, x_1x_n, \ldots)$. Show that the ring $A = k[x_1, x_2, \ldots]/J$ is not coherent.

4 Ringed spaces

In most ordinary commutative geometries, spaces are ‘visualized’ by means of locally ringed spaces (as referred to as standard visualization in Introduction). Homological algebra of $\mathcal{O}_X$-modules, especially that of quasi-coherent and coherent sheaves, is an important tool for analyzing the spaces. Coherent sheaves are particularly useful if the structure sheaf $\mathcal{O}_X$ itself is coherent. So the coherence of the structure sheaf is one of the fundamental conditions for (locally) ringed spaces.
In §4.1 we discuss ringed spaces satisfying this condition in general, which we call coherent ringed spaces.

In §4.2 we begin the study of sheaves of modules and their cohomologies in the context of filtered projective limits. Note that most of the results in these sections are, in fact, rehashes of what are already done in [9], Exposé VI, in the topos-theoretic language.

4.1 Generalities

4.1. (a) Ringed spaces and locally ringed spaces. A ringed space is a couple \((X, \mathcal{O}_X)\) consisting of a topological space \(X\) and a sheaf of rings \(\mathcal{O}_X\) on \(X\), called the structure sheaf. Given two ringed spaces \(X = (X, \mathcal{O}_X)\) and \(Y = (Y, \mathcal{O}_Y)\), a morphism of ringed spaces from \(X\) to \(Y\) is a pair \((f, \varphi)\) consisting of a continuous map \(f: X \to Y\) and a morphism \(\varphi: f^{-1}\mathcal{O}_Y \to \mathcal{O}_X\) of sheaves of rings on \(X\) or, equivalently by adjunction, a morphism \(\mathcal{O}_Y \to f_*\mathcal{O}_X\) of sheaves of rings on \(Y\). We denote by \(\text{Rsp}\) the category of ringed spaces.

A ringed space \((X, \mathcal{O}_X)\) is said to be a locally ringed space if for any point \(x \in X\) the ring \(\mathcal{O}_{X,x}\) is a local ring. In this case we denote by \(m_{X,x}\) the maximal ideal of the local ring \(\mathcal{O}_{X,x}\). A morphism \((f, \varphi): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) of ringed spaces, where \(X = (X, \mathcal{O}_X)\) and \(Y = (Y, \mathcal{O}_Y)\) are locally ringed spaces, is said to be local if for any \(x \in X\) the map \(\varphi_x: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}\) is a local homomorphism (that is, \(\varphi_x(m_{Y,f(x)}) \subseteq m_{X,x}\)). We denote by \(\text{LRsp}\) the category of locally ringed spaces and local morphisms.

Convention. In the sequel, whenever we deal with locally ringed spaces, by a morphism we mean simply a local morphism, unless otherwise clearly stated.

Let \((X, \mathcal{O}_X)\) be a ringed space. An open (ringed) subspace of \((X, \mathcal{O}_X)\) is a ringed space of the form \((U, \mathcal{O}_X|_U)\), where \(U\) is an open subset of \(X\). An open immersion is a morphism of ringed spaces \((Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)\) that factors through the canonical map from an open subspace \((U, \mathcal{O}_U)\) of \((X, \mathcal{O}_X)\) by an isomorphism \((Y, \mathcal{O}_Y) \sim (U, \mathcal{O}_U)\).

A ringed space \((X, \mathcal{O}_X)\) is said to be reduced if for any point \(x \in X\) the ring \(\mathcal{O}_{X,x}\) has no non-zero nilpotent element; in other words, if \(\mathcal{N}_X\) is the subsheaf of \(\mathcal{O}_X\) consisting of nilpotent sections, we have \(\mathcal{N}_X = 0\).

4.1. (b) Generization map. Let \(X\) be a topological space, and \(\mathcal{F}\) a sheaf (of sets) on \(X\). If \(y \in G_x\) is a generization of a point \(x \in X\), then any open neighborhood of \(x\) is an open neighborhood of \(y\). Hence we have a canonical map

\[
\mathcal{F}_x \longrightarrow \mathcal{F}_y
\]
between the stalks, which we call the *generization map*. In particular, if $X = (X, \mathcal{O}_X)$ is a ringed space, then the generization map

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,y}$$

is a ring homomorphism. If, moreover, $X$ is a locally ringed space, then the generization map induces a local homomorphism

$$(\mathcal{O}_{X,x})_{q_y} \longrightarrow \mathcal{O}_{X,y},$$

where $q_y$ is the pull-back of the maximal ideal $m_{X,y}$ at $y$.

**Examples 4.1.1.** (1) If $X$ is a scheme, then $(*)$ is an isomorphism.

(2) If $X$ is a locally Noetherian formal scheme ([54], I, (10.4.2)) or, more generally, a *locally universally rigid-Noetherian formal scheme* (defined later in II.2.1.7 below), then $(*)$ is faithfully flat (Exercise 0.4.1).

### 4.1. (c) Sheaves of modules.

For a ringed space $X = (X, \mathcal{O}_X)$ we denote by $\text{Mod}_X$ the category of $\mathcal{O}_X$-modules. This is an abelian category with tensor products and internal Hom’s. Any morphism

$$(f, \varphi): (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

of ringed spaces induces two functors, adjoint to each other:

$$\text{Mod}_X \xrightarrow{f^*} \text{Mod}_Y;$$

here $f^* \mathcal{G}$ for an $\mathcal{O}_Y$-module $\mathcal{G}$ is defined as

$$f^* \mathcal{G} = f^{-1} \mathcal{G} \otimes_{f^{-1} \mathcal{O}_Y, \varphi} \mathcal{O}_X.$$

**Definition 4.1.2.** Let $(f, \varphi): (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, and $\mathcal{J}$ an ideal sheaf of $\mathcal{O}_Y$. The *ideal pull-back of $\mathcal{J}$* is the ideal sheaf of $\mathcal{O}_X$ generated by the image of $f^{-1} \mathcal{J}$ by the map $\varphi$.

The ideal pull-back is denoted by $(f^{-1} \mathcal{J}) \mathcal{O}_X$ or more simply by $\mathcal{J} \mathcal{O}_X$.

**Definition 4.1.3.** (1) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and $n \in \mathbb{Z}$ a non-negative integer. We say that $\mathcal{F}$ is of *(finite) n-presentation* or *n-presented*, if for any point $x \in X$ there exists an open neighborhood $U$ of $x$ on which there exists an exact sequence of $\mathcal{O}_U$-modules

$$\mathcal{E}^n \longrightarrow \cdots \longrightarrow \mathcal{E}^1 \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{F}|_U \longrightarrow 0,$$

where each $\mathcal{E}^i$ ($0 \leq i \leq n$) is a free $\mathcal{O}_U$-module of finite rank.
(2) An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be of finite type if it is of $0$-presentation. If it is of $1$-presentation, we say that $\mathcal{F}$ is of finite presentation or finitely presented. Finally, if $\mathcal{F}$ is $n$-presented for any non-negative integer $n$, we say that $\mathcal{F}$ is of $\infty$-presentation or $\infty$-presented.

Note that if $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, then the functor $f^* : \text{Mod}_Y \to \text{Mod}_X$ maps $\mathcal{O}_Y$-modules of finite type (resp. of finite presentation) into $\mathcal{O}_X$-modules of finite type (resp. of finite presentation).

**Definition 4.1.4.** (1) An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be quasi-coherent, if for any $x \in X$ there exists an open neighborhood $U$ of $x$ on which we have an exact sequence of $\mathcal{O}_X|_U$-modules of the form

$$\mathcal{O}_X^{\oplus I}|_U \to \mathcal{O}_X^{\oplus J}|_U \to \mathcal{F}|_U \to 0,$$

where $\mathcal{O}_X^{\oplus I}$ for a set $I$ (and similarly $\mathcal{O}_X^{\oplus J}$) denotes the direct sum of copies of $\mathcal{O}_X$ indexed by $I$.

(2) An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be coherent if

(a) $\mathcal{F}$ is of finite type and

(b) the kernel of any morphism $\mathcal{O}_X^{\oplus n}|_U \to \mathcal{F}|_U$, where $U \subseteq X$ is an open subset and $n \geq 0$, is of finite type.

Finitely presented $\mathcal{O}_X$-modules are quasi-coherent, but in general finite type $\mathcal{O}_X$-modules are not quasi-coherent. Coherent $\mathcal{O}_X$-modules are finitely presented and hence quasi-coherent, but finitely presented $\mathcal{O}_X$-modules are not necessarily coherent. We denote by $\text{QCoh}_X$ (resp. $\text{Coh}_X$) the full subcategory of $\text{Mod}_X$ consisting of quasi-coherent (resp. coherent) $\mathcal{O}_X$-modules. These are abelian thick (§C.5) subcategories of $\text{Mod}_X$ (cf. [53], (1.4.7)). Note that, if $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, then the functor $f^* : \text{Mod}_Y \to \text{Mod}_X$ maps $\text{QCoh}_Y$ to $\text{QCoh}_X$. Note also that, if $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then so are $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

**Definition 4.1.5.** Let $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

(1) An $\mathcal{O}_X$-module $\mathcal{F}$ is $f$-flat (or $Y$-flat) at a point $x \in X$ if $\mathcal{F}_x$ is flat as a module over $\mathcal{O}_{Y,f(x)}$. If $\mathcal{F}$ is $f$-flat at all points of $X$, we simply say that $\mathcal{F}$ is $f$-flat (or $Y$-flat). In particular, if $(X, \mathcal{O}_X) = (Y, \mathcal{O}_Y)$ and $f = \text{id}$, then we say $\mathcal{F}$ is flat.

(2) If $\mathcal{O}_X$ is $f$-flat, that is, if the morphism $\varphi_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat for any point $x \in X$, then the the morphism $(f, \varphi)$ is flat.

If $(f, \varphi)$ is flat, then the functor $f^* : \text{Mod}_Y \to \text{Mod}_X$ is exact.
Proposition 4.1.6. Let \((X, \mathcal{O}_X)\) be a ringed space, and \(\mathcal{F}\) an \(\mathcal{O}_X\)-module. Let \(\{\mathcal{G}_i\}_{i \in I}\) be a filtered inductive system of \(\mathcal{O}_X\)-modules indexed by a directed set \(I\). Consider the natural map

\[
\Phi: \lim_{\longrightarrow \atop i \in I} \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_i) \longrightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \lim_{\longrightarrow \atop i \in I} \mathcal{G}_i).
\]

1. If \(X\) is quasi-compact and \(\mathcal{F}\) is of finite type, then \(\Phi\) is injective.
2. If \(X\) is coherent (2.2.1) and \(\mathcal{F}\) is finitely presented, then \(\Phi\) is bijective.

Similarly to 3.1.8, this proposition can be seen as a special case of [9], Exposée VI, Théorème 1.23. One can prove it in a similar way to the proof of 3.1.8, and the checking is left to the reader.

4.1. (d) Cohesive ringed spaces. For a ringed space \(X\) the structure sheaf \(\mathcal{O}_X\) is always quasi-coherent of finite type, but not necessarily coherent.

Definition 4.1.7. A ringed space \(X = (X, \mathcal{O}_X)\) is said to be cohesive if the structure sheaf \(\mathcal{O}_X\) is coherent as an \(\mathcal{O}_X\)-module.

Then one can readily establish the following proposition (cf. 3.3.3).

Proposition 4.1.8. Let \(X = (X, \mathcal{O}_X)\) be a cohesive ringed space. Then an \(\mathcal{O}_X\)-module \(\mathcal{F}\) is coherent if and only if it is finitely presented. (In this case, moreover, \(\mathcal{F}\) admits an \(\infty\)-presentation.)

Corollary 4.1.9. Let \((f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) be a morphism of ringed spaces, and suppose \((X, \mathcal{O}_X)\) is cohesive. Then the functor \(f^*\) maps \(\text{Coh}_Y\) to \(\text{Coh}_X\).

4.1. (e) Filtered projective limit of ringed spaces. We will need to consider filtered projective limits of (locally) ringed spaces. In fact, as one can easily check, such a limit always exists in the category of (locally) ringed spaces.

Proposition 4.1.10. Let \(\{X_i = (X_i, \mathcal{O}_{X_i})\}_{i \in I}\) be a projective system of ringed (resp. locally ringed) spaces indexed by a directed set \(I\). Then the projective limit \(X = \lim_{\longleftarrow \atop i \in I} X_i\) exists in the category of ringed (resp. locally ringed) spaces. Moreover, the underlying topological space of \(X\) is isomorphic to the projective limit of the underlying topological spaces of \(X_i\)’s, and \(\mathcal{O}_{X,x}\) for any \(x \in X\) is canonically isomorphic to the inductive limit \(\lim_{\longrightarrow \atop i \in I} \mathcal{O}_{X_i,p_i(x)}\), where \(p_i : X \to X_i\) for each \(i \in I\) is the canonical projection.

That is to say, the topological space \(X = \lim_{\longleftarrow \atop i \in I} X_i\) coupled with the inductive limit sheaf \(\mathcal{O}_X = \lim_{\longrightarrow \atop i \in I} p_i^{-1}\mathcal{O}_{X_i}\), which is a sheaf of local rings if \(\{X_i\}_{i \in I}\) is a projective system of locally ringed spaces (3.1.7), gives the desired projective limit.
Note that in the local-ringed case, it is due to presence of the filtering that the underlying topological space of the limit coincides with the limit of the underlying topological spaces. In fact, this is not in general the case for projective limits that are not filtered. For instance, fiber products of schemes are taken in the category of locally ringed spaces (cf. [53], (3.2.1)) and, as is well known, their underlying topological spaces do not necessarily coincide with the fiber products of the underlying topological spaces.

**Corollary 4.1.11.** Let \( X = \{X_i = (X_i, \mathcal{O}_{X_i})\}_{i \in I} \) and \( U = \{U_i = (U_i, \mathcal{O}_{U_i})\}_{i \in I} \) be two projective systems of ringed (resp. locally ringed) spaces indexed by a directed set \( I \), and \( i: U \to X \) a morphism \( \{i_i\} \) of projective systems consisting of open immersions \( i_i: U_i \hookrightarrow X_i \). Suppose we have \( U_i \times_{X_i} X_j \to U_j \) for each pair \((i, j)\) with \( i \leq j \). Then the induced map

\[
\lim_{i \in I} i_i: \lim_{i \in I} U_i \longrightarrow \lim_{i \in I} X_i
\]

is an open immersion, and its image coincides with \( p_i^{-1}(U_i) \) for any \( i \), where \( p_i: \lim_{j \in I} X_j \to X_i \) is the projection map.

### 4.2 Sheaves on limit spaces

**4.2. (a) Finitely presented sheaves on limit spaces.** Let us consider

- a filtered projective system of ringed spaces \( \{X_i = (X_i, \mathcal{O}_{X_i}), p_{ij}\}_{i \in I} \) indexed by a directed set \( I \).

We set \( X = \lim_{i \in I} X_i \), and denote by \( p_i: X \to X_i \) the canonical projection for each \( i \in I \). As we saw in §4.1.(e), the ringed space \( X \) is supported on the projective limit of the underlying topological spaces of \( X_i \)'s and has \( \mathcal{O}_X = \lim_{i \in I} p_i^{-1}\mathcal{O}_{X_i} \) as its structure sheaf. We assume that

(a) for any \( i \in I \) the underlying topological space of \( X_i \) is coherent (2.2.1) and sober (§2.1.(b));

(b) for any \( i \leq j \) the underlying continuous mapping of the transition map \( p_{ij}: X_j \to X_i \) is quasi-compact (2.1.4 (2)).

By 2.2.10 (1) the underlying topological space of the limit \( X \) is coherent and sober, and each canonical projection map \( p_i: X \to X_i \) for \( i \in I \) is quasi-compact.
Theorem 4.2.1. (1) For any finitely presented \( O_X \)-module \( F \) there exist an index \( i \in I \) and a finitely presented \( O_{X_i} \)-module \( \mathcal{F}_i \) such that \( F \cong p_i^* \mathcal{F}_i \).

(2) For any morphism \( \varphi: F \to G \) of finitely presented \( O_X \)-modules, there exist an index \( i \in I \) and a morphism \( \varphi_i: F_i \to G_i \) of finitely presented \( O_{X_i} \)-modules such that \( \varphi \cong p_i^* \varphi_i \). Moreover, if \( \varphi \) is an isomorphism (resp. epimorphism), then one can take \( \varphi_i \) to be an isomorphism (resp. epimorphism).

The theorem is a consequence of the following result.

Theorem 4.2.2. Let \( 0 \in I \) be an index, and \( F_0 \) and \( G_0 \) two \( O_{X_0} \)-modules. Suppose \( F_0 \) is finitely presented. Then the canonical map

\[
\lim_{i \geq 0} \text{Hom}_{O_{X_i}}(p_{0i}^* F_0, p_{0i}^* G_0) \to \text{Hom}_{O_X}(p_0^* F_0, p_0^* G_0)
\]

is bijective.

Note that the last map is the inductive limit of the maps

\[
\text{Hom}_{O_{X_i}}(p_{0i}^* F_0, p_{0i}^* G_0) \to \text{Hom}_{O_X}(p_0^* F_0, p_0^* G_0),
\]

which are simply induced by the canonical projections \( p_i: X \to X_i \) for \( i \geq 0 \).

Remark 4.2.3. In 2-categorical language Theorems 4.2.1 and 4.2.2 mean that the category of finitely presented \( O_X \)-modules is equivalent to the ‘inductive limit category’

\[
\text{Lim} \{ \text{finitely presented } O_{X_i} \text{-modules} \}
\]

in the sense of [9], Exposé VI, (6.3). Note that there exists an essentially unique fibered category over \( I \) such that each fiber over \( i \in I \) is the category of \( O_{X_i} \)-modules of finite presentation and that Cartesian morphisms are given by pullbacks; see [52], Exposé VI, §8, for the construction.

Now let us show that the first theorem follows from the second one.

Proof of Theorem 4.2.2 \( \implies \) Theorem 4.2.1. To show (1), we first consider the case where \( F \) has a finite presentation

\[
\mathcal{O}_X^{\oplus p} \to \mathcal{O}_X^{\oplus q} \to F \to 0
\]

over \( X \). By 4.2.2 the morphism \( \mathcal{O}_X^{\oplus p} \to \mathcal{O}_X^{\oplus q} \) is the pull-back of a morphism \( \mathcal{O}_{X_i}^{\oplus p} \to \mathcal{O}_{X_i}^{\oplus q} \) for some \( i \geq 0 \). Let \( \mathcal{F}_i \) be its cokernel. Then we have \( F \cong p_i^* \mathcal{F}_i \).

In general, we take a finite open covering \( X = \bigcup_{k=1}^n U_k \) by quasi-compact open subsets such that \( F|_{U_k} \) for each \( k = 1, \ldots, n \) has a finite presentation. By 2.2.9, for each \( k = 1, \ldots, n \) there exists an index \( i_k \in I \) and a quasi-compact
open subset $V_k \subseteq X_{ik}$ such that $p_{ik}^{-1}(V_k) = U_k$. Taking an upper bound $i$ of 
$i_1, \ldots, i_n$ and replacing each $V_k$ by $p_{ik}^{-1}(V_k)$, we may assume that $V_k$ is a quasi-
compact open subset of $X_i$ for any $k = 1, \ldots, n$. We have $\bigcup_{k=1}^n V_k \subseteq X_i$ and 
$X = \bigcup_{k=1}^n p_{ik}^{-1}(V_k)$. Hence, replacing $i$ by a larger index, we may assume that 
$X_i = \bigcup_{k=1}^n V_k$ (Exercise 0.2.8).

By what we have seen above, replacing $i$ by a larger index if necessary, there exists a finitely presented $\mathcal{O}_{V_k}$-module $\mathcal{F}_k$ such that $p_i^* \mathcal{F}_k \cong \mathcal{F}|_{U_k}$ for any $k = 1, \ldots, n$. Since the $p_i^* \mathcal{F}_k$’s patch together on $X$, thanks to 4.2.2 there exists an 
index $j \geq i$ such that $p_{ij}^* \mathcal{F}_k$’s patch to a finitely presented $\mathcal{O}_{X_j}$-module $\mathcal{F}_j$. Since 
$p_j^* \mathcal{F}_j \cong \mathcal{F}$, we have shown (1).

(2) follows from (1) and 4.2.2; to show that we can take $\varphi_i$ to be an epimorphism, 
we observe that if $\mathcal{H}_i$ is finitely presented $\mathcal{O}_X$-module such that $p_i^* \mathcal{H}_i = 0$, then 
there exists an index $j \in I$ with $i \leq j$ such that $p_{ij}^* \mathcal{H}_i = 0$. This follows 
from 4.2.2. \hfill \Box

We now turn to the proof of 4.2.2.

**Lemma 4.2.4.** Let $Z = (Z, \mathcal{O}_Z)$ be a ringed space, and $\{g_i; X_i \to Z\}_{i \in I}$ a 
collection of morphisms of ringed spaces such that $g_j = g_i \circ p_{ij}$ whenever $i \leq j$. Then 
for any $\mathcal{O}_Z$-module $\mathcal{G}$ we have a canonical isomorphism

$$
\lim_{i \geq 0} p_i^{-1} g_i^* \mathcal{G} \xrightarrow{\sim} g^* \mathcal{G}
$$

of $\mathcal{O}_X$-modules, where $g = \lim_{i \in I} g_i$.

Note that the domain of the last isomorphism can be regarded as an $\mathcal{O}_X$-module via the equality 
$\mathcal{O}_X = \varprojlim_{i \geq 0} p_i^{-1} \mathcal{O}_{X_i}$.

**Proof.** There exists a canonical map $p_i^{-1} g_i^* \mathcal{G} \to p_i^* g_i^* \mathcal{G} = g^* \mathcal{G}$, for $i \in I$, which 
yields the morphism $\lim_{i \geq 0} p_i^{-1} g_i^* \mathcal{G} \to g^* \mathcal{G}$. In order to show that this is an 
isomorphism, we only need to check stalkwise. Let $x \in X$. We have

$$(\lim_{i \geq 0} p_i^{-1} g_i^* \mathcal{G})_x = \lim_{i \geq 0} (g_i^* \mathcal{G})_{p_i(x)} = \lim_{i \geq 0} \mathcal{G}_x \otimes_{\mathcal{O}_{Y,g(x)}} \mathcal{O}_{X_i,p_i(x)}.$$

By 3.1.4,

$$
\lim_{i \geq 0} \mathcal{G}_x \otimes_{\mathcal{O}_{Y,g(x)}} \mathcal{O}_{X_i,p_i(x)} = \mathcal{G}_x \otimes_{\mathcal{O}_{Y,g(x)}} \lim_{i \geq 0} \mathcal{O}_{X_i,p_i(x)}
$$

$$
= \mathcal{G}_x \otimes_{\mathcal{O}_{Y,g(x)}} \mathcal{O}_{X,x},
$$

whence the result. (Note that here we do not use conditions (a) and (b) in the 
beginning of this subsection.) \hfill \Box
Lemma 4.2.5. Suppose in the situation as in 4.2.4 that the underlying topological space of $Z$ is coherent and sober and that the underlying continuous mapping of each $g_i: X_i \to Z$ is quasi-compact. Then the canonical morphism
\[
\lim_{i \in I} g_i^* \mathcal{G} \longrightarrow g^* \mathcal{G}
\]
is an isomorphism of $\mathcal{O}_X$-modules.

Proof. By 4.2.4, one can apply 3.1.13 to the case when $Y_i = Z$ for all $i \in I$ and $\mathcal{F}_i = f_i^* \mathcal{G}$ (hence $\mathcal{F} = f^* \mathcal{G}$). (Note that here we need to use conditions (a) and (b).)

Proof of Theorem 4.2.2. We have the equalities (up to canonical isomorphisms)
\[
\text{Hom}_{\mathcal{O}_X} (p_0^* \mathcal{F}_0, p_0^* \mathcal{G}_0) = \text{Hom}_{\mathcal{O}_{X_0}} (\mathcal{F}_0, p_0^* p_0^* \mathcal{G}_0) = \text{Hom}_{\mathcal{O}_{X_0}} (\mathcal{F}_0, \lim_{i \geq 0} p_{0i}^* p_{0i}^* \mathcal{G}_0),
\]
where the last equality follows from 4.2.5. Now by 4.1.6 (2) we have
\[
\text{Hom}_{\mathcal{O}_{X_0}} (\mathcal{F}_0, \lim_{i \geq 0} p_{0i}^* p_{0i}^* \mathcal{G}_0) = \lim_{i \geq 0} \text{Hom}_{\mathcal{O}_{X_i}} (p_{0i}^* \mathcal{F}_0, p_{0i}^* \mathcal{G}_0),
\]
as desired.

4.2. (b) Limits and direct images. Next, in addition to the data fixed in the beginning of previous subsection, we fix

- another filtered projective system of ringed spaces $\{Y_i, q_{ij}\}_{i \in I}$ indexed by the same directed set $I$ that satisfies conditions similar to (a) and (b) as in the beginning of §4.2. (a);
- a map $\{f_i\}_{i \in I}$ of projective systems from $\{X_i, p_{ij}\}_{i \in I}$ to $\{Y_i, q_{ij}\}_{i \in I}$, that is, a collection of morphisms $f_i: X_i \to Y_i$ such that $q_{ij} \circ f_j = f_i \circ p_{ij}$ whenever $i \leq j$.

We set $Y = \lim_{\leftarrow i \in I} Y_i$, and denote the canonical projection by $q_i: Y \to Y_i$ for each $i \in I$. We have by passage to the projective limits the continuous map
\[
f = \lim_{\leftarrow i \in I} f_i: X \longrightarrow Y.
\]
We moreover assume that

(c) for any $i \in I$ the underlying continuous mapping of $f_i$ is quasi-compact.

Note that the underlying continuous map of $f$ is quasi-compact due to 2.2.13 (1).

**Proposition 4.2.6.** For any $\mathcal{O}_X$-module $\mathcal{F}$ the canonical morphism

$$\lim_{i \in I} q_i^* f_i_* p_i^* \mathcal{F} \longrightarrow f_* \mathcal{F}$$

induced by the canonical morphisms $q_i^* f_i_* p_i^* \mathcal{F} \rightarrow f_* \mathcal{F}$ for $i \in I$, defined by adjunction from $f_i_* p_i^* \cong q_i*, f_*$, is an isomorphism.

The proposition follows from 3.1.15 and the following lemma, which can be viewed as a corollary of 4.2.4.

**Lemma 4.2.7.** Suppose we are given a system $\{\mathcal{F}_i, \varphi_{ij}\}$ consisting of

(a) an $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ for each $i \in I$,

(b) a morphism $\varphi_{ji} : p_{ji}^* \mathcal{F}_i \rightarrow \mathcal{F}_j$ of $\mathcal{O}_{X_j}$-modules for each $j \geq i$,

such that $\varphi_{kj} \circ p_{kj}^* \varphi_{ji} = \varphi_{ki}$ for $k \geq j \geq i$. Then the canonical morphism

$$\lim_{i \in I} p_i^{-1} \mathcal{F}_i \longrightarrow \lim_{i \in I} p_i^* \mathcal{F}_i$$

is an isomorphism of $\mathcal{O}_X$-modules.

Here we regard $\{p_i^{-1} \mathcal{F}_i\}$ and $\{p_i^* \mathcal{F}_i\}$ as filtered inductive systems in an obvious manner. Note that we do not use in the following proof conditions (a) and (b) for $\{X_i, p_{ij}\}_{i \in I}$ in the beginning of §4.2.(a).

**Proof.** By 4.2.4,

$$\lim_{i \in I} p_i^* \mathcal{F}_i = \lim_{i \in I} \lim_{j \geq i} p_{ji}^* \mathcal{F}_i$$

up to canonical isomorphisms, where the last double inductive limit can be seen as a single inductive limit taken along the directed set $J = \{(i, j) : j \geq i\}$ where $(i, j) \leq (i', j')$ with the order $i \leq i'$ and $j \leq j'$ (cf. Exercise 0.1.1). Since the diagonal subset $\{(i, i)\}$ is evidently cofinal, we have

$$\lim_{i \in I} \lim_{j \geq i} p_{ji}^* \mathcal{F}_i = \lim_{i \in I} p_i^{-1} \mathcal{F}_i,$$

as desired. \qed
Applying 4.2.6 to the case when \( X_i = Y_i \) and \( f_i = \text{id}_{X_i} \) \((i \in I)\), we have the following corollary.

**Corollary 4.2.8.** For any \( \mathcal{O}_X \)-module \( \mathcal{F} \) the canonical morphism

\[
\lim_{i \in I} p_i^* p_i^* \mathcal{F} \rightarrow \mathcal{F}
\]

is an isomorphism.

### 4.3 Cohomologies of sheaves on ringed spaces

#### 4.3. (a) Derived category formalism.**

Let \( X = (X, \mathcal{O}_X) \) be a ringed space. We denote by \( \mathbf{D}^*(X) \) (where \( * = \text{``-''}, +, - \)) the derived category associated to the abelian category \( \mathbf{Mod}_X \) of \( \mathcal{O}_X \)-modules (cf. §C.4). Inside \( \mathbf{Mod}_X \) are thick (§C.5) abelian full subcategories \( \mathbf{QCoh}_X \) and \( \mathbf{Coh}_X \) of quasi-coherent sheaves and coherent sheaves, respectively. We denote by \( \mathbf{D}^\ast_{\text{qcoh}}(X) \) (resp. \( \mathbf{D}^\ast_{\text{coh}}(X) \)) the full subcategory of \( \mathbf{D}^*(X) \) consisting of objects \( F \) such that the cohomology sheaves \( \mathcal{H}^k(F) \) are quasi-coherent (resp. coherent) for all \( k \in \mathbb{Z} \) (where \( \mathcal{H}^0 \) is the canonical cohomology functor on \( \mathbf{D}^*(X) \) (cf. C.4.4)). The full subcategories \( \mathbf{D}^\ast_{\text{qcoh}}(X) \) and \( \mathbf{D}^\ast_{\text{coh}}(X) \) are triangulated subcategories of \( \mathbf{D}^*(X) \) with the induced cohomology functor \( \mathcal{H}^0 \) and the induced \( t \)-structure (cf. C.5.1).

#### 4.3. (b) Calculation of cohomologies.**

In this and next subsections we make a few small, but perhaps at least need-to-know, remarks on cohomology groups of \( \mathcal{O}_X \)-modules.

Let \( X = (X, \mathcal{O}_X) \) be a ringed space, and consider the commutative diagram of functors

\[
\begin{array}{ccc}
\mathbf{Mod}_X & \xrightarrow{\mu} & \mathbf{ASh}_X \\
\Gamma_X & \downarrow & \Gamma_X \\
\mathbf{Ab} & & \\
\end{array}
\]

where \( \Gamma_X \)'s are the global section functors and \( \mu: \mathbf{Mod}_X \rightarrow \mathbf{Ab} \) is the forgetful functor. In this situation, one finds that there are at least two ways for obtaining the cohomology groups \( H^q(X, \mathcal{F}) \) of an \( \mathcal{O}_X \)-module \( \mathcal{F} \): one is by \( R^q \Gamma_X(\mathcal{F}) \), that is, by applying directly the right derived functors of \( \Gamma_X: \mathbf{Mod}_X \rightarrow \mathbf{Ab} \), and the other by \( R^q \Gamma_X(\mu(\mathcal{F})) \), calculated from the right derived functors of \( \Gamma_X: \mathbf{ASh}_X \rightarrow \mathbf{Ab} \) applied to the underlying abelian sheaf of \( \mathcal{F} \). These two approaches lead, indeed, to the same result, but for a non-trivial reason.
The functor \( u \), being clearly exact, induces the exact functor
\[
D^+(u) : D^+(X) \rightarrow D^+(\text{ASh}_X),
\]
see C.4.6, defined simply by term-by-term application of \( u \) to complexes of \( \mathcal{O}_X \)-modules. The issue lies in comparison of the exact functors \( R^+\Gamma_X \) and \( R^+\Gamma_X \circ D^+(u) \).

**Proposition 4.3.1.** There exists a canonical isomorphism
\[
R^+\Gamma_X \cong R^+\Gamma_X \circ D^+(u)
\]
of exact functors \( D^+(X) \rightarrow D^+(\text{Ab}) \).

**Proof.** The key point is that the functor \( u \) maps injective objects of \( \text{Mod}_X \), that is, injective \( \mathcal{O}_X \)-modules, to flasque (but not necessarily injective) abelian sheaves. We have by [34], C.D., §2, Proposition 3.1, the canonical isomorphism
\[
R^+(\Gamma_X \circ u) \cong R^+\Gamma_X \circ R^+u.
\]
Now, since \( u \) is exact, the right derived functor \( R^+u \) coincides with the induced functor \( D^+(u) \), and thus we get the desired isomorphism of functors. \( \square \)

A similar remark can be made also for higher direct images. Let
\[
f : X = (X, \mathcal{O}_X) \rightarrow Y = (Y, \mathcal{O}_Y)
\]
be a morphism of ringed spaces, and consider the commutative diagram
\[
\begin{array}{ccc}
\text{Mod}_X & \xrightarrow{u} & \text{ASh}_X \\
\downarrow f_* & & \downarrow f_* \\
\text{Mod}_Y & \xrightarrow{u} & \text{ASh}_Y
\end{array}
\]
Then by [34], C.D., §2, Proposition 3.1, and the fact that \( f_* \) maps flasque sheaves to flasque sheaves, we have the following result.

**Proposition 4.3.2.** There exists a canonical isomorphism
\[
D^+(u) \circ R^+f_* \cong R^+f_* \circ D^+(u)
\]
of exact functors \( D^+(X) \rightarrow D^+(\text{ASh}_Y) \).

That is to say, the underlying abelian sheaves of the higher direct images \( R^qf_*\mathcal{F} \) of an \( \mathcal{O}_X \)-module \( \mathcal{F} \) taken in \( D^+(Y) \) coincide up to isomorphisms with the higher direct images of \( \mathcal{F} \) regarded as an abelian sheaf.
4. (c) Module structures on cohomologies. Let \( X = (X, \mathcal{O}_X) \) be a ringed space. Once given a ring homomorphism \( A \to \Gamma_X(\mathcal{O}_X) \) from a ring \( A \), one can equip with the canonical \( A \)-module structure the global sections \( \Gamma_X(\mathcal{F}) \) of an arbitrary \( \mathcal{O}_X \)-module \( \mathcal{F} \), and thus obtain \( \Gamma_X: \text{Mod}_X \to \text{Mod}_A \) and the associated right derived functor

\[
R^+\Gamma_X: \mathbf{D}^+(X) \to \mathbf{D}^+(\text{Mod}_A).
\]

One has, on the other hand, the commutative diagram

\[
\begin{array}{ccc}
\text{Mod}_X & \xrightarrow{\Gamma_X} & \text{Mod}_A \\
\downarrow{\Gamma_X} & & \downarrow{u} \\
\text{Mod}_Y & \xrightarrow{\Gamma_Y} & \text{Ab}
\end{array}
\]

consisting of global section functors and the forgetful functor; by [34], C.D., §2, Proposition 3.1, we have a canonical isomorphism

\[
R^+\Gamma_X \cong \mathbf{D}^+(u) \circ R^+\Gamma_X
\]

of exact functors \( \mathbf{D}^+(X) \to \mathbf{D}^+(\text{Ab}) \). This amounts to saying that the cohomology groups \( H^q(X, \mathcal{F}) \) of an \( \mathcal{O}_X \)-module \( \mathcal{F} \), defined originally as abelian groups, carry the canonical \( A \)-module structure induced from the above-fixed ring homomorphism \( A \to \Gamma_X(\mathcal{O}_X) \).

Let \( f: X = (X, \mathcal{O}_X) \to Y = (Y, \mathcal{O}_Y) \) be a morphism of ringed spaces, and suppose we are given a ring homomorphism \( A \to \Gamma_Y(\mathcal{O}_Y) \) from a ring \( A \); the homomorphism \( \Gamma_Y(\mathcal{O}_Y) \to \Gamma_X(\mathcal{O}_X) \) induces a ring homomorphism \( A \to \Gamma_X(\mathcal{O}_X) \). As above, the global section functors yield

\[
\Gamma_X: \text{Mod}_X \to \text{Mod}_A \quad \text{and} \quad \Gamma_Y: \text{Mod}_Y \to \text{Mod}_A,
\]

sitting in the commutative diagram

\[
\begin{array}{ccc}
\text{Mod}_X & \xrightarrow{\Gamma_X} & \text{Mod}_A \\
\downarrow{f_*} & & \downarrow{\Gamma_Y} \\
\text{Mod}_Y & \xrightarrow{\Gamma_Y} & \text{Mod}_A
\end{array}
\]

Just similarly to 4.3.2 one has the following result.

**Proposition 4.3.3.** There exists a canonical isomorphism

\[
R^+\Gamma_X \cong R^+\Gamma_Y \circ R^+f_*
\]

of exact functors \( \mathbf{D}^+(X) \to \mathbf{D}^+(\text{Mod}_A) \).

In other words, the two approaches to computing cohomology groups \( H^q(X, \mathcal{F}) \) of an \( \mathcal{O}_X \)-module \( \mathcal{F} \), together with the \( A \)-module structure, one being done directly on \( X \), and the other via \( Y \) by way of the higher direct images of \( f \), lead to the same result.
4.4 Cohomologies of module sheaves on limit spaces

In this subsection we consider the data fixed in the beginning of §4.2.(a) and §4.2.(b) and, furthermore

- for each \( i \in I \), an \( \mathcal{O}_{X_i} \)-module \( \mathcal{F}_i \) and
- for each \( i \leq j \), a morphism \( \varphi_{ij} : p_j^* \mathcal{F}_i \to \mathcal{F}_j \) of \( \mathcal{O}_{X_j} \)-modules such that \( \varphi_{ik} = \varphi_{jk} \circ p_k^* \varphi_{ij} \) whenever \( i \leq j \leq k \).

Then one has the inductive system \( \{ p_i^* \mathcal{F}_i \}_{i \in I} \) of \( \mathcal{O}_X \)-modules indexed by \( I \) and the \( \mathcal{O}_X \)-module

\[
\mathcal{F} = \lim_{i \in I} p_i^* \mathcal{F}_i.
\]

Moreover, we consider

- a ring \( A \) and
- a collection of ring homomorphisms \( A \to \Gamma(X_i, \mathcal{O}_{X_i}) \) for \( i \in I \) such that the following diagram commutes whenever \( i \leq j \):

\[
\begin{array}{ccc}
\Gamma(X_j, \mathcal{O}_{X_j}) & \xrightarrow{\varphi_{ij}} & \Gamma(X_i, \mathcal{O}_{X_i}) \\
\downarrow & & \downarrow \\
A & \xrightarrow{\varphi_{ij}} & \Gamma(X_i, \mathcal{O}_{X_i}).
\end{array}
\]

By 3.1.19, 4.2.7, and what we have establish in §4.3.(c) (canonicity of the \( A \)-module structure on the cohomologies) we have the following proposition.

**Proposition 4.4.1.** The canonical map

\[
\lim_{i \in I} H^q(X_i, \mathcal{F}_i) \longrightarrow H^q(X, \mathcal{F})
\]

is an isomorphism of \( A \)-modules for \( q \geq 0 \).

**Corollary 4.4.2.** Let \( \mathcal{H} \) be an \( \mathcal{O}_X \)-module. Then the canonical morphism

\[
\lim_{i \in I} H^q(X_i, p_i^* \mathcal{H}) \longrightarrow H^q(X, \mathcal{H})
\]

is an isomorphism of \( A \)-modules for \( q \geq 0 \).
Corollary 4.4.3. Let \( Z \) be a ringed space with a coherent underlying topological space, and \( \{g_i: X_i \to Z\}_{i \in I} \) a system of morphisms such that \( g_j = g_i \circ p_{ij} \) whenever \( i \leq j \). Suppose that the underlying continuous mapping of each \( g_i \) is quasi-compact. Then for any \( \mathcal{O}_Z \)-module \( \mathcal{G} \) the canonical map
\[
\lim_{i \in I} H^q(X_i, g_i^* \mathcal{G}) \longrightarrow H^q(X, g^* \mathcal{G}),
\]
where \( g = \lim_{i \in I} g_i \), is an isomorphism of \( A \)-modules for \( q \geq 0 \).

Corollary 4.4.4. Let \( \{X_i, p_{ij}\}_{i \in I}, \{Y_i, q_{ij}\}_{i \in I} \), and \( f_i \)'s be as in \( \S 4.2 \) (with conditions (a) and (b) for \( \{X_i, p_{ij}\}_{i \in I} \) and \( \{Y_i, q_{ij}\}_{i \in I} \) and condition (c) for \( f_i \)'s), and let \( \mathcal{F}_i \) and \( \mathcal{F} \) be as above. Then the canonical morphism of \( \mathcal{O}_Y \)-modules
\[
\lim_{i \in I} q_i^{-1} R^q f_{i*} \mathcal{F}_i \longrightarrow R^q f_* \mathcal{F}
\]
is an isomorphism for \( q \geq 0 \).

This follows from 3.1.22 and 4.2.7. One can, moreover, show the following results in a similar manner.

Corollary 4.4.5. Let \( Z \) be a ringed space with a coherent underlying topological space, and \( \{g_i: X_i \to Z\}_{i \in I} \) a system of morphisms such that \( g_j = g_i \circ p_{ij} \) whenever \( i \leq j \). Suppose that the underlying continuous mapping of each \( g_i \) is quasi-compact. Then for any \( \mathcal{O}_Z \)-module \( \mathcal{G} \) the canonical morphism of \( \mathcal{O}_Z \)-modules
\[
\lim_{i \in I} R^q g_i^* (g_i^* \mathcal{G}) \longrightarrow R^q g^* (g^* \mathcal{G}),
\]
where \( g = \lim_{i \in I} g_i \), is an isomorphism for \( q \geq 0 \).

Corollary 4.4.6. Let \( \{X_i, p_{ij}\}_{i \in I}, \{Y_i, q_{ij}\}_{i \in I} \), and \( f_i \)'s be as in 4.4.4, and \( \mathcal{H} \) an \( \mathcal{O}_X \)-module. Then the canonical morphism of \( \mathcal{O}_Y \)-modules
\[
\lim_{i \in I} q_i^* R^q f_i^* (p_{i*} \mathcal{H}) \longrightarrow R^q f_* \mathcal{H}
\]
is an isomorphism for \( q \geq 0 \).

Exercises

Exercise 0.4.1. Let \( X \) be a locally Noetherian formal scheme, \( x \in X \) a point, and \( y \in G_x \) a generization of \( x \). Show that the map
\[
(\mathcal{O}_{X,x})_{q_y} \longrightarrow \mathcal{O}_{X,y}
\]
induced by the generization map (\( \S 4.1. (b) \)), where \( q_y \) is the pull-back of the maximal ideal \( \mathfrak{m}_{X,y} \) of \( \mathcal{O}_{X,y} \), is faithfully flat.
Exercise 0.4.2. (1) Let \( \{X_i, p_{ij}\}_{i \in I} \) be a filtered projective system of ringed spaces indexed by a directed set \( I \), and \( X = \lim_{\leftarrow i \in I} X_i \). Suppose that for any \( i \leq j \) the transition map \( p_{ji} : X_j \to X_i \) is flat. Show that for each \( i \in I \) the canonical projection \( p_i : X \to X_i \) is flat.

(2) Let \( \{X_i, p_{ij}\}_{i \in I} \) and \( \{Y_i, q_{ij}\}_{i \in I} \) be two filtered projective system of ringed spaces indexed by a directed set, and \( \{f_i\}_{i \in I} \) a projective system of morphisms \( f_i : X_i \to Y_i \) of ringed spaces. Let \( f : X \to Y \) be the limits, and \( p_i : X \to X_i \) and \( q_i : Y \to Y_i \) the canonical maps for each \( i \in I \). Suppose that for any \( i \in I \) the map \( f_i \) is flat. Show that the map \( f \) is flat.

Exercise 0.4.3. Let \( \{X_i, p_{ij}\}_{i \in I} \) be a filtered projective system of ringed spaces indexed by a directed set \( I \), and \( X = \lim_{\leftarrow i \in I} X_i \). Suppose that the underlying topological space of each \( X_i \) is coherent sober, and the underlying continuous map of each \( p_{ij} \) is quasi-compact. Suppose, moreover, that each \( X_i = (X_i, \mathcal{O}_{X_i}) \) is cohesive and that each \( p_{ij} \) is flat. Then show that \( X = (X, \mathcal{O}_X) \) is cohesive.

5 Schemes and algebraic spaces

In [54] coherent sheaves are always discussed under the assumption that the involved schemes are locally Noetherian. However, many non-Noetherian schemes may have coherent structure sheaves, to which, therefore, one can apply many of the results on locally Noetherian schemes. In §5.1 and §5.2, we will introduce the notions of universally cohesive schemes and algebraic spaces; a scheme is said to be universally cohesive if any scheme that is locally of finite presentation over it has coherent structure sheaf. Noetherian schemes are of course universally cohesive. Some of the non-trivial examples of universally cohesive schemes will appear in §8.5.(e).

Unlike the case of Noetherian schemes, quasi-coherent sheaves of finite type on universally cohesive schemes are not necessarily coherent.

In §5.3 we discuss some fundamental topics on the calculation of cohomology in the derived categories. Among these, for example, are the comparison of two derived categories \( D^*(\text{Coh}_X) \), the derived category of the category of coherent sheaves, and \( D^*_{\text{coh}}(X) \), the full subcategory of the derived category of \( \mathcal{O}_X \)-modules consisting of objects having coherent cohomologies.

This section ends with a collection of known facts on cohomologies of quasi-coherent sheaves (§5.4) and on generalities of algebraic spaces (§5.5).
5. Schemes and algebraic spaces

5.1 Schemes

5.1. (a) Schemes. We denote by \( \text{Sch} \) the category of schemes, and by \( \text{Sch}_S \) (where \( S \) is a scheme) the category of \( S \)-schemes. Note that \( \text{Sch} \) is a full subcategory of \( \text{LRsp} \), the category of locally ringed spaces.

For an affine scheme \( X = \text{Spec} \ A \) and an \( A \)-module \( M \), we denote, as usual, by \( \tilde{M} \) the associated quasi-coherent \( \mathcal{O}_X \)-module. As is well known ([53], §1.4), \( M \mapsto \tilde{M} \) gives an exact equivalence of abelian categories

\[
\tilde{\cdot}: \text{Mod}_A \to \text{Qcoh}_X
\]

preserving the tensor products and internal hom’s.

5.1. (b) Universally cohesive schemes

Definition 5.1.1. A scheme \( X \) is said to be universally cohesive if any \( X \)-scheme locally of finite presentation is cohesive (4.1.7) as a ringed space.

Proposition 5.1.2. Let \( A \) be a ring, and \( X = \text{Spec} \ A \). Then \( X \) is universally cohesive if and only if \( A \) is universally coherent (3.3.7).

Proof. Let \( X \) be universally cohesive, and let \( B \) be a finitely presented \( A \)-algebra. Suppose we are given an exact sequence

\[
0 \to K \to B^\oplus m \to B.
\]

We need to show that \( K \) is a finitely generated \( B \)-module. By [53], (1.3.11), we have the exact sequence

\[
0 \to \tilde{K} \to \mathcal{O}_Y^\oplus m \to \mathcal{O}_Y,
\]

where \( Y = \text{Spec} \ B \). Since \( \mathcal{O}_Y \) is coherent, \( \tilde{K} \) is of finite type. Then by [53], (1.4.3), we deduce that \( K \) is finitely generated.

Conversely, suppose \( A \) is universally coherent, and let \( Y \to X \) be an \( X \)-scheme locally of finite presentation. We need to show that for any open subset \( U \subseteq Y \) and any exact sequence of the form

\[
0 \to \mathcal{K} \to \mathcal{O}_U^\oplus m \to \mathcal{O}_U,
\]

the quasi-coherent sheaf \( \mathcal{K} \) on \( U \) is of finite type. To this end, we may assume that \( U \) is affine, \( U = \text{Spec} \ B \), where \( B \) is an \( A \)-algebra of finite presentation and hence is universally coherent. Set \( \mathcal{K} = \tilde{K} \), where \( K \) is a (uniquely determined up to isomorphism) \( B \)-module. Then by [53], (1.3.11), \( K \) is finitely generated, and by [53], (1.4.3), we conclude that \( \mathcal{K} \) is of finite type, as desired. \( \Box \)
**Proposition 5.1.3.** Consider an affine scheme $X = \text{Spec } A$, and suppose $A$ is universally coherent. Then the functor $\sim$ induces an exact equivalence of abelian categories

$$\sim: \text{Coh}_A \sim \text{Coh}_X.$$ 

**Proof.** By [53], (1.4.3), the functor $\sim$ gives a categorical equivalence from the category of finitely presented $A$-modules to the category of finitely presented $\mathcal{O}_X$-modules, which thanks to 4.1.8 are precisely the coherent $\mathcal{O}_X$-modules. $\square$

### 5.2 Algebraic spaces

#### 5.2. (a) Conventions.** Our basic reference to the general theory of algebraic spaces is Knutson’s book [72]. Accordingly, we adopt here the following

**Convention.** In this book, all algebraic spaces are assumed to be quasi-separated.

This assumption will be particularly useful when we compare algebraic spaces with schemes, using several effective étale descent tools ([72], II.3) and several local constructions such as open complement of closed subspaces ([72], II.5). Note that, with this assumption, the two definitions of algebraic spaces, the one as a sheaf on the large étale site of affine schemes and the other as the quotient of schemes by an étale equivalence relation, coincide ([72], II.1). In particular, if $X$ is a quasi-separated scheme, we have the following facts.

- In view of étale descent of quasi-coherent sheaves, the category of quasi-coherent sheaves on $X$, regarded as an algebraic space (with respect to étale topology), and the category of quasi-coherent sheaves on $X$ (considered, as usual, with respect to Zariski topology) are canonically equivalent.

- The cohomologies of quasi-coherent sheaves computed on the scheme $X$ by means of Zariski topology coincide up to canonical isomorphism with those computed from sheaves on $X$ regarded as an algebraic space with respect to étale topology (cf. [9], Exposé VII, Proposition 4.3).

By these facts, as long as we are concerned with quasi-coherent sheaves and their cohomologies, we do not have to distinguish between the two standpoints for a quasi-separated scheme $X$, the one on which $X$ is regarded as a scheme (Zariski topologized) and the other on which $X$ is regarded as an algebraic space (considered with étale topology).

**Convention.** When we say ‘$X$ is an algebraic space,’ we always mean either (a) $X$ is a scheme (not necessarily quasi-separated), or (b) $X$ is an algebraic space (necessarily quasi-separated). When we like to exclude the non-quasi-separated schemes, we say ‘$X$ is an algebraic space or a quasi-separated scheme.’
Accordingly, in case (a), all quasi-coherent sheaves on \( X \) and their cohomologies are considered with respect to the Zariski topology (unless otherwise clearly stated) and, in case (b), they are considered with respect to the étale topology.

In some situations, where algebraic spaces and schemes are mixed, e.g., when taking fiber products of algebraic spaces with schemes, one has to (and we do) assume that the schemes are quasi-separated, even if not explicitly stated.

In the sequel we denote by \( \text{As} \) the category of algebraic spaces and by \( \text{As}_S \) (where \( S \) is an algebraic space) the category of \( S \)-algebraic spaces.

5.2. (b) Basic notions. A morphism \( f: X \to Y \) of algebraic spaces is said to be coherent if it is quasi-compact and quasi-separated. When we say that an algebraic space \( X \) is coherent we always mean that \( X \) is coherent over Spec \( \mathbb{Z} \).

Let \( X \) be an algebraic space, and \( \mathcal{B} \) a quasi-coherent \( \mathcal{O}_X \)-algebra sheaf. Then as in [72], II.5, we have the algebraic space \( \text{Spec} \mathcal{B} \) affine over \( X \); if \( X = \text{Spec} A \) and \( \Gamma(X, \mathcal{B}) = \mathcal{B} \), then it is the associated algebraic space to the affine scheme \( \text{Spec} B \). The existence of \( \text{Spec} \mathcal{B} \) in general follows from effective étale descent of affine maps (cf. [52], Exposé VIII, 2) and the well-known fact on local constructions (1.4.10).

Similarly, for a quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{E} \) of finite type, one has the algebraic space \( \mathcal{P}(\mathcal{E}) \) the projective space over \( X \), together with the invertible sheaf \( \mathcal{O}_{\mathcal{P}(\mathcal{E})}(1) \). The construction in the case where \( X \) is scheme is due to [54], II, (4.1.1); the general case follows from 1.4.10 and effective étale descent of schemes with relatively ample invertible sheaves (cf. [52], Exposé VIII, 7.8).

5.2. (c) Universally cohesive algebraic spaces

**Definition 5.2.1.** An algebraic space \( X \) is said to be universally cohesive if for any algebraic space \( Y \) locally of finite presentation over \( X \), \( \mathcal{O}_Y \) is a coherent \( \mathcal{O}_Y \)-module.

This definition is consistent with 5.1.1 above, because of the following ‘fppf (fidèlement plate de présentation finie) descent of cohesiveness.’

**Proposition 5.2.2.** Let \( f: Y \to X \) be a faithfully flat and finitely presented morphism of schemes. Then \( X \) is universally cohesive if and only if so is \( Y \).

**Proof.** The ‘only if’ part is obvious. To show the ‘if’ part, it suffices to show that \( X \) is cohesive if so is \( Y \), which follows easily from [52], Exposé VIII, 1.10.

5.3 Derived category calculus

5.3. (a) Quasi-coherent sheaves on affine schemes. The following proposition is easy to verify (cf. C.4.6).
Proposition 5.3.1. Let $A$ be a ring, and set $X = \text{Spec} \ A$.

(1) The exact equivalence $\text{Mod}_A \xrightarrow{\sim} \text{Qcoh}_X$ by $M \mapsto \tilde{M}$ induces an exact (cf. §C.1) equivalence

$$D^*(\text{Mod}_A) \xrightarrow{\sim} D^*(\text{Qcoh}_X)$$

of triangulated categories.

(2) If $A$ is universally coherent, then the exact equivalence $\text{Coh}_A \xrightarrow{\sim} \text{Coh}_X$, $M \mapsto \tilde{M}$ induces an exact equivalence

$$D^*(\text{Coh}_A) \xrightarrow{\sim} D^*(\text{Coh}_X)$$

of triangulated categories.

Let us denote the composite functor

$$D^*(\text{Mod}_A) \xrightarrow{\sim} D^*(\text{Qcoh}_X) \xrightarrow{\delta^*} D^*_{\text{qcoh}}(X)$$

(cf. §C.5) by

$$M \mapsto M_X.$$

Proposition 5.3.2. Let $A$ be a universally coherent ring, and set $X = \text{Spec} \ A$. Let $\delta^b$ be the canonical exact functor

$$\delta^b : D^b(\text{Coh}_X) \longrightarrow D^b_{\text{coh}}(X).$$

Then $\delta^b$ is a categorical equivalence.

Proof. By C.5.4 it suffices to show that $\delta^b$ is fully faithful. Since

$$\text{Hom}_{D^b(X)}(K, L) = \mathcal{H}^0(\text{R Hom}_{\text{Coh}_X}(K, L)),$$

etc., it suffices to show that for any objects $K, L$ of $D^b(\text{Coh}_X)$ we have

$$\text{R Hom}_{\text{Coh}_X}(K, L) = \text{R Hom}_{\text{Coh}_X}(\delta^b(K), \delta^b(L)) \quad (*)$$

up to isomorphism in $D^+_{\text{Ab}}$. By 5.3.1, there exists an object $M$ of $D^b(\text{Coh}_A)$ such that $M_X = K$ (up to isomorphism). By 3.3.5 and [101], Chapter III, (2.4.1) (b), we have a finite free resolution $F \in \text{obj}(D^-(\text{Coh}_A))$ of $M$. Then, by 5.4.2 (1), both sides of $(*)$ are isomorphic to $\text{Hom}_{\text{Coh}_X}(F_X, L)$ in $D^+_{\text{Ab}}$, whence the claim. \qed

Corollary 5.3.3. Let $A$ be a universally coherent ring, and $X = \text{Spec} \ A$. Then the functor

$$D^b(\text{Coh}_A) \longrightarrow D^b_{\text{coh}}(X), \quad M \mapsto M_X,$$

is an exact equivalence of triangulated categories.
In other words, any object $M$ of $D^b(X)$ whose cohomologies are all coherent can be represented (in the sense as in C.4.8) by a complex consisting of coherent sheaves, and hence by a complex consisting of finitely presented $A$-modules.

Finally, we include here the related result quoted from [15].

**Proposition 5.3.4** ([15], Exposé II, Corollaire 2.2.2.1). Let $X$ be a Noetherian scheme. Then the canonical exact functor

$$\delta^b : D^b(\text{Coh}_X) \rightarrow D^b_{\text{coh}}(X)$$

is an equivalence of triangulated categories.

### 5.3. (b) Permanence of coherence

**Proposition 5.3.5.** Let $X$ be a universally cohesive algebraic space.

1. For $F, G \in \text{obj}(D^-_{\text{coh}}(X))$, $F \otimes^L_{\mathcal{O}_X} G$ belongs to $D^-_{\text{coh}}(X)$.

2. For $F \in \text{obj}(D^-_{\text{coh}}(X))$ and $G \in \text{obj}(D^+_{\text{coh}}(X))$, $R\text{Hom}_{\mathcal{O}_X}(F, G)$ belongs to $D^+_{\text{coh}}(X)$.

**Proof.** Replacing $X$ by an étale or Zariski covering, one can reduce to the case where $X$ is affine $X = \text{Spec} A$. By 5.1.2 the ring $A$ is universally coherent. (In the following, we may work with either étale topology or Zariski topology, without essential differences.)

1. By suitable shifts we may assume that $\mathcal{H}^k(F) = \mathcal{H}^k(G) = 0$ for $k > 0$. Let $n$ be a negative integer, and consider the natural morphism

$$f^n : F \otimes^L_{\mathcal{O}_X} G \rightarrow \tau^{\geq n} F \otimes^L_{\mathcal{O}_X} \tau^{\geq n} G.$$ 

Then $\mathcal{H}^k(f^n)$ is an isomorphism for $k \geq n$. Hence, to show that $F \otimes^L_{\mathcal{O}_X} G$ has coherent cohomologies, we can assume that $F$ and $G$ belong to $D^b_{\text{coh}}(X)$. By 5.3.3, $F$ and $G$ are represented by bounded complexes of finitely presented (that is, coherent) $A$-modules $M^\bullet$ and $N^\bullet$, respectively. Then the assertion follows from 3.3.6 and an easy argument of homological algebra.

2. Similarly, we can assume that $F$ and $G$ are in $D^b_{\text{coh}}(X)$ so that they are represented by bounded complexes of coherent $A$-modules. Then apply 3.3.6 in a similar way. 

$\square$
**Proposition 5.3.6.** Let \( f : X \to Y \) be a morphism of universally cohesive algebraic spaces. Then the functor \( Lf^* \) maps \( \text{D}^-_{\text{coh}}(Y) \) to \( \text{D}^-_{\text{coh}}(X) \).

**Proof.** Recall that the functor \( Lf^* : \text{D}^-(Y) \to \text{D}^-(X) \) is given by the composition
\[
\text{D}^-(Y) \longrightarrow \text{D}^-(\text{Mod}_{(X, f^{-1}\mathcal{O}_Y)}) \longrightarrow \text{D}^-(X)
\]
mapping \( M \mapsto f^{-1}M \mapsto f^{-1}M \otimes^{L}_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \), where the first functor is the one obtained from the exact functor \( f^{-1} : \text{Mod}_Y \to \text{Mod}_{(X, f^{-1}\mathcal{O}_Y)} \). Since the first functor preserves the canonical \( t \)-structures, we have
\[
f^{-1}\tau^{\geq n}M = \tau^{\geq n}f^{-1}M
\]
for \( n \in \mathbb{Z} \). By this and reasoning in much the same way as in the proof of 5.3.5, we may assume that \( M \) lies in \( \text{D}^b_{\text{coh}}(Y) \). We may also assume, as in the proof of 5.3.5, that \( Y \) is affine. In this situation, \( M \) has a finite free resolution, and hence \( f^{-1}M \) has a free \( f^{-1}\mathcal{O}_Y \)-resolution. Hence, a standard homological algebra argument shows that \( Lf^*M \) is coherent. \( \square \)

### 5.4 Cohomology of quasi-coherent sheaves

#### 5.4.1 Cohomologies on affine schemes.

We first include the following well-known facts.

**Theorem 5.4.1.** Let \( A \) be a ring, and set \( X = \text{Spec} \, A \).

1. ([54], I, (1.4.1)) The following conditions for an \( \mathcal{O}_X \)-module \( \mathcal{F} \) are equivalent:
   
   a. \( \mathcal{F} \) is quasi-coherent;
   
   b. there exists an \( A \)-module \( M \) such that \( \mathcal{F} \cong \mathcal{M} \);
   
   c. there exist a finite covering
   
   \[
   X = \bigcup_i U_i
   \]
   
   by open sets of the form
   
   \[
   U_i = D(f_i), \quad \text{with } f_i \in A,
   \]
   
   and, for each \( i \) an \( A_{f_i} \)-module \( M_i \), such that \( \mathcal{F}|_{U_i} \cong \mathcal{M}_i \).

2. ([54], I, (1.3.7)) For any \( A \)-module \( M \) we have \( \Gamma(X, \mathcal{M}) \cong M \).

3. ([53], (1.4.3)) For any \( A \)-module \( M \) the quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{M} \) is of finite type (resp. of finite presentation) if and only if \( M \) is finitely generated (resp. finitely presented) over \( A \).
Theorem 5.4.2 ([54], III, (1.3.1) and (1.3.2), and [72], II.4.8). (1) Let $A$ be a ring, and $\mathcal{F}$ a quasi-coherent sheaf on $X = \text{Spec} A$. Then for $q > 0$ we have

$$H^q(X, \mathcal{F}) = 0.$$ 

(2) Let $f : X \to Y$ be an affine morphism between algebraic spaces, and $\mathcal{F}$ a quasi-coherent sheaf on $X$. Then for $q > 0$ we have

$$R^q f_* \mathcal{F} = 0.$$ 

Note that, as mentioned in §5.2. (a), the vanishing as in (1) is also true for the cohomology calculated in terms of étale topology.

Let $X$ be an algebraic space, and $\mathcal{F}^\bullet$ a complex of $\mathcal{O}_X$-modules such that $\mathcal{F}^q = 0$ for $q \ll 0$. Then $F = Q^+ (h^+(\mathcal{F}^\bullet))$ is clearly an object of $\textbf{D}^+(X)$ (see §C.3. (a) and §C.4. (a) for the notation). We write

$$R^+ \Gamma_X(\mathcal{F}^\bullet) = R^+ \Gamma_X(F),$$

which is an object of $\textbf{D}^+(\textbf{Ab})$. If $X = \text{Spec} A$, then $R^+ \Gamma_X(\mathcal{F}^\bullet)$ is canonically regarded as an object of $\textbf{D}^+(\textbf{Mod}_A)$ (§4.3. (c)). If $f : X \to Y$ is a morphism of algebraic spaces, then we write

$$R^+ f_* \mathcal{F}^\bullet = R^+ f_* F,$$

which is an object of $\textbf{D}^+(Y)$.

Theorem 5.4.3. (1) Let $A$ be a ring, and $\mathcal{F}^\bullet$ a complex of quasi-coherent sheaves on $X = \text{Spec} A$ such that $\mathcal{F}^q = 0$ for $q \ll 0$ (resp. $|q| \gg 0$). Then $H^q(X, \mathcal{F}^\bullet) = 0$ for $q \ll 0$ (resp. $|q| \gg 0$), and the object $R^+ \Gamma_X(\mathcal{F}^\bullet)$ of $\textbf{D}^+(\textbf{Mod}_A)$ is represented (C.4.8) by the complex $\Gamma_X(\mathcal{F}^\bullet)$.

(2) Let $f : X \to Y$ be an affine morphism between algebraic spaces, and $\mathcal{F}^\bullet$ a complex of quasi-coherent sheaves on $X$ such that $\mathcal{F}^q = 0$ for $q \ll 0$ (resp. $|q| \gg 0$). Then $R^q f_* \mathcal{F}^\bullet = 0$ for $q \ll 0$ (resp. $|q| \gg 0$), and the object $R^+ f_* \mathcal{F}^\bullet$ of $\textbf{D}^+(Y)$ is represented by the complex $f_* \mathcal{F}^\bullet$.

Proof. We only present the proof of (1), for (2) can be shown similarly. First we deal with the bounded case. By a suitable shift we may assume that the complex $\mathcal{F}^\bullet$ is of the form

$$\mathcal{F}^\bullet = (\cdots \to 0 \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \to \mathcal{F}^{l-1} \to 0 \to \cdots).$$
Consider the distinguished triangle

$\sigma^\geq n F^\bullet \longrightarrow F^\bullet \longrightarrow F^0 \xrightarrow{+1}$

in $K^b(QCoh_X)$, where $\sigma^\geq n$ denotes the stupid truncation (§C.2.(d)). Taking the cohomology exact sequence, we find by induction with respect to the length of $F^\bullet$ that we may assume $F$ is a single sheaf; but then the theorem in this case is nothing but 5.4.2.

The general case can be reduced to the bounded case similarly by the stupid truncation; if $F^\bullet = (\cdots \longrightarrow 0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots)$, then to detect $H^q(X, F^\bullet)$ for a fixed $q$, we may replace $F^\bullet$ by $\sigma^{\leq q+1} F^\bullet$.

5.4. (b) Some finiteness results. The following result is a corollary of 5.4.3 (2).

Corollary 5.4.4. Let $X$ be a universally cohesive algebraic space, and $i : Y \hookrightarrow X$ a closed immersion of finite presentation (hence $Y$ is again universally cohesive). Then $R^+ i_* \mapsto D^b_{coh}(Y)$ to $D^b_{coh}(X)$.

The proof is easy. First reduce to the affine situation $Y = \text{Spec } B \hookrightarrow X = \text{Spec } A$, where $A \twoheadrightarrow B$ is a surjective homomorphism with finitely generated kernel; then apply 5.4.3 (2) with the aid of the following easy lemma.

Lemma 5.4.5. Let $A \twoheadrightarrow B$ be a surjective ring homomorphism, and $M$ a $B$-module.

1. If $M$ is finitely presented as an $A$-module, then it is so as a $B$-module.

2. Suppose that the kernel of the map $A \twoheadrightarrow B$ is a finitely generated ideal of $A$. Then, if $M$ is finitely presented over $B$, it is finitely presented over $A$.

Proposition 5.4.6. Let $f : X \rightarrow Y$ be a coherent morphism between algebraic spaces (§5.2.(b)). Then $R^+ f_* \mapsto \text{an object of } D^+_{qcoh}(X)$ to an object of $D^+_{qcoh}(Y)$.

Proof. Here we present the proof for the case where $X$ and $Y$ are schemes. (This is actually enough, due to the standard technique (cohomological étale descent; cf. [9], Exposé Vbis, (4.2.1)).) Let $F$ be an object of $D^+_{qcoh}(X)$. We need to show that $R^q f_* F$ is a quasi-coherent sheaf on $Y$ for any $q \geq 0$. But, to show it for a fixed $q$, we may replace $F$ by $\tau^{\leq q+1} F$, and thus we may assume that $F$ belongs to $D^b_{qcoh}(X)$. By shifting we may assume $\mathcal{H}^q(F) = 0$ for $q < 0$. Then by the distinguished triangle

$F \longrightarrow \tau^{\geq 1} F \longrightarrow G \longrightarrow +1$,

we may assume by induction that $F$ is concentrated in degree 0 (cf. C.4.9 (2)) and hence that $F$ is represented by a single quasi-coherent sheaf $F$ on $X$ (C.4.10). What we need to prove is that $R^q f_* F$ is quasi-coherent for every $q$, which is shown in [54], III, (1.4.10), (cf. [54], IV, (1.7.21)).
Corollary 5.4.7 (cf. [54], III, (1.4.11)). In the situation as in 5.4.6, let \( i: V \to Y \) be an étale morphism from an affine scheme \( V \). Denote by \( j: U = X \times_Y V \to X \) the induced morphism. Let \( \mathcal{F}^\bullet \) be a complex of quasi-coherent sheaves on projective spaces. We fix the following notation. Let \( A \) be a ring, \( r \) a positive integer, and \( S = A[T_0, \ldots, T_r] \) the polynomial ring over \( A \). Let \( T = \mathbb{P}_A^r = \text{Proj } S \). For \( n > 0 \) we set \( T^n = \{ T_0^n, \ldots, T_r^n \} \), and let \( (T^n) \) be the ideal of \( S \) generated by the set \( T^n \). For \( 0 \leq n \leq m \) we have the map \( \varphi_{nm}: S/(T^n) \to S/(T^m) \) defined by the multiplication by \( (T_0 \cdots T_r)^{m-n} \), and thus we get an inductive system \( \{ S/(T^n), \varphi_{nm} \} \) indexed by the set of non-negative integers. For any \( (r+1) \)-tuple \( (p_0, \ldots, p_r) \) of positive integers and an integer \( n \) such that \( n \geq \sup_{0 \leq i \leq r} p_i \), we denote by \( \xi(n)_{p_0 \cdots p_r} \) the modulo \( (T^n) \) class of the monomial \( T_{r-p_0} \cdots T_{r-p_r} \). Clearly, the sequence \( \{ \xi(n)_{p_0 \cdots p_r} \} \) defines a unique element of the inductive limit \( \lim_{\to n} S/(T^n) \), which we denote by \( \xi_{p_0 \cdots p_r} \).

Theorem 5.4.8 ([54], III, (2.1.12)). Set

\[
H^q(X, \mathcal{O}_X(*)) = \bigoplus_{n \in \mathbb{Z}} H^q(X, \mathcal{O}_X(n))
\]

for any \( q \geq 0 \), and regard it as a graded \( A \)-module.

1. We have \( H^q(X, \mathcal{O}_X(*)) = 0 \) for \( q \neq 0, r \).

2. There exists a canonical isomorphism \( S = H^0(X, \mathcal{O}_X(*)) \) of graded \( A \)-modules, where \( S \) is regarded as a graded \( A \)-module in the standard way.

3. The graded \( A \)-module \( H^r(X, \mathcal{O}_X(*)) \) is canonically isomorphic to the inductive limit \( \lim_{\to n} S/(T^n) \), which is free with the basis \( \{ \xi_{p_0 \cdots p_r} \}_{p_0 \cdots p_r > 0} \) and equipped with the grading such that the degree of \( \xi_{p_0 \cdots p_r} \) is \( -(p_0 + \cdots + p_r) \).

Corollary 5.4.9 ([54], III, (2.1.13)). In the situation as above, the cohomology group \( H^q(X, \mathcal{O}_X(n)) \) is a free \( A \)-module of finite type; it is non-zero if and only if either one of the following conditions holds:

(a) \( q = 0 \) and \( n \geq 0 \);

(b) \( q = r \) and \( n \leq -(r + 1) \).

Proposition 5.4.10 ([54], II, (2.7.9)). Let \( A \) be a ring, and \( \mathcal{F} \) be a quasi-coherent sheaf of finite type on \( X = \mathbb{P}_A^r \). Then there exists an integer \( N \) such that for any \( n \geq N \) the sheaf \( \mathcal{F}(n) \) is generated by global sections; more precisely, there exists a surjective morphism \( \mathcal{O}_X^\oplus_k \to \mathcal{F}(n) \), where \( k \) is a positive integer depending on \( n \).
5.4. (d) **Ample and very ample sheaves.** Let us briefly recall the definitions of ample and very ample sheaves (cf. [54], I I, (4.4.2) and (4.6.11)). Let \( f : X \to Y \) be a morphism of finite type between coherent algebraic spaces, and \( \mathcal{L} \) an invertible sheaf on \( X \).

- We say that \( \mathcal{L} \) is **very ample relative to** \( f \) if there exist a quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{E} \) of finite type and a factorization

\[
\begin{array}{ccc}
X & \xrightarrow{i} & \mathbb{P}(\mathcal{E}) \\
& & \xrightarrow{\pi} \mathcal{G} \\
\end{array}
\]

of \( f \) through an immersion \( i \) such that \( \mathcal{L} \) is isomorphic to \( i^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \).

- We say that \( \mathcal{L} \) is **ample relative to** \( f \) (or \( f \)-ample) if for any quasi-compact open subset \( V \) of \( Y \), there exists a positive integer \( k \) such that \( \mathcal{L}^k |_{f^{-1}(V)} \) is very ample relative to \( f_V : f^{-1}(V) \to V \).

5.5 **More basics on algebraic spaces**

Let us include here some useful facts on coherent algebraic spaces, which will be used in our later discussion.

5.5. (a) **The stratification by subschemes.** The following theorem, which shows that coherent algebraic spaces are ‘tangible,’ is very useful in reducing many arguments concerning algebraic spaces to scheme cases, and thus has a lot of important applications.

**Theorem 5.5.1** ([89], Première partie, (5.7.6)). Let \( X \) be a coherent algebraic space. Then there exists a finite sequence \( Z_1, \ldots, Z_r \) of locally closed subspaces of \( X \) with the following properties:

(a) \( Z_i \) for each \( i = 1, \ldots, r \) is reduced and quasi-compact;

(b) the \( Z_i \)'s are pairwise disjoint and cover \( X \), that is, \( X = \bigcup_{i=1}^r Z_i \);

(c) \( Y_i = \bigcup_{j \geq i} Z_j \) for each \( i = 1, \ldots, r \) is an open subspace of \( X \);

(d) for each \( i = 1, \ldots, r \) there exists a separated and quasi-compact elementary étale neighborhood \( Y'_i \) of \( Z_i \) in \( Y_i \) such that the image of \( Y'_i \) in \( X \) coincides with \( Y_i \).

Here, for an algebraic space \( Y \) and a closed subspace \( Z \) of \( Y \), an **elementary étale neighborhood** of \( Z \) in \( Y \) is an étale map \( u : Y' \to Y \) from a scheme \( Y' \) such that the induced map \( Y' \times_Y Z \to Z \) is an isomorphism. In particular, if an elementary étale neighborhood of \( Z \) in \( Y \) exists, then \( Z \) is a scheme, since it is a closed subscheme of \( Y' \).
5.5. (b) Affineness criterion

**Theorem 5.5.2** (affineness criterion; cf. [72], III.2.3). Let $X$ be a coherent algebraic space. Then the following conditions are equivalent.

(a) The global section functor

\[ \Gamma_X : \text{QCoh}_X \longrightarrow \text{Ab} \]

from the category of quasi-coherent sheaves on $X$ is exact and faithful, that is, for any quasi-coherent sheaf $\mathcal{F}$ on $X$, $\Gamma_X(\mathcal{F}) \neq 0$ whenever $\mathcal{F} \neq 0$.

(b) $X$ is an affine scheme.

**Remark 5.5.3.** In [72], III.2.3, the theorem is proved under the slightly stronger hypothesis that $X$ is quasi-compact and separated. One can modify the proof therein to show the above version of the theorem, replacing ‘quasi-compact and separated’ for the map $\gamma : X \rightarrow \text{Spec } A$ by ‘coherent.’ In view of 5.4.6 this is enough for the rest of the proof.

The same remark shows that one can similarly drop ‘separated’ in [72], III.2.5, as follows.

**Theorem 5.5.4** (Serre criterion; cf. [72], III.2.5). Let $X$ be a Noetherian algebraic spaces. Then the following conditions are equivalent.

(a) the global section functor

\[ \Gamma_X : \text{Coh}_X \longrightarrow \text{Ab} \]

from the category of coherent sheaves on $X$ is exact.

(b) $X$ is an affine scheme.

**Corollary 5.5.5.** Let $X$ be a coherent algebraic space, and $X_0$ the closed subspace of $X$ defined by a nilpotent quasi-coherent ideal $\mathfrak{J}$. If $X_0$ is a scheme, then $X$ is a scheme.

**Proof.** By induction with respect to $s \geq 1$ with $\mathfrak{J}^s = 0$, we reduce to the case $\mathfrak{J}^2 = 0$. We may also assume that $X_0$ is affine. For any quasi-coherent sheaf $\mathcal{F}$ on $X$ there exists an exact sequence

\[ 0 \longrightarrow \mathfrak{J} \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} / \mathfrak{J} \mathcal{F} \longrightarrow 0, \]

where the first and the third sheaves can be regarded as quasi-coherent sheaves on $X_0$. Since $X_0$ is affine, we have $H^1(X, \mathfrak{J} \mathcal{F}) = 0$ and $H^1(X, \mathcal{F} / \mathfrak{J} \mathcal{F}) = 0$ (5.4.2 (1)). Hence $H^1(X, \mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$ on $X$. In particular, the functor $\Gamma_X$ is exact. To show that $\Gamma_X$ is faithful, suppose $\Gamma(X, \mathcal{F}) = 0$. Then $\Gamma(X, \mathfrak{J} \mathcal{F}) = \Gamma(X, \mathcal{F} / \mathfrak{J} \mathcal{F}) = 0$ by the exactness. Hence, we have $\mathfrak{J} \mathcal{F} = 0$ and $\mathcal{F} / \mathfrak{J} \mathcal{F} = 0$, which means $\mathcal{F} = 0$. Now by 5.5.2 we deduce that $X$ is an affine scheme. $\square$
5.5. (c) Limit theorem

**Theorem 5.5.6** (Raynaud [89], Première partie, (5.7.8)). Let $X$ be a coherent algebraic space. Then any quasi-coherent sheaf $\mathcal{F}$ on $X$ is the inductive limit $\lim_{\longrightarrow a \in I} \mathcal{F}_a$ of quasi-coherent subsheaves $\mathcal{F}_a \subseteq \mathcal{F}$ of finite type.

This theorem has been proved by D. Knutson [72], III.1.1, in the case when $X$ is a Noetherian locally separated algebraic space. The proof for the general case uses 5.5.1. By this and by an argument similar to the one in the proof of [54], I, (9.4.3), one has the following theorem.

**Corollary 5.5.7** (extension theorem). Let $X$ be a coherent algebraic space, and $U$ a quasi-compact open subspace of $X$. Then for any quasi-coherent sheaf $\mathcal{F}$ of finite type on $U$, there exists a quasi-coherent sheaf $\mathcal{G}$ of finite type on $X$ such that $\mathcal{G}|_U = \mathcal{F}$.

The first author of this book has proved in [40] the following absolute affine limit theorem for algebraic spaces, which generalizes [98], C.9.

**Theorem 5.5.8** (affine limit theorem). Let $S$ be a coherent algebraic space, and $f : X \rightarrow S$ a coherent morphism of algebraic spaces.

1. There exists a projective system $\{X_i\}_{i \in I}$ of $S$-schemes indexed by a category $I$ such that
   a. each $X_i$ is coherent and finitely presented over $S$,
   b. for any arrows $i \rightarrow j$ in $I$ the transition map $X_j \rightarrow X_i$ is affine, and
   c. $X \cong \lim_{\leftarrow i \in I} X_i$.

2. If $X$ is a scheme, then the algebraic spaces $X_i$ can be taken to be schemes.

3. If $S$ is Noetherian, then the index category $I$ can be replaced by a directed set in such a way that each transition map $X_j \rightarrow X_i$ is scheme-theoretically dominant.

By this theorem and 5.5.5 we have the following corollary.

**Corollary 5.5.9.** Let $X$ be a quasi-separated algebraic space. If $X_{\text{red}}$ is a scheme (resp. an affine scheme), then so is $X$. 
Exercises

Exercise 0.5.1. Let $X$ be a coherent scheme, and $U \subseteq X$ an open subset. Show that $U$ is quasi-compact if and only if $X \setminus U$ is the support of a closed subscheme of $X$ of finite presentation.

Exercise 0.5.2. Let $X$ be a scheme, $I \subseteq \mathcal{O}_X$ a nilpotent quasi-coherent ideal of finite type, and $Z$ the closed subscheme of $X$ defined by $I$. Let $f : Y \to X$ be a morphism of schemes such that $f_Z : Y \times_X Z \to Z$ is a closed immersion. Show that $f$ is a closed immersion.

Exercise 0.5.3. Let $A$ be a universally coherent ring, and $X$ a projective finitely presented $A$-scheme. Show that the canonical exact functor

$$\delta^b : D^b(\text{Coh}_X) \to D^b_{\text{coh}}(X)$$

is a categorical equivalence.

6 Valuation rings

In this section, we give a brief overview of the theory of valuation rings. Our basic references for valuation rings are [109], Chapter VI, and [27], Chapter VI. We also refer to [100] as a useful concise survey. Since almost all what we need to know about valuation rings is already well documented in these references, we will most of the time be sketchy, and omit many of the proofs.

In §6.1 we will discuss two prerequisites, totally ordered commutative groups and invertible ideals. The basic definitions and first properties of valuations and valuation rings will be given in §6.2, which also includes the definitions of height (also called ‘rank’ in some literature) and rational rank. As described in §6.3, the spectrum of a valuation ring is a path-like object; this might suggest that valuation rings are algebro-geometric analogue of paths, perhaps more precisely ‘long paths,’ reflecting the fact that valuations may possible be of large height. This subsection also gives a detailed description of valuation rings of finite height, especially of height one, together with the concept of the so-called non-Archimedean norms. In §6.4 we discuss composition and decomposition of valuation rings, which are two of the most characteristic features of valuation rings and their spectra. In the next two subsections §6.5 and §6.6, we recall some techniques for studying structures of valuation rings, which enable us to give a rough classification of them, which we do in the end of §6.6.

The valuation rings discussed in §6.7 are in fact the most important for our purpose, namely, valuation rings equipped with a separated adic topology defined by a principal ideal. Such valuation rings have many significant features, which will be of fundamental importance in our later studies of formal and rigid geometries.
6.1 Prerequisites

6.1. (a) Totally ordered commutative group. An abelian group is a totally ordered commutative group if it is endowed with an ordering $(\Gamma, \geq)$ (cf. §1.1.(b)) such that

(a) if $a \geq b$ for $a, b \in \Gamma$, then $a + c \geq b + c$ for any $c \in \Gamma$ and

(b) for any $a, b \in \Gamma$ either $a \geq b$ or $b \geq a$ holds; that is, $(\Gamma, \geq)$ is a totally ordered set (cf. §1.1.(b)).

An ordered homomorphism of totally ordered commutative groups is a group homomorphism that is also an ordered map (cf. §1.1.(b)).

Let $\Gamma$ be a totally ordered commutative group. An element $a \in \Gamma$ is said to be positive (resp. negative) if $a > 0$ (resp. $a < 0$). A non-empty subset $\Delta$ of $\Gamma$ is said to be a segment if for any element $a \in \Delta$, any $b \in \Gamma$ with $-a \leq b \leq a$ or $a \leq b \leq -a$ belongs to $\Delta$. If a subgroup $\Delta$ of $\Gamma$ is a segment, then it is called an isolated subgroup. Note that, unlike in [100], §3, we allow $\Gamma$ itself to be an isolated subgroup.

The basic role of isolated subgroups is explained in the following ‘homomorphism theorem’ for totally ordered commutative groups.

Proposition 6.1.1 ([27], Chapter VI, §4.2, Proposition 3). Let $\Gamma$ be a totally ordered commutative group.

(1) The kernel of an ordered homomorphism of $\Gamma$ to an ordered group is an isolated subgroup of $\Gamma$.

(2) Conversely, for an isolated subgroup $\Delta \subseteq \Gamma$ the quotient $\Gamma/\Delta$ is again a totally ordered commutative group by the induced ordering, and the canonical map $\Gamma \to \Gamma/\Delta$ is an ordered homomorphism.

Moreover, it is easy to see that the canonical map $\Gamma \to \Gamma/\Delta$ in (2) induces a bijection between the set of all isolated subgroups of $\Gamma$ containing $\Delta$ and the set of all isolated subgroups of $\Gamma/\Delta$.

For a totally ordered commutative group $\Gamma$ we denote by Isol($\Gamma$) the set of all proper isolated subgroups of $\Gamma$. Then Isol($\Gamma$) together with the inclusion order is a totally ordered set; indeed, if there were two isolated subgroups $\Delta$ and $\Delta'$ such that none of $\Delta \subseteq \Delta'$ and $\Delta \supseteq \Delta'$ holds, then there would exist positive elements $a \in \Delta \setminus \Delta'$ and $a' \in \Delta' \setminus \Delta$; if, for example, $a \geq a'$, then $a'$ must belong to $\Delta$, for $\Delta$ is isolated, which is absurd. The order type (cf. §1.1.(b)) of Isol($\Gamma$) is called the height of $\Gamma$ and is denoted by $\text{ht}(\Gamma)$.
If it is finite and equal to $n$ (the order type of the totally ordered set $\{0, 1, \ldots, n-1\}$ with the obvious ordering), then we say that $\Gamma$ is of finite height $n$. Otherwise, $\Gamma$ is said to be of infinite height.

Note that the height 0 totally ordered commutative group is the trivial group $\{0\}$. As for height one groups, we have the following characterization.

**Proposition 6.1.2** ([27], Chapter VI, §4.5, Proposition 8). The following conditions for a totally ordered commutative group $\Gamma$ are equivalent.

(a) $\text{ht}(\Gamma) = 1$;

(b) for any $a, b \in \Gamma$ with $a > 0$ and $b \geq 0$ there exists an integer $n \geq 0$ such that $b \leq na$.

(c) $\Gamma$ is ordered isomorphic to a non-zero subgroup of $\mathbb{R}$, the additive group of real numbers endowed with the usual order.

To discuss totally ordered commutative groups of higher height, the following construction will be useful.

**Example 6.1.3.** Let $h$ be a positive integer, and $\Gamma_i$ totally ordered commutative groups for $i = 1, \ldots, h$. Consider the direct sum

$$\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_h,$$

endowed with the so-called lexicographical order: let $a = (a_1, \ldots, a_h)$ and let $b = (b_1, \ldots, b_h)$ two elements of $\Gamma$, then

$$a \leq b \iff \begin{cases} \text{the first entries } a_i \text{ and } b_i \text{ in } a \text{ and } b \text{ from the left} \\ \text{that are different from each other satisfy } a_i \leq b_i. \end{cases}$$

For $j = 1, \ldots, h$ the subgroup of $\Gamma$ of the form

$$\Gamma_j \oplus \cdots \oplus \Gamma_h \ (= 0 \oplus \cdots \oplus 0 \oplus \Gamma_j \oplus \cdots \oplus \Gamma_h)$$

is an isolated subgroup, and the quotient of $\Gamma$ by this subgroup is ordered isomorphic to $\Gamma_1 \oplus \cdots \oplus \Gamma_{j-1}$. Hence by induction one sees that $\text{ht}(\Gamma) = \sum_{i=1}^{h} \text{ht}(\Gamma_i)$.

In particular, if all $\Gamma_i$ are of height one, the resulting $\Gamma$ as above gives a totally ordered commutative group of height $h$. While it is not true that any totally ordered commutative group of finite height is of this form, we have the following useful fact.

**Proposition 6.1.4** ([3], Chapter II, Proposition 2.10). Let $\Gamma$ be a totally ordered commutative group of height $n < \infty$. Then $\Gamma$ is ordered isomorphic to a subgroup of $\mathbb{R}^n = \mathbb{R} \oplus \cdots \oplus \mathbb{R}$. Suppose, moreover, that $\Gamma$ satisfies the following condition: for any $a \in \Gamma$ and any non-zero integer $m$ there exists $b \in \Gamma$ such that $a = mb$. Then there exists a subgroup $\Gamma_1$ of $\mathbb{R}$ such that $\Gamma$ is order isomorphic to $\Gamma_1 \oplus \cdots \oplus \Gamma_n$ with the lexicographical order.
The **rational rank** of a totally ordered commutative group $\Gamma$ is the dimension of the $\mathbb{Q}$-vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ and is denoted by

$$\text{rat-rank}(\Gamma).$$

**Proposition 6.1.5.** Let $\Gamma$ be a totally ordered commutative group, and suppose the rational rank of $\Gamma$ is finite. Then the height of $\Gamma$ is finite and

$$\text{ht}(\Gamma) \leq \text{rat-rank}(\Gamma).$$

For the proof, see [27], Chapter VI, §10.2, or [109], Chapter VI, §10, Note.

**Example 6.1.6.** Consider the totally ordered commutative group $\Gamma$ as in 6.1.3, where each $\Gamma_i$ is of height one. Then the rational rank of $\Gamma$ is finite if and only if the rational rank of each $\Gamma_i$ ($i = 1, \ldots, h$) is finite. In this case we have

$$\text{rat-rank}(\Gamma) = \sum_{i=1}^{h} \text{rat-rank}(\Gamma_i) \geq h = \text{ht}(\Gamma).$$

### 6.1. (b) Invertible ideals.

Let $A$ be a ring and $F = \text{Frac}(A)$ the total ring of fractions of $A$. An $A$-submodule $I \subseteq F$ of $F$ is said to be **non-degenerate** if $F \cdot I = F$. Hence, in particular, an ideal $I$ of $A$ is non-degenerate if and only if $I$ contains a non-zero-divisor of $A$. It can be shown that for a non-degenerate $A$-submodule $I \subseteq F$ the following conditions are equivalent (cf. [27], Chapter II, §5.6, Theorem 4).

1. There exists an $A$-submodule $J$ of $F$ such that $I \cdot J = A$.
2. $I$ is projective.
3. $I$ is finitely generated, and for any maximal ideal $m \subseteq A$ the $A_m$-module $I_m$ is principally generated.

If these conditions are satisfied, we say that $I$ is an **invertible fractional ideal**. In case $I = (a)$ is a principal fractional ideal, then $I$ is invertible if and only if $a$ is a non-zero-divisor of $F$. If an invertible fractional ideal $I$ is an ideal of $A$, we say that $I$ is an **invertible ideal** of $A$.

**Proposition 6.1.7.** Let $I \subseteq F$ be an invertible fractional ideal. Then the $A$-submodule $J$ as in (a) is unique and is given by

$$J = (A : I) = \{ x \in F : xI \subseteq A \}.$$

In particular, if $I = aA$ for $a \in F$ is invertible, $J = (A : I) = (a^{-1})$. The set of all invertible fractional ideals of $F$ forms a group by multiplication.

**Lemma 6.1.8.** Let $A$ be a ring, and $I, J$ ideals of $A$. Then both $I$ and $J$ are invertible if and only if $IJ$ is invertible.
Proof. Suppose $K = IJ$ is invertible. Since $IF \subseteq F = IJF \subseteq IF$, we have $IF = F$, that is, $I$ is non-degenerate. Take the fractional ideal $L$ such that $KL = A$. Then $JL$ gives the inverse of $I$, and hence $I$ is invertible. One sees similarly that $J$ is invertible. The converse is clear.

The case where $A$ is a local domain will be of particular importance. In this situation, by virtue of (c), any invertible ideal is principal.

**Proposition 6.1.9.** If $A$ is a local domain, then there exists a canonical bijection between the set of all invertible fractional ideals of $F = \text{Frac}(A)$ and the set $F^\times / A^\times$; the bijection is established by $(a) \mapsto [a] = (a \mod A^\times)$. Moreover, if we order the former set by inclusion and $F^\times / A^\times$ by $[x] \leq [y] \iff x = zy$ for some $z \in A$, then this bijection gives an isomorphism of ordered groups.

### 6.2 Valuation rings and valuations

#### 6.2. (a) Valuation rings

Let $B$ be a local ring, and $A$ a subring of $B$ that is again a local ring. We say that $B$ dominates $A$, written $A \preceq B$, if $m_A \subseteq m_B$ or, equivalently, $m_A = A \cap m_B$. For a field $K$ the relation $\preceq$ gives an ordering on the set of all local subrings of $K$.

**Definition 6.2.1.** Let $V$ be an integral domain, and $K$ a field containing $V$ as a subring. Then $V$ is a valuation ring for $K$ if it satisfies one of the following equivalent conditions:

(a) $V$ is a maximal with respect to $\preceq$ in the set of all local subrings of $K$;

(b) for any $x \in K \setminus \{0\}$, either $x$ or $x^{-1}$ belongs to $V$;

(c) $\text{Frac}(V) = K$ and the set of all ideals of $V$ ordered by inclusion order is totally ordered;

(d) $\text{Frac}(V) = K$ and the set of all principal ideals of $V$ ordered by inclusion order is totally ordered.

For the equivalence of these conditions, see the references mentioned at the beginning of this section. When we just say $V$ is a valuation ring, we always mean that $V$ is a valuation ring for its field of fractions. By (b), we have the following easy but useful fact: any subring of a field $K$ that contain at least one valuation ring for $K$ is again a valuation ring for $K$.

Note that, according to our definition of valuation rings, we allow fields to be valuation rings. This case is usually ruled out from the notion of valuation rings, but in some places this convention will be useful for the sake of formality.
As it will turn out in §6.4, this convention is consistent with the fact that we allowed in §6.1. (a) a totally ordered commutative group $\Gamma$ itself to be an isolated subgroup of itself.

**Proposition 6.2.2.** (1) Any valuation ring is integrally closed.

(2) Any finitely generated ideal of a valuation ring is principal.

*Proof.* By [27], Chapter V, §2.1, Theorem 1, we can find a prime ideal $p$ of the integral closure $\bar{V}$ of $V$ in $K$ that lies over $m_V$, that is, $V \cap p = m_V$. If $\bar{V} \neq V$, then $\bar{V}_p$ would be a local subring of $K$ strictly larger than and dominating $V$, and we reached a contradiction. Thus we conclude that $\bar{V} = V$, and so (1) holds.

(2) follows easily from 6.2.1 (c). □

**Proposition 6.2.3.** Let $V$ be a valuation ring and $I \subseteq V$ a finitely generated ideal not equal to $V$. Then there exists the minimal prime ideal $p \subseteq V$ among the prime ideals containing $I$; more explicitly, $p = \sqrt{I}$.

*Proof.* By 6.2.2 (2), we have $I = (a)$ for $a \in m_V$. It suffices to show that the ideal $\sqrt{(a)}$ is prime. Suppose $bc \in \sqrt{(a)}$ and $b \notin \sqrt{(a)}$. This implies that there exists $n \geq 0$ such that $(bc)^n = ad$ for some $d \in V$ and that $a/b^m \in m_V$ for any $m \geq 0$. Then $c^{2n} = a \cdot (a/b^{2n}) \cdot d^2 \in (a)$, and hence $c \in \sqrt{(a)}$. □

6.2. (b) **Valuations.** Let $V$ be a valuation ring and $K = \text{Frac}(V)$. Since $V$ is a local domain, the ordered group of all invertible fractional ideals is isomorphic to $K^\times / V^\times$ (6.1.9). By 6.2.1 (d), this is a totally ordered commutative group. We set $\Gamma_V = K^\times / V^\times$ and write the group operation additively. Consider the mapping

$$v: K \longrightarrow \Gamma_V \cup \{\infty\}, \quad v(x) = \begin{cases} [x] (= x \mod V^\times) & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}$$

As it is easily verified, $v$ enjoys the following properties:

(a) $v(xy) = v(x) + v(y)$ for $x, y \in K$;

(b) $v(x+y) \geq \inf\{v(x), v(y)\}$ for $x, y \in K$;

(c) $v(1) = 0$ and $v(0) = \infty$.

**Definition 6.2.4.** Let $A$ be a ring and $\Gamma$ a totally ordered commutative group. A mapping

$$v: A \longrightarrow \Gamma \cup \{\infty\}$$

is called a valuation on $A$ with values in $\Gamma$ if it maps non-zero elements to elements in $\Gamma$ and satisfies (a), (b), and (c) as above (with $V$ replaced by $A$). In this situation, we call the totally ordered group $\Gamma$ the value target group of the valuation $v$. 
Thus any valuation ring \( V \) induces the canonical valuation \( v: K \to \Gamma_V \) on its fractional field; we call this the **valuation associated to \( V \)** and call the totally ordered commutative group \( \Gamma_V \) the **value group** of \( V \). Note that in this situation we have \( V = \{ x \in K : v(x) \geq 0 \} \).

**Proposition 6.2.5.** Let \( v: K \to \Gamma \cup \{ \infty \} \) be a valuation on a field with values in a totally ordered commutative group \( \Gamma \). Then \( V = \{ x \in K : v(x) \geq 0 \} \) is a valuation ring for \( K \), and \( \{ x \in K : v(x) > 0 \} \) is the maximal ideal of \( V \).

Moreover, the group \( \Gamma \) contains an isomorphic copy of \( \Gamma_V \), namely, the image of \( v \). Note that, if \( K = V \), the corresponding valuation \( v \) maps all elements in \( K^\times \) to 0. Such a valuation is called the **trivial valuation**.

### 6.2. (c) Height and rational rank of valuation rings

**Definition 6.2.6.** Let \( V \) be a valuation ring, \( v \) the associated valuation, and \( \Gamma_V \) the corresponding value group.

1. The height of the value group \( \Gamma_V \) (§6.1.(a)) is called the **height** of \( V \) (or of \( v \)) and is denoted by
   \[
   ht(V) = ht(v) = ht(\Gamma_V).
   \]

2. The rational rank of the value group \( \Gamma_V \) (§6.1.(a)) is called the **rational rank** of \( V \) (or of \( v \)) and is denoted by
   \[
   \text{rat-rank}(V) = \text{rat-rank}(v) = \text{rat-rank}(\Gamma_V).
   \]

**Remark 6.2.7.** We prefer to use the term ‘height’ following [27], while [109] uses the term ‘rank.’

**Proposition 6.2.8** (cf. 6.1.5). Let \( V \) be a valuation ring, and suppose the rational rank of \( V \) is finite. Then the height of \( V \) is finite, and the following inequality holds:

\[
ht(V) \leq \text{rat-rank}(V).
\]

**Proposition 6.2.9.** Let \( V \) be a valuation ring with \( K \) and \( \Gamma_V \) as above. Then there exist canonical order-preserving bijections among the following sets:

1. the set of all prime ideals of \( V \) with the inclusion order;
2. the set of all subrings \((\neq V)\) lying between \( V \) and \( K \) (which are automatically valuation rings) with the reversed inclusion order;
3. the set of all proper isolated subgroups of \( \Gamma_V \) with the reversed inclusion order.
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The bijections are established as follows.

- **From the set (a) to the set (b):** to a prime ideal $p \subseteq V$ we associate the local ring $V_p$; note that by 6.2.1 (b) any local ring lying in between $V$ and $K$ is a valuation ring.

- **From the set (b) to the set (c):** for $W$ as in (b) we consider the subgroup $v(W^\times)$ of $\Gamma_V$, where $v: K \rightarrow \Gamma_V \cup \{\infty\}$ is the valuation associated to $V$; then it is easy to see that $v(W^\times)$ is a segment.

See [109], Chapter VI, Theorem 15, and [27], Chapter VI, §4.3, Proposition 4, for more details. From this proposition it follows that the cardinality of $\text{ht}(V)$ coincides with the Krull dimension of $V$. In particular, a valuation ring of height 0 is nothing but a field.

### 6.3 Spectrum of valuation rings

#### 6.3. (a) General description.

From 6.2.1 (c) it follows in particular that prime ideals of a valuation ring $V$ are totally ordered with respect to the inclusion order. Hence the spectrum $\text{Spec} V$ can be understood as a ‘path’ with two extremities $(0)$ and $m_V$; all the other points lie between these points in such a way that, if $p \subseteq q$, then $p$ lies between $(0)$ and $q$; see Figure 1. The ‘length,’ so to speak, of the path is (the cardinality of) the height of $V$.

![Figure 1. Spec V and a closed set](image)

By 6.2.9, such a linear pattern is reflected in the set of all subrings of $K$ containing $V$, and also in the set of all isolated subgroups of the value group $\Gamma_V$.

The Zariski topology on Spec $V$ can also be understood intuitively in this picture. For two ideals $I, J$ of $V$ with $I \subseteq J$, define the subsets $[I, J)_V$, $(I, J)_V$, $[I, J)_V$, and $(I, J)_V$ as follows:

- $[I, J)_V = \{ p \in \text{Spec} V : I \subseteq p \subseteq J \}$,
- $(I, J)_V = \{ p \in \text{Spec} V : I \subsetneq p \subsetneq J \}$,
- $[I, J)_V = \{ p \in \text{Spec} V : I \subseteq p \subsetneq J \}$,
- $(I, J)_V = \{ p \in \text{Spec} V : I \subsetneq p \subseteq J \}$. 
Then for any ideal \( I \subseteq V \) the set \([I, m_V]_V\) is exactly the closed set \( V(I) \), and all closed sets are of this form. Hence open sets are exactly the subsets of the form \([0), I)\_V\). For any point \( p \in \text{Spec} \ V \) the Zariski closure of \( \{p\} \) is given by \([p, m_V]_V\); in other words, a point is the specialization of points lying to the left and is the generization of points lying to the right, when \( \text{Spec} \ V \) is oriented as in Figure 1.

**Proposition 6.3.1.** The underlying topological space of \( \text{Spec} \ V \), where \( V \) is a valuation ring, is a valuative space (2.3.1).

**Proof.** First note that the underlying topological spaces of affine schemes are coherent and sober. For any point \( p \in \text{Spec} \ V \) the set \( G_p \) of all generizations of \( p \) is totally ordered, as described above. \(\square\)

6.3. (b) Valuation rings of finite height. The valuation ring \( V \) is of finite height if and only if \( \text{Spec} \ V \) consists of finitely many points (Figure 2).

![Figure 2. Spectrum of a valuation ring of finite height](image)

In this case, one can speak about the adjacent points, that is to say, the ‘next’ generization or specialization of an arbitrary given point of \( \text{Spec} \ V \). This feature allows us, aided by composition and decomposition of valuation rings (explained later in §6.4), to carry out inductive arguments with respect to height, reducing many situations to the height-one case. It is therefore important to study the case \( \text{ht}(V) = 1 \), the case where \( \text{Spec} \ V \) consists only of \((0), \) the open point, and \( m_V, \) the closed point.

**Proposition 6.3.2** (cf. 6.1.2). Let \( V \) be a valuation ring of non-zero height. Then the following conditions are equivalent.

(a) The height of \( V \) is 1.

(b) For any \( x \in m_V \setminus \{0\} \) and \( y \in V \setminus \{0\} \) there exist an integer \( n \geq 0 \) and an element \( z \in V \) such that \( yz = x^n \).

(c) The value group \( \Gamma_V \) is isomorphic to a non-zero subgroup of the ordered additive group \( \mathbb{R} \) of real numbers.

6.3. (c) Non-Archimedean norms. By 6.3.2 (3), the associated valuation to a height-one valuation ring \( V \) is of the form \( v: K \to \mathbb{R} \cup \{\infty\} \), where \( K = \text{Frac}(V) \).
In this situation, the function $|\cdot|: K \to \mathbb{R}_{\geq 0}$ defined by $|x| = e^{-\nu(x)}$ (where $e > 1$ is a fixed real number) is more a familiar object, the associated (non-Archimedean) norm. In the literature the pair $(K, |\cdot|)$ is called a non-Archimedean valued field, which, in our language, is equivalent to fraction field of a valuation ring of height one. For more details of norms we refer to some of the first chapters of [18].

**Proposition 6.3.3** ([109], Chapter VI, Theorem 16). Let $V$ be a valuation ring of non-zero height. Then $V$ is Noetherian if and only if the value group $\Gamma_V$ is isomorphic to $\mathbb{Z}$ (and hence, in particular, $V$ is of height one). (In this case $V$ is called a discrete valuation ring (acronym: DVR).)

### 6.4 Composition and decomposition of valuation rings

Let $V$ be a valuation ring. As mentioned in §6.3.(a), $\text{Spec} V$ looks like a path with the extremities $(0)$ and $m_V$. Any prime ideal $p \subseteq V$ divides $\text{Spec} V$ into two segments, $[(0), p]$ and $[p, m_V]$. The first subsegment corresponds to $\text{Spec} V_p$ and the second to $\text{Spec} V/p$ (Figure 3).

![Figure 3. Subdivision of Spec V into two spectra of valuation rings](image)

**Proposition 6.4.1.** (1) The ring $V_p$ is a valuation ring for $K = \text{Frac}(V)$, and $V/p$ is a valuation ring for the residue field at $p$.

(2) The value group of $V/p$ is the isolated subgroup $\Delta$ corresponding to $p$ under the correspondence in 6.2.9, and the value group of $V_p$ is $\Gamma_V / \Delta$.

The proof is easy; see [109], Chapter VI, §10 and [100], Proposition 4.1. In this situation, one recovers $V$ by the formula

$$V = \{x \in V_p: (x \mod pV_p) \in V/p\}.$$ 

Note that $pV_p = p$ (shown easily using 6.2.1 (b)).

**Proposition 6.4.2.** If $\tilde{V}$ is a valuation ring and $W$ is a valuation ring for the residue field of $\tilde{V}$, then the subring $V$ of $\tilde{V}$ consisting of elements $x$ such that $(x \mod m_{\tilde{V}}) \in W$ is a valuation ring. Moreover, $p = m_{\tilde{V}}$ is the prime ideal of $V$ such that $\tilde{V} \simeq V_p$ and $W \simeq V/p$. 
The proof is easy and left to the reader. The valuation ring $V$ in this situation is said to be the composite of the valuation rings $\tilde{V}$ and $W$. Schematically, composition amounts to gluing two segments $\text{Spec } V_p$ and $\text{Spec } V/\mathfrak{p}$ at their endpoints to make a new segment $\text{Spec } V$. The following proposition is clear.

**Proposition 6.4.3.** If the valuation ring $V$ is the composite of $\tilde{V}$ and $W$, then we have $\text{ht}(V) = \text{ht}(\tilde{V}) + \text{ht}(W)$ and $\text{rat-rank}(V) = \text{rat-rank}(\tilde{V}) + \text{rat-rank}(W)$.

Consistently, we have the following result.

**Proposition 6.4.4 ([109], Chapter VI, Theorem 17).** If a valuation ring $V$ is the composite of two valuation rings $\tilde{V}$ and $W$ as in 6.4.2, then there exists a canonical exact sequence

$$0 \longrightarrow \Gamma_W \longrightarrow \Gamma_V \longrightarrow \Gamma_{\tilde{V}} \longrightarrow 0$$

of totally ordered commutative groups.

**Proof.** We may suppose $\tilde{V} = V_p$ and $W = V/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of $V$. Since $\Gamma_V = K^\times/V^\times$ and $\Gamma_{\tilde{V}} = K^\times/V_p^\times$ (where $K = \text{Frac}(V)$), we have the canonical surjection $\Gamma_V \rightarrow \Gamma_{\tilde{V}}$. Hence it is enough to show that the kernel $V_p^\times/V^\times$ is isomorphic to $k^\times/W^\times$, where $k$ is the residue field of $V_p$. But this is clear, since $V = \{x \in V_\mathfrak{p} : (x \mod \mathfrak{p}V_\mathfrak{p}) \in W\}$. \hfill \Box

**Proposition 6.4.5.** Let $V$ and $W$ be valuation rings, and $f : V \rightarrow W$ be a ring homomorphism.

(1) The induced map $\text{Spec } f : \text{Spec } W \rightarrow \text{Spec } V$ maps $\text{Spec } W$ surjectively onto the set $[\ker(f), f^{-1}(m_W)]$ preserving the ordering by inclusion.

(2) $f$ is local if and only if $\text{Spec } f$ maps $\text{Spec } W$ onto the set $[\ker(f), m_V]$.

(3) $f$ is injective if and only if $\text{Spec } f$ maps $\text{Spec } W$ onto $[0, f^{-1}(m_W)]$.

(4) $f$ is injective and local (that is, $W$ dominates $V$) if and only if the map $\text{Spec } f$ is surjective.

The proof uses the following easy fact: an inclusion $V \hookrightarrow W$ of valuation rings is local if and only if $W \cap \text{Frac}(V) = V$.

**Proof.** First note that (2) and (3) are immediate consequences of (1). To show (1), consider the valuation ring $V' = V/\ker(f)$ and the prime ideal $q = (f^{-1}(m_W))V'$ of $V'$. Set $V'' = V'_q$. Then the morphism $\text{Spec } V'' \rightarrow \text{Spec } V$ maps $\text{Spec } V''$ surjectively onto the subset $[\ker(f), f^{-1}(m_W)]$ of $\text{Spec } V$; since $V \rightarrow W$ induces a local injective homomorphism $V'' \rightarrow W$, (1) follows from (4).
Thus it suffices to show (4). The ‘if’ part is easy. Suppose that $V \hookrightarrow W$ is a local injective homomorphism and $p$ is a prime ideal of $V$. We need to find a prime ideal $q$ of $W$ such that $q \cap V = p$. Consider the localization $V' = V_p$, and set $W' = W \otimes_V V_p$. Since $W \subseteq W' \subseteq \text{Frac}(W)$, $W'$ is a valuation ring for $\text{Frac}(W)$. Now it suffices to show that $W'$ dominates $V'$. Since $W$ dominates $V$, we have $W \cap \text{Frac}(V) = V$. Consequently, $W' \cap \text{Frac}(V') = W' \cap \text{Frac}(V) = V'$, and thus we conclude that $W'$ dominates $V'$.

### 6.5 Center of a valuation and height estimates for Noetherian domains

Let $R$ be an integral domain, and $K = \text{Frac}(R)$. Let $v: K \to \Gamma \cup \{\infty\}$ be a valuation on $K$ (6.2.4), and $R_v = \{x \in K: v(x) \geq 0\}$ and $m_v = \{x \in K: v(x) > 0\}$ the corresponding valuation ring and the maximal ideal, respectively. Suppose $R \subseteq R_v$, that is, the valuation $v$ takes non-negative values on $R$. The prime ideal $p = m_v \cap R$ is called center of the valuation $v$ in $R$. Note that the valuation ring $R_v$ dominates the local ring $R_p$.

**Proposition 6.5.1.** Given an integral domain $R$ and a prime ideal $p$, we can always find a valuation $v$ on $K = \text{Frac}(R)$ whose center is $p$.

The proof uses Zorn’s lemma; see [27], Chapter VI, §1.2.

Let the situation be as above, and set $k = R_p/pR_p$ and $k_v = R_v/m_v$. Clearly, $k$ is a subfield of $k_v$. Set

$$\text{tr.deg}_k v = \text{tr.deg}_k k_v.$$

**Theorem 6.5.2** (Abhyankar). Let $R$ be a Noetherian local domain, $K = \text{Frac}(R)$ the fractional field, and $k = R/m_R$ the residue field. Suppose $v$ is a valuation on $K$ that dominates $R$.

1. We have the inequality

$$\text{rat-rank}(v) + \text{tr.deg}_k v \leq \dim(R),$$

where $\dim(R)$ denotes the Krull dimension of $R$.

2. If the equality $\text{rat-rank}(v) + \text{tr.deg}_k v = \dim(R)$ holds, then the value group $\Gamma$ of $v$ is isomorphic as a group to $\mathbb{Z}^d$ (for some $d \geq 0$), and $k_v$ is a finitely generated extension of $k$.

3. If, moreover, the stronger equality $\text{ht}(v) + \text{tr.deg}_k v = \dim(R)$ holds, then $\Gamma$ is isomorphic as an ordered group to $\mathbb{Z}^d$ (with $d = \text{ht}(R)$) equipped with the lexicographical order (cf. 6.1.3).

For the proof we refer to [2] and [93]; see also [100], §9. We will use the following relative version of the inequality; see [100], §5, for the proof.
6. Valuation rings

**Proposition 6.5.3.** Let $K \subseteq K'$ be an extension of fields, $v'$ a valuation on $K'$, and $V' \subseteq K'$ the valuation ring of $v'$. Consider the restriction $v = v|_K$ and the associated valuation ring $V = V' \cap K$. Let $k' = V'/\mathfrak{m}_{V'}$ and $k = V/\mathfrak{m}_V$ be the respective residue fields. Then

$$\dim \mathbb{Q}(\Gamma_{V'/\Gamma_V}) \otimes \mathbb{Q} + \text{tr.deg}_k k' \leq \text{tr.deg}_K K'.$$

If we have equality and if $K'$ is a finitely generated field over $K$, then the group $\Gamma_{V'/\Gamma_V}$ is a finitely generated $\mathbb{Z}$-module, and $k'$ is a finitely generated field over $k$.

### 6.6 Examples of valuation rings

Theorem 6.5.2 is a powerful tool for classifying possible valuations centered in a given Noetherian ring. Here we exhibit the well-known classification of valuations on an algebraic function field of dimension $\leq 2$. The classification in this case was already known to Zariski [104]. For the details and proofs, we refer to [109], Chapter VI, §15, [3], and [93].

#### 6.6. (a) Divisorial valuations.

Let $R$ be a Noetherian local domain, and $v$ a valuation on $K = \text{Frac}(R)$ centered in $R$. The valuation $v$ is said to be **divisorial** if $\text{ht}(v) = 1$ and $\text{tr.deg}_k v = \dim(R) - 1$. By 6.5.2 (3), the value group $\Gamma$ for a divisorial $v$ is isomorphic to $\mathbb{Z}$ with the usual ordering.

Divisorial valuations appear mostly in geometric context as follows. Let

$$X \longrightarrow \text{Spec } R$$

be a birational morphism, and $D$ a height-one regular point of $X$ in the closed fiber. Then $\mathcal{O}_{X,D}$ carries canonically the discrete valuation $v$, whose valuation ring coincides with $\mathcal{O}_{X,D}$ and dominates $R$. The valuation $v$ thus obtained is clearly a divisorial valuation on $K$ and, in fact, all divisorial valuations arise in this way.

Now let $R$ be a Noetherian local regular domain with the fraction field $K$, and $v$ a non-trivial valuation on $K$ dominating $R$. Other notations are as in §6.5.

#### 6.6. (b) The case $\dim(R) = 1$.

Since $v$ is non-trivial, the height of $v$ must be positive and hence is 1. As $\text{tr.deg}_k v = 0$, the valuation $v$ is divisorial and hence is discrete; $R$ is a discrete valuation ring, and $v$ is the associated valuation. Note that $v$ is characterized by the formula

$$v(f) = \max\{n \geq 0 : f \in \mathfrak{m}_R^n \}$$

for any non-zero $f \in R$. 
6.6. (c) The case dim(\(R\)) = 2. Since \(v\) is not trivial, we have \(\text{tr.deg}_k v \leq 2\).

(1) Divisorial case. Suppose \(\text{tr.deg}_k v = 1\). Since the valuation \(v\) is non-trivial, we have \(\text{rat-rank}(v) = 1\). Hence \(v\) is divisorial.

(2) Subject-to-divisorial case. If \(\text{ht}(v) = \text{rat-rank}(v) = 2\), we apply 6.5.2 (3) and deduce that the value group \(\Gamma\) is isomorphic to \(\mathbb{Z}^{\oplus 2}\) with the lexicographical order. Let \(p \subseteq R_v\) be the prime ideal lying in between \((0)\) and \(m_{R_v}\), and \(\Delta\) the corresponding isolated subgroup of \(\Gamma\). Then the valuation ring \(R_{v,p}\), with value group \(\Gamma/\Delta\), is of height one. Let \(v'\) be the valuation associated to \(R_{v,p}\). Then \(\text{tr.deg}_k v' = 1\), and hence \(v'\) is divisorial. The valuation \(v\) is the composition of \(v'\) and a valuation \(v''\) of the residue field \(k'\) of \(R_{v,p}\). Since \(\Delta \cong \mathbb{Z}\) is the value group of \(v''\), \(v''\) is also divisorial. As a result, the valuation \(v\) in this case is the composition of two divisorial valuations.

Geometrically, such a valuation comes through the following picture. Let us take \(X \to \text{Spec } R\) and \(D\) as before such that \(\mathcal{O}_{X,D}\) carries the valuation \(v'\). Let \(E \to \bar{D}\) be a birational morphism to the closed subscheme \(\bar{D}\), the closure of the point \(D\) in \(X\), and \(x\) be a regular closed point of \(E\). As \(x\) is a height-one point of \(E\), we can consider the natural discrete valuation \(v''\) on \(\mathcal{O}_{E,x}\) and the composition \(v\) with \(v'\).

In fact, it is a general feature of valuations having lexicographically ordered group \(\mathbb{Z}^d\) as value groups to have such an inductive structure: they are obtained by successive composition of divisorial valuations.

(3) Irrational case. If \(\text{rat-rank}(v) = 2\) and \(\text{ht}(v) = 1\), the value group \(\Gamma\) is a subgroup of \(\mathbb{R}\) of the form \(\mathbb{Z} + \mathbb{Z}\tau\), where \(\tau\) is an irrational number.

(4) Limit case. If \(\text{rat-rank}(v) = \text{ht}(v) = 1\) and \(\text{tr.deg}_k v = 0\), then the value group is a subgroup of \(\mathbb{Q}\).

For the actual construction of the valuations of type (3) and (4), see [104], §6.

**Remark 6.6.1.** We will find in II.11.1. (c) below a similar list of valuations when classifying points of the unit disk in rigid geometry; see II.11.1.4.

### 6.7 a-adically separated valuation rings

**Proposition 6.7.1.** Let \(V\) be a valuation ring, and \(a \in m_V \setminus \{0\}\). Then the ideal \(J = \bigcap_{n \geq 1} (a^n)\) is a prime ideal of \(V\).

**Proof.** Since \(J \neq V\), it suffices to show that \(bc \in J\) and \(c \not\in J\) imply \(b \in J\). The assumption says that \(bc/a^n \in V\) for any \(n \geq 1\) and that \(a^m/c \in m_V\) for some \(m \geq 1\). If \(b \not\in J\), then \(a^n/b \in m_V\) for some \(n \geq 1\), and hence \(a^{n+m}/bc \in m_V\), which is absurd. \(\square\)
This proposition implies that $V/\bigcap_{n \geq 1}(a^n)$, the associated $a$-adically separated ring, is again a valuation ring (6.4.1 (1)).

**Proposition 6.7.2.** Let $V$ be a valuation ring of arbitrary height, and $a \in \mathfrak{m}_V \setminus \{0\}$. Then the following conditions are equivalent.

(a) $V$ is $a$-adically separated, that is, $\bigcap_{n \geq 1}(a^n) = 0$.

(b) $V[\frac{1}{a}]$ is a field (hence is the fraction field of $V$).

**Proof.** First let us show (a) $\implies$ (b). Let $K$ be the fraction field of $V$, and take $x \in K \setminus V$. We have $x^{-1} \in \mathfrak{m}_V$. Suppose $x^{-1}$ does not divide $a^n$ for any $n \geq 1$. Then all $a^n$ divide $x^{-1}$, which would imply the absurd $x^{-1} = 0$. Hence, there exists $n \geq 1$ such that $(x^{-1}) \supseteq (a^n)$, thereby $xa^n \in V$.

Next, we show (b) $\implies$ (a). Set $J = \bigcap_{n \geq 1}(a^n)$. Suppose there exists a non-zero $b \in J$. The element $b^{-1}$ can be written as $c/a^m$ for some $c \in V$ and $m \geq 0$. Since $b \in J$, there exists $d \in V$ such that $b = a^{m+1}d$. Consequently, $a^m = bc = a^{m+1}cd$, which implies $acd = 1$. But this contradicts $a \in \mathfrak{m}_V$. \qed

An $a$-adically separated valuation ring has the following remarkable property, which will be of particular importance in our later discussion.

**Proposition 6.7.3.** Suppose $V$ is an $a$-adically separated valuation ring for $a \in \mathfrak{m}_V \setminus \{0\}$. Then $V$ has a unique height-one prime ideal; explicitly, it is $\mathfrak{p} = \sqrt{(a)}$.

**Proof.** By 6.2.3 $\mathfrak{p} = \sqrt{(a)}$ is a prime ideal. To see it is the unique height-one prime ideal, it suffices to show that it is actually the ‘minimum’ among all the non-zero prime ideals of $V$. For this it in turn suffices to show that for any non-zero $b \in V$ there exists $n \geq 1$ such that $a^n \in (b)$. Otherwise, we have $a^n/b \notin V$ and hence $b \in (a^n)$ for any $n$. But, since $V$ is $a$-adically separated, this means that $b = 0$, which is absurd. \qed

By the proposition one can depict the spectrum of an $a$-adically separated valuation ring as in Figure 4. As the figure shows, if $V$ is $a$-adically separated, then the generic point $(0)$ has the ‘adjacent’ specialization $\mathfrak{p} = \sqrt{(a)}$. In particular, any $a$-adically separated valuation ring $V$ is the composite of the height-one valuation ring $V_\mathfrak{p}$ and the valuation ring $V/\mathfrak{p}$.

![Figure 4. Spec V for an a-adically separated valuation ring V](image-url)
Definition 6.7.4. Let $V$ be an $a$-adically separated valuation ring for $a \in \mathfrak{m}_V \setminus \{0\}$. We call the prime ideal $p = \sqrt{(a)}$ the associated height-one prime ideal of $V$ (or of the pair $(V, a)$).

Proposition 6.7.5. Let $V$ be an $a$-adically separated valuation ring for $a \in \mathfrak{m}_V \setminus \{0\}$. Then every non-zero prime ideal of $V$ is open with respect to the $a$-adic topology.

Proof. By 6.7.3, every non-zero prime ideal $q$ of $V$ contains the associated height-one prime ideal $p = \sqrt{(a)}$, which contains $a$.

Proposition 6.7.6. Let $f : V \to W$ be a ring homomorphism between valuation rings, and $a \in \mathfrak{m}_V \setminus \ker(f)$. Suppose that $V$ is $a$-adically separated and that $W$ is $f(a)$-adically separated. Then the map $f$ is injective, and the map

$$\text{Spec } f : \text{Spec } W \longrightarrow \text{Spec } V$$

maps the set of all $a$-adically open prime ideals surjectively onto the set $[\sqrt{(a)}, f^{-1}(\mathfrak{m}_W)]$. If, moreover, $f$ is local (that is, $W$ dominates $V$), then it maps the set of all non-zero prime ideals surjectively onto the set of all non-zero prime ideals.

Proof. The map $f$ induces $V\left[\frac{1}{a}\right] \to W\left[\frac{1}{f(a)}\right]$. By the assumption, this is a non-zero map between the fraction fields (6.7.2) and hence is injective. It follows that $f$ is injective. If we show that $\sqrt{f(a)} \cap V = \sqrt{(a)}$, then the other assertions follow from 6.4.5 (3) and 6.4.5 (4). It suffices to check that $\sqrt{f(a)} \cap V \subseteq \sqrt{(a)}$. Let $b \in \sqrt{f(a)} \cap V$ and suppose $b \neq 0$; there exists $n \geq 1$ such that $b^n = ac$ with $c \in W$. Since $c = b^n/a \in \text{Frac}(V)$, $c$ or $c^{-1}$ lies in $V$. If $c \in V$, then $b \in \sqrt{(a)}$. Suppose $c^{-1} \in V$. Then $a = b^n c^{-1} \in \sqrt{(a)}$; since $\sqrt{(a)}$ is a prime ideal, $b^n$ or $c^{-1}$ lies in $\sqrt{(a)}$. If $c^{-1} \in \sqrt{(a)}$, then $c^{-1} \in \sqrt{f(a)} \subseteq \mathfrak{m}_W$, which is absurd. Hence $b^n \in \sqrt{(a)}$, that is, $b \in \sqrt{(a)}$.

Exercises

Exercise 0.6.1. Let $A$ be a ring and $v : A \to \Gamma \cup \{\infty\}$ a valuation on $A$ with values in a totally ordered commutative group $\Gamma$. Show that for $x, y \in A$ with $v(x) \neq v(y)$ we have $v(x + y) = \inf\{v(x), v(y)\}$.

Exercise 0.6.2. Let $V$ be a valuation ring, and $K = \text{Frac}(A)$.

1. A $V$-module $M$ is flat if and only if it is torsion free.
2. If $Z \subseteq \mathbb{P}^1_K$ is finite over $K$, then the closure $Z'$ in $\mathbb{P}^1_V$ is finite flat over $V$.
3. For any finitely generated flat $V$-algebra $A$ such that $A \otimes_V K$ is finite over $K$, $A$ is quasi-finite and finitely presented over $V$. 
Exercise 0.6.3. Let $V$ be an $a$-adically separated valuation ring for $a \in \mathfrak{m}_V \setminus \{0\}$. Show that a $V$-module $M$ is flat if and only if $M$ is $a$-torsion free, that is, for any $x \in M \setminus \{0\}$ and any $n \geq 0$ we have $a^n x \neq 0$.

Exercise 0.6.4. Let $V$ be a valuation ring such that $0 < \text{ht}(V) < +\infty$.

1. Show that there exists $a \in \mathfrak{m}_V \setminus \{0\}$ such that $V$ is $a$-adically separated.

2. Show that for any non-maximal prime ideal $p \subseteq V$, there exists $b \in V$ such that $p = \bigcap_{n \geq 0} b^n V$.

Exercise 0.6.5. Show that if $V$ is a valuation ring of height one, then $V$ is $a$-adically separated for any $a \in \mathfrak{m}_V \setminus \{0\}$.

7 Topological rings and modules

In this section we give a quick but reasonably detailed overview of the theory of linearly topologized rings and modules. First in §7.1 we recall the general definitions and treatments of linearly topologized rings and modules. In 7.1.(c), we discuss Hausdorff completions.

From §7.2 onward, we will be mainly interested in the so-called adic topologies. One of the most important topics in this subsection is the notion of $I$-adic completion which is at first defined by a ring-theoretic mapping universality. The existence of the $I$-adic completion is a delicate matter, especially when dealing with non-Noetherian rings. Indeed, the $I$-adic completion exists under a mild condition, but in general, as we will see below, it may fail to exist.

After briefly discussing Henselian and Zariskian rings in §7.3, we proceed in §7.4 to one of the most delicate issues, the preservation of adicness on passage to subspace topologies. Classically, this is guaranteed in the Noetherian case by the well-known Artin–Rees lemma. As it will be necessary for us to treat non-Noetherian rings, we consider analogous conditions, slightly weaker than the Artin–Rees condition, and prove many useful results.

7.1 Topology defined by a filtration

7.1.(a) Filtrations. Let $A$ be a ring and $M$ an $A$-module. We consider a descending filtration by $A$-submodules $F^* = \{F^\lambda\}_{\lambda \in \Lambda}$, that is, a collection of $A$-submodules $F^\lambda \subseteq M$ indexed by a directed set $\Lambda$, such that for $\lambda, \mu \in \Lambda$

$$\lambda \geq \mu \implies F^\lambda \subseteq F^\mu.$$ 

Such an $F^* = \{F^\lambda\}_{\lambda \in \Lambda}$ is said to be separated (resp. exhaustive) if

$$\bigcap_{\lambda \in \Lambda} F^\lambda = \{0\} \quad (\text{resp. } \bigcup_{\lambda \in \Lambda} F^\lambda = M).$$
Let \( f : M \to L \) be a morphism of \( A \)-modules, and suppose that \( M \) is endowed with a descending filtration by \( A \)-submodules \( F^\bullet = \{F^\lambda\}_{\lambda \in \Lambda} \). Then one has the descending filtration by \( A \)-submodules on \( L \) given by

\[
f(F^\bullet) = \{f(F^\lambda)\}_{\lambda \in \Lambda},
\]

which we call the induced filtration. If \( L \) is of the form \( L = M/N \) for an \( A \)-submodule \( N \subseteq M \) and \( f \) is the canonical projection, then we have \( f(F^\bullet) = \{(N + F^\lambda)/N\}_{\lambda \in \Lambda} \).

Similarly, for a morphism \( g : N \to M \) of \( A \)-modules, where \( M \) is endowed with a descending filtration by \( A \)-submodules as above, one has the descending filtration by \( A \)-submodules on \( N \) given by \( g^{-1}(F^\bullet) = \{g^{-1}(F^\lambda)\}_{\lambda \in \Lambda} \), which we also call the induced filtration. If \( N \) is an \( A \)-submodule of \( M \) and \( g \) is the canonical inclusion, then we have \( g^{-1}(F^\bullet) = \{N \cap F^\lambda\}_{\lambda \in \Lambda} \).

7.1. (b) Topology defined by a filtration

**Proposition 7.1.1** (cf. [24], Chapter III, §1.2). Let \( A \) be a ring and \( M \) an \( A \)-module endowed with a descending filtration by \( A \)-submodules \( F^\bullet = \{F^\lambda\}_{\lambda \in \Lambda} \) indexed by a directed set \( \Lambda \). Then there exists a unique topology on \( M \) satisfying the following conditions:

(a) the topology is compatible with the additive group structure, that is, the addition \( M \times M \to M \) is continuous;

(b) \( \{F^\lambda\}_{\lambda \in \Lambda} \) gives a fundamental system of open neighborhoods of \( 0 \in M \).

Moreover, for any \( a \in A \) the self-map \( x \mapsto ax \) of \( M \) is a continuous endomorphism with respect to this topology.

The topology on \( M \) characterized as in 7.1.1 is called the topology defined by the filtration \( F^\bullet \). It is explicitly described as follows. A subset \( U \subseteq M \) is open if for any \( x \in U \) there exists \( \lambda \in \Lambda \) such that \( x + F^\lambda \subseteq U \). In particular, an \( A \)-submodule \( N \subseteq M \) is open if and only if it contains some \( F^\lambda \). In more formal terms, the topology thus obtained is the one defined by the uniform structure (cf. [24], Chapter II, §1.2) having \( \{\tilde{F}^\lambda\}_{\lambda \in \Lambda} \), where

\[
\tilde{F}^\lambda = \{(x, y) \in M \times M : x - y \in F^\lambda\}
\]

for \( \lambda \in \Lambda \), as its fundamental system of entourages (cf. [24], Chapter III, §3.1); note that since \( M \) is a commutative group by addition, the left and right uniformities coincide.

The following facts are easy to verify, and we leave the proofs to the reader.
Proposition 7.1.2. Let \( M \) be an \( A \)-module endowed with a descending filtration by \( A \)-submodules \( F^\bullet = \{ F^\lambda \}_{\lambda \in \Lambda} \) indexed by a directed set \( \Lambda \).

1. Let \( f : M \to L \) be a morphism of \( A \)-modules. Then \( f \) is continuous with respect to the topology on \( M \) defined by \( F^\bullet \) and the topology on \( L \) defined by \( f(F^\bullet) \) (§7.1.(a)). If, moreover, \( f \) is surjective, then the topology on \( L \) coincides with the quotient topology, that is, the strongest topology on \( L \) such that the map \( f \) is continuous.

2. Let \( g : N \to M \) be a morphism of \( A \)-modules. Then \( g \) is continuous with respect to the topology on \( N \) defined by \( g^{-1}(F^\bullet) \) (§7.1.(a)) and the topology on \( M \) defined by \( F^\bullet \). If, moreover, \( g \) is injective, then the topology on \( N \) coincides with the subspace topology, that is, the weakest topology on \( N \) such that the map \( g \) is continuous.

Proposition 7.1.3. Let \( f : M \to N \) be a morphism of \( A \)-modules, and consider descending filtrations \( \{ F^\lambda \}_{\lambda \in \Lambda} \) and \( \{ G^\sigma \}_{\sigma \in \Sigma} \) by \( A \)-submodules on \( M \) and \( N \), respectively. We equip \( M \) and \( N \) with the topologies defined by these filtrations.

1. The map \( f \) is continuous if and only if for any \( F^\lambda \) there exists \( G^\sigma \) such that \( f(F^\lambda) \subseteq G^\sigma \).

2. The map \( f \) is an open map if and only if for any \( G^\sigma \) there exists \( F^\lambda \) such that \( G^\sigma \subseteq f(F^\lambda) \).

Corollary 7.1.4. Let \( M \) be an \( A \)-module, and consider two descending filtrations by \( A \)-submodules \( \{ F^\lambda \}_{\lambda \in \Lambda} \) and \( \{ G^\sigma \}_{\sigma \in \Sigma} \). Then these filtrations define the same topology on \( M \) if and only if any \( F^\lambda \) contains some \( G^\sigma \) and any \( G^\sigma \) contains some \( F^\lambda \).

Proposition 7.1.5. Let \( M \) be an \( A \)-module endowed with a descending filtration by \( A \)-submodules \( F^\bullet = \{ F^\lambda \}_{\lambda \in \Lambda} \) indexed by a directed set \( \Lambda \). Then the topology on \( M \) defined by \( F^\bullet \) is Hausdorff if and only if the filtration \( F^\bullet \) is separated (§7.1.(a)).

Let \( M \) be a module over a ring \( A \) endowed with a descending filtration of \( A \)-submodules \( F^\bullet = \{ F^\lambda \}_{\lambda \in \Lambda} \). We leave the proof of the following (easy) proposition to the reader as an exercise (Exercise 0.7.1).

Proposition 7.1.6. Let \( N \subseteq M \) be an \( A \)-submodule. Then the closure \( \bar{N} \) of \( N \) in \( M \) with respect to the topology defined by the filtration \( F^\bullet \) is given by

\[
\bar{N} = \bigcap_{\lambda \in \Lambda} (N + F^\lambda).
\]
7.1. (c) **Hausdorff completion.** Let us first briefly recall some generalities on uniform spaces ([24], Chapter II). A uniform space $X$ is said to be Hausdorff complete if the topology is Hausdorff and any Cauchy filter on $X$ is a convergent filter. It is known ([24], Chapter II, §3.7) that any uniform space has the Hausdorff completion $\hat{X}$ and the canonical uniformly continuous mapping $i_X: X \to \hat{X}$ in such a way that the pair $(\hat{X}, i_X)$ is uniquely characterized, up to canonical isomorphisms, by a universal property with respect to uniformly continuous mappings to Hausdorff complete uniform spaces. As a set, $\hat{X}$ is the set of all minimal Cauchy filters on $X$. In particular, $X$ is Hausdorff complete if and only if the canonical map $i_X$ is an isomorphism of uniform spaces.

Let $A$ be a ring and $M$ an $A$-module equipped with a descending filtration by $A$-submodules $F^* = \{F^\lambda\}_{\lambda \in \Lambda}$ indexed by a directed set $\Lambda$. Since the topology on $M$ defined by the filtration $F^*$ is a uniform topology, one can consider the Hausdorff completion, which we denote by $\hat{M}^\wedge_{\bullet}$.

The Hausdorff completion $\hat{M}^\wedge_{\bullet}$ is canonically a commutative group ([24], Chapter III, §3.5, Theorem 2); moreover, since any continuous additive endomorphisms of $M$, which is automatically uniformly continuous, lifts uniquely to one of $\hat{M}^\wedge_{\bullet}$, one sees that $\hat{M}^\wedge_{\bullet}$ is an $A$-module and that the canonical map $i_M: M \to \hat{M}^\wedge_{\bullet}$ is an $A$-module homomorphism. Note that the notion of Hausdorff completeness depends only on the topologies induced from the uniform structures, and hence the Hausdorff completion $\hat{M}^\wedge_{\bullet}$ depends, up to isomorphisms, only on the topologies defined by filtrations.

By [24], Chapter III, §7.3, Corollary 2, the Hausdorff completion $\hat{M}^\wedge_{\bullet}$ is canonically identified (as an $A$-module) with the filtered projective limit

$$\lim_{\lambda \in \Lambda} \frac{M}{F^\lambda}$$

and the canonical map $i_M$ with the map induced by the canonical projections $M \to \frac{M}{F^\lambda}$ for $\lambda \in \Lambda$; the uniform structure on the projective limit $\lim_{\lambda \in \Lambda} \frac{M}{F^\lambda}$ is the one induced by the descending filtration $\hat{F}^* = \{\hat{F}^\lambda\}_{\lambda \in \Lambda}$ (called the induced filtration) given by $\hat{F}^\lambda = \ker(M^\wedge_{\bullet} \to \frac{M}{F^\lambda})$ for each $\lambda \in \Lambda$, where $M^\wedge_{\bullet} = \lim_{\lambda \in \Lambda} \frac{M}{F^\lambda} \to \frac{M}{F^\lambda}$ is the canonical projection map. In other words, it is the uniform structure by which the induced topology is the projective limit topology, where each $\frac{M}{F^\lambda}$ is considered with the discrete topology; more briefly, the topology on $\hat{M}^\wedge_{\bullet}$ is the weakest one such that all projection maps $M^\wedge_{\bullet} \to \frac{M}{F^\lambda}$ are continuous. (Note that since the composition $M \xrightarrow{i_M} \hat{M}^\wedge_{\bullet} \to \frac{M}{F^\lambda}$ is surjective, the canonical projection $M^\wedge_{\bullet} \to \frac{M}{F^\lambda}$ is surjective.)
Note that, since \( M^F = yF \), we have \( i_M(M) + \hat{F}^\lambda = M^\wedge_\bullet \) for any \( \lambda \in \Lambda \). In particular, in view of 7.1.6, the image \( i_M(M) \) is dense in \( M^\wedge_\bullet \). More generally, we have \( \hat{F}^\lambda / \hat{F}^\mu \cong F^\lambda / F^\mu \) for \( \lambda \leq \mu \), and hence \( i_M(F^\lambda) + \hat{F}^\mu = \hat{F}^\lambda \); in particular, \( \hat{F}^\lambda \) coincides with the closure of \( i_M(F^\lambda) \) in \( M^\wedge_\bullet \):

\[
\hat{F}^\lambda = i_M(F^\lambda).
\]

**Proposition 7.1.7.** For any \( \lambda \in \Lambda \) the submodule \( \hat{F}^\lambda \) is the closure of \( i_M(F^\lambda) \) in \( M^\wedge_\bullet \) and coincides, up to canonical isomorphisms, with the Hausdorff completion of \( F^\lambda \) with respect to the filtration induced by \( F^\bullet \).

**Proof.** The first statement has been already shown above. For \( \mu \geq \lambda \) we have the exact sequence

\[
0 \longrightarrow F^\lambda / F^\mu \longrightarrow M / F^\mu \longrightarrow M / F^\lambda \longrightarrow 0.
\]

Since the subset \( \{ \mu \in \Lambda : \mu \geq \lambda \} \) of \( \Lambda \) is cofinal, applying the projective limits \( \lim_{\leftarrow \mu} \), we have the exact sequence

\[
0 \longrightarrow \lim_{\leftarrow \mu} F^\lambda / F^\mu \longrightarrow M^\wedge_\bullet \longrightarrow M / F^\lambda \longrightarrow 0;
\]

note that the passage to projective limits is a left-exact functor and that the canonical projection \( M^\wedge_\bullet \to M / F^\lambda \) is surjective. This shows that \( \hat{F}^\lambda \) coincides with \( \lim_{\leftarrow \mu} F^\lambda / F^\mu \), which is nothing but the Hausdorff completion of \( F^\lambda \) with respect to the induced filtration. \( \square \)

**Proposition 7.1.8.** Let \( M \) be an \( A \)-module endowed with a descending filtration by \( A \)-submodules \( F^\bullet = \{ F^\lambda \}_{\lambda \in \Lambda} \) indexed by a directed set \( \Lambda \), \( M^\wedge_\bullet \) the associated Hausdorff completion, and \( \hat{F}^\bullet = \{ \hat{F}^\lambda \}_{\lambda \in \Lambda} \) the induced filtration on \( M^\wedge_\bullet \).

(1) For any \( \lambda \in \Lambda \) we have \( F^\lambda = i_M^{-1}(\hat{F}^\lambda) \), that is, the induced filtration \( i_M^{-1}(\hat{F}^\bullet) \) coincides with the original one \( F^\bullet \).

(2) The \( A \)-module \( M^\wedge_\bullet \) is Hausdorff complete with respect to the topology defined by the induced filtration \( \hat{F}^\bullet \).

**Proof.** Since \( M^\wedge_\bullet / \hat{F}^\lambda = M / F^\lambda \) for \( \lambda \in \Lambda \), we have

\[
\lim_{\leftarrow \lambda \in \Lambda} M^\wedge_\bullet / \hat{F}^\lambda = \lim_{\leftarrow \lambda \in \Lambda} M / F^\lambda = M^\wedge_\bullet,
\]

which shows (2).
(1) follows easily from the commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{i} & M_{\mathcal{F}}^* \\
\downarrow & & \downarrow \\
M/F^\lambda & \xrightarrow{} & M_{\mathcal{F}}^*/F^\lambda,
\end{array}
\]

for any \( \lambda \in \Lambda \), where the vertical arrows are the canonical projections. \( \square \)

One can easily show the following mapping universality by using [24], Chapter II, \S 3.7, Theorem 3, aided by the fact that a product of Hausdorff complete uniform spaces is Hausdorff complete ([24], Chapter II, \S 3.5, Proposition 10) and the fact that the addition \((x,y) \mapsto x+y\), the inversion \(x \mapsto -x\), and the scalar multiplication \(x \mapsto ax \) (for \( a \in A \)) are uniformly continuous.

**Proposition 7.1.9** (mapping universality of Hausdorff completions). Let \( M \) be an \( A \)-module equipped with a descending filtration \( \mathcal{F}^* = \{F^\lambda\}_{\lambda \in \Lambda} \) indexed by a directed set \( \Lambda \), and consider the canonical map \( i_M: M \to M_{\mathcal{F}}^* \). Let \( N \) be another \( A \)-module, Hausdorff complete with respect to a descending filtration by \( A \)-submodules. Then for any continuous \( A \)-module homomorphism \( M \to N \), there exists a unique continuous \( A \)-module homomorphism \( M_{\mathcal{F}}^* \to N \) such that the resulting diagram

\[
\begin{array}{ccc}
M_{\mathcal{F}}^* & \xrightarrow{i_M} & N \\
\downarrow & & \downarrow \\
M & \xrightarrow{} & N
\end{array}
\]

commutes.

**Remark 7.1.10.** (1) An \( A \)-module \( M \) linearly topologized by a descending filtration \( \mathcal{F}^* = \{F^\lambda\}_{\lambda \in \Lambda} \) by \( A \)-submodules is separated if and only if the canonical map \( i_M: M \to M_{\mathcal{F}}^* \) is injective.

(2) If a ring \( A \) is linearly topologized by a descending filtration \( \mathcal{F}^* = \{F^\lambda\}_{\lambda \in \Lambda} \) by ideals, then the Hausdorff completion \( A_{\mathcal{F}}^* \) is canonically a ring, and the canonical map \( i_A: A \to A_{\mathcal{F}}^* \) is a ring homomorphism.

\[7.1. (d)\] **Hausdorff completion and exact sequences.** Let us consider an exact sequence

\[
0 \to N \xrightarrow{g} M \xrightarrow{f} L \to 0
\]

of \( A \)-modules and a descending filtration \( \mathcal{F}^* = \{F^\lambda\}_{\lambda \in \Lambda} \) by \( A \)-submodules of \( M \). We consider the induced filtrations on \( N \) and \( L \) (cf. \S 7.1. (a)); the one on \( N \) is given by \( g^{-1}(\mathcal{F}^*) = \{N \cap F^\lambda\}_{\lambda \in \Lambda} \) (where we regard \( N \) as an \( A \)-submodule of \( M \)), and
the one on $L$ is $f(F^*) = \{(N + F^\lambda)/N\}_{\lambda \in \Lambda}$ (where $L$ is identified with $M/N$). For each $\lambda \in \Lambda$ we have the induced exact sequence

$$0 \rightarrow N/N \cap F^\lambda \rightarrow M/F^\lambda \rightarrow M/(N + F^\lambda) \rightarrow 0.$$ 

Now we suppose that

- the directed set $\Lambda$ has a cofinal and at most countable subset.

Then by 3.2.8 the induced sequence of Hausdorff completions

$$0 \rightarrow N\wedge_{g-1(F^*)} \rightarrow M\wedge_{F^*} \rightarrow L\wedge_{f(F^*)} \rightarrow 0$$

is exact.

To proceed, let us set for brevity $G^\lambda = g^{-1}(F^\lambda) = N \cap F^\lambda$ and $E^\lambda = f(F^\lambda)$ for $\lambda \in \Lambda$. Let $\hat{G}^\bullet$ (resp. $\hat{F}^\bullet$, resp. $\hat{E}^\bullet$) be the induced filtration on the Hausdorff completion $N\wedge_{G^\bullet}$ (resp. $M\wedge_{F^\bullet}$, resp. $L\wedge_{E^\bullet}$) (cf. §7.1 (c)).

**Proposition 7.1.11.** The filtration $\hat{G}^\bullet$ (resp. $\hat{F}^\bullet$, resp. $\hat{E}^\bullet$) coincides with the one induced from $\hat{F}^\bullet$ by the map $N\wedge_{G^\bullet} \rightarrow M\wedge_{F^\bullet}$ (resp. $M\wedge_{F^\bullet} \rightarrow L\wedge_{E^\bullet}$) as in §7.1 (a).

**Proof.** For each $\lambda \in \Lambda$ we have the commutative diagram with exact rows

$$0 \rightarrow N\wedge_{G^\lambda} \rightarrow M\wedge_{F^\lambda} \rightarrow L\wedge_{E^\lambda} \rightarrow 0.$$

The assertion on the filtration $\hat{G}^\lambda$ follows easily. One can also show the other assertion easily by diagram chasing, using the fact that the left-hand vertical arrow is surjective (3.2.2).

**Proposition 7.1.12.** (1) The image of $N\wedge_{G^\bullet} \rightarrow M\wedge_{F^\bullet}$ coincides with the closure of the image of $N$ by the canonical map $i_M: M \rightarrow M\wedge_{F^\bullet}$.

(2) The closure $\bar{N}$ of $N$ in $M$ coincides with the pull-back of $N\wedge_{G^\bullet}$ by the canonical map $i_M: M \rightarrow M\wedge_{F^\bullet}$.

**Proof.** (1) Consider the $A$-submodule $i_M(N) + \hat{F}^\lambda$ for each $\lambda \in \Lambda$. Observe that $i_M(N) + \hat{F}^\lambda$ is the pull-back of the image of $N/N \cap F^\lambda$ by the canonical projection $M\wedge_{F^\bullet} \rightarrow M/F^\lambda$. Therefore, we have the exact sequence

$$0 \rightarrow i_M(N) + \hat{F}^\lambda \rightarrow M\wedge_{F^\bullet} \rightarrow L/f(F^\lambda) \rightarrow 0.$$ 

Applying the projective limits $\lim_{\leftarrow \lambda \in \Lambda} (3.2.4)$, we have the exact sequence

$$0 \rightarrow \bigcap_{\lambda \in \Lambda} (i_M(N) + \hat{F}^\lambda) \rightarrow M\wedge_{F^\bullet} \rightarrow L\wedge_{E^\bullet} \rightarrow 0.$$ 

Now the assertion follows from 7.1.6.
(2) By 7.1.8 (1), one sees easily that the equality
\[ i^{-1}_M(i_M(N) + \hat{F}^\lambda) = N + F^\lambda \]
holds for any \( \lambda \in \Lambda \). Now the assertion follows from (1) and 7.1.6. \( \square \)

7.1. (e) Completeness of sub and quotient modules. Consider as before an exact sequence
\[ 0 \longrightarrow N \xrightarrow{g} M \xrightarrow{f} L \longrightarrow 0 \]
of \( A \)-modules, and a descending filtration \( F^\bullet = \{ F^\lambda \}_{\lambda \in \Lambda} \) by \( A \)-submodules of \( M \), which yields the induced filtrations \( G^\bullet = N \cap F^\bullet \) and \( E^\bullet = f(F^\bullet) \) on \( N \) and \( L \), respectively. Furthermore, we continue with the assumption that the directed set \( \Lambda \) contains a cofinal and at most countable subset.

**Proposition 7.1.13.** Suppose \( M \) is Hausdorff complete with respect to the topology defined by the filtration \( F^\bullet \). Then the following conditions are equivalent:

(a) \( N \) is closed in \( M \) with respect to the topology defined by \( F^\bullet \);

(b) \( N \) is Hausdorff complete with respect to the topology defined by \( G^\bullet \);

(c) \( L \) is Hausdorff complete with respect to the topology defined by \( E^\bullet \).

**Proof.** Since \( M \) is Hausdorff complete, the exact sequence
\[ 0 \longrightarrow N \xrightarrow{g} M \xrightarrow{f} L \longrightarrow 0. \]
Suppose \( N \) is closed in \( M \). Since the induced filtration \( \hat{F}^\bullet \), defined as in §7.1.(c), coincides with \( F^\bullet \), we deduce by 7.1.12 (1) that \( N = N_{G^\bullet}^\wedge \), which shows that \( N \) is Hausdorff complete with respect to the topology defined by \( G^\bullet \). The above exact sequence now shows that \( L = L_{E^\bullet}^\wedge \), that is, \( L \) is Hausdorff complete with respect to the topology defined by \( E^\bullet \). Finally, if \( L \) is complete with respect to the topology defined by \( E^\bullet \), then again by the above exact sequence we know that \( N = N_{G^\bullet}^\wedge \), which is closed in \( M \) by 7.1.12 (1). \( \square \)

7.2 Adic topology

7.2. (a) Adic filtration and adic topology. Let \( A \) be a ring, \( I \subseteq A \) an ideal, and \( M \) an \( A \)-module. One has the descending filtration on \( M \) by \( A \)-submodules given by
\[ I^\bullet M = \{ I^nM \}_{n \geq 0}. \]
We call this filtration the \( I \)-adic filtration on \( M \). The \( I \)-adic filtration \( \{ I^n \}_{n \geq 0} \) in the case \( M = A \) is simply denoted by \( I^\bullet \).
Let $A$ be a ring equipped with the $I$-adic filtration by an ideal $I \subseteq A$. Then an ideal $J \subseteq A$ is called an ideal of definition if there exist positive integers $m, n > 0$ such that

$$I^m \subseteq J^n \subseteq I.$$  

The following proposition is an immediate consequence of 7.1.4.

**Proposition 7.2.1.** Let $A$ be a ring equipped with the $I$-adic filtration by an ideal $I \subseteq A$.

(1) Let $J \subseteq A$ be an ideal of definition of $A$. Then for any $A$-module $M$ the $I$-adic filtration $I^n M$ and the $J$-adic filtration $J^m M$ define the same topology.

(2) Let $J \subseteq A$ be an ideal. If the filtrations $J^n$ and $I^n$ define the same topology on $A$, then $J$ is an ideal of definition of $A$.

Let $A$ be a ring, and $M$ a linearly topologized $A$-module. If the topology on $M$ is the same as the one defined by the $I$-adic filtration for an ideal $I$ of $A$, we say that the topology is an adic topology; if we like to spell out the ideal $I$, we say it is the $I$-adic topology. Explicitly, a topology on $M$ defined by a descending filtration $F^\bullet = \{F^\lambda\}_{\lambda \in \Lambda}$ by $A$-submodules indexed by a directed set $\Lambda$ is $I$-adic if and only if for any $\lambda \in \Lambda$ there exists $n \geq 0$ such that $I^n M \subseteq F^\lambda$, and for any $n \geq 0$ there exists $\lambda \in \Lambda$ such that $F^\lambda \subseteq I^n M$ (7.1.4).

**Remark 7.2.2.** According to EGA terminology (cf. [54], 0, §7),

$$\text{preadic} + \text{ separated and complete} = \text{adic.}$$

The terminology ‘preadic’ is, however, not commonly used nowadays. In this book, too, we avoid this terminology and consider that ‘adic’ does not imply ‘separated and complete,’ except for adic ring, which is already customarily considered to be separated and complete with respect to an adic topology; cf. I.1.1.3 (2).

Let $A$ and $B$ be rings with adic topologies. A ring homomorphism $f: A \rightarrow B$ is said to be adic if for some ideal of definition $I$ of $A$ the ideal $IB = f(I)B$ is an ideal of definition of $B$. It is, in fact, easy to see that the condition for $f$ to be adic is equivalent $IB$ being an ideal of definition of $B$ for any ideal of definition $I$ of $A$. Note that, whereas ‘adic’ implies ‘continuous,’ the converse is not true. For example, any ring homomorphism $A \rightarrow B$ is continuous, if $A$ is equipped with the $0$-adic topology (= discrete topology), but is not adic unless $(0)$ is an ideal of definition of $B$.

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be ring homomorphisms between adically topologized rings. Then one can easily see the following:

- if $f$ and $g$ are adic, then so is the composite $g \circ f$;
- if $g \circ f$ and $f$ are adic, then so is $g$. 

Let \( A \) be a ring, \( I \subseteq A \) an ideal, and \( M \) an \( A \)-module considered with the \( I \)-adic topology. We say that

- \( M \) is \( I \)-adically separated, if \( \bigcap_{n \geq 0} I^n M = \{0\} \) or, equivalently, the canonical map \( i_M : M \to \lim_{\leftarrow n \geq 0} M/I^n M \) is injective;
- \( M \) is \( I \)-adically complete, if it is Hausdorff complete with respect to the \( I \)-adic topology, that is, the canonical map \( i_M : M \to \lim_{\leftarrow n \geq 0} M/I^n M \) is an isomorphism.

Note that what we mean by ‘complete’ with respect to \( I \)-adic topology is ‘Hausdorff complete,’ that is, in our convention ‘\( I \)-adically complete’ implies ‘\( I \)-adically separated.’

**Lemma 7.2.3.** Let \( A \) be a ring, \( I \subseteq A \) an ideal, and \( M \) an \( A \)-module considered with the \( I \)-adic topology. Suppose there exists an integer \( n \geq 1 \) such that \( I^n M = 0 \). Then \( M \) is \( I \)-adically complete.

The following proposition will be frequently used later.

**Proposition 7.2.4 (81, Theorem 8.4).** Let \( A \) be a ring, \( I \subseteq A \) an ideal, and \( M \) an \( A \)-module. Suppose that \( A \) is \( I \)-adically complete and that \( M \) is \( I \)-adically separated. Let \( \{x_1, \ldots, x_r\} \subseteq M \). If \( \{(x_1 \mod IM), \ldots, (x_r \mod IM)\} \) generates \( M/IM \) over \( A/I \), then \( \{x_1, \ldots, x_r\} \) generates \( M \) over \( A \). In particular, \( M \) is finitely generated if and only if \( M/IM \) is finitely generated over \( A/I \).

**Proposition 7.2.5.** Let \( A \) be a ring topologized by a descending filtration \( F^* = \{F^\lambda\}_{\lambda \in \Lambda} \) by ideals, and suppose it is Hausdorff complete. Let \( I \subseteq A \) be a topologically nilpotent ideal, that is, for any \( \lambda \in \Lambda \) there exists \( n \geq 0 \) such that \( I^n \subseteq F^\lambda \). Then \( A \) is \( I \)-adically complete if either one of the following conditions is satisfied.

(a) \( I^n \) is closed in \( A \) for any \( n \geq 0 \).

(b) \( I \) is finitely generated.

**Proof.** (a) We have \( I^n = \bigcap_{\lambda \in \Lambda} (I^n + F^\lambda) \) for any \( n \geq 0 \). Take for any \( \lambda \in \Lambda \) an integer \( N_\lambda \geq 0 \) such that \( I^n \subseteq F^\lambda \) for \( n \geq N_\lambda \). Then

\[
\lim_{\leftarrow n \geq 0} A/I^n = \lim_{\leftarrow n \geq 0} \lim_{\leftarrow \lambda \in \Lambda} A/(I^n + F^\lambda)
\]

\[
= \lim_{\leftarrow n, \lambda} A/(I^n + F^\lambda)
\]

\[
= \lim_{\lambda} \lim_{\leftarrow n \geq N_\lambda} A/F^\lambda = A
\]
(up to canonical isomorphisms), where the second equality is due to (dual of) Exercise 0.1.1, and the third equality is obtained by replacing the index set by a cofinal one.

(b) Clearly, \( A \) is \( I \)-adically separated. Let \( \{x_n\}_{n \geq 0} \) be a sequence in \( A \) such that for \( n \leq m \) we have \( x_n \equiv x_m \mod I^n \). There exists \( x \in A \) such that \( x \equiv x_n \mod F^\lambda \) for sufficiently large \( \lambda \) (depending on \( n \)). We want to show that \( x \) is the limit of \( \{x_n\} \) with respect to the \( I \)-adic topology. Set \( I = (a_1, \ldots, a_r) \), and let \( k \geq 1 \) be an arbitrary fixed positive integer. Let \( \xi_1, \ldots, \xi_s \) be the monomials in \( a_i \)'s of degree \( k \). Then one finds inductively \( s \) elements \( y_{l,j} \in A \) \((l \geq 0, j = 1, \ldots, s)\) such that

- \( x_{k+l} - x_k = \sum_{j=1}^{s} y_{l,j} \xi_j \);
- \( y_{l,j} \equiv y_{l',j} \mod I^l \) for \( l' \geq l \) and \( j = 1, \ldots, s \).

Since \( y_{l,j} \) for each \( j \) converges in \( A \) to an element \( y_j \), we have \( x = x_k + \sum_{j=1}^{s} y_j \xi_j \), which belongs to \( I^k \), as desired. \( \square \)

In connection to the above proposition, we remark here that there is a flaw in [54], 0, (7.2.4), which has been repaired in the Springer version [53], 0, (7.2.4).

7.2. (b) \( I \)-adic completion. Our definition of \( I \)-adic completions is given by using the universal mapping property and, a priori, presented independently from the notion of Hausdorff completions.

**Definition 7.2.6.** Let \( A \) be a ring, \( I \subseteq A \) an ideal, and \( M \) an \( A \)-module. An \( I \)-adic completion of \( M \) is an \( A \)-module \( \hat{M} \) together with an \( A \)-module homomorphism \( i_M : M \to \hat{M} \) such that the following conditions are satisfied:

(a) \( \hat{M} \) is \( I \)-adically complete;

(b) for any \( I \)-adically complete \( A \)-module \( N \) and any \( A \)-module homomorphism \( f : M \to N \), there exists uniquely an \( A \)-module homomorphism \( \hat{f} : \hat{M} \to N \) such that the resulting diagram

\[
\begin{array}{ccc}
\hat{M} & \xrightarrow{\hat{f}} & N \\
i_M \uparrow & & \downarrow f \\
M & \xrightarrow{f} & N
\end{array}
\]

commutes.

It is clear by definition that an \( I \)-adic completion of \( M \) is, if it exists, unique up to canonical isomorphisms.
As we will see in 7.2.10 below, the \( I \)-adic completion thus defined may fail to exist in general. It exists, however, under a mild condition; see 7.2.15 and 7.2.16 below.

Here the subtlety concerning the existence can be more clearly illustrated as follows. Consider the Hausdorff completion of \( M \) with respect to the \( I \)-adic topology, which we denote in what follows simply by

\[
M_{I^*} = \lim_{\leftarrow} M/I^n M.
\]

Then, as we are soon going to see,

- if \( M_{I^*} \) is \( I \)-adically complete, then this together with the canonical map \( M \to M_{I^*} \) actually gives an \( I \)-adic completion of \( M \);
- conversely, if the \( I \)-adic completion \( \hat{M} \) of \( M \) exists, then it is isomorphic to \( M_{I^*} \).

Hence the clue for the existence of the \( I \)-adic completion lies in whether or not the Hausdorff completion with respect to the \( I \)-adic topology is \( I \)-adically complete or not. The above dangerous bend says that, actually, there exists an example in which the Hausdorff completion with respect to the \( I \)-adic topology is not \( I \)-adically complete.

Suppose that the \( I \)-adic completion \( i_{M}: M \to \hat{M} \) of \( M \) exists, and consider the canonical projections

\[
p_n: M \to M/I^n M, \quad q_n: \hat{M} \to \hat{M}/I^n \hat{M},
\]

for \( n \geq 0 \). The \( A \)-module homomorphism \( i_{M}: M \to \hat{M} \) induces for each \( n \geq 0 \) a map \( i_n: M/I^n M \to \hat{M}/I^n \hat{M} \) such that the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{i_M} & \hat{M} \\
\downarrow{p_n} & & \downarrow{q_n} \\
M/I^n M & \xrightarrow{i_n} & \hat{M}/I^n \hat{M}
\end{array}
\]

commutes. Note that, since \( M/I^n M \) is \( I \)-adically complete by 7.2.3, we have the unique maps \( \hat{p}_n: \hat{M} \to M/I^n M \) such that \( \hat{p}_n \circ i_M = p_n \) for \( n \geq 0 \), which then induce by passage to the projective limits the morphism

\[
p: \hat{M} \to M_{I^*} = \lim_{\leftarrow} M/I^n M
\]

such that \( p \circ i_M = p_n \).

\[\text{\footnote{The notation } M_{I^*} \text{ is the abbreviation of } M_{I^* M}, \text{ the Hausdorff completion of } M \text{ with respect to the } I \text{-adic topology on } \hat{M}.}\]
Proposition 7.2.7. The morphism \( p \) is an isomorphism.

Since \( \hat{M} \) is \( I \)-adically complete, the canonical morphism

\[
\hat{M} \rightarrow \lim_{n \geq 0} \hat{M} / I^n \hat{M}
\]

is an isomorphism. Hence the proposition follows from the following lemma.

Lemma 7.2.8. For each \( n \geq 0 \) the map \( i_n: M / I^n M \rightarrow \hat{M} / I^n \hat{M} \) is an isomorphism.

Proof. The \( A \)-module homomorphism \( \hat{p}_n \) induces \( j_n: \hat{M} / I^n \hat{M} \rightarrow M / I^n M \) such that \( j_n \circ q_n = \hat{p}_n \). We want to show that the maps \( i_n \) and \( j_n \) are inverse to each other.

Consider first the composition \( j_n \circ i_n \). Since

\[
j_n \circ i_n \circ \hat{p}_n \circ i_M = j_n \circ i_n \circ p_n = j_n \circ q_n \circ i_M = \hat{p}_n \circ i_M,
\]

we have

\[
j_n \circ i_n \circ \hat{p}_n = \hat{p}_n
\]

by the mapping universality of \( I \)-adic completions. But since \( \hat{p}_n \) is surjective (because \( p_n \) is surjective), we deduce that

\[
j_n \circ i_n = \text{id}_{M / I^n M}.
\]

Next we discuss \( i_n \circ j_n \). We first note that \( \hat{M} / I^n \hat{M} \) is \( I \)-adically complete (by 7.2.3), and hence the canonical projection \( q_n \) is the unique map that makes the above diagram commute. We have

\[
i_n \circ j_n \circ q_n \circ i_M = i_n \circ \hat{p}_n \circ i_M = i_n \circ p_n = q_n \circ i_M,
\]

and hence

\[
i_n \circ j_n \circ q_n = q_n.
\]

Since \( q_n \) is surjective, we deduce that

\[
i_n \circ j_n = \text{id}_{\hat{M} / I^n \hat{M}},
\]

as desired. \( \Box \)

Let \( A \) be a ring, \( I \subseteq A \) an ideal, and \( M \) an \( A \)-module. Consider the Hausdorff completion \( M_{I^\infty} \) of \( M \) with respect to the \( I \)-adic topology. Similarly, we set

\[
\hat{F}^k = \lim_{n \geq 0} I^k M / I^{k+n} M \quad \text{for each } k \geq 1,
\]
which is the Hausdorff completion of $I^k M$ with respect to the $I$-adic topology. As in 7.1.7, $\{\hat{F}^k\}_{k\geq 1}$ gives the descending filtration by $A$-submodules on $M_{I^\bullet}$ induced from the $I$-adic filtration on $M$. The following statement, which is a corollary of 7.2.7 (and 7.2.8), gives an existence criterion of the $I$-adic completion of $M$.

**Corollary 7.2.9.** The following conditions are equivalent:

(a) an $I$-adic completion $i_M: M \to \hat{M}$ exists;

(b) the $A$-module $M_{I^\bullet}$ is $I$-adically complete;

(c) $I^k M_{I^\bullet}$ is closed in $M_{I^\bullet}$ with respect to the topology defined by the filtration $\{\hat{F}^n\}_{n\geq 1}$ for any $k \geq 1$;

(d) $\hat{F}^k = I^k M_{I^\bullet}$ for any $k \geq 1$.

**Proof.** Implication (a) $\implies$ (b) follows from 7.2.7. Conversely, if (b) holds, then one can check that $M_{I^\bullet}$ together with the canonical morphism $M \to M_{I^\bullet}$ gives an $I$-adic completion, thanks to the mapping universality of projective limits, whence the implication (b) $\implies$ (a).

Next we show that (c) $\iff$ (d). Implication (d) $\implies$ (c) follows immediately. By 7.1.7, $\hat{F}^k$ is the closure of the image of $I^k M$ under the map $M \to M_{I^\bullet}$; since $I^k M_{I^\bullet}$ contains the image, and since $I^k M_{I^\bullet} \subseteq \hat{F}^k$, $\hat{F}^k$ is the closure of $I^k M_{I^\bullet}$. Hence we have the other implication (c) $\implies$ (d).

Suppose that (a) holds. Since $\hat{F}^k$ is the kernel of the map $M_{I^\bullet} \to M/I^k M$, we have $\hat{F}^k = I^k M_{I^\bullet}$ for any $k \geq 1$ by 7.2.7 (and 7.2.8), whence the implication (a) $\implies$ (d). Finally, if (d) holds, then

$$M_{I^\bullet} = \lim_{k \to \infty} M_{I^\bullet}/\hat{F}^k = \lim_{k \to \infty} M_{I^\bullet}/I^k M_{I^\bullet},$$

which shows that $M_{I^\bullet}$ is $I$-adically complete, whence (d) $\implies$ (b). \qed

**Example 7.2.10.** Let $A = k[x_1, x_2, x_3, \ldots]$ be the polynomial ring of countably many variables over a field $k$, and $I = (x_1, x_2, x_3, \ldots)$ the ideal of $A$ generated by all the indeterminacies. We claim that the $I$-adic completion of $A$ does not exist. Indeed, assuming the contrary, $B = A_{I^\bullet} = \lim_{n \to \infty} A/I^n$ would be $IB$-adically complete and hence $IB = \hat{F}^1 = \ker(B \to A/I)$. But one can show that the infinite sum $x_1 + x_2^2 + x_3^3 + \cdots = \sum_{i=1}^\infty x_i^i$ does not converge with respect to the $IB$-adic topology as follows. Indeed, consider the partial sums $s_k = \sum_{i=1}^k x_i^i$ for $k \geq 1$ and suppose that $\{s_k\}_{k \geq 1}$ has the limit $s \in B$ with respect to the $IB$-adic topology; since $s - s_k$ lies in $\hat{F}^{k+1} = \ker(B \to A/I^{k+1})$ for each $k$, $s$ is also the limit of $\{s_k\}_{k \geq 1}$ with respect to the topology defined by the filtration $\{\hat{F}^k\}$; consequently $s \in \hat{F}^1 = IB$, which is absurd.
7. (c) Criterion for adicness. It is in general difficult to determine whether a given topology defined by a filtration is adic or not. Let us state some criteria for adicness. The following proposition is a rehash of some contents in [53], 0, §7.2, and [27], Chapter III, §2.11.

Proposition 7.2.11. Let $A$ be a ring endowed with a descending filtration by ideals \( \{F^{(n)}\}_{n \geq 0} \) indexed by non-negative integers and such that $F^{(0)} = A$. Set $I = F^{(1)}$. Suppose that the following conditions are satisfied.

(a) $A$ is Hausdorff complete with respect to the topology defined by the filtration $\{F^{(n)}\}_{n \geq 0}$. In other words, the canonical map

\[
i: A \longrightarrow \lim_{n \geq 0} A/F^{(n)}
\]

is an isomorphism.

(b) For any $n > 0$ the induced filtration on $A/F^{(n)}$ (cf. §7.1. (a)) is $I$-adic; in other words,

\[
I^m (A/F^{(n)}) = (F^{(1)}/F^{(n)})^m = F^{(m)}/F^{(n)}
\]

for any $m$ and $n$ with $0 \leq m \leq n$.

(c) $F^{(1)}/F^{(2)}$ is finitely generated as an ideal of $A/F^{(2)}$.

Then we have

\[
F^{(n)} = I^n
\]

for any $n \geq 0$, and thus the filtration $\{F^{(n)}\}_{n \geq 0}$ is $I$-adic. In particular, $A$ is $I$-adically complete, and $I$ is finitely generated.

As a first corollary of the proposition, we have the following useful statement.

Proposition 7.2.12. Let $B$ be a ring, and $J \subseteq B$ a finitely generated ideal. Suppose that $B$ is $J$-adically complete, that is,

\[
B = \lim_{k \geq 0} B_k,
\]

where $B_k = B/J^{k+1}$ for $k \geq 0$. Let $\{A_k\}_{k \geq 0}$ be a projective system of $B$-algebras such that for $k \leq l$ the transition map $A_l \rightarrow A_k$ is surjective, with the kernel equal to $J^{k+1}A_l$. Let $A = \lim_{k \geq 0} A_k$, and consider the descending filtration $\{F^{(n)}\}_{n \geq 0}$ by the ideals $F^{(0)} = A$ and $F^{(n)} = \ker(A \rightarrow A_{n-1})$ for $n \geq 1$. Set $I = F^{(1)}$. Then the ring $A$ is $I$-adically complete and $I = JA$.

To show this, we first need the following elementary fact.
Lemma 7.2.13. Let $A$ be a ring, and $I \subseteq A$ an ideal, and suppose $A$ is $I$-adically complete. Then $1 + I \subseteq A^\infty$. In particular, $I$ is contained in the Jacobson radical of $A$.

Proof. The inverse of $1-a$ since $a \in I$ is given by $\sum_{n=0}^{\infty} a^n$, which belongs to $A$, for $A$ is $I$-adically complete. \hfill \qed

Proof of Proposition 7.2.12. First note that each projection map $A \rightarrow A_{n-1}$ for $n \geq 1$ is surjective (3.2.2). Note also that $J^m A \subseteq F^{(m)}$ for any $m \geq 1$. We have

$$A_{n-1}/IA_{n-1} = A/I = A_0 = A_{n-1}/JA_{n-1},$$

which shows that $I = JA + F^{(n)}$. This implies that $IA_{n-1} = JA_{n-1}$ and hence that

$$I^m A_{n-1} = J^m A_{n-1} = F^{(m)} A_{n-1} = F^{(m)}/F^{(n)}$$

for $0 \leq m \leq n$. Hence we can apply 7.2.11 to conclude that $I^n = F^{(n)}$ for $n \geq 1$ and that $A$ is $I$-adically complete. Moreover, in view of 7.2.13, the equality $JA + I^2 = I$ implies $I = JA$, by Nakayama’s lemma. \hfill \qed

Proposition 7.2.14. Let $A$ be a ring, $I \subseteq A$ a finitely generated ideal, and $M$ an $A$-module endowed with a descending filtration by $A$-submodules of the form $\{F^{(n)}\}_{n \geq 0}$ with $F^{(0)} = M$ such that $I^n M \subseteq F^{(n)}$ for any $n \geq 0$. Suppose that the following conditions are satisfied:

(a) $A$ is $I$-adically complete, and $M$ is Hausdorff complete with respect to the topology defined by the filtration $F^\bullet$. In other words, the canonical map

$$i : M \longrightarrow \lim_{\leftarrow \; n \geq 0} M/F^{(n)}$$

is an isomorphism;

(b) for any $n > 0$ the induced filtration on $M/F^{(n)}$ (cf. §7.1. (a)) is $I$-adic. In other words, we have

$$I^m (M/F^{(n)}) = F^{(m)}/F^{(n)}$$

for any $0 \leq m \leq n$.

Then we have

$$F^{(n)} = I^n M$$

for any $n \geq 0$, and thus the filtration $\{F^{(n)}\}_{n \geq 0}$ is $I$-adic. In particular, $M$ is $I$-adically complete. If, moreover,

(c) $M/F^{(1)}$ is finitely generated as an $A$-module,

then $M$ is finitely generated as an $A$-module.
Proof. Consider $B = A \oplus M$ and regard it as an $A$-algebra with respect to the multiplication $(a, x) \cdot (a', x') = (aa', ax' + a'x)$ for $a, a' \in A$ and $x, x' \in M$; this is a ring with the square-zero ideal $M \subseteq B$. Consider a descending filtration $\{J(n)\}_{n \geq 0}$ by the ideals $J(n) = I^n \oplus F(n)$, $n \geq 0$, and set $J = J(1)$. By (a), the ring $B$ is Hausdorff complete with respect to the topology defined by this filtration, since, clearly, Hausdorff completion commutes with direct sums. Moreover, $I^n B \subseteq J(n)$ for $n \geq 0$ by the assumption. By (b) with $m = 1$, one has $I(M/F(n)) = F(1)/F(n)$ and hence $I^m(M/F(n)) = I^{m-1}(F(1)/F(n))$ for any $m \geq 1$. Consequently,

$$J^m(B/J(n)) = I^m/I^n \oplus I^{m-1}(F(1)/F(n))$$

$$= I^m(A/I^n \oplus M/F(n))$$

$$= I^m \oplus F(m)/I^n \oplus F(n)$$

$$= J^m/J^n.$$

(Note that the first line shows that $J^m(B/J(n)) = I^m(B/J(n))$.) Moreover,

$$J(1)/J(2) = I(A/I^2 \oplus M/F(2)) = I(B/J(2))$$

is finitely generated. Hence it follows from 7.2.11 that $J(n) = J^n$ for $n \geq 1$, that $B$ is $J$-adically complete, and that $J \subseteq B$ is finitely generated. Now, setting $B_k = B/J^{k+1}$ for $k \geq 0$, we see that the kernel of the surjective map $B_l \to B_k$ for $k \geq l$ is given by $J^{k+1}/J^{l+1} = I^{k+1}B_l$ and so that by 7.2.12 we have $J = IB$, and hence $J^n = I^n B$ for $n \geq 1$. In particular, $I^n M$ is closed in $M$, and hence $F(n) = I^n M$, as desired. If $M/F(1) = M/IM$ is finitely generated, $M$ is finitely generated due to 7.2.4. \qed

7.2. (d) Existence of $I$-adic completions. As indicated in 7.2.10, it is highly non-trivial whether or not the $I$-adic completion exists for a given adically topologized module. The following propositions shows that the existence holds if the ideal $I$ is finitely generated.

**Proposition 7.2.15.** Let $A$ be a ring and $I \subseteq A$ a finitely generated ideal. Then the Hausdorff completion

$$A_{I}^\wedge = \lim_{\leftarrow} A/I^n$$

of $A$ with respect to the $I$-adic topology is $I$-adic complete. In particular, the $I$-adic completion of $A$ exists.

**Proposition 7.2.16.** Let $A$ be a ring, $I \subseteq A$ a finitely generated ideal, and $M$ an $A$-module. Then the Hausdorff completion $M_{I}^\wedge$ of $M$ with respect to the $I$-adic topology is $I$-adically complete. In particular, the $I$-adic completion $\hat{M}$ of $M$ exists. If, moreover, $M/IM$ is finitely generated, then $\hat{M}$ is finitely generated over the $I$-adic completion $\hat{A}$ of $A$. 
Proof of Proposition 7.2.15. Consider the Hausdorff completion $B = A^\wedge_i$ of $A$ with respect to the $I$-adic topology, and set $F^{(k)} = \varprojlim_{n \geq 0} I^k / I^{k+n}$ for $k \geq 1$. By 7.2.9, it suffices to show the equality $F^{(k)} = I^k B$ for each $k \geq 0$. To this end, we need to check $F^{(1)} = IB$ and to verify (a), (b), and (c) in 7.2.11 (with $A$ replaced by $B$). Condition (a) is clear. Since $F^{(m)}/F^{(n)} = I^m/I^n$ for $0 \leq m \leq n$, (b) is also verified. Finally, by the assumption, $F^{(1)}/F^{(2)} = I/I^2$ is finitely generated as an ideal of $B/F^{(2)} = A/I^2$, and thus (c) holds. Thus we have shown that $B$ is $J$-adically complete, where $J = F^{(1)}$. Moreover, by 7.2.4 we deduce that $J$ is finitely generated, since $J/J^2 = F^{(1)}/F^{(2)} = I/I^2$ is finitely generated as an ideal of $B/F^{(2)} = A/I^2$.

Next we show that the equality $J = IB$ holds. Since the image of $I$ in $B$ is dense in $J$, the finitely generated ideal $IB$ is dense in $J$. In particular, $IB + J^2 = J$. Then the desired equality follows, in view of 7.2.13, from Nakayama’s lemma.  

Proof of Proposition 7.2.16. By 7.2.15, we already know that the $I$-adic completion $\hat{A}$ of $A$ exists. Let $N = M^\wedge_i$ be the Hausdorff completion of $M$ with respect to the $I$-adic topology. We consider the filtration $\{F^{(k)}\}_{k \geq 1}$ on $N$ given by

$$F^{(k)} = \varprojlim_{n \geq 0} I^k M / I^{k+n} M, \quad k \geq 1.$$ 

In view of 7.2.9 we only need to show that $N$ is $I$-adically complete and, to this end, to check the conditions in 7.2.14 with $M$ replaced by $N$, $A$ by $\hat{A}$, and $I$ by $I \hat{A}$. Condition (a) is clear. Since $N/F^{(n)} = M/I^n M$ for $n \geq 0$ and $F^{(m)}/F^{(n)} = I^m M/I^n M$ for $0 \leq m \leq n$, (b) also holds.  

7.3 Henselian rings and Zariskian rings

7.3. (a) Henselian rings. Recall that (cf. [33], [50], and [75]) a ring $A$ endowed with an $I$-adic topology ($I \subseteq A$) is said to be Henselian with respect to $I$ or $I$-adically Henselian if it satisfies either one of the following equivalent conditions:

(a) for any étale morphism $X \to \text{Spec } A$, any section $\sigma_0$ of the morphism $X_0 (= X \times_{\text{Spec } A} \text{Spec } A_0) \to \text{Spec } A_0$, where $A_0 = A/I$, lifts to a section $\sigma$ of $X \to \text{Spec } A$: 

$$
\begin{array}{ccc}
X_0 & \to & X \\
\sigma_0 \downarrow & \sigma \downarrow & \downarrow \\
\text{Spec } A_0 & \to & \text{Spec } A \\
\end{array}
$$
(b) the ideal \( I \) is contained in the Jacobson radical of \( A \), and for any monic polynomial \( F(T) \in A[T] \) such that \( F(0) \equiv 0 \mod I \) and \( F'(0) \) is invertible in \( A/I \), there exists \( a \in I \) such that \( F(a) = 0 \) (that is, Hensel’s lemma holds for monic polynomials).

(It is easy to see that the property ‘Henselian’ is actually a topological one.) Similarly to completion, there is the notion of Henselization

\[
i_A: A \to A^h,
\]

characterized up to isomorphisms by a universal mapping property similar to that for \( I \)-adic completions (7.2.6). The Henselization \( A^h \) always exists, even when the ideal \( I \) is not finitely generated. Here is a rough sketch of the construction: the pair \((A^h, IA^h)\) is the inductive limit of all pairs \((B, IB)\) with \( B \) an étale \( A \)-algebra such that \( B/IB \cong A/I \); see [50] and [75], §2.8, for other (equivalent) constructions.

**Proposition 7.3.1.** The property ‘Henselian’ is preserved by filtered inductive limits. More precisely, if \( \{A_\lambda\}_{\lambda \in \Lambda} \) is an inductive system of rings with adic topologies with adic transition maps (§7.2. (a)), and if all \( A_\lambda \) are Henselian, then the inductive limit

\[
A = \lim_{\to \lambda \in \Lambda} A_\lambda
\]

with the induced adic topology is Henselian.

The proof is easy and left to the reader (cf. [50], 3.6). Note that the analogous statement for ‘adically complete’ is not true.

**7.3. (b) Zariskian rings.** The following proposition is easy, and the proof is left to the reader.

**Proposition 7.3.2.** The following conditions for a ring \( A \) and an ideal \( I \subseteq A \) are equivalent:

(a) for any \( a \in I \) the element \( 1 + a \) is invertible in \( A \), that is, \( 1 + I \subseteq A^\times \);

(b) an element \( a \in A \) is invertible if and only if \( a \mod I \) is invertible in \( A/I \);

(c) \( I \) is contained in the Jacobson radical of \( A \).

A ring \( A \) endowed with the \( I \)-adic topology defined by an ideal \( I \subseteq A \) is Zariskian with respect to \( I \), or \( I \)-adically Zariskian, if it satisfies the equivalent conditions in 7.3.2. We have already seen in 7.2.13 that any \( I \)-adically complete ring is \( I \)-adically Zariskian. Note that, due to condition (c) in 7.3.2, ‘Zariskian’ is a topological property.

For an arbitrary ring \( A \) and an ideal \( I \subseteq A \) one can construct the associated Zariskian ring \( A^{\text{Zar}} \) simply by setting \( A^{\text{Zar}} = S^{-1}A \), where \( S = 1 + I \) is the multiplicative subset consisting of all elements of the form \( 1 + a \), with \( a \in I \). It is clear that this construction gives a unique solution to the universal mapping property similar to the universal mapping properties of completion and of Henselization.
**Proposition 7.3.3.** Let $A$ be a ring, and $I \subseteq A$ an ideal. Then the following conditions are equivalent:

(a) $A$ is $I$-adically Zariskian;

(b) every maximal ideal of $A$ is open with respect to the $I$-adic topology, and for any $I$-adically open prime ideal $p \subseteq A$, the localization $A_p$ is $IA_p$-adically Zariskian;

(c) for any maximal ideal $m \subseteq A$, $A_m$ is $IA_m$-adically Zariskian.

**Proof.** If $A$ is $I$-adically Zariskian, then any maximal ideal $m$ contains $I$ and hence is open. Let $p$ be an open prime ideal. Since $p$ contains $I^n$ for some $n > 0$, it contains $I$. We need to show that any element of $A$ of the form $f + a$, where $f \notin p$ and $a \in I$, is invertible in $A_p$. Suppose it is not. Then $f + a \in pA_p$. Since $f \notin pA_p$, we have $a \notin pA_p$. But this is absurd, since $I \subseteq p$. Hence we have implication (a) $\Rightarrow$ (b). Implication (b) $\Rightarrow$ (c) is trivial and implication (c) $\Rightarrow$ (a) is easy to verify.

**Remark 7.3.4.** The term ‘Zariskian’ is coined from the already widespread term ‘Zariski ring’ ([109], Chapter VIII, §4). The required condition $1 + I \subseteq A^\times$ is, if $A$ is Noetherian, equivalent to several other conditions as in [109], Chapter VIII, Theorem 9. When $A$ is not Noetherian, however, it is not necessarily equivalent to all of them; in fact, an examination of the proof of [109], Chapter VIII, Theorem 9, leads one to the question of validity of Artin–Rees lemma, which will be at the center of our later observation in §7.4; cf. 7.4.16 below.

### 7.3 (c) Interrelation of the conditions

**Proposition 7.3.5.** Let $A$ be a ring endowed with an adic topology defined by an ideal $I \subseteq A$.

(1) The following implications hold:

- ‘complete’ $\Rightarrow$ ‘Henselian’ $\Rightarrow$ ‘Zariskian.’

(2) There exists a unique adic homomorphism (cf. §7.2 (a)) $A^{\text{Zar}} \rightarrow A^h$ such that the diagram

\[
\begin{array}{ccc}
A^{\text{Zar}} & \longrightarrow & A^h \\
\downarrow & & \downarrow \\
A & \longrightarrow & A^h
\end{array}
\]

commutes. If the $I$-adic completion $\hat{\hat{A}}$ of $A$ exists, then there exists a unique adic homomorphism $A^h \rightarrow \hat{\hat{A}}$ such that the diagram

\[
\begin{array}{ccc}
A^h & \longrightarrow & \hat{\hat{A}} \\
\downarrow & & \downarrow \\
A & \longrightarrow & \hat{\hat{A}}
\end{array}
\]

commutes.
7. Topological rings and modules

Proof. By Hensel’s lemma (cf. [27], Chapter III, §4.3, Theorem 1), we know that $I$-adically complete rings are $I$-adically Henselian. It is clear from the definition (cf. §7.3 (a)) that $I$-adically Henselian rings are $I$-adically Zariskian. This proves (1).

(2) follows immediately from the universal mapping properties. □

In particular, it follows that, for example, the $IA^h$-adic completion of $A^h$ coincides up to canonical isomorphism with $\hat{A}$, and the $IA^{\text{Zar}}$-adic Henselization of $A^{\text{Zar}}$ with $A^h$, etc.

In the rest of this subsection we collect some basic facts on the canonical maps $A^{\text{Zar}} \to A^h$ and $A^{\text{Zar}} \to \hat{A}$, etc. concerning flatness.

Lemma 7.3.6. Let $A$ and $B$ be rings with adic topologies, $I \subseteq A$ an ideal of definition of $A$, and $f : A \to B$ an adic homomorphism. Suppose that $A$ is $I$-adically Zariskian and that $B$ is $A$-flat. Then the following conditions are equivalent.

(a) The induced morphism $A/I \to B/IB$ is faithfully flat.

(b) The morphism $f : A \to B$ is faithfully flat.

Proof. Implication (b) $\implies$ (a) is clear. To show the converse, let $N$ be a finitely generated $A$-module such that $N \otimes_A B = 0$. We have $N \otimes_A B/IB = 0$, and hence $N/I N = N \otimes_A A/I = 0$ due to (a). Since $A$ is $I$-adically Zariskian, we deduce that $N = 0$ by Nakayama’s lemma. □

Proposition 7.3.7. Let $A$ be a ring with an adic topology, and $A^h$ the Henselization. Then the map $A \to A^h$ is flat.

Proof. As $A^h$ is isomorphic to the inductive limit of a system of rings étale over $A$, $A^h$ is flat over $A$. □

Note that the analogous statement for ‘completion’ may fail to hold; the canonical map $A \to \hat{A}$ may not be flat in general (cf. Exercise 0.7.7).

Proposition 7.3.8. Let $A$ be a ring endowed with an adic topology defined by an ideal $I \subseteq A$.

(1) The canonical map $A^{\text{Zar}} \to A^h$ (7.3.5 (2)) is faithfully flat.

(2) If the $I$-adic completion $\hat{A}$ exists and if the canonical map $A \to \hat{A}$ is flat, then the canonical map $A^{\text{Zar}} \to \hat{A}$ (7.3.5 (2)) is faithfully flat.

Proof. (1) It follows from 7.3.7 that the map $A^{\text{Zar}} \to A^h$ is flat. By the construction of the Henselization we know that $A^{\text{Zar}}/IA^{\text{Zar}} \cong A^h/IA^h$. Hence the assertion follows from 7.3.6.

(2) Since $A^{\text{Zar}}/IA^{\text{Zar}} \cong \hat{A}/\hat{I}$, the assertion follows immediately from 7.3.6. □
7.4 Preservation of adicness

7.4. (a) General observation. Let $A$ be a ring, and consider an exact sequence of $A$-modules:

$$0 \longrightarrow M \xrightarrow{f} L \longrightarrow 0. \quad (\star)$$

Let $I \subseteq A$ be an ideal of $A$, and consider the $I$-adic filtration $I^n M = \{I^n M\}_{n \geq 0}$ on $M$. This filtration induces, as in §7.1. (a), the filtration $\{G^{(n)} = N \cap I^n M\}_{n \geq 0}$ on $N$ and the filtration $\{E^{(n)} = (N + I^n M)/N\}_{n \geq 0}$ on $L$. It will be important in many places in the sequel to know whether or not these induced topologies are the $I$-adic ones. In order to prepare for such situations, let us here make a general observation.

To this end, let us temporarily consider the following situation. Let $A$ be a ring, $I \subseteq A$ an ideal, and $M$ an $A$-module equipped with a descending filtration $F^\bullet = \{F^{(n)}\}_{n \in \mathbb{Z}}$ by $A$-submodules indexed by the directed set of all integers. We consider the following conditions for the filtration:

1. $(F_1)$ $IF^{(m)} \subseteq F^{(m+1)}$ for any $m \in \mathbb{Z}$, and
2. $(F_2)$ there exist $p \geq 0$ and $q \in \mathbb{Z}$ such that $I^p M \subseteq F^{(q)}$.

Note that the second condition is satisfied if, for example, there exists $q \in \mathbb{Z}$ such that $F^{(q)} = M$.

**Lemma 7.4.1.** The topology on $M$ is $I$-adic if the filtration $F^\bullet$ satisfies $(F_1)$, $(F_2)$, and

1. $(F_3)$ there exist $c \geq 0$ and $d \in \mathbb{Z}$ such that $F^{(n+d)} \subseteq I^n M$ for any $n \geq c$.

**Proof.** First note that $(F_1)$ and $(F_2)$ imply that $I^{m+p} M \subseteq F^{(m+q)}$ for any $m \geq 0$. Set $e = \max\{c, p\}$. Then we have the inclusions

$$F^{(n+d)} \subseteq I^n M \subseteq F^{(n+q-p)} \quad \text{and} \quad I^{n+d+p} M \subseteq F^{(n+d+q)} \subseteq I^{n+q} M$$

for any $n \geq e$. Hence the assertion follows from 7.1.4. \qed

**Definition 7.4.2.** Let $A$ be a ring, $I \subseteq A$ an ideal, and $M$ an $A$-module equipped with a descending filtration by $A$-submodules $F^\bullet = \{F^{(n)}\}_{n \in \mathbb{Z}}$. We say that the filtration $F^\bullet$ is $I$-good if it satisfies $(F_1)$ and

1. $(F_4)$ there exists an integer $N$ such that $IF^{(n)} = F^{(n+1)}$ for $n \geq N$.

**Lemma 7.4.3.** If $F^\bullet$ is $I$-good, then it satisfies $(F_3)$. In particular, if $F^\bullet$ is $I$-good and satisfies $(F_2)$, then the topology on $M$ defined by the filtration $F^\bullet$ is $I$-adic.

**Proof.** If $F^\bullet$ is $I$-good, we deduce by induction that $I^n F^{(d)} = F^{(n+d)}$ for any $d \geq N$ and $n \geq 0$. Then we have $F^{(n+d)} = I^n F^{(d)} \subseteq I^n M$. The last assertion follows from 7.4.1. \qed
7.4. (b) $I$-adicness of quotient topologies. Now we turn back to the exact sequence (★) in §7.4. (a). First we study the filtration $E^\bullet$ on the quotient module $L$ induced by the $I$-adic filtration $I^\bullet M$ on $M$. In fact, one can readily show that the topology on $L$ is always $I$-adic, since it is completely elementary to check that $(N + I^nM)/N = I^n(M/N)$ for $n \geq 0$.

**Lemma 7.4.4.** The induced filtration $E^\bullet$ on the quotient module $L = M/N$ coincides with the $I$-adic filtration $I^\bullet L$.

**Proposition 7.4.5.** Let $A$ be a ring, $I \subseteq A$ an ideal, and $f : M \to L$ a surjective morphism of $A$-modules.

(1) The induced map $f_{I^\bullet} : M_{I^\bullet}^\wedge \to L_{I^\bullet}^\wedge$ between the Hausdorff completions with respect to the $I$-adic topologies is also surjective.

(2) If $N = \ker(f)$, then
\[ \ker(f_{I^\bullet}) = \varprojlim_{n \geq 0} N/(N \cap I^nM), \]
which coincides with the closure (with respect to the topology defined by the filtration $\{\hat{F}^{(n)} = \ker(M_{I^\bullet}^\wedge \to M/I^nM)\}_{n \geq 0}$ of the image of $N$ in $M_{I^\bullet}^\wedge$ under the canonical map $M \to M_{I^\bullet}^\wedge$.

**Proof.** Since the topology on $L$ defined by the induced filtration $E^\bullet$ is $I$-adic, we have a canonical isomorphism $L_{f^{\bullet}}^\wedge \cong L_{I^\bullet}^\wedge$ between the Hausdorff completion with respect to the topology defined by the filtration $E^\bullet$ and the Hausdorff completion with respect to the $I$-adic topology. Hence we get as in §7.1. (d) the exact sequence
\[ 0 \to N_{I^\bullet}^\wedge \to M_{I^\bullet}^\wedge \to L_{I^\bullet}^\wedge \to 0. \]
In particular, we have (1). By 7.1.12 (1), we also have (2). \qed

**Corollary 7.4.6.** Let $A$ be an $I$-adically complete ring with respect to an ideal $I \subseteq A$, and $M$ an $I$-adically complete $A$-module. Let $N \subseteq M$ be an $A$-submodule of $M$. Then the quotient $M/N$ is $I$-adically complete if and only if $N$ is closed in $M$ with respect to the $I$-adic topology.

**Corollary 7.4.7.** Let $A$ be an $I$-adically complete ring with respect to an ideal $I \subseteq A$. Then any finitely generated $I$-adically separated $A$-module is $I$-adically complete.

**Proof.** Let $M$ be a finitely generated $A$-module, and write $M \cong A^{\oplus m}/K$ for some $m > 0$ and an $A$-submodule $K \subseteq A^{\oplus m}$. By 7.4.5, $M_{I^\bullet}^\wedge \cong A^{\oplus m}/\bar{K}$, where $\bar{K}$ is the closure of $K$ in $A^{\oplus m}$ with respect to the $I$-adic topology; note that $A^{\oplus m}$ is $I$-adically complete. Hence the canonical map $M \to M_{I^\bullet}^\wedge$ is surjective. If, furthermore, $M$ is $I$-adically separated, it is injective. \qed
Since any $A$-submodule of an $I$-adically separated $A$-module is $I$-adically separated, we have the following immediate corollary.

**Corollary 7.4.8.** Let $A$ be an $I$-adically complete ring with respect to an ideal $I \subseteq A$, and $M$ an $I$-adically separated $A$-module. Then any finitely generated $A$-submodule $N \subseteq M$ is $I$-adically complete.

Note that the submodule $N$ in 7.4.8 is not necessarily closed in $M$, whereas it is $I$-adically complete; in fact, the subspace topology on $N$ induced by the $I$-adic topology on $M$ may fail to coincide with the $I$-adic topology on $N$. Consequently, the quotient $M/N$ is not necessarily $I$-adically complete.

7.4. (c) $I$-adicness of subspace topologies. Let $A$ be a ring, $I \subseteq A$ an ideal, $M$ an $A$-module, and $N \subseteq M$ an $A$-submodule of $M$. We consider the $I$-adic topology on $M$. It is usually a very delicate problem to determine whether the subspace topology on $N$ induced by the $I$-adic topology on $M$ or, what amounts to the same, the topology defined by the induced filtration $G^* = \{N \cap I^n M\}_{n \geq 0}$ (cf. 7.1.2 (2)), is $I$-adic or not.

Let us start with the following lemma, which follows immediately from 7.1.4 and the obvious inclusions $I^n N \subseteq N \cap I^n M$ ($m \geq 0$).

**Lemma 7.4.9.** The topology on $N$ defined by the induced filtration

$$G^* = \{N \cap I^n M\}_{n \geq 0}$$

is $I$-adic if and only if

($\ast$) for any $n \geq 0$ there exists $m \geq 0$ such that $N \cap I^m M \subseteq I^n N$.

**Proposition 7.4.10.** Let $A$ be a ring and $I \subseteq A$ an ideal. Consider an exact sequence

$$0 \longrightarrow N \overset{g}{\longrightarrow} M \overset{f}{\longrightarrow} L \longrightarrow 0$$

of $A$-modules. If $N \subseteq M$ satisfies ($\ast$) in 7.4.9, then the sequence

$$0 \longrightarrow N^\wedge_{G^*} \overset{g^\wedge_{G^*}}{\longrightarrow} M^\wedge_{f^*} \overset{f^\wedge_{G^*}}{\longrightarrow} L^\wedge_{I^*} \longrightarrow 0$$

consisting of Hausdorff completions with respect to the $I$-adic topologies is exact.

**Proof.** We have already obtained, in the proof of 7.4.5, the exact sequence

$$0 \longrightarrow N^\wedge_{G^*} \longrightarrow M^\wedge_{f^*} \longrightarrow L^\wedge_{I^*} \longrightarrow 0.$$

Since the topology defined by the filtration $G^*$ on $N$ is $I$-adic by 7.4.9, we have $N^\wedge_{G^*} \cong N^\wedge_{f^*}$. \qed
It is easy to see that, in the situation as in 7.4.9, (*) is satisfied if \( N \) is an open submodule of \( M \). Indeed, if \( I^s M \subseteq N \) (\( s \geq 0 \)), then for any \( n \geq 0 \) one has \( N \cap I^{n+s} M = I^{n+s} M \subseteq I^n N \). However, for a general \( N \), (*) in 7.4.9 is not necessarily satisfied. In this connection, it is useful to axiomatize some practical conditions for \( I \)-adicness of subspace topologies. Let \( A \) be a ring and \( I \subseteq A \) an ideal:

- (AP) any \( A \)-submodule \( N \subseteq M \) of a finitely generated \( A \)-module \( M \) satisfies (*) in 7.4.9;
- (APf) any finitely generated \( A \)-submodule \( N \subseteq M \) of a finitely generated \( A \)-module \( M \) satisfies (*) in 7.4.9.

**Proposition 7.4.11.** Let \( A \) be a ring, and \( I \subseteq A \) a finitely generated ideal. Suppose that \( A \) with the \( I \)-adic topology satisfies (APf). Then if

\[
N \xrightarrow{g} M \xrightarrow{f} L
\]

is an exact sequence of finitely generated \( A \)-modules, the induced sequence of \( I \)-adic completions

\[
\widehat{N} \xrightarrow{\hat{g}} \widehat{M} \xrightarrow{\hat{f}} \widehat{L}
\]

is exact. Hence the \( I \)-adic completion functor \( M \mapsto \widehat{M} \) is exact on the full subcategory of \( \text{Mod}_A \) consisting of finitely generated \( A \)-modules.

**Proof.** Let \( N_1 \) (resp. \( L_1 \)) be the image of the map \( g: N \rightarrow M \) (resp. \( f: M \rightarrow L \)). Then 7.4.10 implies that

\[
0 \rightarrow \widehat{N}_1 \rightarrow \widehat{M} \rightarrow \widehat{L}_1 \rightarrow 0
\]

is exact. We need to show that

- the map \( \widehat{N} \rightarrow \widehat{N}_1 \) is surjective, and
- the map \( \widehat{L}_1 \rightarrow \widehat{L} \) is injective.

The former assertion follows promptly from 7.4.5 (1). For the latter we apply 7.4.10 to the exact sequence \( 0 \rightarrow L_1 \rightarrow L \rightarrow L/L_1 \rightarrow 0 \), where \( L_1 \subseteq L \) satisfies (*) in 7.4.9.

**Proposition 7.4.12.** Let \( A \) be a ring, \( I \subseteq A \) an ideal, \( M \) an \( A \)-module, and \( N \subseteq M \) an \( A \)-submodule of \( M \). Suppose that

- (i) \( M \) is \( I \)-adically complete, and
- (ii) \( N \subseteq M \) satisfies (*) in 7.4.9.
The following conditions are equivalent.

(a) \( N \) is closed in \( M \) with respect to the \( I \)-adic topology.

(b) \( N \) is \( I \)-adically complete.

(c) \( M/N \) is \( I \)-adically complete.

This follows immediately from 7.1.13 in view of the fact that the filtration on \( M/N \) induced by the \( I \)-adic filtration on \( M \) is \( I \)-adic (due to 7.4.4) and the induced topology on \( N \) is \( I \)-adic (due to the assumption). Note that assumption (ii) is automatic in either of the following cases:

- \( M \) is finitely generated and \( A \) satisfies (AP);
- \( M \) and \( N \) are finitely generated and \( A \) satisfies (APf);
- \( N \) is open in \( M \) with respect to the \( I \)-adic topology.

**Proposition 7.4.13.** Let \( A \) be a ring and \( I \subseteq A \) a finitely generated ideal. Consider a finitely generated \( A \)-module \( M \) and an \( A \)-submodule \( N \subseteq M \). Suppose that either one of the following conditions is satisfied.

- \( A \) with the \( I \)-adic topology and the \( I \)-adic completion \( \hat{A} \) with the \( I \hat{A} \)-adic topology satisfy (AP).
- \( N \) is open in \( M \) with respect to the \( I \)-adic topology.

Then the \( I \)-adic completion \( \hat{N} \) coincides with the closure of the image of \( N \) in \( \hat{M} \). Moreover, we have the exact sequence

\[
0 \longrightarrow \hat{N} \longrightarrow \hat{M} \longrightarrow \hat{M}/\hat{N} \longrightarrow 0.
\]

**Proof.** By the assumption, the \( I \)-adic completion \( \hat{N} \) coincides with the Hausdorff completion of \( N \) with respect to the induced filtration \( \{ N \cap I^n M \}_{n \geq 0} \). Hence, the canonical map \( \hat{N} \to \hat{M} \) is injective, and, by 7.1.12, its image coincides with the closure of the image of \( N \) in \( \hat{M} \). We obtain the desired exact sequence thanks to the observation in §7.1.(d) and 7.4.4. \( \square \)

Finally, let us mention here that any adically topologized Noetherian ring \( A \) satisfies (AP) (which amounts to the same as (APf)).

**Proposition 7.4.14.** Let \( A \) be a Noetherian ring endowed with the \( I \)-adic topology defined by an ideal \( I \subseteq A \). Then \( A \) satisfies (AP).

The proposition is classical, verified by Artin–Rees lemma, which we will mention below (8.2.11).
7. Topological rings and modules

7.4. (d) Useful consequences of the conditions

**Proposition 7.4.15.** Let $A$ be a ring and $I \subseteq A$ a finitely generated ideal. Suppose that the ring $A$ together with the $I$-adic topology satisfy (APf) in §7.4.(c). Then for any finitely presented $A$-module $M$ the canonical morphism

$$M \otimes_A \hat{A} \rightarrow \hat{M}$$

is an isomorphism.

**Proof.** Take a finite presentation

$$A^{\oplus q} \rightarrow A^{\oplus p} \rightarrow M \rightarrow 0,$$

and consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
A^{\oplus q} \otimes_A \hat{A} & \rightarrow & A^{\oplus p} \otimes_A \hat{A} & \rightarrow & M \otimes_A \hat{A} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\hat{A}^{\oplus q} & \rightarrow & \hat{A}^{\oplus p} & \rightarrow & \hat{M} & \rightarrow & 0.
\end{array}$$

Here the exactness of the second row is a consequence of (APf) via 7.4.11. Since the first two vertical arrows are clearly isomorphisms, the third is also an isomorphism.

**Proposition 7.4.16.** Let $A$ be a ring with the $I$-adic topology defined by an ideal $I \subseteq A$ satisfying (APf) in §7.4.(c). Then the following conditions are equivalent:

(a) $A$ is $I$-adically Zariskian;
(b) any finitely generated $A$-module is $I$-adically separated;
(c) any $A$-submodule of a finitely generated $A$-module $M$ is closed in $M$ with respect to the $I$-adic topology;
(d) any maximal ideal of $A$ is closed with respect to the $I$-adic topology.

**Proof.** The proposition can be shown similarly to [27], Chapter III, §3.3, Proposition 6; we include the proof for the reader’s convenience.

First we prove (a) $\implies$ (b). Let $M$ be a finitely generated $A$-module, and take $x \in \bigcap_{n \geq 0} I^n M$. We want to show that $x = 0$. Consider the submodule $Ax$ of $M$. By virtue of (APf), the topology on $Ax$ induced by $M$ is the $I$-adic topology, which is, however, the coarsest topology on $Ax$, for $Ax$ is contained in all neighborhoods of $M$. Hence, $Ix = Ax$, which means $(1 - a)x = 0$ for some $a \in I$. Since $1 + I \subseteq A^\times$, we have $x = 0$, as desired.
Next we show (b) \(\implies\) (c). Let \(N\) be an \(A\)-submodule of \(M\). Then \(M/N\) is \(I\)-adically separated, whence (c) (cf. Exercise 0.7.1).

As implication (c) \(\implies\) (d) is clear, it only remains to show (d) \(\implies\) (a). Let \(m\) be a maximal ideal of \(A\). We need to show that \(m\) contains \(I\). Consider the field \(A/m\), which is \(I\)-adically separated, that is, \(I(A/m) \neq A/m\). Hence we have \(I(A/m) = 0\), that is, \(I \subseteq m\).

\[\square\]

**Corollary 7.4.17.** Let \(A\) be a ring and \(I \subseteq A\) an ideal. Suppose \(A\) is \(I\)-adically complete and satisfies \((\text{APf})\). Then any finitely generated \(A\)-module \(M\) is \(I\)-adically complete.

**Proof.** Since \(A\) is \(I\)-adically Zariskian, by 7.4.16 any finitely generated \(A\)-module is \(I\)-adically separated. Then the corollary follows from 7.4.7.

By this, 7.1.13, and 7.4.12, we have the following corollary.

**Corollary 7.4.18.** Let \(A\) be an \(I\)-adically complete ring with respect to an ideal \(I \subseteq A\), and \(M\) a finitely generated \(A\)-module. Suppose \(A\) with the \(I\)-adic topology satisfies \((\text{APf})\). Then any \(A\)-submodule \(N \subseteq M\) is closed in \(M\) with respect to the \(I\)-adic topology. If, moreover, \(A\) satisfies \((\text{AP})\), then \(N\) is \(I\)-adically complete.

Here is another useful corollary of 7.4.16.

**Corollary 7.4.19.** Let \(A\) be a ring and \(I \subseteq A\) an ideal. Suppose \(A\) is \(I\)-adically Zariskian and satisfies \((\text{AP})\). Then the following conditions for a finitely generated \(A\)-module \(M\) are equivalent.

(a) \(M\) is finitely presented over \(A\).

(b) \(M/I^n M\) is finitely presented over \(A/I^n\) for any \(n \geq 1\).

**Proof.** Implication (a) \(\implies\) (b) is obvious. Suppose that (b) holds, and write \(M \cong A^{\oplus m}/K\). We need to show that \(K\) is finitely generated. We have for any \(n > 0\) the exact sequence

\[0 \longrightarrow K/K \cap I^n A^{\oplus m} \longrightarrow (A/I^n)^{\oplus m} \longrightarrow M/I^n M \longrightarrow 0.\]

Take an \(n > 0\) such that \(K \cap I^n A^{\oplus m} \subseteq IK\) (here we use \((\text{AP})\)). Since \(K/K \cap I^n A^{\oplus m}\) is finitely generated (cf. [27], Chapter I, \S 2.8, Lemma 9), we deduce that \(K/IK\) is finitely generated.

Take a finitely generated \(A\)-submodule \(K' \subseteq K\) that is mapped surjectively onto \(K/IK\). Then \(K/K' = I(K/K')\). On the other hand, since \(K/K'\) is a finitely generated \(A\)-submodule of \(A^{\oplus m}/K'\), it is \(I\)-adically separated by 7.4.16. Consequently, \(K/K' = 0\), that is, \(K = K'\) is finitely generated.

\[\square\]
Finally, let us summarize some of the important consequences of (AP) and (APf). Let $A$ be a ring and $I \subseteq A$ an ideal. Suppose $A$ is $I$-adically complete.

- Condition (APf) for $A$ implies that
  
  (a) any finitely generated $A$-module is $I$-adically complete, and
  
  (b) any $A$-submodule $N$ of a finitely generated $A$-module $M$ is closed in $M$.

- If, moreover, $A$ satisfies (AP), then
  
  (c) any $A$-submodule $N$ of a finitely generated $A$-module $M$ is $I$-adically complete.

### 7.5 Rees algebra and $I$-goodness

Let $A$ be a ring and $I \subseteq A$ an ideal. In this situation, the associated Rees algebra is the graded algebra

$$R(A, I) = \bigoplus_{n \geq 0} I^n$$

(where $I^0 = A$) over $A$. Clearly, if the ideal $I$ is finitely generated, then $R(A, I)$ is an $A$-algebra of finite type. Let $M$ be an $A$-module equipped with a descending filtration $F^\bullet = \{F(n)\}_{n \in \mathbb{Z}}$ by $A$-submodules. Suppose that (F1) in §7.4. (a) is satisfied, that is,

$$I^q F(n) \subseteq F(n+q)$$

holds for any $q \geq 0$ and $n \in \mathbb{Z}$. Then for any $k \in \mathbb{Z}$

$$N_{\geq k} = \bigoplus_{n \geq k} F(n)$$

is a graded $R(A, I)$-module.

**Proposition 7.5.1.** Suppose in the above situation that the ideal $I$ is finitely generated. If $N_{\geq k}$ is finitely generated as an $R(A, I)$-module for some $k \in \mathbb{Z}$, then the filtration $F^\bullet$ is $I$-good (7.4.2).

**Proof.** By shift of indices we may assume that $k = 0$; write $N = N_{\geq 0}$. Let \{\(l_1, \ldots, l_s\)\} be a set of generators of $N$ consisting of homogeneous elements, and set $d_i = \deg(l_i)$ for $1 \leq i \leq s$. Let $c = \max\{d_1, \ldots, d_s\}$, and let $a_1, \ldots, a_r$ generate $I$. Suppose $n > c$. Any $x \in F(n)$ is written as

$$x = \sum_{i=1}^s f_i(a_1, \ldots, a_r)l_i,$$
where $f_i$ for each $i$ is a homogeneous polynomial of degree $n - d_i$. If $h(a_1, \ldots, a_r)$ is a monomial of degree $n - d_i$, factor $h = h_1h_2$ with $\deg(h_1) = n - c$ and $\deg(h_2) = c - d_i$. Since $c - d_i \geq e$, we have $h_2(a)l_i \in I^{n-c}F(c)$. Hence, $h(a)l_i \in I^{n-c}F(c)$ and thus have the inclusion $F^{(n)} \subseteq I^{n-c}F(c)$, whence the equality $F^{(n)} = I^{n-c}F(c)$. This yields (F4) in §7.4.(a). \qed

It is sometimes useful to consider the so-called *conormal cone* associated to $R(A, I)$:

$$\text{gr}_I^*(A) = \bigoplus_{n \geq 0} I^n/I^{n+1}.$$ If $F^\bullet$ is a descending filtration as above, then

$$\text{gr}_F^*(M) = \bigoplus_{n \in \mathbb{Z}} F^{(n)}/F^{(n+1)}$$

is a graded $\text{gr}_I^*(A)$-module.

**Proposition 7.5.2** (cf. [27], Chapter III, §3.1, Proposition 3). Suppose that the ideal $I$ is finitely generated, that $A$ is $I$-adically complete, and that the filtration $F^\bullet$ is separated and exhaustive (cf. §7.1.(a)). Then the following conditions are equivalent:

(a) $\text{gr}_F^*(M)$ is a finitely generated $\text{gr}_I^*(A)$-module.

(b) $M$ and $F^{(n)}$ for all $n \in \mathbb{Z}$ are finitely generated as $A$-modules, and the filtration $F^\bullet$ is $I$-good.

**Proof.** We want to prove (a) $\implies$ (b) (the converse (b) $\implies$ (a) is easy (cf. [27], Chapter III, §3.1, Proposition 3)). Suppose $\text{gr}_F^*(M)$ is finitely generated over $\text{gr}_I^*(A)$. Since $\text{gr}_I^*(A) = \bigoplus_{m \geq 0} I^m/I^{m+1}$ is positively graded, any homogeneous element $x$ of $\text{gr}_F^*(M)$ a linear combination of the generators of degree less than or equal to the degree of $x$. Hence $F^{(m)} = M$ for $m$ sufficiently small, as $F^\bullet$ is exhaustive. Renumbering the indices if necessary, we may assume $F^{(0)} = M$. Since $I$ is finitely generated, $\bigoplus_{0 \leq m \leq k} I^m/I^{m+1}$ is a finitely generated $A$-module, and hence $\bigoplus_{0 \leq m \leq k} F^{(m)}/F^{(m+1)}$ is also finitely generated over $A$ for any $k$.

Hence, to prove the assertion, it is enough to show that there exists an integer $N \geq 0$ such that $F^{(n+1)}$ for $n \geq N$ is finitely generated $A$-module and that $IF^{(n)} = F^{(n+1)}$.

We take $N$ such that $\text{gr}_F^*(M)$ is generated by homogeneous elements of degree less than $N$. If $n \geq N$, then $\text{gr}_F^*(M) = F^{(n)}/F^{(n+1)}$ is the image of $I^{n-N}F(N)$. Consider $F^{(n+1)}$ with the filtration $F'$ induced from $F^\bullet$. Then $\text{gr}_F^*(F^{(n+1)}) = \bigoplus_{m \geq n+1} F^{(m)}/F^{(m+1)}$ is generated by $IF^{(n)}/F^{(n+2)}$ as a $\text{gr}_I^*(A)$-module. Since
the $A$-modules $F^{(n)}/F^{(n+1)}$ and $I$ are finitely generated, we can find a finite set $\{u_j\}_{j \in J}$ of elements of $IF^{(n)}$ that generates $IF^{(n)}/F^{(n+2)}$. Then by [27], Chapter III, §2.9, Proposition 12, $\{u_j\}_{j \in J}$ generates $F^{(n+1)}$. Hence $F^{(n+1)}$ is finitely generated and, moreover, we deduce that $F^{(n+1)} \subseteq IF^{(n)}$, as desired.

\[ \square \]

**Exercises**

Exercise 0.7.1. Let $M$ be a module over a ring $A$, and consider the topology on $M$ defined by a descending filtration $F^\bullet = \{ F^\lambda \}_{\lambda \in \Lambda}$ of $A$-submodules of $M$. Show that for an $A$-submodule $N \subseteq M$ the closure $\overline{N}$ of $N$ is given by

$$\overline{N} = \bigcap_{\lambda \in \Lambda} (N + F^\lambda).$$

Exercise 0.7.2. Let $A$ be a ring, $\{ I^{(\lambda)} \}_{\lambda \in \Lambda}$ a descending filtration by ideals, and $g \in A$ an element. Suppose that $A$ is complete with respect to the topology defined by $\{ I^{(\lambda)} \}_{\lambda \in \Lambda}$; $(g \mod I^{(\lambda)})$ is a non-zero-divisor in $A/I^{(\lambda)}$ for any $\lambda \in \Lambda$.

Show that the principal ideal $(g) \subseteq A$ is closed.

Exercise 0.7.3. Let $A$ be a ring, $I \subseteq A$ an ideal, and $B$ a faithfully flat $A$-algebra. Let $\{ J^{(\lambda)} \}_{\lambda \in \Lambda}$ be a descending filtration by ideals of $A$. The topology on $B$ defined by the filtration $\{ J^{(\lambda)}B \}_{\lambda \in \Lambda}$ is $IB$-adic if and only if the topology on $A$ defined by the filtration $\{ J^{(\lambda)} \}_{\lambda \in \Lambda}$ is $I$-adic.

Exercise 0.7.4. Let $A$ be a ring, $I = (a_1, \ldots, a_r) \subseteq A$ a finitely generated ideal, and $M$ an $A$-module. Show that if $M$ is $a_i$-adically complete for each $i = 1, \ldots, r$, then $M/\bigcap_{n \geq 0} I^n M$ is $I$-adically complete. In particular, if $M$ is $I$-adically separated, then $M$ is $I$-adically complete if and only if it is $a_i$-adically complete for each $i = 1, \ldots, r$.

Exercise 0.7.5. Let $M$ be a module over a ring $A$ and $N \subseteq M$ an $A$-submodule. Consider a descending filtration $F^\bullet = \{ F^{(n)} \}_{n \geq 0}$ by $A$-submodules of $N$, and topologize $M$ and $N$ by this filtration. Let $N^\wedge_{F^\bullet}$ and $M^\wedge_{F^\bullet}$ be the Hausdorff completions of $N$ and $M$, respectively, with respect to this topology.

1. The $A$-module $N^\wedge_{F^\bullet}$ is canonically an $A$-subgroup of $M^\wedge_{F^\bullet}$, and the canonical map $M/N \to M^\wedge_{F^\bullet}/N^\wedge_{F^\bullet}$ is an isomorphism.

2. If the filtration $F^\bullet$ is separated, then $N^\wedge_{F^\bullet} \cap M = N$.

Exercise 0.7.6. Let $A$ be a ring endowed with a descending filtration $\{ I^{(\lambda)} \}_{\lambda \in \Lambda}$ by ideals. Let $M$ (resp. $N$) be an $A$-module endowed with a descending filtration $F^\bullet = \{ F^\alpha \}_{\alpha \in \Sigma}$ (resp. $G^\bullet = \{ G^\beta \}_{\beta \in \mathcal{T}}$) by $A$-submodules. We suppose that for
any $\alpha \in \Sigma$ (resp. $\beta \in T$) there exists $\lambda \in \Lambda$ such that $I^{(\lambda)}M \subseteq F^\alpha$ (resp. $I^{(\lambda)}N \subseteq G^\beta$). Consider the tensor product $M \otimes_A N$ together with the descending filtration $\{H^{\alpha,\beta}\}_{(\alpha, \beta) \in \Sigma \times T}$ by $A$-submodules given by

$$H^{\alpha,\beta} = \text{image}(F^\alpha \otimes_A N \rightarrow M \otimes_A N) + \text{image}(M \otimes_A G^\beta \rightarrow M \otimes_A N)$$

for any $(\alpha, \beta) \in \Sigma \times T$; here we regard $\Sigma \times T$ as a directed set by the ordering

$$(\alpha, \beta) \leq (\alpha', \beta') \iff \alpha \leq \alpha', \beta \leq \beta'.$$

The Hausdorff completion of $M \otimes_A N$ with respect to the topology defined by this filtration is denoted by $M \widehat{\otimes}_A N$ and is called the complete tensor product. Show that there exist canonical isomorphisms

$$M \widehat{\otimes}_A N \sim M_{\hat{F}^*} \otimes_{\hat{A}^*} N_{\hat{G}^*} \sim M_{\hat{F}^*} \otimes_{\hat{A}^*_I} N_{\hat{G}^*}.$$  

**Exercise 0.7.7.** Let $V$ be a valuation ring of height 2 and $a \in V$ an element such that $\bigcap_{n \geq 0}(a^n)$ is the prime ideal of height 1 (cf. 6.7.1). Show that the $a$-adic completion $\hat{V}$ is not flat over $V$.

**Exercise 0.7.8.** Let $A$ be a ring, and $I \subseteq A$ an ideal. Show that (AP) (resp. (APf)) in §7.4. (c) is equivalent to the following condition: for any finitely generated $A$-module $M$ and any $A$-submodule (resp. finitely generated $A$-submodule) $N \subseteq M$ such that $I^n N = 0$ for some $n \geq 0$, there exists $m \geq 0$ such that $N \cap I^m M = 0$.

**Exercise 0.7.9.** Let $A$ be a ring, and $I \subseteq A$ a finitely generated ideal. Suppose that the $I$-adic completion $\hat{A}$ with the $I \hat{A}$-adic topology satisfies (AP) (resp. (APf)). Show that for any finitely generated (resp. finitely presented) $A$-module $M$ the canonical map $M \otimes_A \hat{A} \rightarrow \hat{M}$ is an isomorphism.

**Exercise 0.7.10.** Let $T$ be an abelian group topologized by a descending filtration $\{F^{(n)}\}_{n \in \mathbb{Z}}$ by subgroups.

1. Suppose that the filtration $F^\bullet$ is separated. For a fixed real number $0 < c < 1$, define a function $d : T \times T \rightarrow \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ c^a & \text{if } x \neq y, \text{ where } a = \min\{n : x - y \in F^{(n)}\}. \end{cases}$$

Show that $d$ gives a metric on $T$. 
(2) Show that $T$ is complete with respect to the filtration $F^\bullet$ if and only if the metric space $(T, d)$ is complete.

(3) (Baire’s category theorem) Suppose that $T$ is complete. Let $U_1, U_2, \ldots$ be sequence of dense open subsets of $T$. Show that the intersection $\bigcap_{i=1}^{\infty} U_i$ is non-empty and dense in $T$ (cf. [25], IX, §5.3, Theorem 1).

(4) Suppose $T$ is complete. Show that, if $T_1, T_2, \ldots$ is a sequence of closed subgroups such that $T = \bigcup_{i=1}^{\infty} T_i$, then $T_i$ for some $i$ is an open subgroup.

8 Pairs

By a pair we simply mean throughout this book a couple $(A, I)$ consisting of a commutative ring $A$ and an ideal $I \subseteq A$. Given a pair $(A, I)$, one has the $I$-adic topology ($\S7.2$. (a)) on $A$-modules. In fact, according to our definition of morphisms between pairs, the notion of pairs is equivalent to the notion of adically topologized rings. Accordingly, the theory of pairs, which we develop in this section, can be seen as a continuation of what we have done in the previous section, being more focused on adic topology, especially in the context of homological algebra.

After briefly discussing generalities on pairs in $\S8.1$, we discuss in $\S8.2$ the so-called bounded torsion condition (BT) on pairs, which will turn out to be closely related to preservation of adicness (e.g. (AP)) as discussed in $\S7.4$. (c). This subsection contains (without proof) a significant result by Gabber $(8.2.19)$, which says that any $I$-adically complete Noetherian-outside-$I$ ring satisfies both (BT) and (AP). In $\S8.4$, we recall the so-called restricted formal power series rings, important objects in the theory of formal schemes.

In $\S8.5$, we will introduce the new notions of (pseudo-) adhesive and universally (pseudo-) adhesive pairs. It will turn out that adhesive pairs satisfy (AP), hence enjoy many useful properties as topological rings. Moreover, interestingly enough, universal adhesiveness guarantees some ring-theoretic properties of the underlying ring. For example, if $(A, I)$ is universally adhesive and $A$ is $I$-torsion free, then the ring $A$ is universally coherent ($3.3.7$), as we will see in $\S8.5$. (e). This fact suggests that the notion of universally adhesive pairs provides a good generalization of Noetherian rings.

Subsection $\S8.7$ discusses a different matter, the so-called $I$-valuative rings, which, as we shall see later, are the prototype for the local rings attached to the visualization of rigid spaces.

In the final subsection, $\S8.8$, we collect several useful results on homological algebra in interplay with $I$-adic topology and related filtrations, which provide a technical background for our later calculations of cohomologies on formal schemes.
8.1 Pairs

8.1. (a) Generalities. Recall that by a pair we mean a couple \((A, I)\) consisting of a commutative ring \(A\) and an ideal \(I\) of \(A\). When \(I\) is principal, say \(I = (a)\), then we often write \((A, a)\) in place of \((A, I)\). A morphism of pairs \(f : (A, I) \to (B, J)\) is a ring homomorphism \(f : A \to B\) such that there exists an integer \(n \geq 1\) such that \(I^n \subseteq f^{-1}(J)\) or, equivalently, \(I^n B \subseteq J\) holds.

Given a pair \((A, I)\), one can consider the \(I\)-adic topology (§7.2. (a)) on \(A\).

Let \((A, I)\) be a pair and \(J \subseteq A\) an ideal. Then \(J\) is said to be an ideal of definition, if the identity map \(id_A\) gives an isomorphism of pairs between \((A, I)\) and \((A, J)\) or, equivalently, there exist positive integers \(m, n\) such that \(I^m \subseteq J^n \subseteq I\).

A morphism \((A, I) \to (B, J)\) of pairs is said to be adic if \(IB\) is an ideal of definition of \((B, J)\); this definition is consistent with the definition of adic maps already given in §7.2. (a).

A pair \((A, I)\) is said to be complete (resp. Henselian, resp. Zariskian) if the ring \(A\) is \(I\)-adically complete (resp. Henselian, resp. Zariskian). Given an arbitrary pair \((A, I)\), one can construct the Henselian pair \((A^h, IA^h)\) (cf. §7.3. (a)), called the Henselization of \((A, I)\), and the Zariskian pair \((A^\text{Zar}, IA^\text{Zar})\) (cf. §7.3. (b)), called the associated Zariskian pair of \((A, I)\).

8.1. (b) Pairs of finite ideal type. A pair \((A, I)\) is said to be of finite ideal type if there exists a finitely generated ideal of definition. If \((A, I)\) is a pair of finite ideal type, then replacing it by an isomorphic one, we may assume that \(I\) is finitely generated. Note that if a pair \((A, I)\) is of finite ideal type, then one can construct the complete pair \((\hat{A}, I \hat{A})\), the so-called completion (7.2.16), which is again of finite ideal type.

The following proposition says that the property ‘of finite ideal type’ is local with respect to the flat topology.

**Proposition 8.1.1.** Let \((A, I)\) be a pair and \(B\) a faithfully flat \(A\)-algebra. If \((B, IB)\) is of finite ideal type, then so is \((A, I)\).

**Proof.** Since \(\text{Spec } B \setminus V(IB)\) is quasi-compact, so is \(\text{Spec } A \setminus V(I)\). Hence, there exists a finitely generated ideal \(J \subseteq A\) such that \(V(J) = V(I)\). Since \(\sqrt{J} = \sqrt{I}\), and since \(J\) is finitely generated, there exists \(n > 0\) such that \(J^n \subseteq I\). On the other hand, since \((B, IB)\) is of finite ideal type, there exists a finitely generated subideal \(K \subseteq IB\) such that \(I^m B \subseteq K\) for some \(m > 0\). Since we have \(V(JB) = V(K)\), there exists \(l > 0\) such that \(K^l \subseteq JB\). Hence we have \(I^{lm} B \subseteq JB\) and thus \(I^{lm} \subseteq J\). \(\square\)
**Definition 8.1.2.** Let \((A, I)\) be a pair of finite ideal type. An ideal \(J \subseteq A\) is said to be \(I\)-admissible or, briefly, admissible, if \(J\) is finitely generated and there exists an integer \(n \geq 1\) such that \(I^n \subseteq J\).

In other words, \(I\)-admissible ideals are precisely the finitely generated open ideals with respect to the \(I\)-adic topology.

8.1. (c) Torsions and saturation. Let \(A\) be a ring, \(a \in A\) an element, and \(M\) an \(A\)-module. An element \(x \in M\) is said to be an \(a\)-torsion if there exists an integer \(n > 0\) such that \(a^n x = 0\). The subset of all \(a\)-torsion elements in \(M\), denoted by \(M_{a\text{-tor}}\), is an \(A\)-submodule, called the \(a\)-torsion part of \(M\).

Let \(I \subseteq A\) be an ideal. An element \(x \in M\) is called an \(I\)-torsion if it is \(a\)-torsion for all \(a \in I\). The \(A\)-submodule of all \(I\)-torsion elements, the so-called \(I\)-torsion part, denoted by \(M_{I\text{-tor}}\), is the intersection of all \(M_{a\text{-tor}}\) for \(a \in I\). It is easy to see that if \(I\) is generated by \(a_1, \ldots, a_r \in A\), then \(M_{I\text{-tor}}\) coincides with the intersection of all \(a_i\)-torsion parts for \(i = 1, \ldots, r\). We say that \(M\) is \(I\)-torsion free (resp. \(I\)-torsion) if \(M_{I\text{-tor}} = 0\) (resp. \(M_{I\text{-tor}} = M\)).

Let \(N \subseteq M\) be an \(A\)-submodule. The \(I\)-saturation of \(N\) in \(M\) is the \(A\)-submodule

\[
\tilde{N} = \{ x \in M : \text{for any } a \in I \text{ there exists } n \geq 0 \text{ such that } a^n x \in N \},
\]
or, equivalently, the inverse image of \((M/N)_{I\text{-tor}}\) by the canonical map

\[
M \longrightarrow M/N.
\]

We say that \(N\) is \(I\)-saturated in \(M\) if \(N = \tilde{N}\). It is clear that the \(I\)-saturation \(\tilde{N}\) is the smallest \(I\)-saturated \(A\)-submodule containing \(N\). Note that these notions are topological, that is, stable under change of ideals of definition. Note also that the \(I\)-saturation of \(\{0\}\) in \(M\) is nothing but \(M_{I\text{-tor}}\), and hence that the \(I\)-torsion part \(M_{I\text{-tor}}\) is always \(I\)-saturated.

If \((A, I)\) is of finite ideal type, then these notions can be described in terms of schemes as follows. Set \(X = \text{Spec } A\) and \(U = X \setminus V(I)\), and let \(j: U \hookrightarrow X\) be the open immersion. Let \(\mathcal{M}\) (resp. \(\mathcal{N}\)) be the quasi-coherent sheaf on \(X\) associated to \(M\) (resp. \(N\)). Then \(M_{I\text{-tor}}\) corresponds to the kernel of the canonical morphism \(\mathcal{M} \rightarrow j_* j^* \mathcal{M}\), and \(\tilde{N}\) to the pull-back of \(j_* j^* \mathcal{N}\) by the same morphism. Note that, since \((A, I)\) is of finite ideal type, the morphism \(j\) is quasi-compact and hence that these sheaves are quasi-coherent (cf. [54], I, (9.2.2)).

**Definition 8.1.3.** Let \((A, I)\) be a pair and \(M\) an \(I\)-torsion \(A\)-module. We say that \(M\) is of bounded \(I\)-torsion if there exists an integer \(n \geq 1\) such that \(I^n M = 0\).

Clearly, if \(I\) is a finitely generated ideal, then any finitely generated \(I\)-torsion module is bounded. We have already stated in 7.2.3 that any \(A\)-module of bounded \(I\)-torsion is \(I\)-adically complete; the following lemma is a slightly enhanced version.
Lemma 8.1.4. Let \((A, I)\) be a pair, and \(M\) a bounded \(I\)-torsion \(A\)-module. Then \(M\) is \(I\)-adically complete. Moreover, the canonical map \(M \rightarrow M \otimes_A \hat{A}_I\) is an isomorphism, where \(\hat{A}_I\) is the Hausdorff completion of \(A\) with respect to the \(I\)-adic topology.

Proof. The first part of the lemma is clear, since \(I^nM = 0\) for \(n \gg 1\). Since \(M\) is automatically an \(\hat{A}_I\)-module, and any \(A\)-module homomorphism from \(M\) to an \(\hat{A}_I\)-module is automatically an \(\hat{A}_I\)-module homomorphism, we have \(M \cong M \otimes_A \hat{A}_I\) by the universality of tensor products.

Definition 8.1.5. Let \((A, I)\) be a pair.

(1) We say \(A\) is locally Noetherian outside \(I\) if the scheme \(\text{Spec} \ A \setminus V(I)\) is locally Noetherian; if, moreover, \((A, I)\) is of finite ideal type, then \(A\) is said to be Noetherian outside \(I\).

(2) An \(A\)-module \(M\) is said to be finitely generated (resp. finitely presented) outside \(I\) if the quasi-coherent sheaf \(M|_U\) on \(U = \text{Spec} \ A \setminus V(I)\) (cf. §5.1 (a) for the notation) is of finite type (resp. of finite presentation).

The following lemma will be useful later.

Lemma 8.1.6. Let \((A, I)\) be a pair of finite ideal type, and \(M\) an \(A\)-module.

(1) If \(M\) is finitely generated outside \(I\), then there exists a finitely generated submodule \(N \subseteq M\) such that \(M/N\) is \(I\)-torsion.

(2) Suppose \(M\) is finitely generated as an \(A\)-module. If \(M\) is finitely presented outside \(I\), then there exist a finitely presented \(A\)-module \(N\) and a surjective morphism \(N \rightarrow M\) whose kernel is \(I\)-torsion.

Proof. Set \(X = \text{Spec} \ A, U = X \setminus V(I),\) and \(\mathcal{F} = \tilde{M}\).

(1) Since \(\mathcal{F}|_U\) is a quasi-coherent sheaf of finite type, by [54], I, (9.4.7), and IV, (1.7.7), there is a quasi-coherent subsheaf \(\mathcal{G} \subseteq \mathcal{F}\) of finite type such that \(\mathcal{G}|_U = \mathcal{F}|_U\). If \(N \subseteq M\) is the finitely generated \(A\)-submodule such that \(\mathcal{G} = \tilde{N}\), then \(M/N\) is \(I\)-torsion, since \(\mathcal{G}|_U = \mathcal{F}|_U\).

(2) Take a surjective morphism \(A^n \rightarrow M\), and let \(K\) be its kernel. Then \(K\) is finitely generated outside \(I\) (cf. [27], Chapter I, §2.8, Lemma 9). Hence by (1) there exists a finitely generated \(A\)-submodule \(K_0 \subseteq K\) such that \(K/K_0\) is \(I\)-torsion. Set \(N = A^n/K_0\), which is a finitely presented \(A\)-module, and consider the surjective map \(N \rightarrow M\). Its kernel is \(K/K_0\), which is \(I\)-torsion.

\(\square\)
8.2 Bounded torsion condition and preservation of adicness

8.2. (a) Bounded torsion condition. The following conditions for a pair \((A, I)\), called bounded torsion conditions, will be important in what follows.

(BT) for any finitely generated \(A\)-module \(M\), \(M_{I} -\text{tor}\) is of bounded \(I\)-torsion.

(UBT) for any \(A\)-algebra \(B\) of finite type, \((B, IB)\) satisfies (BT).

The following proposition is easy to see.

Proposition 8.2.1. Let \((A, I)\) be a pair satisfying (BT) (resp. (UBT)). If \(B\) is a finite \(A\)-algebra (resp. an \(A\)-algebra of finite type), then \((B, IB)\) satisfies (BT) (resp. (UBT)).

Corollary 8.2.2. For a pair \((A, I)\) to satisfy (UBT) it is necessary and sufficient that the polynomial ring pairs \((A[X_1, \ldots, X_n], IA[X_1, \ldots, X_n])\) satisfy (BT) for any \(n \geq 0\).

Proposition 8.2.3. (1) Let \((A, I)\) and \((B, J)\) be pairs. Then \((A, I)\) and \((B, J)\) satisfy (BT) (resp. (UBT)) if and only if \((A \times B, I \times J)\) satisfies (BT) (resp. (UBT)).

(2) Let \((A, I)\) be a pair and \(B\) a faithfully flat \(A\)-algebra. If \((B, IB)\) satisfies (BT) (resp. (UBT)), then \((A, I)\) satisfies (BT) (resp. (UBT)).

Proof. (1) The ‘if’ part follows from 8.2.1. To show the converse, note that any finitely generated \(A \times B\)-module \(M\) is a product of finitely generated modules \(M = M_A \oplus M_B\), and we clearly have \(M_{I \times J -\text{tor}} = (M_A)_{I -\text{tor}} \times (M_B)_{J -\text{tor}}\). Hence, if \((A, I)\) and \((B, J)\) satisfy (BT), then (BT) for \(A \times B\) follows immediately. A similar argument works for finitely generated algebras, and hence (UBT) for \(A \times B\) follows from that for \(A\) and \(B\).

(2) Clearly, it suffices to check the case of (BT). Let \(M\) be a finitely generated \(A\)-module, and consider \(M \otimes_A B\). We have \(I^n (M \otimes_A B)_{I -\text{tor}} = 0\) for some \(n > 0\). Since \(B\) is flat over \(A\), we have \( (M \otimes_A B)_{I -\text{tor}} \subseteq M_{I -\text{tor}} \otimes_A B\), but since the right-hand module is clearly \(I\)-torsion, we have \((M \otimes_A B)_{I -\text{tor}} = M_{I -\text{tor}} \otimes_A B\). Since \(B\) is faithfully flat over \(A\), we have \(I^n M_{I -\text{tor}} = 0\).

Proposition 8.2.4. Let \((A, I)\) be a pair of finite ideal type. Then \((A, I)\) satisfies (BT) if and only if so does the associated Zariskian pair \((A^{Zar}, IA^{Zar})\).

Proof. We may assume that \(I\) is finitely generated. Suppose \((A, I)\) satisfies (BT). For a finitely generated \(A^{Zar}\)-module \(N\) one can find a finitely generated \(A\)-module \(M\) such that \(M \otimes_A A^{Zar} \cong N\). For any element \(x/(1 + a) \in N_{I -\text{tor}}\) \((x \in M, a \in I)\), since \(I\) is finitely generated, one can find \(b \in I\) such that \((1 + b)x \in M_{I -\text{tor}}\). Then one can find \(n \geq 0\), independent of the element \((1 + b)x\), such that \(I^n \cdot (1 + b)x = 0\), thereby \(I^n \cdot x/(1 + a) = 0\).
Conversely, suppose \((A^{Zar}, IA^{Zar})\) satisfies \((BT)\), and for a finitely generated \(A\)-module \(M\), consider \(M \otimes_A A^{Zar}\). Take any \(x \in M_{I\text{-tor}}\). Since \(I\) is finitely generated, there exists \(a \in I\) such that \((1-a)x\) is annihilated by \(I^n\) (where \(n\) is independent of \(x\)). For any \(b \in I\) we have \((1-a)b^n x = 0\) and thus \(b^n x = ab^n x = a^2b^n x = \cdots = 0\), which shows that \(I^n \cdot x = 0\).

**Proposition 8.2.5.** The following conditions for a pair \((A, I)\) are equivalent.

(a) \((A, I)\) satisfies \((BT)\) (resp. \((UBT)\)).

(b) \(A_{I\text{-tor}}\) is of bounded \(I\)-torsion, and \((A/A_{I\text{-tor}}, I(A/A_{I\text{-tor}}))\) satisfies \((BT)\) (resp. \((UBT)\)).

**Proof.** Since \((A/A_{I\text{-tor}})[X] \cong A[X]/A[X]_{I\text{-tor}}\), it suffices to show the assertion for \((BT)\). (a) \(\implies\) (b) is obvious by 8.2.1. To show the converse, let \(M\) be a finitely generated \(A\)-module, and set \(M' = M \otimes_A (A/A_{I\text{-tor}}) = M/A_{I\text{-tor}} M\). Then it is easy to verify that we have the exact sequence

\[
0 \rightarrow A_{I\text{-tor}} M \rightarrow M_{I\text{-tor}} \rightarrow M'_{I\text{-tor}} \rightarrow 0.
\]

Since \(A_{I\text{-tor}} M\) and \(M'_{I\text{-tor}}\) are of bounded \(I\)-torsion, so is \(M_{I\text{-tor}}\).

**Proposition 8.2.6.** Let \((A, I)\) be a pair of finite ideal type, and suppose that

(a) \((\hat{A}, I \hat{A})\) satisfies \((BT)\), and

(b) \(A \rightarrow \hat{A}\) is flat.

Then \((A, I)\) satisfies \((BT)\).

**Proof.** We may assume that \(I\) finitely generated. Since \(A \rightarrow \hat{A}\) is flat, \(A^{Zar} \rightarrow \hat{A}\) is faithfully flat (7.3.8 (2)). By 8.2.3 (2), the pair \((A^{Zar}, IA^{Zar})\) satisfies \((BT)\), and hence the assertion follows from 8.2.4.

**8.2. (b) Preservation of adicness.** We say that a pair \((A, I)\) satisfies \((AP)\) if \(A\) endowed with the \(I\)-adic topology satisfies \((AP)\) in §7.4. (c). We will also consider the following condition for pairs \((A, I)\):

\((UAP)\) for any \(A\)-algebra \(B\) of finite type, the induced pair \((B, IB)\) satisfies \((AP)\).

**Proposition 8.2.7.** Let \((A, I)\) be a pair satisfying \((AP)\) (resp. \((UAP)\)). If \(B\) is a finite \(A\)-algebra (resp. an \(A\)-algebra of finite type), then \((B, IB)\) satisfies \((AP)\) (resp. \((UAP)\)).

**Corollary 8.2.8.** For a pair \((A, I)\) to satisfy \((UAP)\) it is necessary and sufficient that the polynomial ring pairs \((A[X_1, \ldots, X_n], IA[X_1, \ldots, X_n])\) satisfy \((AP)\) for \(n \geq 0\).
One can show the following proposition by an argument similar to that in 8.2.3.

**Proposition 8.2.9.** (1) Let \((A, I)\) and \((B, J)\) be pairs. Then \((A, I)\) and \((B, J)\) satisfy (AP) (resp. (UAP)) if and only if \((A \times B, I \times J)\) satisfies (AP) (resp. (UAP)).

(2) Let \((A, I)\) be a pair and \(B\) a faithfully flat \(A\)-algebra. If \((B, IB)\) satisfies (AP) (resp. (UAP)), then \((A, I)\) satisfies (AP) (resp. (UAP)).

**Proposition 8.2.10.** Let \((A, I)\) be a pair of finite ideal type, and suppose that the following conditions are satisfied.

(a) \((\hat{A}, I \hat{A})\) satisfies (AP).

(b) \(A \to \hat{A}\) is flat.

(c) \(A\) is Noetherian outside \(I\). (8.1.5 (1))

Then \((A, I)\) satisfies (AP).

**Proof.** We may assume that \(I\) is finitely generated. Since \(A \to \hat{A}\) is flat, \(A^{\text{Zar}} \to \hat{A}\) is faithfully flat (7.3.8 (2)). By 8.2.9 (2) the pair \((A^{\text{Zar}}, IA^{\text{Zar}})\) satisfies (AP). On the other hand, since \(I\) is finitely generated, the open set \(\text{Spec } A \setminus V(I)\) is the union of finitely many affine open subsets of the form \(D(f_i)\) (\(i = 1, \ldots, r\)), where \(f_1, \ldots, f_r\) generate \(I\). Since each \(A_{f_i}\) is Noetherian, the pair \((A_{f_i}, IA_{f_i})\) clearly satisfies (AP) (cf. 8.2.11). Let \(B = A^{\text{Zar}} \times A_{f_1} \times \cdots \times A_{f_r}\). Then the pair \((B, IB)\) satisfies (AP) (8.2.9 (1)), and \(A \to B\) is faithfully flat. Then again by 8.2.9 (2) the pair \((A, I)\) satisfies (AP). \(\square\)

For a pair \((A, I)\) condition (AP) can be refined as follows. In classical commutative algebra (*) in 7.4.9 is usually verified by means of the \(I\)-goodness (cf. 7.4.2) of the induced filtration \(G^* = \{N \cap I^n M\}_{n \geq 0}\). For an \(A\)-module \(M\) and an \(A\)-submodule \(N \subseteq M\), the \(I\)-goodness in question is equivalently recast as

\[(**) \text{ there exists a non-negative integer } c \text{ such that for any } n > c \text{ we have } N \cap I^n M = I^{n-c}(N \cap I^c M).\]

Similarly to (AP) and (APf), accordingly, we can consider the following conditions for a pair \((A, I)\):

\[(\text{AR}) (**) \text{ holds for any finitely generated } A\text{-module } M \text{ and any } A\text{-submodule } N \subseteq M;\]

\[(\text{ARf}) (**) \text{ holds for any finitely generated } A\text{-module } M \text{ and any finitely generated } A\text{-submodule } N \subseteq M.\]

Obviously, these conditions, together with (AP) and (APf), sit in the diagram of implications

\[\begin{array}{c}
\text{(AR)} \implies \text{(AP)} \\
\text{(ARf)} \implies \text{(APf)}
\end{array}\]
It is well known that (AR) is always satisfied if $A$ is Noetherian. This is exactly what the classical Artin–Rees lemma asserts.

**Proposition 8.2.11** (classical Artin–Rees lemma; cf. e.g. [81], Theorem 8.5). Any pair $(A, I)$ with $A$ Noetherian satisfies (AR).

Conditions (AR) and (ARf) depend on the choice of the ideal of definition $I \subseteq A$; hence, when considering these conditions for adically topologized rings, one has to specify an ideal of definition.

### 8.2 (c) The properties (BT) and (AP)

**Proposition 8.2.12.** (AP) implies (BT).

**Proof.** Suppose a pair $(A, I)$ satisfies (AP), and let $M$ be a finitely generated $A$-module. Define $F_k$ for any $k \geq 1$ to be the $A$-submodule of $M$ consisting of the elements annihilated by $I^k$. Clearly, $M_{1\text{-tor}} = \bigcup_{k \geq 1} F_k$. By Exercise 0.7.8, there exists $m > 0$ such that $F_1 \cap I^m M = 0$. Then for $k \geq m + 1$ we have $I^{k-1} F_k \subseteq F_1 \cap I^m M = 0$ and hence $F_m = F_{m+1} = F_{m+2} = \cdots$. Consequently, $M_{1\text{-tor}} = F_m$, which is of bounded $I$-torsion.

**Proposition 8.2.13.** Let $(A, I)$ be a pair with $I = (a)$ principal. Suppose $(A, I)$ satisfies (BT). Then it satisfies (AR) too.

The main part of the proof relies on the following lemma.

**Lemma 8.2.14.** Let $(A, I)$ be a pair with $I = (a)$ principal, $M$ an $A$-module, and $N \subseteq M$ an $A$-submodule. Suppose $a^n (M/N)_{a\text{-tor}} = 0$ for some $n \geq 0$. Then for any $m \geq 0$ we have
\[
N \cap a^{n+m} M = a^m (N \cap a^n M),
\]
that is, condition (**) in §7.4.(c) is satisfied.

**Proof.** Set $L = M/N$, and denote the canonical projection $M \to L$ by $\pi$. The inclusion $a^m (N \cap a^n M) \subseteq N \cap a^{n+m} M$ is trivial. To show the converse inclusion, take $x = a^m y \in N \cap a^{n+m} M$ ($y \in a^n M$). Since $a^m \pi(y) = \pi(x) = 0$ in $L$, we have $\pi(y) \in a^n L \cap L_{a\text{-tor}}$. But since $a^n L \cap L_{a\text{-tor}} = a^n L_{a\text{-tor}} = 0$, we have $y \in N$, that is, $x = a^m y \in a^m (N \cap a^n M)$, as desired.

**Proof of Proposition 8.2.13.** Let $M$ be a finitely generated $A$-module and $N \subseteq M$ an $A$-submodule. Condition (BT) implies $a^n (M/N)_{a\text{-tor}} = 0$ for some $n \geq 0$. Hence, by 8.2.14, condition (**) in §7.4.(c) is satisfied.
Proposition 8.2.15. Let $A$ be a ring and $I = (a_1, \ldots, a_r) \subseteq A$ a finitely generated ideal. Suppose that for each $i = 1, \ldots, r$ the pair $(A, a_i)$ satisfies (BT). Then the pair $(A, I)$ satisfies (BT) and (AP).

Proof. For any finitely generated $A$-module $M$ the $a_i$-torsion part of $M$ is bounded for each $i = 1, \ldots, r$. As the $I$-torsion part of $M$ is the intersection of the $a_i$-torsion parts for $1 \leq i \leq r$, $M$ is of bounded $I$-torsion. Thus $(A, I)$ satisfies (BT).

To show that $(A, I)$ satisfies (AP), we apply induction with respect to $r$. If $r = 1$, the assertion follows from 8.2.13. Set $a = a_1$ and $J = (a_2, \ldots, a_r)$, and let $N \subseteq M$ be an $A$-submodule such that $I^n M = 0$. We want to show that $N \cap I^m M = 0$ for some $m \geq 0$ (cf. Exercise 0.7.8). Take $s \geq 0$ so that $a^s$ annihilates the $a$-torsion part of $M$. Since $N$ is $a$-torsion, we have $N \cap a^s M = 0$. Hence one can regard $N$ as a submodule of $\overline{M} = M/a^s M$. Note that the $I$-adic topology on $\overline{M}$ coincides with the $J$-adic topology. By induction, we know that $N \cap I^t \overline{M} = 0$ for some $t \geq 0$ (cf. Exercise 0.7.8). Then for $m \geq \max\{s, t\}$ one has $N \cap I^m M = 0$, as desired. \hfill $\Box$

Proposition 8.2.16. Let $(A, I)$ be a pair of finite ideal type, and suppose that $A$ is Noetherian outside $I$ (8.1.5 (1)). If $(A, I)$ satisfies (BT), then for any finitely generated subideal $J \subseteq I$, $(A, J)$ satisfies (BT) and (AP).

Proof. In view of 8.2.15 we only have to show that for any principal subideal $J = (a) \subseteq I$, $(A, a)$ satisfies (BT). Let $M$ be a finitely generated $A$-module. Since Spec $A \setminus V(I)$ is Noetherian, there exists a finitely generated submodule $N$ of $M_{a\text{-tor}}$ such that $M_{a\text{-tor}}/N$ is supported on $V(I)$ (8.1.6 (1)). Now the $a$-torsion part of $M/N$ is $M_{a\text{-tor}}/N$, which is nothing but the $I$-torsion part of $M/N$. Hence $M_{a\text{-tor}}/N$ is of bounded $a$-torsion. Since $N$ is finitely generated and hence of bounded $a$-torsion, we deduce that $M_{a\text{-tor}}$ is of bounded $a$-torsion. \hfill $\Box$

Corollary 8.2.17. Let $(A, I)$ be a pair of finite ideal type. If $A$ is Noetherian outside $I$ and $(A, I)$ satisfies (BT), then it also satisfies (AP).

Proposition 8.2.18. Let $(A, I)$ be a pair of finite ideal type satisfying (BT), and suppose $A$ is Noetherian outside $I$.

(1) For any finitely generated $A$-module $M$ the canonical map $M \otimes_A \hat{A} \to \hat{M}$ is an isomorphism.

(2) The canonical map $A \to \hat{A}$ is flat.

Proof. (1) By 8.1.6 (2), there exists a surjective morphism $N \to M$ of $A$-modules such that $N$ is finitely presented and the kernel $K$ is $I$-torsion. By (BT), the kernel
$K$ is bounded $I$-torsion. Consider the exact sequence $0 \to K \to N \to M \to 0$, which yields the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \to & \widehat{K} & \to & \widehat{N} & \to & \widehat{M} & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & K \otimes_A \widehat{A} & \to & N \otimes_A \widehat{A} & \to & M \otimes_A \widehat{A} & \to & 0.
\end{array}
$$

Note that the exactness of the first row is due to 7.4.11 (here we use 8.2.17). The first vertical arrow is an isomorphism due to 8.1.4, and the second one is an isomorphism due to 7.4.15. Hence the last vertical arrow is an isomorphism, which is what we wanted to show.

(2) Let $a \subseteq A$ be a finitely generated ideal. We want to show that the map $a \otimes_A \widehat{A} \to \widehat{A}$ is injective. By (1) it follows that $a \otimes_A \widehat{A} = \widehat{a}$. By 7.4.11, on the other hand, $\widehat{a}$ is an ideal of $\widehat{A}$, as desired.  

\[ \Box \]

8.2. (d) Bounded torsion condition for complete pairs. For the proof of the following significant theorem by Gabber, we refer to [41].

**Theorem 8.2.19** (O. Gabber; [41], Theorem 5.1.2). Let $(A, I)$ be a complete pair of finite ideal type, and suppose that $A$ is Noetherian outside $I$. Then $(A, I)$ satisfies (BT) and (AP).

In particular, by 7.4.17 and 7.4.18 we have the following corollary.

**Corollary 8.2.20.** Let $(A, I)$ be a complete pair of finite ideal type, and suppose that $A$ is Noetherian outside $I$. Then any finitely generated $A$-module is $I$-adically complete. Moreover, if $M$ is a finitely generated $A$-module, then any $A$-submodule $N \subseteq M$ is closed in $M$ with respect to the $I$-adic topology.

8.3 Pairs and flatness

8.3. (a) gluing of flatness

**Theorem 8.3.1** (gluing of flatness (I)). Let $(A, I)$ be a pair of finite ideal type, and $M$ an $A$-module. Then $M$ is flat over $A$ if and only if the following conditions hold.

(a) $\text{Tor}_q^A(M, N) = 0$ for any $q \geq 1$ and any $A$-module $N$ supported in $V(I) \subseteq \text{Spec } A$.

(b) $\tilde{M}$ is flat over $\text{Spec } A \setminus V(I)$.
Proof. We may assume that the ideal $I \subseteq A$ is finitely generated; let $a_1, \ldots, a_n$ generate $I$ ($n \geq 0$). For $k = 0, \ldots, n$ we set $I_k = (a_1, \ldots, a_k)$. Consider the condition

$$(*)_k \quad \text{Tor}^A_q(M, N) = 0 \text{ for any } q \geq 1 \text{ and any } A\text{-module } N \text{ supported in } V(I_k) \subseteq \text{Spec } A.$$  

Condition $(*)_n$ is nothing but $(a)$, and what we want to establish is $(*)_0$ (cf. [27], Chapter I, §4, Proposition 1). It suffices therefore to show $(*)_k$ by descending induction with respect to $k$.

Suppose $(*)_k \Rightarrow (b)$ is true, and let $N$ be an $A$-module supported in $V(I_k)$. Set $a = a_{k+1}$, and consider the exact sequence

$$0 \longrightarrow N_{a\text{-tor}} \longrightarrow N \longrightarrow N/N_{a\text{-tor}} \longrightarrow 0.$$

Since $N_{a\text{-tor}}$ is supported in $V(I_{k+1})$, by induction we have $\text{Tor}^A_q(M, N_{a\text{-tor}}) = 0$ for $q \geq 1$. Hence it suffices to show $\text{Tor}^A_q(M, N/N_{a\text{-tor}}) = 0$ for $q \geq 1$, and thus we may assume that $N$ is $a$-torsion free. Consider the exact sequence

$$0 \longrightarrow N \longrightarrow N \left[ \frac{1}{a} \right] \longrightarrow C \longrightarrow 0.$$

Here the $A$-module $C$ is supported in $V(I_{k+1})$. Since $N \left[ \frac{1}{a} \right]$ is flat over $A \left[ \frac{1}{a} \right]$ (by (b)), we have $\text{Tor}^A_q(M, N) = 0$ for $q \geq 1$, as desired.

Corollary 8.3.2. Let $(A, a)$ be a pair with $a \in A$ being a non-zero divisor, and $M$ an $A$-module. Then the following conditions are equivalent.

(a) $M$ is $A$-flat.

(b) $M \left[ \frac{1}{a} \right]$ is $A \left[ \frac{1}{a} \right]$-flat, $M/aM$ is $(A/aA)$-flat, and $M$ is $a$-torsion free.

Proof. (a) $\Rightarrow$ (b) is obvious. To show the converse, in view of 8.3.1 it suffices to show that $\text{Tor}^A_q(M, N)$ vanishes for $q \geq 1$ and for any $A$-module $N$ supported in $V((a))$. Since the functor $\text{Tor}^A_q(M, -)$ commutes with inductive limits, we may assume that $N$ is annihilated by some $a^n$. But if so, we may further reduce the situation where $N$ is annihilated even by $a$ by the inductive argument using the filtration $(a^mN)_{m \geq 0}$.

Since $a$ is a non-zero-divisor and $M$ is $a$-torsion free, $\text{Tor}^A_q(M, A/aA) = 0$ for $q \geq 1$. This implies the last isomorphism in the relations

$$M \otimes_A^L N \cong M \otimes_A^L (A/aA) \otimes_{A/aA}^L N \cong (M/aM) \otimes_{A/aA}^L N.$$  

Now, since $M/aM$ is $(A/aA)$-flat, we deduce that $\text{Tor}^A_q(M, N) = 0$ for $q \geq 1$, as desired.
**Proposition 8.3.3** (gluing of flatness (II)). Let \((A, I)\) be a pair of finite ideal type, \(B\) an \(A\)-algebra, and \(M\) a \(B\)-module. Suppose that

(a) \(B\) and \(M\) are flat over \(A\), and

(b) \(M/IM\) is flat over \(B/IB\), and \(\tilde{M}\) is flat over \(\text{Spec } B \setminus V(IB)\).

Then \(M\) is \(B\)-flat.

**Proof.** In case the ideal \(I\) is principal, the assertion follows immediately from [44], 5.2.1. In general, in order to apply induction with respect to the number of generators, set \(I = (a_1, \ldots, a_s, b)\) and \(J = (a_1, \ldots, a_s)\). Since \(\bar{A} = A/bA\), \(B/bB\), and \(M/bM\) together with the ideal \(\bar{I} = J A\) satisfy the conditions, we deduce by induction that \(M/bM\) is \(B/bB\)-flat. Considering next the situation with \(A\), \(B\), \(M\), and \((b) \subseteq A\), we conclude that \(M\) is \(B\)-flat, as desired. \(\Box\)

8.3. (b) Local criterion of flatness

**Proposition 8.3.4** (local criterion of flatness). Let \((A, I)\) be a pair and \(M\) an \(A\)-module. Suppose that following conditions are satisfied.

(i) For any finitely generated ideal \(\alpha\) of \(A\), the topology on \(\alpha\) induced by that of \(A\) coincides with the \(I\)-adic topology; that is, for any \(n \geq 0\) there exists \(k \geq 0\) such that

\[ I^k \cap \alpha \subseteq I^n \alpha. \]

(ii) \(M\) is idealwise separated for \(I\), that is, for any finitely generated ideal \(\alpha\) of \(A\), the \(A\)-module \(\alpha \otimes_A M\) is \(I\)-adically separated.

We write \(A_k = A/I^{k+1}\) and \(M_k = M/I^{k+1}M\) for any \(k \geq 0\). Then the following conditions are equivalent.

(a) \(M\) is \(A\)-flat.

(b) for any \(A_0\)-module \(N\) we have \(\text{Tor}_1^A(N, M) = 0\).

(c) \(M_0\) is \(A_0\)-flat, and we have \(\text{Tor}_1^A(A_0, M) = 0\).

(d) \(M_k\) is \(A_k\)-flat for any \(k \geq 0\).

The main idea of the proof is borrowed from [27], Chapter III, §5.3.

**Proof.** Implications (a) \(\implies\) (b) \(\implies\) (c) \(\implies\) (d) are shown in [27], Chapter III, §5.2, Theorem 1, and here we omit the proofs. We are going to show (d) \(\implies\) (a) by verifying that for any finitely generated ideal \(\alpha\) of \(A\), the map \(\iota: \alpha \otimes_A M \to M\) is injective.
Take $x \in \ker(i)$. Since $\bigcap_n I^n (a \otimes_A M) = 0$ by hypothesis (ii), it suffices to show that $x \in I^n (a \otimes_A M)$ for any $n$. By (i), there exists $k$ such that $I^{k+1} \cap a \subseteq I^n a$. Hence it is enough to show that $x$ belongs to the image of the canonical map $(I^{k+1} \cap a) \otimes_A M \rightarrow a \otimes_A M$. To this end, we consider the commutative diagram with exact top row

$$
\begin{array}{cccccc}
(I^{k+1} \cap a) \otimes_A M & \rightarrow & a \otimes_A M & \rightarrow & (a/a \cap I^{k+1}) \otimes_A M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
M & \rightarrow & A_k \otimes_A M.
\end{array}
$$

Here by the assumption, the first vertical arrow maps $x$ to 0. Hence $x$ is mapped to 0 in $A_k \otimes_A M$. On the other hand, the second vertical arrow coincides with the morphism $(a/a \cap I^{k+1}) \otimes_{A_k} M_k \rightarrow M_k$, which is injective by our assumption that $M_k$ is $A_k$-flat. Hence $x$ belongs to the kernel of the map $(\ast)$, which is nothing but the image of the first horizontal arrow in the top row. But this is what we wanted to prove.

**Corollary 8.3.5.** Let $(A, I)$ be a pair, and $M$ an $A$-module. We suppose that (i) and (ii) in 8.3.4 are satisfied and that $(A, I)$ is a Zariskian pair. Then the following conditions are equivalent.

(a) $M$ is faithfully flat over $A$.

(b) $M_k$ is faithfully flat over $A_k$ for any $k \geq 0$.

**Proof.** Implication (a) $\Rightarrow$ (b) is clear. Conversely, by 8.3.4 we know that $M$ is $A$-flat. To show that $M$ is faithfully flat, it suffices to show that for any maximal ideal $m$ of $A$ we have $M \otimes_A (A/m) \neq 0$. Since $(A, I)$ is Zariskian, we have $I \subseteq m$. Hence $M \otimes_A (A/m) = M_0 \otimes_{A_0} (A_0/m_0)$, where $m_0 = m/I$. Since $M_0$ is faithfully flat over $A_0$, the last module is non-zero, as desired.

**Remark 8.3.6.** Conditions (i) and (ii) in 8.3.4 are trivially satisfied if $I$ is a nilpotent ideal. In this case, moreover, one can easily show (cf. [52], Exposé IV, Proposition 5.1) that conditions (a) $\sim$ (d) are equivalent to

(e) $M_0$ is $A_0$-flat, and the canonical surjective morphism

$$
gr^0_I(M) \otimes_{A_0} \gr^\bullet_I(A) \rightarrow \gr^\bullet_I(M)
$$

is an isomorphism.

In particular, we have the following result (cf. [52], Exposé IV, Corollary 5.9).
Proposition 8.3.7. Let \((A, I)\) be a pair where \(I\) is nilpotent, \(B\) a flat \(A\)-algebra, and \(M\) a \(B\)-module. Then the following conditions are equivalent.

(a) \(M\) is flat over \(B\).

(b) \(M\) is flat over \(A\) and \(M_0 = M/IM\) is flat over \(B_0 = B/IB\).

Proof. (a) \(\Rightarrow\) (b) is trivial. Suppose that (b) holds. In order to apply 8.3.4, we check (e) in 8.3.6. For \(n \geq 0\) the \(n\)-th graded piece of \(\text{gr}^0_{IB}(M) \otimes_{B_0} \text{gr}^*_{IB}(B)\) is \(M \otimes_B (I^nB/I^{n+1}B)\). Since \(B\) is \(A\)-flat, \(I^nB/I^{n+1}B = (I^n/I^{n+1}) \otimes_A B\). Since \(M\) is \(A\)-flat, we have \(M \otimes_B (I^nB/I^{n+1}B) = M \otimes_A (I^n/I^{n+1}) = I^nM/I^{n+1}M\), which is the \(n\)-th graded piece of \(\text{gr}^*_{IB}(M)\).

Finally, we give a useful sufficient condition to verify (i) and (ii) in 8.3.4.

Proposition 8.3.8. Let \((A, I)\) be a pair, \(B\) an \(A\)-algebra, and \(M\) a finitely generated \(B\)-module. Suppose that \((A, I)\) and \((B, IB)\) satisfy (APf) and that \(B\) is \(IB\)-adically Zariskian. Then (i) and (ii) in 8.3.4 are satisfied, and therefore (a)–(d) in 8.3.4 are all equivalent.

Proof. Clearly, (i) in 8.3.4 is satisfied. To verify (ii), let \(a \subseteq A\) be a finitely generated ideal, and consider \(N = a \otimes_A M\). Since \(N\) is finitely generated over \(B\), it is \(I\)-adically separated due to 7.4.16.

Corollary 8.3.9. Let \((A, I)\) and \((B, IB)\) be as in 8.3.8. Then \(A \to B\) is flat if and only if \(A_k = A/I^{k+1} \to B_k = B/I^{k+1}B\) is flat for any \(k \geq 0\).

8.3. (c) Formal fpqc descent of ‘Noetherian outside \(I\).’ The following proposition, which gives a formal fpqc patching principle for the property ‘Noetherian outside \(I\),’ will be of fundamental importance in our later discussion.

Proposition 8.3.10 (O. Gabber; [41], Proposition 5.2.1). Let \((A, I) \to (B, IB)\) be an adic morphism between complete pairs of finite ideal type such that for any \(k \geq 0\) the induced map \(A/I^{k+1} \to B/I^{k+1}B\) is faithfully flat. Suppose \(B\) is Noetherian outside \(IB\) (8.1.5 (1)). Then \(A\) is Noetherian outside \(I\). Moreover, the map \(A \to B\) is faithfully flat.

Proof. To prove the first assertion, we want to show that any ideal \(J \subseteq A\) is finitely generated outside \(I\) (that is, the associated quasi-coherent ideal \(\overline{J}\) on \(\text{Spec } A\) is of finite type over \(\text{Spec } A \setminus V(I)\)). Considering an approximation of \(J\) by finitely generated subideals of \(J\), one has a finitely generated subideal \(J_0 \subseteq J\) such that \(J_0B\) and \(JB\) coincides outside \(I\). This means that \(JB/J_0B\) is \(I\)-torsion; since the \(I\)-torsion part of \(B/J_0B\) is bounded (due to 8.2.19), there exists \(n > 0\) such that
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$I^n J B \subseteq J_0 B$. Moreover, by the assumption, $I^n J A / I^{k+1} \subseteq J_0 A / I^{k+1}$ for any $k \geq 0$, that is,

$$\bigcap_{k \geq 0} (I^n J + I^{k+1}) \subseteq \bigcap_{k \geq 0} (J_0 + I^{k+1}) = J_0$$  \hspace{1cm} (\ast)$$

(the closure of $J_0$ in $A$).

We want to show the inclusion $I^n J \subseteq J_0$, for this implies that $J$ and $J_0$ coincide outside $I$. By (\ast), it suffices that $J_0$ is closed in $A$. As $J_0$ is $I$-adically complete (7.4.7), it is enough to show that the subspace topology on $J_0$ induced by the $I$-adic topology on $A$ is $I$-adic.

To show this, we first use (AP) for $B$ to deduce that for any $i > 0$ there exists $m = m(i) > i$ such that

$$J_0 B \cap I^m(i) B \subseteq I^i J_0 B.$$  

Again, by the assumption,

$$J_0 \cap I^m(n) \subseteq I^n J_0.$$  \hspace{1cm} (\ast\ast)$$

We want to show that the left-hand side is actually contained in $I^n J_0$. Suppose $x$ lies in the left-hand side. By (\ast\ast), $x$ is decomposed as

$$x = z_1 + x_1, \quad z_1 \in I^n J_0, \quad x_1 \in I^m(n+1).$$

As $x_1$ also lies in $J_0$, we again apply (\ast\ast) to decompose $x_1$ into the sum of $z_2 \in I^{n+1} J_0$ and $x_2 \in I^m(n+2)$. One can repeat this procedure to get sequences $\{z_k\}$ and $\{x_k\}$ such that

$$x_k = z_{k+1} + x_{k+1}, \quad z_{k+1} \in I^{n+k} J_0, \quad x_{k+1} \in I^m(n+k+1).$$

Hence $x$ is equal to the infinite series $\sum_{k \geq 1} z_k$, which converges in the $I$-adically complete $I^n J_0$ (7.4.7). This means that $x \in I^n J_0$, which shows the first assertion of the proposition. The other assertion follows from 8.3.8, 8.2.19, and 8.3.5. \hfill \Box

8.4 Restricted formal power series ring

Let $(A, I)$ be a complete pair of finite ideal type, and $M$ an $I$-adically complete $A$-module. We denote by

$$M \llangle X_1, \ldots, X_n \rrangle$$

the set of all formal power series $f = \sum a_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n}$ with all coefficients $a_{i_1, \ldots, i_n}$ in $M$ such that for any $m \geq 1$ there exists $N \geq 1$ such that $a_{i_1, \ldots, i_n} \in I^m M$ whenever $i_1 + \cdots + i_n > N$. This is an $I$-adically complete $A$-module.
In particular, $A \langle X_1, \ldots, X_n \rangle$ is an $I$-adically complete $A$-algebra, the so-called restricted formal power series ring ([54], 0.1, (7.5.1)), which is isomorphic to the $I$-adic completion of the polynomial ring $A[X_1, \ldots, X_n]$ and yields the complete pair $(A \langle X_1, \ldots, X_n \rangle, IA \langle X_1, \ldots, X_n \rangle)$ of finite ideal type. Clearly, we have

$$M \hat{\otimes}_A A \langle X_1, \ldots, X_n \rangle \cong M \langle X_1, \ldots, X_n \rangle,$$

and if $Y_1, \ldots, Y_m$ is another set of variables, then

$$M \langle X_1, \ldots, X_n \rangle \hat{\otimes}_A A \langle Y_1, \ldots, Y_m \rangle \cong M \langle X_1, \ldots, X_n, Y_1, \ldots, Y_m \rangle.$$

**Definition 8.4.1.** Let $(A, I)$ be a complete pair of finite ideal type. An $I$-adically complete $A$-algebra $B$ is said to be a topologically finitely generated $A$-algebra or an $A$-algebra topologically of finite type if $B$ is isomorphic to an $A$-algebra of the form $A \langle X_1, \ldots, X_n \rangle/a$. If, moreover, $a$ is finitely generated, we say that $B$ is a topologically finitely presented $A$-algebra or an $A$-algebra topologically of finite presentation.

In this book, as indicated in the above definition, all topologically finitely generated/presented algebras are assumed to be complete (and hence the ideal $a$ as above has to be closed due to 7.4.6.) As we will see soon (8.4.4), this hypothesis is practically not restrictive.

**Proposition 8.4.2.** Let $(A, I)$ be a complete pair with a finitely generated ideal $I \subseteq A$, and $B$ an $I$-adically complete $A$-algebra. Then the following conditions are equivalent.

(a) $B$ is topologically finitely generated over $A$.

(b) $B/IB$ is an $(A/I)$-algebra of finite type.

**Proof.** Implication (a) $\implies$ (b) is clear. Suppose (b) holds, and take $c_1, \ldots, c_n \in B$ whose images in $B/IB$ generates $B/IB$ as an $(A/I)$-algebra. Since $B$ is $IB$-adically complete, there exists a morphism $A' = A \langle X_1, \ldots, X_n \rangle \to B$ that maps $X_i$ to $c_i$ for $i = 1, \ldots, n$. By 7.2.4, this map is surjective.

**Definition 8.4.3.** Let $(A, I)$ be a pair of finite ideal type. We say that $A$ is topologically universally Noetherian outside $I$ if it is Noetherian outside $I$ (8.1.5 (1)) and for any $n \geq 0$, $\hat{A} \langle X_1, \ldots, X_n \rangle$ (the $I$-adic completion of $A[X_1, \ldots, X_n]$) is Noetherian outside $IA \langle X_1, \ldots, X_n \rangle$.

It follows from the definition that if $A$ is topologically universally Noetherian outside $I$, then any topologically finitely generated $\hat{A}$-algebra $B$ is topologically universally Noetherian outside $IB$. 
Proposition 8.4.4. Let \((A, I)\) be a complete pair of finite ideal type, and suppose that \(A\) is topologically universally Noetherian outside \(I\). Then for any \(n \geq 0\) any ideal \(\mathfrak{a} \subseteq A\langle X_1, \ldots, X_n \rangle\) is closed (hence any ring of the form \(A\langle X_1, \ldots, X_n \rangle/\mathfrak{a}\) is topologically finitely generated over \(A\) in our sense).

Proof. Let \(B = A\langle X_1, \ldots, X_n \rangle\). Since \(B\) is Noetherian outside \(IB\), it satisfies (AP) (8.2.19). Then by 7.4.18 any ideal of \(A\langle X_1, \ldots, X_n \rangle\) is closed. □

Proposition 8.4.5. Let \((A, I)\) be a complete pair of finite ideal type, and suppose that \(A\) is topologically universally Noetherian outside \(I\). Let \(B\) be an \(I\)-adically complete \(A\)-algebra. Then if \(B/I^n B\) is an \(A/I^n\)-algebra of finite presentation for any \(n \geq 1\), \(B\) is topologically finitely presented over \(A\).

Proof. By 8.4.2, \(B\) is isomorphic to an \(A\)-algebra of the form \(A\langle X_1, \ldots, X_n \rangle/\mathfrak{a}\). It then follows from 7.4.19 that \(B\) is finitely presented as an \(A\langle X_1, \ldots, X_n \rangle\)-module. □

Proposition 8.4.6. Let \((A, I)\) be a complete pair of finite ideal type, and suppose that \(A\) is Noetherian outside \(I\).

1. Any finitely generated \(A\)-module is \(I\)-adically complete, and for any finitely generated \(A\)-module \(M\) the canonical map
\[
M \otimes_A A\langle X_1, \ldots, X_n \rangle \longrightarrow M\langle X_1, \ldots, X_n \rangle
\]
is an isomorphism.

2. The map \(A \rightarrow A\langle X_1, \ldots, X_n \rangle\) is flat.

Proof. First note that, by 8.2.19, \((A, I)\) satisfies (AP) and (BT). To show (1), we first note that the completeness of \(M\) was already proved in 8.2.20. To show the other statement in (1), we first assume that \(M\) is finitely presented. Take a presentation \(A^{\oplus q} \rightarrow A^{\oplus p} \rightarrow M \rightarrow 0\), and let \(K\) be the image of \(A^{\oplus q} \rightarrow A^{\oplus p}\). Then the subspace topology on \(K\) coincides with the \(I\)-adic topology (by (AP)), and \(K\) is complete. Hence we have the exact sequence
\[
0 \longrightarrow K\langle X_1, \ldots, X_n \rangle \longrightarrow A\langle X_1, \ldots, X_n \rangle^{\oplus p} \longrightarrow M\langle X_1, \ldots, X_n \rangle \longrightarrow 0.
\]
Thus we get an exact sequence
\[
A\langle X_1, \ldots, X_n \rangle^{\oplus q} \longrightarrow A\langle X_1, \ldots, X_n \rangle^{\oplus p} \longrightarrow M\langle X_1, \ldots, X_n \rangle \longrightarrow 0.
\]
On the other hand, since \(A^{\oplus p} \otimes_A A\langle X_1, \ldots, X_n \rangle \cong A\langle X_1, \ldots, X_n \rangle^{\oplus p}\), one can show the desired isomorphism by an argument similar to that in the proof of 7.4.15.
In general, we first observe that, since \( M \) is finitely presented outside \( I \), we have a surjective map \( N \to M \) from a finitely presented \( A \)-module such that the kernel is \( I \)-torsion (8.1.6 (2)). We can apply the same argument as in the proof of 8.2.18 (1), once we know that the assertion is true for bounded \( I \)-torsion modules (here we use (BT)). But the assertion in this case is easy, for both \( M \otimes_A A\langle X_1, \ldots, X_n \rangle \) and \( M\langle X_1, \ldots, X_n \rangle \) are isomorphic to \( M[X_1, \ldots, X_n] \).

The second assertion can be shown by an argument similar to that in the proof of 8.2.18 (2).

**Proposition 8.4.7.** Let \((A, I)\) be a complete pair of finite ideal type, and consider the restricted power series ring \( A\langle X \rangle \) in one variable. Suppose that \( A \) is Noetherian outside \( I \). Then \( A\langle X \rangle \) is flat over \( A[X] \).

**Proof.** First we note that both \( A\langle X \rangle \) and \( A\langle X \rangle \) are flat over \( A \) (8.4.6 (2)). Since \( A\langle X \rangle / I A\langle X \rangle \cong (A/I)[X] \), in view of 8.3.3 we only need to show the flatness over the points outside \( I \). To this end, take \( x \in U = \text{Spec } A \setminus V(I) \), and set \( R = O_{U,x} \), which is a Noetherian local ring. Then it suffices to show that the map

\[
R[X] \longrightarrow A\langle X \rangle \otimes_A R
\]

is flat.

**Claim.** If \((\otimes)\) is flat outside the maximal ideal \( m_R \) of \( R \), then \((\otimes)\) is flat.

Indeed, again applying 8.3.3 with \( A \) replaced by \( R \) and the ideal \( I \) by \( m_R \), we find that we only have to show that the induced map \((\otimes)_R \) is flat, where \( k \) is the residue field of \( R \). Let \( p \subseteq A \) be the prime ideal corresponding to \( x \). By 8.4.6 (1), \( A\langle X \rangle \otimes_A (A/p) \cong (A/p)[X] \). Hence \( A\langle X \rangle \otimes_A k \) is regarded as a subring of \( k[X] \). Since \( k[X] \) is torsion free as a \( k[X] \)-module, \( A\langle X \rangle \otimes_A k \) is also torsion free. But since \( k[X] \) is PID, this means that \( A\langle X \rangle \otimes_A k \) is flat over \( k[X] \), as desired.

Now we want to show the flatness of \((\otimes)\) by induction with respect to \( \dim(R) \). If \( \dim(R) = 0 \), then \( \text{Spec } R \setminus V(m_R) \) is empty, and there is nothing to prove. Hence the desired flatness follows from the claim.

If \( \dim(R) > 0 \), by induction with respect to \( \dim(R) \) the map \((\otimes)\) replaced by \( R_p \) for any non-maximal prime ideal \( p \) (that is, the local ring at the point \( y \in U \) corresponding to \( p \)) is flat. In particular, \((\otimes)\) is flat outside \( m_R \). Hence again by the claim we deduce that \((\otimes)\) is flat.

**Theorem 8.4.8.** Let \((A, I)\) be a complete pair of finite ideal type. Suppose that \( A \) is topologically universally Noetherian outside \( I \). Then for any \( n \geq 0 \) the pair \((A\langle X_1, \ldots, X_n \rangle, I A\langle X_1, \ldots, X_n \rangle)\) satisfies (UBT) and (UAP) (§8.2. (b)).

**Proof.** By 8.2.2, 8.2.8, and 8.2.16 we only need to check that a pair of the form

\[
(A\langle X_1, \ldots, X_r \rangle[Y_1, \ldots, Y_s], I A\langle X_1, \ldots, X_r \rangle[Y_1, \ldots, Y_s])
\]
satisfies (BT). We claim that the map

$$A \langle X_1, \ldots, X_r \rangle [Y_1, \ldots, Y_s] \longrightarrow A \langle X_1, \ldots, X_r, Y_1, \ldots, Y_s \rangle$$

is flat. The case $s = 1$ follows from 8.4.7. The general case follows by induction with respect to $s$ from the factorization

$$A \langle X_1, \ldots, X_r \rangle [Y_1, \ldots, Y_s] \longrightarrow A \langle X_1, \ldots, X_r, Y_1, \ldots, Y_{s-1} \rangle [Y_s] \longrightarrow A \langle X_1, \ldots, X_r, Y_1, \ldots, Y_s \rangle.$$

Now apply 8.2.19 and 8.2.6 to deduce the desired result.

\[\square\]

### 8.5 Adhesive pairs

#### 8.5. (a) Adhesive pairs and universally adhesive pairs

**Definition 8.5.1.** A pair $(A, I)$ of finite ideal type is pseudo-adhesive if

(a) $A$ is Noetherian outside $I$ (8.1.5), and

(b) $(A, I)$ satisfies the condition (BT) in §8.2. (a).

A pseudo-adhesive pair $(A, I)$ is said to be adhesive if it satisfies the following condition, stronger than (b):

(c) the $I$-torsion part $M_{I,\text{tor}}$ of any finitely generated $A$-module $M$ is finitely generated.

It is clear that the pseudo-adhesiveness and the adhesiveness depend only on the topology on $A$, and not on the ideal $I$ itself. In the sequel we often say that the ring $A$ is $I$-adically adhesive (resp. $I$-adically pseudo-adhesive) to mean that the pair $(A, I)$ is adhesive (resp. pseudo-adhesive). Note that the theorem of Gabber (8.2.19) implies the following result.

**Proposition 8.5.2.** A complete pair $(A, I)$ of finite ideal type is pseudo-adhesive if and only if $A$ is Noetherian outside $I$.

**Proposition 8.5.3.** Let $(A, I)$ be a pair of finite ideal type. Then the following conditions are equivalent:

(a) the pair $(A, I)$ is adhesive;

(b) for any finitely generated $A$-module $M$, $M/M_{I,\text{tor}}$ is finitely presented;

(c) for any finitely generated $A$-module $M$ and any $A$-submodule $N$ of $M$, the $I$-saturation $\bar{N}$ of $N$ in $M$ is finitely generated.
Proof. We may assume that the ideal $I \subseteq A$ is finitely generated. First we show the equivalence of (b) and (c). Suppose that (b) holds, and let $M$ and $N$ be as in (c). The $I$-saturation $\widetilde{N}$ (§8.1.(c)) sits in the following exact sequence:

$$0 \to \widetilde{N} \to M \to (M/N)/(M/N)_{I,\text{tor}} \to 0.$$ 

By [27], Chapter I, §2.8, Lemma 9, we deduce $\widetilde{N}$ is finitely generated, whence (c). The converse ($(c) \implies (b)$) follows from the fact that if $\phi: F \to M/M_{I,\text{tor}}$ is a surjective morphism from a finitely generated $A$-module, then $\ker(\phi)$ is saturated.

Next let us show the equivalence of (a) and (c). Suppose that (c) holds, and let $M$ be a finitely generated $A$-module. Since $M_{I,\text{tor}}$ is $I$-saturated, it is finitely generated. To show that $\text{Spec } A \setminus V(I)$ is a Noetherian scheme, it suffices to show that $A[a^{-1}]$ is Noetherian for any $a \in I$. Let $J$ be an ideal of $A[a^{-1}]$, and $J'$ the pull-back of $J$ by $A \to A[a^{-1}]$. Then $J'$ is easily seen to be $I$-saturated, and we have $J'A[a^{-1}] = J$. Since $J'$ is finitely generated, so is $J$. Conversely, suppose that (a) holds. Let $N$ be a submodule of a finitely generated $A$-module $M$. By 8.1.6 (1) we can find a finitely generated submodule $N'$ of $N$ such that $N/N'$ is $I$-torsion. Since we have $\widetilde{N'} = \widetilde{N}$, we can replace $N$ by $N'$ and hence assume that $N$ is finitely generated. Then the exact sequence

$$0 \to N \to \widetilde{N} \to (M/N)_{I,\text{tor}} \to 0$$

gives that $\widetilde{N}$ is finitely generated.

**Definition 8.5.4.** A pair $(A, I)$ is said to be universally adhesive (resp. universally pseudo-adhesive) if for any $n \geq 0$ the pair $(A[X_1, \ldots, X_n], IA[X_1, \ldots, X_n])$ is adhesive (resp. pseudo-adhesive).

In this situation we also say that the ring $A$ is $I$-adically universally adhesive (resp. $I$-adically universally pseudo-adhesive). Note that for a pseudo-adhesive pair $(A, I)$ to be universally pseudo-adhesive it is necessary and sufficient that $(A, I)$ satisfies (UBT) (cf. 8.2.2).

**Proposition 8.5.5.** (1) Let $(A, I)$ be an adhesive pair (resp. universally adhesive). Then for any finitely generated subideal $J \subseteq I$ the pair $(A, J)$ is adhesive (resp. universally adhesive).

(2) Let $I = (a_1, \ldots, a_n) \subseteq A$ be a finitely generated ideal, and suppose that $(A,a_i)$ is adhesive (resp. universally adhesive) for any $i = 1, \ldots, n$. Then $(A,I)$ is adhesive (resp. universally adhesive).

Proof. Clearly, it is enough to prove the assertions only in the ‘adhesive’ case.

(1) Any finitely generated $J$-torsion free $A$-module is $I$-torsion free, which verifies (b) of 8.5.3.
(2) By induction with respect to \( n \) we can reduce to the case \( I = (a, b) \); suppose \((A, a)\) and \((A, b)\) are adhesive. It is clear that \( A \) is Noetherian outside \( I \), for it is Noetherian outside \((a)\) and \((b)\). For a finitely generated \( A \)-module \( M \), the \( a \)-torsion part \( M_{a\text{-tor}} \) is finitely generated. Hence its \( b \)-torsion part \( (M_{a\text{-tor}})_{b\text{-tor}} \), which is nothing but \( M_{I\text{-tor}} \), is finitely generated, so (c) in 8.5.1 is verified.

**Proposition 8.5.6.** (1) If \((A, I)\) and \((B, J)\) are adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive) pairs, then also the pair \((A \times B, I \times J)\) is adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive).

(2) Let \((A, I)\) be a pair and \( B \) a faithfully flat \( A \)-algebra. If \((B, IB)\) is adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive), then so is \((A, I)\).

**Proof.** (1) This can be shown by an argument similar to that in the proof of 8.2.3 (1) in view of the fact that \( A \times B \) is Noetherian outside \( I \times J \) if and only if \( A \) is Noetherian outside \( I \) and \( B \) is Noetherian outside \( J \).

(2) It suffices to verify the assertion only in the ‘adhesive’ and ‘pseudo-adhesive’ cases. By 8.1.1, we see that \((A, I)\) is of finite ideal type. Since

\[
\text{Spec } B \setminus V(IB) \longrightarrow \text{Spec } A \setminus V(I)
\]

is faithfully flat, it follows that \( A \) is Noetherian outside \( I \). If \((B, IB)\) is pseudo-adhesive, then it follows from 8.2.3 (2) that \((A, I)\) is pseudo-adhesive. Suppose \((B, IB)\) is adhesive and let \( M \) be an \( I \)-torsion free finitely generated \( A \)-module. Take a surjection \( A^{\oplus m} \rightarrow M \) and denote by \( K \) its kernel. Since \( B \) is \( A \)-flat, \( M \otimes_A B \) is \( IB \)-torsion free, and we have the exact sequence

\[
0 \longrightarrow K \otimes_A B \longrightarrow B^{\oplus m} \longrightarrow M \otimes_A B \longrightarrow 0.
\]

By the assumption, \( K \otimes_A B \) is a finitely generated \( B \)-module (here we used [27], Chapter I, §2.8, Lemma 9). By [27], Chapter I, §3.1, Proposition 2, we deduce that \( K \) is finitely generated and hence that \((A, I)\) is adhesive.

**Proposition 8.5.7.** Let \((A, I)\) be an adhesive (resp. a pseudo-adhesive) pair:

(1) for any multiplicative subset \( S \subseteq A \) the induced pair \((S^{-1}A, IS^{-1}A)\) is adhesive (resp. pseudo-adhesive);

(2) for any quasi-finite \( A \)-algebra \( B \) the induced pair \((B, IB)\) is adhesive (resp. pseudo-adhesive).
Proof. (1) As Spec $S^{-1}A \setminus V(S^{-1}I)$ is clearly Noetherian, it suffices to check that for any finitely generated $S^{-1}A$-module $M$, its $S^{-1}I$-torsion part is finitely generated (resp. bounded). Let $x_1, \ldots, x_n$ be generators of $M$. Set

$$M' = Ax_1 + \cdots + Ax_n;$$

we have $M' \otimes_A S^{-1}A = M$. As one can easily verify, $M_{S^{-1}I}$, the $S^{-1}I$-torsion part of $M$, coincides with $M'_{S^{-1}I} \otimes_A S^{-1}A$. Since $M'_{S^{-1}I}$ is finitely generated (resp. bounded $I$-torsion), $M_{S^{-1}I}$ is finitely generated over $S^{-1}A$ (resp. bounded $S^{-1}I$-torsion).

(2) The assertion is clear if $B$ is finite over $A$. In general, we apply Zariski’s Main Theorem ([54], IV, (18.12.13)) to reduce to this case, using (1) and 8.5.6 as follows: Spec $B \to$ Spec $A$ is the composition of an open immersion followed by a finite morphism. Hence there exists a finite open covering Spec $A = \bigcup_{i \in I}$ Spec $A_i$ such that for each $i \in I$

- $A_i$ is of the form $S_i^{-1}A$ for a multiplicative subset $S_i \subseteq A$, and
- there exists a finite $A$-algebra $B'$ such that $B_i = B \otimes_A A_i$ is isomorphic to an $A$-algebra of the form $T_i^{-1}B'$ for a multiplicative subset $T_i \subseteq B'$.

By (1), each $(A_i, IA_i)$ is adhesive (resp. pseudo-adhesive), and hence $(B_i, IB_i)$ is adhesive (resp. pseudo-adhesive). Now by 8.5.6 (1) and (2) applied to

$$\bigsqcup \text{Spec } B_i \longrightarrow \text{Spec } B,$$

it follows that $(B, IB)$ is adhesive (resp. pseudo-adhesive).

Proposition 8.5.8. Let $(A, I)$ be a universal adhesive (resp. universally pseudo-adhesive) pair:

1. for any multiplicative subset $S \subseteq A$ the induced pair $(S^{-1}A, S^{-1}I)$ is universally adhesive (resp. universally pseudo-adhesive);

2. for any $A$-algebra $B$ of finite type the induced pair $(B, IB)$ is universally adhesive (resp. universally pseudo-adhesive).

The meaning of the second assertion is that universally-(pseudo-)adhesiveness is stable under finite type extensions.

Proof. (1) follows easily from 8.5.7 (1). To show (2), note first that any polynomial ring over $A$ is universally adhesive (resp. universally pseudo-adhesive) and apply 8.5.7 (2).


**Proposition 8.5.9.** Let \((A, I)\) be a pair of finite ideal type. The following two conditions are equivalent:

(a) \((A, I)\) is adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive);

(b) \(A_{I\text{-tor}}\) is finitely generated over \(A\) (resp. bounded \(I\)-torsion, resp. finitely generated over \(A\), resp. bounded \(I\)-torsion) and \((A/A_{I\text{-tor}}, I(A/A_{I\text{-tor}}))\) is adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive).

*Proof.* Since \((A/A_{I\text{-tor}})[X] \cong A[X]/A[X]_{I\text{-tor}}\), it suffices to show the assertion in the ‘adhesive’ and ‘pseudo-adhesive’ cases. As (a) \(\implies\) (b) is obvious by 8.5.7 (2), we show the converse.

First we claim that the scheme \(\text{Spec } A \setminus V(I)\) is Noetherian. For any \(f \in I\) the canonical map \(A \to A_f\) factors through \(A/A_{I\text{-tor}}\), whence \((A/A_{I\text{-tor}})_f \cong A_f\). This means that \(A_f\) for any \(f \in I\) is Noetherian and hence that \(\text{Spec } A \setminus V(I)\) is Noetherian.

Let \(M\) be a finitely generated \(A\)-module, and set

\[ M' = M \otimes_A (A/A_{I\text{-tor}}) = M/A_{I\text{-tor}}M. \]

We have the exact sequence

\[ 0 \to A_{I\text{-tor}}M \to M_{I\text{-tor}} \to M'_{I\text{-tor}} \to 0. \]

Since \(A_{I\text{-tor}}M\) and \(M'_{I\text{-tor}}\) are finitely generated over \(A\) in the ‘adhesive’ case or are bounded \(I\)-torsion in the ‘pseudo-adhesive’ case, so is \(M_{I\text{-tor}}\).

\[\square\]

**Proposition 8.5.10.** Let \((A, I)\) be a pair of finite ideal type. Consider an inductive system \(\{B_i, f_{ij}: B_i \to B_j\}\) of \(A\)-algebras indexed by a directed set \(J\), and set \(B = \lim_{i \in I} B_i\). Suppose that

(a) \((B_i, IB_i)\) is adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive) for each \(i\),

(b) for each pair of indices \(i, j\) such that \(i \leq j\), the morphism \(f_{ij}\) is flat, and

(c) \(\text{Spec } B \setminus V(IB)\) is Noetherian.

Then the pair \((B, IB)\) is adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive).
Proof. It suffices to check the proposition for the ‘adhesive’ and ‘pseudo-adhesive’ cases. Let $M$ be a finitely generated $B$-module, and consider an exact sequence

$$0 \longrightarrow N \longrightarrow B^\oplus m \longrightarrow M \longrightarrow 0.$$  

Let $\widetilde{N}$ be the $I$-saturation of $N$ in $B^\oplus m$. Then we have $\widetilde{N}/N \cong M_{I\text{-tor}}$. Since $N$ is finitely generated outside $I$, one can find a finitely generated submodule $N' \subseteq N$ such that its $I$-saturation in $B^\oplus m$ coincides with $\widetilde{N}$ (8.1.6 (1)).

Take an index $i \in J$ and a finitely generated $B_i$-submodule $N_0$ of $B^\oplus m_i$ such that $N_0 \otimes_{B_i} B = N'$ (note that by (b) the map $B_i \rightarrow B$ is flat). Take the $I$-saturation $\widetilde{N}_i'$ of $N_i$ in $B^\oplus m_i$. Since

$$\widetilde{M}_i' = B^\oplus m_i/\widetilde{N}_i'$$  

is an $I$-torsion free $B_i$-module isomorphic outside $I$ to $M_i' = B^\oplus m_i/N_0'$, it follows from the flatness of $B_i \rightarrow B$ that $\widetilde{M}_i' \otimes_{B_i} B$ is an $I$-torsion free $B$-module isomorphic outside $I$ to

$$M' = M_i' \otimes_{B_i} B \cong B^\oplus m / N'$$  

and hence also isomorphic outside $I$ to $M$. This implies that $\widetilde{M}_i' \otimes_{B_i} B$ coincides with $\widetilde{M} = B^\oplus m / \widetilde{N}$ and hence that $\widetilde{N}_i' \otimes_{B_i} B = \widetilde{N}$. Now the assertion in each case follows from the surjection

$$(\widetilde{N}_i'/N_0') \otimes_{B_i} B \cong \widetilde{N}/N' \longrightarrow \widetilde{N}/N \cong M_{I\text{-tor}}$$  

and the fact that $\widetilde{N}_i'/N_0' \cong (M_i')_{I\text{-tor}}$. \hfill $\Box$

The following proposition can be proved by an argument similar to that in 8.2.10, using 8.5.6.

Proposition 8.5.11. Let $(A, I)$ be a pair of finite ideal type. Suppose that

(a) $(\hat{A}, I \hat{A})$ is adhesive (resp. pseudo-adhesive), where $\hat{A}$ is the $I$-adic completion of $A$,

(b) $A \rightarrow \hat{A}$ is flat, and

(c) $A$ is Noetherian outside $I$.

Then the pair $(A, I)$ is adhesive (resp. pseudo-adhesive).

Remark 8.5.12. We do not know whether the converse of 8.5.11 holds or not, that is: if $(A, I)$ is adhesive (resp. pseudo-adhesive), then is $(\hat{A}, I \hat{A})$ adhesive (resp. pseudo-adhesive), too?
8.5. (b) Some examples. Here we collect a few examples of adhesive pairs, which will be of particular importance in our later arguments.

Example 8.5.13 (adhesive pairs of type (N)). Needless to say, any Noetherian ring $A$ is $I$-adically universally adhesive for any ideal $I \subseteq A$. Note that a ring $A$ is $1$-adically adhesive if and only if $A$ is an Noetherian ring.

Example 8.5.14 (adhesive pairs of type (V)). One of the most interesting examples of adhesive pairs is given by a pair $(V, a)$ consisting of a valuation ring (of arbitrary height) and an element $a \in \mathfrak{m}_V \setminus \{0\}$ such that $V$ is $a$-adically separated (cf. §6.7).

Let us prove the last-mentioned fact.

Proposition 8.5.15. Let $V$ be a valuation ring, and $a \in \mathfrak{m}_V \setminus \{0\}$. Then the following conditions are equivalent:

(a) $V$ is $a$-adically adhesive;

(b) $V$ is $a$-adically pseudo-adhesive;

(c) $V$ is $a$-adically separated;

(d) $V[\frac{1}{a}]$ is a field (= Frac($V$)).

Proof. Implication (a) $\Rightarrow$ (b) is clear. Let us show (b) $\Rightarrow$ (c). Suppose that $V$ is $a$-adically pseudo-adhesive, and consider the ideal $J = \bigcap_{n \geq 1} (a^n)$. Take a finitely generated subideal $J_0 \subseteq J$ such that $J/J_0$ is $a$-torsion. As $J/J_0$ is contained in $V/J_0$, $J/J_0$ is bounded $a$-torsion. This means that there exists $n \geq 0$ such that $a^n J \subseteq J_0$; but since $a^n J = J$, it follows that $J$ itself is finitely generated and hence is principal. Then we easily see that $J = (0)$.

We have already shown (c) $\Rightarrow$ (d) in 6.7.2. To show (d) $\Rightarrow$ (a), first note that (d) implies that any non-zero element $b \in V \setminus \{0\}$ divides some power $a^n (n \geq 1)$ of $a$. Hence, for any $V$-module $M$, $M$ is torsion free if and only if it is $a$-torsion free (that is, $M_{a\text{-tor}} = 0$). In particular, any $a$-torsion free finitely generated $V$-module $M$ is $V$-flat and hence is a free $V$-module.

We will see later in §9.2 that, if $V$ as above is, moreover, $a$-adically complete, then for any $n \geq 0$ the pair $(V\langle X_1, \ldots, X_n \rangle, a)$ is universally adhesive (9.2.7).

8.5. (c) Preservation of adicness. By 8.2.16 and 8.2.13, we have the following proposition.

Proposition 8.5.16. Let $(A, I)$ be a pseudo-adhesive pair. Then $(A, I)$ satisfies (AP). If, moreover, $I$ is principal, then $(A, I)$ satisfies (AR).
Hence, by what we have already seen before, a pseudo-adhesive pair \((A, I)\) enjoys the following properties.

- The functor \(M \mapsto \hat{M}\) given by \(I\)-adic completion on the full subcategory of \(\text{Mod}_A\) consisting of finitely generated \(A\)-modules is exact (7.4.11).
- \(M \otimes_A \hat{A} \cong \hat{M}\) for any finitely generated \(A\)-module \(M\) (8.2.18 (1)).
- the \(I\)-adic completion map \(A \to \hat{A}\) is flat (8.2.18 (1)).

If, moreover, \((A, I)\) is complete, then (§7.4.(d)) we have the following properties.

- Any finitely generated \(A\)-module is \(I\)-adically complete.
- Any \(A\)-submodule \(N\) of a finitely generated \(A\)-module \(M\) is closed in \(M\) and \(I\)-adically complete.

8.5. (d) Topologically universally adhesive pairs

**Definition 8.5.17.** We say that a pair \((A, I)\) is *topologically universally adhesive* (resp. *topologically universally pseudo-adhesive*), or that \(A\) is \(I\)-adically topologically universally adhesive (resp. \(I\)-adically topologically universally pseudo-adhesive) if \((A, I)\) is universally adhesive (resp. universally pseudo-adhesive) and for any \(n \geq 0\) the \(I\)-adic completion of \((A[X_1, \ldots, X_n], IA[X_1, \ldots, X_n])\) is again universally adhesive (resp. universally pseudo-adhesive).

We will often shorten the lengthy name ‘topologically universally adhesive’ (resp. ‘topologically universally pseudo-adhesive’) to ‘t.u. adhesive’ (resp. ‘t.u. pseudo-adhesive’).\(^{12}\) Similarly to (pseudo-)adhesiveness and universal (pseudo-)adhesiveness, these notions depend only on the topology on \(A\). Note that if \((A, I)\) is t.u. pseudo-adhesive, then \(A\) is topologically universally Noetherian outside \(I\) (8.4.3). Moreover, by 8.4.8 we have the following result.

**Proposition 8.5.18.** A complete pair \((A, I)\) of finite ideal type is t.u. pseudo-adhesive if and only if it is topologically universally Noetherian outside \(I\).

\(^{12}\)As indicated in the definition, the property ‘\(I\)-adically t.u. adhesive’ does not imply ‘\(I\)-adically complete’; but later in 1, §2.1.(a) we define for the sake of terminological brevity the notion of *t.u. adhesive rings*, which are complete by definition. Compare to the terminology *adic rings*; in [54], 0, §7.1, adic rings are complete by definition, whereas ‘\(I\)-adic’ does not imply completeness; see 7.2.2 and the warning after 1.1.1.3.
Proposition 8.5.19. Let \( (A, I) \) be t.u. adhesive (resp. t.u. pseudo-adhesive):

(1) the completion \( (\hat{A}, I \hat{A}) \) is t.u. adhesive (resp. t.u. pseudo-adhesive);

(2) for any \( A \)-algebra \( B \) of finite type, the induced pair \( (B, IB) \) is t.u. adhesive (resp. t.u. pseudo-adhesive);

(3) for any \( \hat{A} \)-algebra \( B \) topologically of finite type (8.4.1), the induced pair \( (B, IB) \) is t.u. adhesive (resp. t.u. pseudo-adhesive).

Proof. (1) is clear.

(2) follows easily from 8.5.8 (2). In the situation as in (3), the \( I \)-adic completion of a polynomial ring over \( B \) is finite over an \( A \)-algebra of the form \( \hat{A}\langle Y_1, \ldots, Y_m \rangle \) and hence is universally adhesive (resp. universally pseudo-adhesive) by (2). □

Theorem 8.5.20. Let \( (A, I) \) be a complete pair of finite ideal type. Then the following conditions are equivalent.

(a) \( (A, I) \) is t.u. adhesive (resp. t.u. pseudo-adhesive).

(b) \( (A\langle X_1, \ldots, X_n \rangle, IA\langle X_1, \ldots, X_n \rangle) \) is adhesive (resp. pseudo-adhesive) for any \( n \geq 0 \).

Proof. The proof is done in a similar way to that of 8.4.8. Only (b) \( \implies \) (a) needs to be shown, namely that a pair of the form

\[
(A\langle X_1, \ldots, X_r \rangle[Y_1, \ldots, Y_s], IA\langle X_1, \ldots, X_r \rangle[Y_1, \ldots, Y_s])
\]

is adhesive (resp. pseudo-adhesive). Since \( A\langle X_1, \ldots, X_n \rangle \) is Noetherian outside \( I \), one can verify as in the proof of 8.4.8 that the map

\[
A\langle X_1, \ldots, X_r \rangle[Y_1, \ldots, Y_s] \rightarrow A\langle X_1, \ldots, X_r, Y_1, \ldots, Y_s \rangle
\]

is flat. Then we apply 8.5.11 to deduce the desired result. □

8.5. (e) Adhesiveness and coherence

Proposition 8.5.21. Let \( (A, I) \) be an adhesive pair, and suppose \( A \) is \( I \)-torsion free. Then the ring \( A \) is coherent (3.3.1 (2)).

Proof. We verify (e) in 3.3.3. By 3.3.4 (1) and by easy homological algebra we may assume \( M = A^\oplus m \) (for some \( m \geq 0 \)) without loss of generality. Then \( L \) is an \( I \)-torsion free finitely generated \( A \)-module, and hence is finitely presented. □
Definition 8.5.22. Let \((A, I)\) be a t.u. pseudo-adhesive pair. Then the ring \(A\) is said to be \emph{topologically universally coherent with respect to} \(I\) if any topologically finitely presented \(A\)-algebra is universally coherent (3.3.7), where \(\hat{A}\) denotes the \(I\)-adic completion of \(A\).

Proposition 8.5.23. Let \(A\) be a topologically universally coherent ring with respect to an ideal \(I \subseteq A\).

(1) Any finitely presented \(A\)-algebra \(B\) is topologically universally coherent with respect to \(IB\).

(2) Any topologically finitely presented \(\hat{A}\)-algebra \(B\) is topologically universally coherent with respect to \(IB\).

Proof. To show (1), write \(B = A[Y_1, \ldots, Y_m]/\mathfrak{a}\) for a finitely generated \(\mathfrak{a}\). Then \((B, IB)\) is t.u. pseudo-adhesive (8.5.19). Let \(R = A[Y_1, \ldots, Y_m, X_1, \ldots, X_n]\). Then \(R/\mathfrak{a}R = B[X_1, \ldots, X_n]\). Since \(\hat{R}\) satisfies (AP), the ideal \(\mathfrak{a}\hat{R}\) is closed in \(\hat{R}\) due to 7.4.18, and we have \(\hat{B} \langle X_1, \ldots, X_n \rangle = \hat{R}/\mathfrak{a}\hat{R}\). Therefore \(\hat{B} \langle X_1, \ldots, X_n \rangle\) is a topologically finitely presented \(\hat{A}\)-algebra and hence is universally coherent. It follows that \(B\) is topologically universally coherent with respect to \(IB\).

(2) can be shown in a similar manner. \(\square\)

Proposition 8.5.24. Let \(A\) be \(I\)-adically t.u. adhesive and topologically universally coherent with respect to \(I \subseteq A\). Then \(A\) is universally coherent.

Proof. Since \((A, I)\) is t.u. adhesive, the canonical map \(A^{\text{Zar}} \to \hat{A}\) is faithfully flat (8.2.18, 7.3.8 (2)). We may assume that \(I\) is finitely generated; set \(I = (f_1, \ldots, f_r)\). Then \(\text{Spec } A \setminus V(I)\) is the union of the open subsets \(\text{Spec } A_{f_i}\) \((i = 1, \ldots, r)\). Set \(B = A^{\text{Zar}} \times A_{f_1} \times \cdots \times A_{f_r}\), which is a faithfully flat algebra over \(A\). Since each ring \(A_{f_i}\) is Noetherian, it is clearly coherent. By the assumption, the ring \(\hat{A}\) is coherent. Then by 3.3.2 (1) and (2) we deduce that \(A\) is coherent. Since, as we saw above, ‘topologically universally coherent’ is closed under finitely presented extension, we have the assertion. \(\square\)

Theorem 8.5.25. Let \((A, I)\) be a pair.

(1) Suppose \(A\) is \(I\)-adically universally adhesive and \(I\)-torsion free. Then \(A\) is universally coherent.

(2) Suppose \(A\) is \(I\)-adically complete, \(I\)-adically t.u. adhesive, and \(I\)-torsion free. Then \(A\) is topologically universally coherent with respect to \(I\).

Hence in (1) any finitely presented \(A\)-algebra \(B\) is again universally coherent. Likewise, in (2) any finitely presented algebra \(B\) over a topologically finitely presented \(A\)-algebra is again topologically universally coherent. Note that in both cases \(B\) may not be \(I\)-torsion free.
Proof. (1) Let $B$ be a finitely presented $A$-algebra. We are going to check (e) in 3.3.3. Take a surjective map $B' \to B$ with finitely generated kernel, where $B'$ is a polynomial ring over $A$. Note that $B'$ is $IB'$-torsion free. Since for a $B$-module to be finitely presented over $B$ is equivalent to being finitely presented over $B'$ (cf. 5.4.5), we may assume $B = B'$. But then the assertion in this case is nothing but 8.5.21.

(2) Let $B$ be a topologically finitely presented $A$-algebra, and take a surjective map $B' = A \langle X_1, \ldots, X_n \rangle \to B$ with finitely generated kernel. Note that $B'$ is $IB'$-torsion free. Then the rest of the proof goes similarly to that of assertion (1).

\[ \square \]

8.6 Scheme-theoretic pairs

Let us mention briefly that the notion of pairs has an obvious interpretation into the language of schemes. A (scheme-theoretic) pair is a couple $(X, Y)$ consisting of a scheme $X$ and a closed subscheme $Y$ of $X$. More generally, one can consider a pair $(X, Y)$ consisting of an algebraic space $X$ and a closed subspace $Y$. By a morphism of scheme-theoretic pairs $f: (X, Y) \to (Z, W)$, we mean a morphism $f: X \to Z$ of schemes (or algebraic spaces) such that $I^n_X \mathcal{O}_X \subseteq I_Y$ for some $n \geq 1$, where $I_Y$ (resp. $I_W$) is the defining ideal of $Y$ (resp. $W$) in $X$ (resp. $Z$).

An appropriate scheme-theoretic counterpart of complete pairs is provided by adic formal schemes.

Definition 8.6.1. A formal pair is a couple $(\mathfrak{X}, Y)$ consists of an adic formal scheme $\mathfrak{X}$ and its closed subscheme $Y$ defined by an ideal of definition.

(See I.1.1.14 for the definition of adic formal schemes.) For the practical use, however, a more handy definition of formal pairs may be the following one: a formal pair is a pair $(\mathfrak{X}, I)$ consisting of an adic formal scheme $\mathfrak{X}$ and an ideal of definition $I$. If, moreover, we are only interested in properties that do not depend on particular choices of ideals of definition, which is most frequently the case in the sequel, we even do not have to spell out $I$, and just consider an adic formal scheme $\mathfrak{X}$ itself as a scheme-theoretic counterpart of complete pairs.

The above definition of formal pairs fits in with the so-called formal completion ([54], I, §10.8): for a scheme $X$ and a closed subscheme $Y \subseteq X$ of finite presentation, the formal completion

$$\hat{X}|_Y$$

of $X$ along $Y$ is an adic formal scheme (cf. 7.2.15); the associated pair $(\hat{X}|_Y, Y)$ is a formal pair.

As for Henselian pairs, the correct substitution is given by Henselian schemes (cf. [75], Chapter 7).
Definition 8.6.2. A Henselian pair is a couple \((X, Y)\) consisting of a Henselian scheme \(X\) and a closed subscheme \(Y\) defined by an ideal of definition.

Similarly to the case of formal pairs, it might be more useful in practice to think of Henselian schemes themselves as the scheme-theoretic counterpart of Henselian pairs.

To consider the scheme-theoretic counterpart of Zariskian pairs, we need the notion of Zariskian schemes. For a scheme-theoretic pair \((X, Y)\) we denote by \(Z = X^{\text{Zar}}|_Y\) the locally ringed space supported on the underlying topological space of \(Y\) and with the structure sheaf \(\mathcal{O}_Z = i^*\mathcal{O}_X\), where \(i : Y \to X\) is the closed immersion. The sheaf \(\mathcal{O}_Z\) comes with the topology induced by the \(I\)-adic topology on \(\mathcal{O}_X\), where \(I\) is the defining ideal of \(Y\); moreover, \(I\) gives rise to a quasi-coherent sheaf of ideals of \(\mathcal{O}_Z\), denoted again by \(I\). Note that we have \(1 + I \subseteq \mathcal{O}_Z\).

A quasi-coherent ideal \(J \subseteq \mathcal{O}_Z\) is said to be an ideal of definition if, locally, there exists positive integers \(n, m \in \mathbb{Z}\) such that \(I^n \subseteq J^m \subseteq I\).

Definition 8.6.3. A Zariskian scheme is a topologically locally ringed space \(X = (X, \mathcal{O}_X)\) that is locally isomorphic (as a topologically locally ringed space) to a Zariskian scheme associated to a pair.

In I, §B we give more generalities on Zariskian schemes.

Definition 8.6.4. A Zariskian pair is a pair \((X, Y)\) consisting of a Zariskian scheme \(X\) and a closed subscheme \(Y\) defined by an ideal of definition.

For any pair \((X, Y)\) we have the associated Zariskian pair
\[
X^{\text{Zar}}|_Y = (X^{\text{Zar}}|_Y, Y)
\]
and the Henselization
\[
X^h|_Y = (X^h|_Y, Y).
\]

By 8.5.7 (1) and 8.5.8 (1) one can obviously talk about (pseudo-)adhesiveness and universally (pseudo-) adhesiveness for pairs of schemes.

Definition 8.6.5. A scheme-theoretic pair \((X, Y)\) is said to be adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive) if there exists an affine open covering \(\{U_i = \text{Spec} A_i\}_{i \in I}\) of \(X\) such that for each \(i\) the pair \((A_i, I_i)\) is adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive), where \(I_i\) is the ideal defining the closed subscheme \(Y \cap U_i \in U_i\).

The following proposition follows immediately from 8.5.8.

Proposition 8.6.6. Let \((X, Y)\) be a universally adhesive (resp. universally pseudo-adhesive) pair of schemes, and \(X' \to X\) an \(X\)-scheme locally of finite type. Then \((X', X' \times_X Y)\) is universally adhesive (resp. universally pseudo-adhesive).
As the notion of (universally-) adhesiveness is local with respect to étale topology (8.5.6 (2) and 8.5.7 (2)), (universal, pseudo-)adhesiveness for pairs \((X, Y)\) of algebraic spaces (where \(Y\) is a closed subspace of \(X\)) can be defined in the obvious way (see §5.2 (a) for what we mean precisely by algebraic spaces); we leave the details to the reader.

From 8.1.1, 8.5.6, 8.5.7 (1), and 8.5.8 (1) one easily derives the following result (and its obvious analogue for pairs of algebraic spaces).

**Proposition 8.6.7.** A pair \((X, Y)\) of schemes is adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive) if and only if for any étale neighborhood \(U = \text{Spec} \, A\) of \(X\) the induced pair \((A, I)\) is adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive), where \(I\) is the ideal of \(A\) corresponding to the closed subscheme \(Y \times_X U\) of \(U\).

Finally, let us include a useful fact on coherence of structure sheaves, which follows immediate from 8.5.25 (1) and 5.1.2.

**Proposition 8.6.8.** Let \((X, Y)\) be a universally adhesive pair of algebraic spaces such that \(\mathcal{O}_X\) is \(I\)-torsion free, where \(I = I_Y\) is the defining ideal of the closed subspace \(Y\). Then \(X\) is universally cohesive (5.1.1).

### 8.7 \(I\)-valuative rings

#### 8.7.(a) \(I\)-valuative rings. Let \((A, I)\) be a pair of finite ideal type. Recall that an ideal \(J\) of \(A\) is said to be \(I\)-admissible if it is finitely generated and contains a power of \(I\) (cf. 8.1.2).

**Definition 8.7.1.** Let \((A, I)\) be a pair of finite ideal type. The ring \(A\) is said to be \(I\)-valuative if any \(I\)-admissible ideal is invertible (cf. §6.1 (b)).

If one replaces \(I\) by a finitely generated ideal of definition (without loss of generality), the above definition requires that also the ideal \(I\) itself is invertible. If \(I\) can be taken to be a principal ideal \(I = (a)\), we will often say that \(A\) is \(a\)-valuative.

The notion of \(I\)-valuative rings becomes particularly simple in the case of local rings: a local ring \(A\) is \(I\)-valuative if and only if \(I\) is a principal ideal \(I = (a)\) generated by a non-zero-divisor \(a \in A\) and every \(I\)-admissible ideal is principal. For example, valuation rings are \(I\)-valuative for any non-zero finitely generated ideal \(I\). The following propositions are useful for reducing many situations to the case of local rings.

**Proposition 8.7.2.** Let \((A, I)\) be a pair of finite ideal type. If \(A\) is \(I\)-valuative, then for any multiplicative subset \(S \subseteq A\) the localization \(B = S^{-1}A\) is \(IB\)-valuative.
Proof. Let $J$ be a finitely generated ideal of $B$. Then one can take a finitely generated ideal $J'$ of $A$ such that $J' B = J$. Suppose $I^n B \subseteq J$ for some $n > 0$. Then one can replace $J'$ by $J' + I^n$, and hence we may assume that $J'$ is $I$-admissible. Since $J'$ is invertible, $J = J' B$ is invertible. \hfill $\Box$

**Proposition 8.7.3.** Let $(A, I)$ be a pair of finite ideal type. The following conditions are equivalent.

(a) $A$ is $I$-valuative.

(b) $A_p$ is $IA_p$-valuative for any prime ideal $p$ of $A$.

(c) $A_m$ is $IA_m$-valuative for any maximal ideal $m$ of $A$.

**Proof.** Implication $(a) \implies (b)$ follows from 8.7.2, and $(b) \implies (c)$ is trivial. Implication $(c) \implies (a)$ follows easily from [27], Chapter II, §5.6, Theorem 4. \hfill $\Box$

By 8.7.3, $I$-valuativeness has an obvious translation into the language of schemes.

**Definition 8.7.4.** Let $(S, T)$ be a scheme-theoretic pair (§8.6) such that the defining ideal $I$ of $T$ is of finite type. We say that $S$ is $I$-valuative (synonymously, $S$ is $T$-valuative or $(S, T)$ is valuative) if for any $x \in S$ the local ring $\mathcal{O}_{S, x}$ is $I_x$-valuative.

**Proposition 8.7.5.** Let $S$ be a scheme, and $I$ a quasi-coherent ideal of finite type. Suppose that $S$ is $I$-valuative, and set $U = S \setminus V(I)$.

(1) The scheme $S$ is integral if and only if $U$ is integral.

(2) The scheme $S$ is integrally closed if and only if $U$ is integrally closed.

**Proof.** (1) The ‘only if’ part is clear. Suppose $U$ is integral. Let $j : U \hookrightarrow S$ be the open immersion. Since $I$ is invertible, $\mathcal{O}_S \twoheadrightarrow j_* \mathcal{O}_U$ is injective. Let $\xi \in U$ be the generic point, and $i : \{\xi\} \hookrightarrow \mathcal{O}_S$ the inclusion. Since $U$ is integral, $\mathcal{O}_U \twoheadrightarrow i_* \mathcal{O}_{U, \xi}$ is injective. It follows that $\mathcal{O}_S \twoheadrightarrow i_* \mathcal{O}_{S, \xi}$ is injective, which shows that $S$ is integral.

(2) Only the ‘if’ part calls for a proof. We may assume that $S$ is affine $S = \text{Spec} A$ and that $I$ comes from a principal ideal $I = (a)$ with $a$ being a non-zero-divisor. We may further assume that $A$ is a local ring. Suppose $A[\frac{1}{a}]$ is integrally closed (in its total ring of fractions). Let $f(x) = x^n + \alpha_1 x^{n-1} + \cdots + \alpha_{n-1} x + \alpha_n$ be a monic polynomial in $A[x]$, and suppose an element $x = c/b$, where $b$ is a non-zero-divisor, satisfies $f(x) = 0$. Since $A[\frac{1}{a}]$ is integrally closed, one can take $c$ and $b$ such that $b$ is invertible in $A[\frac{1}{a}]$ and hence that $(b)$ is $I$-admissible. Since $(b, c)$ is also $I$-admissible and since $A$ is a local $I$-valuative ring, $(b, c)$ is a principal ideal $(d)$ generated by a non-zero-divisor $d$. Replacing $b$ and $c$ respectively by $b/d$ and $c/d$, we may assume that $(b, c) = (1)$. To show that $b$ is invertible, we suppose $b \in m_A$. Since $(b, c) = (1)$, $c$ must be a unit. But then by the equation $b^n f(c/b) = 0$, $c^n$ is divisible by $b$, which is absurd. \hfill $\Box$
Proposition 8.7.6. Let $S$ be an $I$-valuative scheme, and $T$ an integral subscheme of $S$. Suppose that $T$ is not contained in $V(I)$. Then the scheme $T$ is $I \otimes T$-valuative.

Proof. Using 8.7.3, we reduce to the case $S = \text{Spec } A$ with $A$ local, $T = \text{Spec } A/\mathfrak{p}$ where $\mathfrak{p} \subset A$ is a prime ideal, and $I = \langle a \rangle$ with $a \not\in \mathfrak{p}$. Then the assertion is straightforward.

Proposition 8.7.7. Let $A$ be an $a$-valuative local ring with $a \in \mathfrak{m}_A$, and suppose that $A[\frac{1}{a}]$ is a field. Then $A$ is an $a$-adically separated valuation ring. Conversely, if $A$ is a valuation ring, separated with respect to the adic topology defined by a finitely generated ideal $I$, then $\lim_{n \geq 1} \text{Hom}(I^n, A)$ is a field.

Note that $\lim_{n \geq 1} \text{Hom}(I^n, A) \cong A[\frac{1}{a}]$ by the choice of the generator $a \in I$. Hence, the second part of the proposition was already shown in 6.7.2.

Proof. By 8.7.5 (1), $A$ is an integral domain, and $A[\frac{1}{a}]$ is the fraction field of $A$. Let $b/a^n \in A[\frac{1}{a}]$. Since $(b, a^n)$ is invertible, we set $(b, a^n) = (d)$ and set $b = b'd$ and $a^n = c'd$. If $c'$ is invertible in $A$, then $b/a^n = b'/c' \in A$. If not, since $(b', c') = (1)$, $b'$ is a unit, and hence $(b/a^n)^{-1} \in A$. We have thus verified (b) in 6.2.1 and have shown that $A$ is a valuation ring. To show it is $a$-adically separated, suppose $J = \bigcap_{n \geq 1} \langle a^n \rangle \neq 0$, and take $f \in J \setminus \{0\}$. Since $f$ is invertible in $A[\frac{1}{a}]$, we can write $f^{-1} = c/a^n$ for $c \in A$, that is, $a^n$ is divisible by $f$. But since $f$ is divisible by all $a^m (m \geq 1)$ at the same time, this is absurd.

8.7. (b) Structure theorem. The most important feature of $I$-valuative local rings is that they are ‘composites’ of local rings and valuation rings.

Theorem 8.7.8. (1) Let $A$ be an $I$-valuative local ring, where $I \subset A$ is a non-zero proper finitely generated ideal. Set $J = \bigcap_{n \geq 1} I^n$. Then

(a) $B = \lim_{n \geq 1} \text{Hom}(I^n, A)$ is a local ring, and $V = A/J$ is an $\bar{a}$-adically separated valuation ring (where $IV = (\bar{a})$) for the residue field $K$ of $B$;

(b) $A = \{ f \in B : (f \mod \mathfrak{m}_B) \in V \}$;

(c) $J = \mathfrak{m}_B$.

(2) Conversely, for a local ring $B$ and an $\bar{a}$-adically separated valuation ring $V$ for the residue field $K$ of $B$ with $\bar{a} \neq 0$, the subring $A$ of $B$ defined in (b) above is an $I$-valuative local ring for any finitely generated ideal $I$ such that $IV = (\bar{a})$. Moreover, $B = \lim_{n \geq 1} \text{Hom}(I^n, A)$. 

Note that in (1), since \( I \subseteq \mathfrak{m}_A \), we have \( I \neq I^2 \) and hence \( \tilde{a} \neq 0 \). Needless to say, composition/decomposition of valuation rings is the basic example of the theorem; recall that a valuation ring \( V \) is \( I \)-valuative for any finitely generated ideal \( I \subseteq V \) and that \( J = \bigcap_{n \geq 1} I^n \) is a prime ideal (6.7.1); the theorem in this particular situation is equivalent to 6.4.1 (1) and 6.4.2.

**Proof.** (1) Let \( I = (a) \); we have \( B = A\left[\frac{1}{a}\right] \). Let \( x \) be a closed point of \( \text{Spec} \, B \), and \( S \) the Zariski closure of \( \{x\} \) in \( \text{Spec} \, A \). If \( \mathfrak{m} \) denotes the maximal ideal of \( B \) corresponding to \( x \), we have \( \mathfrak{m} = \text{Spec} \, A/\mathfrak{p} \) with \( \mathfrak{p} = A \cap \mathfrak{m} \). By 8.7.6, the ring \( A/\mathfrak{p} \) is \( I(A/\mathfrak{p}) \)-valuative. Since \( (A/\mathfrak{p})[\frac{1}{a}] = B/\mathfrak{m} \) is a field, \( A/\mathfrak{p} \) is an \( \bar{a} \)-adically separated valuation ring due to 8.7.7, where \( \bar{a} \) is the image of \( a \) in \( A/\mathfrak{p} \).

**Claim 1.** \( V = A/\mathfrak{p} \) or, equivalently, \( \mathfrak{p} = J \).

Indeed, since \( A/\mathfrak{p} \) is \( \bar{a} \)-adically separated, \( (J \mod \mathfrak{p}) = 0 \) and thus \( J \subseteq \mathfrak{p} \). For \( f \in \mathfrak{p} \) consider the admissible ideal \( (a^n, f) \) of \( A \) for each \( n \geq 1 \), and set \( (d) = (a^n, f) \) with \( d \in A \). Since \( f \) vanishes at \( x \), \( d \) cannot be a unit. We need to show \( f \in (a^n) = I^n \), and for this it is enough to show \( d \in (a^n) \). Since \( a^n \in (d) \), there exists \( b \in A \) such that \( a^n = bd \). On the other hand, since \( f \in \mathfrak{p} \), \( (b \mod \mathfrak{p}) \) is a unit in \( A/\mathfrak{p} \). Since \( A \) is a local ring, this implies that \( b \) is a unit of \( A \) and hence that \( d \in (a^n) \), as desired.

**Claim 2.** \( B \) is a local ring.

We want show that any \( f \in B \setminus \mathfrak{m} \) is invertible in \( B \). Since \( a \) is invertible in \( B \), we may assume \( f \in A \). Let \( \bar{f} \) be the image of \( f \) under the canonical map \( A \to A/\mathfrak{p} = V \). Since \( V \) is an \( \bar{a} \)-adically separated valuation ring, \( \bar{a}^n \in (\bar{f}) \) for some \( n \geq 1 \). Let \( H = (a^n, f) \), which is an admissible ideal of \( A \) such that \( HV = (\bar{f}) \). There exists \( h \in A \) such that \( H = (h) \), and hence \( f = gh \) for some \( g \in A \). Since the image of \( g \) in \( V \) is a unit, \( g \) is a unit of \( A \), and hence \( a^n \in (f) \). But this means that \( fB = B \) and hence that \( f \) is a unit of \( B \), as desired.

Now since \( V = A/(A \cap \mathfrak{m}_B) \) (\( \mathfrak{m}_B = \mathfrak{m} \)), one has \( V[\frac{1}{a}] = K \), where \( K = B/\mathfrak{m}_B \); that is, the residue field of \( B \) is the fraction field of \( V \). Therefore, (a) is proved.

Next, let us show (b). The inclusion \( A \subseteq \{f \in B; (f \mod \mathfrak{m}_B) \in V\} \) is clear. Let \( f \) be an element in the right-hand side, and set \( f = g/a^n \) with \( g \in A \). There exists \( h \in A \) such that \( (h \mod J) = (f \mod \mathfrak{m}_B) \), that is, \( h - g/a^n \in \mathfrak{m}_B \), and hence that \( a^n h - g \in J \). In particular, one can find \( s \in A \) such that \( a^n h - g = a^n s \), which gives \( f = h - s \in A \), as desired. By (b), in particular, we know that \( \mathfrak{m}_B \subseteq A \). Hence we have \( J = \mathfrak{p} = A \cap \mathfrak{m}_B = \mathfrak{m}_B \), whence (c).
(2) Let \( A = \{ f \in B : (f \mod m_B) \in V \} \) and take a finitely generated \( I \) such that \( IV = (\tilde{a}) \). Set \( J = m_B \cap A \). We have \( A/J = V \). First we claim that \( A \) is a local ring. Set \( q = \{ x \in A : (x \mod J) \in m_V \} \), and take \( x \in A \setminus q \). Then \( \tilde{x} = x \mod J \) is a unit in \( V \), which implies \( x \in B^X \). But since \( (x^{-1} \mod m_B) \) belongs to \( V \), we deduce \( x^{-1} \in A \). Hence \( A \) is a local ring, and \( m_A = q \).

Take \( a \in I \) such that \( (a \mod J) = \tilde{a} \). We claim that \( A[\frac{1}{a}] = B \). The inclusion \( A[\frac{1}{a}] \subseteq B \) is clear. Take any \( f \in B \), and set \( \hat{f} = (f \mod m_B) \). By 6.7.2, which yields exists \( n \geq 0 \) such that \( f\hat{a}^n \in V \) and hence that \( fa^n \in A \). Hence \( A[\frac{1}{a}] \supseteq B \), as desired.

Next we claim that \( \bigcap_{n \geq 1} (a^n) = J \). We have \( \bigcap_{n \geq 1} (a^n) \subseteq J \), as \( V \) is \( \tilde{a} \)-adically separated. Suppose \( x \in A \setminus \bigcap_{n \geq 1} (a^n) \), that is, \( x \notin (a^n) \) for some \( n \geq 1 \). Since \( x/a^n \notin A \), we have \( \tilde{a}^n/\tilde{x} \in m_V \). There exists \( y \in m_A \) such that \( \tilde{y} = \tilde{a}^n/\tilde{x} \), that is, \( \tilde{x} \tilde{y} = \tilde{a}^n \). Since \( \tilde{a} \neq 0 \), \( x \notin J \). Hence \( \bigcap_{n \geq 1} (a^n) \supseteq J \), thereby the claim. In particular, since \( (a) \subseteq I \) and \( V \) is \( \tilde{a} \)-adically separated (which implies \( \bigcap_{n \geq 1} I^n \subseteq J \)), we have \( \bigcap_{n \geq 1} I^n = J \).

Now, since all \( I \)-admissible ideals of \( A \) contain \( J \), they are in inclusion-preserving one-to-one correspondence with the finitely generated (hence principal) ideals of \( V \). Let \( H \supseteq I^n \) be an \( I \)-admissible ideal, and \((\tilde{b})\) the corresponding ideal of \( V \), where \( b \in H \). To show \( H = (b) \), it suffices to show that \((b)\) contains \( J = \bigcap_{n \geq 1} (a^n) \). Since \((\tilde{b}) \supseteq (\tilde{a}^n)\), there exists \( c \in A \) such that \( a^n - bc \in J \). Hence there exists \( e \in 1 + J \) such that \( bc = a^n e \). As \( e \) is a unit in \( A \), we deduce that \((a^n) \subseteq \tilde{b} \) and so \( J \subseteq \tilde{b} \), as desired. Hence all \( I \)-admissible ideals of \( A \) are principal and are invertible, since \( a \) is a non-zero-divisor. (In particular, we have \( I = (a) \).) Therefore, \( A \) is \( I \)-valuative, and now all the assertions are proved.

\[ \square \]

8.7. (c) **Patching method.** Let \( A \) be an \( I \)-valuative local ring with \( I = (a) \subseteq m_A \), and \( B \) and \( V \) the corresponding local ring and valuation ring, respectively, determined as in 8.7.8. \( V \) is a valuation ring for the residue field \( K = B/m_B \) of \( B \) and is \( \tilde{a} \)-adically separated, where \( \tilde{a} = (a \mod J) \). In particular, we have \( K = V[\frac{1}{a}] \) (6.7.2).

Let \( X \) be an object over \( A \), e.g., a scheme, algebra, module, etc. Then, by base change, it induces the objects \( X_B \) over \( B \) and \( X_V \) over \( V \) together with an isomorphism \( \phi : X_B \otimes K \tilde{\to} X_V \otimes K \). In many situations, it is essential to regard \( X \) as being obtained by ‘patching’ \( X_B \) and \( X_V \) along \( \phi \). While it will turn out in many situations that the functor \( X \mapsto (X_B, X_V, \phi) \) is essentially surjective, that is to say, any triple \((X_B, X_V, \phi)\) can be anyway patched together to an object over \( A \), we would like to have a more precise picture of the patching; for example, we would like to ask for an oppositely-oriented functor, so to speak, the ‘patching functor,’ from the category of the triples as above to the category of \( A \)-objects.
Let us formulate the situation more precisely. For a ring $R$ we denote by $\text{Mod}_R$ the category of $R$-modules. By base change, we have the commutative (more precisely, 2-commutative) diagram of categories

\[
\begin{array}{ccc}
\text{Mod}_A & \longrightarrow & \text{Mod}_B \\
\downarrow & & \downarrow \\
\text{Mod}_V & \longrightarrow & \text{Mod}_K,
\end{array}
\]

which gives rise to a functor

\[
\beta: \text{Mod}_A \longrightarrow \text{Mod}_B \times_{\text{Mod}_K} \text{Mod}_V,
\]

where the right-hand side denotes the 2-fiber product of the categories, that is, the category of triples $(L, M, \iota)$ consisting of

- a $B$-module $L$,
- a $V$-module $M$,
- an isomorphism $\iota: L \otimes_B K \xrightarrow{\sim} M \otimes_V K$ of $K$-modules,

and where a morphism

\[(L_1, M_1, \iota_1) \longrightarrow (L_2, M_2, \iota_2)\]

is a pair $(f, g)$ consisting of a $B$-morphism $f: L_1 \to L_2$ and a $V$-morphism $g: M_1 \to M_2$ such that the following diagram commutes:

\[
\begin{array}{ccc}
L_1 \otimes_B K & \xrightarrow{f \otimes B \text{id}_K} & L_2 \otimes_B K \\
\downarrow{\iota_1} & & \downarrow{\iota_2} \\
M_1 \otimes_V K & \xrightarrow{g \otimes V \text{id}_K} & M_2 \otimes_V K.
\end{array}
\]

We say that $(f, g)$ is injective (resp. surjective) if both $f$ and $g$ are injective (resp. surjective). The functor $\beta$ is then given by

\[
\beta(N) = (N \otimes_A B, N \otimes_A V, \text{can})
\]

for any $A$-module $N$, where $\text{can}$ is the canonical isomorphism.

Now we define the patching functor

\[
\alpha: \text{Mod}_B \times_{\text{Mod}_K} \text{Mod}_V \longrightarrow \text{Mod}_A
\]

by

\[
\alpha(L, M, \iota) = \{(x, y) \in L \times M: \iota(x \mod m_B L) = y \otimes 1 \text{ in } M \otimes_V K\}.
\]
Thus we have two functors between these categories,

\[
\text{Mod}_A \xrightarrow{\beta} \text{Mod}_B \times_{\text{Mod}_K} \text{Mod}_V.
\]

These functors can be similarly defined between the corresponding categories of algebras:

\[
\text{Alg}_A \xrightarrow{\beta} \text{Alg}_B \times_{\text{Alg}_K} \text{Alg}_V.
\]

**Theorem 8.7.9.** (1) The functor \( \beta \) is the left-adjoint to the functor \( \alpha \), and the adjunction morphism \( \beta \circ \alpha \to \text{id} \) is a natural equivalence. In particular, \( \beta \) is essentially surjective, and \( \alpha \) is fully faithful.

(2) The essential image of \( \alpha \) consists of \( A \)-modules \( N \) such that \( JN \) is \( a \)-torsion free.

Moreover, the similar assertions hold for algebras.

**Proof.** Clearly, it suffices to show the theorem in the case of modules.

(1) First we are to show that the adjunction map \( \beta \circ \alpha(L, M, t) \to (L, M, t) \) is an isomorphism. This map is defined by the pair of morphisms \( N \otimes_A B \to L \) and \( N \otimes_A V \to M \), where \( N = \alpha(L, M, t) \subseteq L \times M \), induced respectively by the first and second projections. It is easy to see that these maps are injective and that \( N \otimes_A V \to M \) is surjective. To show that \( N \otimes_A B \to L \) is surjective, take \( x \in L \) and set \( \bar{x} = x \mod m_B L \). Since \( K = \text{V}[\frac{1}{a}] \) (6.7.2), we can find \( n \geq 0 \) such that \( t(a^n x) \) is of the form \( y \otimes 1 \). Then one has the element \( (a^n x, y) \otimes a^{-n} \) in \( N \otimes_A B \), which maps to \( x \), as desired. Now, by means of the natural transformation \( \beta \circ \alpha \to \text{id} \) thus obtained, we have the canonical map

\[
\text{Hom}_{\text{Mod}_A}(N, \alpha(L, M, t)) \to \text{Hom}_{\text{Mod}_B \times_{\text{Mod}_K} \text{Mod}_V}(\beta(N), (L, M, t)),
\]

which is bijective due to the presence of the other adjunction map \( N \to \alpha \circ \beta(N) \) defined in an obvious way.

(2) Let us first show that the adjunction map \( N \to \alpha \circ \beta(N) \) is surjective. Every element of \( \alpha \circ \beta(N) \) is of the form \( (x \otimes a^{-n}, y \otimes 1) \in N_B \times N_V \) such that \( \bar{x} = \bar{a}^n \bar{y}, \) where \( \bar{\cdot} \) denotes the \( \text{mod-}JN \) class. Since \( x - a^n y \in JN \), there exists \( z \in N \) such that \( x = a^n z \) and \( \bar{z} = \bar{y}. \) Hence we have \( (x \otimes a^{-n}, y \otimes 1) = (z \otimes 1, z \otimes 1), \) which is the image of \( z \) under the adjunction map, thereby the claim. Note that the kernel of this map is \( JN \cap N_{a-\text{tor}} = (JN)_{a-\text{tor}} \). The assertion follows from these observations, combined with the fact that the other adjunction map \( \beta \circ \alpha(L, M, t) \to (L, M, t) \) is an isomorphism. \( \square \)
Proposition 8.7.10. Let $N$ be an $A$-module. Then $N$ lies in the essential image of the functor $\alpha$ if $\text{Tor}^B_1(N \otimes_A B, K) = 0$ or, sufficiently, if $N$ is flat outside $I$.

Proof. The exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow V \longrightarrow 0$$

gives rise to the exact sequence

$$0 \longrightarrow \text{Tor}^A_1(N, V) \longrightarrow J \otimes_A N \longrightarrow N \longrightarrow N/JN \longrightarrow 0.$$

Since $\text{Tor}^A_1(N, V) \otimes K \cong \text{Tor}^A_1(N, K) \cong \text{Tor}^B_1(N \otimes_A B, K) = 0$, $\text{Tor}^A_1(N, V)$ is an $a$-torsion module. But since $J \otimes_A N$ is $a$-torsion free (as $x \mapsto ax$ is bijective on $J$), we have $\text{Tor}^A_1(N, V) = 0$. Consequently, $J \otimes_A N \cong JN$ and that $JN$ is $a$-torsion free. \qed

Proposition 8.7.11. (1) Let $(L, M, \iota)$ be an object of $\text{Mod}_B \times_{\text{Mod}_K} \text{Mod}_V$, and set $N = \alpha(L, M, \iota)$. Suppose $L$ is flat over $B$. The following conditions are equivalent.

(a) $N$ is a finitely generated $A$-module.

(b) $L$ is a finitely generated $B$-module, and $M$ is a finitely generated $V$-module.

(2) Let $(P, Q, \iota)$ be an object of $\text{Alg}_B \times_{\text{Alg}_K} \text{Alg}_V$, and set $R = \alpha(P, Q, \iota)$. Suppose $P$ is flat over $B$. Then the following conditions are equivalent.

(a) $R$ is a finitely generated (resp. finitely presented) $A$-algebra.

(b) $P$ is a finitely generated (resp. finitely presented) $B$-algebra, and $Q$ is a finitely generated (resp. finitely presented) $V$-algebra.

Proof. Both in (1) and in (2) the direction (a) $\Rightarrow$ (b) is easy. We want to show the converse. The reasoning for proving (b) $\Rightarrow$ (a) in (1) and (2) (for ‘finitely generated’) are quite similar, and we confine ourselves to present it only in (2).

Suppose $P$ (resp. $Q$) is a finitely generated algebra over $B$ (resp. $V$), and take the generators $x_1, \ldots, x_s$ (resp. $\tilde{y}_1, \ldots, \tilde{y}_t$). Since $P = R \otimes_A B$, multiplying by a power of $a$ if necessary, we may assume that all $x_i$ belong to $R$. Similarly, since $Q = R \otimes_A V = R/JR$, we can take $y_i \in R$ that lifts $\tilde{y}_i$ for $i = 1, \ldots, t$. Then we want to show that $\{x_1, \ldots, x_s, y_1, \ldots, y_t\}$ generates $R$. To this end, we consider the morphism $f: C = A[X_1, \ldots, X_s, Y_1, \ldots, Y_t] \rightarrow R$ given by $f(X_i) = x_i$ and $f(Y_j) = y_j$. 

Claim 1. The induced map \( f : JC \to JR \) is surjective.

It is clear that, if the claim is settled, we can prove that \( R \) is finitely generated; indeed, \( Q \) is finitely generated, and \( Q = R/JR \).

To prove the claim, we first note that the flatness assumption implies that

\[
\text{Tor}_1^B(R \otimes_A B, K) = 0.
\]

The same reasoning as in the proof of 8.7.10 shows that \( \text{Tor}_1^B(R, V) = 0 \) and \( J \otimes_A R = JR \). Similarly, \( J \otimes_A C = JC \). But since \( \text{id}_J \otimes f : J \otimes_A C \to J \otimes_A R \) is surjective, the claim follows.

Next, suppose \( P \) (resp. \( Q \)) is a finitely presented algebra over \( B \) (resp. \( V \)). To show that \( R \) is finitely presented, it suffices to show that for any surjection \( A[X] = A[X_1, \ldots, X_s] \to R \) from a polynomial ring its kernel \( H \) is finitely generated. Since \( H \otimes_A B \) is the kernel of the induced map \( B[X_1, \ldots, X_s] \to P = R \otimes_A B \) and since \( \text{Tor}_i^B(P, K) = 0 \) for all \( i \geq 1 \), we have \( \text{Tor}_1^B(H \otimes_A B, K) = 0 \). Hence \( H \) lies in the essential image of \( \alpha \) by 8.7.10. In particular, \( H = \alpha \circ \beta(H) \).

Claim 2. \( \beta(H) \to \beta(A[X]) \) is injective.

Indeed, since \( \text{Tor}_1^A(R, V) = 0 \) as above, we see that \( H \otimes_A V \to A[X] \otimes_A V \) is injective. Since the other map \( H \otimes_A B \to B[X] \) is clearly injective, the claim is settled.

The claim implies that \( H \otimes_A V \) is the kernel of \( V[X] \to Q \). By the assumption \( H \otimes_A B \) (resp. \( H \otimes_A V \)) is a finitely generated \( B[X] \) (resp. \( V[X] \))-module. Hence, by (1), \( H = \alpha \circ \beta(H) \) is a finitely generated \( A \)-module.

Proposition 8.7.12. Let \( (P, Q, \iota) \) be an object of \( \text{Alg}_B \times_{\text{Alg}_K} \text{Alg}_V \), and set \( R = \alpha(P, Q, \iota) \). Then the following conditions are equivalent:

(a) \( R \) is flat (resp. smooth, resp. \( \acute{e} \)tale) over \( A \);

(b) \( P \) is flat (resp. smooth, resp. \( \acute{e} \)tale) over \( B \), and \( Q \) is flat (resp. smooth, resp. \( \acute{e} \)tale) over \( V \).

Proof. (a) \( \Rightarrow \) (b) is clear. We need to show the converse. We have the diagram

\[
\begin{array}{ccc}
\text{Spec } Q & \to & \text{Spec } R \\
\downarrow & & \downarrow \\
\text{Spec } V & \to & \text{Spec } A \\
\end{array}
\]

where the squares are Cartesian (due to 8.7.9 (1)) and the right-hand (resp. left-hand) inclusions are open (resp. closed) immersions. Then in view of 8.7.11 (2) and [54], IV, (17.5.1) and (17.6.1), it is enough to show that, if \( P \) is \( B \)-flat and \( Q \) is
V-flat, then R is A-flat. Since \( P = R[a^{-1}] \) and \( Q = R/JR \), it suffices to show that R is a-torsion free; indeed, since \( Q \) is V-flat, \( R/aR \) is flat over \( V/aV = A/aA \), and hence we can apply 8.3.2.

Let \( x \in R \) and suppose \( \overline{a^n}x = 0 \) for some \( n > 0 \). Then in V we have \( \overline{a^n}x = 0 \); since \( \overline{a} \neq 0 \) and V is an integral domain, \( x \in JR \). But by 8.7.9 (2) we know that \( JR \) is a-torsion free and so \( x = 0 \), as desired. \( \square \)

**Proposition 8.7.13.** Let \( A \) be an I-valuative local ring for a non-zero \( I \neq A \), and \( B, J, V, \) and \( K \) as in 8.7.8. Then the following conditions are equivalent.

(a) \( A \) is \( I \)-adically Henselian.

(b) \( V \) is \( IV \)-adically Henselian, and \( B \) is \( m_B \)-adically Henselian.

**Proof.** First we prove (a) \( \Rightarrow \) (b). It is obvious that \( V \) is Henselian, for it is a quotient of \( A \). To see that \( B \) is Henselian, we verify the following: for any \( B \)-algebra \( P \) étale over \( B \) such that \( P \otimes_B F = P/m_B P = K \), there exists a homomorphism \( P \to B \) such that the composition \( B \to P \to B \) is the identity map.

If \( P \) is as above, then \( (P, V, i) \) (for some \( i \)) is an object of \( Alg_B \times_{Alg_K} Alg_V \), and hence we have an \( A \)-algebra \( R = \alpha(P, V, i) \). By 8.7.12 \( R \) is an étale \( A \)-algebra, and we know that \( R/IR = V/IV = A/I \). Hence there exists a homomorphism \( R \to A \) such that the composition \( A \to R \to A \) is the identity map. Since \( R[a^{-1}] = P \), we have the desired morphism \( P \to B \) by base change.

Next, we show (b) \( \Rightarrow \) (a). Let \( R \) be an étale \( A \)-algebra such that \( R/IR = A/I \). Then by base change (that is, by the functor \( \beta \)), we have the étale \( B \)-algebra \( P = R[a^{-1}] \) and the étale \( V \)-algebra \( Q = R/JR \). Since \( V \) is Henselian with respect to the \( IV \)-adic topology, the obvious equalities \( Q/IQ = R/IR = V/IV \) yield a morphism \( Q \to V \); similarly, as \( J = m_B \), we have a morphism \( P/JP = Q[a^{-1}] \to V[a^{-1}] = K \) (cf. 6.7.2), which gives rise to a map \( P \to B \). Hence we have homomorphisms \( P \to B \) and \( Q \to V \) that give respectively the right-inverses of \( B \to P \) and \( V \to Q \). These maps form a morphism \( (P, Q, \text{can}) \to (B, V, \text{can}) \) in \( Alg_B \times_{Alg_K} Alg_V \), giving a right-inverse to the structural map

\[
(B, V, \text{can}) \to (P, Q, \text{can}).
\]

Then by 8.7.10 the functor \( \alpha \) gives rise to a morphism \( R \to A \) such that the composition \( A \to R \to A \) is the identity map, whence the claim. \( \square \)

**8.8 Pairs and complexes**

**8.8. (a) Set-up.** Let us first fix notations that we will use throughout this subsection. Let \( (A, I) \) be a pair, and

\[
K^\bullet : \cdots \longrightarrow K^{q-1} \overset{d^{q-1}}{\longrightarrow} K^q \overset{d^q}{\longrightarrow} K^{q+1} \overset{d^{q+1}}{\longrightarrow} \cdots
\]
a complex of $A$-modules. We consider the $I$-adic filtration $\{I^nK^*\}_{n \geq 0}$ on the complex $K^*$ and define the complex $K^*_k$ by the exact sequence

$$0 \longrightarrow I^{k+1}K^* \longrightarrow K^* \longrightarrow K^*_k \longrightarrow 0 \quad \text{(*)}$$

of complexes for each $k \geq 0$. This induces the cohomology long exact sequence

$$\cdots \longrightarrow H^q(I^{k+1}K^*) \xrightarrow{\iota^q_k} H^q(K^*) \xrightarrow{\sigma^q_k} H^q(K^*_k) \xrightarrow{\delta^q_k} H^{q+1}(I^{k+1}K^*) \longrightarrow \cdots.$$ 

**Definition 8.8.1.** For $q \in \mathbb{Z}$ we define the filtrations $F^nH^q(I^{k+1}K^*)_{n \geq 0}$ and $F^nH^q(K^*_k)_{n \geq 0}$ on the cohomologies $H^q(I^{k+1}K^*)$ and $H^q(K^*_k)$, respectively, for each $k \geq 0$ by

$$F^nH^q(I^kK^*) = \text{image}(H^q(I^nK^* \cap I^kK^*) \longrightarrow H^q(I^kK^*));$$

$$F^nH^q(K^*_k) = \text{image}(H^q(K^*/I^nK^* \cap I^{k+1}K^*) \longrightarrow H^q(K^*_k)).$$

We call these filtrations the *induced filtrations on the cohomologies*.

**8.8. (b) Results in case $I$ is finitely generated**

**Lemma 8.8.2.** Let $(A, I)$ be a complete pair of finite ideal type, and $q \in \mathbb{Z}$. Suppose that the following conditions are satisfied.

(a) $K^q$ is $I$-adically separated, and $K^{q-1}$ is $I$-adically complete.

(b) $H^q(K^*)/IH^q(K^*)$ is finitely generated as an $A$-module.

Then the $A$-module $H^q(K^*)$ is finitely generated. More precisely, if $\beta_1, \ldots, \beta_m$ are elements of $H^q(K^*)$ that generate $H^q(K^*)/IH^q(K^*)$, then $\beta_1, \ldots, \beta_m$ generate $H^q(K^*)$ as an $A$-module.

**Proof.** We may assume that $I$ is finitely generated: $I = (a_1, \ldots, a_r)$. Consider the projection $p: \ker(d^q) \rightarrow H^q(K^*)$ and for each $j = 1, \ldots, m$ an element $\alpha_j \in \ker(d^q)$ such that $p(\alpha_j) = \beta_j$. Set $M = A^{\oplus m}$, and let $\Phi: M \rightarrow \ker(d^q)$ be the map given by the elements $\alpha_1, \ldots, \alpha_m$. Set $\varphi = p \circ \Phi$. We have

$$H^q(K^*) = \varphi(M) + I H^q(K^*).$$

Hence, in particular,

$$\ker(d^q) = \Phi(M) + \text{image}(d^{q-1}) + I \ker(d^q).$$
For $x = x_0 \in \ker(d^q)$ we have $y_1 \in M$, $z_1 \in K^{q-1}$, and $x_1^{(1)}, \ldots, x_1^{(r)} \in \ker(d^q)$ such that

$$x = \Phi(y_1) + d^{q-1}(z_1) + \sum_{i=1}^{r} a_i x_1^{(i)}.$$  

We do the same for each $x_1^{(i)}$ to get a similar equalities

$$x_1^{(i)} = \Phi(y_2^{(i)}) + d^{q-1}(z_2^{(i)}) + \sum_{j=1}^{r} a_j x_2^{(ij)}.$$  

Then we have

$$x = \Phi(y_2) + d^{q-1}(z_2) + \sum_{i,j=1}^{r} a_i a_j x_2^{(ij)},$$

where $y_2 = y_1 + \sum_{i=1}^{r} a_i y_2^{(i)}$ and $z_2 = z_1 + \sum_{i=1}^{r} a_i z_2^{(i)}$.

We repeat this to obtain the sequences $\{y_n\}_{n \geq 1}$, where $y_n$ is a polynomial in $a_1, \ldots, a_r$ of degree $n - 1$ with the coefficients in $M$, and $\{z_n\}$, where $z_n$ is a polynomial in $a_1, \ldots, a_r$ of degree $n - 1$ with the coefficients in $K^{q-1}$, such that, for any $n \geq 1$, $x - (\Phi(y_n) + d^{q-1}(z_n))$ is a homogeneous polynomial in $a_1, \ldots, a_r$ of degree $n$ with the coefficients in $\ker(d^q)$, hence belonging to $I^n \ker(d^q)$. Set $y = \lim y_n \in M$ and $z = \lim z_n \in K^{q-1}$. Since $K^q$, and hence $\ker(d^q)$ also, is $I$-adically separated, we have

$$x = \Phi(y) + d^{q-1}(z).$$

Thus we have shown that $\ker(d^q) = \Phi(M) + \text{image}(d^{q-1})$, that is, the map $\varphi: M \to H^q(K^\bullet)$ is surjective. \qed

**Remark 8.8.3.** If we further assume in 8.8.2 that $I$ is principal, $I = (a)$, and that $H^q(K^\bullet)_{a\text{-tor}}$ is bounded $a$-torsion, then the consequence of 8.8.2 follows by a simpler argument: by 7.2.4, it suffices to show that $H^q(K^\bullet)$ is $a$-adically separated, which is verified by an easy argument (cf. Exercise 0.8.2).

**Lemma 8.8.4.** Let $(A, I)$ be a complete pair of finite ideal type, and $q \in \mathbb{Z}$ an integer. Suppose that the following conditions are satisfied:

(a) $K^q$ is $I$-adically separated, and $K^{q-1}$ is $I$-adically complete;

(b) the topology on $H^q(K^\bullet)$ defined by $\{F^n H^q(K^\bullet)\}_{n \geq 0}$ is $I$-adic;

(c) for any $k \geq 0$ the image of the map $H^q(K^\bullet) \to H^q(K^\bullet_k)$ is finitely generated as an $A$-module.

Then $H^q(K^\bullet)$ is finitely generated as an $A$-module.
Proof. By (c), the \( A \)-module \( \text{H}^q(K^\bullet)/\text{F}^{k+1}\text{H}^q(K^\bullet) \) is finitely generated for any \( k \geq -1 \). By (b), there exists an integer \( s = s(k) \geq 0 \) such that \( \text{F}^{k+s+1}\text{H}^q(K^\bullet) \subseteq \text{I}^{k+1}\text{H}^q(K^\bullet) \) for any \( k \geq 0 \). Hence \( \text{H}^q(K^\bullet)/\text{I}\text{H}^q(K^\bullet) \) is a quotient of the finitely generated \( A \)-module \( \text{H}^q(K^\bullet)/\text{F}^{s(0)+1}\text{H}^q(K^\bullet) \), and hence is finitely generated. Now, applying \ref{lem8.8.2}, we deduce the assertion. \( \square \)

Lemma \ref{lem8.8.5}. Suppose that the pair \((A, I)\) is pseudo-adhesive \ref{eq8.5.1}.

1. We have the equality
   \[
   (\delta_k^q)^{-1}(\text{F}^n\text{H}^{q+1}(I^{k+1}K^\bullet)) = \text{F}^n\text{H}^q(K_k^\bullet)
   \]
   for any \( k, n \geq 0 \) and \( q \in \mathbb{Z} \).

2. For a fixed \( q \in \mathbb{Z} \) suppose that \( \text{H}^{q+1}(I^{k+1}K^\bullet) \) is finitely generated and that the topology on it defined by the induced filtration \( \{\text{F}^n\text{H}^{q+1}(I^{k+1}K^\bullet)\}_{n \geq 0} \) is \( I \)-adic for any \( k \geq 0 \). Then the projective system \( \{\text{H}^q(K_k^\bullet)\}_{k \geq 0} \) satisfies (ML).

Proof. To show (1), we may assume \( n \leq k \). Consider the commutative diagram

\[
\begin{array}{ccccc}
\text{H}^q(K^\bullet) & \longrightarrow & \text{H}^q(K_n^\bullet) & \overset{\delta_k^q}{\longrightarrow} & \text{H}^{q+1}(I^{n+1}K^\bullet) \\
\downarrow & & \downarrow & & \downarrow \\
\text{H}^q(K^\bullet) & \longrightarrow & \text{H}^q(K_k^\bullet) & \overset{\delta_k^q}{\longrightarrow} & \text{H}^{q+1}(I^{k+1}K^\bullet)
\end{array}
\]

with the exact rows. What to prove is the equality \((\delta_k^q)^{-1}(\text{image}(\beta)) = \text{image}(\alpha)\), which follows from an easy diagram chasing.

To show (2), first note that

\[
(\delta_k^q)^{-1}(\text{F}^n\text{H}^{q+1}(I^{k+1}K^\bullet) \cap \text{H}^{q+1}(I^{k+1}K^\bullet)_{I\text{-tor}}) = \text{F}^n\text{H}^q(K_k^\bullet)
\]

due to (1). Since \((A, I)\) is pseudo-adhesive, \( \text{H}^{q+1}(I^{k+1}K^\bullet)_{I\text{-tor}} \) is of bounded \( I \)-torsion. Since the filtration \( \{\text{F}^n\text{H}^{q+1}(I^{k+1}K^\bullet)\}_{n \geq 0} \) defines the \( I \)-adic topology and since \((A, I)\) is pseudo-adhesive, we have in view of \(\ref{eq8.5.16} \)

\[
\text{F}^n\text{H}^{q+1}(I^{k+1}K^\bullet) \cap \text{H}^{q+1}(I^{k+1}K^\bullet)_{I\text{-tor}} = 0
\]

for sufficiently large \( n \). This implies that the filtration \( \{\text{F}^n\text{H}^q(K_k^\bullet)\}_{n \geq 0} \) is stationary. In other words, for any \( k \geq -1 \) there exists \( l \geq k \) such that for any \( m \geq l \) the maps \( \text{H}^q(K_k^\bullet) \rightarrow \text{H}^q(K_k^\bullet) \) and \( \text{H}^q(K_m^\bullet) \rightarrow \text{H}^q(K_k^\bullet) \) have the same image, which is nothing but (ML). \( \square \)
8.8. (c) **Results in case $I$ is principal.** When the ideal of definition $I$ is principal $I = (a)$, one can prove stronger results.

**Lemma 8.8.6.** Consider the situation as in §8.8 (a), and suppose that $I$ is principal, $I = (a)$. Let $q \in \mathbb{Z}$ be an integer.

1. Suppose $a^n K^{q+1}_{a_{\text{tor}}} = 0$ for some $n \geq 0$. Then for any $k \geq n$ we have
   \[ a^{k+1} H^q(K^\bullet) \subseteq F^{k+1} H^q(K^\bullet) \subseteq a^{k-n+1} H^q(K^\bullet). \]

2. Suppose, moreover, that $A$ and $K^{q-1}$ is $a$-adically complete and $K^q$ is $a$-adically separated. Let $\beta_1, \ldots, \beta_m \in H^q(K^\bullet)$ be elements such that for some $k \geq n$ the elements $\sigma^q_k(\beta_1), \ldots, \sigma^q_k(\beta_m)$ generate $\sigma^q_K(H^q(K^\bullet))$ as an $A$-module. Then $H^q(K^\bullet)$ is generated by the elements $\beta_1, \ldots, \beta_m$ over $A$. In particular, the $A$-module $H^q(K^\bullet)$ is finitely generated, and the topology on $H^q(K^\bullet)$ defined by the induced filtration
   \[ \{F^l H^q(K^\bullet)\}_{l \geq 0} \]
   is $I$-adic.

**Proof.** Applying 8.2.14 to
   \[ 0 \longrightarrow \ker(d^q) \longrightarrow K^q \longrightarrow \text{image}(d^q) \longrightarrow 0, \]
one obtains the equality $\ker(d^q) \cap a^{k+1} K^q = a^{k-n+1} (\ker(d^q) \cap a^n K^q)$. Hence, one has the inclusions $a^{k+1} \ker(d^q) \subseteq \ker(d^q) \cap a^{k+1} K^q \subseteq a^{k-n+1} \ker(d^q)$, whence (1). This means that the topology on $H^q(K^\bullet)$ defined by the filtration $\{F^l H^q(K^\bullet)\}_{l \geq 0}$ coincides with the $a$-adic topology, which shows the last assertion of (2). Then the first assertion of (2) follows from 8.8.2. \qed

**Remark 8.8.7.** Similarly to Remark 8.8.3, the proof of 8.8.6 can be much simplified if we further assume, for example, that the $a$-torsion part $K^q_{a_{\text{tor}}}$ is bounded $a$-torsion (which we can assume in most of the later applications), since in this situation one can show that the cohomology $H^q(K^\bullet)$ is $a$-adically separated.

**Lemma 8.8.8.** Consider the situation as in §8.8 (a), where the ideal $I$ is assumed to be principal $I = (a)$, and suppose, moreover, that $A$ is $a$-adically complete t.u. adhesive (8.5.17) and $a$-torsion free (hence, in particular, $A$ is topologically universally coherent with respect to $I$ (8.5.22) by 8.5.25). Let $q \in \mathbb{Z}$, and suppose there exists $n \geq 0$ such that $a^n K^q_{a_{\text{tor}}} = 0$.

1. If $K^{q-1}$ is $a$-adically complete, $K^q$ is $a$-adically separated, $H^q(K^\bullet)$ is finitely generated as an $A$-module, and $H^{q+1}(a^{k+1} K^\bullet)$ is a coherent $A$-module for any $k \geq n$, then $H^q(K^\bullet)$ is a finitely generated $A$-module.
8. Pairs

(2) If $H^q(K^\bullet)$ is finitely generated as an $A$-module and $H^q(K^\bullet_k)$ is a coherent $A$-module for any $k \geq n$, then $H^q(K^\bullet)$ is a coherent $A$-module.

Proof. (1) Since $\delta^q_k(H^q(K^\bullet_k))$ is finitely presented, $\ker(\delta^q_k) = \text{image}($ is finitely generated. Then the assertion follows from 8.8.6 (2).

(2) Set $T = H^q(K^\bullet)_{a-tor}$. Since $(A, a)$ is adhesive, $H^q(K^\bullet)/ T$ is finitely presented, and hence is coherent due to 3.3.3. In view of 3.3.4 (2), it suffices to show that $T$ is finitely presented. Since $T$ is finitely generated, there exists $m \geq 0$ such that $a^{m+1}T = T \cap a^{m+1}H^q(K^\bullet) = 0$. Let $k = m + n$. Then, by 8.8.6 (1), $\ker(\sigma^q_k) = \text{image}(i^q_k) \subseteq a^{m+1}H^q(K^\bullet)$. Hence the map $T \to H^q(K^\bullet_k)$ given by $\sigma^q_k$ is injective, and thus $T$ can be regarded as a finitely generated $A$-module of the coherent $A$-module $H^q(K^\bullet_k)$. Hence $T$ is finitely presented, as desired. \qed

The following lemma will be used later.

Lemma 8.8.9. Consider the situation as in §8.8. (a), where the ideal $I$ is assumed to be principal, $I = (a)$, and $A$ is a-adically complete. Let $B$ be an a-adically complete flat $A$-algebra such that the pair $(B, IB)$ satisfies (APf). Let $L^\bullet$ be a complex of $B$-modules, and $K^\bullet \to L^\bullet$ a morphism of complexes of $A$-modules. By the commutative diagram

$$
\begin{array}{cccc}
0 & \to & a^{k+1}K^\bullet & \to & K^\bullet & \to & K^\bullet_k & \to & 0 \\
0 & \to & a^{k+1}L^\bullet & \to & L^\bullet & \to & L^\bullet_k & \to & 0 \\
\end{array}
$$

of complexes with exact rows, one has the commutative diagram

$$
\begin{array}{cccc}
\cdots & \to & H^q(a^{k+1}K^\bullet)_B & \overset{i^q_k}{\to} & H^q(K^\bullet)_B & \overset{\sigma^q_k}{\to} & H^q(K^\bullet_k)_B & \overset{\delta^q_k}{\to} & H^q+1(a^{k+1}K^\bullet)_B & \to & \cdots \\
\vphantom{\cdots} & \overset{\phi^q}{\downarrow} & \vphantom{\cdots} & \vphantom{\cdots} & \vphantom{\cdots} & \vphantom{\cdots} & \vphantom{\cdots} & \vphantom{\cdots} & \vphantom{\cdots} & \vphantom{\cdots} & \vphantom{\cdots} \\
\cdots & \to & H^q(a^{k+1}L^\bullet)_B & \overset{i^q_k}{\to} & H^q(L^\bullet)_B & \overset{\epsilon^q_k}{\to} & H^q(L^\bullet_k)_B & \overset{\epsilon^q_k}{\to} & H^q+1(a^{k+1}L^\bullet)_B & \to & \cdots \\
\end{array}
$$

of $B$-modules with exact rows (due to the flatness of $A \to B$). Here the exact sequence in the first row is the one obtained via base change by $A \to B$ of the cohomology exact sequence as in §8.8. (a). Let $q \in \mathbb{Z}$ and suppose that $H^q(K^\bullet)$ is finitely generated as an $A$-module.

(1) If $a^n K^q_{a-tor} = 0$ for some $n \geq 0$ and $\Phi^q_k$ is injective for any $k \geq n$, then $\varphi^q$ is injective.

(2) Suppose that $K^{q-1}$ is a-adically complete, that $L^q$ is a-adically separated, and that $a^n L_{a-tor}^{q+1} = 0$ for some $n \geq 0$. If for any $k \geq n$ the map $\Phi^q_k$ is surjective and the map $\varphi^q_k$ is injective, then $\varphi^q$ is surjective.
Proof. (1) For any \(k \geq n\) we have \(\ker(\varphi^q) \subset \ker(\tau_k^q \circ \varphi^q) = \ker(\Phi_k^q \circ \sigma_{k,B}^q) = \ker(\sigma_{k,B}^q) = \text{image}(\tau_k^q) \subset \text{image}(\tau_k^q) \otimes_A B \subset a^{k-n+1}(H^q(K^\bullet) \otimes_A B)\), where the last inclusion is due to 8.8.6 (1). By 7.4.16, \(H^q(K^\bullet) \otimes_A B\) is \(a\)-adically separated. Hence \(\ker(\varphi^q) = 0\).

(2) By an easy diagram chasing one deduces that \(\tau_k^q(\varphi^q(H^q(K^\bullet) \otimes_A B)) = \tau_k^q(H^q(L^\bullet))\) for \(k \geq n\). Let \(\alpha_1, \ldots, \alpha_m\) be elements of \(H^q(K^\bullet)\) that generate \(H^q(K^\bullet)\) as an \(A\)-module, and set \(\beta_j = \varphi^q(\alpha_j \otimes 1_B)\) for each \(j = 1, \ldots, m\). We know that the elements \(\tau_k^q(\beta_1), \ldots, \tau_k^q(\beta_m)\) generate \(\tau_k^q(H^q(L^\bullet))\) as a \(B\)-module. Using 8.8.6 (2), we deduce that \(H^q(L^\bullet)\) is generated by \(\beta_1, \ldots, \beta_m\), that is, \(\varphi^q\) is surjective. \(\square\)

Exercises

Exercise 0.8.1. Let \((A, I)\) be a Zariskian pair, and \(f_1, \ldots, f_r \in A\) a finite collection of elements such that \((f_1, \ldots, f_r) = A\). Show that

\[
\bigcap_{i=1}^r \text{Spec } (A_f)^{\text{Zar}} \longrightarrow \text{Spec } A
\]

gives a flat covering of \(\text{Spec } A\).

Exercise 0.8.2. Let \((A, I)\) be a complete pair with \(I = (a)\) principal, and let the map \(f: M \rightarrow N\) be a morphism of \(A\)-modules. Suppose that

(a) \(M\) is \(a\)-adically complete, and \(N\) is \(a\)-adically separated, and

(b) \((\text{coker}(f))_{a\text{-tor}}\) is bounded \(a\)-torsion.

Show that \(\text{coker}(f)\) is \(a\)-adically separated.

Exercise 0.8.3. Let \((A, I)\) be a pair, \(a\) an ideal of \(A\), and \(f \in A\) an element of \(A\) such that the ideal \((f)\) is \(I\)-adically open. Show that, if \(a\) is \(f\)-saturated, then so is the closure \(\bar{a}\) in \(A\) with respect to the \(I\)-adic topology.

Exercise 0.8.4. Let \(W\) be a discrete valuation ring, and \(V\) a valuation ring for the residue field of \(W\) with \(0 < \text{ht}(V) < \infty\). Let \(\tilde{V}\) be the composite of \(W\) and \(V\). Show that there exists \(a \in m_V\) such that \(\tilde{V}\) is Noetherian outside \(I = (a)\) and \(a\)-torsion free, but is not \(a\)-adically separated. In particular, \(\tilde{V}\) is not \(a\)-adically adhesive.

Exercise 0.8.5. (1) Show that adhesiveness (resp. pseudo-adhesiveness) is stable under étale extensions of rings.

(2) If \((A, I)\) is adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive) and \(\text{Spec } A^h \setminus V(I^h)\) is Noetherian, then show that \((A^h, I^h)\) is adhesive (resp. pseudo-adhesive, resp. universally adhesive, resp. universally pseudo-adhesive).
Exercise 0.8.6. Let \((A, I)\) be an adhesive pair, \(M\) an \(I\)-torsion free \(A\)-module and \(N, P \subseteq M\) finitely generated \(A\)-submodules. Show that \(N \cap P\) is finitely generated.

Exercise 0.8.7. Let \(A\) be a ring, and \(I \subseteq A\) a finitely generated ideal.

1. Show that the following conditions are equivalent (cf. 8.5.3):
   
   (a) for any finitely generated \(A\)-module \(M\) that is finitely presented outside \(I\), \(M_{I\text{-tor}}\) is finitely generated.

   (b) for any finitely generated \(A\)-module \(M\) that is finitely presented outside \(I\), \(M/M_{I\text{-tor}}\) is finitely presented.

   (c) for any finitely generated \(A\)-module \(M\) and any finitely generated outside \(I\) \(A\)-submodule \(N\), the \(I\)-saturation \(\overline{N}\) is finitely generated.

If one of (and hence all) these conditions are satisfied, then we say that the pair \((A, I)\) is pre-adhesive. If, moreover, any polynomial rings over \(A\) together with the ideal induced by \(I\) is pre-adhesive, we call \((A, I)\) universally pre-adhesive. Clearly, a pair \((A, I)\) is adhesive (resp. universally adhesive) if and only if it is pre-adhesive (resp. universally pre-adhesive) and \(A\) is Noetherian outside \(I\).

2. Show that the statements analogous to 8.5.7 and 8.5.8 hold; in particular, universally pre-adhesiveness is stable under finite type extensions.

3. Consider an inductive system \(\{B_i, f_{ij}: B_i \to B_j\}\) of \(A\)-algebras indexed by a directed set \(J\), and set \(B = \lim_{\to i \in I} B_i\). Suppose that
   
   (i) \((B_i, IB_i)\) is pre-adhesive (resp. universally pre-adhesive) for each \(i\) and

   (ii) for each pair of indices \(i, j\) such that \(i \leq j\) the morphism \(f_{ij}\) is flat.

Show that \((B, IB)\) is pre-adhesive (resp. universally pre-adhesive) (cf. 8.5.10).

4. Show that if \((A, I)\) is pre-adhesive (resp. universally pre-adhesive), then so is the Henselization \((A^h, I^h)\).

Exercise 0.8.8. Let \(A\) (resp. \(A'\)) be \(a\)-valuative (resp. \(a'\)-valuative) local ring, where \(a \in m_A\) (resp. \(a' \in m_{A'}\)), and \(h: A \to A'\) a local homomorphism that is adic with respect to the \(a\)-adic and \(a'\)-adic topologies. Let \(J = \cap_{n \geq 1} a^n A, B = A[\frac{1}{a}]\), \(V = A/J\), and \(K = \text{Frac}(V)\) be as in 8.7.8, and similarly, \(J' = \cap_{n \geq 1} a'^n A', B' = A'[\frac{1}{a'}], V' = A'/J',\) and \(K' = \text{Frac}(V')\).

1. Show that the map \(h\) induces a local injection \(V \hookrightarrow V'\) and, moreover, we have \(V = K \cap V'\) in \(K'\).

2. Show that the map \(\hat{h}\) induces a local homomorphism \(g = h[\frac{1}{a}]: B \to B'\) and, moreover, we have \(g^{-1}(A') = A\).
9 Topological algebras of type (V)

By a topological algebra of type (V), we mean a topologically finitely generated algebra over an $a$-adically complete valuation ring. The terminology echoes a notion that will be introduced, the rigid spaces of type (V) (II, §2.5.(a)), which will be defined consistently as rigid spaces locally of finite type over an $a$-adically complete valuation ring. Similarly to how finitely generated algebras over a field are ‘coordinate rings’ and play a dominating role in classical algebraic geometry, topological algebras of type (V), especially those over an $a$-adically complete valuation ring of height one, are the central object in the classical rigid geometry of Tate and Raynaud. More precisely, if $A$ is a topologically finitely generated algebra over an $a$-adically complete valuation ring $V$, then the algebra $A[\frac{1}{a}]$ is an affinoid algebra in Tate’s rigid analytic geometry (see [94]). Many of the known properties of affinoid algebras, ring-theoretic (e.g. Noetherianness) or Banach algebra-theoretic, are, as we will see in §9.2.(a), actually consequences of the fact that $V$ is $a$-adically topologically universally adhesive (8.5.17).

In §9.1 we give a general fact concerning $a$-adically complete valuation rings and $a$-adic completion of valuation rings. The main theorem (9.1.1) describes in detail the structure of the $a$-adic completion of a valuation ring, which will be at the basis of the discussions in the subsequent subsections.

In §9.2.(b) we will discuss the Noether normalization theorem for topologically finitely generated algebra over an $a$-adically complete valuation ring of height one, which is one of the ‘special techniques’ that work only in the height-one situation.

The final subsection, §9.3, surveys several important and basic properties of affinoid algebras.

Convention. In this book, affinoid algebras obtained as $A[\frac{1}{a}]$ from a topologically finitely generated algebra $A$ over an $a$-adically complete valuation ring are called classical affinoid algebras, in order to distinguish them from more general algebras that arise from affinoids (II, §6) in our approach to rigid geometry.

9.1 $a$-adic completion of valuation rings

9.1.(a) Fundamental structure theorem. Throughout this section we fix a valuation ring $V$ and a non-zero element $a \in m_V \setminus \{0\}$. (Note that the valuation ring $V$ is therefore of non-zero height, that is, not a field.) We consider the $a$-adic topology on $V$ and denote by

$$\hat{V} = \lim_{\leftarrow n} V/a^n V$$

the $a$-adic completion of $V$ (cf. 7.2.15). Furthermore, we set $K = \text{Frac}(V)$. 
Theorem 9.1.1. Suppose \( V \) is \( a \)-adically separated.

(1) The \( a \)-adic completion \( \hat{V} \) is a valuation ring of non-zero height, and \( \text{Frac}(\hat{V}) \) is canonically isomorphic to the Hausdorff completion \( \hat{K} \) of \( K \) with respect to the filtration \( \{a^n V\}_{n \geq 0} \).

(2) The canonical map \( V \hookrightarrow \hat{V} \) is a local homomorphism, that is, \( m_{\hat{V}} \cap V = m_V \). Moreover, the canonical map \( V/m_V \rightarrow \hat{V}/m_{\hat{V}} \) is an isomorphism.

(3) Let \( p \) be a non-zero prime ideal of \( V \). Then the localization \( V_p \) is \( a \)-adically separated, and \( \hat{V} \) is isomorphic to the composite of \( \hat{V}_p \), the \( a \)-adic completion of \( V_p \), and \( V/p \).

(4) For any prime ideal \( p \) of \( V \), \( p\hat{V} \) is a prime ideal of \( \hat{V} \), and \( p\hat{V} \cap V = p \).

(5) The underlying continuous mapping of the canonical morphism

\[
\text{Spec } \hat{V} \longrightarrow \text{Spec } V
\]

is a homeomorphism; in particular, \( \text{ht}(\hat{V}) = \text{ht}(V) \). Moreover, the canonical map

\[
\Gamma_V = K^\times / V^\times \longrightarrow \Gamma_{\hat{V}} = \hat{K}^\times / \hat{V}^\times
\]

between the value groups is an isomorphism of totally ordered commutative groups.

Let us make the following small remark before proceeding to the proof. In the situation as in the theorem, we have the commutative square

\[
\begin{array}{ccc}
\hat{V} & \longrightarrow & \hat{K} \\
\uparrow & & \uparrow \\
V & \longrightarrow & K
\end{array}
\]

consisting of injective ring homomorphisms. The injectivity of the vertical arrows follows from the fact that the filtration \( \{a^n V\}_{n \geq 0} \) is separated, and the injectivity of the upper horizontal arrow follows from Exercise 0.7.5. It follows, moreover, from the same exercise that the above square is Cartesian, that is, \( V = \hat{V} \cap K \), and that \( K/V \cong \hat{K}/\hat{V} \). In particular, we have \( \hat{V} \neq \hat{K} \).
9.1. (b) Proof of Theorem 9.1.1. We first show assertions (1) and (2) of the theorem. To show that the completion \( \hat{K} \) of \( K \) is a field, take \( x \in \hat{K} \). Then \( x \) is the limit of a Cauchy sequence \( \{x_n\}_{n \geq 0} \) in \( K \). If \( x \neq 0 \), replacing \( \{x_n\}_{n \geq 0} \) by a cofinal subsequence, we may assume that there exists \( N \geq 1 \) such that \( x_n \not\in a^N V \) for any \( n \geq 0 \). Let us consider the sequence \( \{x_n^{-1}\}_{n \geq 0} \), which we want to prove to be a Cauchy sequence. For any \( y \) in \( V \), the maximal ideal of \( V \), and that the inclusion map \( V \to V \) is of non-zero height. Thus we have shown all assertions in (1) and (2) of the theorem.

To continue the proof, we need some auxiliary results.

Lemma 9.1.2. Let \( V \) be an \( \alpha \)-adically separated valuation ring, and \( S \subseteq V \) a multiplicative subset such that \( S \cap \sqrt{a} = \emptyset \). Then for any \( n \geq 0 \) we have

\[
\alpha^{n+1} V_S \subseteq \alpha^n V \subseteq \alpha^n V_S.
\]

In particular, the \( \alpha \)-adic topology on the ring \( V_S \) coincides with the one given by the filtration \( \{\alpha^n V\}_{n \geq 0} \).
9. Topological algebras of type (V)  

Proof. The inclusion $a^n V \subseteq a^n V_S$ is obvious. Let $a^{n+1}b/c \in a^{n+1}V_S$ where $c \in S$. Since $c \notin \langle a \rangle$, there exists $d \in V$ such that $a = cd$. Hence $a^{n+1}b/c = a^n bd \in a^n V$, thereby the other inclusion. \hfill \qedsymbol

Corollary 9.1.3. Let $V$ be an a-adically separated valuation ring, and $\mathfrak{p}$ a non-zero prime ideal of $V$. Then $V_\mathfrak{p}$ is a-adically separated, and ${\text{Frac}}(\hat{V}) = {\text{Frac}}(\hat{V}_\mathfrak{p})$, where $\hat{V}_\mathfrak{p}$ denotes the a-adic completion of $V_\mathfrak{p}$.

Proof. In view of 6.7.3 we know that $\mathfrak{p}$ contains $\sqrt{\langle a \rangle}$; applying 9.1.2, we have
\[ \bigcap_{n \geq 0} a^n V_\mathfrak{p} = \bigcap_{n \geq 0} a^n V = 0, \]
which shows that $V_\mathfrak{p}$ is a-adically separated. Moreover, by 9.1.2 and Exercise 0.7.5, $\hat{V} \subseteq \hat{V}_\mathfrak{p} \subseteq \hat{K} = {\text{Frac}}(\hat{V})$, whence the other assertion. \hfill \qedsymbol

Lemma 9.1.4. For any $x \in \hat{K}$ there exists $u \in \hat{V}^\times$ such that $ux \in K$. Moreover, if $x \in \hat{V}$, then $ux \in V$.

Proof. We may assume $x \neq 0$. Then $x$ is the limit of a Cauchy sequence $\{x_n\}_{n \geq 0}$ of elements in $K$; since $x \neq 0$, we may assume that $x_n \neq 0$ for any $n \geq 0$. Since the sequence $\{x_n/x\}_{n \geq 0}$ converges to 1, one can find a sufficiently large $n$ such that $u = x_n/x$ belongs to the open neighborhood $1 + a\hat{V}$ of 1. Since $1 + a\hat{V} \subseteq 1 + m\hat{\mathfrak{p}}$, we have $u \in \hat{V}^\times$, and thus the first claim is shown. If $x \in \hat{V}$, then $ux \in \hat{V} \cap K \neq \emptyset$. \hfill \qedsymbol

Now we proceed to the proof of assertions (3), (4), and (5) of 9.1.1. Let $\mathfrak{p}$ be a non-zero prime ideal of $V$. By 9.1.3, we already know that $V_\mathfrak{p}$ is a-adically separated. By (2), the residue field of $\hat{V}_\mathfrak{p}$ is equal to that of $V_\mathfrak{p}$, and hence one can take the composite valuation ring $\hat{V}$ of $V_\mathfrak{p}$ and $V/\mathfrak{p}$. By 9.1.3, $\hat{V}$ is a valuation ring for ${\text{Frac}}(\hat{V}) = \hat{K}$. Moreover, by the construction, we have the canonical inclusion map $\hat{V} \hookrightarrow \hat{V}$, which is local. Hence, by the characterization (a) in 6.2.1, $\hat{V} = \hat{V}$, whence (3).

To show (4), let us consider the completion $\hat{\mathfrak{p}}$ of $\mathfrak{p}$ with respect to the filtration $\{a^n V\}_{n \geq 1}$ (note that $a^n V$ is contained in $\mathfrak{p}$ for any $n \geq 1$ (6.7.3)). By Exercise 0.7.5, $\hat{\mathfrak{p}}$ is an ideal of $\hat{V}$; since $\hat{V}/\hat{\mathfrak{p}} \cong V/\mathfrak{p}$, it is a prime ideal; moreover, $\hat{\mathfrak{p}} \cap V = \mathfrak{p}$. Now, by the construction we have the canonical inclusion $\mathfrak{p}\hat{V} \subseteq \hat{\mathfrak{p}}$. Moreover, since $\mathfrak{p}\hat{V}$ contains $a\hat{V}$, we have $V/\mathfrak{p} \cong \hat{V}/\mathfrak{p}\hat{V}$. Hence, $\mathfrak{p}\hat{V} = \hat{\mathfrak{p}}$, which shows (4).

Since $\hat{V}$ is a-adically separated, any non-zero prime ideal contains $a$. Hence, to show the first assertion of (5), it suffices to show that the map
\[ \text{Spec } \hat{V}/a\hat{V} \longrightarrow \text{Spec } V/aV \]
is a homeomorphism, which is clear, since $\hat{V}/a\hat{V} \cong V/aV$. This shows the equality $\text{ht}(\hat{V}) = \text{ht}(V)$ between the heights.
Finally, let us show $\Gamma V \cong \Gamma \hat{V}$. Since $\hat{V} \cap K = V$ and $m_{\hat{V}} \cap V = m_V$, it is easy to see that the map $k^\times/V^\times \to \hat{k}^\times/\hat{V}^\times$ is injective. The surjectivity follows from 9.1.4.

9.1. (c) Corollaries. Theorem 9.1.1 has several useful corollaries.

**Corollary 9.1.5.** Let $V$ be a valuation ring and $a \in m_V \setminus \{0\}$. Then the $a$-adic completion $\hat{V} = \lim_{\leftarrow n \geq 0} V/a^n V$ is a valuation ring of non-zero height.

*Proof.* By 6.7.1, $J = \bigcap_{n \geq 0} a^n V$ is a prime ideal, and hence $V/J$ is a valuation ring (6.4.1 (1)). Since $\hat{V}$ is isomorphic to the $a$-adic completion of $V/J$ and $a \not\in J$, the assertion follows from 9.1.1 (1).

**Remark 9.1.6.** Here is another proof of 9.1.5. Since we have a canonical bijection between the set of all $a$-admissible ideals (8.1.2) of $\hat{V} = \lim_{\leftarrow n \geq 0} V/a^n V$ and the set of $a$-admissible ideals of $V$, we see that $\hat{V}$ is $a$-valuative (8.7.1). Hence, by 8.7.8, $\hat{V} = \hat{V} / \bigcap_{n \geq 0} a^n \hat{V}$ is a valuation ring.

**Corollary 9.1.7.** Let $V$ be an $a$-adically separated valuation ring with $a \in m_V \setminus \{0\}$, and $S \subseteq V$ a multiplicative subset such that $S \cap \sqrt(a) = \emptyset$. Then the canonical map

$$(\hat{V})_S \longrightarrow \hat{V}_S$$

is an isomorphism, where the left-hand ring is the ring of fractions of $\hat{V}$ with respect to the image of $S$ under the canonical map $V \hookrightarrow \hat{V}$, and the right-hand ring is the $a$-adic completion of $V_S$.

*Proof.* By 9.1.2 and Exercise 0.7.5 the canonical map $\hat{V} \to \hat{V}_S$ is injective, and hence the map in question is injective. To show it is surjective, we write each element $x$ of $\hat{V}_S$ as a power series in $a$

$$x = \sum_{n \geq 0} b_n s_n a^n,$$

where $b_n \in V$ and $s_n \in S$ for $n \geq 0$. Since $s_n \not\in (a)$, there exists $t_n \in V$ such that $a = s_n t_n$. Hence,

$$x = \frac{b_0}{s_0} + \sum_{n \geq 1} b_n t_n a^{n-1},$$

which belongs to $(\hat{V})_S$. \qed
Corollary 9.1.8. If $V$ is an $a$-adically complete valuation ring ($a \in \mathfrak{m}_V \setminus \{0\}$) and $S \subseteq V$ is a multiplicative subset such that $S \cap \sqrt{a} = \emptyset$, then $V_S$ is $a$-adically complete.

Corollary 9.1.9. Let $V$ be an $a$-adically separated valuation ring ($a \in \mathfrak{m}_V \setminus \{0\}$), and $p$ a non-zero prime ideal of $V$. Then $(\hat{V})_p = V_p$.

Proof. In view of 9.1.7 it suffices to show that the localization of $\hat{V}$ by $S = V \setminus p$ is equal to the localization by $\hat{V} \setminus p\hat{V}$. By 9.1.4, for any $x \in \hat{V} \setminus p\hat{V}$ there exists $u \in \hat{V}$ such that $ux \in V \cap (\hat{V} \setminus p\hat{V})$. Since $p\hat{V} \cap V = p$ (9.1.1 (4)), $ux \in S$. The claim follows immediately from this.

Corollary 9.1.10. Let $V$ be an $a$-adically separated valuation ring ($a \in \mathfrak{m}_V \setminus \{0\}$). Let $\mathfrak{p} = \sqrt{aV}$ (resp. $\hat{\mathfrak{p}} = \sqrt{a\hat{V}}$) be the associated height-one prime of $V$ (resp. $\hat{V}$) (cf. 6.7.4). Then $(\hat{V})_{\hat{\mathfrak{p}}} = \hat{V}_p$.

Proof. In view of 9.1.9 it suffices to show that $\hat{\mathfrak{p}} = p\hat{V}$. Since $p\hat{V} \cap V = p$ (9.1.1 (4)) and since Spec $\hat{V} \to \text{Spec} V$ is a homeomorphism (9.1.1 (5)), both $p\hat{V}$ and $\hat{\mathfrak{p}}$ are height one primes of $\hat{V}$. Hence we have $\hat{\mathfrak{p}} = p\hat{V}$.

9.2 Topologically finitely generated $V$-algebras

9.2. (a) Adhesiveness. We consider in the sequel of this section an $a$-adically complete valuation ring $V$ with $a \in \mathfrak{m}_V \setminus \{0\}$. As defined in 8.4.1, a topologically finitely generated $V$-algebra is an $a$-adically complete $V$-algebra of the form

$$A = V \llangle X_1, \ldots, X_n \rrangle / a,$$

where $a$ is a closed ideal of the restricted formal power series ring $V \llangle X_1, \ldots, X_n \rrangle$.

We have already seen in 8.5.15 that the an $a$-adically complete valuation ring is $a$-adically adhesive. Our aim here is to show a much stronger result, namely, that $(V, a)$ is topologically universally adhesive (9.2.7). Due to 8.5.20 it suffices to show the following theorem.

Theorem 9.2.1. The pair $(V \llangle X_1, \ldots, X_n \rrangle, a)$ is adhesive. (Hence any topologically finitely generated algebra (cf. 8.4.1) over $V$ is $a$-adically adhesive.)

The proof of the theorem will be done in two steps. The first step deals with the special case where $V$ is of finite height, while the second treats the general case. The argument of the first step is almost the same as [38], Lemma 1.1.2, the proof of which is only valid when the height of $V$ is finite. Our proof here fills in this gap and is valid for arbitrary height.

Before the proof we include here for the reader’s convenience a useful result by Raynaud and Gruson.
Theorem 9.2.2 ([89], Première partie, Théorème (3.4.6)). Let $f : X \to S$ be a locally of finite presentation morphism of schemes, and $M$ an $\mathcal{O}_X$-module of finite type. Suppose that the set of associated primes $\text{Ass}(S)$ of $S$ to $\mathcal{O}_S$, that is, the set of point $s \in S$ such that $\mathfrak{m}_{S,s}$ is up to radical the annihilator of an element of $\mathcal{O}_{S,s}$, is locally finite. Then the set $U$ of all points of $X$ where $M$ is $S$-flat is open, and $M|_U$ is a finitely presented $\mathcal{O}_U$-module.

Corollary 9.2.3 ([89], Première partie, Corollaire (3.4.7)). Any flat of finite type algebra over an integral domain is finitely presented.

Proposition 9.2.4. Let $A$ be an $a$-adically complete $V$-algebra, and $S \subseteq V$ a multiplicative subset such that $S \cap \sqrt{aV} = \emptyset$. Then the ring $A_S$ is $a$-adically complete, and hence we have the equalities

$$A_S = A \otimes_V V_S = A \hat{\otimes}_V V_S$$

up to canonical isomorphisms.

Proof. By a similar reasoning as in the proof of 9.1.2, for any $s \in S$ there exists $t \in V$ such that $st = a$. Hence, for $n \geq 1$ the subset $a^nA_S$ of $A_S$ is contained in the image of $a^{n-1}A$ under the canonical map $A \to A_S$. This shows that $A_S$ is $a$-adically separated, and hence that the canonical map $A_S \to \hat{A}_S$ to the $a$-adic completion is injective. To show that this map is bijective, observe that any element $x$ of $\hat{A}_S$ is written as a power series $x = \sum_{n \geq 0} b_n a^n$ as in the proof of 9.1.7, where $b_n \in A$ and $s_n \in S$ for each $n \geq 0$. Then by the similar reasoning one finds that $x$ lies in $A_S$.

Proof of Theorem 9.2.1. Set $A = V \langle X_1, \ldots, X_n \rangle$, and let $M$ be an $a$-torsion free finitely generated $A$-module. We need to show that $M$ is finitely presented (cf. 8.5.3 (b)). Take a surjection $A^{\oplus n} \to M$ and consider the exact sequence

$$0 \to L \to A^{\oplus n} \to M \to 0.$$  

We want to prove is that $L$ is finitely generated. This will be shown in two steps.

Step 1. Suppose that the height of $V$ is finite. Thanks to 7.2.4 it is enough to show that $L/aL$ is a finitely generated $A/aA$-module or, equivalently, that $M/aM$ is a finitely presented $A/aA$-module [27], Chapter I, §2.8, Lemma 9. Since $M$ has no $a$-torsion, it is $V$-flat and hence $M/aM$ is $(V/aV)$-flat. Now, $A/aA$ is the polynomial ring $A/aA \cong (V/aV)[X_1, \ldots, X_n]$, and $(V/aV)_{\text{red}}$ is a valuation ring (possibly a field), since $p = \sqrt{(a)}$ is a prime. Hence we can apply 9.2.2 (here we use the hypothesis that the height is finite). Since $M/aM$ is a finitely generated flat $(A/aA)$-module, it follows that $M/aM$ is of finite presentation as desired, and the proof in this case is done.
Step 2. In general, consider the associated height-one prime \( p = \sqrt{(a)} \) (6.7.4), and set \( V' = V_p \). By 9.1.10, \( V' \) is an \( a \)-adically complete valuation ring of height one. Moreover, by 9.2.4, \( A \otimes_V V' \) is isomorphic to \( V' \langle X_1, \ldots, X_n \rangle \). Hence one can apply the argument of Step 1 to conclude that \( M \otimes_V V' \) is an \( A \otimes_V V' \)-module of finite presentation and hence that \( L \otimes_V V' \) is finitely generated over \( A \otimes_V V' \). Take \( x_1, \ldots, x_d \in L \) that generate \( L \otimes_V V' \). Consider the exact sequence induced by (*)..

\[
0 \longrightarrow L/pL \longrightarrow (A/pA)^{\oplus N} \longrightarrow M/pM \longrightarrow 0 \quad (**)
\]

and note that it is exact because \( L \) is \( a \)-saturated (and hence \( M \) is \( V \)-flat). Since (**) is an exact sequence of modules over a polynomial ring \( A/pA \cong (V/p)[X_1, \ldots, X_n] \), and since \( M/pM \) is flat over \( V/p \), it follows from 9.2.3 that \( L/pL \) is finitely generated over \( (A/pA) \). Hence one can take \( y_1, \ldots, y_e \in L \) that generate \( L/pL \).

Now we claim that \( x_1, \ldots, x_d, y_1, \ldots, y_e \) generate \( L \) as an \( A \)-module. Take any \( z \in L \). There exist \( \alpha_1, \ldots, \alpha_e \in A \) such that

\[
z = (\alpha_1 y_1 + \cdots + \alpha_e y_e) \in pL.
\]

Set

\[
y = z - (\alpha_1 y_1 + \cdots + \alpha_e y_e) \quad (y \in p, y \in L).
\]

We can find \( \beta_1, \ldots, \beta_d \in A \otimes_V V' \) such that

\[
y = \beta_1 x_1 + \cdots + \beta_d x_d.
\]

Since \( pV' \subseteq V \), we have \( y \beta_i \in A \) for \( i = 1, \ldots, d \), and hence

\[
z = \alpha_1 y_1 + \cdots + \alpha_e y_e + (y \beta_1)x_1 + \cdots + (y \beta_d)x_d
\]

expresses \( z \) as an \( A \)-linear combination. \( \square \)

**Remark 9.2.5.** It worth pointing out that the above proof has the following basic feature, which seems applicable to many other situations. It is divided into two parts, and this division arises from taking the associated height-one prime \( p = \sqrt{(a)} \) of the valuation ring \( V \); i.e., from exploiting the decomposition of the valuation ring \( V \) into \( V_p \) and \( V/p \). Unlike the first part, which discusses things over the height-one generalization \( V_p \), the second part deals with the situation over \( V/p \) in which, since \( a = 0 \), the issues usually come down to classical algebraic geometry or classical commutative algebra. In this sense the first part is usually the essential part of the proof and much harder than the second part. It is, however, manageable in principle and can be done with the aid of the finiteness of height; moreover, we have several special techniques that work in the height-one situation. It is by this reason that it is important to develop specialized techniques for dealing with topological algebras of type (V) over a height one-base; in \( \S 9.2. \) (b) and Appendix \( \S A \), we will explain some of these techniques. For example, by using some of the results in \( \S A \), we can give an alternative argument for Step 1 of the above proof that does not use the result of Raynaud and Gruson; see A.2.5.
Let us mention that there is a shorter proof of 9.2.1 due to O. Gabber. In this proof one only needs to argue as in Step 1 of the above proof; to show that \( M/aM \) is finitely presented (even in case \( V \) is of infinite height), we first note that \( M \otimes_V (V/aV)_{\text{red}} \) is finitely presented by 9.2.3 and apply the following lemma, which complements results by Raynaud and Gruson.

**Lemma 9.2.6.** Let \( A \) be a ring, \( B = A[X_1 \ldots X_n] \) a polynomial ring, and \( M \) a finitely generated \( B \)-module. Suppose \( M \) is \( A \)-flat. Then \( M \) is finitely presented over \( B \) if and only if \( M \otimes_A A_{\text{red}} \) is finitely presented over \( B \otimes_A A_{\text{red}} \).

**Proof.** Take a finitely presented \( B \)-module \( N \) and a surjective map \( \varphi: N \to M \) such that \( N \otimes_A A_{\text{red}} \cong M \otimes_A A_{\text{red}} \). To obtain these, write \( M \cong B^{\oplus n}/K \) and take a finitely generated \( B \)-submodule \( K_0 \subseteq K \) such that

\[
K_0 \otimes_A A_{\text{red}} = K \otimes_A A_{\text{red}};
\]

then

\[
N = B^{\oplus n}/K_0 \longrightarrow M
\]
gives the desired map. By [89], Première partie, Théorème (3.4.1), \( M_p \) for any prime ideal \( p \subseteq B \) is finitely presented over \( B_p \). Hence if \( L = \ker(\varphi) \), \( L_p \) is finitely generated over \( B \). Since \( M \) is flat over \( A \), we have the exact sequence

\[
0 \longrightarrow L_p \otimes_A A_{\text{red}} \longrightarrow N_p \otimes_A A_{\text{red}} \longrightarrow M_p \otimes_A A_{\text{red}} \longrightarrow 0,
\]

whence \( L_p \otimes_A A_{\text{red}} = 0 \). But since \( L_p \) is finitely generated, one can apply Nakayama’s lemma to deduce \( L_p = 0 \). Since this holds for any prime ideal \( p \) of \( B \), we conclude that \( \varphi \) is an isomorphism. \( \square \)

Now by 8.5.20 and 9.2.1 we have the following corollary.

**Corollary 9.2.7** (O. Gabber). Let \( V \) be an \( a \)-adically complete valuation ring of arbitrary height. Then \( V \) is \( a \)-adically topologically universally adhesive. (Hence any topologically finitely generated \( V \)-algebra is \( a \)-adically topologically universally adhesive.)

From this and 8.5.25 we deduce the following corollary.

**Corollary 9.2.8.** Let \( V \) be an \( a \)-adically complete valuation ring of arbitrary height. Then any algebraic space locally of finite presentation over \( V \) is universally cohesive.

Moreover, since any \( V \)-flat algebra is \( a \)-torsion free, we have the following corollary.

**Corollary 9.2.9.** Let \( V \) be an \( a \)-adically complete valuation ring of arbitrary height. Then any \( V \)-flat topologically finitely generated algebra is topologically finitely presented.
9. Topological algebras of type (V)

9.2. (b) Noether normalization

Theorem 9.2.10 (Noether normalization theorem for topologically finitely generated V-algebras). Let V be an a-adically complete (for a ∈ mV \ {0}) valuation ring of height one, and A a V-flat topologically finitely generated V-algebra. Then there exists a finite injective V-morphism

\[ V \langle X_1, \ldots, X_d \rangle \hookrightarrow A \]

with V-flat cokernel.

Before proceeding to the proof of the theorem, let us introduce a useful notion. Consider the restricted formal power series ring \( B = V \langle Y_1, \ldots, Y_n \rangle \) (where V is not necessarily of height one). Each element \( f \in B \) can be written as a power series

\[ f = \sum_{v_1, \ldots, v_n \geq 0} b_{v_1, \ldots, v_n} Y_1^{v_1} \cdots Y_n^{v_n}, \]

where \( b_{v_1, \ldots, v_n} \in V \) for any \( v_1, \ldots, v_n \geq 0 \). The content ideal \( \text{cont}(f) \) of \( f \) is the ideal of \( V \) generated by all the coefficients \( b_{v_1, \ldots, v_n} \). This ideal is actually finitely generated (Exercise 0.9.3) and hence is principal. Note that, if \( \text{cont}(f) = (b) \) and \( f \neq 0 \), then \( f \) is divisible by \( b \) and \( \text{cont}(f/b) = V \).

Lemma 9.2.11. Suppose \( V \) is of height one, and let \( \alpha \subseteq V \langle Y_1, \ldots, Y_n \rangle \) be an \( a \)-saturated ideal. If \( \alpha \neq 0 \), then there exists a non-zero \( f \in \alpha \) such that

\( \text{cont}(f) = V. \)

Proof. Take \( f \in \alpha \setminus \{0\} \), and consider the content ideal \( \text{cont}(f) = (b) \). Since \( V \) is of height one, there exist \( m \geq 0 \) and \( c \in V \) such that \( bc = a^m \); see (6.3.2). Since \( \alpha \) is \( a \)-saturated, the element \( cf/a^m \) belongs to \( \alpha \) and \( \text{cont}(cf/a^m) = V. \) □

Proof of Theorem 9.2.10. Let \( k = V/m_V \) be the residue field of \( V \), and consider \( A_0 = A \otimes_V k \), which is a finite type algebra over \( k \). By the classical Noether normalization theorem ([84], Chapter I, §14) we can find elements \( \tilde{x}_1, \ldots, \tilde{x}_d \in A_0 \) algebraically independent over \( k \) such that the map \( k[\tilde{x}_1, \ldots, \tilde{x}_d] \hookrightarrow A_0 \) is finite. Take \( x_1, \ldots, x_d \in A \) such that \( \tilde{x}_i = (x_i \mod m_V) \) for \( i = 1, \ldots, d \), and consider the \( V \)-subalgebra \( A' \subseteq A \) topologically generated by them; \( A' \) is the image of the unique morphism \( \varphi: V \langle X_1, \ldots, X_d \rangle \to A \) mapping each \( X_i \) to \( x_i \) (i = 1, \ldots, d) (cf. B.1.8 in the appendix). Clearly, \( A' = A' \otimes_V k \cong k[\tilde{x}_1, \ldots, \tilde{x}_d] \subseteq A_0 \).

We first claim that the map \( \varphi \) is injective. Indeed, if not, \( \ker(\varphi) \) is a non-zero \( a \)-saturated ideal (since \( A \) and \( A' \) are \( a \)-torsion free by our hypothesis). Then, by 9.2.11, there exists \( f \in \ker(\varphi) \) such that \( \tilde{f} = (f \mod m_V) \neq 0 \); but this would imply that the map \( k[\tilde{x}_1, \ldots, \tilde{x}_d] \to A_0 \) has non-zero kernel, contradicting that \( \tilde{x}_1, \ldots, \tilde{x}_d \) are algebraically independent over \( k \). Now, since \( \varphi_0 = \varphi \otimes_V k: A'_0 \hookrightarrow A_0 \) is finite and since \( A' \) and \( A \) are \( a \)-adically complete, we readily deduce that \( \varphi \) is finite by 7.2.4.
Finally, let us show that the cokernel $A/A'$ is $a$-torsion free. Since $V$ is of height one, one has $m_V = \sqrt{(a)}$. Then one verifies easily that for an $V$-module $M$ to be $a$-torsion free it is necessary and sufficient that $\text{Tor}_1^V(M, k) = 0$. Hence, to verify the claim, it suffices to invoke that the map $A'_0 = A' \otimes_V k \rightarrow A_0 = A \otimes_V k$ is injective.

**Corollary 9.2.12.** Let $V$ be as in 9.2.10, and $A$ a $V$-flat quotient of $V\langle X_1, \ldots, X_n \rangle$ such that the image of the closed immersion

$$\text{Spec } A[\frac{1}{a}] \hookrightarrow \text{Spec } V\langle X_1, \ldots, X_n \rangle[\frac{1}{a}]$$

is a finite set of closed points. Then $A$ is finite over $V$.

**Proof.** We have a finite injective $V$-morphism

$$V\langle Y_1, \ldots, Y_d \rangle \hookrightarrow A.$$

Since Spec $A[\frac{1}{a}]$ consists of a single point, we have $d = 0$. \qed

### 9.3 Classical affinoid algebras

**9.3. (a) Tate algebra and classical affinoid algebras.** As in the previous subsections $V$ denotes an $a$-adically complete valuation ring with $a \in m_V \setminus \{0\}$, and $K = \text{Frac}(V)$ the field of fractions of $V$.

**Definition 9.3.1.** A classical affinoid algebra is a $K$-algebra of the form

$$\mathcal{G} = A \otimes_V K \left( = A\left[\frac{1}{a}\right]\right)$$

(cf. 6.7.2), where $A$ is a topologically finitely generated $V$-algebra.

Let $p = \sqrt{a V}$ be the associated height-one prime of $V$ (6.7.4), and $V' = V_p$ the associated height-one localization of $V$. Since $A \otimes_V K = (A \otimes_V V') \otimes_{V'} K$ and since $A \otimes_V V'$ is $a$-adically complete (hence is a topologically finitely generated $V'$-algebra) due to 9.2.4, we may always assume, whenever discussing classical affinoid algebras, that $V$ is of height one. This reduction will be very helpful in developing generalities on classical affinoid algebras, for then one can use special techniques valid only in the height-one case, such as Noether normalization (9.2.10).

The first general result on classical affinoid algebras reads as follows.

**Proposition 9.3.2.** Any classical affinoid algebra is a Noetherian ring.

**Proof.** Let $\mathcal{G} = A \otimes_V K$ be a classical affinoid algebra. By 9.2.1 we know that $A$ is $a$-adically adhesive and hence is Noetherian outside $(a)$. \qed
9. Topological algebras of type (V)

Let us mention a special kind of affinoid algebras; this is the case where $A$ as above is the ring $V\langle X_1, \ldots, X_n \rangle$ of restricted formal power series. In this case the corresponding classical affinoid algebra $A \otimes_V K$ is usually called the Tate algebra and is denoted by $K\langle X_1, \ldots, X_n \rangle$; explicitly,

$$K\langle X_1, \ldots, X_n \rangle = \left\{ \sum_{v_1, \ldots, v_n \geq 0} a_{v_1, \ldots, v_n} X_1^{v_1} \cdots X_n^{v_n} \in K[X_1, \ldots, X_n] \mid |a_{v_1, \ldots, v_n}| \to 0 \text{ as } v_1 + \cdots + v_n \to \infty \right\}.$$ 

Here the function $|\cdot|: K \to \mathbb{R}_{\geq 0}$ denotes a non-Archimedean valuation associated to the height-one localization $V'$ (cf. §6.3. (c)). That $A$ is $a$-adically complete is then interpreted as the fact that $K\langle X_1, \ldots, X_n \rangle$ is a $K$-Banach algebra with respect to the norm

$$\| \sum_{v_1, \ldots, v_n \geq 0} a_{v_1, \ldots, v_n} X_1^{v_1} \cdots X_n^{v_n} \| = \sup_{v_1, \ldots, v_n \geq 0} |a_{v_1, \ldots, v_n}|,$$

called the Gauss norm. Indeed, we have the following lemma.

**Lemma 9.3.3.** The topology on $K\langle X_1, \ldots, X_n \rangle$ given by the Gauss norm $\| \cdot \|$ is equivalent to the $a$-adic topology, that is, the topology given by the filtration \{a^n V\langle X_1, \ldots, X_n \rangle\}_{n \geq 0}.

**Proof.** By 9.1.2, we can replace $V$ by the height-one localization and thus assume that $V$ is of height one. Let

$$A = V\langle X_1, \ldots, X_n \rangle$$

and

$$B = A[\frac{1}{a}] = K\langle X_1, \ldots, X_n \rangle.$$ 

Set $\alpha = |a|$. Since $a \in \mathfrak{m}_V \setminus \{0\}$, $0 < \alpha < 1$. Since $a^n A \subseteq \{ f \in B : \| f \| \leq \alpha^n \}$, it suffices to show that for any $n \geq 0$ there exists $\beta > 0$ such that $\{ f \in B : \| f \| \leq \beta \} \subseteq a^n A$. If $g \in B$ satisfies $\| g \| \leq \alpha^n$, then every coefficient of $g$ has to be divisible by $a^n$. Therefore,

$$\{ f \in B : \| f \| \leq \alpha^n \} = a^n A. \quad \Box$$

**Proposition 9.3.4.** Any ideal $\mathfrak{a}$ of $K\langle X_1, \ldots, X_n \rangle$ is closed.

**Proof.** Since $K\langle X_1, \ldots, X_n \rangle$ is Noetherian, any ideal $\mathfrak{a}$ is finitely generated, and hence one can choose generators of $\mathfrak{a}$ from $V\langle X_1, \ldots, X_n \rangle$. Thus we have a finitely generated ideal $\tilde{\mathfrak{a}}$ of $V\langle X_1, \ldots, X_n \rangle$ such that

$$\mathfrak{a} = \tilde{\mathfrak{a}} K\langle X_1, \ldots, X_n \rangle;$$
replacing $\tilde{a}$ by its $a$-saturation, which is again finitely generated due to the adhesiveness of $V \langle X_1, \ldots, X_n \rangle$, we may assume that $\tilde{a}$ is an $a$-saturated ideal. Since $V \langle X_1, \ldots, X_n \rangle$ is $a$-adically adhesive, we know by 8.5.16 and 7.4.18 that $\tilde{a}$ is closed with respect to the $a$-adic topology, that is,

$$
\bigcap_{n \geq 0} (\tilde{a} + a^n V \langle X_1, \ldots, X_n \rangle) = \tilde{a}
$$

(cf. Exercise 0.7.1). Now, to show the assertion, in view of 9.3.3, we only need to check the inclusion $\bigcap_{n \geq 0} (a + a^n V \langle X_1, \ldots, X_n \rangle) \subseteq a$ (since the other inclusion is obvious). Since $\tilde{a}$ is $a$-saturated, it is easy to see that $V \langle X_1, \ldots, X_n \rangle \cap \bigcap_{n \geq 0} (\tilde{a} + a^n V \langle X_1, \ldots, X_n \rangle)

= \bigcap_{n \geq 0} (\tilde{a} + a^n V \langle X_1, \ldots, X_n \rangle)

= \tilde{a}.

Now for any $f \in \bigcap_{n \geq 0} (a + a^n V \langle X_1, \ldots, X_n \rangle)$ there exists $n \geq 0$ such that $a^n f \in V \langle X_1, \ldots, X_n \rangle$, and hence $a^n f \in \tilde{a}$; consequently $f \in a$.

The proposition implies that any $K$-algebra $\mathcal{G}$ of the form

$$
\mathcal{G} = K \langle X_1, \ldots, X_n \rangle / a,
$$

where $a$ is an ideal of $K \langle X_1, \ldots, X_n \rangle$, is a $K$-Banach algebra while the induced norm

$$
\|f\| = \inf_{F \mapsto f} \|F\|
$$

for $f \in \mathcal{G}$, where $F$ varies among all elements $F \in K \langle X_1, \ldots, X_n \rangle$ that are mapped to $f$ by the canonical surjection $K \langle X_1, \ldots, X_n \rangle \to \mathcal{G}$.

Moreover, any $K$-algebra $\mathcal{G}$ of this form is a classical affinoid algebra; indeed, as in the proof of 9.3.4, one can take a finitely generated ideal $\tilde{a}$ of $V \langle X_1, \ldots, X_n \rangle$ such that, if we set $A = V \langle X_1, \ldots, X_n \rangle / \tilde{a}$ (which is topologically finitely generated over $V$), we have $\mathcal{G} = A \otimes_V K = A[\frac{1}{a}]$. Replacing $\tilde{a}$ by its $a$-saturation (as in the proof of 9.3.4), we may moreover find such an $A$ to be $a$-torsion free, hence flat over $V$ (cf. Exercise 0.6.3).

**Definition 9.3.5.** Let $V$ be an $a$-adically complete valuation ring for $a \in m_V \setminus \{0\}$ (not necessarily of height one), and $\mathcal{G}$ a classical affinoid algebra over $K = \text{Frac}(V).

1. A **formal model** of $\mathcal{G}$ over $V$ is a topologically finitely generated $V$-algebra $A$ such that $A \otimes_V K \cong \mathcal{G}$.

2. A formal model $A$ of $\mathcal{G}$ is said to be **distinguished** if $A$ is $a$-torsion free.
The term ‘distinguished’ is coined in order to maintain consistency with our later terminology (cf. II, §2.1.(b)). Distinguished formal models may also be called flat formal models, since $a$-torsion freeness is equivalent to $V$-flatness (Exercise 0.6.3). By what we have seen above, any classical affinoid algebra has a distinguished formal model. Note that due to the adhesiveness of $V\langle X_1, \ldots, X_n \rangle$ any distinguished formal model is topologically finitely presented (cf. proof of 9.3.4).

9.3.(b) Ring-theoretic properties. Finally, let us discuss some ring-theoretic properties of classical affinoid algebras. We assume, without loss of generality, that the $a$-adically complete valuation ring $V$ is of height one.

First, by the Noether normalization theorem for topologically finitely presented $V$-algebras (9.2.10), we get a similar theorem for classical affinoid algebras.

**Theorem 9.3.6** (Noether normalization theorem for classical affinoid algebras). For any classical affinoid algebra $\mathfrak{A}$ over $K$ there exists an injective finite map of the form

$$K\langle T_1, \ldots, T_d \rangle \hookrightarrow \mathfrak{A}.$$ 

**Corollary 9.3.7.** For any maximal ideal $m$ of the Tate algebra $K\langle X_1, \ldots, X_n \rangle$, the residue field $K\langle X_1, \ldots, X_n \rangle/m$ is a finite extension of $K$.

**Corollary 9.3.8** (weak Hilbert Nullstellensatz). Suppose that $K$ is algebraically closed. Then maximal ideals of $K\langle X_1, \ldots, X_n \rangle$ are precisely the ideals of the form $(X_1-a_1, \ldots, X_n-a_n)$ with $a_1, \ldots, a_n \in V$.

**Proof.** It is clear that the ideals of the form $(X_1-a_1, \ldots, X_n-a_n)$ with $a_1, \ldots, a_n \in V$ are maximal. Let $m \subseteq K\langle X_1, \ldots, X_n \rangle$ be any maximal ideal. Since $K$ is algebraically closed, $K\langle X_1, \ldots, X_n \rangle/m$ is isomorphic to $K$ as topological rings. For $i = 1, \ldots, n$, let $a_i \in K$ be the image of $X_i$. Since $X_i$ is power-bounded (cf. §B.1.(b) in the appendix), so is $a_i$, and hence $a_i \in V$. Since $m$ contains $(X_1-a_1, \ldots, X_n-a_n)$, which is maximal, we have $m = (X_1-a_1, \ldots, X_n-a_n)$. $\square$

**Proposition 9.3.9.** For any closed point $z$ of $\text{Spec } K\langle X_1, \ldots, X_n \rangle$,

$$\dim_z(\text{Spec } K\langle X_1, \ldots, X_n \rangle) = n.$$ 

In particular, the Krull dimension of the ring $K\langle X_1, \ldots, X_n \rangle$ is equal to $n$.

**Proof.** Consider the canonical map

$$f: \text{Spec } K\langle X_1, \ldots, X_n \rangle \longrightarrow \text{Spec } K[X_1, \ldots, X_n],$$

which is flat since $V[X_1, \ldots, X_n] \hookrightarrow V\langle X_1, \ldots, X_n \rangle$ is flat due to 8.2.18 (2). By Exercise 0.9.4 (1) and [54], IV, (6.1.2),

$$\dim_z(\text{Spec } K\langle X_1, \ldots, X_n \rangle) = \dim_{f(z)}(\text{Spec } K[X_1, \ldots, X_n]) = n,$$

as desired. The last part of the proposition follows from [54], IV, (5.1.4). $\square$
Proposition 9.3.10. Classical affinoid algebras are Jacobson.

For the proof we need the following lemma.

Lemma 9.3.11. Let \( F \in V \langle X_1, \ldots, X_n \rangle \) be such that \( \text{cont}(F) = V \) or, what amounts to the same, \( \| F \| = 1 \), where \( \| \cdot \| \) is the Gauss norm. Then the following conditions are equivalent.

(a) \( F \) is invertible in \( K \langle X_1, \ldots, X_n \rangle \).

(b) \( F \) is invertible in \( V \langle X_1, \ldots, X_n \rangle \).

(c) The constant term \( F_0 \) is invertible in \( V \), and \( \text{cont}(F - F_0) \neq V \).

Proof. Since \( \| \cdot \| \) is a norm, the inverse \( F^{-1} \) of \( F \) in \( K \langle X_1, \ldots, X_n \rangle \) satisfies \( \| F^{-1} \| = 1 \), that is, \( F^{-1} \in V \langle X_1, \ldots, X_n \rangle \), whence the equivalence of (a) and (b). Suppose \( F \) is invertible in \( V \langle X_1, \ldots, X_n \rangle \). Then the image of \( F \) under the residue map \( V \langle X_1, \ldots, X_n \rangle \to V \langle X_1, \ldots, X_n \rangle / m_V V \langle X_1, \ldots, X_n \rangle = k[X_1, \ldots, X_n] \) modulo the maximal ideal \( m_V = \sqrt{(a)} \) of \( V \), where \( k \) is the residue field of \( V \), is invertible, and this shows implication \( (b) \implies (c) \). If, conversely, \( (c) \) is satisfied, then we may assume that \( F_0 = 1 \), and then \( F \) belongs to the subset \( 1 + a^r V \langle X_1, \ldots, X_n \rangle \), where \( (a^r) = \text{cont}(F - F_0) \). Since \( V \langle X_1, \ldots, X_n \rangle \) is \( a \)-adically complete and hence is \( a \)-adically Zariskian, we deduce that \( F \) is invertible.

Proof of Proposition 9.3.10. It is enough to show the proposition for Tate algebras \( \mathcal{A} = K \langle X_1, \ldots, X_n \rangle \) ([54], IV, (10.4.6)). The proof is done by induction with respect to \( n \). If \( n = 0 \), then \( \mathcal{A} = K \) is a field, which is obviously Jacobson. In general, we need to prove that for any prime ideal \( p \subseteq \mathcal{A} \) the intersection of all maximal ideals containing \( p \) coincides with \( p \). Suppose \( p \neq 0 \), and consider \( \mathfrak{B} = \mathcal{A}/p \). Since \( p \) has positive height, the dimension of the ring \( \mathfrak{B} \) is strictly less than \( n \), and hence by 9.3.6 there exists an injective finite map \( K \langle Y_1, \ldots, Y_d \rangle \hookrightarrow \mathfrak{B} \) with \( d < n \). By induction, \( K \langle Y_1, \ldots, Y_d \rangle \) is Jacobson, and hence so is \( \mathfrak{B} \) by [54], IV, (10.4.6); in other words, \( p \) is the intersection of all maximal ideals that contain \( p \).

It remains to show that the intersection of all maximal ideals of \( \mathcal{A} \) is \( (0) \). Suppose \( F \neq 0 \) belongs to the intersection of all maximal ideals of \( \mathcal{A} \). We may assume that \( F \in V \langle X_1, \ldots, X_n \rangle \) and that \( \text{cont}(F) = V \). Let \( F_0 \) be the constant term of \( F \). Then \( F - F_0 \) belongs to the maximal ideal \( (X_1, \ldots, X_n) \) of \( \mathcal{A} \), and hence so does \( F_0 = F + (F_0 - F) \). Consequently, \( F_0 = 0 \). But then \( 1 + F \) is not invertible, since \( \text{cont}(F) = V \) (9.3.11). Hence there exists a maximal ideal \( m \) of \( \mathcal{A} \) such that \( 1 + F \in m \). But again this is absurd, since we would have \( 1 = (1 + F) - F \in m \). □

Finally, let us include without proofs a few more ring-theoretic facts, which will be needed in our later discussion.
Proposition 9.3.12 (Exercise 0.9.4 and [18], (5.2.6/1)). *Tate algebras are regular and factorial.*

Theorem 9.3.13 (R. Kiehl [71]). *Any classical affinoid algebra is excellent.*

Exercises

Exercise 0.9.1. Let $V \hookrightarrow V'$ be an inclusion of $a$-adically separated valuation rings, where $a \in V$. Show that the induced map $\widehat{V} \to \widehat{V}'$ between the $a$-adic completions is injective.

Exercise 0.9.2. Let $V$ be a valuation ring for $K = \text{Frac}(V)$.

1. Show that $V$ is the filtered inductive limit of subrings $V = \lim_{\lambda \in \Lambda} V_{\lambda}$ with each $V_{\lambda} \subseteq V$ being a valuation ring of finite height.

2. If, moreover, $V$ is $a$-adically separated (resp. complete) for $a \in m_V \setminus \{0\}$, then one can find the inductive system of subrings $\{V_{\lambda}\}_{\lambda \in \Lambda}$ as above consisting of $a_\lambda$-adically separated (resp. complete) valuation rings of finite height.

Exercise 0.9.3. Let $V$ be an $a$-adically complete valuation ring, and consider the restricted formal power series ring $V \langle X_1, \ldots, X_n \rangle$. Let $f \in V \langle X_1, \ldots, X_n \rangle$. Show that the content ideal $\text{cont}(f)$ ($\S 9.2. (b)$) is finitely generated.

Exercise 0.9.4. Consider the Tate algebra $\mathcal{Q} = K \langle X_1, \ldots, X_n \rangle$ and the canonical inclusion $A_0 = K[X_1, \ldots, X_n] \hookrightarrow \mathcal{Q}$.

1. Show that for any maximal ideal $m \subseteq \mathcal{Q}$ its contraction $m_0 = m \cap A_0$ is a maximal ideal of $A_0$ such that $\mathcal{Q}/m^{k+1} = A_0/m_0^{k+1}$ and $m_0^{k+1} \mathcal{Q} = m^{k+1}$ for any $k \geq 0$.

2. Show that $\mathcal{Q} = K \langle X_1, \ldots, X_n \rangle$ is a regular ring.

A Appendix: Further techniques for topologically of finite type algebras

A.1 Nagata’s idealization trick

The so-called *Nagata’s idealization trick*,\(^\text{13}\) stated in his book [84], p. 2, reduces many situations involving finitely generated modules to the special case of finitely generated ideals. Let us first state its general principle.

\(^{13}\) Nagata’s idealization is the method of representing a given quasi-coherent sheaf on an affine scheme as the conormal sheaf of a closed immersion, viz. the zero section of the affine cone of the sheaf, thus regarding it as an ideal of the first infinitesimal neighborhood.
A ring homomorphism \( f : S \to R \) is said to be a \emph{thickening of order} \( \leq 1 \) if it is surjective and \( K = \ker(f) \) satisfies \( K^2 = (0) \). Let \( R \) be a ring, and \( M \) a finitely generated \( R \)-module. We set \( S = \text{Sym}_R M/(M^2) \); \( S \) is the direct sum \( R \oplus M \) as an \( R \)-module with the multiplication by \((r + m)(r' + m') = rr' + rm' + r'm\). Then \( S \) is a finite \( R \)-algebra containing \( M \) as an ideal. Moreover, the projection map \( S = R \oplus M \to R \) is a thickening of order \( 1 \). Hence we get the diagram

\[
\text{Spec } R \xrightarrow{\sigma} \text{Spec } S,
\]

where \( \sigma \) is finite and \( \tau \) is the ‘zero section’ to \( \sigma \). Now the quasi-coherent sheaf \( \hat{M} \) of finite type on \( \text{Spec } R \) coincides with the pull-back of the quasi-coherent ideal \( \hat{M} \) of finite type on \( \text{Spec } S \); indeed, the two homomorphisms between \( R \) and \( S \) give rise to the isomorphism between \( M \) and \( M \otimes_S R \) as \( R \)-modules.

Thanks to this observation the following proposition is now clear.

**Proposition A.1.1** (Nagata’s idealization trick). \( A \) be a ring, and consider a property \( P(R, M) \) involving an \( A \)-algebra \( R \) of finite type and a finitely generated \( R \)-module \( M \). Suppose that for any thickening \( S \to R \) of order \( \leq 1 \) between \( A \)-algebras of finite type, we have the implication \( P(S, M) \implies P(R, M \otimes_S R) \). Then the following conditions are equivalent.

(a) \( P(R, I) \) holds for any \( R \) and a finitely generated ideal \( I \subseteq R \).

(b) \( P(R, M) \) holds for any \( R \) and \( M \).

**A.2 Standard basis and division algorithm**

The notion of standard (or Gröbner) basis, useful both in theories and calculations in algebraic geometry, is also useful in dealing with the algebras \( V \langle X_1, \ldots, X_n \rangle \) over an \( a \)-adically complete valuation ring \( V \) of height one. As H. Hironaka has first envisaged in complex analytic situation, the resulting division theorem gives a generalization of the Weierstrass preparation-division theorem; in our situation it also gives an analogous division theorem, which will be of broad use as a theoretical and computational device.

**A.2. (a) Setting.** We consider \( \mathbb{N}^n \) as an additive monoid in the standard way, and we equip it with a \emph{term ordering} (see, e.g., [4], 1.4), for example, the lexicographical order (cf. 6.1.3). For \( \nu = (v_1, \ldots, v_n) \in \mathbb{N}^n \) we write \( X^\nu = X_1^{v_1} \cdots X_n^{v_n} \).

Let \( R \) be a ring and \( I = (a_1, \ldots, a_r) \subseteq R \) a finitely generated ideal, and suppose that \( R \) is \( I \)-adically complete. For a restricted power series

\[
(\ast) \quad f = \sum_{\nu \in \mathbb{N}^n} a^\nu X^\nu \in R \langle X_1, \ldots, X_n \rangle,
\]

\[
\text{Spec } R \xrightarrow{\sigma} \text{Spec } S,
\]
the content ideal of \( f \), denoted by \( \text{cont}(f) \), is the ideal of \( R \) generated by all the coefficients \( a_v \) of \( f \). It is easy to see that \( \text{cont}(f) \) is actually finitely generated (cf. Exercise 0.9.3). We say that \( f \) is primitive if \( \text{cont}(f) = R \).

If \( R = V \) is a valuation ring, complete with respect to the \( a \)-adic topology \((I = (a))\), then the content ideal \( \text{cont}(f) \) is always principal, and we can set

\[
v(f) = \begin{cases} v(f_0 \text{ mod } m_V), & \text{where } f_0 = f/\alpha \text{ with } \text{cont}(f) = (\alpha), \text{ if } f \neq 0, \\ -\infty, & \text{otherwise.} \end{cases}
\]

Here the leading degree \( v(g) \) of a polynomial \( g \in A[X_1, \ldots, X_n] \) over any ring \( A \) is the maximal (with respect to the term order) multidegree among those appear in non-zero terms in \( g \). Note that the definition of \( v(f) \) for \( f \neq 0 \) is also given by the following:

\[
v(f) = \sup \{v : a_v \text{ generates cont}(f)\}
\]

for \( f \) presented as \((*)\). Define, furthermore,

\[
\text{LT}(f) = a_{v(f)}X^{v(f)},
\]

and call it the leading term of \( f \).

A.2. (b) Division algorithm. As in the previous paragraph, we consider an \( I \)-adically complete ring \( R \), where \( I = (a_1, \ldots, a_r) \subseteq R \) is finitely generated.

**Theorem A.2.1** (division lemma). Let \( \{g_{\lambda}\}_{\lambda \in \Lambda} \) be a finite collection of elements in \( R\langle X_1, \ldots, X_n \rangle \) such that, for each \( \lambda \in \Lambda \), \( g_{0\lambda} = g_\lambda \text{ mod } IR\langle X_1, \ldots, X_n \rangle \) is a monic polynomial in \( (R/I)[X_1, \ldots, X_n] \). Set \( v_\lambda = v(g_{0\lambda}) \) for any \( \lambda \in \Lambda \), and \( M = \bigcup_{\lambda \in \Lambda} (v_\lambda + \mathbb{N}^n) \). Then, for any \( f \in R\langle X_1, \ldots, X_n \rangle \), there exists \( g \in R\langle X_1, \ldots, X_n \rangle \) with no exponents in \( M \) such that \( f - g \) belongs to \( \sum_{\lambda \in \Lambda} g_\lambda R\langle X_1, \ldots, X_n \rangle \).

**Proof.** By performing the division algorithm by \( \{g_{0\lambda}\}_{\lambda \in \Lambda} \) in the polynomial ring \((R/I)[X_1, \ldots, X_n]\), there exists an expression

\[
f = g' + \sum_{\lambda \in \Lambda} a_\lambda g_\lambda + \sum_{j=1}^r a_j f_j
\]

in \( R\langle X_1, \ldots, X_n \rangle \), where \( g' \) has no exponents in \( M \). Repeating this, by induction, one deduces that, for any \( m \geq 1 \),

\[
f = g'_m + \sum_{\lambda \in \Lambda} a_{\lambda, m} g_\lambda + \sum_{h \in P_m} h(a_1, \ldots, a_r) f_h,
\]

where the last sum is taken over all monomials \( h \) of \( r \) variables of degree \( m \), such that
• \( g'_m \) has no exponents in \( M \);
• \( g'_{m+1} - g'_m \) and \( \alpha_{\lambda,m+1} - \alpha_{\lambda,m} \) belong to \( I^m R \langle X_1, \ldots, X_n \rangle \).

Then, with \( g = \lim_{m \to \infty} g'_m \) and \( \alpha = \lim_{m \to \infty} \alpha_{\lambda,m} \), we have

\[
f = g + \sum_{\lambda \in \Lambda} \alpha_{\lambda} g_{\lambda},
\]

where \( g \) has no exponents in \( M \).

\[\square\]

**A.2. (c) Standard bases.** Set \( R = V \langle X_1, \ldots, X_n \rangle \), where \( V \) is an \( a \)-adically complete valuation ring \((a \in m_V \setminus \{0\})\). For a non-zero ideal \( I \subseteq R \) we denote by \( \text{LT}(I) \) the ideal of \( R \) generated by the leading terms of all non-zero elements in \( I \).

**Definition A.2.2.** Let \( I \) be a non-zero ideal of \( R \), and \( g_1, \ldots, g_d \in I \) non-zero primitive elements. The set \( \{g_1, \ldots, g_d\} \) is said to be a **standard basis** for \( I \) if

\[
\text{LT}(I) = (\text{LT}(g_1), \ldots, \text{LT}(g_d)).
\]

**Corollary A.2.3.** If \( \{g_1, \ldots, g_d\} \) is a standard basis of a non-zero ideal \( I \) of \( A \), we have \( I = (g_1, \ldots, g_d) \).

**Proof.** We only need to check the inclusion \( I \subseteq (g_1, \ldots, g_d) \). Let \( f \in I \), and take \( q_1, \ldots, q_d \in R \) such that \( h = f - \sum_{i=1}^{d} q_i g_i \) has no exponent in

\[
M = \bigcup_{i=1}^{d} (v(g_i) + \mathbb{N}^n),
\]

as in A.2.1. Suppose \( h \neq 0 \). As \( h \in I \), we have \( \text{LT}(h) \in \text{LT}(I) \). But since the leading degree \( v(h) \) does not belong to \( M \), this is absurd. Hence \( h = 0 \) and thus \( f \in (g_1, \ldots, g_d) \). \(\square\)

Since \( R \) is not necessarily Noetherian, it is not always the case that an ideal \( I \subseteq R \) has a standard basis; however, we have the following result.

**Proposition A.2.4.** Let \( I \subseteq R \) be a non-zero and \( a \)-saturated (§8.1.(c)) ideal. Then \( I \) has a standard basis (and hence is finitely generated).

**Proof.** Define the subset \( L \subset \mathbb{N}^n \) by

\[
L = \{v(f) : f \in I \setminus \{0\}\}.
\]

Then \( L \) is an ideal of the monoid \( \mathbb{N}^n \), that is, for any \( v \in L \) and \( \mu \in \mathbb{N}^n \) we have \( v + \mu \in L \). By Dickson’s lemma (cf. e.g. [4], Exercise 1.4.12) \( L \) is finitely generated, that is, there exist \( v_1, \ldots, v_r \in L \) such that

\[
L = \bigcup_{i=1}^{r} (v_i + \mathbb{N}^n).
\]
Take \( g_1, \ldots, g_r \in I \) such that \( v(g_i) = v_i \) for \( i = 1, \ldots, r \). Since \( I \) is \( a \)-saturated, we may assume that the coefficient of \( LT(g_i) \) is 1 for each \( i \). Then for \( f \in I \) there exist \( i \) and a monomial \( h \) such that \( LT(f) = hLT(g_i) \). Hence we have \( LT(I) = (LT(g_1), \ldots, LT(g_r)) \). \( \square \)

Finally, let us mention one application of the Nagata’s trick and the division algorithm. We give an elementary proof, without referring to the result by Raynaud and Gruson (9.2.2), of 9.2.1 in the case \( V \) is of height one.

**Proposition A.2.5.** Let \( V \) be an \( a \)-adically complete valuation ring of height one. Then any topologically finitely generated \( V \)-algebra is \( a \)-adically adhesive.

**Proof.** By 8.5.7 (2), it suffices to show that \( V \langle X_1, \ldots, X_n \rangle \) is \( a \)-adically adhesive. We apply Nagata’s trick as in A.1.1. For any \( R = V \langle X_1, \ldots, X_n \rangle / J \), for some \( n \geq 0 \) and some ideal \( J \), and any finitely generated \( R \)-module \( M \), let \( P(R, M) \) be the property that for any \( R \)-submodule \( N \subseteq M \) its \( a \)-saturation \( \tilde{N} \) is finitely generated over \( R \). By A.1.1 we only have to prove \( P(R, I) \) for any finitely generated ideal \( I \) of \( R \); moreover, one reduces to \( P(R, R) \), that is, that for any ideal \( I \) of \( R \) its \( a \)-saturation \( \tilde{I} \) is again finitely generated. We may further assume that \( R = V \langle X_1, \ldots, X_n \rangle \). But the assertion in this case is proved in A.2.4. \( \square \)

**Exercises**

**Exercise 0.A.1.** Show that a complete pair of finite ideal type \((A, I)\) is adhesive if the following condition holds: if \( B \) is an \( I \)-adically complete finite \( A \)-algebra, any \( IB \)-saturated ideal \( J \subseteq B \) is finitely generated.

**Exercise 0.A.2.** Let \( V \) be an \( a \)-adically complete valuation ring, and \( V' \) the associated height one valuation ring (§6.7).

1. Let \( A \) be a topologically finitely generated \( V \)-algebra such that \( A\left[\frac{1}{a}\right] \) is finite over \( V\left[\frac{1}{a}\right] \). Show that \( A \) is finite type over \( V \) and that \( (A/A_{a\text{-tor}}) \otimes_V V' \) is finite over \( V' \).

2. Show that, conversely, a finite type \( V \)-algebra \( A \) is \( a \)-adically complete if and only if \( (A/A_{a\text{-tor}}) \otimes_V V' \) is finite over \( V' \).

3. Show that any finite type \( V \)-algebra \( A \) such that \( A\left[\frac{1}{a}\right] \) is finite over \( V\left[\frac{1}{a}\right] \) is canonically decomposed as \( A = A' \times A'' \), where \( A' \) is \( a \)-adically complete and \( A'' \otimes_V (V/aV) = 0 \).
Exercise 0.A.3 (Weierstrass preparation theorem). Let $V$ be an $a$-adically complete valuation ring of height one, and $f \in V \langle X_1, \ldots, X_n \rangle$ ($n \geq 1$) a primitive (that is, $\text{cont}(f) = V$) element. Consider the lexicographical order (6.1.3) as the term order for the exponents of monomials, and set $\nu(f) = (v_1(f), \ldots, v_n(f))$. Suppose that, if we write

$$f = \sum_{n=0}^{\infty} f_n(X_2, \ldots, X_n)X_1^n,$$

then $f_{v_1(f)}$ is a unit in $V \langle X_2, \ldots, X_n \rangle$. Then show that there exists a unique monic polynomial $g \in V \langle X_2, \ldots, X_n \rangle[X_1]$ of degree $v_1(f)$ in $X_1$ and a unique unit element $u \in V \langle X_1, \ldots, X_n \rangle^\times$ such that $g = u \cdot f$.

B Appendix: f-adic rings

In [59] R. Huber introduced the notion of ‘f-adic rings’ in an attempt to give a broad generalization of the notion of (classical) affinoid algebras and thus to develop a new geometry that contains the classical rigid analytic geometry as a special case. This appendix gives a brief survey of generalities on f-adic rings and thus prepares for our later discussion on Huber’s adic spaces in II, §A.

B.1 f-adic rings

B.1. (a) Extension of adic topologies. Let $A$ be a ring, $B \subseteq A$ a subring, and $I \subseteq B$ a finitely generated ideal of $B$. The $I$-adic filtration $\{I^n\}_{n \geq 1}$ (cf. §7.1. (b)) gives rise to a linear topology on $A$ regarded as a $B$-module.

Proposition B.1.1. The underlying ring structure makes the topological $B$-module $A$ with the topology defined by the filtration $\{I^n\}_{n \geq 1}$ into a topological ring if and only if

$$A = \bigcup_{n \geq 0} [B : I^n].$$

In this situation, moreover,

$$\text{Spec } A \setminus V(I A) = \text{Spec } B \setminus V(I).$$

Proof. Suppose $A$ is a topological ring, and take $x \in A$. Since the selfmap $y \mapsto xy$ on $A$ is continuous, there exists $n \geq 0$ such that $I^n x \subseteq B$, that is, $x \in [B : I^n]$. The ‘if’ part is easy to see. Finally, in this situation, the inclusion $B \hookrightarrow A$ has $I$-torsion cokernel, and so $\text{Spec } A \setminus V(I A) = \text{Spec } B \setminus V(I)$. \qed
Aside from the trivial case $B = A$, in which the topology in question is nothing but the $I$-adic topology, there are plenty of examples of the situations described in B.1.1. For example, if $B$ is $a$-torsion free for an element $a \in B$, then $A = B \left[ \frac{1}{a} \right]$ with the topology defined by the $a$-adic filtration on $B$ is a topological ring. This, needless to say, gives grounds for justifying the topology on classical affinoid algebras, already discussed in §9.3 (cf. 9.3.3).

B.1. (b) f-adic rings

Definition B.1.2. An f-adic ring is a topological ring $A$ that admits an open subring $A_0 \subseteq A$ such that the induced topology on $A_0$ is an adic topology defined by a finitely generated ideal $I_0$ of $A_0$.

In this situation, the subring $A_0$ is called a ring of definition, and the ideal $I_0$ is called an ideal of definition of $A$. If, for example, $I \subseteq B \subseteq A$ are as in B.1.1, then $A$ is an f-adic ring if and only if $A = \bigcup_{n \geq 0} [B : I^n]$; moreover, in this situation, $B$ is a ring of definition, and $I$ is an ideal of definition of $A$. It can be shown, moreover, that

- a subring $B \subseteq A$ of an f-adic ring $A$ is a ring of definition of $A$ if and only if it is open and bounded ([59], Proposition 1 (ii));
- every ring of definition of an f-adic ring has at least one finitely generated ideal of definition ([59], Proposition 1 (iii)).

Here, a subset $S$ of a topological ring $A$ is said to be bounded if for any neighborhood $U$ of 0 in $A$ there exists a neighborhood $V$ of 0 in $A$ such that $V \cdot S \subseteq U$.

In general, for a topological ring $A$,

- an element $a \in A$ is said to be power-bounded if the subset $\{a^n\}_{n \geq 0}$ is bounded;
- an element $a \in A$ is said to be topologically nilpotent if for any open neighborhood $V$ of 0 there exists $N \geq 0$ such that $a^N \in V$ whenever $n \geq N$.

We denote by $A^o$ (resp. $N(A)$) the subset of $A$ consisting of all power-bounded (resp. topologically nilpotent) elements. The following propositions are easily proved.

Proposition B.1.3. Let $A$ be an f-adic ring.

1. Let $A_0 \subseteq A$ be a ring of definition. Then $a \in A$ is power-bounded if and only if $A_0[a] \subseteq A$ is a ring of definition of $A$.

2. The set of power-bounded elements $A^o$ coincides with the union of all rings of definition of $A$. 
Proposition B.1.4. Let $A$ be an $f$-adic ring, $A_0$ a ring of definition of $A$, and $I_0$ an ideal of definition of $A_0$.

(1) $A^o$ is an open subring of $A$, and $N(A)$ is an open ideal of $A^o$.

(2) Any element of $I_0$ is topologically nilpotent. Conversely, for any topologically nilpotent element $a$ of $A$ there exists $n > 0$ such that $a^n \in I_0$.

Let $A$ be an $f$-adic ring, $A_0$ a ring of definition, and $I_0$ an ideal of definition. Then every open subring $A'_0 \subseteq A_0$ is a ring of definition of $A$. Moreover, due to B.1.4 (2), there exists $n \geq 0$ such that $I_0^n \subseteq A'_0$, and hence the topology on $A'_0$ is $I_0^n$-adic.

Let $\varphi: A \to B$ be a continuous ring homomorphism between $f$-adic rings, and $A_0 \subseteq A$ and $B_0 \subseteq B$ respective rings of definition. Then $A'_0 = A_0 \cap \varphi^{-1}(B_0)$ is a ring of definition of $A$, since it is open and bounded. Thus, whenever we are given a continuous ring homomorphism as above, we can always take rings of definition $A_0$ and $B_0$ in such a way that $\varphi(A_0) \subseteq B_0$.

If the restriction $\varphi|_{A_0}: A_0 \to B_0$ is an adic homomorphism (0, §7.2. (a)), we say that the morphism $\varphi: A \to B$ is adic. Note that this notion does not depend on the choice of the rings of definition; indeed, for another choice $A'_0 \subseteq A$ and $B'_0 \subseteq B$ with $\varphi(A'_0) \subseteq B'_0$, taking the intersections $A_0 \cap A'_0$ and $B_0 \cap B'_0$, which are again rings of definitions, we may assume that $A'_0 \subseteq A_0$ and $B'_0 \subseteq B_0$ and thus reduce to showing that $A_0 \to B_0$ is adic if and only if $A'_0 \to B'_0$, which is easy to see.

B.1. (c) Extremal $f$-adic rings. The notion of $f$-adic rings contains, as a special case, usual linearly topologized rings with adic topology defined by finitely generated ideals. Also, as remarked at the end of §B.1. (a), classical affinoid algebras are another example of $f$-adic rings. These special cases indicate that there are two interesting classes of $f$-adic rings, as follows.

One of them is comprised of $f$-adic rings that themselves are bounded (hence ‘bounded $f$-adic rings’ should be the logical name); if $A$ is such an $f$-adic ring, then $A$ itself is a ring of definition, and hence $f$-adic rings of this type are nothing but rings with adic topology defined by finitely generated ideals.

The other interesting type of $f$-adic rings is the following one:

- an $f$-adic ring $A$ is said to be extremal if it has an ideal of definition $I$ (of a ring of definition) such that $IA = A$.

(It can be easily shown that this notion does not depend on the choice of $I$.) If $A$ is extremal $f$-adic and $A_0 \subseteq A$ is a ring of definition with an ideal of definition $I_0 \subseteq A_0$, then it follows from B.1.1 that $\text{Spec } A = \text{Spec } A_0 \setminus V(I_0)$. Conversely, if $B$ is a ring endowed with an adic topology defined for a finitely generated ideal
$I \subseteq B$ such that Spec $B \setminus V(I)$ is affine, then $A$ with Spec $A = \text{Spec} B \setminus V(I)$ is an extremal f-adic ring by the topology defined by the filtration $\{I^n\}_{n \geq 1}$.

The f-adic rings of the type mentioned at the end of §B.1.(a), and hence all classical affinoid algebras, are a further special type of extremal f-adic rings:

- an extremal f-adic ring is called a Tate ring if it has a principal ideal of definition;

equivalently, Tate rings are f-adic rings having topologically nilpotent units ([59], §1, Definition (ii)).

**Proposition B.1.5.** An f-adic ring $A$ is extremal f-adic if and only if it has a ring of definition $A_0$ and finitely generated topologically nilpotent ideal $J_0 \subseteq A_0$ such that $J_0 A = A$. In this situation, moreover, the ideal $J_0$ is open in $A_0$ and the topology on $A_0$ is $J_0$-adic.

**Proof.** The ‘only if’ part is trivial. To show the converse, replacing $J_0$ by an ideal of the form $J_0^n$, we may assume that $J_0$ is contained in an ideal of definition $I_0$. Then obviously $I_0 A = A$. Since $V(I_0) \subseteq V(J_0)$, we have the chain of morphisms

$$\text{Spec } A = \text{Spec } A \setminus V(J_0 A) \longrightarrow \text{Spec } A_0 \setminus V(J_0) \longleftarrow \text{Spec } A \setminus V(I_0) = \text{Spec } A$$

of which the composition is equal to the identity map of Spec $A$. Hence we deduce that $V(I_0) = V(J_0)$ and that there exists $n \geq 0$ such that $I_0^n \subseteq J_0$. 

**Corollary B.1.6** (cf. [59], Proposition 1.10). Let $A$ and $B$ be f-adic rings, and let the map $\varphi: A \to B$ be a continuous homomorphism of rings. Suppose $A$ is an extremal f-adic (resp. Tate) ring. Then $B$ is also an extremal f-adic (resp. Tate) ring, and the map $\varphi$ is adic.

**Proof.** Take rings of definition $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $\varphi(A_0) \subseteq B_0$. Let $I_0 \subseteq A_0$ be an ideal of definition, and set $J_0 = I_0 B_0$. Then $J_0$ is a finitely generated topologically nilpotent ideal such that $J_0 B = B$, and hence $B_0$ is extremal f-adic due to B.1.5; moreover, $J_0$ is an ideal of definition of $B_0$. 

**B.1.(d) Complete f-adic rings.** Let $A$ be an f-adic ring, $B$ a ring of definition, and $I \subseteq B$ a finitely generated ideal of definition of $B$. By B.1.1, $J = \bigcap_{n \geq 1} I^n$ is an ideal of $A$, and $A$ is separated if and only if $J = 0$. In general, the associated separated f-adic ring is the one given by $A/J$, which is again an f-adic ring with an ideal of definition $B/J$.

The f-adic ring $A$ is said to be complete if it is separated and complete. Note that if $A$ is a complete f-adic ring, then any ring of definition $A_0 \subseteq A$ is $I_0$-adically complete for a finitely generated ideal of definition $I_0 \subseteq A_0$, since $A_0$ is an open subring of $A$. 


Let $A$, $B$, and $I$ be as above. Then $\hat{B}$, the $I$-adic completion of $B$ (cf. 0.7.2.15), can be seen as a subring of the completion $\hat{A}$ of $A$ (cf. §7.1.(c)).

**Proposition B.1.7.** The canonical map

$$\hat{B} \otimes_B A \longrightarrow \hat{A}$$

is an isomorphism of rings.

*Proof.* The inverse map is constructed as follows. Any element $x$ of $\hat{A}$ is represented by a Cauchy sequence $\{x_i\}_{i=0}^{\infty}$ of elements of $A$. We may assume that for any $i, j$ we have $x_i - x_j \in B$. Then $\{x_i - x_0\}_{i=0}^{\infty}$ is a Cauchy sequence in $B$, and hence defines a unique element $y \in \hat{B}$. Then the inverse mapping in question is given by $x \mapsto y + x_0 \in \hat{B} \otimes_B A$. For more details of the proof, see [59], 1.6. $\Box$

By 7.2.15, the topology on $\hat{B}$ is the $I\hat{B}$-adic topology, and hence the completion $\hat{A}$ is again an $f$-adic ring having $\hat{B}$ and $I\hat{B}$ as a ring of definition and an ideal of definition, respectively. It is clear that, if $A$ is extremal $f$-adic (resp. Tate) then so is $\hat{A}$.

Let $A$ be a complete $f$-adic ring, and $n \geq 0$ an integer. We denote by

$$A\langle X_1, \ldots, X_n \rangle$$

the restricted formal power series ring with coefficients in $A$, that is, the completion of $A[X_1, \ldots, X_n]$; if $A_0 \subseteq A$ is a ring of definition, then

$$A\langle X_1, \ldots, X_n \rangle = A_0\langle X_1, \ldots, X_n \rangle \otimes_{A_0} A,$$

where $A_0\langle X_1, \ldots, X_n \rangle$ is the ring defined as in §8.4. Note that $A\langle X_1, \ldots, X_n \rangle$ is again an $f$-adic ring having $A_0\langle X_1, \ldots, X_n \rangle$ as a ring of definition. The following lemma is easy to establish.

**Lemma B.1.8.** Let $A \rightarrow B$ be an adic map between complete $f$-adic rings, and $b_1, \ldots, b_n \in B^o$ power-bounded elements of $B$. Then there exists an adic $A$-algebra homomorphism $A\langle X_1, \ldots, X_n \rangle \rightarrow B$ that maps each $T_i$ to $b_i$ ($i = 1, \ldots, n$).

Let $A \rightarrow B$ be an adic map between complete $f$-adic rings, and $b_1, \ldots, b_n \in B$. Suppose that the $A$-algebra homomorphism $A[X_1, \ldots, X_n] \rightarrow B$ that maps each $X_i$ to $b_i$ ($i = 1, \ldots, n$) extends to an adic map $A\langle X_1, \ldots, X_n \rangle \rightarrow B$. In this case we say that the image of the last map is the subring weakly generated by $b_1, \ldots, b_n$ over $A$. If, moreover, the induced subspace topology on the image coincides with the quotient topology induced by the topology on $A\langle X_1, \ldots, X_n \rangle$, we say that the image is generated by $b_1, \ldots, b_n$ over $A$. 


B. Appendix: f-adic rings

B.1. (e) Banach f-adic rings and classical affinoid algebras. Let $V$ be an $a$-adically complete valuation ring of height one, where $a$ is a non-zero element of $m_V$; then in view of 6.7.3 we have $m_V = \sqrt{(a)}$. The fraction field of $V$ is denoted by $K = \text{Frac}(V)$. Then as in §6.3. (c) we have a valuation $| \cdot |: K \rightarrow \mathbb{R}_{\geq 0}$ such that $|a| < 1$; here the valuation is written multiplicatively.

A $K$-Banach algebra is a pair $(A, \| \cdot \|)$ consisting of a $K$-algebra and a non-Archimedean ring norm $\| \cdot \|$ such that for any $x \in A$ and $u \in K$ we have $\|ux\| \leq |u|\|x\|$ and that $A$ is complete with respect to $\| \cdot \|$.

**Proposition B.1.9.** Let $A = (A, \| \cdot \|)$ be a $K$-Banach algebra, and set $A_0 = \{x \in A: \|x\| \leq 1\}$. Then $A_0$ is an open subring, and the induced topology on $A_0$ is $a$-adic. In particular, $A$ is a complete Tate ring.

The proof is easy and is left to the reader. An f-adic ring $A$ obtained in this way is called a Banach f-adic ring.

Now let us consider a topologically finitely generated algebra $A$ over $V$ (8.4.1) and the related classical affinoid algebra $\mathfrak{G} = A[\frac{1}{a}]$ (0, §9.3. (a)). In case where $A = V \langle T_1, \ldots, T_n \rangle$, the associated classical affinoid algebra $A[\frac{1}{a}]$ is the Tate algebra $K \langle T_1, \ldots, T_n \rangle$ equipped with the Gauss norm (§9.3. (a))

$$\left\| \sum_{v_1, \ldots, v_n \geq 0} a_{v_1, \ldots, v_n} T_1^{v_1} \cdots T_n^{v_n} \right\| = \sup_{v_1, \ldots, v_n \geq 0} |a_{v_1, \ldots, v_n}|.$$

Obviously, we have $K \langle T_1, \ldots, T_n \rangle^0 = V \langle T_1, \ldots, T_n \rangle$.

In general, a classical affinoid algebra $\mathfrak{G}$, written $\mathfrak{G} = K \langle T_1, \ldots, T_n \rangle / a$, is a $K$-Banach algebra with respect to the norm induced by the Gauss norm, the so-called residue norm, defined as follows. For $f \in \mathfrak{G}$,

$$\| f \|_{\text{res}} = \sup_{F \mapsto f} \| F \|,$$

where $F$ ranges over the set of all elements of $K \langle T_1, \ldots, T_n \rangle$ that are mapped to $f$ by the quotient map.

**Lemma B.1.10.** The topology induced by the residue norm $\| \cdot \|_{\text{res}}$ on $\mathfrak{G}$ coincides with the $a$-adic topology. In particular, the topology on $\mathfrak{G}$ induced by the residue norm does not depend on the choice of presentation of $\mathfrak{G}$ as a quotient of a Tate algebra.
Proof. As in §7.4.(b), the \( a \)-adic topology on \( \mathcal{O} \) is the one induced by the filtration \( \{ a + a^n V \langle T_1, \ldots, T_n \rangle \}_{n \geq 0} \) by the subgroups consisting of modulo \( a \) residue classes. Since \( a + a^n V \langle T_1, \ldots, T_n \rangle = \{ f \in A : \| f \|_{\text{res}} \leq |a^n| \} \) for each \( n \geq 0 \), we have the lemma. \( \Box \)

### B.2 Modules over f-adic rings

**B.2. (a) Topological modules.** Let \( A \) be an f-adic ring with a ring of definition \( A_0 \subseteq A \) and a finitely generated ideal of definition \( I_0 \subseteq A_0 \). Let \( M \) be an \( A \)-module. To topologize \( M \), let \( M_0 \subseteq M \) be an \( A_0 \)-submodule, and consider the \( I_0 \)-adic filtration \( \{ I^n_0 M_0 \}_{n \geq 1} \) (cf. §7.1.(b)).

**Proposition B.2.1.** The topological group \( M \) endowed with the topology given by the filtration \( \{ I^n_0 M_0 \}_{n \geq 1} \) is a topological \( A \)-module if and only if

\[
M = \bigcup_{n \geq 0} [M_0 : I^n_0].
\]

The proof is similar to that of B.1.1; one only has to verify that the scalar multiplication \( A \times M \to M \) is continuous if and only if the above equality holds.

In general, a subset \( W \subseteq M \) of a topological \( A \)-module \( M \) is said to be bounded if for any open neighborhood \( U \subseteq M \) of 0 in \( M \) there exists an open neighborhood \( V \subseteq A \) of 0 in \( A \) such that \( V \cdot W \subseteq U \).

**Proposition B.2.2.** Let \( M \) be a topological \( A \)-module, and \( M_0 \subseteq M \) an \( A_0 \)-submodule. Then the subspace topology on \( M_0 \) induced by that of \( M \) coincides with the \( I_0 \)-adic topology if and only if \( M_0 \) is open and bounded in \( M \).

**Proof.** If the subspace topology on \( M_0 \) is \( I_0 \)-adic, then \( M_0 \) is clearly open in \( M \). Since the map \( A \times M \to M \) is continuous, for any open neighborhood \( U \subseteq M \) of 0 there exists \( n \geq 1 \) such that \( I^n M_0 \subseteq U \), which shows that \( M_0 \) is bounded. Conversely, if \( M_0 \) is open and bounded, then so is \( I^n_0 M_0 \) for any \( n \geq 1 \). Hence \( \{ I^n_0 M_0 \}_{n \geq 1} \) gives a fundamental system of open neighborhoods of \( M \). \( \Box \)

**B.2. (b) Open mapping theorem**

**Theorem B.2.3** (open mapping theorem). Let \( A \) be a complete f-adic ring, and \( A_0 \subseteq A \) a ring of definition. Let \( M \) and \( N \) be complete topological \( A \)-modules, and \( M_0 \subseteq M \) and \( N_0 \subseteq N \) open and bounded \( A_0 \)-submodules. Then a continuous \( A \)-linear homomorphism \( \varphi : M \to N \) is a topological isomorphism if and only if \( \varphi \) is bijective.
In the following proof of the theorem we regard $M$ and $N$ as metric spaces as in Exercise 0.7.10, defined by the decreasing filtration

$$E(n) = \begin{cases} 
I_0^n M_0, & n \geq 0, \\
|M_0 : I_0^n|, & n < 0, 
\end{cases}$$

and similarly for $N$, where $I_0 \subseteq A_0$ is a finitely generated ideal of definition. Note that $M$ and $N$ are complete as metric spaces.

**Proof of Theorem B.2.3.** Suppose that $\varphi$ is bijective. Due to the continuity of $\varphi$ we may assume $\varphi(M_0) \subseteq N_0$. By identifying $M$ with the image $\varphi(M)$, we may assume that $M = N$ as an $A$-module and that $\varphi$ is the identity map.

We need to show that $M_0$ is open in $M$ with respect to the topology defined by $N_0$. To this end, let us regard $M$ as a topological group with respect to the topology induced by that of $N_0$. The resulting $M$ is a complete metric space.

Since

$$M = \bigcup_{n \geq 1} [M_0 : I_0^n] \subseteq \bigcup_{n \geq 1} [M_0 : I_0^n],$$

by Baire’s category theorem (Exercise 0.7.10 (3)), there exist integers $m, n \geq 1$ such that $I_0^m N_0 \subseteq [M_0 : I_0^n]$. It follows that

$$I_0^{m+n} N_0 \subseteq M_0 + I_0^{n+k} N_0$$

for any $k \geq 0$. For $k = m + 1$ and $c = m + n$,

$$I_0^c N_0 \subseteq M_0 + I_0 I_0^c N_0,$$

and so for any $x \in I_0^c N_0$ and $l \geq 0$ there exists an element $y_l \in M_0$ such that $x - y_l \in I_0^l \cdot I_0^c N_0$ and $y_{l+1} - y_l \in I_0^{l+1} M_0$. Since $M_0$ is $I_0$-adically complete, the sequence $\{y_l\}_{l \geq 0}$ converges to an element $y \in M_0$. On the other hand, the same sequence converges with respect to the $I_0 N_0$-adic topology to the element $x$, equal to $y$. Therefore, $I_0^c N_0 \subseteq M_0$, and $M_0$ is open with respect to the topology defined by $N_0$, as desired. \(\square\)

It follows from B.2.1 that for a continuous morphism $\varphi: A \to B$ of f-adic rings, $B$ can be regarded as a topological $A$-module if $\varphi$ is adic. Hence we have the following corollary.

**Corollary B.2.4** (open mapping theorem for complete f-adic rings). Let $\varphi: A \to B$ be an adic homomorphism between complete f-adic rings. Then $\varphi$ is a topological isomorphism if and only if it is bijective.
By B.1.6 we have the following corollary.

**Corollary B.2.5.** Let $\varphi: A \to B$ be a continuous homomorphism between two complete $f$-adic rings. Suppose $A$ is extremal $f$-adic. Then $\varphi$ is a topological isomorphism if and only if it is bijective.

## Appendix: Addendum on derived categories

We refer to Verdier’s expositions [34], C.D., and [101] as our basic references for generalities of derived categories. In this appendix we collect some (but not all) of the facts and materials on derived categorical calculi used in our discussion. The proofs of almost all assertions below can be found, in addition to the above-mentioned references, also in [10] and [66].

### C.1 Prerequisites on triangulated categories

In the sequel we write triangulated categories as

$$\mathcal{D} = (\mathcal{D}, T, \mathcal{T}),$$

where $\mathcal{D}$ is the underlying additive category, $T$ is the automorphism of $\mathcal{D}$ (the *shift operator*), and $\mathcal{T}$ is the collection of all distinguished triangles. An additive functor $F: \mathcal{D} \to \mathcal{D}' = (\mathcal{D}', T', \mathcal{T}')$ between triangulated categories is said to be *exact* if there exists an isomorphism of functors $p: F \circ T \to T' \circ F$ such that for any distinguished triangle $(X, Y, Z, u, v, w)$ of $\mathcal{D}$

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{p(X) \circ F(w)} T'(F(X))$$

is a distinguished triangle of $\mathcal{D}'$.

Let $\mathcal{D} = (\mathcal{D}, T, \mathcal{T})$ be a triangulated category, and $\mathcal{A}$ an abelian category. A functor $\mathcal{H}: \mathcal{D} \to \mathcal{A}$ is called a *cohomology functor* if for any distinguished triangle $(X, Y, Z, u, v, w)$ of $\mathcal{D}$ the sequence

$$\mathcal{H}(X) \xrightarrow{\mathcal{H}(u)} \mathcal{H}(Y) \xrightarrow{\mathcal{H}(v)} \mathcal{H}(Z) \xrightarrow{\mathcal{H}(w)} \mathcal{H}(T(X))$$

is exact. In this situation, we have the long exact sequence in $\mathcal{A}$

$$\cdots \to \mathcal{H}^k(X) \xrightarrow{\mathcal{H}^k(u)} \mathcal{H}^k(Y) \xrightarrow{\mathcal{H}^k(v)} \mathcal{H}^k(Z) \xrightarrow{\mathcal{H}^k(w)} \mathcal{H}^{k+1}(X) \to \cdots,$$

where $\mathcal{H}^k = \mathcal{H} \circ T^k$. For example, for any object $X$ of $\mathcal{D}$ the functors

$$\text{Hom}_\mathcal{D}(X, \cdot): \mathcal{D} \to \mathcal{Ab}$$
and

\[ \operatorname{Hom}_D(\cdot, X) : D^{\text{opp}} \to \text{Ab} \]

are cohomology functors ([101], Chapter II, (1.2.1)).

Let \((D^{\leq 0}, D^{\geq 0})\) be a pair of full subcategories of a triangulated category \(D\). We set
\[ D^{\leq n} = T^{-n}(D^{\leq 0}) \quad \text{and} \quad D^{\geq n} = T^{-n}(D^{\geq 0}). \]

\((D^{\leq 0}, D^{\geq 0})\) is a \(t\)-structure on \(D\) if the following conditions are satisfied:

(a) \(D^{\leq -1} \subseteq D^{\leq 0}\) and \(D^{\geq 1} \subseteq D^{\geq 0}\);

(b) if \(X \in \text{obj}(D^{\leq 0})\) and \(Y \in \text{obj}(D^{\geq 1})\), then \(\operatorname{Hom}_D(X, Y) = 0\);

(c) for any \(X \in \text{obj}(D)\) there exists a distinguished triangle of the form
\[ X_0 \to X \to X_1 \to +1, \]
where \(X_0 \in \text{obj}(D^{\leq 0})\) and \(X_1 \in \text{obj}(D^{\geq 1})\).

If \((D^{\leq 0}, D^{\geq 0})\) is a \(t\)-structure, we set
\[ A = D^{\leq 0} \cap D^{\geq 0} \]
and call it the core of the \(t\)-structure. The core \(A\) is an abelian category.

In this book, triangulated categories, such as derived categories, are almost always equipped with the ‘canonical’ cohomology functor and the ‘canonical’ \(t\)-structure (cf. §C.4.(b)), and accordingly, exact functors between them preserve these structures in the following sense. Let \(D = (D, T, T')\) and \(D' = (D', T', T')\) be triangulated categories having respective cohomology functors \(\mathcal{H} : D \to A\) and \(\mathcal{H}' : D' \to A'\) and respective \(t\)-structures \((D^{\leq 0}, D^{\geq 0})\) and \((D'^{\leq 0}, D'^{\geq 0})\). Let \(F : D \to D'\) be an exact functor. Then,

- \(F\) \emph{preserve the cohomology functors} with respect to an exact functor
  \[ q : A \to A' \]
  of abelian categories if there exists an isomorphism
  \[ q \circ \mathcal{H} \sim \mathcal{H}' \circ F \]
  of the functors;

- \(F\) \emph{preserve the \(t\)-structures} if \(F\) maps \(D^{\leq 0}\) (resp. \(D^{\geq 0}\)) to \(D'^{\leq 0}\) (resp. \(D'^{\geq 0}\)).

\textbf{Convention.} In the sequel, when discussing triangulated categories endowed with the ‘canonical’ cohomology functors and ‘canonical’ \(t\)-structures that are clear from the context, all exact functors are assumed to preserve the cohomology functors and \(t\)-structures, unless otherwise clearly stated.
C.2 The category of complexes

C.2. (a) Definitions. Let $\mathcal{A}$ be an additive category.

- A complex with entries in $\mathcal{A}$ is a collection
  \[ F^\bullet = \{F^n, d^n_F\}_{n \in \mathbb{Z}} \]
  consisting of objects $F^n$ of $\mathcal{A}$ and arrows
  \[ d^n_F : F^n \to F^{n+1}, \]
  such that
  \[ d^{n+1}_F \circ d^n_F = 0, \quad n \in \mathbb{Z}. \]

- A morphism of complexes
  \[ f^\bullet : F^\bullet \to G^\bullet \]
  is a collection of arrows
  \[ f^n : F^n \to G^n \]
  in $\mathcal{A}$ such that
  \[ d^n_G \circ f^n = f^{n+1} \circ d^n_F, \quad n \in \mathbb{Z}. \]

We denote by $\mathcal{C}(\mathcal{A})$ the category of all complexes in $\mathcal{A}$. $\mathcal{C}(\mathcal{A})$ is an additive category. If $\mathcal{A}$ is an abelian category, then $\mathcal{C}(\mathcal{A})$ is an abelian category.

A complex $F^\bullet$ is said to be bounded (resp. bounded below, resp. bounded above) if $F^n = 0$ for $|n| \gg 0$ (resp. $n \ll 0$, resp. $n \gg 0$). The full subcategory of $\mathcal{C}(\mathcal{A})$ consisting of bounded complexes (resp. complexes bounded below, resp. complexes bounded above) is denoted by $\mathcal{C}^b(\mathcal{A})$ (resp. $\mathcal{C}^+(\mathcal{A})$, resp. $\mathcal{C}^-(\mathcal{A})$).

There exists a canonical functor $\mathcal{A} \to \mathcal{C}^b(\mathcal{A})$ that maps an object $F$ of $\mathcal{A}$ to the complex $F^\bullet$, where $F^n = 0$ unless $n = 0$ and $F^0 = F$. In this way $\mathcal{A}$ can be regarded as a full subcategory of $\mathcal{C}^b(\mathcal{A})$, and thus we get the diagram of inclusions of categories

\[ \mathcal{A} \hookrightarrow \mathcal{C}^b(\mathcal{A}) \quad \mathcal{C}^+(\mathcal{A}) \quad \mathcal{C}^-(\mathcal{A}) \quad \mathcal{C}(\mathcal{A}) \]

The category $\mathcal{C}^*(\mathcal{A})$ ($* = \text{"\ }, \text{b, +, -}$) is canonically endowed with

- the shift functor
  \[ [k] : \mathcal{C}^*(\mathcal{A}) \to \mathcal{C}^*(\mathcal{A}), \quad k \in \mathbb{Z}. \]
If $\mathcal{A}$ is abelian, $\mathbf{C}^*(\mathcal{A})$ has, moreover, the following structures:

- the cohomology functor
  
  \[ \mathcal{H}^0: \mathbf{C}^*(\mathcal{A}) \to \mathcal{A}; \]

- the truncations
  \[ \tau_{\leq n}, \tau_{\geq n}: \mathbf{C}^*(\mathcal{A}) \to \mathbf{C}^*(\mathcal{A}). \]

C.2. (b) Shifts. For any object $F^\bullet$ of $\mathbf{C}(\mathcal{A})$, the shift (by $k \in \mathbb{Z}$) $F[k]^\bullet$ is the complex defined as

\[ F[k]^n = F^{n+k} \quad \text{and} \quad d_F^n = (-1)^k d_F^{n+k}, \]

for any $n \in \mathbb{Z}$. Any morphism $f^\bullet: F^\bullet \to G^\bullet$ in $\mathbf{C}(\mathcal{A})$ canonically induces

\[ f[k]^\bullet: F[k]^\bullet \to G[k]^\bullet \]

given by

\[ f[k]^n = f^{n+k} \]

for any $n \in \mathbb{Z}$.

C.2. (c) Cohomology functor. Suppose $\mathcal{A}$ is an abelian category. For an object $F^\bullet$ of $\mathbf{C}(\mathcal{A})$ we set

\[ \mathcal{H}^0(F^\bullet) = \ker(d_F^0)/\text{image}(d_F^{-1}). \]

For an arrow $f^\bullet: F^\bullet \to G^\bullet$ in $\mathbf{C}(\mathcal{A})$,

\[ \mathcal{H}^0(f^\bullet): \mathcal{H}^0(F^\bullet) \to \mathcal{H}^0(G^\bullet) \]

is defined in the obvious manner. For $k \in \mathbb{Z}$ we set

\[ \mathcal{H}^k(F^\bullet) = H^0(F[k]^\bullet). \]

C.2. (d) Truncations. The stupid truncations are the operators $\sigma_{\leq n}$ and $\sigma_{\geq n}$ ($n \in \mathbb{Z}$) defined as follows. For an object $F^\bullet$ in $\mathbf{C}(\mathcal{A})$,

\[ \sigma_{\leq n} F^\bullet = (\ldots \to F^{n-1} \to F^n \to 0 \to 0 \to \ldots), \]

and

\[ \sigma_{\geq n} F^\bullet = (\ldots \to 0 \to 0 \to F^n \to F^{n+1} \to \ldots). \]
Any arrow $f^\bullet: F^\bullet \to G^\bullet$ in $C(\mathcal{A})$ induces

$$\sigma^{\leq n} f^\bullet: \sigma^{\leq n} F^\bullet \to \sigma^{\leq n} G^\bullet$$

(and similarly $\sigma^{\geq n} f^\bullet$) in the obvious way.

The stupid truncations provide a handy device to manipulate complexes, but since they change cohomologies of the complexes, they do not conform with other structures. Conceptually more important are the following truncations. Suppose $\mathcal{A}$ is an abelian category. For an object $F^\bullet$ of $C(\mathcal{A})$,

$$\tau^{\leq n} F^\bullet = (\cdots \to F^{n-2} \to F^{n-1} \to \ker(d^n_F) \to 0 \to \cdots),$$

and

$$\tau^{\geq n} F^\bullet = (\cdots \to 0 \to \coker(d^{n-1}_F) \to F^{n+1} \to F^{n+2} \to \cdots).$$

For an arrow $f^\bullet: F^\bullet \to G^\bullet$ in $C(\mathcal{A})$,

$$\tau^{\leq n} f^\bullet: \tau^{\leq n} F^\bullet \to \tau^{\leq n} G^\bullet \quad \text{and} \quad \tau^{\geq n} f^\bullet: \tau^{\geq n} F^\bullet \to \tau^{\geq n} G^\bullet$$

are defined in the obvious way. There exists obvious arrows

$$\tau^{\leq n} F^\bullet \to F^\bullet \quad \text{and} \quad F^\bullet \to \tau^{\geq n} F^\bullet$$

in $C(\mathcal{A})$. In the sequel we often write

$$\tau^{< n} = \tau^{\leq n-1} \quad \text{and} \quad \tau^{> n} = \tau^{\geq n+1}$$

and similarly for the stupid truncations.

**Proposition C.2.1.** Let $F^\bullet$ be an object of $C(\mathcal{A})$. Then

$$\epsilon^k(\tau^{\leq n} F^\bullet) = \begin{cases} \epsilon^k(F^\bullet), & k \leq n, \\ 0, & k > n, \end{cases}$$

and

$$\epsilon^k(\tau^{\geq n} F^\bullet) = \begin{cases} 0, & k < n, \\ \epsilon^k(F^\bullet), & k \geq n, \end{cases}$$

for $k, n \in \mathbb{Z}$. 
C. Appendix: Addendum on derived categories

C.3 The triangulated category $K(\mathcal{A})$

C.3. (a) Homotopies. We say that a morphism $f : F^\bullet \to G^\bullet$ in $\mathbf{C}(\mathcal{A})$ is homotopic to zero if there exists a collection of arrows

$$\{s^n : F^n \to G^{n-1}\}_{n \in \mathbb{Z}}$$

such that

$$f^n = s^{n+1} \circ d^n_F + d^{n-1}_G \circ s^n$$

for $n \in \mathbb{Z}$. Two arrows $f^\bullet, g^\bullet : F^\bullet \to G^\bullet$ are homotopic if $f - g$ is homotopic to zero.

We denote by $\text{Htp}(F^\bullet, G^\bullet)$ the set of all homotopic-to-zero arrows $F^\bullet \to G^\bullet$. This is a subgroup of $\text{Hom}_{\mathbf{C}(\mathcal{A})}(F^\bullet, G^\bullet)$; moreover, the composition map

$$\text{Hom}_{\mathbf{C}(\mathcal{A})}(F^\bullet, G^\bullet) \times \text{Hom}_{\mathbf{C}(\mathcal{A})}(G^\bullet, H^\bullet) \to \text{Hom}_{\mathbf{C}(\mathcal{A})}(F^\bullet, H^\bullet)$$

maps $\text{Htp}(F^\bullet, G^\bullet) \times \text{Hom}_{\mathbf{C}(\mathcal{A})}(G^\bullet, H^\bullet)$ and $\text{Hom}_{\mathbf{C}(\mathcal{A})}(F^\bullet, G^\bullet) \times \text{Htp}(G^\bullet, H^\bullet)$ to $\text{Htp}(F^\bullet, H^\bullet)$.

Definition C.3.1. We define the category $K(\mathcal{A})$ as follows.

- The objects of $K(\mathcal{A})$ are the same as those of $\mathbf{C}(\mathcal{A})$:

$$\text{obj}(K(\mathcal{A})) = \text{obj}(\mathbf{C}(\mathcal{A})).$$

- For two objects $F^\bullet$ and $G^\bullet$ we set

$$\text{Hom}_{K(\mathcal{A})}(F^\bullet, G^\bullet) = \text{Hom}_{\mathbf{C}(\mathcal{A})}(F^\bullet, G^\bullet) / \text{Htp}(F^\bullet, G^\bullet).$$

The categories $K^*(\mathcal{A})$ for $* = +, -, b$ are defined similarly; they are canonically regarded as full subcategories of $K(\mathcal{A})$. By definition, we easily see that $K^*(\mathcal{A})$ ($* = "\ "$, $b, +, -$) is canonically an additive category. We denote the canonical functor $\mathbf{C}^*(\mathcal{A}) \to K^*(\mathcal{A})$ by

$$h^* : \mathbf{C}^*(\mathcal{A}) \to K^*(\mathcal{A})$$

for $* = "\ "$, $b, +, -$. The canonical functor $\mathcal{A} \to K(\mathcal{A})$, the composition of the inclusion $\mathcal{A} \hookrightarrow \mathbf{C}(\mathcal{A})$ followed by $h$, is fully faithful. Hence we have the diagram of categories consisting of fully faithful arrows

$$\begin{array}{ccc}
\mathcal{A} & \hookrightarrow & K^b(\mathcal{A}) \\
& \nearrow & \searrow \\
K^+(\mathcal{A}) & & K(\mathcal{A}) \\
& \searrow & \nearrow \\
& & K^-(\mathcal{A})
\end{array}$$
Proposition C.3.2. Let $* = \{+, -, \}$. Let $D, b, C, NUL$.

1. The shift operator $[k]$ on $C(A)$ maps homotopic arrows to homotopic arrows. Consequently, there exists a unique self-functor $[k]$ on $K^*(A)$ such that
   \[ h^* \circ [k] = [k] \circ h^*. \]

2. Suppose that $A$ is abelian. If $f^*: F^* \to G^*$ is a morphism in $C(A)$ homotopic to zero, then $\mathcal{H}^0(f^*)$ is a zero map. Consequently, there exists a unique functor
   \[ \mathcal{H}^0: K^*(A) \to A \]
   such that
   \[ \mathcal{H}^0 \circ h^* = \mathcal{H}^0. \]

3. Suppose $A$ is abelian. Let $f^*: F^* \to G^*$ be a morphism in $C(A)$, and $n$ an integer. Then if $f^*$ is homotopic to zero, so are $\tau \leq n f^*$ and $\tau \geq n g^*$. Consequently, there exists unique self-functors $\tau \leq n, \tau \geq n$ on $K^*(A)$ such that
   \[ \tau \leq n \circ h^* = h^* \circ \tau \leq n \quad \text{and} \quad \tau \geq n \circ h^* = h^* \circ \tau \geq n. \]

C.3. (b) Mapping cones. For a morphism $f: F^* \to G^*$ in $C(A)$, the mapping cone of $f$, denoted by cone($f$)$^*$, is the object of $C(A)$ defined as follows:

\[ \text{cone}(f)^n = F[1]^n \oplus G^n, \quad d^n_{\text{cone}(f)} = \begin{bmatrix} d^n_{F[1]} & 0 \\ f^{n+1} & d^n_G \end{bmatrix}. \]

This complex admits the canonical morphisms

\[ q_f^*: G^* \to \text{cone}(f)^*, \quad p_f^*: \text{cone}(f)^* \to F[1]^*, \]

given respectively by the collection of canonical inclusions $q_f^* = \{q_f^n\}$ and by the collection of canonical projections $p_f^* = \{p_f^n\}$. Note that if $f^*$ is an arrow in $C^*(A)$, then cone($f$)$^*$ belongs to $C^*(A)$.

Proposition C.3.3. Let $f^*, g^*: F^* \to G^*$ be two homotopic morphisms in $C(A)$. Then there exists an isomorphism

\[ r^*: \text{cone}(f)^* \sim \text{cone}(g)^* \]

in $C(A)$ such that the following diagram commutes:
Proof. If \( \{ s^n: F^n \to G^{n-1} \} \) gives the homotopy from \( f^\bullet \) to \( g^\bullet \), that is
\[
f^n - g^n = s^{n+1} \circ d_F^n + d_G^{n-1} \circ s^n
\]
for \( n \in \mathbb{Z} \), then the map \( F[1]^n \oplus G^n \ni (u, v) \mapsto (u, v + s^{n+1}(u)) \in F[1]^n \oplus G^n \) induces a morphism \( r^\bullet: \text{cone}(f)^\bullet \to \text{cone}(g)^\bullet \). Clearly, it has an inverse map and hence is an isomorphism of complexes. It is then straightforward to check that the map \( r^\bullet \) thus constructed have the desired property.

By C.3.3, the mapping cone \( \text{cone}(f)^\bullet \) together with the canonical arrows \( q_f^\bullet \) and \( p_f^\bullet \) can be defined for an arrow \( f^\bullet \) in \( K(\mathcal{A}) \). Thus for any morphism
\[
f^\bullet: F^\bullet \to G^\bullet
\]
in \( K(\mathcal{A}) \) we have a triangle
\[
F^\bullet \xrightarrow{f^\bullet} G^\bullet \xrightarrow{q_f^\bullet} \text{cone}(f)^\bullet \xrightarrow{q_f^\bullet} F[1]^\bullet.
\]

**Proposition C.3.4** ([101], Chapter II, (1.3.2)). Let \( * = \{ \cdot, b, +, - \} \). Let \( \mathcal{T}^* \) be the family of triangles in \( K^*(\mathcal{A}) \) that are isomorphic to a triangle of the form (●). Then \( K^*(\mathcal{A}) = (K^*(\mathcal{A}), [1], \mathcal{T}^*) \) is a triangulated category.

**C.4 The derived category \( D(\mathcal{A}) \)**

**C.4. (a) Definition and first properties.** Let \( \mathcal{A} \) be an abelian category, and consider the category \( K^*(\mathcal{A}) \), where \( * = \{ \cdot, b, +, - \} \). Recall that a complex \( F^\bullet \in \text{obj}(C(\mathcal{A})) \) is said to be acyclic if \( \mathcal{H}^k(F^\bullet) = 0 \) for any \( k \in \mathbb{Z} \). By C.3.2 (2), one can also define the acyclicity of objects in \( K(\mathcal{A}) \). We denote by \( \text{Ac}^*(\mathcal{A}) \) the full subcategory of \( K^*(\mathcal{A}) \) consisting of acyclic objects. This is a saturated full subcategory ([101], Chapter II, (2.1.5)) of \( K^*(\mathcal{A}) \).

**Definition C.4.1.** A map \( f^\bullet: F^\bullet \to G^\bullet \) in \( K(\mathcal{A}) \) is called a quasi-isomorphism if there exists a distinguished triangle of the form
\[
F^\bullet \xrightarrow{f^\bullet} G^\bullet \xrightarrow{+1} H^\bullet,
\]
where \( H^\bullet \) is acyclic.

In other words, a quasi-isomorphism is a morphism \( f^\bullet: F^\bullet \to G^\bullet \) such that for any \( k \in \mathbb{Z} \) the induced morphism \( \mathcal{H}^k(F^\bullet) \to \mathcal{H}^k(G^\bullet) \) is an isomorphism in \( \mathcal{A} \). We denote by \( \text{Qis}^*(\mathcal{A}) \) the set of all quasi-isomorphisms in \( K^*(\mathcal{A}) \). This is a multiplicative system of the triangulated category \( K^*(\mathcal{A}) \) compatible with the triangulation and is the one corresponding to \( \text{Ac}^*(\mathcal{A}) \) under the correspondence described in [101], Chapter II, (2.1.8).
**Definition C.4.2.** We set

\[ D^*(\mathcal{A}) = K^*(\mathcal{A})/Ac^*(\mathcal{A}) \]

for \( \, \ast = \{ \cdot, b, +, - \} \) and call them the derived categories of \( \mathcal{A} \).

We denote by

\[ Q^*: K^*(\mathcal{A}) \to D^*(\mathcal{A}) \]

the quotient functor. \( D^*(\mathcal{A}) \) is a triangulated category as follows.

- The shift operator \([k]\), which simply comes from the shift operator \([k]\) of \( K^*(\mathcal{A}) \).
- The set of all distinguished triangles \( \mathcal{T}^* \), that is, the image of the set of all distinguished triangles of \( K^*(\mathcal{A}) \) under the quotient functor \( Q \).

**Proposition C.4.3.** The canonical functors form a diagram of categories

\[
\begin{array}{ccc}
\mathcal{A} & \to & D^b(\mathcal{A}) \\
\downarrow & & \downarrow \\
D^+(\mathcal{A}) & \to & D(\mathcal{A}) \\
\end{array}
\]

consisting of fully faithful functors.

**C.4. (b) Canonical cohomology functor and canonical \( t \)-structure.** The triangulated category \( D^*(\mathcal{A}) \) possesses the following canonical structures:

- the canonical cohomology functor

\[ \mathcal{H}^0: D^*(\mathcal{A}) \to \mathcal{A}; \]

- the canonical \( t \)-structure

\[ (D^*(\mathcal{A})_{\leq 0}, D^*(\mathcal{A})_{\geq 0}). \]

**Proposition C.4.4.** Let \( \, \ast = \{ \cdot, b, +, - \} \). The functor \( \mathcal{H}^0: K^*(\mathcal{A}) \to \mathcal{A} \) whose existence is established in C.3.2 (2) gives rise to a cohomology functor

\[ \mathcal{H}^0: D^*(\mathcal{A}) \to \mathcal{A}. \]
Proposition C.4.5. Let $* = -, +, 0$.

1. Consider the full subcategory $D^*(\mathcal{A})^{\leq 0}$ (resp. $D^*(\mathcal{A})^{\geq 0}$) of $D^*(\mathcal{A})$ consisting of objects $F$ such that $\mathcal{H}^k(F) = 0$ for $k > 0$ (resp. $k < 0$). Then $(D^*(\mathcal{A})^{\leq 0}, D^*(\mathcal{A})^{\geq 0})$ gives a $t$-structure on $D^*(\mathcal{A})$.

2. Let $\tau^{\leq n}, \tau^{\geq n} : K^*(\mathcal{A}) \to K^*(\mathcal{A})$ be as in C.3.2 (3). Then $\tau^{\leq n}$ and $\tau^{\geq n}$ map $Ac^*(\mathcal{A})$ to itself. Consequently, they induce self-functors

$$\tau^{\leq n}, \tau^{\geq n} : D^*(\mathcal{A}) \to D^*(\mathcal{A}),$$

respectively.

3. The functor $\tau^{\leq n}$ (resp. $\tau^{\geq n}$) gives a right-adjoint (resp. left-adjoint) to the inclusion functor $D^*(\mathcal{A})^{\leq n} \to D^*(\mathcal{A})$ (resp. $D^*(\mathcal{A})^{\geq n} \to D^*(\mathcal{A})$).

Proposition C.4.6 (cf. [101], Chapter III, 1.2.7). Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ an exact functor. Then $F$ induces canonically the commutative diagram

$$
\begin{array}{ccc}
C^*(\mathcal{A}) & \xrightarrow{C^*(F)} & C^*(\mathcal{B}) \\
\downarrow h^* & & \downarrow h^* \\
K^*(\mathcal{A}) & \xrightarrow{K^*(F)} & K^*(\mathcal{B}) \\
\downarrow Q^* & & \downarrow Q^* \\
D^*(\mathcal{A}) & \xrightarrow{D^*(F)} & D^*(\mathcal{B}).
\end{array}
$$

Moreover, $D^*(F)$ is exact and preserves the canonical cohomology functors (with respect to $F$) and the canonical $t$-structures (cf. §C.1).

C.4. (c) Representation by complexes and amplitude

Proposition C.4.7. Let $* = -, +, 0$. The canonical functor $\mathcal{A} \to D^*(\mathcal{A})$ gives an exact categorical equivalence between $\mathcal{A}$ and the core of the $t$-structure, that is, the full subcategory of $D^*(\mathcal{A})$ consisting of the objects $F$ such that

$$\mathcal{H}^k(F) = 0$$

unless $k = 0$.

Definition C.4.8. Let $* = -, +, 0$.

1. Let $F$ be an object of $D^*(\mathcal{A})$. A complex $F^\bullet$ in $C^*(\mathcal{A})$ represents $F$ if $F$ and $Q^* \circ h^*(F^\bullet)$ are isomorphic in $D^*(\mathcal{A})$. 

(2) Let \( f: F \to G \) be an arrow in \( \mathsf{D}^*(\mathcal{A}) \). A morphism \( f^*: F^* \to G^* \) of complexes represents \( f \) if \( f \) and \( Q^* \circ h^*(f^*) \) are isomorphic in \( \mathsf{D}^*(\mathcal{A}) \).

**Definition C.4.9.** (1) Let \( F \) be an object of \( \mathsf{D}^b(\mathcal{A}) \). The amplitude of \( F \), denoted \( \text{amp}(F) \), is the number (if it exists)

\[
\text{amp}(F) = \sup\{k: \mathcal{H}^k(F) \neq 0\} - \inf\{k: \mathcal{H}^k(F) \neq 0\}.
\]

(2) An object \( F \) of \( \mathsf{D}(\mathcal{A}) \) is concentrated in degree \( n \) if \( \mathcal{H}^k(F) = 0 \) unless \( k = n \).

**Proposition C.4.10.** Let \( F \) be an object of \( \mathsf{D}(\mathcal{A}) \). Then \( F \) is concentrated in degree \( 0 \) if and only if it is represented by a complex contained in the image of \( \mathcal{A} \hookrightarrow \mathsf{C}(\mathcal{A}) \).

### C.5 Subcategories of \( \mathsf{D}(\mathcal{A}) \)

Let \( \mathcal{A} \) be an abelian category, and \( \mathcal{B} \) an abelian full subcategory of \( \mathcal{A} \). We say \( \mathcal{B} \) is *thick* in \( \mathcal{A} \) if for any exact sequence of the form

\[
X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow X_4
\]

in \( \mathcal{A} \) with \( X_0, X_1, X_3, X_4 \in \text{obj}(\mathcal{B}) \), we have \( X_2 \in \text{obj}(\mathcal{B}) \).

Let \( \mathcal{A} \) be an abelian category, and \( \mathsf{D}^*(\mathcal{A}) \) the associated derived category where \( * = \text{``} \), \( b, +, - \). We consider the canonical cohomology functor \( \mathcal{H}^0 \) (C.4.4) and the canonical \( t \)-structure (C.4.5). For an abelian full subcategory \( \mathcal{B} \) of \( \mathcal{A} \) we denote by

\[
\mathsf{D}^*_\mathcal{B}(\mathcal{A})
\]

the full subcategory of \( \mathsf{D}^*(\mathcal{A}) \) consisting of objects \( F \) such that \( \mathcal{H}^k(F) \in \text{obj}(\mathcal{B}) \) for any \( k \in \mathbb{Z} \).

**Proposition C.5.1.** Suppose \( \mathcal{B} \) is thick in \( \mathcal{A} \). Then \( \mathsf{D}^*_\mathcal{B}(\mathcal{A}) \) together with the shift operator \( [k]|_{\mathsf{D}^*_\mathcal{B}(\mathcal{A})} \) and the set of distinguished triangles of \( \mathsf{D}^*(\mathcal{A}) \)

\[
F \longrightarrow G \longrightarrow H \overset{+1}{\longrightarrow}
\]

such that \( F, G, H \in \text{obj}(\mathsf{D}^*_\mathcal{B}(\mathcal{A})) \) is a triangulated category. Moreover

(1) the composition

\[
\mathcal{H}^0: \mathsf{D}^*_\mathcal{B}(\mathcal{A}) \hookrightarrow \mathsf{D}^*(\mathcal{A}) \overset{\mathcal{H}^0}{\longrightarrow} \mathcal{A}
\]

is a cohomology functor (again called the canonical cohomology functor) and
(2) if we set
\[ D^*_B(A)^{\leq 0} = D^*(A)^{\leq 0} \cap D^*_B(A) \quad \text{and} \quad D^*_B(A)^{\geq 0} = D^*(A)^{\geq 0} \cap D^*_B(A), \]
then \((D^*_B(A)^{\leq 0}, D^*_B(A)^{\geq 0})\) gives a t-structure on \(D^*_{B(A)}\) (again called the canonical t-structure).

In the sequel we fix an abelian category \(A\) and a thick abelian full subcategory \(B\) of \(A\). By the construction of the categories \(D^*_{B(A)}\) and \(D^*(A)\), we have a natural functor \(\delta^*: D^*(B) \to D^*(A)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
C^*(B) & \longrightarrow & C^*(A) \\
Q^*_{oh} & \downarrow & Q^*_{oh} \\
D^*(B) & \longrightarrow & D^*(A).
\end{array}
\]

Clearly, this functor maps \(D^*(B)\) to \(D^*_B(A)\). Thus we get the functor
\[
\delta^*: D^*(B) \longrightarrow D^*_B(A),
\]
which is clearly exact.

**Proposition C.5.2** (cf. [66], 1.7.11 and 1.7.12). (1) Suppose that
- for any monomorphism \(f: G \to F\) of \(A\) such that \(G \in \text{obj}(B)\) there exists a morphism \(g: F \to H\) with \(H \in \text{obj}(B)\) such that \(g \circ f: G \to H\) is a monomorphism.

Then the functors \(\delta^+\) and \(\delta^b\) are equivalences.

(2) Suppose that
- for any epimorphism \(f: F \to G\) of \(A\) such that \(G \in \text{obj}(B)\) there exists a morphism \(g: H \to F\) with \(H \in \text{obj}(B)\) such that \(f \circ g: H \to G\) is an epimorphism.

Then the functors \(\delta^-\) and \(\delta^b\) are equivalences.

**Corollary C.5.3.** (1) If \(B\) has enough \(A\)-injectives (that is, for any object \(F\) of \(A\) there exists a monomorphism \(F \to G\), where \(G \in \text{obj}(B)\) and is an injective object in \(A\)), then \(\delta^+\) and \(\delta^b\) are equivalences.

(2) If \(B\) has enough \(A\)-projectives (that is, for any object \(F\) of \(A\) there exists an epimorphism \(G \to F\), where \(G \in \text{obj}(B)\) and is a projective object in \(A\)), then \(\delta^-\) and \(\delta^b\) are equivalences.

**Proposition C.5.4.** If the functor \(\delta^b\) is fully faithful, then it is an equivalence.
Proof. Let $F \in \text{obj}(\mathbf{D}_B^b(\mathcal{A}))$. We are going to show that $F$ belongs to the essential image of $\delta^b$ by induction with respect to the amplitude $\text{amp}(F)$. If $\text{amp}(F) = 0$, we may assume by suitable shifts (which does not change the nature of our assertion) that $F$ is concentrated in degree 0. Then the claim follows from C.4.10. In the general case, consider the distinguished triangle

$$
\tau^{\geq n+1} F[-1] \longrightarrow \tau^{\leq n} F \longrightarrow F \longrightarrow +1
$$

induced by $\tau^{\leq n} F \rightarrow F \rightarrow \tau^{\geq n+1} F \rightarrow +1$. By induction, we may assume that $\tau^{\geq n+1} F[-1] = \delta^b(L)$ and $\tau^{\leq n} F = \delta^b(M)$ for $L, M \in \text{obj}(\mathbf{D}_b(B))$. Since $\delta^b$ is fully faithful, there exists a unique arrow $L \rightarrow M$ in $\mathbf{D}_b(B)$ that is mapped to the arrow $\tau^{\geq n+1} F[-1] \rightarrow \tau^{\leq n} F$. Take a distinguished triangle

$$
L \longrightarrow M \longrightarrow N \longrightarrow +1
$$

of $B$. Then since $\delta^b$ is exact, $F$ is isomorphic to $\delta^b(N)$. \qed
Chapter I

Formal geometry

This chapter is devoted to formal geometry, the geometry of formal schemes. As we pointed out in the introduction, it is for us essential to treat non-Noetherian formal schemes, e.g., finite type formal schemes over an \( a \)-adically complete valuation ring of arbitrary height, the importance of which stems from the requirement for the functoriality of taking fibers of finite type morphisms in rigid geometry. Our aim in this chapter is to present a sufficiently general theory of formal schemes, including GFGA theorems, which is not restricted to the locally Noetherian situation.

Section 1 collects basic notions of formal geometry. Our central objects in this and the following sections are the so-called adic formal schemes of finite ideal type, that is, formal schemes that are locally isomorphic to the formal spectrum \( \text{Spf} \, A \) for an adic ring \( A \) having a finitely generated ideal of definition. In §2 we will introduce the so-called universally rigid-Noetherian and universally adhesive formal schemes, corresponding respectively to topologically universally pseudo-adhesive (via Gabber’s theorem) and topologically universally adhesive rings (0, §8.5), which sit in the following hierarchy of classes of formal schemes:

\[
\{ \text{univ. adhesive formal schemes} \} \subseteq \{ \text{univ. rigid-Noetherian formal schemes} \} \subseteq \{ \text{adic formal schemes of finite ideal type} \}.
\]

For example, all finite type formal schemes over an \( a \)-adically complete valuation ring, including those that have been called admissible formal schemes in Tate–Raynaud’s classical rigid analytic geometry, are all universally adhesive.

Sections 3–5 are devoted largely to basic aspects of formal geometry. Among them, what we do in Section 3 is worth noting; we give a systematic treatment of what we call adically quasi-coherent sheaves, which seems missing in the previous literature, even in [53], (0, §10). An adically quasi-coherent sheaf on an adic formal scheme \( X \) with an ideal of definition \( I \) is an \( O_X \)-module \( F \) that is

1. complete, i.e., \( F \cong \lim \limits_{\leftarrow k \geq 0} F/I^{k+1}F \), and

2. such that \( F_k = F/I^{k+1}F \) for every \( k \geq 0 \) is a quasi-coherent sheaf on the scheme \( X_k = (X, O_X/I^{k+1}) \).
It will be shown (Theorem 3.2.8) that, if \( X = \text{Sp} A \) is affine, then an \( \mathcal{O}_X \)-module \( \mathcal{F} \) is adically quasi-coherent if and only if it is isomorphic to the \( \mathcal{O}_X \)-module given by the ‘\( \Delta \)-construction’, that is, \( M^\Delta \) for a complete \( A \)-module \( M \) (cf. [54], I, (10.10.1)); moreover, in this situation we have \( M = \Gamma(X, \mathcal{F}) \). The functor \( M \mapsto M^\Delta \) from the category of finitely generated \( A \)-modules is exact if \( X = \text{Sp} A \) is universally rigid-Noetherian (Theorem 3.5.6), and this property improves significantly the homological algebra of adically quasi-coherent sheaves of finite type on locally universally rigid-Noetherian formal schemes. This section, moreover, contains the following fundamental result (Corollary 3.7.12): any coherent (\( = \) quasi-compact and quasi-separated) formal scheme of finite type admits an ideal of definition of finite type.

In §6 we develop the theory of formal algebraic spaces, more precisely, \textit{quasi-separated adic formal algebraic spaces of finite ideal type}. It seems that, in the previous literature, only [72], Chapter 5, is a general and systematic reference for formal algebraic spaces. But the class of formal algebraic spaces covered in this reference consists only of separated and Noetherian formal algebraic spaces, which is evidently not general enough for our purpose. It is therefore necessary to implement the whole theory from the very beginning, and this is what we do in this section. Note that, as we have explained in Introduction, there are important reasons for us to deal with formal algebraic spaces, not only formal schemes, in our framework of rigid geometry.

The rest of the chapter, (§7–§11), gives the main body of formal geometry, which consists, roughly speaking, of generalizations of the contents of [54], III, including GFGA theorems. The results in these sections generalize the classical results in the following two ways:

- we replace the Noetherianity assumption with a weaker one;
- the theorems (finiteness theorem and GFGA theorems) are formulated and proved in the language of derived categories.

As for the first, more precisely, the GFGA theorems will be proved in the universally adhesive situation and subsequently generalized to the universally rigid-Noetherian situation in Appendix §C.

1 \textbf{Formal schemes}

In this section we survey the fundamental concepts in the theory of formal schemes. The section is, therefore, mostly a rehash of the already well-written accounts such as [54], I, §10, and [53], §10. Our main objects of study are so-called \textit{adic formal schemes of finite ideal type} (1.1.14, 1.1.16), that is, adic formal schemes that admit, Zariski locally, ideals of definition of finite type.
In §1.1 we collect some basic definitions and properties of formal schemes and ideals of definition. As discussed in §1.2, the category of formal schemes has fiber products, which extends the notion of fiber products of schemes.

The above-mentioned references define the notion of adic morphisms only for morphisms between locally Noetherian formal schemes (cf. [53], §10.12). In §1.3 we extend the definition to morphisms between general adic formal schemes of finite ideal type.

In §1.4 we discuss formal completions of schemes. Then, after briefly discussing several categories of formal schemes in §1.5, we finish this section by introducing locally of finite type morphisms between formal schemes in §1.7.

1.1 Formal schemes and ideals of definition

1.1. (a) Admissible rings. We basically refer to [54], I, §7, and [54], I, §10, for most of the fundamental notions in the theory of formal schemes. Here we recall some of them.

Let $A$ be a ring endowed with the topology defined by a descending filtration $F^\bullet = \{F^\lambda\}_{\lambda \in \Lambda}$ by ideals (cf. I, §7.1. (a)).

**Definition 1.1.1.** An ideal $I \subseteq A$ is said to be an *ideal of definition* of the topological ring $A$ if

(a) $I$ is open, that is, there exists $\lambda \in \Lambda$ such that $F^\lambda \subseteq I$, and

(b) $I$ is topologically nilpotent, that is, for any $\mu \in \Lambda$ there exists $n \geq 0$ such that $I^n \subseteq F^\mu$.

Clearly, any open ideal contained in an ideal of definition is again an ideal of definition. Hence, if $A$ admits at least one ideal of definition, it has a fundamental system of open neighborhoods of 0 consisting of ideals of definition, called a fundamental system of ideals of definition.

**Lemma 1.1.2.** Let $A$ be a ring endowed with the topology defined by a descending filtration $F^\bullet = \{F^\lambda\}_{\lambda \in \Lambda}$ by ideals, and $I \subseteq A$ an ideal. Then the following conditions are equivalent.

(a) The topology on $A$ is $I$-adic (I, §7.2. (a)).

(b) $I$ is an ideal of definition, and $I^n$ is open for any $n \geq 0$.

(c) $I^n$ is an ideal of definition for any $n \geq 1$.

(d) $\{I^n\}_{n \geq 0}$ is a fundamental system of open neighborhoods of 0.

Moreover, if these conditions are fulfilled, then for any ideal of definition $J \subseteq A$ the topology on $A$ is $J$-adic.
Proof. The equivalence of (a) and (d) follows from the definition of adic topologies. (a) is equivalent to the two conditions

(i) for any \( \lambda \in \Lambda \) there exists \( n \geq 0 \) such that \( I^n \subseteq F^\lambda \) and

(ii) for any \( n \geq 0 \) there exists \( \mu \in \Lambda \) such that \( F^\mu \subseteq I^n \).

The first condition says exactly that \( I \) is topologically nilpotent, and the second one says that all \( I^n \) are open, thereby the equivalence of (a) and (b). The equivalence of (b) and (c) is clear. If the topology on \( A \) is \( I \)-adic and \( J \) is an ideal of definition, then there exists \( n \geq 0 \) such that \( I^n \subseteq J \); consequently, \( I^{nm} \subseteq J^m \), that is, \( J^m \) for any \( m \geq 0 \) is open. \( \square \)

Definition 1.1.3. Let \( A \) be a ring endowed with the topology defined by a descending filtration \( F^\bullet = \{F^\lambda\}_{\lambda \in \Lambda} \) by ideals.

(1) We say that \( A \) is an admissible ring if

(a) \( A \) admits an ideal of definition, and

(b) \( A \) is Hausdorff complete (cf. 0, §7.1. (c)).

(2) An admissible ring \( A \) is said to be an adic ring if the topology on \( A \) is \( I \)-adic for some ideal \( I \subseteq A \).

Note that giving an adic ring \( A \) amounts to the same as giving an isomorphism class of complete pairs \((A, I)\) (0, §8.1. (a)).

In 0, §7.2. (a) we used the terminology ‘adic topology’ even when the topologies in question are not necessarily Hausdorff complete. However, adic rings are always required to be Hausdorff complete; in other words,

adic ring = a ring with Hausdorff complete adic topology,

which the reader should always keep in mind; cf. 0.7.2.2.

Example 1.1.4. Let \( A \) be a ring, and \( I \subseteq A \) an ideal. We consider the \( I \)-adic topology on \( A \) (0, §7.2. (a)). Then the Hausdorff completion \( A_I^{\wedge} \) of \( A \) with respect to the \( I \)-adic topology is an admissible ring, and the closure \( J \) of the image of \( I \) in \( A_I^{\wedge} \) (cf. 0, §7.1. (c)) is an ideal of definition ([54], 0, (7.2.2)).

Definition 1.1.5. Let \( A \) and \( B \) be admissible rings.

(1) A morphism of admissible rings \( f : A \rightarrow B \) is a continuous ring homomorphism.

(2) Suppose \( A \) and \( B \) are adic rings. Then a morphism \( f : A \rightarrow B \) of admissible rings is said to be adic if there exists an ideal of definition \( I \) of \( A \) such that \( IB \) is an ideal of definition of \( B \) (cf. 0, §7.2. (a)).
Note that the continuity in (1) is equivalent to the property that for any ideal of
definition \( J \) of \( B \) there exists an ideal of definition \( I \) of \( A \) such that \( IB \) is contained
in \( J \) (cf. 0.7.1.3 (1)). It is easy to see that in the situation as in (2) a morphism
\( f : A \to B \) is adic if and only if \( IB \) is an ideal of definition of \( B \) for any ideal of
definition \( I \) of \( A \).

**Definition 1.1.6.** An admissible ring \( A \) is said to be of finite ideal type if it has a
fundamental system of ideals of definition consisting of finitely generated ideals.

If the topology on \( A \) is adic, then in view of 1.1.2 the condition is equivalent to
\( A \) having at least one finitely generated ideal of definition.

**Example 1.1.7.** Suppose in the situation as in 1.1.4 that the ideal \( I \) is finitely gen-
erated. Then \( \hat{A} = A_f^\ast \) is the I-adic completion of \( A \) (0.7.2.15), and hence is an
adic ring of finite ideal type. Note that in this situation the closure \( J \) of the image
of \( I \) in \( \hat{A} \) coincides with \( I \hat{A} \).

Let \( A \) be an admissible ring, \( \{F^\lambda\}_{\lambda \in \Lambda} \) a fundamental system of ideals of definition,
and \( S \subseteq A \) a multiplicative subset. Consider the ring of fractions \( A_S \) endowed
with the topology defined by \( \{F^\lambda A_S\}_{\lambda \in \Lambda} \). Let \( A_{\{S\}} \) denote the Hausdorff completion:
\[
A_{\{S\}} = \lim_{\lambda \in \Lambda} A_S / F^\lambda A_S.
\]
It has the induced filtration \( \{\hat{F}^\lambda_S\}_{\lambda \in \Lambda} \) defined as in 0, §7.1.(c); each \( \hat{F}^\lambda_S \) is the
closure of the image of \( F^\lambda A_S \) under the canonical map \( A_S \to A_{\{S\}} \) (0.7.1.7).

**Proposition 1.1.8 ([54], 01, (7.6.11)).** (1) The topological ring \( A_{\{S\}} \) is an admissi-
ble ring, and \( \{\hat{F}^\lambda_S\}_{\lambda \in \Lambda} \) gives a fundamental system of ideals of definition.

(2) Suppose \( A \) is adic of finite ideal type, and let \( I \subseteq A \) be a finitely generated
ideal of definition. Then \( A_{\{S\}} \) is again adic of finite ideal type, and \( IA_{\{S\}} \) is an
ideal of definition.

Note that (2) follows from (1) and 0.7.2.11; indeed, if one defines \( J^{(n)} \subseteq A_{\{S\}} \)
\((n \geq 1)\) to be the closure of the image of \( I^n A_S \) in \( A_{\{S\}} \), then 0.7.2.11 implies that
\( J^{(n)} = I^n A_{\{S\}} \) for each \( n \geq 0 \).

The admissible rings \( A_{\{S\}} \) in the case \( S = \{f^n : n \geq 0\} \), denoted simply by
\( A_{\{f\}} \), will be frequently used. If \( A \) is an adic ring of finite ideal type, then the ring
\( A_{\{f\}} \) allows a more explicit description as follows. Since \( A_f = A[f^{-1}] \), we have a morphism
\[
A\langle\langle T \rangle\rangle \to A_{\{f\}}
\]
that sends \( X \) to \( f^{-1} \), where \( A\langle\langle T \rangle\rangle \) denotes the restricted formal power series
ring (0, §8.4) in the variable \( T \).
Lemma 1.1.9. Let $A$ be an adic ring of finite ideal type. Then the above-defined morphism induces an isomorphism

$$A \langle \langle T \rangle \rangle/(fT - 1) \sim A\{f\}$$

of adic rings.

Proof. See [17], 7.1, Remark 10. \qed

1.1. (b) Formal spectrum. Let $A$ be an admissible ring. The formal spectrum $\text{Spf} A$ is a topologically locally ringed space with the underlying set consisting of all open prime ideals of $A$. Note that a prime ideal $p \subseteq A$ is open if and only if it contains at least one (hence all) ideal of definition ([54], 0I, (7.1.5)). Hence, $\text{Spf} A$ as a pointset is nothing but the closed subset $V(I)$ of $\text{Spec} A$ defined by an ideal of definition $I$ and, in fact, the topology (the Zariski topology) of $\text{Spf} A$ is the subspace topology induced by that of $\text{Spec} A$. Moreover, $X = \text{Spf} A$ is endowed with the sheaf of topological rings (considered with the pseudo-discrete topology ([54], 0I, (3.8.1)))

$$\mathcal{O}_X = \lim_{\leftarrow I} A/I|_X,$$

where $I$ runs through all ideals of definition of $A$ (with the reversed inclusion order). Here the projective limit is filtered, and any fundamental system of ideals of definition is cofinal in the set of all ideals of definition. In particular, if $A$ is adic, the collection $\{I^{k+1}\}_{k \geq 0}$ provided by an ideal of definition $I$ is cofinal, and thus the above sheaf coincides with the projective limit of $A_k|_X$ ($k \geq 0$), where $A_k = A/I^{k+1}$ for $k \geq 0$.

Definition 1.1.10. (1) A topologically locally ringed space isomorphic to $\text{Spf} A$ for an admissible ring $A$ is called an affine formal scheme.

(2) A morphism $f: X \to Y$ between two affine formal schemes is a morphism of topologically locally ringed spaces.

Like in the case of schemes, the functor

$$A \mapsto \text{Spf} A$$

gives rise to a categorical equivalence between the opposite category of the category of admissible rings and the category of affine formal schemes ([54], I, §10.2); in particular, from $X = \text{Spf} A$ we recover $A = \Gamma(X, \mathcal{O}_X)$ as a topological ring ([54], I, (10.1.3)).

Let $A$ be an admissible ring, and consider the formal spectrum $X = \text{Spf} A$. As $X$ is a subspace of the topological space $\text{Spec} A$, the subsets of the form $\mathcal{D}(f) = D(f) \cap X$ for $f \in A$ give an open basis of $X$; moreover, since $X$ is closed
in Spec $A$, such an open subset is quasi-compact. The space $\mathcal{D}(f)$, equipped with the topologically locally ringed structure as an open subspace of $X$ (cf. 0, §4.1.(a)), is an affine formal scheme isomorphic to Spf $A\langle f \rangle$.

**Lemma 1.1.11.** The following conditions for finitely many elements $f_1, \ldots, f_r \in A$ are equivalent.

(a) The open sets $\mathcal{D}(f_i) (i = 1, \ldots, r)$ cover $X$.

(b) Let $\tilde{f}_i = (f_i \mod I), i = 1, \ldots, r$. For any ideal of definition $I$ of $A$ the open sets $D(\tilde{f}_i)$ cover $\text{Spec } A/I$.

(c) The open sets $D(f_i) (i = 1, \ldots, r)$ cover $\text{Spec } A$.

(d) The ideal generated by $f_1, \ldots, f_r$ coincides with $A$.

**Proof.** The implications (a) $\iff$ (b) $\iff$ (c) $\iff$ (d) are clear; the first equivalence is due to the fact that $V(I) = \text{Spec } A/I$ and Spf $A$ are homeomorphic to each other. If (b) holds, then the ideal $(f_1, \ldots, f_r)$ contains an element of $1 + I$. Then we deduce (d) from the fact $1 + I \subset A^\times$ (that is, $A$ is $I$-adically Zariskian; cf. 0.7.2.13).

**Proposition 1.1.12.** An open subset $U$ of $X = \text{Spf } A$ is quasi-compact if and only if $U = X \setminus V(a)$ (in Spec $A$) for a finitely generated ideal $a \subseteq A$.

**Proof.** The ‘if’ part follows from [53], (1.1.4), and the fact that $X$ is a closed subset of Spec $A$. If $U$ is quasi-compact, then $U = \bigcup_{i=1}^{r} \mathcal{D}(f_i)$ for $f_1, \ldots, f_r \in A$. Set $V = \bigcup_{i=1}^{r} D(f_i)$. Then $V$ is a quasi-compact open set of Spec $A$ and is equal to $\text{Spec } A \setminus V(a)$, where $a = (f_1, \ldots, f_r)$. Hence, $U = V \cap X = X \setminus V(a)$. □

1.1. (c) Formal schemes

**Definition 1.1.13.** (1) A *formal scheme* is a topologically locally ringed space that is locally isomorphic to an affine formal scheme.

(2) Let $X$ and $Y$ be formal schemes. A *morphism* $f : X \to Y$ of formal schemes is a morphism of topologically locally ringed spaces.

An *open formal subscheme* of a formal scheme $X$ is a formal scheme of the form $(U, \mathcal{O}_X|_U)$, where $U$ is an open subset of the underlying topological space of $X$. An *open immersion* of formal schemes is defined in a similar way as in the case of ringed spaces; cf. 0, §4.1.(a). An open formal subscheme $U \subseteq X$ is said to be *affine* if it is an affine formal scheme. Thus any formal scheme $X$ admits an open covering $X = \bigcup_{a \in L} U_a$ consisting of affine open formal subschemes; an open covering of this form is called an *affine (open) covering*. 
**Definition 1.1.14.** A formal scheme $X$ is said to be *adic* if it admits an affine open covering $X = \bigcup_{\alpha \in \mathcal{L}} U_{\alpha}$ such that each $U_{\alpha}$ is isomorphic to $	ext{Spf} A_{\alpha}$ for an adic ring $A_{\alpha}$ (1.1.3 (2)).

**Remark 1.1.15.** Any scheme can be regarded as an adic formal scheme. Indeed, any ring is an adic ring for the ideal $(0)$ (hence a 0-adic ring). Hence, schemes are naturally regarded as 0-adic formal schemes. Thus the category of all formal schemes contain as a full subcategory the category of schemes. Most importantly, the category of all formal schemes has a final object $\text{Spec } \mathbb{Z}$.

**Definition 1.1.16.** An adic formal scheme $X$ is said to be *of finite ideal type* if there exists an affine open covering $X = \bigcup_{\alpha \in \mathcal{L}} U_{\alpha}$ such that each $U_{\alpha}$ is isomorphic to $	ext{Spf} A_{\alpha}$ for an adic ring $A_{\alpha}$ of finite ideal type (1.1.6).

The following proposition follows from 1.1.8 (2).

**Proposition 1.1.17.** Let $X$ be an adic formal scheme of finite ideal type. Then there exists open basis of the topology on $X$ consisting of affine open subschemes of the form $\text{Spf} A$ by adic rings $A$ of finite ideal type.

We will see later in 3.7.13 that, if $X = \text{Spf } A$ is an affine adic formal scheme of finite ideal type, then $A$ itself is an adic ring of finite ideal type.

**1.1. (d) Ideals of definition.** Let $A$ be an admissible ring. For any open ideal $J \subseteq A$ one defines the sheaf $J^\Delta$ on $X = \text{Spf } A$ by

$$J^\Delta = \lim_{\overset{\rightarrow}{I \subseteq J}} \frac{J}{I}$$

(cf. 0, §5.1 (a)), where $I$ runs through all ideals of definition contained in $J$. Since $\frac{J}{I}$ is an ideal of $\frac{A}{I} = \mathcal{O}_{\text{Spec } A/I}$, $J^\Delta$ is a sheaf of ideals of $\mathcal{O}_X$ (cf. 0.3.2.4). Note that, in view of [54], I, (1.3.7), and the definition of projective limit sheaves (cf. 0, §3.2 (c)), we have

$$\Gamma(X, J^\Delta) = \lim_{\overset{\rightarrow}{I \subseteq J}} \Gamma(X, \frac{J}{I}) = \lim_{\overset{\rightarrow}{I \subseteq J}} J/I = J,$$

where the last equality follows from the fact that $J$ is open (and hence is closed); cf. Exercise 0.7.5.

**Definition 1.1.18.** (1) Let $A$ be an admissible ring, and $X = \text{Spf } A$. A sheaf of ideals $I$ of $\mathcal{O}_X$ is said to be an *ideal of definition*, if for any $x \in X$ there exists an open neighborhood $U$ of $x$ of the form $U = \mathcal{O}(f)$ ($f \in A$) such that $I|_U = J^\Delta$ for some ideal of definition $J$ of the admissible ring $A_{(f)}$. 
(2) Let $X$ be a formal scheme, and $\mathcal{I}$ an ideal sheaf of $\mathcal{O}_X$. We say that $\mathcal{I}$ is an ideal of definition of $X$, if for any $x \in X$ there exists an affine open neighborhood $U \cong \text{Spf} \, A$ of $x$ such that $\mathcal{I}|_U$ is isomorphic to an ideal of definition of $\text{Spf} \, A$ as in (1). An ideal of definition $\mathcal{I}$ is said to be of finite type if it is of finite type as an $\mathcal{O}_X$-module.

**Proposition 1.1.19** ([54], I, (10.3.5)). Any ideal of definition of $\text{Spf} \, A$ is of the form $I^\Delta$ for a uniquely determined ideal of definition $I$ of $A$.

Thus, if $\mathcal{I}$ is an ideal of definition of a formal scheme $X$ and $X = \bigcup_{\alpha \in L} U_\alpha$ is an affine open covering with $U_\alpha \cong \text{Spf} \, A_\alpha$ for each $\alpha \in L$, then $\mathcal{I}|_{U_\alpha} \cong I^\Delta_\alpha$ for $\alpha \in L$, where $I_\alpha \subseteq A_\alpha$ is an ideal of definition of the admissible ring $A_\alpha$. In particular, the locally ringed space $(X, \mathcal{O}_X/\mathcal{I})$ is a scheme, which has $(U_\alpha, \mathcal{O}_U/\mathcal{I}|_{U_\alpha}) = \text{Spec} \, A/I_\alpha$ for each $\alpha \in L$ as an affine open covering.

In this setting, a collection $\{I^{(\lambda)}\}_{\lambda \in \Lambda}$ of ideals of definition of $X$ indexed by a directed set is called a fundamental system of ideals of definition of $X$, if for each $\alpha \in L$ the collection of ideals $\{I^{(\lambda)}_\alpha\}_{\lambda \in \Lambda}$ given by $I^{(\lambda)}|_{U_\alpha} \cong (I^{(\lambda)}_\alpha)^\Delta$ for each $\lambda \in \Lambda$ is a fundamental system of ideals of definition of $A_\alpha$ (cf. [54], I, (10.3.7) and (10.5.1)).

**Proposition 1.1.20.** Let $A$ be an adic ring of finite ideal type (1.1.6) and $I \subseteq A$ a finitely generated ideal of definition. Then

$$I^\Delta = I \mathcal{O}_X.$$

**Proof.** For any $f \in A$ the ring $A_{\{f\}}$ coincides with the $IA_f$-adic completion of $A_f$ (0.7.2.15). Then by 0.7.2.9 we see that the closure of the image of $IA_f$ in $A_{\{f\}}$ coincides with $IA_{\{f\}}$. The assertion follows from this.

**Corollary 1.1.21.** Under the assumptions of 1.1.20, the associated ideal of definition $I^\Delta$ of $X = \text{Spf} \, A$ is a sheaf of ideals of $\mathcal{O}_X$ of finite type and, moreover,

$$(I^\Delta)^n = (I^n)^\Delta$$

for any $n \geq 1$ (cf. [54], I, (10.3.6)). In particular, $\{I^n\}_{n \geq 1}$ with $I = I^\Delta$ gives a fundamental system of ideals of definition of $X = \text{Spf} \, A$.

**Proposition 1.1.22.** Let $A$ be an adic ring of finite ideal type and $\mathcal{I}$ an ideal of definition of finite type on $X = \text{Spf} \, A$. Then there exists a finitely generated ideal of definition $I \subseteq A$ such that $\mathcal{I} = I^\Delta$.

The following lemma is useful not only for proving this proposition, but also in the sequel.
Lemma 1.1.23. Let $X$ be a formal scheme and $\{I^{(\lambda)}\}_{\lambda \in \Lambda}$ a fundamental system of ideals of definition, and suppose that the directed set $\Lambda$ has a final and at most countable subset. Let $\{F_\lambda\}_{\lambda \in \Lambda}$ be a projective system of $\mathcal{O}_X$-modules such that

(a) for any $\lambda \in \Lambda$ we have $I^{(\lambda)}F_\lambda = 0$, and $F_\lambda$ is a quasi-coherent sheaf on the scheme $X_\lambda = (X, \mathcal{O}_X/I^{(\lambda)})$ and

(b) the projective system $\{F_\lambda\}_{\lambda \in \Lambda}$ is strict, that is, all transitions maps $F_\mu \to F_\lambda$

for $\lambda \leq \mu$ are surjective.

(1) For $q \geq 1$ we have

$$\lim_{\lambda \in \Lambda}^{(q)} F_\lambda = 0.$$

(2) If, moreover, $X$ is affine, then for $q \geq 1$ we have

$$H^q(X, F) = 0.$$

Proof. In view of 0.3.2.13 (2) the projective system $\{F_\lambda\}_{\lambda \in \Lambda}$ satisfies (E1) and (E2) in 0, §3.2.(e). Hence, (1) follows from 0.3.2.14 (1). To show (2), we first note that for $\lambda \leq \mu$ the surjection $F_\mu \to F_\lambda$ of quasi-coherent sheaves on the affine scheme $X_\mu = \text{Spec} \ A/I^{(\mu)}$ (where $(I^{(\mu)})^\Delta = I^{(\mu)}$) induces the surjective map $\Gamma(X, F_\mu) \to \Gamma(X, F_\lambda)$; that is, the projective system $\{\Gamma(X, F_\lambda)\}_{\lambda \in \Lambda}$ is strict. Hence the assertion follows from 0.3.2.16 and 0.5.4.2 (1).

Proof of Proposition 1.1.22. Take the unique $I \subseteq A$ such that $I^\Delta = I$ (1.1.19). We want to show that $I$ is finitely generated. Take a finitely generated ideal of definition $J \subseteq A$ such that $J \subseteq I$, and set $J = J^\Delta$. Consider the exact sequence

$$0 \to J/J^n \to \overline{I/J^n} \to \overline{I/J} \to 0$$

for any $n > 0$. By 1.1.23 (1), $\overline{I/J} \cong I^\Delta/J^\Delta$, which is a quasi-coherent sheaf on the scheme Spec $A/J$ of finite type. Hence $I/J$ is finitely generated, and thus $I$ is finitely generated, as desired.

Corollary 1.1.24. Let $X$ be an adic formal scheme of finite ideal type and $I$ an ideal of definition of finite type. Then $\{I^n\}_{n \geq 1}$ is a fundamental system of ideals of definition of $X$.

Proof. Obviously, we may assume that $X$ is affine of the form $X = \text{Spf} \ A$, where $A$ is an adic ring of finite ideal type. Then, by 1.1.22, $I$ is of the form $I^\Delta$ for a finitely generated ideal of definition $I \subseteq A$. The assertion in this case has already been shown in 1.1.21.
1.1. (e) Noetherian formal schemes

**Definition 1.1.25** (cf. [54], I, (10.4.2)). A formal scheme $X$ is said to be *locally Noetherian* if it has an affine open covering $X = \bigcup_{\alpha \in \Lambda} U_{\alpha}$ such that each $U_{\alpha}$ is isomorphic to $\text{Spf} \ A_{\alpha}$ for a Noetherian adic ring $A_{\alpha}$. A locally Noetherian formal scheme is said to be *Noetherian* if the underlying topological space is quasi-compact.

Thus any locally Noetherian formal scheme is, by definition, an *adic* formal scheme of finite ideal type. One of the most remarkable properties of locally Noetherian formal schemes is that they always have a global ideal of definition.

**Proposition 1.1.26** ([54], I, (10.5.4)). Any locally Noetherian formal scheme $X$ has a unique ideal of definition $I$ of finite type such that the induced scheme $(X, \mathcal{O}_X/I)$ is reduced.

1.2 Fiber products

1.2. (a) **Complete tensor product of admissible rings.** Consider the diagram of admissible rings

$$
\begin{array}{c}
B & \xleftarrow{f} & A & \xrightarrow{g} & C, \\
& & \uparrow & & \uparrow \\
& & f & & g \\
& & A & \xleftarrow{g} & C.
\end{array}
$$

and let $\{I^{(\lambda)}\}_{\lambda \in \Lambda}$ (resp. $\{J^{(\alpha)}\}_{\alpha \in \Sigma}$, resp. $\{K^{(\beta)}\}_{\beta \in T}$) be a fundamental system of ideals of definition of $A$ (resp. $B$, resp. $C$). Since $f$ and $g$ are continuous, for any $\alpha \in \Sigma$ and $\beta \in T$ there exists $\lambda \in \Lambda$ such that $I^{(\lambda)}B \subseteq J^{(\alpha)}$ and $I^{(\lambda)}C \subseteq K^{(\beta)}$.

We consider the complete tensor product $B \hat{\otimes}_A C$ sitting in the diagram

$$
\begin{array}{c}
B & \xrightarrow{f} & B \hat{\otimes}_A C \\
& \uparrow & \uparrow \\
A & \xleftarrow{g} & C.
\end{array}
$$

The ring $B \hat{\otimes}_A C$ is the Hausdorff completion of the tensor product $B \otimes_A C$ with respect to the topology defined by the filtration $\{H^{\alpha,\beta}\}_{(\alpha, \beta) \in \Sigma \times T}$, where

$$
H^{\alpha,\beta} = f \otimes g(J^{(\alpha)} \otimes_A C) + f \otimes g(B \otimes_A K^{(\beta)})
$$

for $(\alpha, \beta) \in \Sigma \times T$ (cf. Exercise 0.7.6). Let $\hat{H}^{\alpha,\beta}$ for each $(\alpha, \beta) \in \Sigma \times T$ be the closure of the image of $H^{\alpha,\beta}$ in $B \hat{\otimes}_A C$; then $B \hat{\otimes}_A C$ is Hausdorff complete with respect to the topology defined by the filtration $\{\hat{H}^{\alpha,\beta}\}_{(\alpha, \beta) \in \Sigma \times T}$ (cf. 0, §7.1. (c), 0.7.1.8 (2)). More explicitly,

$$
B \hat{\otimes}_A C = \lim_{(\alpha, \beta) \in \Sigma \times T} B / J^{(\alpha)} \otimes_A C / K^{(\beta)} = \lim_{(\lambda, \alpha, \beta) \in L} B / J^{(\alpha)} \otimes_{A/I^{(\lambda)}} C / K^{(\beta)},
$$
where $\Sigma \times T$ is considered with the ordering defined by

$$(\alpha, \beta) \leq (\alpha', \beta') \iff \alpha \leq \alpha' \text{ and } \beta \leq \beta',$$

and $L$ is the directed set given by

$$L = \{((\lambda, \alpha, \beta) \in \Lambda \times \Sigma \times T : I(\lambda)B \subseteq J(\alpha), I(\lambda)C \subseteq K(\beta)\}$$

considered with the similar ordering; note that the map $L \to \Sigma \times T$ given by the canonical projection is cofinal.

**Proposition 1.2.1** ([54], 0I, (7.7.7)). The topological ring $B \hat{\otimes}_A C$ is admissible, and $\{\hat{H}^{\alpha, \beta}\}_{(\alpha, \beta) \in \Sigma \times T}$ is a fundamental system of ideals of definition.

**Lemma 1.2.2.** If $B$ and $C$ are adic of finite ideal type, then so is $B \hat{\otimes}_A C$. If $J$ (resp. $K$) is a finitely generated ideal of definition of $B$ (resp. $C$) and if we set

$$H^{m,n} = f \otimes g(J^m \otimes_A C) + f \otimes g(B \otimes_A K^n)$$

for $m, n \geq 0$, then $H = H^{1,1}(B \hat{\otimes}_A C)$ is a finitely generated ideal of definition of $B \hat{\otimes}_A C$.

**Proof.** Clearly, the filtration $\{H^{n,n}\}_{n \geq 0}$ gives a fundamental system of neighborhoods of $0$ for the ring $B \otimes_A C$, for the diagonal map $\mathbb{N} \hookrightarrow \mathbb{N}^2$ is cofinal. For $k \geq 0$,

$$(H^{1,1})^{2k} = [f \otimes g(J \otimes_A C) + f \otimes g(B \otimes_A K) \otimes A C]^{2k}$$

$$\subseteq \sum_{i=0}^{2k} f \otimes g(J^{2k-i} \otimes_A C) \cdot f \otimes g(B \otimes_A K^i)$$

$$\subseteq f \otimes g(J^k \otimes_A C) + f \otimes g(B \otimes_A K^k)$$

$$= H^{k,k}.$$

On the other hand, we clearly have $H^{2k,2k} \subseteq (H^{1,1})^{2k}$. Hence

$$H^{2k,2k} \subseteq (H^{1,1})^{2k} \subseteq H^{k,k}$$

holds for any $k \geq 0$, and thus the topology on $B \otimes_A C$ given by $\{H^{m,n}\}_{m,n \geq 0}$ is $H^{1,1}$-adic. Now since $H^{1,1}$ is a finitely generated ideal of definition, the Hausdorff completion $B \hat{\otimes}_A C$ is actually the $H^{1,1}$-adic completion (0.7.2.15), and hence $H = H^{1,1}(B \hat{\otimes}_A C)$ is an ideal of definition (0.7.2.9).
Corollary 1.2.3. Under the assumptions of Lemma 1.2.2, suppose furthermore that $A$ is adic. Let $I$ be an ideal of definition of $A$ such that $IB \subseteq J$ and $IC \subseteq K$. Then

$$B \otimes_A C = \lim_{\rightarrow k} B_k \otimes_{A_k} C_k,$$

where $A_k = A/I^{k+1}$, $B_k = B/J^{k+1}$, and $C_k = C/K^{k+1}$ for $k \geq 0$. Moreover, $B \otimes_A C$ is an adic ring with a finitely generated ideal of definition $H$ generated by the images of $J \otimes_A C \to B \otimes_A C$ and $B \otimes_A K \to B \otimes_A C$, and we have $B \otimes_A C/H^{k+1} \cong B_k \otimes_{A_k} C_k$ for any $k \geq 0$.

1.2. (b) Fiber products of formal schemes

Theorem 1.2.4. The category of formal schemes has fiber products.

As in [54], I, §10.7, the general construction of fiber products reduces to the case of affine formal schemes, and for a diagram

$$\text{Spf } B \to \text{Spf } A \leftarrow \text{Spf } C$$

of affine formal schemes the fiber product is given by $\text{Spf } B \otimes_A C$. The following statements are corollaries of 1.2.2 and 1.2.3.

Corollary 1.2.5. Let $Y \to X \leftarrow Z$ be a diagram of formal schemes, where $Y$ and $Z$ are adic formal schemes of finite ideal type. Then the fiber product $Y \times_X Z$ is an adic formal scheme of finite ideal type.

Corollary 1.2.6. Let $Y \to X \leftarrow Z$ be a diagram of formal schemes, where $Y$ and $Z$ are schemes. Then the fiber product $Y \times_X Z$ is a scheme.

The last corollary shows, in particular, that fiber products of schemes taken in the category of formal schemes coincide with the ones taken in the category of schemes.

1.2. (c) Fiber products and open immersions

Proposition 1.2.7. Let $f : X \to Y$ be a morphism of formal schemes and $V \hookrightarrow Y$ an open immersion. Then the Cartesian diagram

$$V \times_Y X \leftarrow X \quad g \quad \downarrow \quad \downarrow f 
\quad V \hookrightarrow Y$$

of formal schemes remains Cartesian on the underlying topological spaces. In particular,

$$\text{image}(g) = \text{image}(f) \cap V.$$
Chapter I. Formal geometry

Proof. By the construction of fiber products, one reduces to the affine situation $X = \text{Spf } A$, $Y = \text{Spf } B$, and $V = \text{Spf } B_{(h)}$ for some $h \in B$. The assertion in this case is easy to verify.

Proposition 1.2.8. (1) Let \( f: X \to Y \) and \( g: Y \to Z \) be morphisms of formal schemes. If \( f \) and \( g \) are open immersions, then so is \( g \circ f \). If \( g \circ f \) and \( g \) are open immersions, then so is \( f \).

(2) If \( S \) is a formal scheme and if \( f: X \to X' \) and \( g: Y \to Y' \) are two \( S \)-open immersions of formal schemes, then

\[
 f \times_S g: X \times_S Y \longrightarrow X' \times_S Y'
\]

is an open immersion.

(3) If \( S \) is a formal scheme and if \( f: X \to Y \) is an \( S \)-open immersion between formal schemes, then for any morphism \( S' \to S \) of formal schemes the induced morphism

\[
 f_{S'}: X \times_S S' \longrightarrow Y \times_S S'
\]

is an open immersion.

Proof. (1) is clear. By 0.1.4.1, (2) and (3) follow from the special case of (3) with \( Y = S \), which is already shown in 1.2.7.

1.3 Adic morphisms

1.3. (a) Adic morphisms

Definition 1.3.1. A morphism \( f: X \to Y \) of adic formal schemes of finite ideal type is said to be adic if there exists an open covering \( \{V_\alpha\}_{\alpha \in L} \) of \( Y \) and for each \( \alpha \in L \) an ideal of definition \( I_\alpha \) of \( V_\alpha \) of finite type, such that for each \( \alpha \in L \) the pull-back ideal \( I_\alpha \otimes_{f^{-1}(V_\alpha)} I f^{-1}(V_\alpha) \) (0.4.1.2) is an ideal of definition of the open formal subscheme \( f^{-1}(V_\alpha) \subset X \).

By an argument similar to that in [54], I, (10,12,1), \( I_\alpha \otimes_{f^{-1}(V_\alpha)} I f^{-1}(V_\alpha) \) is an ideal of definition of \( f^{-1}(V_\alpha) \) for any ideal of definition of finite type \( I_\alpha \) on \( V_\alpha \). Hence, in particular, if \( Y \) itself has an ideal of definition \( I \) of finite type, then \( I \otimes_X \) is an ideal of definition of \( X \). (This implies, in particular, that our definition of adic morphisms agrees with the one in [54], I, (10,12,1), in the locally Noetherian case.)

Note, cf. 0.1.4.8 (1), that the property ‘adic’ is local on the target under the Zariski topology.

Proposition 1.3.2. Let \( \varphi: A \to B \) be a morphism of adic rings of finite ideal type and \( f: Y = \text{Spf } B \to X = \text{Spf } A \) the induced morphism. Then \( f \) is adic if and only if \( \varphi \) is adic (1.1.5 (2)).
Proof. Suppose \( \varphi \) is adic, and let \( I \subseteq A \) be a finitely generated ideal of definition. Then \( \mathcal{I} = I \mathcal{O}_X \) is an ideal of definition of \( X \) of finite type (1.1.20, 1.1.21). Since \( IB \) is an ideal of definition of \( B \), \( I \mathcal{O}_Y = I \mathcal{O}_Y \) is an ideal of definition, which shows that \( f \) is adic. Conversely, if \( f \) is adic, then \( I \mathcal{O}_Y \) is an ideal of definition. Since \( I \mathcal{O}_Y = (IB)^{\Delta} \), we see that \( IB \) is an ideal of definition (cf. [54], I, (10.3.5)). \( \square \)

Proposition 1.3.3. Let \( X, Y, \) and \( Z \) be adic formal schemes of finite ideal type and \( Y \to X \) and \( Z \to X \) morphisms of formal schemes. Suppose \( Y \to X \) is adic. Then the fiber product \( Y \times_X Z \) is an adic formal scheme of finite ideal type, and the morphism \( Y \times_X Z \to Z \) is adic.

This proposition follows from the following lemma.

Lemma 1.3.4. Under the assumptions of Proposition 1.2.3, suppose that the morphism \( A \to B \) is adic. Then the adic ring \( B \widehat{\otimes}_A C \) has \( H \mathcal{D}_{B \widehat{\otimes}_A C} \) as an ideal of definition, and the map \( C \to B \widehat{\otimes}_A C \) is adic.

Proof. We may assume \( J = IB \). Then the ideal \( H \) of \( B \widehat{\otimes}_A C \) as in 1.2.3 coincides with \( (B \otimes_A K)B \widehat{\otimes}_A C \), which is clearly the one generated by the image of \( K \) under the map \( C \to B \widehat{\otimes}_A C \). \( \square \)

By 1.3.4 and 1.2.3, we have the following corollary.

Corollary 1.3.5. Let \( X \to Y \) and \( Z \to Y \) be morphisms of adic formal schemes of finite ideal type. Suppose that \( X \to Y \) is adic and that there exist ideals of definition \( \mathcal{I} \) and \( \mathcal{K} \) of finite type of \( Y \) and \( Z \), respectively, such that \( \mathcal{I} \mathcal{O}_Z \subseteq \mathcal{K} \). Set \( Y_k = (Y, \mathcal{O}_Y/\mathcal{I}^{k+1}) \), \( X_k = (X, \mathcal{O}_X/\mathcal{I}^{k+1}\mathcal{O}_X) \), and \( Z_k = (Z, \mathcal{O}_Z/\mathcal{K}^{k+1}) \) for \( k \geq 0 \), and \( W = X \times_Y Z \). Then for any \( k \geq 0 \) the scheme \( W_k = (W, \mathcal{O}_W/\mathcal{K}^{k+1}\mathcal{O}_W) \) is isomorphic to the fiber product \( X_k \times_Y Z_k \) of schemes.

Proposition 1.3.6. (1) Let \( f: X \to Y \) and \( g: Y \to Z \) be morphisms of adic formal schemes of finite ideal type. If \( f \) and \( g \) are adic, then so is the composition \( g \circ f \). If \( g \circ f \) and \( g \) are adic, then so is \( f \).

(2) Let \( S \) be an adic formal scheme of finite ideal type, and \( f: X \to X' \) and \( g: Y \to Y' \) two adic \( S \)-morphisms of adic formal schemes of finite ideal type over \( S \). Then

\[
 f \times_S g: X \times_S Y \longrightarrow X' \times_S Y'
\]

is adic.

(3) Let \( S \) be an adic formal scheme of finite ideal type, and \( f: X \to Y \) an adic \( S \)-morphism between adic formal schemes of finite ideal type over \( S \). Then for any morphism \( S' \to S \) of adic formal schemes of finite ideal type,

\[
 f_{S'}: X \times_S S' \longrightarrow Y \times_S S'
\]

is adic.
Proof. (1) is easy to see. Applying 0.1.4.1 to the category of adic formal schemes (with morphisms being not necessarily adic), we deduce that (2) and (3) follows from the special case of (3) with $S = Y$, which was already shown in 1.3.3. 

Note that in the statements (2) and (3) of 1.3.6 the formal schemes $X$, $Y$, $X'$, $Y'$, and $S'$ are adic formal scheme over $S$, but not necessarily adic over $S$, that is, the structural maps such as $X \to S$ are not necessarily adic.

In the sequel we employ the following convention, which remains in force throughout this book, in order to distinguish adicness of morphisms from that of formal schemes alone.

Convention. For adic formal schemes $X$ and $S$ of finite ideal type,

- by ‘$X$ over $S$’ we only mean that $X$ is simply considered with a morphism $X \to S$, not necessarily adic, of formal schemes;
- if, however, we say ‘$X$ adic over $S$’, then we mean that the structural map $X \to S$ is adic.

1.3. (b) Adicness of diagonal maps

Proposition 1.3.7. Let $f : X \to Y$ be a morphism of formal schemes, and suppose that $X$ is adic of finite ideal type. Then the diagonal map

$$\Delta_X : X \longrightarrow X \times_Y X$$

is an adic morphism.

Note that thanks to 1.2.5 the formal scheme $X \times_Y X$ is adic of finite ideal type; note also that here we do not assume that the map $f$ or $Y$ are adic.

Proof. We may assume that $X$ and $Y$ are affine: $X = \text{Spf} \ B \to Y = \text{Spf} \ A$. This places us in the case 1.2.2, where $B = C$. Using the notation therein, we know that $B \hat{\otimes}_A B$ has the ideal $H$ generated by $f \otimes g(J \otimes_A B) + f \otimes g(B \otimes_A J)$ as an ideal of definition. The diagonal map $\Delta_X$ as above is induced by the codiagonal map $B \hat{\otimes}_A B \to B$, and so we clearly have $HB = J$, thereby the assertion. 

1.4 Formal completion

1.4. (a) Formal schemes as inductive limits of schemes. We have already seen in §1.1. (d) that for any formal scheme $X$ and any ideal of definition $I$, the locally ringed space $(X, \mathcal{O}_X / I)$ is a scheme.
Proposition 1.4.1 ([54], I, (10.6.2)). Let $X$ be a formal scheme, and suppose there exists a fundamental system of ideals of definition $\{I^{(\lambda)}\}_{\lambda \in \Lambda}$, and set

$$X_\lambda = (X, \mathcal{O}_X / I^{(\lambda)}), \quad \lambda \in \Lambda.$$ 

Then $\{X_\lambda\}_{\lambda \in \Lambda}$ with the canonical closed immersions is an inductive system of schemes, and

$$X = \lim_{\lambda \in \Lambda} X_\lambda$$

in the category of formal schemes.

Proposition 1.4.2. Let $X$ be a topological space, and consider a projective system $\{\mathcal{O}_i, u_{ij}\}_{i \in I}$ of sheaves of rings on $X$ indexed by a directed set $I \neq \emptyset$ that admits an at most countable final subset. Fix $0 \in I$, and for any $i \geq 0$ let $I_i$ be the kernel of $u_{0i}: \mathcal{O}_i \rightarrow \mathcal{O}_0$. Suppose that

(a) for any $i \geq 0$ the ringed space $(X, \mathcal{O}_i)$ is a scheme,

(b) for any $i \geq 0$ and any $x \in X$ there exists an open neighborhood $U_i$ of $x$ in $X$ such that the sheaf $I_i$ restricted to $U_i$ is nilpotent, and

(c) for any $i \geq j \geq 0$ the morphism $u_{ij}$ is surjective.

Then the ringed space $(X, \mathcal{O})$ equipped with the projective limit sheaf $\mathcal{O} = \lim_{\leftarrow i \in I} \mathcal{O}_i$ is a formal scheme. The canonical morphism $u_i: \mathcal{O} \rightarrow \mathcal{O}_i$ for $i \geq 0$ is surjective. If one denotes by $I^{(i)}$ the kernel of $u_i$, then $\{I^{(i)}\}_{i \geq 0}$ is a fundamental system of ideals of definition, and $I^{(0)}$ coincides with the projective limit $\lim_{\leftarrow i \geq 0} I_i$.

Proof. We may assume without loss of generality that $I = \mathbb{N}$ (Exercise 0.1.2). Then the proposition in this case is nothing but [54], I, (10.6.3).

It is often delicate to judge whether a given formal scheme, given as an inductive limit of schemes as above, is adic or not; the situation is somewhat similar to judging adicness of a given topological ring (0, §7.2. (c)). But similarly to 0.7.2.11, one has a criterion for the property ‘adic of finite ideal type’, as follows.

Proposition 1.4.3. In the situation as in 1.4.2 with $I = \mathbb{N}$ we assume that for $i \geq j$ the kernel of the morphism $u_{ji}$ coincides with $I^{(i)}_{j+1}$ and that $I_1/I_2^2$ is of finite type over $\mathcal{O}_0 = \mathcal{O}_1/I_1$. Then the formal scheme $X$ is adic. If one denotes by $I^{(k)}$ the kernel of the map $\mathcal{O}_X \rightarrow \mathcal{O}_k$ and puts $I = I^{(0)}$, then $I^{(k)} = I^{(k+1)}$ and $I/I^2$ is isomorphic to $I_1$. In particular, $I$ is an ideal of definition of the adic formal scheme $X$.

We refer to [54], I, (10.6.4), for the proof. Note that, in this situation, the ideal of definition $I$ is of finite type, and hence the formal scheme $X$ thus obtained is an adic formal scheme of finite ideal type (Exercise I.1.4).
1.4. (b) **Formal completion of schemes.** Let $X$ be a scheme and $Y \subseteq X$ a closed subscheme. We denote by $I_Y$ the quasi-coherent sheaf of ideals of $\mathcal{O}_X$ that defines $Y$. For each $k \geq 0$ we set $X_k = (Y, \mathcal{O}_X/I_Y^{k+1})$. Then the inductive limit
\[
\hat{X}|_Y = \lim_{\longrightarrow} X_k
\]
is represented by a formal scheme (1.4.2), called the *formal completion of $X$ along $Y* (cf. 0, §8.6 and [54], I, §10.8). The formal scheme $\hat{X}|_Y$ is sometimes denoted simply by $\hat{X}$, when $Y$ is clear from the context.

In the affine situation $X = \text{Spec} A$ and $Y = \text{Spec} A/I$, where $I$ is an ideal of $A$, the formal completion $\hat{X}|_Y$ is nothing but the affine formal scheme $\text{Spf} A_I^{\hat{\bullet}}$, where $A_I^{\hat{\bullet}}$ is the Hausdorff completion of $A$ with respect to the topology defined by $I^\bullet = \{I^n\}_{n \geq 0}$, which is an admissible ring (1.1.4).

The following proposition is clear; cf. 1.1.7.

**Proposition 1.4.4.** Suppose that the closed subscheme $Y$ is of finite presentation. Then the formal scheme $\hat{X} = \hat{X}|_Y$ is adic of finite ideal type and has an ideal of definition $I_Y = I_Y\mathcal{O}_{\hat{X}}$ of finite type.

Here we include a basic fact on the formal completion of morphisms of schemes.

**Proposition 1.4.5 ([53], (10.9.9)).** Let $Y$ be a scheme, $Z \subseteq Y$ a closed subscheme, and $f: X \to Y$ a morphism of schemes. Let $W = f^{-1}(Z)$, and set $\hat{X} = \hat{X}|_W$ and $\hat{Y} = \hat{Y}|_Z$. Then the commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{j} & \hat{X} \\
\downarrow f & & \downarrow \hat{f} \\
Y & \xleftarrow{i} & \hat{Y}
\end{array}
\]
is Cartesian in the category of formal schemes.

The following proposition follows from 1.3.5.

**Proposition 1.4.6.** Let $X \to Y$ and $Z \to Y$ be morphisms of schemes and $W \subseteq Y$ a closed subscheme of $Y$ of finite presentation. Let $\hat{X}$ (resp. $\hat{Y}$, resp. $\hat{Z}$) be the formal completion along the $W$ (resp. $W_X$, resp. $W_Z$). Then there exists a canonical isomorphism
\[
\hat{X} \times_P \hat{Z} \xrightarrow{\sim} X \times_Y Z,
\]
where $X \times_Y Z$ denotes the formal completion of $X \times_Y Z$ along $W_X \times_Y W_Z$. 
1. Formal schemes

1.4. (c) Formal completion of quasi-coherent sheaves. Let $X$ be a scheme, $Y \subseteq X$ a closed subscheme with the defining ideal sheaf $I_Y$, and $F$ a quasi-coherent $\mathcal{O}_X$-module. Then the formal completion of $F$ along $Y$, denoted by $\widehat{F}|_Y$ or by $\widehat{F}$, is defined as $([53], (10.8.2))$

$$\widehat{F}|_Y = \varprojlim_n F \otimes_{\mathcal{O}_X} (\mathcal{O}_X/I^n_Y) = \varprojlim_n F/I^n_Y F.$$ 

This is an $\mathcal{O}_{\widehat{X}|_Y}$-module; we will see later in §3 that $\widehat{F}|_Y$ thus obtained, in case $Y$ is a closed subscheme of finite presentation, is an example of what we will later call adically quasi-coherent sheaves (3.1.5). Note that in the affine situation $X = \text{Spec } A$, $Y = \text{Spec } A/I$, and $F = \widetilde{M}$, we have

$$\Gamma(\widehat{X}|_Y, \mathcal{F}) = \Gamma(Y, \varprojlim_n F/I^n_Y F) = \varprojlim_n M/I^n M = M_f^\wedge.$$ 

**Proposition 1.4.7.** Let $(X, Y)$ be a pseudo-adhesive pair of schemes (0.8.6.5), and set $\widehat{X} = \widehat{X}|_Y$.

1. For any quasi-coherent sheaf $\mathcal{F}$ on $X$ of finite type, the canonical morphism $i^* \mathcal{F} \rightarrow \widehat{F} = \widehat{F}|_Y$,

where $i: \widehat{X} \rightarrow X$ is the canonical morphism, is an isomorphism.

2. The map $i: \widehat{X} \rightarrow X$ of locally ringed spaces is flat, that is, for any $x \in \widehat{X}$ the ring $\mathcal{O}_{\widehat{X}, x}$ is flat over $\mathcal{O}_{X, i(x)}$ (cf. 0.4.1.5 (2)).

**Proof.** Since the question is local, we may assume that $X = \text{Spec } A$, where we are given a pseudo-adhesive pair $(A, I)$ (0.8.6.7), and hence that $\widehat{X} = \text{Spf } \widehat{A}$. Denote the canonical map $A \rightarrow \widehat{A}$ by $j$. 

1. Take the finitely generated $A$-module $M$ such that $\mathcal{F} = \widetilde{M}$. By Exercise I.1.6 and the equality $\widehat{A}_f = \widehat{A}_{\{j(f)\}}$ ([54], 0.1, (7.6.2)), we need to prove the following statement: for any $f \in A$ the canonical morphism $M_f \otimes_{A_f} \widehat{A}_f \rightarrow \widehat{M}_f$ is an isomorphism. Since $M_f$ is finitely generated over $A_f$ and since $A_f = A[T]/(fT - 1)$ is $IA_f$-adically pseudo-adhesive (0.8.5.7), this follows from 0.8.2.18 (1).

2. By 0.8.2.18 (2), the canonical map $j_f: A_f \rightarrow \widehat{A}_f$ is flat. By Exercise I.1.6 and the equality $\widehat{A}_f = \widehat{A}_{\{j(f)\}}$, the map between stalks $\mathcal{O}_{X, i(x)} \rightarrow \mathcal{O}_{\widehat{X}, x}$ is the filtered inductive limit of the maps of the form $j_f$ as above. Since flatness is preserved by filtered inductive limits, the assertion follows. \qed
1.5 Categories of formal schemes

1.5. (a) Notation. First of all, we set

- $\mathbf{F}_s$ = the category of all formal schemes.

This category has fiber products and the final object $\text{Spec} \, \mathbb{Z}$. As mentioned before (1.1.15), the category $\mathbf{F}_s$ contains the category $\mathbf{Sch}$ of schemes as the full subcategory consisting of 0-adic formal schemes. Moreover, it has a strict initial object (the empty scheme $\emptyset = \text{Spec} \, 0$) and disjoint sums (cf. 0.1.4.6 and Exercise I.1.2).

The category $\mathbf{F}_s$ has the following full subcategories:

- $\mathbf{Aff}_s$ = the category of affine formal schemes,
- $\mathbf{AcF}_s$ = the category of adic formal schemes of finite ideal type,
- $\mathbf{AfAcF}_s$ = the category of affine adic formal schemes of finite ideal type.

Note that, here, the condition ‘adic’ is only put on objects and not on morphisms. We set

- $\mathbf{AcF}_s^*$ = the category of adic formal schemes of finite ideal type with adic morphisms,
- $\mathbf{AfAcF}_s^*$ = the category of affine adic formal schemes of finite ideal type with adic morphisms,

and will follow the principle that the superscript ‘*’ always means that the morphisms between the objects under consideration are only adic morphisms.

The categories $\mathbf{Aff}_s$, $\mathbf{AcF}_s$, and $\mathbf{AfAcF}_s$ have fiber products (1.2.5), strictly initial objects, disjoint sums, and final objects. The categories $\mathbf{AcF}_s^*$ and $\mathbf{AfAcF}_s^*$ have fiber products (1.3.3), strictly initial objects, and disjoint sums.

For any formal scheme $S$ we denote by $\mathbf{F}_s S$, $\mathbf{AcF}_s S$, and $\mathbf{AcF}_s^* S$ etc., the respective categories of objects over $S$. Here, even in case $S$ is adic of finite ideal type, objects of the category $\mathbf{AcF}_s S^*$ are not necessarily adic over $S$ (that is, the structural maps $X \to S$ are not assumed to be adic; see the general convention at the end of §1.3. (a)). To specify the categories of adic formal schemes of finite ideal type that are adic over a fixed adic formal scheme $S$ of finite ideal type, we set

- $\mathbf{AcF}_s^*/S$ = the category of adic formal schemes of finite ideal type adic over $S$,
- $\mathbf{AfAcF}_s^*/S$ = the category of affine adic formal schemes of finite ideal type adic over $S$.

In these categories all arrows are automatically adic due to 1.3.6 (1), and hence $\mathbf{AcF}_s^*/S$ and $\mathbf{AfAcF}_s^*/S$ are full subcategories of $\mathbf{F}_s S$, $\mathbf{AcF}_s S$, and $\mathbf{AcF}_s^* S$, etc.
The categories $\mathbf{Fs}_S$, $\mathbf{AfFs}_S$, $\mathbf{AcFs}_S$, $\mathbf{AfAcFs}_S$, $\mathbf{AcFs}^*_S$, and $\mathbf{AfAcFs}^*_S$ have fiber products, strictly initial objects, and disjoint sums; moreover, $\mathbf{Fs}_S$ and $\mathbf{AcFs}^*_S$ have final objects, and $\mathbf{AfFs}_S$ (resp. $\mathbf{AcFs}_S$, resp. $\mathbf{AfAcFs}_S$) has final objects if $S$ is affine (resp. adic of finite ideal type, resp. affine adic of finite ideal type).

1.5. (b) Properties of morphisms in $\mathbf{Fs}$. Consider a property $P$ for morphisms of formal schemes such that

(I) any isomorphism satisfies $P$, and

(C) if $f: X \to Y$ and $g: Y \to Z$ are arrows satisfying $P$, then $g \circ f$ satisfies $P$.

As in §0.1.4.1 we consider the subcategory $\mathcal{D} = \mathcal{D}_P$ of $\mathbf{Fs}$ consisting of morphisms of formal schemes satisfying $P$. Then, as we have seen in §0.1.4.1, the following conditions are equivalent, and when they are fulfilled, we say that the property $P$ is base-change stable:

(B₁) for any $Z$-morphisms $f: X \to Y$ and $g: X' \to Y'$ satisfying $P$ of formal schemes over a formal scheme $Z$, the induced morphism

$$f \times_Z g: X \times_Z Y \to X' \times_Z Y'$$

satisfies $P$;

(B₂) for any $Z$-morphism $f: X \to Y$ satisfying $P$ of formal schemes over a formal scheme $Z$ and for any morphism $Z' \to Z$ of formal schemes, the induced morphism

$$f_{Z'}: X \times_Z Z' \to Y \times_Z Z'$$

satisfies $P$;

(B₃) for any morphism $f: X \to Y$ satisfying $P$ and any morphism $Y' \to Y$ of formal schemes, the induced arrow

$$f_{Y'}: X \times_Y Y' \to Y'$$

satisfies $P$.

1.5. (c) Properties of morphisms in $\mathbf{AcFs}$. Let $P$ be a property of morphisms of adic formal schemes of finite ideal type that satisfies (I) and (C) in §1.5. (b). We assume that $P$ implies that the morphisms in question are adic, that is to say, that the corresponding subcategory $\mathcal{D}_P$ is contained in $\mathbf{AcFs}^*$. Then the conditions in §0.1.4.1, which are equivalent, are written in the following way.
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(B₁) For any adic \( Z \)-morphisms \( f : X \to Y \) and \( g : X' \to Y' \) satisfying \( P \) of adic formal schemes of finite ideal type over an adic formal scheme of finite ideal type \( Z \), the induced morphism

\[
f \times_Z g : X \times_Z Y \to X' \times_Z Y'
\]
satisfies \( P \) (note that \( f \times_Z g \) is automatically adic).

(B₂) For any adic \( Z \)-morphism \( f : X \to Y \) satisfying \( P \) of adic formal schemes of finite ideal type over an adic formal scheme of finite ideal type \( Z \) and for any morphism \( Z' \to Z \) of adic formal schemes of finite ideal type, the induced morphism

\[
f_{Z'} : X \times_Z Z' \to Y \times_Z Z'
\]
satisfies \( P \).

(B₃) For any adic morphism \( f : X \to Y \) satisfying \( P \) and any morphism \( Y' \to Y \) of adic formal schemes of finite ideal type, the induced arrow

\[
f_{Y'} : X \times_Y Y' \to Y'
\]
satisfies \( P \).

Similarly to the warning at the end of §1.3. (a), the morphisms that are not being spelled out to be adic are not assumed to be adic. For example, \( X, Y, X', \) and \( Y' \) in (B₁) are not assumed to be adic over \( Z \).

In the category \( \mathbf{AcFs} \), we can, moreover, consider the following conditions.

(R) For any adic morphism \( f : X \to Y \), where \( Y \) has an ideal of definition \( I \) of finite type, the following conditions are equivalent.

(a) \( f \) satisfies \( P \).

(b) \( f_k \) satisfies \( P \) for any \( k \geq 0 \).

Here, for any integer \( k \geq 0 \) we set \( X_k = (X, \mathcal{O}_X/I^{k+1}) \) and \( Y_k = (Y, \mathcal{O}_Y/I^{k+1}) \) and set \( f_k : X_k \to Y_k \) to be the induced map of schemes.

(F) For any morphism \( f : X \to Y \) of schemes satisfying \( P \) and any closed subscheme \( Z \subseteq Y \) of finite presentation, the induced map between formal completions \( \widehat{f}_{|Z} : \widehat{X}_{|f^{-1}(Z)} \to \widehat{Y}_{|Z} \) satisfies \( P \).

**Proposition 1.5.1.** Let \( P \) be a property that satisfies (R) and is local on the target under the Zariski topology (cf. 0.1.4.8 (1)). Then \( P \) is base-change stable in \( \mathbf{AcFs} \) if and only if the property \( P \) restricted to morphisms of schemes is base-change stable in \( \mathbf{Sch} \).
Proof. The ‘only if’ part is trivial. Suppose the property $P$ restricted to morphisms of schemes is base-change stable in $\textbf{Sch}$. Let $f : X \to Y$ be an adic morphism of adic formal schemes of finite ideal type that satisfies $P$, and $Y' \to Y$ a (not necessarily adic) morphism of adic formal schemes of finite ideal type. Let $Y = \bigcup_{\alpha \in L} U_{\alpha}$ be an open covering of $Y$. Then $Y'$ is covered by open subsets $Y' \times_Y U_{\alpha} = Y'_\alpha$. Since $P$ is local on the target under the Zariski topology, it suffices to show the property $P$ for the base change $f'_\alpha : X'_\alpha \to Y'_\alpha$, where $X'_\alpha = X \times_Y Y'_\alpha$. Hence we may assume that $Y$ has an ideal of definition $I$ of finite type. Similarly, since $Y'$ is covered by open subsets that have ideals of definition of finite type, we may further assume that $Y'$ has an ideal of definition $\mathcal{J}$ of finite type. We can also consider that $Y$ and $Y'$ are affine, and so that $I \mathcal{O}_{Y'} \subseteq \mathcal{J}$.

Now let $f_k : X_k \to Y_k$ for any integer $k \geq 0$ be the induced morphism of schemes as above, and set $Y'_k = (Y', \mathcal{O}_{Y'}/\mathcal{J}^{k+1})$. We have the induced morphism $Y'_k \to Y_k$ for any $k \geq 0$. By 1.3.5, the similarly defined $(X \times_Y Y')_k$ is isomorphic to $X_k \times_{Y_k} Y'_k$, and $(f \times_Y Y')_k = f_k \times_{Y_k} Y'_k$ for any $k \geq 0$. By the assumption, the map $f_k$ satisfies $P$, and hence $(f \times_Y Y')_k = f_k \times_{Y_k} Y'_k$ satisfies $P$. But this implies that $f \times_Y Y'$ satisfies $P$, as desired. \qed

**Proposition 1.5.2.** Let $P$ be a property that satisfies (R) and is base-change stable restricted to the morphisms of schemes. Then $P$ satisfies (F).

**Proof.** Suppose that $f : X \to Y$ is a morphism of schemes that satisfies $P$ and that $Z$ is a closed subscheme of $Y$ of finite presentation. For any $k \geq 0$ the scheme $(\hat{X}|_Z)_k$ (defined as above) is the closed subscheme of $X$ of finite presentation $I_k$, where $I$ is the defining ideal of $Z$ in $X$. By the base-change stability for morphisms of schemes, each $(\hat{f}|_Z)_k$ satisfies $P$. Then (R) implies that $\hat{f}|_Z$ satisfies $P$. \qed

### 1.5. (d) Adicalization.

Let $P$ be a property of morphisms of schemes that satisfies (I) and (C) in §1.5. (b) and is stable under the Zariski topology (0.1.4.8 (1)). Let $f : X \to Y$ be a morphism of adic formal schemes of finite ideal type. Then we say that $f$ satisfies adically $P$ if

(a) $f$ is adic and

(b) there exists an open covering $Y = \bigcup_{\alpha \in L} V_\alpha$ and for each $\alpha$ an ideal of definition of finite type $I_\alpha$ on $V_\alpha$, such that for any $\alpha \in L$ and $k \geq 0$ the induced morphism of schemes

$$U_{\alpha,k} = (U_\alpha, \mathcal{O}_{U_\alpha}/I_\alpha^{k+1} \mathcal{O}_{U_\alpha}) \longrightarrow V_{\alpha,k} = (V_\alpha, \mathcal{O}_{V_\alpha}/I_\alpha^{k+1})$$

satisfies $P$, where $U_\alpha = f^{-1}(V_\alpha)$.
Proposition 1.5.3. Let \( f : X \to Y \) be an adic morphism of adic formal schemes of finite ideal type. Then \( f \) satisfies adically \( P \) if and only if for any open subspace \( V \subseteq Y \) and any ideal of definition of finite type \( \mathcal{I} \) on \( V \), the induced morphism

\[
U_0 = (U, \mathcal{O}_U / \mathcal{I} \mathcal{O}_U) \longrightarrow V_0 = (V, \mathcal{O}_V / \mathcal{I})
\]

of schemes satisfies \( P \), where \( U = f^{-1}(V) \).

Proof. The ‘if’ part is clear. Suppose \( f \) satisfies adically \( P \). Then we have an open covering \( Y = \bigcup_{\alpha \in L} V_\alpha \) as above. Take for any \( \alpha \in L \) a sufficiently large \( k \geq 0 \) such that \( \mathcal{I}_\alpha^{k+1} \subseteq \mathcal{I} \) on \( V \cap V_\alpha \). Then on \( V \cap V_\alpha \) the morphism \( U_0 \to V_0 \) in question is obtained by base change of \( U_\alpha,k \to V_\alpha,k \); since \( P \) is stable under the Zariski topology, we deduce that \( U_0 \to V_0 \) satisfies \( P \). \( \square \)

Proposition 1.5.4. The property ‘adically \( P \)’ satisfies \((I), (C)\) (in the category \( \text{AcFs} \)), \( (R) \), and \( (F) \). In particular, it is base-change stable, if the property \( P \) is base-change stable.

Proof. It is clear that \((I)\) is satisfied. By 1.5.3, \((C)\) and \((R)\) are also satisfied. The other assertion follows from 1.5.1 and 1.5.2. \( \square \)

In some cases, one can drop ‘adically’ from ‘adically \( P \)’ without major problems; for example, ‘adically of finite type’ is, due to 1.7.3 below, the same as what we appropriately call ‘of finite type’. A similar example is ‘adically affine’, which will turn out to be just ‘affine (and adic)’ (§4.1.\((d)\)). But in some other cases, it is important to distinguish ‘adically \( P \)’ from ‘\( P \)’, as in the following examples.

Examples 1.5.5. (1) \( P = \text{‘flat’} \): adically flat morphisms will be of essential importance, since ‘flat’ in formal geometry in general is not such a reasonable notion. Adically flat morphisms are discussed in more detail in §4.8.\((c)\) below.

(2) \( P = \text{‘quasi-affine’} \): adically quasi-affine morphisms will be of technical importance in §6.3.\((a)\), since for them adically flat effective descent holds.

1.6 Quasi-compact and quasi-separated morphisms

1.6.\((a)\) Quasi-compact morphisms and some preliminary facts on diagonal morphisms

Definition 1.6.1. A morphism \( f : X \to Y \) of formal schemes is said to be quasi-compact if, for any quasi-compact open subset \( U \) of \( Y \), \( f^{-1}(U) \) is quasi-compact.
Here, observe (cf. 0.2.1.4 (2)) that ‘quasi-compactness’ is a topological condition. In particular, if \( f : X \to Y \) is an adic morphism between adic formal schemes of finite ideal type and \( Y \) has an ideal of definition \( I \) of finite type, then \( f \) is quasi-compact if and only if the induced morphism of schemes

\[
f_0 : X_0 = (X, \mathcal{O}_X / I \mathcal{O}_X) \longrightarrow Y_0 = (Y, \mathcal{O}_Y / I)
\]

is quasi-compact.

In order to define quasi-separated morphisms, we need to prove some preparatory results.

**Lemma 1.6.2.** Let \( f : X \to Y \) be a morphism of formal schemes, and consider the diagonal map \( \Delta_X : X \to X \times_Y X \). Then for any open immersion \( U \hookrightarrow X \) the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\Delta_U} & U \times_Y U \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Delta_X} & X \times_Y X
\end{array}
\]

is Cartesian in the category of formal schemes.

**Proof.** Let \( W = X \times_{(X \times_Y X)} (U \times_Y U) \). Since \( U \times_Y U \to X \times_Y X \) is an open immersion (1.2.8 (2)), so is \( W \to X \). We have a morphism \( \varphi : U \to W \) such that the resulting diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\Delta_U} & U \times_Y U \\
\downarrow \varphi & & \downarrow \\
W & \xrightarrow{\Delta_X} & X \times_Y X
\end{array}
\]

commutes. Then by 1.2.8 (1) the morphism \( \varphi \) is an open immersion. Consider the compositions

\[
W \longrightarrow U \times_Y U \xrightarrow{\text{pr}_i} U,
\]

for \( i = 1, 2 \), and denote them by \( \psi_i \). If we denote by \( \alpha \) the open immersion \( W \hookrightarrow X \), then we have \( \alpha \circ \varphi \circ \psi_i = \alpha \). Then by 1.2.8 (1) the morphism \( \psi_i \) is an open immersion. But since \( \psi_i \circ \varphi = \text{id}_U \), \( \varphi \) must be an isomorphism. \( \square \)

**Corollary 1.6.3.** Let \( f : X \to Y \) be a morphism of formal schemes. Then the diagonal map \( \Delta_X : X \to X \times_Y X \) maps the underlying topological space of \( X \) homeomorphically onto its image \( \Delta_X(X) \) endowed with the subspace topology induced by the topology on \( X \times_Y X \).
Proof. Since \( \text{pr}_1 \circ \Delta X = \text{id}_X \), the diagonal map \( \Delta X \) is clearly injective. Hence it suffices to show that for any open subset \( U \subseteq X \) the image \( \Delta_X(U) \) is open in \( \Delta_X(X) \). By 1.6.2 and 1.2.7, identifying \( U \times_Y U \) with the open subset of \( X \times_Y X \) by the open immersion \( U \times_Y U \to X \times_Y X \), we have

\[
\Delta_X(U) = \Delta_U(U) = \Delta_X(X) \cap U \times_Y U,
\]

whence the result. \( \square \)

Corollary 1.6.4. The following conditions for a morphism \( f: X \to Y \) of formal schemes are equivalent.

(a) The diagonal morphism \( \Delta: X \to X \times_Y X \) is quasi-compact.

(b) The inclusion \( \Delta(X) \hookrightarrow X \times_Y X \) of the underlying topological spaces is quasi-compact (0.2.1.4 (2)).

1.6. (b) Quasi-separated morphisms and coherent morphisms

Definition 1.6.5. A morphism \( f: X \to Y \) of formal schemes is said to be quasi-separated if it satisfies one of the (equivalent) conditions in 1.6.4. A formal scheme \( X \) is said to be quasi-separated if it is quasi-separated over \( \text{Spec} \mathbb{Z} \).

Definition 1.6.6. A quasi-compact and quasi-separated morphism (resp. formal scheme) is said to be coherent.

For example, any affine formal scheme \( X = \text{Spf} A \) is coherent. Indeed, it is quasi-compact, as it is a closed subset of the affine scheme \( \text{Spec} A \); since the diagonal map \( \Delta: X \to X \times_Z X \) comes from the surjective map \( A \otimes_{\mathbb{Z}} A \to A \), one can easily show that it is quasi-compact (by an argument similar to that in 1.3.7, one can show that this map satisfies the assumption in Exercise I.1.1).

Proposition 1.6.7. (1) The composition of two quasi-compact (resp. quasi-separated, resp. coherent) maps of formal schemes is quasi-compact (resp. quasi-separated, resp. coherent).

(2) If \( f: X \to X' \) and \( g: Y \to Y' \) are two quasi-compact (resp. quasi-separated, resp. coherent) \( S \)-morphisms of formal schemes, then

\[
f \times_S g: X \times_S Y \to X' \times_S Y'
\]
is quasi-compact (resp. quasi-separated, resp. coherent).
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(3) If \( f : X \to Y \) is a quasi-compact (resp. quasi-separated, resp. coherent) \( S \)-morphism of formal schemes and \( S' \to S \) is a morphism, then

\[
f_{S'} : X \times_S S' \to Y \times_S S'
\]

is quasi-compact (resp. quasi-separated, resp. coherent).

(4) If the composition \( g \circ f \) of two morphisms of formal schemes is quasi-compact and \( f \) is surjective, then \( g \) is quasi-compact. If \( g \circ f \) is quasi-compact and \( g \) is quasi-separated, then \( f \) is quasi-compact. If \( g \circ f \) is quasi-separated, then \( f \) is quasi-separated. If, moreover, \( f \) is quasi-compact and surjective, then \( g \) is quasi-separated.

Proof. First let us prove (1), (2), and (3) for quasi-compactness. (1) is clear. As we have seen in §1.5. (b), (2) and (3) follow from the special case of (3) with \( S = Y \). Let \( f : X \to Y \) be quasi-compact, and \( g: Y' \to Y \) any morphism. We set \( f' : X' \to Y' \) to be the base change; we want to show that \( f' \) is quasi-compact. Let \( U' \subseteq Y' \) be a quasi-compact open subset, and \( s' \in U' \) a point. Take an affine neighborhood \( T \) in \( Y \) of the point \( g(s') \) and an affine neighborhood \( W \) in \( U' \cap g^{-1}(T) \) of \( s' \). Since it suffices to show that \( f^{-1}(W) \) is quasi-compact, we may assume \( Y \) and \( Y' \) are affine; moreover, it is enough to show that the formal scheme \( X' = X \times_Y Y' \) is quasi-compact. Since \( X \) is quasi-compact, it is covered by a finite collection of affine open sets \( V_1, \ldots, V_n \). Then \( X' \) is covered by the affine open subsets \( V_i \times_Y Y' \) (1.2.8 (3)), and thus \( X' \) is quasi-compact, as desired.

Now we proceed to show (1), (2), and (3) for quasi-separatedness. Let both \( f : X \to Y \) and \( g: Y \to Z \) be quasi-separated morphisms of formal schemes. We have the commutative diagram with the Cartesian square

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta_g} & X \times Y \\
\downarrow{\Delta_f} & & \downarrow{g} \\
X \times_Z X & \xrightarrow{f \times_Z f} & Y \times_Z Y
\end{array}
\]

Since \( \Delta_g \) is quasi-compact, the upper arrow in the square is quasi-compact. Since \( \Delta_f \) is quasi-compact, the composition \( \Delta_{g \circ f} \) is quasi-compact, which implies that \( g \circ f \) is quasi-separated. Thus (1) for quasi-separatedness is proved.
Similarly to the quasi-compactness case, (2) and (3) follow from the special case of (3) with \( S = Y \). Let \( f : X \to Y \) be quasi-separated, and let \( g : Y' \to Y \) be a morphism, and \( f' : X' \to Y' \) the base change. We have the commutative diagram

\[
\begin{array}{ccc}
X' \times_{Y'} X' & \xrightarrow{(X \times_Y X) \times_Y Y'} & X \times_Y X \\
\Delta_{f'} & & \Delta_f \\
X' & \xrightarrow{X \times_Y Y'} & X.
\end{array}
\] (**)

Since \( \Delta_f \) is quasi-compact, \( \Delta_{f'} = \Delta_f \times_Y Y' \) is quasi-compact by (3) for quasi-compactness, which we have already shown; that is, \( f' \) is quasi-separated, as desired.

Finally let us prove (4). The first assertion is easy to see; the second one can be shown similarly to [53], (6.1.5) (v). Let \( f : X \to Y \) and \( g : Y \to Z \) be such that \( g \circ f \) is quasi-separated. The morphism \( f \) coincides with the composition

\[
X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y,
\]

where \( \Gamma_f \) is the graph of \( f \). The projection \( p_2 \) coincides with \((g \circ f) \times_Z \text{id}_Y\), and hence is quasi-separated. The diagonal morphism of the graph \( \Gamma_f \) is isomorphic to \( \text{id}_X \), and hence \( \Gamma_f \) is quasi-separated. Thus \( f = p_2 \circ \Gamma_f \) is quasi-separated. Suppose that \( g \circ f \) is quasi-separated and that \( f \) is quasi-compact and surjective. In diagram (*) above we know that the maps \( \Delta_{g \circ f} \) and \( f \times_Z f \) are quasi-compact. Hence the composition \( \Delta_g \circ q \circ \Delta_f = \Delta_g \circ f \) is quasi-compact. Since \( f \) is surjective, we deduce that \( \Delta_g \) is quasi-compact, that is, \( g \) is quasi-separated.

**Proposition 1.6.8.** Let \( f : X \to Y \) be a morphism of formal schemes, and let \( Y = \bigcup_{\alpha \in L} V_\alpha \) be an open covering of \( Y \). Then \( f \) is quasi-compact (resp. quasi-separated, resp. coherent) if and only if the induced map

\[
f_\alpha : X_\alpha = X \times_Y V_\alpha \to V_\alpha
\]

is quasi-compact (resp. quasi-separated, resp. coherent) for any \( \alpha \in L \).

**Proof.** The ‘only if’ part follows from 1.6.7 (3). Suppose \( f_\alpha \) is quasi-compact for all \( \alpha \in L \), and let \( V \subset Y \) be a quasi-compact open subset of \( Y \). Then \( V \) is covered by finitely many quasi-compact open subsets, each of which is contained in some \( V_\alpha \). Thus \( f^{-1}(V) \) is covered by finitely many quasi-compact open subsets and hence is quasi-compact.
Next, suppose $f_\alpha$ is quasi-separated for all $\alpha \in L$. Since the diagonal map

$$\Delta_{X \times_Y V_\alpha} : X \times_Y V_\alpha \to (X \times_Y V_\alpha) \times_{V_\alpha} (X \times_Y V_\alpha) \cong (X \times_Y X) \times_Y V_\alpha$$

coincides with the base change of $\Delta_X$ by the open immersion

$$(X \times_Y X) \times_Y V_\alpha \hookrightarrow X \times_Y X$$

(1.6.2), the assertion follows from the assertion on quasi-compactness.

\begin{proposition}
\textbf{1.6.9.} Let $f : X \to Y$ be an adic morphism of adic formal schemes of finite ideal type, and suppose $Y$ has an ideal of definition $I$ of finite type. For any $k \geq 0$ set $X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X)$ and $Y_k = (Y, \mathcal{O}_Y/I^{k+1})$, and let $f_k : X_k \to Y_k$ be the induced morphism of schemes. Then the following conditions are equivalent.

\begin{enumerate}
    \item[(a)] $f$ is quasi-compact (resp. quasi-separated, resp. coherent).
    \item[(b)] $f_k$ is quasi-compact (resp. quasi-separated, resp. coherent) for any $k \geq 0$.
    \item[(c)] $f_0$ is quasi-compact (resp. quasi-separated, resp. coherent).
\end{enumerate}

\textbf{Proof.} The assertion for quasi-compact morphisms is clear. As for quasi-separated morphisms, the assertion follows easily from 1.3.5.

By 1.5.2, we have the following corollary.

\begin{corollary}
\textbf{1.6.10.} Let $f : X \to Y$ be a morphism of schemes, and $Z$ a closed subscheme of $Y$ of finite presentation. If $f$ is quasi-compact (resp. quasi-separated, resp. coherent), then the formal completion (§1.4. (b))

$$\hat{f} : \hat{X}_{f^{-1}(Z)} \to \hat{Y}_{\hat{Z}}$$

is quasi-compact (resp. quasi-separated, resp. coherent).

\end{corollary}

\textbf{1.6. (c) Notation.} In the sequel we write

- $\text{CFS}_S$ = the category of formal schemes coherent over $S$.

We employ the similar rule as in §1.5. (a) for the notations of categories of coherent formal schemes; for example

- $\text{AcCFS}_S$ = the category of adic formal schemes of finite ideal type coherent over $S$;
- $\text{AcCFS}_S^*$ = the category of adic formal schemes of finite ideal type coherent over $S$ with adic $S$-morphisms;
• $\text{AcCFs}^*_S = \text{the category of adic formal schemes of finite ideal type coherent and adic over } S$;

When $S = \text{Spec } \mathbb{Z}$, we denote these categories without reference to $S$; e.g. $\text{CFs} = \text{CFs}_{\text{Spec } \mathbb{Z}}$.

1.7 Morphisms of finite type

Definition 1.7.1. A morphism $f : X \rightarrow Y$ of adic formal schemes of finite ideal type is said to be locally of finite type if

(a) the morphism $f$ is adic (1.3.1), and

(b) there exist an affine open covering $\{V_\alpha\}_{\alpha \in L}$ of $Y$ with $V_\alpha = \text{Spf } B_\alpha$, where each $B_\alpha$ is an adic ring of finite ideal type, and for each $\alpha \in L$ an affine open covering $\{U_{\alpha, \lambda}\}_{\lambda \in A_\alpha}$ of $f^{-1}(V_\alpha)$ with $U_{\alpha, \lambda} = \text{Spf } A_{\alpha, \lambda}$, where each $A_{\alpha, \lambda}$ is an adic ring topologically finitely generated over $B_\alpha$ (0.8.4.1).

The morphism $f$ is said to be of finite type if it is locally of finite type and quasi-compact (1.6.1).

Proposition 1.7.2. (1) An open immersion of adic formal schemes of finite ideal type is locally of finite type.

(2) The composition of two morphisms locally of finite type (resp. of finite type) is again locally of finite type (resp. of finite type). If the composition $g \circ f$ of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ of formal schemes is locally of finite type and $g$ is adic, then $f$ is locally of finite type.

(3) Let $S$ be an adic formal scheme of finite ideal type, and $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ two adic $S$-morphisms of adic formal schemes of finite ideal type. Suppose that $f$ and $g$ are locally of finite type (resp. of finite type). Then

$$f \times_S g : X \times_S Y \longrightarrow X' \times_S Y'$$

is locally of finite type (resp. of finite type).

(4) Let $S$ be an adic formal scheme of finite ideal type, and $f : X \rightarrow Y$ an adic $S$-morphism between adic formal schemes of finite ideal type. Suppose $f$ is locally of finite type (resp. of finite type). Then for any morphism $S' \rightarrow S$ of adic formal schemes of finite ideal type,

$$f_{S'} : X \times_S S' \longrightarrow Y \times_S S'$$

is locally of finite type (resp. of finite type).
Proof. (1) and (2) are easy to check. As we have seen in §1.5 (c), (3) and (4) follow from the special case of (4) with \( S = Y \). To show (4) in that case, we may assume that all formal schemes are affine of the form

\[
f : X = \text{Spf } B \llbracket X_1, \ldots, X_n \rrbracket \longrightarrow Y = \text{Spf } B
\]

and \( Y' = S' = \text{Spf } R \to Y \). Let \( I \) be a finitely generated ideal of definition of \( B \), and \( J \) a finitely generated ideal of definition of \( R \) such that \( IR \subseteq J \). Then we need to prove that the adic ring \( B \llbracket X_1, \ldots, X_n \rrbracket \widehat{\otimes}_B R \) is topologically finitely generated over \( R \). But it is easy to see that the admissible ring in question is isomorphic to \( R \llbracket X_1, \ldots, X_n \rrbracket \), since it is the \( J \)-adic completion of \( B[X_1, \ldots, X_n] \otimes_B R = R[X_1, \ldots, X_n] \).

If \( X \) and \( Y \) are locally Noetherian, it can be shown that our definition of ‘of finite type’ agrees with that of [54], I, (10.13.1), due to [54], I, (10.13.4); that is, ‘of finite type’ means that the induced morphism \( (X, \mathcal{O}_X / (f^{-1}\mathcal{I}) \mathcal{O}_X) \to (Y, \mathcal{O}_Y / \mathcal{J}) \) (where \( \mathcal{J} \) is an ideal of definition of \( Y \)) is of finite type. In fact, we have the following more general result.

**Proposition 1.7.3.** Let \( f : X \to Y \) be an adic morphism of adic formal schemes of finite ideal type. Suppose that \( Y \) has an ideal of definition \( \mathcal{I} \) of finite type. Set \( X_k = (X, \mathcal{O}_X / \mathcal{I}^{k+1} \mathcal{O}_X) \) and \( Y_k = (Y, \mathcal{O}_Y / \mathcal{I}^{k+1}) \) for \( k \geq 0 \), and denote by \( f_k : X_k \to Y_k \) the induced morphism of schemes. Then the following conditions are equivalent.

(a) \( f \) is locally of finite type (resp. of finite type).

(b) \( f_k \) is locally of finite type (resp. of finite type) for \( k \geq 0 \).

(c) \( f_0 \) is locally of finite type (resp. of finite type).

The proposition follows immediately from the following lemma.

**Lemma 1.7.4.** Let \( A \) be an adic ring with a finitely generated ideal of definition \( \mathcal{I} \) and \( B \) an \( IB \)-adically complete \( A \)-algebra. Then the following conditions are equivalent.

(a) the morphism \( \text{Spf } B \to \text{Spf } A \) is of finite type.

(b) the morphism \( \text{Spec } B / IB \to \text{Spec } A / I \) is of finite type.

(c) \( B \) is topologically finitely generated over \( A \).

**Proof.** Implication (c) \( \Rightarrow \) (b) is clear; the converse follows from [54], I, (6.3.3), and 0.8.4.2. Implication (c) \( \Rightarrow \) (a) is also clear. Since for any topologically finitely generated \( A \)-algebra \( C \), \( C / IC \) is an \( (A / I) \)-algebra of finite type, (a) \( \Rightarrow \) (b) follows. \( \square \)
Corollary 1.7.5. Let \( f : X \to Y \) be a morphism of schemes, and \( Z \) a closed subscheme of \( Y \) of finite presentation. If \( f \) is locally of finite type (resp. of finite type), then the formal completion

\[
\hat{f} : \hat{X} |_{f^{-1}(Z)} \to \hat{Y} |_{Z}
\]

is locally of finite type (resp. of finite type).

Exercises

Exercise I.1.1. Let \( \varphi : A \to B \) be a continuous homomorphism between admissible rings. Suppose there exists an ideal of definition \( I \subseteq A \) such that \( IB \) is an ideal of definition of \( B \) (e.g., \( A \) and \( B \) are adic rings of finite ideal type and \( \varphi \) is an adic morphism). Show that the square

\[
\begin{array}{ccc}
\text{Spf } B & \longrightarrow & \text{Spec } B \\
\downarrow & & \downarrow \\
\text{Spf } A & \longrightarrow & \text{Spec } A
\end{array}
\]

is Cartesian in the category of topological spaces.

Exercise I.1.2. Let \( \{X_\lambda\}_\lambda \in \Lambda \) be a collection of formal schemes. Show that the functor \( \text{Fs} \to \text{Sets} \) defined by \( Y \mapsto \prod_{\lambda \in \Lambda} \text{Hom}_{\text{Fs}}(X_\lambda, Y) \) is representable; in other words, the disjoint sum \( \bigsqcup_{\lambda \in \Lambda} X_\lambda \) exists in the category \( \text{Fs} \). Show, moreover, that if each \( X_\lambda \) is adic, then \( \bigsqcup_{\lambda \in \Lambda} X_\lambda \) is adic.

Exercise I.1.3. Let \( X \) be a formal scheme. Show that, if \( I, I' \subseteq \mathcal{O}_X \) are ideals of definition of \( X \), then \( I + I' \) are ideals of definition.

Exercise I.1.4. Let \( X \) be an adic formal scheme and \( I \) an ideal of definition of \( X \). Suppose that \( I/I^2 \) is a quasi-coherent ideal sheaf of finite type on the scheme \((X, \mathcal{O}_X/I)\). Show that \( I \) is of finite type.

Exercise I.1.5. Let \( \{X_\lambda\}_{\lambda \in \Lambda} \) and \( \{Y_\mu\}_{\mu \in M} \) be collections of formal schemes over a fixed formal scheme \( S \), and \( X = \bigsqcup_{\lambda \in \Lambda} X_\lambda \) and \( Y = \bigsqcup_{\mu \in M} Y_\mu \) the respective disjoint sums. Show that the fiber product \( X \times_S Y \) is canonically isomorphic to the disjoint sum of \( \{X_\lambda \times_S Y_\mu\}_{(\lambda, \mu) \in \Lambda \times M} \).

Exercise I.1.6. Let \( (A, I) \) be a pair of finite ideal type, and consider the canonical maps \( j : A \to \hat{A} \) and \( i : \text{Spf } \hat{A} \to \text{Spec } A \). Show that for any \( f \in \hat{A} \) there exists \( g \in A \) such that \( \mathcal{D}(f) = \mathcal{D}(j(g)) = i^{-1}D(g) \).
Exercise I.1.7. Let $S$ be an adic formal scheme with an ideal of definition $\mathcal{I}$ of finite type, and consider the category $\text{AcFs}_{/S}^*$ of adic formal schemes of finite ideal type adic over $S$. For any object $X$ of $\text{AcFs}_{/S}^*$ and any integer $k \geq 0$, we set $X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X) = X \times_S S_k$ and define the functor $\text{AcFs}_{/S}^* \to \text{Sch}_{S_k}$ by $X \mapsto X_k$. Varying $k$, we get the canonical functor

$$\text{AcFs}_{/S}^* \to \varprojlim_k \text{Sch}_{S_k},$$

where the latter category is the 2-categorical limit of the categories $\text{Sch}_{S_k}$ for $k \geq 0$. Show that this functor is fully faithful, that is, for any formal schemes $X, Y$ adic over $S$ the canonical map

$$\text{Hom}_{\text{AcFs}_{/S}^*} (X, Y) \to \lim_k \text{Hom}_{S_k} (X_k, Y_k)$$

is a bijection.

Exercise I.1.8. Let $Y$ be an adic formal scheme of finite ideal type, and $I$ an ideal of definition of finite type of $Y$. Let $Y_k = (Y, \mathcal{O}_Y/I^{k+1})$ for $k \geq 0$. Suppose we are given an inductive system of schemes $\{X_k\}_{k \geq 0}$ over $Y$ such that

- for $k \leq l$ the map $X_k \to X_l$ is a closed immersion whose underlying continuous mapping is a homeomorphism, and
- for $k \leq l$ the kernel of $\mathcal{O}_{X_l} \to \mathcal{O}_{X_k}$ coincides with $I^{k+1}\mathcal{O}_{X_l}$.

Show that $X = \lim_{\to k \geq 0} X_k$ is an adic formal schemes of finite ideal type and that the morphism $X \to Y$ is adic, that is, $I\mathcal{O}_X$ is an ideal of definition of finite type of $X$.

Exercise I.1.9. Let $X$ be a formal scheme, $A$ an admissible ring, and $\{I^{(\lambda)}\}_{\lambda \in \Lambda}$ a fundamental system of ideals of definition on $X$. Consider on the ring $\Gamma(X, \mathcal{O}_X)$ the topology induced by the filtration $\{\Gamma(X, I^{(\lambda)})\}_{\lambda \in \Lambda}$ by ideals. Show that there exists a canonical bijection between the set of all continuous homomorphisms $A \to \Gamma(X, \mathcal{O}_X)$ and the set of all morphisms $X \to \text{Spf} A$ of formal schemes (cf. [54], I, (2.2.4)).

2 Universally rigid-Noetherian formal schemes

In this section, we introduce two new classes of adic formal schemes, the so-called (locally) universally rigid-Noetherian formal schemes and (locally) universally adhesive formal schemes. The former is based on the ring-theoretic notion of ‘topologically universally Noetherian outside $I$’ (0.8.4.3), and the latter on the notion of ‘topologically universally adhesive’ (0.8.5.17). Hence the notion of (locally) universally adhesive formal schemes is more restrictive than that of (locally) universally rigid-Noetherian formal schemes.
rigid-Noetherian formal schemes. As we will see later, this kind of formal schemes enjoy many of the nice properties that locally Noetherian formal schemes possess and thus provide a good generalization of the notion of locally Noetherian formal schemes, which is often too restrictive for developing general rigid geometry.

First in §2.1. (a) we introduce the ring-theoretic notions of t.u. (= topologically universally) rigid-Noetherian rings and t.u. adhesive rings, before giving the definition in §2.1. (b) of (locally) universally rigid-Noetherian formal schemes and universally adhesive formal schemes. The notion of locally universally rigid-Noetherian formal schemes allows one to define locally of finite presentation morphisms, which we discuss in §2.2.

It turns out that the category of (locally) universally adhesive formal schemes contains the category of the so-called admissible formal schemes, which play a central role in the classical rigid geometry. We briefly recall the definition of admissible formal schemes in §2.3. (a), and discuss the interrelations of the several notions of formal schemes introduced so far in §2.3. (b).

### 2.1 Universally rigid-Noetherian and universally adhesive formal schemes

#### 2.1. (a) t.u. rigid-Noetherian rings and t.u. adhesive rings

**Definition 2.1.1.** Let \( A \) be an adic ring of finite ideal type and \( I \subseteq A \) an ideal of definition.

1. The topological ring \( A \) is called a *rigid-Noetherian ring* if it is Noetherian outside \( I \) (0.8.1.5 (1)); if it is, furthermore, topologically universally Noetherian outside \( I \) (0.8.4.3), then it is called a *topologically universally* (abbreviated as t.u.) *rigid-Noetherian ring*.

2. The ring \( A \) is called a *t.u. adhesive ring* if it is \( I \)-adically topologically universally adhesive (0.8.5.17).

It follows immediately from the definition that, if \( A \) is t.u. adhesive (resp. t.u. rigid-Noetherian), then any topologically finitely generated \( A \)-algebra is again t.u. adhesive (resp. t.u. rigid-Noetherian). Since adhesiveness implies Noetherian outside \( I \), it follows that ‘t.u. adhesive’ implies ‘t.u. rigid-Noetherian’:

\[
\text{t.u. adhesive} \implies \text{t.u. rigid-Noetherian} \implies \text{rigid-Noetherian.}
\]

If \( A \) is a t.u. adhesive ring and \( I \subseteq A \) is an ideal of definition, then the pair \((A, I)\) is a complete t.u. adhesive pair. Note that, due to 0.8.2.19, rigid-Noetherian rings are pseudo-adhesive (0.8.5.1). From 1.1.9 and 0.8.4.7 we readily infer the following proposition.
Proposition 2.1.2. Let $A$ be a rigid-Noetherian ring. Then for any $f \in A$ the $I$-adic completion map $A_f \to A_{\{f\}}$ is flat. In particular, the map $\text{Spf} \, A \to \text{Spec} \, A$ of locally ringed spaces is flat (cf. 0.4.1.5 (2)).

Remark 2.1.3. By 0.8.4.8, if $A$ is t.u. rigid-Noetherian ring, then the pair $(A, I)$ is a complete t.u. pseudo-adhesive pair (0.8.5.17). Hence, in particular, a t.u. rigid-Noetherian ring $A$ enjoys the following property:

- for any $n \geq 0$ and $m \geq 0$, any $A$-algebra of the form
  $$A\langle X_1, \ldots, X_n \rangle[Y_1, \ldots, Y_m],$$
  together with the ideal of definition $IA\langle X_1, \ldots, X_n \rangle[Y_1, \ldots, Y_m]$, satisfies (BT) in 0, §8.2 (a) and (AP) in 0, §7.4 (c).

Then by 0.8.2.18 we have that

- if $A$ is t.u. rigid-Noetherian, then any finitely generated $A$-module $M$ is $I$-adically complete and, moreover, any $A$-submodule $N \subseteq M$ is closed in $M$ and $I$-adically complete (0.7.4.18), and

- if $B$ is an $A$-algebra of finite type, then the map $B \to \widehat{B}$ defined by $I$-adic completion is flat.

Examples 2.1.4. (1) Any Noetherian adic ring is t.u. adhesive. Indeed, if $A$ is a Noetherian adic ring with an ideal of definition $I \subseteq A$, then it is $I$-adically universally adhesive (0.8.5.13); if $B$ is an $A$-algebra of finite type (hence Noetherian), then the $I$-adic completion of $B$ is again Noetherian, and hence is $I$-adically universally adhesive.

(2) Let $V$ be an $a$-adically complete valuation ring (of arbitrary height), where $a \in m_V \setminus \{0\}$. By Gabber’s theorem (0.9.2.7), the topological ring $V$ is t.u. adhesive. Hence any topologically finitely generated $V$-algebra, that is, a topological algebra over type (V) (cf. 0, §9), is t.u. adhesive.

Using 2.1.2, one can prove the following proposition similarly to 1.4.7.

Proposition 2.1.5. Let $A$ be a rigid-Noetherian ring and $I \subseteq A$ a finitely generated ideal of definition. Then for $X = \text{Spec} \, A$, $Y = \text{Spec} \, A/I$, and $\widehat{X} = \text{Spf} \, A$, both (1) and (2) in 1.4.7 hold true.

Let us finally mention that the formal fpqc patching principle holds for the properties ‘t.u. rigid-Noetherian’ and ‘t.u. adhesive’.

Proposition 2.1.6. Let $A \to B$ be an adic morphism between adic rings of finite ideal type, and $I \subseteq A$ a finitely generated ideal of definition of $A$. Suppose that for any $k \geq 0$ the induced map $A/I^{k+1} \to B/I^{k+1}$ is faithfully flat. If $B$ is t.u. rigid-Noetherian (resp. t.u. adhesive), then so is $A$. Moreover, in this situation, the map $A\langle X_1, \ldots, X_n \rangle \to B\langle X_1, \ldots, X_n \rangle$ is faithfully flat for any $n \geq 0$. 

Proof. By 0.8.3.10 \( A\langle X_1, \ldots, X_n \rangle \) is Noetherian outside \( IA\langle X_1, \ldots, X_n \rangle \) and the map \( A\langle X_1, \ldots, X_n \rangle \to B\langle X_1, \ldots, X_n \rangle \) is faithfully flat. By 0.8.5.6 (2), if \( B\langle X_1, \ldots, X_n \rangle \) is universally adhesive, then so is \( A\langle X_1, \ldots, X_n \rangle \). \( \square \)

2.1. (b) Universally adhesive and universally rigid-Noetherian formal schemes

Definition 2.1.7. A formal scheme \( X \) is locally universally rigid-Noetherian (resp. locally universally adhesive) if there exists an affine open covering \( X = \bigcup_{\alpha \in L} U_\alpha \) such that each \( U_\alpha \) is isomorphic to \( \text{Spf } A_\alpha \) with \( A_\alpha \) t.u. rigid-Noetherian (resp. t.u. adhesive). If \( X \) is, moreover, quasi-compact, we say that \( X \) is universally rigid-Noetherian (resp. universally adhesive).

Locally universally rigid-Noetherian (resp. locally universally adhesive) formal schemes are, by definition, adic of finite ideal type (1.1.14, 1.1.16). Note also that locally universally adhesive formal schemes are locally universally rigid-Noetherian. Since the properties ‘t.u. adhesive’ and ‘t.u. right-Noetherian’ are closed under topologically finitely generated extension, we have the following result (cf. 1.1.17).

Proposition 2.1.8. Let \( X \) be a locally universally rigid-Noetherian (resp. locally universally adhesive) formal scheme and \( X' \to X \) a locally of finite type morphism of adic formal schemes of finite ideal type. Then \( X' \) is locally universally rigid-Noetherian (resp. locally universally adhesive).

Proposition 2.1.9. An affine formal scheme \( \text{Spf } A \), where \( A \) is an adic ring of finite ideal type, is universally rigid-Noetherian (resp. universally adhesive) if and only if the topological ring \( A \) is t.u. rigid-Noetherian (2.1.1 (1)) (resp. t.u. adhesive (2.1.1 (2))).

Proof. The ‘if’ part is clear. To show the converse, take a finite affine covering \( \prod_{\alpha \in L} \text{Spf } A_\alpha \to \text{Spf } A \) such that each \( A_\alpha \) is t.u. rigid-Noetherian (resp. t.u. adhesive). Applying 2.1.6 to the map \( A \to B = \prod_{\alpha \in L} A_\alpha \) (cf. 0.8.5.6 (1)), we deduce that \( A \) is t.u. rigid-Noetherian (resp. t.u. adhesive), as desired.

Corollary 2.1.10. Let \( A \) be a t.u. rigid-Noetherian (resp. t.u. adhesive) ring, and \( X \) a formal scheme locally of finite type over \( \text{Spf } A \). Then \( X \) is locally universally rigid-Noetherian (resp. locally universally adhesive).

Proposition 2.1.11. Let \( X \xrightarrow{f} Z \xleftarrow{g} Y \) be a diagram of adic formal schemes. Suppose that \( X \) is locally universally rigid-Noetherian (resp. locally universally adhesive) and that \( g \) is locally of finite type. Then the fiber product \( X \times_Z Y \) in the category of formal schemes is locally universally rigid-Noetherian (resp. locally universally adhesive), and the projection \( X \times_Z Y \to X \) is adic.
Proof. The first assertion follows from the fact that $X \times_Z Y$ is locally of finite type over $X$ (1.7.2 (4)). The other assertion follows from 1.3.6 (3).

Examples 2.1.12. (1) Any locally Noetherian formal scheme is locally universally adhesive; this follows from what we have seen in 2.1.4 (1).

(2) Let $V$ be as in 2.1.4 (2), and consider the affine formal scheme $\text{Spf} V$. By 2.1.9, $\text{Spf} V$ is a universally adhesive formal scheme. Hence, by 2.1.10, any formal scheme locally of finite type over $\text{Spf} V$ is universally adhesive.

2.1. (c) Categories of universally rigid-Noetherian formal schemes. We often use the following notations for categories:

- $\text{RigNoeFs} = \text{the category of locally universally rigid-Noetherian formal schemes}$;
- $\text{RigNoeFs}^* = \text{the category of locally universally rigid-Noetherian formal schemes with adic morphisms}$;
- $\text{RigNoeFs}_S^* = \text{the category of locally universally rigid-Noetherian formal schemes adic over } S$.

The similar categories of locally universally adhesive formal schemes are likewise denoted by $\text{AdhFs}$, $\text{AdhFs}^*$, and $\text{AdhFs}_S^*$, respectively.

We also define $\text{RigNoeFs}_S$, $\text{RigNoeFs}_S^*$, $\text{AdhFs}_S$, $\text{AdhFs}_S^*$ etc. in the usual way for any formal scheme $S$; note that in the categories $\text{RigNoeFs}_S^*$ and $\text{AdhFs}_S^*$ all arrows are adic, but neither the base $S$ nor the structural map $X \rightarrow S$ are assumed to be adic. Note also that in $\text{RigNoeFs}_S^*$ and $\text{AdhFs}_S^*$ the base formal scheme $S$, necessarily adic of finite type, is not required to be locally universally adhesive nor locally universally rigid-Noetherian.

We also have the following ‘coherent versions’ of the above categories:

- $\text{RigNoeCFs} = \text{the category of coherent universally rigid-Noetherian formal schemes}$;
- $\text{RigNoeCFs}^* = \text{the category of coherent universally rigid-Noetherian formal schemes with adic morphisms}$;
- $\text{RigNoeCFs}_S^* = \text{the category of coherent universally rigid-Noetherian formal schemes adic over } S$;

and also their ‘affine versions’:

- $\text{AffRigNoeFs} = \text{the category of affine universally rigid-Noetherian formal schemes}$;
- $\text{AffRigNoeFs}^* = \text{the category of affine universally rigid-Noetherian formal schemes with adic morphisms}$;
• \( \text{AfRigNoeFs}_S^* \) = the category of affine universally rigid-Noetherian formal schemes adic over \( S \).

Similar rules of notation are also employed for coherent and affine universally adhesive formal schemes; for example, \( \text{AdhCFs}^* \) denotes the category of coherent universally adhesive formal schemes with adic morphisms, etc.

### 2.2 Morphisms of finite presentation

**Definition 2.2.1.** A morphism \( f: X \to Y \) of locally universally rigid-Noetherian formal schemes is said to be *locally of finite presentation* if

(a) \( f \) is adic (1.3.1) and

(b) there exist an affine open covering \( \{V_\alpha\} \) of \( Y \) with \( V_\alpha = \text{Spf} \ B_\alpha \), where each \( B_\alpha \) is an adic ring of finite ideal type, and for each \( \alpha \) an affine open covering \( \{U_{\alpha\beta}\}_\beta \) of \( f^{-1}(V_\alpha) \) with \( U_{\alpha\beta} = \text{Spf} \ A_{\alpha\beta} \), where each \( A_{\alpha\beta} \) is an adic ring topologically finitely presented over \( B_\alpha \) (0.8.4.1).

The morphism \( f \) is said to be *of finite presentation* if it is locally of finite presentation and quasi-compact (1.6.1).

**Proposition 2.2.2.** (1) An open immersion between locally universally rigid-Noetherian formal schemes is locally of finite presentation.

(2) The composition of two morphisms locally of finite presentation (resp. of finite presentation) is again locally of finite presentation (resp. of finite presentation).

(3) Let \( X, X', Y, \) and \( Y' \) be locally universally rigid-Noetherian formal schemes adic over an adic formal scheme \( S \) of finite ideal type, and let \( f: X \to X' \) and \( g: Y \to Y' \) be two \( S \)-morphisms. Suppose \( X' \times_S Y' \) is locally universally rigid-Noetherian. Then if \( f \) and \( g \) are locally of finite presentation (resp. of finite presentation), so is

\[
\begin{align*}
f \times_S g : X \times_S Y & \longrightarrow X' \times_S Y'.
\end{align*}
\]

(4) If \( S \) is an adic formal scheme of finite ideal type and if \( f: X \to Y \) is an \( S \)-morphism locally of finite presentation (resp. of finite presentation) between locally universally rigid-Noetherian formal schemes adic over \( S \), then for any morphism \( S' \to S \) of adic formal schemes of finite ideal type such that \( Y \times_S S' \) is universally rigid Noetherian,

\[
f_{S'} : X \times_S S' \longrightarrow Y \times_S S'
\]

is locally of finite presentation (resp. of finite presentation).
Proof. For a t.u. rigid-Noetherian ring $A$ and $f \in A$, the ring

$$A_{(f)} \cong A \langle T \rangle / (fT - 1)$$

(1.1.9) is topologically of finitely presentation over $A$, whence (1).

(2) is easy to see.

To show (3), we may work in the affine situation. Let $(R, I)$ be a complete pair, and consider the morphisms of t.u. rigid-Noetherian rings, all adic over $R$,

$$A \rightarrow A\langle X_1, \ldots, X_n \rangle / \alpha, \quad B \rightarrow B\langle Y_1, \ldots, Y_m \rangle / \beta,$$

where $\alpha$ and $\beta$ are finitely generated, hence closed (2.1.3), ideals. We want to show that the closure of the ideal generated by the image of

$$c = \alpha \otimes B\langle Y_1, \ldots, Y_m \rangle + A\langle X_1, \ldots, X_n \rangle \otimes \beta$$

in the ring

$$A\langle X_1, \ldots, X_n \rangle \widehat{\otimes}_R B\langle Y_1, \ldots, Y_m \rangle = (A \widehat{\otimes}_R B)\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m \rangle$$

is finitely generated. But since $A \widehat{\otimes}_R B$ is assumed to be t.u. rigid-Noetherian, the ideal generated by the image of $c$ is already closed (2.1.3), whence (3).

(4) also follows from the affine local observation as follows. Let $S = \text{Spf} \ A$, where $R$ is an adic ring of finite ideal type, and $Y = \text{Spf} \ B$, where $B$ is t.u. rigid-Noetherian and adic over $R$. Let $X = \text{Spf} \ A$ with $A = B\langle X_1, \ldots, X_n \rangle / \alpha$, where $\alpha$ is finitely generated ideal, and $S' = \text{Spf} \ R'$, where $R'$ is an adic ring of finite ideal type that is adic over $R$. We need to prove that the morphism

$$\text{Spf} \ A \widehat{\otimes}_R R' \rightarrow \text{Spf} \ B \widehat{\otimes}_R R'$$

is finitely presented if $B \widehat{\otimes}_R R'$ is t.u. rigid-Noetherian. Set $B' = B \widehat{\otimes}_R R'$. We have $A \widehat{\otimes}_R R' = B'\langle X_1, \ldots, X_n \rangle / \beta$, where $\beta$ is the closure of $\alpha B'\langle X_1, \ldots, X_n \rangle$. Since the ring $A \widehat{\otimes}_R R'$ is t.u. rigid-Noetherian, the ideal $\alpha B'\langle X_1, \ldots, X_n \rangle$ is already closed, and hence $\beta = \alpha B'\langle X_1, \ldots, X_n \rangle$, which is finitely generated. \hfill \Box

**Proposition 2.2.3.** Let $A$ be a t.u. rigid-Noetherian ring, $I \subseteq A$ a finitely generated ideal of definition, and $B$ a topologically finitely generated $A$-algebra. Then the following conditions are equivalent.

(a) The morphism $\text{Spf} \ B \rightarrow \text{Spf} \ A$ is of finite presentation.

(b) The morphism $\text{Spec} \ B/\sqrt{I^{k+1}B} \rightarrow \text{Spec} \ A/\sqrt{I^{k+1}A}$ is of finite presentation for any $k \geq 0$.

(c) $B$ is topologically finitely presented over $A$. 


Corollary 2.2.4. Let $f: X \to Y$ be an adic morphism of locally universally rigid-Noetherian formal schemes. Suppose that $Y$ has an ideal of definition $I$ of finite type. Set $X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X)$ and $Y_k = (Y, \mathcal{O}_Y/I^{k+1})$ for $k \geq 0$, and denote by $f_k: X_k \to Y_k$ the induced morphism of schemes. Then the following conditions are equivalent:

(a) $f$ is locally of finite presentation (resp. of finite presentation).

(b) $f_k$ is locally of finite presentation (resp. of finite presentation) for any $k \geq 0$.

2.3 Relation with other notions

2.3. (a) Admissible formal schemes

Proposition 2.3.1. Let $Y$ be a locally universally adhesive formal scheme and $f: X \to Y$ a morphism locally of finite type (resp. of finite type). Suppose that

(*) there exist an open covering $X = \bigcup_{\alpha \in \Lambda} U_{\alpha}$ and for each $\alpha \in \Lambda$ an ideal of definition $I_{\alpha}$ of finite type on $U_{\alpha}$ such that $\mathcal{O}_{U_{\alpha}}$ is $I_{\alpha}$-torsion free.

Then $f$ is locally of finite presentation (resp. of finite presentation).

Proof. The assertion follows from the following argument. Let $B$ be a t.u. adhesive ring with a finitely generated ideal of definition $I \subseteq B$, and consider $A = B\langle X_1, \ldots, X_n \rangle / \mathfrak{a}$, where $\mathfrak{a}$ is an ideal. Suppose $A$ is $I$-torsion free. Since $B$ is t.u. adhesive, the pair $(B\langle X_1, \ldots, X_n \rangle, IB\langle X_1, \ldots, X_n \rangle)$ is adhesive. Since $A$ is $I$-torsion free, we deduce from 0.8.5.3 that $A$ is finitely presented as a module over $B\langle X_1, \ldots, X_n \rangle$ or, equivalently, that $\mathfrak{a}$ is finitely generated. □

Corollary 2.3.2. Let $V$ be an $a$-adically complete valuation ring (of arbitrary height) where $a \in m_V \setminus \{0\}$, and $X$ a formal scheme locally of finite type over $\text{Spf} V$. Suppose that the structure sheaf $\mathcal{O}_X$ is $a$-torsion free. Then $X$ is locally of finite presentation over $\text{Spf} V$.

Note that a $V$-algebra $A$ is $a$-torsion free if and only if it is flat over $V$ (Exercise 0.6.3). In classical rigid geometry the so-called admissible formal schemes play a central role; cf. [19], §1.

Definition 2.3.3. An admissible formal scheme is a formal scheme $X$ locally of finite type over $\text{Spf} V$, where $V$ is an $a$-adically complete valuation ring (that is $a \in m_V \setminus \{0\}$) of height one, such that the structure sheaf $\mathcal{O}_X$ is $a$-torsion free.
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It follows from 2.3.2 that any admissible formal scheme is locally of finite presentation over $\text{Spf } V$.

**Remark 2.3.4.** Let $V$ be an $a$-adically complete valuation ring (where $a \in m_V \setminus \{0\}$) of arbitrary height, and $X$ a formal scheme locally of finite type over $S = \text{Spf } V$ such that $\mathcal{O}_X$ is $a$-torsion free. Let $p = \sqrt{aV}$ be the associated height-one prime of $V$ (0.6.7.4), and set $V' = V_p$, which is an $a$-adically complete valuation ring of height one (0.9.1.10). Then in view of 0.9.2.4 the base change $X \times_S S'$ (where $S' = \text{Spf } V'$) is an admissible formal scheme.

2.3. (b) Interrelations between the classes. Finally, let us exhibit interrelations between the classes of formal schemes introduced so far. Let $\text{NoeFs}^*$ be the category of locally Noetherian formal schemes with adic morphisms, which is, in fact, a full subcategory of $\text{AdhFs}^*$ (2.1.12 (1)). Hence one can draw the diagram

\[
\begin{array}{ccc}
\text{NoeFs}^* & \hookrightarrow & \text{AdhFs}^* & \hookrightarrow & \text{AcFs}^* & \hookrightarrow & \text{Fs} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\text{Fs}_{\text{fin/DVR}}^* & \hookrightarrow & \text{Fs}_{\text{fin/Val}}^* \\
\end{array}
\]

Here $\text{Fs}_{\text{fin/DVR}}^*$ denotes the full subcategory of $\text{NoeFs}^*$ consisting of formal schemes locally of finite type over a complete discrete valuation ring, and $\text{Fs}_{\text{fin/Val}}^*$ is the full subcategory of $\text{AdhFs}^*$ consisting of formal schemes locally of finite type over an $a$-adically complete valuation ring of arbitrary height (cf. 2.1.12 (2)). Note that the category $\text{RigNoeFs}^*$ of locally universally rigid-Noetherian formal schemes lies between $\text{NoeFs}^*$ and $\text{AdhFs}^*$. Moreover,

- the first two inclusions in the first row are fully faithful, whereas the last one is only faithful;
- only $\text{Fs}$ in (*) has a final object (cf. 1.1.15);

3 Adically quasi-coherent sheaves

From now on, we deal mainly with **adic** formal schemes of **finite ideal type**. On such formal schemes, a reasonable class of $\mathcal{O}_X$-module sheaves is provided by the so-called **adically quasi-coherent** (a.q.c.) **sheaves**, which we are going to discuss in this section. They are complete (3.1.1 (2)) $\mathcal{O}_X$-modules such that the truncated pieces (that is, the sheaves obtained by taking the quotient by the ideals of definition) are quasi-coherent sheaves on the induced schemes. In §3.1 we give the general definition of adically quasi-coherent sheaves and discuss some of their general properties.
In the affine situation, as we will discuss in §3.2, adically quasi-coherent sheaves are obtained by the ‘Δ-construction’ ([54], I, (10.10.1)), that is, the sheaves on \( X = \text{Spf} \; A \) of the form \( M^\Delta \) for an \( A \)-module \( M \). In fact, if \( A \) is an adic ring of finite ideal type (1.1.6), then the Δ-construction gives rise to a categorical equivalence between the category of complete \( A \)-modules and the category of adically quasi-coherent sheaves on \( X = \text{Spf} \; A \) (3.2.8 (2)).

After discussing adically quasi-coherent sheaves as projective limits of quasi-coherent sheaves on schemes in §3.4, we examine the case where the formal schemes are locally universally rigid-Noetherian in §3.5. The notion of adically quasi-coherent sheaves in this particular situation turns out to be much more tractable. For example, the above-mentioned categorical equivalence restricted to finitely generated modules and a.q.c. sheaves of finite type is an exact equivalence if \( X = \text{Spf} \; A \) is universally rigid-Noetherian (3.5.6).

In the final subsection §3.7, we discuss the so-called admissible ideals, which are, so to speak, the sheaf version of \( I \)-admissible ideals (0.8.1.2). This class of ideals contains the ideals of definition and has a lot of nice properties. One of them is the ‘extension property’, which is most significantly shown in 3.7.11 below. In pursuing the extension properties of admissible ideals, we will obtain in §3.7 a technically remarkable result saying that any coherent (1.6.6) adic formal scheme of finite ideal type admits an ideal of definition of finite type (3.7.12), a generalization of the known fact that any Noetherian formal scheme has a coherent ideal of definition (cf. 1.1.26).

### 3.1 Complete sheaves and adically quasi-coherent sheaves

**3.1. (a) Hausdorff completion of \( \mathcal{O}_X \)-modules.** Let \( X \) be an adic formal scheme of finite ideal type, and suppose for the time being that \( X \) has an ideal of definition \( I \) of finite type. In this situation, \( \{I^{k+1}\}_{k \geq 0} \) gives a fundamental system of ideals of definition of \( X \) (1.1.24). For \( k \geq 0 \), we set \( X_k = (X, \mathcal{O}_X/I^{k+1}) \). This is a scheme having the same underlying topological space as \( X \), and \( \{X_k\}_{k \geq 0} \) forms a filtered inductive system of schemes together with the closed immersions \( X_k \hookrightarrow X_l \) as the transition maps for all \( k \leq l \). Moreover, the formal scheme \( X \) coincides with the inductive limit of this system, \( X = \lim_k X_k \) (1.4.1), or equivalently, the structure sheaf \( \mathcal{O}_X \) is the projective limit (cf. 0, §3.2.(c))

\[
\mathcal{O}_X = \lim_k \mathcal{O}_{X_k}
\]

as a sheaf of topological rings on the topological space \( X \).
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For an \( \mathcal{O}_X \)-module \( \mathcal{F} \) we set

\[
\mathcal{F}_k = \mathcal{F} / \mathcal{I}^{k+1} \mathcal{F}
\]

for \( k \geq 0 \). This is an \( \mathcal{O}_{X_k} \)-module on the scheme \( X_k \), and the resulting system \( \{ \mathcal{F}_k \}_{k \geq 0} \) together with the obvious transition morphisms is a filtered projective system of abelian sheaves on the topological space \( X \), which admits the compatible action of the projective system of rings \( \{ \mathcal{O}_{X_k} \}_{k \geq 0} \); that is, each \( \mathcal{F}_k \) is an \( \mathcal{O}_{X_k} \)-module and the transition maps \( \mathcal{F}_l \to \mathcal{F}_k \) \( (k \leq l) \) are compatible with the maps \( \mathcal{O}_{X_l} \to \mathcal{O}_{X_k} \). Consequently, the projective limit

\[
\mathcal{\hat{F}} = \lim_{\longrightarrow} \mathcal{F}_k
\]

has the canonical \( \mathcal{O}_X \)-module structure, and the canonical morphism

\[
i_{\mathcal{F}} : \mathcal{F} \to \mathcal{\hat{F}}
\]

is a morphism of \( \mathcal{O}_X \)-modules. Note that the definition of \( \mathcal{\hat{F}} \) does not depend on the choice of the ideal of definition \( \mathcal{I} \); indeed, the above projective limit coincides with the projective limit \( \lim_{\leftarrow} (\mathcal{F} / \mathcal{J} \mathcal{F}) \), where \( \mathcal{J} \) runs over the filtered set of all ideals of definition on \( X \) ordered by reversed inclusion, and for a fixed ideal of definition \( \mathcal{I} \) of finite type, the collection \( \{ \mathcal{I}^{k+1} \}_{k \geq 0} \) gives a cofinal subset.

By the last-mentioned fact, for a given \( \mathcal{O}_X \)-module \( \mathcal{F} \) we can define \( \mathcal{\hat{F}} \) up to canonical isomorphism even in the case where \( X \) does not have an ideal of definition; indeed, since \( X \) has locally an ideal of definition of finite type, one can define \( \mathcal{\hat{F}} \) locally on \( X \), and then glue together the local version by canonical isomorphisms. The canonical morphism \( \mathcal{F} \to \mathcal{\hat{F}} \) can be likewise defined.

**Definition 3.1.1.** Let \( X \) be an adic formal scheme of finite ideal type, and \( \mathcal{F} \) an \( \mathcal{O}_X \)-module.

1. The above-defined \( \mathcal{O}_X \)-module \( \mathcal{\hat{F}} \) together with the canonical morphism \( i_{\mathcal{F}} : \mathcal{F} \to \mathcal{\hat{F}} \) is called the completion of \( \mathcal{F} \).

2. We say that \( \mathcal{F} \) is complete if \( i_{\mathcal{F}} : \mathcal{F} \to \mathcal{\hat{F}} \) is an isomorphism.

3.1. (b) Adically quasi-coherent (a.q.c.) sheaves

**Lemma 3.1.2.** Let \( X \) be an adic formal scheme of finite ideal type, and \( \mathcal{I} \) an ideal of definition of finite type of \( X \). Let \( \mathcal{F} \) be an \( \mathcal{O}_X \)-module. Then the following conditions are equivalent.

(a) For any \( k \geq 0 \), \( \mathcal{F}_k = \mathcal{F} / \mathcal{I}^{k+1} \mathcal{F} \) is a quasi-coherent sheaf on the scheme \( X_k = (X, \mathcal{O}_X / \mathcal{I}^{k+1}) \).

(b) For any ideal of definition \( \mathcal{J} \) of \( X \), \( \mathcal{F} / \mathcal{J} \mathcal{F} \) is a quasi-coherent sheaf on the scheme \( (X, \mathcal{O}_X / \mathcal{J}) \).
Proof. Implication (b) $\implies$ (a) is trivial. Now suppose that (a) holds. Since the conditions are local on $X$, we may assume that $X$ is affine (hence, in particular, the underlying topological space is quasi-compact). For any ideal of definition $\mathcal{J}$ we can find a positive integer $k$ such that $I^{k+1} \subseteq \mathcal{J}$. Consider the closed immersion $i:(X, \mathcal{O}_X/\mathcal{J}) \hookrightarrow X_k = (X, \mathcal{O}_X/I^{k+1})$. The sheaf $\mathcal{F}/\mathcal{J}\mathcal{F}$ on $(X, \mathcal{O}_X/\mathcal{J})$ coincides with $i^*\mathcal{F}_k$, which is quasi-coherent. 

**Definition 3.1.3.** (1) Let $X$ be an adic formal scheme of finite ideal type, and $\mathcal{F}$ an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is **adically quasi-coherent** (acronym: a.q.c.) if the following conditions are satisfied.

(a) $\mathcal{F}$ is complete.

(b) For any open subset $U \subseteq X$ considered as an open formal subscheme (see §1.1. (c)) and for any ideal of definition $I$ of finite type of $U$, the sheaf $(\mathcal{F}|_U)/I(\mathcal{F}|_U)$ is a quasi-coherent sheaf on the scheme $(U, \mathcal{O}_U/I)$.

(2) An adically quasi-coherent sheaf $\mathcal{F}$ on $X$ is said to be **of finite type** if it is of finite type as an $\mathcal{O}_X$-module.

(3) A **morphism** between adically quasi-coherent sheaves is a morphism of $\mathcal{O}_X$-modules.

By 3.1.2, (b) is equivalent to

(b)' there exist an open covering $X = \bigcup_{\alpha \in L} U_\alpha$ and for each $\alpha \in L$ an ideal of definition $I_\alpha$ of finite type of $U_\alpha$, such that for any $\alpha \in L$ and $k \geq 0$, $(\mathcal{F}|_{U_\alpha})/I_\alpha^{k+1}(\mathcal{F}|_{U_\alpha})$ is a quasi-coherent sheaf on the scheme $(U_\alpha, \mathcal{O}_{U_\alpha}/I_\alpha^{k+1})$.

If $X$ itself has an ideal of definition $I$ of finite type, then again by 3.1.2 the last condition is equivalent to, with the notation as in 3.1.2, $\mathcal{F}_k$ being quasi-coherent on $X_k$ for any $k \geq 0$.

Note that any morphism of adically quasi-coherent sheaves on $X$ is continuous in the following sense. If $X$ has an ideal $I$ of definition of finite type, such a morphism $f: \mathcal{F} \to \mathcal{G}$ induces the morphism $f_k: \mathcal{F}_k \to \mathcal{G}_k$ of quasi-coherent sheaves on $X_k$ for $k \geq 0$ and coincides with the projective limit of $\{f_k\}$.

We denote by $\textbf{AQCoh}_X$ the category of adically quasi-coherent sheaves on an adic formal scheme $X$ of finite ideal type.

**Proposition 3.1.4.** Let $X$ be an adic formal scheme of finite ideal type.

(1) The structure sheaf $\mathcal{O}_X$ is adically quasi-coherent of finite type.

(2) Any ideal of definition $I$ of $X$ is an adically quasi-coherent ideal of $\mathcal{O}_X$.

(3) If $I$ is an ideal of definition and $\mathcal{F}$ is an adically quasi-coherent sheaf, then $I\mathcal{F}$ is again an adically quasi-coherent sheaf.
Proof. (1) is clear. (2) follows from (3) applied to $F = O_X$. To show (3), since the question is local on $X$, we may assume that there exists an ideal of definition $\mathcal{J}$ of finite type such that $I^n \subseteq \mathcal{J} \subseteq I$. We first look at the exact sequence

$$0 \longrightarrow I F / \mathcal{J} \longrightarrow F / \mathcal{J} \longrightarrow I F \longrightarrow 0$$

of $O_X$-modules for $k \geq 1$. Since $\mathcal{F} = J^k C_1 F$ and $\mathcal{F} / I \mathcal{F}$ are quasi-coherent sheaves on the scheme $X_k = (X, O_X / \mathcal{J})$ (3.1.2), we deduce that $I \mathcal{F} / \mathcal{J}$ is quasi-coherent. Then by 1.1.23 (1) we see that

$$0 \longrightarrow \lim_{k \geq 1} I \mathcal{F} / \mathcal{J} \mathcal{F} \longrightarrow I \mathcal{F} / I \mathcal{F} \longrightarrow 0$$

is exact and hence that $I \mathcal{F} = \lim_{k \geq 1} I \mathcal{F} / \mathcal{J} \mathcal{F}$. This shows that $I \mathcal{F}$ is complete, for we have $\mathcal{J} \mathcal{F} \subseteq \mathcal{J} \mathcal{F} \subseteq \mathcal{J} I \mathcal{F}$ for any $k \geq 1$. \qed

The following proposition is clear.

**Proposition 3.1.5.** Let $X$ be a scheme, and $Y \subseteq X$ a closed subscheme of finite presentation. Consider the formal completion $\hat{X}|_Y$ (cf. 1.4.4). If $\mathcal{F}$ is a quasi-coherent sheaf on $X$, then its formal completion $\hat{\mathcal{F}}|_Y$ (§1.4. (c)) is a.q.c.

### 3.2 A.q.c. sheaves on affine formal schemes

**3.2. (a) $\Delta$-sheaves.** We first recall the definition of the sheaf of $O_X$-modules $M^\Delta$ on an adic formal scheme $X = \text{Spf } A$ associated to an $A$-module $M$ ([54], I, (10.10.1)). Let $I \subseteq A$ be an ideal of definition, and set $X_k = \text{Spec } A / I^{k+1}$ for $k \geq 0$; we have $X = \lim_{\rightarrow k} X_k$ (1.4.1). For an $A$-module $M$, $M_k = M / I^{k+1} M$ defines the quasi-coherent sheaf $\hat{M}_k$ on the scheme $X_k$. Then

$$M^\Delta = \lim_{\rightarrow k} \hat{M}_k,$$

which is a sheaf of $O_X$-modules on $X$. Here is an alternative definition of $M^\Delta$, equivalent up to canonical isomorphism to the above one: the sheaf $M^\Delta$ is the formal completion $\hat{M}|_{X_0}$ (§1.4. (c)) of the quasi-coherent sheaf $\hat{M}$ on $\text{Spec } A$ along the closed subscheme $X_0 = \text{Spec } A / I$, that is, the sheaf $\lim_{\rightarrow k} \hat{M} / I^{k+1} \hat{M}$ restricted to the topological space $X$.

Note that for any open ideal $J \subseteq A$, the sheaf $J^\Delta$ thus constructed coincides with the one given in §1.1. (d); indeed, if $I \subseteq A$ is an ideal of definition contained in $J$, then, since $I^{k+1} J \subseteq I^{k+1} \subseteq I^k J$ for any $k \geq 0$, we have $\lim_{\rightarrow k} J / I^{k+1} \cong \lim_{\rightarrow k \geq 0} J / I^{k+1} J$. 
The construction of $M^\Delta$ induces an additive functor

$$\Delta: \text{Mod}_A \to \text{Mod}_X, \quad M \mapsto M^\Delta,$$

from the category of $A$-modules to the category of $\mathcal{O}_X$-modules.

**Proposition 3.2.1.** We have

$$\Gamma(X, M^\Delta) = M_f^\wedge,$$

where $M^\wedge_f$ denotes the Hausdorff completion of $M$ with respect to the $I$-adic topology (0, §7.1 (c)). More generally, for an affine open set $U = \text{Spf } A\langle f \rangle$ with $f \in A$, we have

$$\Gamma(U, M^\Delta) = (M_f^\wedge)_f^\wedge,$$

where $M_f = M \otimes_A A_f$.

**Proof.** Indeed, one calculates

$$\Gamma(U, M^\Delta) = \Gamma(U, \lim_{k \geq 0} \tilde{M}_k) = \lim_{k \geq 0} \Gamma(U, \tilde{M}_k) = \lim_{k \geq 0} M_f/I^{k+1}M_f = (M_f)^{\wedge}_f.$$  

$\square$

### 3.2. (b) Adically quasi-coherent $\Delta$-sheaves

**Proposition 3.2.2.** Let $A$ be an adic ring, and $I \subseteq A$ a finitely generated ideal. Set $X = \text{Spf } A$ and $I = I^\Delta$. Let $M$ be an $A$-module, and consider the sheaf $M^\Delta$ on $X$. We have

$$M^\Delta/I^{k+1}M^\Delta \cong \tilde{M}_k,$$

where $M_k = M/I^{k+1}M$ for $k \geq 0$, and the sheaf $M^\Delta$ is complete (3.1.1 (2)). In particular, $M^\Delta$ is an a.q.c. sheaf on $X$.

To show the proposition, we first prove the following lemma.

**Lemma 3.2.3** (cf. 0.7.4.13; see also 3.5.3 below). Let $A$ be an adic ring with a finitely generated ideal of definition $I$, $M$ an $A$-module, and $N \subseteq M$ an $A$-submodule open with respect to the $I$-adic topology on $M$. Then the sequence

$$0 \to N^\Delta \to M^\Delta \to (M/N)^\Delta \to 0$$

given by the canonical maps is exact.

**Proof.** Take $n \geq 0$ such that $I^{n+1}M \subseteq N$, and consider the exact sequence

$$0 \to \tilde{N}/I^{k+1}M \to \tilde{M}_k \to \tilde{M}/N \to 0$$

of quasi-coherent sheaves on $X_k$ for $k \geq n$. Since $I^{k+1}N \subseteq I^{k+1}M \subseteq I^{k-n}N$, we have $\lim_{\leftarrow k} N/I^{k+1}M \cong N^\Delta$. Moreover, since $N$ is open in $M$, we have $M/N = (M/N)^\Delta$. Hence, taking projective limits along $k$, we get the desired exact sequence by using 1.1.23 (1).

Lemma 3.2.4. Let $A$ be an adic ring with a finitely generated ideal of definition $I \subseteq A$, and $M$ an $A$-module. Set $I = I^\Delta$. Then for any $k \geq 0$ we have

$$(I^{k+1}M)\Delta = I^{k+1}M^\Delta = I^{k+1}M^\Delta$$

as sheaves on $X = \text{Spf } A$.

Proof. We remark that $I^{k+1}M^\Delta$ is the associated sheaf of the presheaf given by $U \mapsto \Gamma(U, I^{k+1}) \cdot \Gamma(U, M^\Delta)$ ([54], 0.4.6)). By 1.1.21, we have $\Gamma(U, I^{k+1}) \cdot \Gamma(U, M^\Delta) = I^{k+1}\hat{M}_f$ for $U = \text{Spf } A_f$, where $\hat{M}_f$ is the $I$-adic completion of $M_f$ (0.7.2.16). Since $I^{k+1}\hat{M}_f$ is a closed subset of $\hat{M}_f$ (0.7.2.9), it is $I$-adically complete (0.7.4.12); it then coincides with the $I$-adic completion of $I^{k+1}M_f$ and hence with $\Gamma(U, (I^{k+1}M^\Delta))$, whence the result. 

Proof of Proposition 3.2.2. By 3.2.3 and 3.2.4, $M^\Delta/I^{k+1}M^\Delta = \hat{M}_k$ for $k \geq 0$. Since

$$\lim_{\leftarrow k \geq 0} M^\Delta/I^{k+1}M^\Delta = \lim_{\leftarrow k \geq 0} \hat{M}_k = M^\Delta,$$

the sheaf $M^\Delta$ is complete, as desired. 

Proposition 3.2.5. Let $A$ be an adic ring of finite ideal type, and $M \to N$ a surjective homomorphism of $A$-modules. Then the induced morphism $M^\Delta \to N^\Delta$ of sheaves on $X = \text{Spf } A$ is surjective.

Proof. Let $I \subseteq A$ be a finitely generated ideal of definition. Since for $k \geq 0$ the induced map $M_k = M/I^{k+1}M \to N_k = N/I^{k+1}N$ is surjective, $\hat{M}_k \to \hat{N}_k$ is surjective. By 3.2.4, we have $\hat{M}_k = (M^\Delta)_k = M^\Delta/I^{k+1}M^\Delta$, etc. Let $\mathcal{K}_k$ be the kernel of the surjection $(M^\Delta)_k \to (N^\Delta)_k$. Consider the commutative diagram
with exact rows and columns

\[
\begin{array}{ccc}
0 & 0 & 0 \\
& I^{k+1}(M^\Delta)_l & I^{k+1}(N^\Delta)_l \\
& \downarrow & \downarrow \\
0 & \mathcal{K}_l & (M^\Delta)_l \\
& \downarrow & \downarrow \\
0 & \mathcal{K}_k & (M^\Delta)_k \\
& \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
\]

for \( k \leq l \). Since the map \((\ast)\) is surjective, the snake lemma shows that the map \( \mathcal{K}_l \to \mathcal{K}_k \) is surjective, that is, the projective system \( \{\mathcal{K}_k\}_{k \geq 0} \) of quasi-coherent sheaves is strict. Hence, by 1.1.23 (1),

\[
M^\Delta = \lim_{k \geq 0} (M^\Delta)_k \to N^\Delta = \lim_{k \geq 0} (N^\Delta)_k
\]

is surjective, as desired. \( \square \)

**Corollary 3.2.6.** Let \( A \) be an adic ring of finite ideal type and \( M \) a finitely generated \( A \)-module. Then \( M^\Delta \) is an a.q.c. sheaf of finite type.

**Proof.** Take a surjection \( A^{\oplus n} \to M \). Then we have the surjection

\[
\mathcal{O}_X^{\oplus n} = (A^{\oplus n})^\Delta \longrightarrow M^\Delta.
\]

\( \square \)

**Proposition 3.2.7.** Let \( A \) be an adic ring with a finitely generated ideal of definition \( I \subseteq A \), and set \( X = \text{Spf} \, A \). Let \( \mathcal{F} \) (resp. \( \mathcal{B} \)) be an \( \mathcal{O}_X \)-module (resp. an \( \mathcal{O}_X \)-algebra). Then the following conditions are equivalent.

(a) There exist an affine open covering \( \{U_\alpha\}_{\alpha \in L} \) of \( X \) with \( U_\alpha = \text{Spf} \, A_{(f_\alpha)} \) and for each \( \alpha \in L \) an \( IA_{(f_\alpha)} \)-adically complete \( A_{(f_\alpha)} \)-module \( M_\alpha \) (resp. \( A_{(f_\alpha)} \)-algebra \( B_\alpha \)), such that \( \mathcal{F}|_{U_\alpha} \cong M_\alpha^\Delta \) (resp. \( \mathcal{B}|_{U_\alpha} \cong B_\alpha^\Delta \)).

(b) There exists an \( I \)-adically complete \( A \)-module \( M \) (resp. \( A \)-algebra \( B \)) such that \( \mathcal{F} \cong M^\Delta \) (resp. \( \mathcal{B} \cong B^\Delta \)).

(c) \( \mathcal{F} \) (resp. \( \mathcal{B} \)) is an a.q.c. sheaf (resp. an a.q.c. \( \mathcal{O}_X \)-algebra) on \( X \).

Moreover, if \( \mathcal{F} \cong M^\Delta \) as in (b), \( \mathcal{F} \) is of finite type if and only if \( M = \Gamma(X, \mathcal{F}) \) is finitely generated.
Proof. First we show the assertion for the sheaf $\mathcal{F}$ of modules. (b) \implies (a) is trivial, and (a) \implies (c) follows from 3.2.2. Let us show (c) \implies (b). Set $I = I^\Delta$ and $\mathcal{F}_k = \mathcal{F}/I^{k+1}\mathcal{F}$ for $k \geq 0$; each $\mathcal{F}_k$ is a quasi-coherent sheaf on the scheme $X_k = \text{Spec} \mathcal{O}/I^{k+1}$. Let $M_k = \Gamma(X_k, \mathcal{F}_k)$. These $A$-modules constitute a strict projective system $\{M_k\}_{k \geq 0}$. Set $M = \lim_{\leftarrow k \geq 0} M_k$, and let $F^{(n)}$ be the kernel of the surjective map $M \to M_{n-1}$ for each $n \geq 1$. Then by 0.7.2.14 we have $F^{(n)} = I^n M$ for $n \geq 1$, and hence $M$ is $I$-adically complete. Moreover, by 3.2.4,

$$\mathcal{F} = \lim_{\leftarrow k \geq 0} \mathcal{F}_k = \lim_{\leftarrow k \geq 0} M_k = \lim_{\leftarrow k \geq 0} M^\Delta/I^{k+1}M = M^\Delta.$$

The last assertion follows from 3.2.6 and the following observation: if $F$ is of finite type, then $M_0 = M/F(1)$ is finitely generated, and hence by the last part of 0.7.2.14 $M$ is finitely generated.

For the sheaf $\mathcal{B}$ of algebras implications (b) \implies (a) and (a) \implies (c) are also clear. To show (c) \implies (b), we set $\mathcal{B}_k = \mathcal{B}/I^{k+1}\mathcal{B}$ and $B_k = \Gamma(X_k, \mathcal{B}_k)$ for each $k \geq 0$ and define $B = \lim_{\leftarrow k \geq 0} B_k$. We know that the map $B_l \to B_k$ for $k \leq l$ is surjective with the kernel equal to $I^{k+1}B_l$. Hence, by 0.7.2.12, $B$ is $IB$-adically complete and $B/I^{k+1} = B_k$ for $k \geq 0$. The rest of the argument is similar to the module case. \qed

**Theorem 3.2.8.** Let $X = \text{Spf} \ A$, where $A$ is an adic ring of finite ideal type, and consider the functor

$$M \longmapsto M^\Delta$$

as in §3.2. (a).

(1) If $M$ is an $I$-adically complete $A$-module, then

$$\Gamma(X, M^\Delta) = M.$$

(2) The functor (*) gives a categorical equivalence between the category of $I$-adically complete $A$-modules (resp. $I$-adically complete finitely generated $A$-modules, resp. $I$-adically complete $A$-algebras) and the category of a.q.c. sheaves (resp. a.q.c. sheaves of finite type, resp. adically quasi-coherent $\mathcal{O}_X$-algebras) on $X$. The quasi-inverse functor is given by

$$\mathcal{F} \longmapsto \Gamma(X, \mathcal{F}).$$

(1) follows from 3.2.1. The other assertion follows from 3.2.7 and the following lemma.
Lemma 3.2.9. In the situation as in 3.2.8, let $M$ and $N$ be $I$-adically complete $A$-modules. Then the canonical map

$$\text{Hom}_A(M, N) \longrightarrow \text{Hom}_{\mathcal{O}_X}(M^\Delta, N^\Delta)$$

is an isomorphism. The analogous statement with $M, N$ replaced by $I$-adically complete $A$-algebras and the morphisms replaced by algebra homomorphisms is also true.

Proof. Given a morphism $\varphi: M^\Delta \rightarrow N^\Delta$, we have $\Gamma_X(\varphi): M \rightarrow N$ and thus $\Gamma_X(\varphi)^\Delta: M^\Delta \rightarrow N^\Delta$. We claim that $\varphi = \Gamma_X(\varphi)^\Delta$. To show this, we first show $\Gamma_X(\varphi_k) = \Gamma_X(\varphi) \otimes_A A_k$ for $k \geq 0$, where $\varphi_k: \tilde{M}_k = M^\Delta / I^{k+1} M^\Delta \rightarrow \tilde{N}_k = N^\Delta / I^{k+1} N^\Delta$ (where $I = I^\Delta$) is the induced map (cf. 3.2.4) and $A_k = A / I^{k+1}$.

By 1.1.23 (2) we have the commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Gamma(X, I^{k+1} M^\Delta) & \longrightarrow & \Gamma(X, M^\Delta) & \longrightarrow & \Gamma(X, \tilde{M}_k) & \longrightarrow & 0 \\
& & \downarrow{\Gamma(\varphi)} & & \downarrow{\Gamma(\varphi_k)} & & \downarrow{\Gamma(\varphi_k)} & & \\
0 & \longrightarrow & \Gamma(X, I^{k+1} N^\Delta) & \longrightarrow & \Gamma(X, N^\Delta) & \longrightarrow & \Gamma(X, \tilde{N}_k) & \longrightarrow & 0.
\end{array}
$$

Here we used the fact that $I^{k+1} M^\Delta$ and $I^{k+1} N^\Delta$ are a.q.c. sheaves on $X$ due to 3.1.4 (3) and hence have vanishing higher cohomologies due to 1.1.23 (2). By 3.2.4, $\Gamma(X, I^{k+1} M^\Delta) = I^{k+1} M$ and $\Gamma(X, I^{k+1} N^\Delta) = I^{k+1} N$ (here we used the assumption that $M$ and $N$ are finitely generated) and hence $\Gamma_X(\varphi_k) = \Gamma_X(\varphi) \otimes_A A_k$, as desired. Then,

$$\Gamma_X(\varphi)^\Delta = \lim_{k \geq 0} \Gamma_X(\varphi) \otimes_A A_k = \lim_{k \geq 0} \Gamma_X(\varphi_k) = \lim_{k \geq 0} \varphi_k = \varphi,$$

which shows our claim. In particular, we have shown that the map (**) in question is surjective. Since for any homomorphism $f: M \rightarrow N$ of $A$-modules we clearly have $\Gamma_X(f^\Delta) = f$ (by 0.3.2.9), (**) is injective. This concludes the proof of the first statement. The other assertion (for morphisms between $I$-adically complete $A$-algebras) is shown similarly.

\section{A.q.c. algebras of finite type}

Definition 3.3.1. Let $X$ be an adic formal scheme of finite ideal type. We say that an adically quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ is of finite type if there exist an open covering $X = \bigcup_{\alpha \in \Lambda} U_\alpha$ and an ideal of definition of finite type $I_\alpha$ on each $U_\alpha$, such that $\mathcal{B} / I_\alpha \mathcal{B}$ for each $\alpha$ is a quasi-coherent algebra of finite type on the scheme $(U_\alpha, \mathcal{O}_{U_\alpha} / I_\alpha)$. 
Using \textbf{0.8.4.2} (applied to the case where $I$ is nilpotent) we readily see that the required condition does not depend on the choice of the open covering and the ideal of definitions. In particular, we have the following proposition.

\textbf{Proposition 3.3.2.} Let $X$ be an adic formal scheme of finite ideal type, and $\mathcal{B}$ an a.q.c. $\mathcal{O}_X$-algebra of finite type. Then for any open subspace $U \subseteq X$ that admits an ideal of definition $I$ of finite type, $\mathcal{B}/I\mathcal{B}$ is a quasi-coherent algebra of finite type on the scheme $(U, \mathcal{O}_U/I)$.

\textbf{Proposition 3.3.3.} Let $A$ be an adic ring of finite ideal type, and set $X = \text{Spf } A$. Let $\mathcal{B}$ be an adically quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{B}$ is of finite type if and only if $B = \Gamma(X, \mathcal{B})$ is a topologically finitely generated $A$-algebra.

\textit{Proof.} In view of \textbf{3.2.8} (2), $B$ is an $I$-adically complete $A$-algebra where $I$ is a finitely generated ideal of definition. Since $B/IB = B/I\mathcal{B}$ on $\text{Spec } A/I$, the result follows from \textbf{0.8.4.2}. \hfill $\square$

\subsection*{3.4 A.q.c. sheaves as projective limits}

\textbf{Proposition 3.4.1.} Let $X$ be an adic formal scheme of finite ideal type and $I$ an ideal of definition of finite type. Set $X_k = (X, \mathcal{O}_X/I^{k+1})$. Suppose we have a projective system $\{\mathcal{F}_k, \varphi_{ij}\}_{i \in \mathbb{N}}$ of $\mathcal{O}_X$-modules such that

(a) for each $k \geq 0$, $I^{k+1}\mathcal{F}_k = 0$ and $\mathcal{F}_k$ is a quasi-coherent sheaf on the scheme $X_k$, and

(b) for any $i \leq j$ the morphism $\varphi_{ij} : \mathcal{F}_j \to \mathcal{F}_i$ is a surjective map with the kernel equal to $I^{i+1}\mathcal{F}_j$.

Then the projective limit $\mathcal{F} = \lim_{\leftarrow k} \mathcal{F}_k$ is an a.q.c. sheaf on $X$ such that

$$\mathcal{F}/I^{k+1}\mathcal{F} \cong \mathcal{F}_k$$

for each $k \geq 0$.

Moreover, if $\mathcal{F}_0$ is of finite type, then $\mathcal{F}$ is of finite type.

\textit{Proof.} We may assume that $X$ is affine, $X = \text{Spf } A$ with $I = I^\Delta$, where $I$ is a finitely generated ideal of definition of $A$. We have $X_k = \text{Spec } A_k$, where $A_k = A/I^{k+1}$ for any $k \geq 0$. Take for each $k$ the $A_k$-module $M_k$ such that $\mathcal{F}_k = \widetilde{M}_k$.

Then we have the projective system $\{M_k, f_{ij}\}$ of $A$-modules such that for each $i \leq j$ the transition map $f_{ij} : M_j \to M_i$ is surjective with the kernel $I^{i+1}M_j$. Set $M = \lim_{\leftarrow k} M_k$ and for each $i$ denote by $f_i : M \to M_i$ the projection map. Set $F^{(n)} = \ker(f_{n-1})$ for $n \geq 1$. Then by \textbf{0.7.2.14} $M$ is an $I$-adically complete finitely generated $A$-module, and $F^{(n)} = I^nM$ for all $n \geq 1$. If $\mathcal{F}_0$ is of finite type, then $M_0 = M/F^{(1)}$ is finitely generated, and hence $M$ is finitely generated.
Now consider the open subset $U = \text{Spf} \, A_{(g)}$ of $X$ for any $g \in A$; we have

$$\Gamma(U, \mathcal{F}) = \lim_k \Gamma(U, \mathcal{F}_k) = \lim_k (M_k \otimes_A A_g) = M \otimes_A A_{(g)},$$

which coincides with $\Gamma(U, M^\Delta)$, as calculated in 3.2.1. Hence we have $\mathcal{F} = M^\Delta$, which is an adically quasi-coherent sheaf (3.2.2). Moreover, by 3.2.4,

$$\mathcal{F} / I^{k+1} \mathcal{F} = M^\Delta / I^{k+1} M^\Delta = \hat{M}_k = \mathcal{F}_k,$$

as desired. \hfill \square

One can show the following analogous statement for adically quasi-coherent algebras; the proof is similar (cf. the proof of 3.2.7).

**Proposition 3.4.2.** Let $X$ be an adic formal scheme of finite ideal type, and $I$ an ideal of definition of finite type. Set $X_k = (X, \mathcal{O}_X / I^{k+1})$. Suppose we have a projective system $\{B_k, \varphi_{ij}\}_{i \in \mathbb{N}}$ of $\mathcal{O}_X$-algebras such that

(a) for each $k \geq 0$, $I^{k+1} B_k = 0$, and the sheaf $B_k$ is a quasi-coherent algebra on the scheme $X_k$, and

(b) for any $i \leq j$, the morphism $\varphi_{ij} : B_j \to B_i$ is a surjective map with the kernel equal to $I^{i+1} B_j$.

Then the projective limit $B = \lim_k B_k$ is an a.q.c. $\mathcal{O}_X$-algebra on $X$ such that $B / I^{k+1} B \cong B_k$ for each $k \geq 0$. Moreover, if $B_0$ is of finite type, then $B$ is an a.q.c. $\mathcal{O}_X$-algebra of finite type.

### 3.5 A.q.c. sheaves on locally universally rigid-Noetherian formal schemes

#### 3.5. (a) $\Delta$-sheaves on affine universally rigid-Noetherian formal schemes

**Proposition 3.5.1.** Let $A$ be a rigid-Noetherian ring (2.1.1 (1)), and set $X = \text{Spf} \, A$ and $Y = \text{Spec} \, A$. Consider the canonical morphism $i : X \to Y$ of locally ringed spaces. Then for any finitely generated $A$-module $M$

$$M^\Delta \cong i^* \hat{M},$$

**Proof.** By 2.1.5,

$$i^* \hat{M} \cong \hat{M} = \lim_{k \geq 0} \hat{M} / I^{k+1} \hat{M} = M^\Delta,$$

where $I = I^\Delta$ and $I \subseteq A$ is a finitely generated ideal of definition. \hfill \square
Proposition 3.5.2. In the situation as in 3.5.1, the functor $M \mapsto M^\Delta$ gives an exact equivalence between the category of finitely generated $A$-modules and the category of a.q.c. sheaves of finite type on $X$.

Proof. In view of 3.2.8 (2) and the fact that any finitely generated $A$-module is $I$-adically complete (2.1.3), only the exactness of the functor is in question. But this follows from 3.5.1 and 2.1.5.

Proposition 3.5.3 (cf. 0.7.4.13; see also 3.2.3). Let $A$ be a rigid-Noetherian ring with a finitely generated ideal of definition $I \subseteq A$ and $M$ be a finitely generated $A$-module. Let $N \subseteq M$ be an $A$-submodule. Then the sequence

$$0 \rightarrow N^\Delta \rightarrow M^\Delta \rightarrow (M/N)^\Delta \rightarrow 0$$

given by canonical maps is exact.

Proof. Let $f \in A$, and consider the affine open subset $U = \text{Spf} A_{\{f\}}$ of $X$. Since $A_f$ is $IA_f$-adically pseudo-adhesive (0.8.5.7 (1)), it satisfies (AP) in 0, §7.4.(c). Hence, by 0.7.4.13, we have the exact sequence

$$0 \rightarrow N \hat{\otimes}_A A_{\{f\}} \rightarrow M \hat{\otimes}_A A_{\{f\}} \rightarrow (M/N) \hat{\otimes}_A A_{\{f\}} \rightarrow 0$$

and, by 3.2.1, the exact sequence

$$0 \rightarrow \Gamma(U, N^\Delta) \rightarrow \Gamma(U, M^\Delta) \rightarrow \Gamma(U, (M/N)^\Delta) \rightarrow 0.$$ 

Since this is valid for any open subsets of the form $U = \text{Spf} A_{\{f\}}$, we have the desired exact sequence.

3.5. (b) A.q.c. sheaves of finite presentation

Proposition 3.5.4. Let $A$ be a rigid-Noetherian ring (2.1.1 (1)), and set $X = \text{Spf} A$. Then any finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ is an a.q.c. sheaf. Moreover, $M = \Gamma(X, \mathcal{F})$ is a finitely presented $A$-module, and $\mathcal{F} = M^\Delta$.

Proof. The sheaf $\mathcal{F}$ is locally isomorphic to the cokernel of a map of the form $\mathcal{O}_X^{\oplus q} \rightarrow \mathcal{O}_X^{\oplus p}$. By 3.5.2, this means that $\mathcal{F}$ can be locally written as a $\Delta$-sheaf of a finitely generated module. Hence $\mathcal{F}$ is a.q.c. of finite type. Moreover, we know that each $\mathcal{F}_k = \mathcal{F} / I^{k+1} \mathcal{F}$ (where $I \subseteq A$ is a finitely generated ideal of definition) is finitely presented on the scheme $\text{Spec} A/I^{k+1}$. This means that $M = \Gamma(X, \mathcal{F})$ is finitely presented due to 0.7.4.19, since $M_k = M/I^{k+1}M$ is finitely presented by [53], (1.4.3). 

□
Corollary 3.5.5. Any finitely presented $\Theta_X$-module on a locally universally rigid-Noetherian formal scheme is a.q.c. of finite type.

We can call such a sheaf $\mathcal{F}$, as usual, a quasi-coherent sheaf of finite presentation (even without ‘adically’), because it is obviously quasi-coherent in the usual sense (cf. 0.4.1.4 (1)).

Theorem 3.5.6. Let $X = \text{Spf } A$, where $A$ is a rigid-Noetherian ring. Then the functor

$$M \mapsto M^\Delta$$

gives an exact categorical equivalence between the category of finitely generated (resp. finitely presented) $A$-modules and the category of a.q.c. sheaves of finite type (resp. of finite presentation) on $X$. The functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ gives the quasi-inverse to $\ast$.

Proof. The assertion for ‘finitely generated’ was already shown in 3.5.2. The other part follows from 3.5.2 and 3.5.4.

Proposition 3.5.7. Let $A$ be a rigid-Noetherian ring, and $M$ and $N$ finitely generated $A$-modules. Set $X = \text{Spf } A$.

1. We have a canonical isomorphism

$$(M \otimes_A N)^\Delta \cong M^\Delta \otimes_{\Theta_X} N^\Delta.$$  

2. If $M$ is finitely presented, then we have a canonical isomorphism

$$x \text{Hom}_A(M, N)^\Delta \cong \text{Hom}_{\Theta_X}(M^\Delta, N^\Delta).$$

Proof. Set $Y = \text{Spec } A$, and let $i : X \to Y$ be the canonical morphism of locally ringed spaces.

1. By 3.5.1, [54], I, (1.3.12) (i), and [54], 0_1, (4.3.3.1), we have

$$(M \otimes_A N)^\Delta \cong \overline{i^* M \otimes_A N}$$

$$\cong \overline{i^*(\overline{M} \otimes_{\Theta_Y} \overline{N})}$$

$$\cong i^* \overline{M} \otimes_{\Theta_X} i^* \overline{N}$$

$$\cong M^\Delta \otimes_{\Theta_X} N^\Delta,$$

as claimed.
(2) By 3.5.1, [54], I, (1.3.12) (ii), and [54], (6.7.6.1), we have
\[
\text{Hom}_A(M, N)^\Delta \cong \overline{i^* \text{Hom}_A(M, N)} \\
\cong i^* \text{Hom}_{\mathcal{O}_Y}(\tilde{M}, \tilde{N}) \\
\cong \text{Hom}_{\mathcal{O}_X}(i^* \tilde{M}, i^* \tilde{N}) \\
\cong \text{Hom}_{\mathcal{O}_X}(M^\Delta, N^\Delta),
\]
as claimed; here we used the fact that \(\tilde{M}\) is of finite presentation (3.5.6).

**Corollary 3.5.8.** Let \(X = \text{Spf} \ A\) where \(A\) is a rigid-Noetherian ring. Then the functor \((*)\) in 3.5.6 defined on the category of finitely presented \(A\)-modules preserves tensor products and internal Hom’s.

**Corollary 3.5.9.** Let \(X\) be a locally universally rigid-Noetherian formal scheme and \(\mathcal{F}, \mathcal{G}\) a.q.c. sheaves of finite type on \(X\). Then \(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}\) is an a.q.c. sheaf of finite type. If, moreover, \(\mathcal{F}\) is of finite presentation, then \(\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\) is an a.q.c. sheaf of finite type on \(X\). Moreover, if both \(\mathcal{F}\) and \(\mathcal{G}\) are finitely presented, then \(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}\) and \(\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\) are of finite presentation.

Finally, let us mention that \(I\)-torsion free a.q.c. sheaves of finite type on *locally universally adhesive* formal schemes are automatically finitely presented.

**Proposition 3.5.10.** Let \(X\) be a locally universally adhesive formal scheme. Suppose that \(X\) has an ideal of definition of finite type \(I \subseteq \mathcal{O}_X\). Then any \(I\)-torsion free a.q.c. sheaf of finite type is finitely presented.

**Proof.** We may assume \(X = \text{Spf} \ A\), where \(A\) is a t.u. adhesive ring with a finitely generated ideal of definition \(I \subseteq A\). Any \(I\)-torsion free a.q.c. sheaf of finite type \(\mathcal{F}\) is of the form \(\mathcal{F} = M^\Delta\), where \(M\) is an \(I\)-torsion free finitely generated \(A\)-module. Since \(A\) is \(I\)-adically adhesive, \(M\) is finitely presented (cf. 0.8.5.3).

3.5. (c) A.q.c. algebras of finite presentation

**Definition 3.5.11.** Let \(X\) be a locally universally rigid-Noetherian formal scheme. We say that an a.q.c. \(\mathcal{O}_X\)-algebra \(\mathcal{B}\) is of finite presentation if there exists an open covering \(X = \bigcup_{\alpha \in L} U_\alpha\) and an ideal of definition of finite type \(I_\alpha\) on each \(U_\alpha\) such that, for each \(\alpha\) and \(k \geq 0\) the sheaf \(\mathcal{B}/I_\alpha^{k+1}\) is a quasi-coherent algebra of finite presentation on the scheme \((U_\alpha, \mathcal{O}_{U_\alpha}/I_\alpha^{k+1})\).

The following two propositions can be established in the same way as we did for 3.3.2 and 3.3.3, using 0.8.4.5 instead of 0.8.4.2.
Proposition 3.5.12. Let $X$ be a locally universally rigid-Noetherian formal scheme, and $B$ an a.q.c. $\mathcal{O}_X$-algebra of finite presentation. Then for any open subspace $U \subseteq X$ that admits an ideal of definition of finite type $I$, the sheaf $B/I\mathcal{B}$ is a quasi-coherent algebra of finite presentation on the scheme $(U, \mathcal{O}_U/I)$.

Proposition 3.5.13. Let $A$ be a t.u. rigid-Noetherian ring, and set $X = \text{Spf} A$. Let $\mathcal{B}$ be an a.q.c. $\mathcal{O}_X$-algebra. Then $\mathcal{B}$ is of finite presentation if and only if $B = \Gamma(X, \mathcal{B})$ is a topologically finitely presented $A$-algebra.

3.6 Complete pull-back of a.q.c. sheaves

Definition 3.6.1. Let $f: X \to Y$ be an adic morphism of adic formal schemes of finite ideal type, and $\mathcal{F}$ an $\mathcal{O}_Y$-module. We define the $\mathcal{O}_X$-module $\widehat{f^*}\mathcal{F}$ by

$$\widehat{f^*}\mathcal{F} = \widehat{f^*}\mathcal{F},$$

where the last sheaf is the completion of the $\mathcal{O}_X$-module $f^*\mathcal{F}$ (3.1.1 (1)); we call this sheaf the complete pull-back of $\mathcal{F}$.

Proposition 3.6.2. Let $f: X \to Y$ be an adic morphism of locally universally rigid-Noetherian formal schemes, and $\mathcal{F}$ an a.q.c. sheaf of finite type (resp. of finite presentation) on $Y$. Then we have

$$\widehat{f^*}\mathcal{F} = f^*\mathcal{F},$$

which is an a.q.c. sheaf of finite type (resp. of finite presentation) on $X$.

Proposition 3.6.3. Let $f: X \to Y$ be an adic morphism of adic formal schemes of finite ideal type, and $\mathcal{F}$ an a.q.c. sheaf (resp. a.q.c. sheaf of finite type, resp. a.q.c. $\mathcal{O}_Y$-algebra) on $Y$. Suppose that $f$ is adically flat (1.5.5 (1)). Then the complete pull-back $\widehat{f^*}\mathcal{F}$ is an a.q.c. sheaf (resp. a.q.c. sheaf of finite type, resp. a.q.c. $\mathcal{O}_X$-algebra) on $X$.

To show the last two propositions, since the question is local, we may work in the affine situation. Then the propositions follow from the following lemma.

Lemma 3.6.4. Let $B \to A$ be an adic homomorphism of adic rings of finite ideal type, and $I \subseteq B$ a finitely generated ideal of definition of $B$.

1) Suppose $A$ and $B$ are t.u. rigid-Noetherian rings, and let $M$ be a finitely generated (resp. finitely presented) $B$-module. Then

$$f^*M^\Delta \cong \widehat{f^*M^\Delta} \cong (M \otimes_B A)^\Delta.$$
3. Adically quasi-coherent sheaves

(2) Suppose that for any \( k \geq 0 \) the induced map \( B/I^{k+1} \rightarrow A/I^{k+1} A \) is flat, and let \( M \) be a \( B \)-module (resp. \( B \)-algebra). Then
\[
\tilde{f}^* M^\Delta \cong (M \otimes_B A)^\Delta.
\]

Proof. (1) follows easily, for in this case \( M^\Delta \) is the pull-back of \( \tilde{M} \) on \( \text{Spec} \ B \) by the map \( \text{Spf} \ B \rightarrow \text{Spec} \ B \) (3.5.1), and thus \( f^* M^\Delta \) is the pull-back of \( M \otimes_B A \) by the map \( \text{Spf} \ A \rightarrow \text{Spec} \ A \), which is already complete.

To show (2), we look at the canonical morphism
\[
f^* I^{k+1} M \longrightarrow I^{k+1} f^* M^\Delta
\]
for \( k \geq 0 \), where \( I = I^\Delta \). We first want to show that this is an isomorphism by checking that the map between stalks at each point is bijective. To this end, it suffices to show that for any \( h \in A \) the map
\[
(I^{k+1} M) \otimes_B A_{(h)} \longrightarrow I^{k+1} (M \otimes_B A_{(h)})
\]
is an isomorphism. This amounts to showing that the descending filtration \( \{J^{(n)}\}_{n \geq 1} \) on \( M \otimes_B A_{(h)} \) defined by
\[
J^{(n)} = \text{image}((I^n M) \otimes_B A_{(h)} \longrightarrow M \otimes_B A_{(h)}), \quad n \geq 1,
\]
is the \( I \)-adic filtration. Since
\[
\lim_{n \geq 1} M \otimes_B A_{(h)}/J^{(n)} = \lim_{n \geq 1} (M/I^n M) \otimes_B A/I^n A[1/n] = M \otimes_B A_{(h)},
\]
it suffices to show that (b) of 0.7.2.14 (resp. 0.7.2.11; cf. 0.7.2.12) holds. But since \( B/I^n \rightarrow A/I^n A \) is flat, the map
\[
(I^m M/I^n M) \otimes_B A/I^n A[1/n] \longrightarrow I^m (M/I^n M) \otimes_B A/I^n A[1/n]
\]
(which is, a priori, surjective) is an isomorphism for \( 0 \leq m \leq n \). Thus we have shown that the morphism (\( \ast \)) is an isomorphism for \( k \geq 0 \).

Next, we have
\[
\tilde{f}^* M^\Delta = \lim_{k \geq 0} f^* M^\Delta / I^{k+1} f^* M^\Delta
\]
\[
= \lim_{k \geq 0} f^* (M^\Delta / I^{k+1} M^\Delta)
\]
\[
= \lim_{k \geq 0} f^* \tilde{M}_k
\]
\[
= \lim_{k \geq 0} \tilde{M}_k \otimes_B A_k,
\]
where \( M_k = M/I^{k+1} M \), which shows that \( \tilde{f}^* M^\Delta = (M \otimes_B A)^\Delta \), as desired. \( \square \)
The following result is a corollary of the proof of 3.6.4 (2).

**Corollary 3.6.5.** *In the situation as in 3.6.3, suppose that $Y$ admits an ideal of definition of finite type $I$. Let*

$$f_k: X_k = (X, \mathcal{O}_X/I^{k+1}) \longrightarrow Y_k = (Y, \mathcal{O}_Y/I^{k+1})$$

*be the induced morphism of schemes for $k \geq 0$. Then*

$$f^*I^{k+1}\mathcal{F} = I^{k+1}f^*\mathcal{F}, \quad k \geq 0,$$

*and*

$$\widehat{f^*\mathcal{F}} = \lim_{\longrightarrow} f_k^*\mathcal{F}_k,$$

*where $\mathcal{F}_k = \mathcal{F}/I^{k+1}\mathcal{F}$ for $k \geq 0$.

### 3.7 Admissible ideals

#### 3.7. (a) Pull-back of quasi-coherent sheaves on closed subschemes

**Proposition 3.7.1.** *Let $X$ be an adic formal scheme, $I \subseteq \mathcal{O}_X$ an ideal of definition of finite type, and $\mathcal{F}$ an a.q.c. sheaf on $X$. Let $\mathcal{G}$ be an a.q.c. subsheaf of $\mathcal{F}$ such that $I\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{F}$. Then $\mathcal{G}/I\mathcal{F}$ is a quasi-coherent sheaf on the scheme $X_0 = (X, \mathcal{O}_X/I)$.*

**Proof.** We may assume that $X$ is affine, $X = \text{Spf} \, A$, where $A$ is an adic ring of finite ideal type, and that $I = I^A$ where $I \subseteq A$ is a finitely generated ideal of definition. Set $N = \Gamma(X, \mathcal{G}) \subseteq M = \Gamma(X, \mathcal{F})$. In view of 3.2.4, $IM \subseteq N \subseteq M$. We see by 3.2.3 that $\mathcal{F}/\mathcal{G} = (M/N)^A$. Since $I(\mathcal{F}/\mathcal{G}) = 0$, we deduce that $\mathcal{F}/\mathcal{G}$ is a quasi-coherent sheaf on the scheme $X_0$. Now we look at the exact sequence

$$0 \longrightarrow \mathcal{G}/I\mathcal{F} \longrightarrow \mathcal{F}/I\mathcal{F} \longrightarrow \mathcal{F}/\mathcal{G} \longrightarrow 0.$$ 

Since $\mathcal{F}/I\mathcal{F}$ and $\mathcal{F}/\mathcal{G}$ are quasi-coherent on $X_0$, we conclude that $\mathcal{G}/I\mathcal{F}$ is quasi-coherent on $X_0$. \qed

**Proposition 3.7.2.** *Let $X$ be an adic formal scheme of finite ideal type with an ideal of definition $I$ of finite type, and $\mathcal{F}$ an a.q.c. sheaf (resp. of finite type) on $X$. Consider the quasi-coherent sheaf $\mathcal{F}_0 = \mathcal{F}/I\mathcal{F}$ on the scheme $X_0 = (X, \mathcal{O}_X/I)$ and a quasi-coherent subsheaf $\mathcal{G}_0 \subseteq \mathcal{F}_0$ (resp. of finite type). Then the inverse image $\mathcal{G} \subseteq \mathcal{F}$ of $\mathcal{G}_0$ by the canonical map $\mathcal{F} \to \mathcal{F}_0$ is an a.q.c. sheaf (resp. of finite type) on $X$.*
Proof. We may assume without loss of generality that $X$ is affine, $X = \text{Spf } A$. Let $I \subseteq A$ be the finitely generated ideal such that $\mathcal{I} = I^\Delta$, and $M$ the $I$-adically complete $A$-module such that $\mathcal{F} = M^\Delta$. The quasi-coherent subsheaf $\mathcal{G}_0 \subseteq \mathcal{F}_0$ corresponds to a submodule $N_0 \subseteq M_0 = M/IM$. Let $N$ be the pull-back of $N_0$ by the canonical map $M \to M_0$. Since $IM \subseteq N$, we have the exact sequence

$$0 \to N^\Delta \to M^\Delta \to (M/N)^\Delta \to 0$$

(3.2.3). But since $(M/N)^\Delta = \widetilde{M_0/N_0} = \mathcal{F}_0/\mathcal{G}_0$ and since

$$0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{F}_0/\mathcal{G}_0 \to 0$$

is exact, $\mathcal{G} = N^\Delta$. Note that since $N$ is open in $M$ and $M$ is $I$-adically complete, $N$ is $I$-adically complete. If $\mathcal{F}$ and $\mathcal{G}_0$ are of finite type, then $M$ and $N_0$ are finitely generated. Since $N/N \cap I^2M = N/I^2M$ is finitely generated due to the exact sequence

$$0 \to IM/I^2M \to N/I^2M \to N/IM = (N/N \cap IM = N_0) \to 0,$$

we deduce that $N/IN$ is finitely generated (for $I^2M \subseteq IN$). By 0.7.2.4, $N$ is finitely generated and hence $\mathcal{G}$ is of finite type.

Corollary 3.7.3. Let $X$ be an adic formal scheme, $\mathcal{I} \subseteq \mathcal{O}_X$ an ideal of definition of finite type, and $\mathcal{F}$ an a.q.c. sheaf (resp. of finite type) on $X$. Consider the following sets.

(a) The set of all a.q.c. subsheaves $\mathcal{G} \subseteq \mathcal{F}$ (resp. of finite type) that contains $\mathcal{I}^\mathcal{F}$.

(b) The set of all quasi-coherent subsheaves of $\mathcal{F}_0 = \mathcal{F}/\mathcal{I}^\mathcal{F}$ (resp. of finite type) on the scheme $X_0 = (X, \mathcal{O}_X/\mathcal{I})$.

Then the map $\mathcal{G} \mapsto \mathcal{G}/\mathcal{I}^\mathcal{F}$ from set (a) to set (b) is a bijection.

3.7. (b) Admissible ideals

Definition 3.7.4. Let $X$ be an adic formal scheme of finite ideal type, and $\mathcal{J}$ an ideal sheaf of $\mathcal{O}_X$. Then $\mathcal{J}$ is said to be an admissible ideal if it satisfies the following conditions.

(a) Finiteness: $\mathcal{J}$ is an a.q.c. ideal of finite type.

(b) Openness: $\mathcal{J}$ contains locally an ideal of definition.

We denote by $\text{AId}_X$ the set of all admissible ideals of $X$. 
By 3.2.3, 3.2.4, and 3.2.8 we have the following result.

**Proposition 3.7.5.** Let $A$ be an adic ring with a finitely generated ideal of definition $I \subseteq A$, and set $X = \text{Spf} A$. Then for any $I$-admissible ideal $J \subseteq A$ (0.8.1.2), $J^\Delta$ is an admissible ideal of $\mathcal{O}_X$. Moreover, any admissible ideal of $\mathcal{O}_X$ is of this form for a uniquely determined $I$-admissible ideal $J$.

**Proposition 3.7.6.** Let $f : Y \to X$ be an adic morphism of adic formal schemes of finite ideal type, and $\mathfrak{I}$ an admissible ideal of $\mathcal{O}_X$. Then the pull-back ideal $\mathfrak{I}\mathcal{O}_Y = (f^{-1}\mathfrak{I})\mathcal{O}_Y$ (0.4.1.2) is an admissible ideal of $\mathcal{O}_Y$.

This yields the mapping $\text{AId}_X \to \text{AId}_Y$ given by $\mathfrak{I} \mapsto \mathfrak{I}\mathcal{O}_Y$.

**Proof.** We can work in the affine situation $X = \text{Spf} A$ and $Y = \text{Spf} B$, where $A$ and $B$ are adic rings of finite ideal type. Let $I \subseteq A$ be a finitely generated ideal of definition of $A$. Set $\mathfrak{I} = J^\Delta$, where $J$ is an $I$-admissible ideal of $A$ (3.7.5). Then $JB$ is an $IB$-admissible ideal of $B$. We need to show that $\mathfrak{I}\mathcal{O}_Y = (JB)^\Delta$. This follows from 3.2.3 because $\mathcal{O}_Y = B^\Delta$ and $\mathcal{O}_Y/\mathfrak{I}\mathcal{O}_Y = B/\mathfrak{I}B$. □

**Proposition 3.7.7** (cf. 4.3.14 below). Let $X$ be an adic formal scheme $X$ of finite ideal type. For an admissible ideal $\mathfrak{I} \subseteq \mathcal{O}_X$, let $Y$ be the support of the sheaf $\mathcal{O}_X/\mathfrak{I}$. Then $Y$ is a closed subset of the underlying topological space of $X$, and the locally ringed space $(Y, \mathcal{O}_X/\mathfrak{I})$ is a closed subscheme of $X$.

**Proof.** We may assume that $X$ has an ideal of definition of finite type $I$ such that $I \subseteq \mathfrak{I}$. Consider the scheme $X_0 = (X, \mathcal{O}_X/I)$. The sheaf of ideals $\mathfrak{I}/I$ on $X_0$ defines the closed subscheme of $X_0$ as in the proposition. □

**Corollary 3.7.8.** Let $X$ be an adic formal scheme, and $\mathfrak{I} \subseteq \mathcal{O}_X$ an admissible ideal. Let $Y$ be the closed subscheme corresponding to $\mathfrak{I}$ (as in 3.7.7), and $i : Y \hookrightarrow X$ the canonical morphism.

1. For any quasi-coherent ideal $\mathcal{K}$ of finite type of $\mathcal{O}_Y$, the inverse image of $i_*\mathcal{K}$ under the map $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is an admissible ideal of $\mathcal{O}_X$.

2. The map $\mathcal{K} \mapsto i^{-1}\mathcal{K}\mathcal{O}_Y$ gives a bijection from the set of all admissible ideals of $\mathcal{O}_X$ containing $\mathfrak{I}$ to the set of all quasi-coherent ideals of finite type of $\mathcal{O}_Y$. The inverse mapping is given by the inverse image under $\mathcal{O}_X \to i_*\mathcal{O}_Y$.

**Proof.** We may assume without loss of generality that $X$ has an ideal of definition $I$ such that $I \subseteq \mathfrak{I}$. We regard quasi-coherent sheaves on $Y$ as quasi-coherent sheaves on the scheme $X_0 = (X, \mathcal{O}_X/I)$ and apply 3.7.3. □
Proposition 3.7.9. Let $X$ be an adic formal scheme. If $\mathcal{J}$ and $\mathcal{J}'$ are admissible ideals on $X$, then $\mathcal{J}\mathcal{J}'$ and $\mathcal{J} + \mathcal{J}'$ are admissible ideals.

Proof. We can assume that $X$ admits an ideal of definition of finite type $I$ contained in $\mathcal{J}$ and $\mathcal{J}'$. Consider the closed subscheme $X_1 = (X, \mathcal{O}_X/I^2)$. One easily sees that $\mathcal{J}\mathcal{J}'$ coincides with the pull-back of $\mathcal{J}\mathcal{O}_{X_1} \cdot \mathcal{J}'\mathcal{O}_{X_1}$. Similarly, $\mathcal{J} + \mathcal{J}'$ is the pull-back of $\mathcal{J}\mathcal{O}_{X_1} + \mathcal{J}'\mathcal{O}_{X_1}$. □

Proposition 3.7.10. Let $f : X \to Y$ be an adically flat (1.5.5 (1)) morphism of adic formal schemes of finite ideal type, and $\mathcal{J} \subseteq \mathcal{O}_Y$ an admissible ideal of $Y$. Then we have

$$\widehat{f^*\mathcal{J}} = \mathcal{J}\mathcal{O}_X.$$ 

In particular, $\widehat{f^*\mathcal{J}}$ is an admissible ideal of $X$ (cf. 3.6.3).

Proof. We can assume that $Y$ has an ideal of definition $I$ of finite type. Let $f_k : X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X) \to Y_k = (Y, \mathcal{O}_Y/I^{k+1})$ be the induced morphism of schemes for $k \geq 0$. By 3.6.5, we have

$$\widehat{f^*\mathcal{J}} = \lim_{k \geq 1} f_k^*\mathcal{J}/I^{k+1}\mathcal{J} = \lim_{k \geq 1} \mathcal{J}\mathcal{O}_{X_k} = \mathcal{J}\mathcal{O}_X,$$

as desired. □

3.7. (c) Extension of admissible ideals. Let us introduce a relation $\sim$ on the set $\text{AId}_X$ as follows: $\mathcal{J} \sim \mathcal{J}'$ for $\mathcal{J}, \mathcal{J}' \in \text{AId}_X$ if there exist $m, n > 0$ such that $\mathcal{J}^m \subseteq \mathcal{J}'^n \subseteq \mathcal{J}$. This gives an equivalence relation compatible with the semigroup structure on $\text{AId}_X$. Note that the set of all ideals of definition of finite type on $X$ (if they exist) forms a single equivalence class.

Proposition 3.7.11. Let $X$ be a coherent (1.6.6) adic formal scheme and $X = \cup_{\alpha \in L} U_\alpha$ a finite covering by quasi-compact open subsets. Suppose that for each $\alpha \in L$ an admissible ideal $I_\alpha$ on $U_\alpha$ is given for which on $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we have $I_\alpha|U_{\alpha\beta} \sim I_\beta|U_{\alpha\beta}$. Then there exists an admissible ideal $I$ on $X$ such that we have $I|U_\alpha \sim I_\alpha$ on each $U_\alpha$.

Proof. By an easy inductive argument, it suffices to show the following statement: let $X = U_1 \cup U_2$ with $U_1$ and $U_2$ quasi-compact, and $I_1$ and $I_2$ admissible ideals on $U_1$ and $U_2$, respectively, such that on $U_{12} = U_1 \cap U_2$ (which is quasi-compact, since $X$ is quasi-separated (1.6.5)) there exists a positive integer $m$ for which $I_1^m \subseteq I_2 \subseteq I_1$; then there exists an admissible ideal $I$ on $X$ such that $I_1^m \subseteq I \subseteq I_1$ on $U_1$ and $I = I_2$ on $U_2$. 


Let $V$ be the closed subscheme of $U_1$ defined by $I_1^m$ (3.7.7). Similarly, let $W$ be the closed subscheme of $U_{12}$ defined by $I_1^m$ (restricted to $U_{12}$). Then since $U_{12}$ is quasi-compact, $W$ is a quasi-compact open subscheme of $V$.

Consider the quasi-coherent ideal $I_2/I_1^m$ of finite type on $W$. By [54], I, (9.4.2) (ii), one can extend this ideal to a quasi-coherent ideal $x_21$ on $V$. Replacing $x_21$ by $x_21/\Gamma$ if necessary, we may assume that the extension $x_21$ is contained in $I_1/I_1^m$. Moreover, by [54], I, (9.4.9), and IV, (1.7.7), there exists a subideal of $x_21$ of finite type that coincides with $I_2/I_1^m$ on $W$ (since $I_2/I_1^m$ is of finite type). Hence, we may further assume that $x_21$ is quasi-coherent of finite type on $V$.

Let $I_{21}$ be the ideal on $U_1$ obtained by taking the inverse image of $x_21$ under the map $O_{U_1} \to O_V$. By 3.7.8 (1), $I_{21}$ is an admissible ideal of $O_{U_1}$. Since $I_{21}/I_1^m$ coincides with $I_2/I_1^m$ on $W$, we have $I_{21} = I_2$ on $U_{12}$ by 3.7.8 (2). Moreover, $I_{21} \subseteq I_1$. Hence we get, by gluing, the admissible ideal $I$ with the desired properties.

Corollary 3.7.12. Let $X$ be a coherent adic formal scheme of finite ideal type. Then $X$ has an ideal of definition $I$ of finite type.

Proof. Take a finite open covering $\{U_\alpha\}_{\alpha \in L}$ such that on each $U_\alpha$ there exists an ideal of definition $I_\alpha$ of finite type. Then apply 3.7.11. □

Corollary 3.7.13. Let $A$ be an admissible ring (1.1.3) such that $X = \text{Spf} A$ is an affine adic formal scheme of finite ideal type. Then $A$ is an adic ring of finite ideal type.

Proof. There exists an ideal of definition of finite type $I$ on $X$ (3.7.12). By 1.1.19, we have the unique finitely generated ideal of definition $I^{(n)} \subseteq A$ such that $I^n = (I^{(n)})^\Delta$ for each $n > 0$. Since $\{I^{(n)}\}_{n>0}$ gives a fundamental system of ideals of definition on $X$ (1.1.24), $\{I^{(n)}\}_{n>0}$ is a fundamental system of ideals of definition of the admissible ring $A$. Set $I = I^{(1)}$. Since each $X_k = (X, O_X/I^{k+1})$ $(k \geq 0)$ is affine, we have for $m \leq n$

$$I^{(m)}/I^{(n)} = \Gamma(X, I^m O_X/I^n) = I^m(A/I^{(n)});$$

moreover, $I^{(1)}/I^{(2)}$ is obviously a finitely generated ideal of $A/I^{(2)}$. Hence, by 0.7.2.11, $I^{(n)} = I^n$ for $n > 0$, and hence $A$ is adic of finite ideal type. □

By this and 2.1.9 we deduce the following result.

Corollary 3.7.14. Let $A$ be an admissible ring such that $X = \text{Spf} A$ is universally adhesive (resp. universally rigid-Noetherian) (2.1.7). Then $A$ is t.u. adhesive (resp. t.u. rigid-Noetherian) (2.1.1).
Proposition 3.7.15 (extension lemma). Let $X$ be a coherent adic formal scheme and $U$ a quasi-compact open subset of $X$. Then for any admissible ideal $\mathcal{J}$ on $U$ there exists an admissible ideal $\widetilde{\mathcal{J}}$ on $X$ such that $\widetilde{\mathcal{J}}|_U = \mathcal{J}$. (That is, the restriction map $\text{AId}_X \rightarrow \text{AId}_U$ is surjective.)

Proof. Let $I$ be an ideal of definition of $X$ (3.7.12). We may assume that $I \subseteq \mathcal{J}$ on $U$, since $U$ is quasi-compact. Let $V$ (resp. $W$) be the closed subscheme of $X$ (resp. $U$) defined by $I$. Then $W$ is a quasi-compact open subset of $V$. We apply [54], I, (9.4.2) (ii), and [54], I, (9.4.9) and IV, (1.7.7), to get a quasi-coherent extension $\mathcal{F}$ of finite type on $X$ and an admissible ideal $\widetilde{\mathcal{J}}$ on $X$ (3.7.8 (1)). Using 3.7.8 (2), one sees that this $\widetilde{\mathcal{J}}$ is a desired extension. \hfill \Box

Exercises

Exercise I.3.1. Let $X$ be an adic formal scheme, $I$ an ideal of definition of finite type of $X$, and $\mathcal{B}$ an a.q.c. $\mathcal{O}_X$-algebra. Show that $\mathcal{B}$ is an adically quasi-coherent $\mathcal{O}_X$-module sheaf of finite type if and only if $\mathcal{B}/I\mathcal{B}$ is a quasi-coherent sheaf of finite type over the scheme $X_0 = (X, \mathcal{O}_X/I)$.

Exercise I.3.2. Let $X$ be a locally universally rigid-Noetherian formal scheme and $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ a morphism of a.q.c. sheaves of finite type on $X$. Show that ker($\varphi$) and coker($\varphi$) are a.q.c. sheaves on $X$. Show, moreover, that if $X$ is coherent and $I$ is an ideal of definition of finite type on $X$, then the $I$-torsion parts of ker($\varphi$) and coker($\varphi$) are bounded $I$-torsion.

Exercise I.3.3 (extension of adically quasi-coherent sheaves). Let $X$ be a coherent adic formal scheme with an ideal of definition $I$ of finite type and $U$ a quasi-compact open subset of $X$. Let $\mathcal{F}$ be an a.q.c. sheaf of finite type and $\mathcal{G}$ an a.q.c. subsheaf of $\mathcal{F}|_U$ of finite type such that $I\mathcal{F}|_U \subseteq \mathcal{G}$. Show that there exists an a.q.c. subsheaf $\mathcal{G}'$ of $\mathcal{F}$ of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.

Exercise I.3.4. Let $X$ be a coherent universally rigid-Noetherian formal scheme with an ideal of definition $I$ of finite type, $\mathcal{F}$ an a.q.c. sheaf of finite type on $X$, and $\mathcal{G} \subseteq \mathcal{F}$ an a.q.c. subsheaf. Show that $\mathcal{G}$ is an inductive limit $\varprojlim \mathcal{H}_\lambda$ of a.q.c. subsheaves of finite type such that for all $\lambda$, $\mathcal{G}/\mathcal{H}_\lambda$ is annihilated by $I^n$ for some $n > 0$.

Exercise I.3.5 (cf. 3.5.3). Let $A$ be a rigid-Noetherian ring with a finitely generated ideal of definition $I$, and $N \subseteq M$ an inclusion of $A$-modules, and suppose that $M$ is contained as an $A$-submodule in a finitely generated $A$-module. Show that the sequence

\[ 0 \rightarrow N^\Delta \rightarrow M^\Delta \rightarrow (M/N)^\Delta \rightarrow 0 \]

given by canonical morphisms is exact.
Exercise I.3.6. Let $X$ be a locally universally rigid-Noetherian formal scheme, and consider an exact sequence of $\mathcal{O}_X$-modules

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0.$$  

1. If $\mathcal{G}$ and $\mathcal{H}$ are a.q.c. of finite type, show that $\mathcal{F}$ is adically quasi-coherent.
2. If $\mathcal{F}$ is adically quasi-coherent and $\mathcal{G}$ is a.q.c. of finite type, show that $\mathcal{H}$ is a.q.c. of finite type.

Exercise I.3.7. Let $X$ be a locally Noetherian formal scheme (1.1.25).

1. Show that the sheaf $\mathcal{O}_X$ is coherent.
2. Show that an $\mathcal{O}_X$-module sheaf $\mathcal{F}$ is coherent if and only if it is a.q.c. of finite type.
3. Show that any a.q.c. ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ is coherent.

Exercise I.3.8. Let $X$ be a locally universally adhesive formal scheme, and $\mathcal{J}, \mathcal{J}' \subseteq \mathcal{O}_X$ admissible ideals. Suppose that $X$ has an ideal of definition $I$ and that $\mathcal{O}_X$ is $I$-torsion free. Show that $\mathcal{J} \cap \mathcal{J}'$ is admissible.

4 Several properties of morphisms

4.1 Affine morphisms

4.1. (a) Definition of affine morphisms. First of all, let us recall the definition of affine morphisms of formal schemes ([53], (10.16.1)).

Definition 4.1.1. A morphism $f : X \rightarrow Y$ of formal schemes is said to be affine if there exists an affine open covering $Y = \bigcup_{\alpha \in L} V_\alpha$ of $Y$ such that for each $\alpha \in L$ the open formal subscheme $f^{-1}(V_\alpha)$ in $X$ is an affine formal scheme.

For example, any morphism between affine formal schemes is affine.

Proposition 4.1.2. Let $f : X \rightarrow Y$ be a morphism of formal schemes and $Y = \bigcup_{\alpha \in L} V_\alpha$ an open covering of $Y$. Then $f$ is affine if and only if the induced map

$$f_\alpha : X_\alpha = X \times_Y V_\alpha \rightarrow V_\alpha$$

is affine for any $\alpha \in L$.

Proof. The ‘if’ part is trivial. The ‘only if’ part follows from the following observation: Let $U = \text{Spf} \ B$ be an affine open subset of $Y$ such that $f^{-1}(U)$ is affine: $f^{-1}(U) = \text{Spf} \ A$. Then $V_\alpha \cap U$ for each $\alpha \in L$ is covered by affine open subsets of the form $W = \text{Spf} \ B_{(g)}$, and so $f^{-1}(W) = \text{Spf} \ A \otimes_\mathbb{B} B_{(g)}$. \qed
4. Several properties of morphisms

4.1. (b) Affine adic morphisms and a.q.c. sheaves

**Proposition 4.1.3.** Let \( f : X \to Y \) be an affine adic (1.3.1) morphism between adic formal schemes of finite ideal type. Then for any a.q.c. sheaf \( \mathcal{F} \) on \( X \), \( f_* \mathcal{F} \) is an a.q.c. sheaf on \( Y \).

Since the question is local on \( Y \), we may assume that \( X \) and \( Y \) are affine \( X = \text{Spf} \ A \) and \( Y = \text{Spf} \ B \), where \( A \) and \( B \) are adic rings of finite ideal type, and that the map \( B \to A \) is adic; we have \( \mathcal{F} = M^\Delta \) by an \( A \)-module \( M \) (3.2.7). Then the proposition follows from the following lemma and 3.2.2.

**Lemma 4.1.4.** In the situation as above, we have

\[
 f_* M^\Delta = M^\Delta_{[B]},
\]

where \( M_{[B]} \) denotes the module \( M \) regarded as a \( B \)-module.

**Proof.** By 3.2.2 and 0.3.2.9 (2),

\[
 f_* M^\Delta = \lim_{k \geq 0} M/I^{k+1}M = \lim_{k \geq 0} f_* M/I^{k+1}M = \lim_{k \geq 0} M_{[B]}/I^{k+1}M_{[B]},
\]

where the last projective limit sheaf is equal to \( M^\Delta_{[B]} \), as desired. \( \square \)

4.1. (c) Formal spectra of a.q.c. algebras. As a corollary of 4.1.3 we have the following corollary.

**Corollary 4.1.5.** Let \( f : X \to Y \) be an affine adic morphism of adic formal schemes of finite ideal type. Then \( f_* \mathcal{O}_X \) is an a.q.c. \( \mathcal{O}_Y \)-algebra. Moreover, we have the following facts.

1. \( f_* \mathcal{O}_X \) is of finite type (3.3.1) if and only if \( f : X \to Y \) is locally of finite type (1.7.1).

2. Suppose \( Y \) is locally universally rigid-Noetherian (2.1.7). Then \( f_* \mathcal{O}_X \) is of finite presentation (3.5.11) if and only if \( f : X \to Y \) is locally of finite presentation (2.2.1).

**Proof.** We may suppose that \( X \) and \( Y \) are affine; if \( f : X = \text{Spf} \ A \to Y = \text{Spf} \ B \) comes from an adic morphism \( B \to A \) of adic rings of finite ideal type, then we have \( f_* \mathcal{O}_X = A^\Delta_{[B]} \). Then apply 3.3.3 and 3.5.13. \( \square \)

**Proposition 4.1.6.** Let \( f : X \to Y \) and \( f' : X' \to Y \) be affine adic morphisms of adic formal schemes of finite ideal type. Then the induced map

\[
 \text{Hom}_Y(X, X') \to \text{Hom}_{\text{Alg}_Y}(f_* \mathcal{O}_{X'}, f_* \mathcal{O}_X)
\]

is bijective.
Proof. We may work in the affine situation: \( Y = \text{Spf} \ B, X = \text{Spf} \ A, \) and \( X' = \text{Spf} \ A', \) and \( f \) and \( f' \) come respectively from adic morphisms \( B \to A \) and \( B \to A' \) of adic rings of finite ideal type. Then by [54], I, (10.1.3), the set \( \text{Hom}_Y (X, X') \) is identified with the set of continuous homomorphisms \( A' \to A \) of topological rings over \( B. \) But since \( A \) and \( A' \) are the adic over \( B, \) the last set is simply the set \( \text{Hom}_{B} (A', A) \) of \( B \)-algebra homomorphisms. By 3.2.9, this is further isomorphic to \( \text{Hom}_{B} (A', A) \). Now, since \( A = f_* \mathcal{O}_X \) etc., we obtain the desired result.

Proposition 4.1.7. Let \( Y \) be an adic formal scheme of finite ideal type, and \( \mathcal{A} \) an a.q.c. \( \mathcal{O}_Y \)-algebra. Then there exists a unique, up to \( Y \)-isomorphisms, affine morphism \( f : X \to Y \) such that \( f_* \mathcal{O}_X \cong \mathcal{A}. \)

Proof. By 3.2.8, for any affine open subsets \( V = \text{Spf} \ B_V \) of \( Y \) (\( B_V \) is an adic ring of finite ideal type due to 3.7.13), \( \mathcal{A}(V) = A_V \) is an \( I_V \)-adically complete \( B_V \)-algebra, where \( I_V \) is an ideal of definition of \( B_V, \) and \( A_{V|V} = \mathcal{A}_{|V} \) holds. Take an affine open covering \( Y = \bigcup_{\alpha \in L} V_\alpha \) of \( Y, \) and set \( X_\alpha = \text{Spf} \ A_{V_\alpha} \) for each \( \alpha \in L. \) By 4.1.6, these formal schemes glue to an adic formal scheme \( X = \bigcup_{\alpha \in L} X_\alpha \) adic over \( Y. \) The map \( f : X \to Y \) thus obtained is affine and adic, and \( f_* \mathcal{O}_X \cong \mathcal{A}. \) Note that the above construction does not depend on the choice of the affine covering \( \{V_\alpha\}_{\alpha \in L} \) (due to 4.1.6).

Let \( Y \) be an adic formal scheme of finite ideal type, and consider the functor

\[
\text{AfAcFs}^{*, \text{opp}}_{Y} \to \text{AQCohAlg}_Y,
\]

where \( \text{AQCohAlg}_Y \) is the category of adically quasi-coherent \( \mathcal{O}_Y \)-algebras (with \( \mathcal{O}_Y \)-algebra homomorphisms), which maps each affine adic morphism \( f : X \to Y \) to \( f_* \mathcal{O}_X. \) By 4.1.6 and 4.1.7, we have the following result.

Theorem 4.1.8. The functor \( (*) \) is a categorical equivalence.

Definition 4.1.9. Let \( X \) be an adic formal scheme of finite ideal type, and \( \mathcal{A} \) an a.q.c. \( \mathcal{O}_X \)-algebra. Then by 4.1.7 there exist a unique, up to isomorphism over \( X, \) an adic formal scheme \( Z \) and an affine adic map \( f : Z \to X \) such that \( f_* \mathcal{O}_Z \cong \mathcal{A}. \) We denote this formal scheme \( Z \) by \( \text{Spf} \ \mathcal{A} \) and call it the formal spectrum of \( \mathcal{A}. \)

Similarly to the scheme case, the formation of the formal spectra is an example of ‘effective local construction’ (with respect to the Zariski topology) in 0.1.4.9.

4.1. (d) Basic properties of affine adic morphisms

Proposition 4.1.10. Let \( f : X \to Y \) be an adic morphism of adic formal schemes of finite ideal type. Then \( f \) is an affine map if and only if for any affine open subset \( V = \text{Spf} \ B \) of \( Y \) the open formal subscheme \( f^{-1}(V) \) of \( X \) is affine.
Proof. The ‘if’ part is trivial. Suppose $f : X \to Y$ is affine adic, and let $\mathcal{A} = f_* \mathcal{O}_X$. By 4.1.8, $X \cong \text{Spf} \mathcal{A}$ over $Y$. Let $V = \text{Spf} B$ be an affine open subset of $Y$. By 3.7.13, $B$ is an adic ring of finite ideal type, and so is $\mathcal{A}(V)$, due to 3.2.8. Then by the construction of $\text{Spf} \mathcal{A}$ we find that $X \times_Y V \cong \text{Spf} \mathcal{A}|_V = \text{Spf} \mathcal{A}(V)$. □

**Proposition 4.1.11.** (1) The composition of two affine adic morphisms of adic formal schemes of finite ideal type is again affine adic.

(2) For any affine adic $S$-morphisms $f : X \to Y$ and $g : X' \to Y'$ of adic formal schemes of finite ideal type over an adic formal scheme $S$ of finite ideal type, the induced morphism

$$f \times_S g : X \times_S Y \longrightarrow X' \times_S Y'$$

is affine adic.

(3) For any affine adic $S$-morphism $f : X \to Y$ of adic formal schemes of finite ideal type over an adic formal scheme $S$ of finite ideal type, and for any morphism $S' \to S$ of adic formal schemes of finite ideal type, the induced morphism

$$f_{S'} : X \times_S S' \longrightarrow Y \times_S S'$$

is affine adic.

Proof. (1) follows immediately from 4.1.10. By 1.5.1, (2) and (3) follow from 4.1.12 below combined with 4.1.2 and [54], II, (1.6.2). □

**Proposition 4.1.12.** Let $f : X \to Y$ be an adic morphism of adic formal schemes of finite ideal type, and suppose $Y$ has an ideal of definition $I$ of finite type. For any integer $k \geq 0$ we denote by

$$f_k : X_k = (X, \mathcal{O}_X/I^{k+1} \mathcal{O}_X) \longrightarrow Y_k = (Y, \mathcal{O}_Y/I^{k+1})$$

the induced morphism of schemes. Then the following conditions are equivalent.

(a) $f$ is affine.

(b) $f_k$ is affine for any $k \geq 0$.

(c) $f_0$ is affine.

Proof. By 4.1.2, we may assume that $Y$ is affine, $Y = \text{Spf} B$, with the ideal $I$ of definition of $B$. Suppose $f$ is affine. Then by 4.1.10 $X$ is an affine formal scheme $X = \text{Spf} A$ and, in this situation, we have $X_k = \text{Spec} A_k$, where $A_k = A/I^{k+1}A$. Hence implication (a) $\implies$ (b) holds. (b) $\implies$ (c) is trivial. Suppose (c) holds. By [53], (2.3.5), we deduce that $f_k$ is affine for any $k \geq 0$. Set $X_k = \text{Spec} A_k$ for each $k$. Then $\{A_k\}_{k \geq 0}$ forms a projective system of rings such that for $k \leq l$ the transition map $A_l \to A_k$ is surjective with the kernel equal to $I^{k+1}A_l$. Then by 0.7.2.12 we see that $A$ is $IA$-adically complete and that $A/I^{k+1} = A_k$ for any $k \geq 0$. Therefore, $X = \text{Spf} A$. □
In view of 1.5.2, 4.1.12 and [54], II, (1.6.2) yield the following result.

**Corollary 4.1.13.** Let \( f : X \to Y \) be a morphism of schemes and \( Z \) a closed subscheme of \( Y \) of finite presentation. If \( f \) is affine, then the formal completion
\[
\hat{f} : \hat{X}|_{f^{-1}(Z)} \to \hat{Y}|_Z
\]
is affine.

### 4.2 Finite morphisms

**Proposition 4.2.1.** The following conditions for an adic morphism \( f : X \to Y \) of adic formal schemes of finite ideal type are equivalent.

(a) There exists an affine open covering \( \{ V_\alpha = \text{Spf } A_\alpha \}_{\alpha \in L} \) of \( Y \), where each \( A_\alpha \) is an adic ring with a finitely generated ideal of definition \( I_\alpha \subseteq A_\alpha \), such that for each \( \alpha \in L \) the induced morphism \( X \times_Y V_{\alpha,0} \to V_{\alpha,0} \) (where \( V_{\alpha,0} = \text{Spec } A_\alpha/I_\alpha \)) of schemes is finite.

(a)' For any affine open set \( V = \text{Spf } A \) of \( Y \), where \( A \) is an adic ring with a finitely generated ideal of definition \( I \subseteq A \), the induced morphism \( X \times_Y V_0 \to V_0 \) (where \( V_0 = \text{Spec } A/I \)) of schemes is finite.

(b) There exists an affine open covering \( \{ V_\alpha = \text{Spf } A_\alpha \}_{\alpha \in L} \) of \( Y \), where each \( A_\alpha \) is an adic ring with a finitely generated ideal of definition \( I_\alpha \subseteq A_\alpha \), such that for each \( \alpha \in L \) the inverse image \( f^{-1}(V_\alpha) \) is affine of the form \( f^{-1}(V_\alpha) = \text{Spf } B_\alpha \), where \( B_\alpha \) is finitely generated as an \( A_\alpha \)-module.

(b)' For any affine open set \( V = \text{Spf } A \) of \( Y \), where \( A \) is an adic ring with a finitely generated ideal of definition \( I \subseteq A \), \( f^{-1}(V) \) is affine of the form \( f^{-1}(V) = \text{Spf } B \), where \( B \) is finitely generated as an \( A \)-module.

**Proof.** Implications (b) \(\Rightarrow\) (a) and (b)' \(\Rightarrow\) (a)' are immediate. Let us show implications (a) \(\Rightarrow\) (b) and (a)' \(\Rightarrow\) (b)' Suppose (a)' holds. We set \( V_k = \text{Spf } A/I^{k+1} \) and \( U_k = X \times_Y V_k \) for \( k \geq 0 \). By [54], I, (5.1.9), each \( X_k \) is an affine scheme, \( X_k = \text{Spec } B_k \), and for \( k \leq l \) we have \( B_k = B_l/I^{k+1}B_l \). Then, by 0.7.2.12, \( B = \lim_{\leftarrow k \geq 0} B_k \) is an IB-adically complete A-algebra such that \( B/I^{k+1}B = B_k \) for \( k \geq 0 \). Since \( B_0 \) is finitely generated as an \( A \)-module, 0.7.2.4 implies that \( B \) is finitely generated as an \( A \)-module. Moreover, since \( f : X \to Y \) is adic, we have \( f^{-1}(V) = \lim_{\leftarrow k \geq 0} U_k = \text{Spf } B \) and thus (b)' holds. The other implication (a) \(\Rightarrow\) (b) can be verified similarly.

It remains to show the equivalence of (a) and (a)' As implication (a)' \(\Rightarrow\) (a) is clear, we want to show the converse. First note that in view of [54], I, (5.1.9), each ideal of definition \( I_\alpha \) of \( A_\alpha \) can be replaced by a power \( I_\alpha^n \) for any \( n \geq 1 \). Note also
that the covering \( \{ V_\alpha = \text{Spf} A_\alpha \}_{\alpha \in L} \) can be replaced by a refinement obtained by replacing each \( V_\alpha = \text{Spf} A_\alpha \) by a finite affine covering by affine open subschemes of the form \( \text{Spf}(A_\alpha)_{(g)} \). Hence the desired implication follows from the fact that finite morphisms of schemes are stable under the Zariski topology (0.1.4.8 (1)). □

**Definition 4.2.2.** Let \( X \) and \( Y \) be adic formal schemes of finite ideal type. Then a morphism \( f : X \to Y \) is said to be finite if it is adic and satisfies the conditions in Proposition 4.2.1.

**Proposition 4.2.3.** Let \( f : X \to Y \) be an adic morphism of adic formal schemes of finite ideal type, and suppose \( Y \) has a ideal of definition \( I \) of finite type. For any integer \( k \geq 0 \) we denote by

\[
  f_k : X_k = (X, \mathcal{O}_X / I^{k+1}) \to Y_k = (Y, \mathcal{O}_Y / I^{k+1})
\]

the induced morphism of schemes. Then the following conditions are equivalent.

(a) \( f \) is finite.

(b) \( f_k \) is finite for any \( k \geq 0 \).

(c) \( f_0 \) is finite.

**Proof.** We may assume that \( Y \) is affine of the form \( Y = \text{Spf} A \), where \( A \) is an adic ring with the finitely generated ideal of definition \( I \) such that \( I = I^A \). Then the equivalence of (b) and (c) follows from [54], I, (5.1.9), and 0.7.2.4. The equivalence of (a) and (c) follows from the definition. □

**Proposition 4.2.4.** (1) Any finite morphism is affine (4.1.1).

(2) The composition of two finite morphisms is again finite.

(3) For any finite \( S \)-morphisms \( f : X \to Y \) and \( g : X' \to Y' \) of adic formal schemes of finite ideal type over an adic formal scheme \( S \) of finite ideal type, the induced morphism

\[
  f \times_S g : X \times_S Y \to X' \times_S Y'
\]

is finite.

(4) For any finite \( S \)-morphism \( f : X \to Y \) of adic formal schemes of finite ideal type over an adic formal scheme \( S \) of finite ideal type, and for any morphism \( S' \to S \) of adic formal schemes of finite ideal type, the induced morphism

\[
  f_{S'} : X \times_S S' \to Y \times_S S'
\]

is finite.

**Proof.** (1) and (2) are clear. In view of 1.5.1, (3) and (4) follow from 4.2.3, 4.2.1, and [54], II, (6.1.5). □
In view of 1.5.2, 4.2.3 and [54], II, (6.1.5) yield the following result.

**Corollary 4.2.5.** Let \( f : X \to Y \) be a morphism of schemes and \( Z \) a closed subscheme of \( Y \) of finite presentation. Then if \( f \) is finite, the formal completion

\[
\hat{f} : \hat{X}_{f^{-1}(Z)} \to \hat{Y}|_Z
\]

is finite.

Since any finite morphism \( f : X \to Y \) is affine, it comes from a formal spectrum \( X \cong \text{Spf } A \to Y \) (4.1.9). By 3.2.8 (2), we have the following proposition (cf. Exercise I.3.1).

**Proposition 4.2.6.** Let \( Y \) be an adic formal scheme of finite ideal type, and \( A \) an a.q.c. \( \mathcal{O}_Y \)-algebra. Then the map \( \text{Spf } A \to Y \) is finite if and only if \( A \) is an a.q.c. \( \mathcal{O}_Y \)-module of finite type.

### 4.3 Closed immersions

**4.3. (a) A preliminary result**

**Proposition 4.3.1.** Let \( X \) be an adic formal scheme of finite ideal type, \( \mathcal{F} \) an a.q.c. sheaf on \( X \), and \( \mathcal{K} \subseteq \mathcal{F} \) an \( \mathcal{O}_X \)-submodule. Suppose that the quotient \( \mathcal{G} = \mathcal{F}/\mathcal{K} \) is an a.q.c. sheaf.

1. Suppose \( X \) has an ideal of definition \( I \) of finite type. Then the morphism

\[
\mathcal{K} \to \lim_{\to} \mathcal{K}/\mathcal{K} \cap I^{k+1} \mathcal{F}
\]

is an isomorphism.

2. For any affine open set \( U = \text{Spf } A \) of \( X \), where \( A \) is an adic ring of finite ideal type, the sequence

\[
0 \to \Gamma(U, \mathcal{K}) \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G}) \to 0
\]

is exact.

**Proof.** Consider the exact sequence

\[
0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{G} \to 0 \quad (\ast)
\]

of \( \mathcal{O}_X \)-modules. Taking \( \otimes \mathcal{O}_X \mathcal{O}_X/I^{k+1} \), we get the exact sequence

\[
0 \to \mathcal{K}/\mathcal{K} \cap I^{k+1} \mathcal{F} \to \mathcal{F}_k \to \mathcal{G}_k \to 0 \quad (\ast)_k
\]
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(where \( F_k = F / I^{k+1} F \), etc.) for any \( k \geq 0 \). The exact sequences \((*)_k\) induce, by passage to the projective limits, the exact sequence

\[
0 \rightarrow \lim_{k \geq 0} \mathcal{K} / \mathcal{K} \cap I^{k+1} F \rightarrow \lim_{k \geq 0} F_k \rightarrow \lim_{k \geq 0} G_k.
\]

Since \( F \) and \( G \) are a.q.c. sheaves, \( F \cong \lim_{k \geq 0} F_k \) and \( G \cong \lim_{k \geq 0} G_k \). Hence, comparing the last exact sequence with \((*)\), we get (1). To show (2), first note that, due to \((*)_k\), \( \mathcal{K} / \mathcal{K} \cap I^{k+1} F \) is a quasi-coherent sheaf on the scheme \( X_k = (X, \mathcal{O}_X / I^{k+1}) \); then the assertion follows from 1.1.23 (2).

\textbf{Corollary 4.3.2.} Let \( A \) be an adic ring of finite ideal type, and \( I \subseteq A \) a finitely generated ideal of definition. Let \( \mathcal{F} \) be an a.q.c. sheaf of finite type on \( X = \text{Spf} \, A \), and \( \mathcal{K} \subseteq \mathcal{F} \) an \( \mathcal{O}_X \)-submodule such that \( \mathcal{G} = \mathcal{F} / \mathcal{K} \) is a.q.c. Then \( \Gamma(X, \mathcal{K}) \) is closed in \( \Gamma(X, \mathcal{F}) \) with respect to the \( I \)-adic topology.

\textbf{Proof.} Since \( \mathcal{F} \) and \( \mathcal{G} = \mathcal{F} / \mathcal{K} \) are a.q.c. of finite type, \( \Gamma(X, \mathcal{F}) \) and \( \Gamma(X, \mathcal{G}) \) are \( I \)-adically complete finitely generated \( A \)-modules (3.2.8 (2)). Then the result follows from 4.3.1 (2) and 0.7.4.6.

\textbf{4.3. (b) Definitions and first properties}

\textbf{Definition 4.3.3.} Let \( X \) be an adic formal scheme of finite ideal type. A \textit{closed formal subscheme} of \( X \) is a formal scheme of the form \( (Y, (\mathcal{O}_X / \mathcal{K})|_Y) \), where \( \mathcal{K} \) is an ideal of \( \mathcal{O}_X \) such that \( \mathcal{O}_X / \mathcal{K} \) is a.q.c. and \( Y \) is the support of the sheaf \( \mathcal{O}_X / \mathcal{K} \).

Note that the subset \( Y \) is closed in \( X \) due to [54], 0.1, (5.2.2). Moreover, one can show that the ringed space \( (Y, (\mathcal{O}_X / \mathcal{K})|_Y) \) is actually an adic formal scheme of finite ideal type. Indeed, let \( U = \text{Spf} \, A \) be an affine open set of \( X \), where \( A \) is an adic ring with a finitely generated ideal of definition \( I \subseteq A \), and set \( K = \Gamma(U, \mathcal{K}) \). Then by 4.3.2 \( B = A / K \) is an \( I \)-adically complete algebra over \( A \) and hence is an adic ring. By 4.3.1 (2) and 3.2.8 (2), we have \( (\mathcal{O}_X / \mathcal{K})|_U = B^\Delta \). Hence \( Y \) is covered by affine formal scheme as of the form \( \text{Spf} \, B \).

Note that we do not assume in 4.3.3 that the ideal \( \mathcal{K} \) itself is a.q.c.

\textbf{Definition 4.3.4.} A morphism \( i : Z \rightarrow X \) of adic formal schemes of finite ideal type is said to be a \textit{closed immersion} if it admits a factorization \( Z \xrightarrow{\sim} Y \xleftarrow{\sim} X \) as an isomorphism onto a closed formal subscheme \( Y \) of \( X \) followed by the canonical morphism.

\textbf{Proposition 4.3.5.} A closed immersion is finite (4.2.2).
We will see in 4.3.13 below that the notions of closed formal subschemes and closed immersions thus defined coincide in the locally Noetherian case with those in [54], I, §10.14.

**Proposition 4.3.6.** Let \( i: Z \to X \) be an adic morphism of adic formal schemes of finite ideal type, and suppose \( X \) has an ideal of definition \( I \) of finite type. For any integer \( k \geq 0 \) we denote by
\[
i_k: Z_k = (Z, \mathcal{O}_Z/I^{k+1}\mathcal{O}_Z) \to X_k = (X, \mathcal{O}_X/I^{k+1})
\]
the induced morphism of schemes. Then the following conditions are equivalent.

(a) \( i \) is a closed immersion.

(b) \( i_k \) is a closed immersion for any \( k \geq 0 \).

(c) \( i_0 \) is a closed immersion.

**Proof.** Let us first show implication (a) \( \implies \) (b). Consider the exact sequence
\[
0 \to \mathcal{K} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0,
\]
which induces for any \( k \geq 0 \) the exact sequence
\[
0 \to \mathcal{K}_k \to \mathcal{O}_{X_k} \to i_k_*\mathcal{O}_{Z_k} \to 0,
\]
where \( \mathcal{K}_k = \mathcal{K}/\mathcal{K} \cap I^{k+1} \). Since \( i_*\mathcal{O}_Z \) is a.q.c. of finite type (4.2.6), \( i_k_*\mathcal{O}_{Z_k} \) is quasi-coherent on \( X_k \). Hence, \( \mathcal{K}_k \) is a quasi-coherent sheaf on \( X \), and \( i_k \) is the closed immersion of schemes corresponding to the quasi-coherent ideal \( \mathcal{K}_k \).

Next we show the converse. By 4.1.12, we already know that \( i \) is affine. By 4.1.3, \( i_*\mathcal{O}_Z \) is an adically quasi-coherent \( \mathcal{O}_X \)-algebra. To show that the map \( \mathcal{O}_X \to i_*\mathcal{O}_Z \) is surjective, let \( \mathcal{K}_k \) be the kernel of \( \mathcal{O}_{X_k} \to i_k_*\mathcal{O}_{Z_k} \) for any \( k \geq 0 \).

It is easy to see that \( \mathcal{K}_k \to \mathcal{K}_l \) is surjective for \( k \geq l \) and hence that the projective system \( \{\mathcal{K}_k\}_{k\geq 0} \) is strict. Hence we have the exact sequence
\[
0 \to \lim_k \mathcal{K}_k \to \mathcal{O}_X \to \lim_k i_k_*\mathcal{O}_{Z_k} \to 0
\]
(see 0.3.2.14 (1) and 0.3.2.13 (2)). On the other hand, \( \lim_k i_k_*\mathcal{O}_{Z_k} = i_*\mathcal{O}_Z \) thanks to 0.3.2.9 (2) and 1.4.1. Hence the map \( \mathcal{O}_X \to i_*\mathcal{O}_Z \) is surjective, as desired. Now since \( i(Z) \) clearly coincides with the support of \( \mathcal{O}_X/\mathcal{K} \) (which is equal to the support of \( \mathcal{O}_{X_k}/\mathcal{K}_k \)), we deduce that \( i \) is a closed immersion. Thus we have shown the equivalence of (a) and (b). The equivalence of (b) and (c) follows from Exercise 0.5.2. \( \square \)

**Corollary 4.3.7.** Let \( A \to B \) be an adic map between adic rings of finite ideal type, and consider the morphism \( f: Y = \text{Spf } B \to X = \text{Spf } A \) of formal schemes. Then the following conditions are equivalent.
(a) $f$ is a closed immersion.

(b) $A \to B$ is surjective.

Proof. Let us first show (a) $\implies$ (b). Suppose $f$ is a closed immersion. Then we have the surjective map $\mathcal{O}_X \to f_*\mathcal{O}_Y$. Since $f$ is finite (4.3.5), $f_*\mathcal{O}_Y$ is an a.q.c. sheaf of finite type on $X$ by 4.2.6. Since $A^\Delta = \mathcal{O}_X$ and $B^\Delta = \mathcal{O}_Y$, the surjectivity of $A \to B$ follows from 4.3.1 (2).

Next we show (b) $\implies$ (a). Let $I$ be a finitely generated ideal of definition of $A$. For any $k \geq 0$ the map $A/I^{k+1} \to B/I^{k+1}B$ is surjective. Hence by 4.3.6 the morphism $f$ is a closed immersion.

Corollary 4.3.8. Let $i: Z \to X$ be a morphism of adic formal schemes of finite ideal type. Then the following conditions are equivalent.

(a) $i$ is a closed immersion.

(b) $i$ is adic and affine, and the morphism $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.

Proof. (a) $\implies$ (b) follows from 4.3.5. Let us show the converse. By 4.1.3, the sheaf $i_*\mathcal{O}_Z$ is a.q.c. on $X$. Let $\mathcal{K}$ be the kernel of $\mathcal{O}_X \to i_*\mathcal{O}_Z$. For any affine open subset $U = \text{Spf} A$ of $X$ (where $A$ is an adic ring of finite ideal type), $i^{-1}(U)$ is affine, $i^{-1}(U) = \text{Spf} B$ (where $B$ is an adic ring), and by 4.3.1 (2) the induced map $A \to B$ is surjective. Hence, by 4.3.7, the base change $f^{-1}(U) \to U$ is a closed immersion. As the underlying morphism of $i$ is, therefore, injective, it suffices to show that the set $i(Z)$ coincides with the support of $\mathcal{O}_X/\mathcal{K} = i_*\mathcal{O}_Y$. But this is clear, since they coincide on each affine open subsets.

Corollary 4.3.9. Let $f: Y \to X$ be a morphism of adic formal schemes of finite ideal type, and $X = \bigcup_{\alpha \in L} V_\alpha$ an open covering of $X$. Then $f$ is a closed immersion if and only if for any $\alpha \in L$ the base change $f^{-1}(V_\alpha) \to V_\alpha$ is a closed immersion.

Proposition 4.3.10. (1) If $f: Z \to Y$ and $g: Y \to X$ are closed immersions, then so is the composition $g \circ f$.

(2) For any $S$-closed immersions $f: X \to Y$ and $g: X' \to Y'$ of adic formal schemes of finite ideal type over an adic formal scheme $S$ of finite ideal type, the induced morphism $f \times_S g: X \times_S Y \to X' \times_S Y'$ is a closed immersion.

(3) For any $S$-closed immersion $f: X \to Y$ of adic formal schemes of finite ideal type over an adic formal scheme $S$ of finite ideal type, and for any morphism $S' \to S$ of adic formal schemes of finite ideal type, the induced morphism $f_{S'}: X \times_S S' \to Y \times_S S'$ is a closed immersion.
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Proof. (1) follows easily from 1.3.6 (1), 4.1.11 (1), and 4.3.8. By 1.5.1, (2) and (3) follow from 4.3.6, 4.3.9, and [53], (4.3.6).

Corollary 4.3.11. Let $f: X \to Y$ be a morphism of schemes, and $Z$ a closed subscheme of $Y$ of finite presentation. If $f$ is a closed immersion, then the formal completion

$$\hat{f}: \hat{X}_{\mathcal{O} X} \to \hat{Y}_{\mathcal{O} Y}$$

is a closed immersion.

Proof. Since closed immersions are closed under composition, we may apply 1.5.2, and the corollary follows from 4.3.6 and [53], (4.3.6).

4.3. (c) Universally rigid-Noetherian case

Proposition 4.3.12. Let $X$ be a locally universally rigid-Noetherian formal scheme, see (2.1.7), and $\mathcal{K} \subseteq \mathcal{O} X$ an a.q.c. ideal (resp. of finite type). Consider the locally ringed space $Y = (Y, \mathcal{O} X / \mathcal{K})$, where $Y$ is the support of the sheaf $\mathcal{O} X / \mathcal{K}$. Then $Y$ is a closed formal subscheme of $X$, and the canonical morphism $i: Y \hookrightarrow X$ is a closed immersion (resp. of finite presentation). Moreover, any closed immersion $i: Z \hookrightarrow X$ (resp. of finite presentation) can be obtained in this way up to isomorphism from a uniquely determined adically quasi-coherent ideal $\mathcal{K}$ (resp. of finite type).

Proof. Let $U = \text{Spf } A$ be an affine open set of $X$, where $A$ is a t.u. rigid-Noetherian ring (2.1.1 (1)) with a finitely generated ideal of definition $I \subseteq A$. Set $K = \Gamma(U, \mathcal{K})$. Then $K$ is an ideal (resp. a finitely generated ideal) of $A$ such that $K^\Delta = \mathcal{K}|_U$. The ring $B = A / K$ is a finitely generated $A$-module, and hence is $I$-adically complete (2.1.3). By 3.5.3, $B^\Delta = (\mathcal{O} X / \mathcal{K})|_U$, and hence $\mathcal{O} X / \mathcal{K}$ is a.q.c. Thus the first half of the proposition has been proved.

Conversely, if $i: Z \to X$ is a closed immersion (resp. of finite presentation), then for any affine open $U = \text{Spf } A$ of $X$, we have $i^{-1}(U) = \text{Spf } B$, where $A \to B$ is surjective (4.3.7). Let $K$ be the kernel, which is a finitely generated ideal if $i$ is of finite presentation (2.2.3). By 3.5.3, $K^\Delta$ is an a.q.c. sheaf (resp. of finite type) isomorphic to the kernel of $\mathcal{O} X \to i_* \mathcal{O} Z$ restricted to $U$. Hence the kernel of $\mathcal{O} X \to i_* \mathcal{O} Z$ is an a.q.c. ideal (resp. of finite type) of $\mathcal{O} X$, as desired.

Remark 4.3.13. Note that, by the proposition, our notion of ‘closed immersion’ coincides in the locally Noetherian case with that of [54], I, §10.14, indeed, if $X$ is locally Noetherian, then $X$ is locally universally adhesive, and any a.q.c. ideal is a coherent ideal (cf. Exercise I.3.7).
4. (d) **Closed immersions and admissible ideals.** As usual, for an adic formal scheme \( X \) with an ideal of definition \( \mathcal{I} \) of finite type, a closed subscheme of \( X \) means a closed subscheme of \( X_k = (X, \mathcal{O}_X / \mathcal{I}^{k+1}) \) for some \( k \geq 0 \). Since the quotient \( \mathcal{O}_X / \mathfrak{J} \) by an admissible ideal \( \mathfrak{J} \subseteq \mathcal{O}_X \) is a quasi-coherent sheaf, we have the following proposition.

**Proposition 4.3.14.** Let \( X \) be an adic formal scheme of finite ideal type. For any admissible ideal \( \mathfrak{J} \), let \( Y \) be the closed formal subscheme of \( X \) corresponding to \( \mathfrak{J} \). Then \( Y \) is a scheme.

**Proposition 4.3.15.** Let \( i: Y \hookrightarrow X \) be a closed immersion of finite presentation between locally universally rigid-Noetherian formal schemes, and \( \mathcal{K} \) an admissible ideal of \( \mathcal{O}_Y \). Let \( \mathfrak{J} \) be the pull-back of \( i_* \mathcal{K} \) by the map \( \mathcal{O}_X \rightarrow i_* \mathcal{O}_Y \). Then \( \mathfrak{J} \) is an admissible ideal of \( \mathcal{O}_X \).

**Proof.** We may work in the affine situation \( X = \text{Spf } B \) and \( Y = \text{Spf } A \), where \( A, B \) are t.u. rigid-Noetherian rings and \( \varphi: B \to A \) is a surjective adic homomorphism with finitely generated kernel. Let \( K \subseteq A \) be a finitely generated open ideal such that \( \mathcal{K} = K^\Delta \), and \( J = \varphi^{-1}(K) \), which is a finitely generated open ideal of \( B \). By 4.1.3, \( i_* \mathcal{K} \) and \( i_* \mathcal{O}_Y \) are a.q.c. sheaves on \( X \) of finite type. Hence, by the exactness of the functor \( \cdot^\Delta \) (3.5.6), we have \( \mathfrak{J} = J^\Delta \).

4.4 **Immersions**

**Definition 4.4.1.** A morphism \( f: Y \to X \) of adic formal schemes of finite ideal type is said to be an immersion if it is a composition \( f = j \circ i \) of a closed immersion \( i \) followed by an open immersion \( j \).

Note that immersions are adic morphisms.

**Proposition 4.4.2.** Let \( f: Y \to X \) be an adic morphism of adic formal schemes of finite ideal type, and suppose \( X \) has an ideal of definition \( \mathcal{I} \) of finite type. For any \( k \geq 0 \) we denote by \( f_k: Y_k = (Y, \mathcal{O}_Y / \mathcal{I}^{k+1}) \to X_k = (X, \mathcal{O}_X / \mathcal{I}^{k+1}) \) the induced morphism of schemes. Then the following conditions are equivalent.

(a) \( f \) is an immersion.

(b) \( f_k \) is an immersion for any \( k \geq 0 \).

**Proof.** Let \( f = j \circ i \) be an immersion, where \( i: Y \hookrightarrow U \) is a closed immersion and \( j: U \hookrightarrow X \) is an open immersion. Let \( U_k \) for \( k \geq 0 \) be the scheme defined similarly to \( X_k \). Then, clearly, \( j_k: U_k \to X_k \) is an open immersion. Due to 4.3.6, one finds that \( i_k: Y_k \to U_k \) is a closed immersion, and thus \( f_k = j_k \circ i_k \) is an immersion.
Conversely, suppose \( f_k \) is an immersion for any \( k \geq 0 \). Take an open subset \( U \) of \( X \) that contains \( f_0(Y) \) as a closed subset. For any \( k \geq 0 \) we have the morphism \( f_k: Y_k \to U_k \) of schemes, which is a closed immersion by [54], I, (4.2.2) (ii). Hence \( f: Y \to U \) is a closed immersion due to 4.3.6; this implies that \( f \) is an immersion, as desired.

**Proposition 4.4.3.** Let \( f: Y \to X \) be a morphism of adic formal schemes, and \( \{V_\alpha\}_{\alpha \in L} \) an open covering of \( X \). Then \( f \) is an immersion if and only if for any \( \alpha \in L \) the base change \( f^{-1}(V_\alpha) \to V_\alpha \) is an immersion.

**Proof.** The ‘only if’ part is clear. Let us show the other part. Take an open subset \( U_\alpha \) of \( V_\alpha \) for each \( \alpha \in L \) such that the immersion \( f^{-1}(V_\alpha) \to V_\alpha \) factors through the closed immersion \( f^{-1}(V_\alpha) \to U_\alpha \). Set \( U = \bigcup_{\alpha \in L} U_\alpha \). Then by 4.3.9 the morphism \( Y \to U \) is a closed immersion.

**Proposition 4.4.4.** Let \( f: X \to Y \) be a morphism of formal schemes, where \( Y \) is adic of finite ideal type, and \( i: Z \to Y \) an immersion. Then underlying topological space of \( X \times_Y Z \) is homeomorphic to \( f^{-1}(i(Z)) \).

**Proof.** The assertion was shown in 1.2.7 in the case where \( i \) is an open immersion. Hence it suffices to show the claim when \( i \) is a closed immersion. Again by 1.2.7 we may assume that \( X \) and \( Y \) have ideals of definition. Then the claim follows from 4.3.6 and [54], I, (4.4.1).

**Lemma 4.4.5.** Let \( Z \to Y \) be an open immersion, and \( Y \to X \) a closed immersion, where \( X, Y, Z \) are adic formal schemes of finite ideal type. Then the composition \( Z \to X \) is an immersion.

**Proof.** Take an open subset \( U \subseteq X \) such that \( Z = Y \cap U \). Then \( Z \to U \) is a closed immersion.

**Proposition 4.4.6.** (1) If \( f: Z \to Y \) and \( g: Y \to X \) are immersions, then so is the composition \( g \circ f \).

(2) For any \( S \)-immersions \( f: X \to Y \) and \( g: X' \to Y' \) of adic formal schemes of finite ideal type over an adic formal scheme \( S \) of finite ideal type, the induced morphism

\[
f \times_S g: X \times_S Y \longrightarrow X' \times_S Y'
\]

is an immersion.

(3) For any \( S \)-immersion \( f: X \to Y \) of adic formal schemes of finite ideal type over an adic formal scheme \( S \) of finite ideal type, and for any morphism \( S' \to S \) of adic formal schemes of finite ideal type, the induced morphism

\[
f_{S'}: X \times_S S' \longrightarrow Y \times_S S'
\]

is an immersion.
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**Proof.** (1) follows from 4.4.5 combined with 1.2.8 (1) and 4.3.10 (1). In view of 1.5.1, (2) and (3) follow from 4.4.2, 4.4.3, and [53], (4.3.6). □

**Corollary 4.4.7.** Let \( f : X \rightarrow Y \) be a morphism of schemes, and \( Z \) a closed subscheme of \( Y \) of finite presentation. If \( f \) is an immersion, then the formal completion

\[
\hat{f} : \hat{X}_{f^{-1}(Z)} \rightarrow \hat{Y}_{\mathcal{O}_X(Z)}
\]

is an immersion.

**Proof.** Since immersions are closed under composition, we may apply 1.5.2, and the corollary follows from 4.4.2 and [53], (4.3.6). □

### 4.5 Surjective, closed, and universally closed morphisms

#### 4.5. (a) Surjective morphisms

**Definition 4.5.1.** A morphism \( f : X \rightarrow Y \) of formal schemes is said to be **surjective** if the underlying continuous mapping is surjective.

If there exist ideals of definition \( I \) and \( J \) of \( X \) and \( Y \), respectively, such that \( J \mathcal{O}_X \subseteq I \), then \( f \) is surjective if and only if the induced morphism

\[
(X, \mathcal{O}_X/I) \rightarrow (Y, \mathcal{O}_Y/J)
\]

of schemes is surjective. The following proposition follows immediately from 1.2.7.

**Proposition 4.5.2.** Let \( f : Y \rightarrow X \) be a morphism of formal schemes, and \( \{V_\alpha\}_{\alpha \in L} \) an open covering of \( X \). Then \( f \) is a surjective if and only if for any \( \alpha \in L \) the base change \( f^{-1}(V_\alpha) \rightarrow V_\alpha \) is surjective.

**Proposition 4.5.3.** (1) The composition of two surjective morphisms of formal schemes is surjective.

(2) If \( S \) is a formal scheme and if \( f : X \rightarrow X' \) and \( g : Y \rightarrow Y' \) are surjective morphisms of \( S \)-formal schemes, then

\[
f \times_S g : X \times_S Y \rightarrow X' \times_S Y'
\]

is surjective.

(3) If \( S \) is a formal scheme and if \( f : X \rightarrow Y \) is a surjective morphism of \( S \)-formal schemes, then for any morphism \( S' \rightarrow S \) the induced morphism

\[
f_{S'} : X \times_S S' \rightarrow Y \times_S S'
\]

is surjective.
The proof uses the following lemma, which can be shown similarly to the scheme case (cf. [53], (3.6.2)).

**Lemma 4.5.4.** Let \( f: X \rightarrow Y \) be a surjective morphism, let \( K \) be a field, and let \( \text{Spec} \ K \rightarrow Y \) be a map of formal schemes. Then \( X \times_Y \text{Spec} \ K \) is non-empty.

**Proof of Proposition 4.5.3.** (1) is clear. As we saw in §1.5.(b), (2) and (3) follow from the special case of (3) with \( S = Y \). Let \( f: X \rightarrow Y \) be a surjective map of formal schemes, and \( Y' \rightarrow Y \) a map of formal schemes. We want to show that \( f': X' = X \times_Y Y' \rightarrow Y' \) is surjective. By 4.5.2, we may assume that \( Y \) and \( Y' \) are affine; set \( Y = \text{Spf} \ B \) and \( Y' = \text{Spf} \ B' \). Let \( q \in Y' \) be an open prime ideal of \( B' \). Let \( K = \text{Frac}(B'/q) \), and \( \text{Spec} \ K \rightarrow Y' \) the natural map. Then by 4.5.4 \( X \times_Y \text{Spec} \ K = X' \times_Y \text{Spec} \ K \) is non-empty, and any point of \( X' \times_Y \text{Spec} \ K \) is mapped to a point of \( X' \) that is mapped to \( q \) by \( f' \).

**4.5. (b) Closed and universally closed morphisms**

**Definition 4.5.5.** Let \( f: X \rightarrow Y \) be a morphism of formal schemes.

1. We say that \( f \) is **closed** if the underlying continuous map of \( f \) is closed.

2. We say that \( f \) is **universally closed** if for any morphism \( S \rightarrow Y \) of formal schemes the induced morphism \( f_S: X \times_Y S \rightarrow S \) is closed.

**Proposition 4.5.6.** Let \( f: Y \rightarrow X \) be a morphism of formal schemes, and \( \{V_\alpha\}_{\alpha \in L} \) an open covering of \( X \). Then \( f \) is universally closed if and only if for any \( \alpha \in L \) the base change \( f^{-1}(V_\alpha) \rightarrow V_\alpha \) is universally closed.

**Proof.** The ‘only if’ part is clear. To show the ‘if’ part, by 1.2.8 (3) it suffices to show that the map \( f: Y \rightarrow X \) is closed. But this follows easily from 1.2.7. \(\□\)

**Proposition 4.5.7.** Let \( f: X \rightarrow Y \) be a morphism of formal schemes, and suppose \( Y \) is a scheme. Then \( f \) is universally closed if and only if for any morphism \( S \rightarrow Y \) of schemes the induced morphism \( f_S: X \times_Y S \rightarrow S \) is closed.

Our notion of ‘universally closed’ restricted to morphisms of schemes coincides with the usual one in scheme theory.

**Proof.** The ‘only if’ part is trivial. Let us show the converse. Let \( S \rightarrow Y \) be a morphism of formal schemes. By 4.5.6, we may assume that \( S \) is affine, \( S = \text{Spf} \ R \), where \( R \) is an admissible ring. Let \( J \subseteq R \) be an ideal of definition, and consider \( S_0 = \text{Spec} \ R_0 \) with \( R_0 = R/J \). Then \( S_0 \) is a scheme, and hence \( f_{S_0}: X \times_Y S_0 \rightarrow S_0 \) is closed. To show that \( f: X \times_Y S \rightarrow S \) is closed, it suffices to show that \( X \times_Y S \) and \( X \times_Y S_0 \) have the same underlying topological space. Indeed, we may assume that \( X \) and \( Y \) are affine \( X = \text{Spf} \ A \) and \( Y = \text{Spec} \ B \). Let \( I \subseteq A \) be
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an ideal of definition. Consider the ideal $H = \text{image}(I \otimes R) + \text{image}(A \otimes J)$ of $A \otimes_B R$, and set $(X \times_Y S)_0 = \text{Spec} A \otimes_B R/H$. Then $X \times_Y S$ and $(X \times_Y S)_0$ have the same underlying topological space. On the other hand, we have $X \times_Y S_0 = \text{Spf} A \otimes_B R/J$. Observe that the kernel of $A \otimes_B R \to A \otimes_B R/H$ is contained in the kernel of $A \otimes_B R \to A \otimes_B R/J$. Hence we have the chain of closed immersions $(X \times_Y S)_0 \hookrightarrow X \times_Y S_0 \hookrightarrow X \times_Y S$, from which the claim follows.

**Proposition 4.5.8.** (1) The composition of two closed (resp. universally closed) morphisms of formal schemes is closed (resp. universally closed).

(2) If $f : X \to X'$ and $g : Y \to Y'$ are two universally closed $S$-morphisms of formal schemes, then

$$f \times_S g : X \times_S Y \longrightarrow X' \times_S Y'$$

is universally closed.

(3) If $f : X \to Y$ is a universally closed $S$-morphism of formal schemes and $S' \to S$ is a morphism, then

$$f_{S'} : X \times_S S' \longrightarrow Y \times_S S'$$

is universally closed.

**Proof.** (1) is clear. (2) and (3) are consequences of the special case of (3) with $S = Y$ (§1.5. (b)), which in turn follows immediately from the definition.

**Corollary 4.5.10.** Let $f : X \to Y$ be an immersion between adic formal schemes of finite ideal type. Then $f$ is a closed immersion if and only if it is closed.
Proof. The ‘only if’ part is clear. Suppose \( f \) is closed. To show that \( f \) is a closed immersion, we may work locally on \( Y \), and thus we may assume that \( Y \) has an ideal of definition \( J \) of finite type. With the notation as in 4.5.9, the morphism \( f_0 \) of schemes is closed, and is an immersion (due to 4.4.2). Hence \( f_0 \) is a closed immersion (e.g. [54], I, (4.2.2)). Then it follows from 4.3.6 that \( f \) is a closed immersion.

Corollary 4.5.11. Let \( f : X \to Y \) be a morphism of schemes, and \( Z \) a closed subscheme of \( Y \) of finite presentation. If \( f \) is universally closed, then the formal completion \( \hat{f} : \hat{X} |_{f^{-1}(Z)} \to \hat{Y} |_Z \) is universally closed.

4.6 Separated morphisms

4.6. (a) Definition and fundamental properties

Proposition 4.6.1. Let \( f : X \to Y \) be a morphism of adic formal schemes, and suppose \( X \) is of finite ideal type. Then the diagonal map \( \Delta_X : X \to X \times_Y X \) is an immersion.

Note that we have already shown in 1.3.7 that the diagonal map \( \Delta_X \) is adic.

Proof. Let \( V \subset Y \) be an open subset, and \( U = f^{-1}(V) \). Then the base change of \( \Delta_X \) by the open immersion \( U \times_Y U \to X \times_Y X \) (1.2.8 (2)) coincides with \( \Delta_U : U \to U \times_Y U \). Since the image of \( \Delta_X \) is contained in the union of open subsets of the form \( U \times_Y U \), by 4.4.3 we may assume that \( Y \) is affine, \( Y = \text{Spf} \, B \). Let \( X = \bigcup_{\alpha \in L} U_\alpha \) be an affine open covering. By 1.2.8 (2), the canonical morphism \( U_\alpha \times_Y U_\beta \to X \times_Y X \) is an open immersion. As the image of \( \Delta_X \) is contained in the open subset \( \bigcup_{\alpha \in L} U_\alpha \times_Y U_\alpha \), it suffices to show that the map \( X \to \bigcup_{\alpha \in L} U_\alpha \times_Y U_\alpha \) is a closed immersion. By [53], 0, (1.4.8), \( \Delta_X^{-1}(U_\alpha \times_Y U_\alpha) = U_\alpha \). By 4.3.9, it suffices to show that for each \( \alpha \in L \) the diagonal map \( \Delta_{U_\alpha} : U_\alpha \to U_\alpha \times_Y U_\alpha \) is a closed immersion. Set \( U_\alpha = \text{Spf} \, A_\alpha \) for \( \alpha \in L \), where \( A_\alpha \) is an adic ring of finite ideal type. Then the diagonal map corresponds to \( A_\alpha \otimes_B A_\alpha \to A_\alpha \), which is clearly surjective. Now by 4.3.7 the proof is complete.

Definition 4.6.2 ([54], I, §10.15). A morphism \( f : X \to Y \) of formal schemes is said to be separated if the image \( \Delta_X(X) \) of the diagonal map \( \Delta_X : X \to X \times_Y X \) is a closed subset of \( X \times_Y X \). A formal scheme \( X \) is said to be separated if it is separated over \( \text{Spec} \, \mathbb{Z} \).

Proposition 4.6.3. Separated morphisms are quasi-separated.
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By 4.6.1 we have the following result.

**Proposition 4.6.4.** Let \( f : X \to Y \) be a morphism of adic formal schemes, where \( X \) is of finite ideal type. Then \( f \) is separated if and only if the diagonal map \( \Delta_X : X \to X \times_Y X \) is a closed immersion.

**Proposition 4.6.5.**

1. The composition of two separated morphisms of adic formal schemes of finite ideal type is separated.

2. If the composition \( g \circ f \) of two morphisms of adic formal schemes of finite ideal type is separated, then \( f \) is separated.

3. Any open immersion is separated.

**Proof.** We use diagram (\( *) \) in the proof of 1.6.7, page 283. Suppose \( f : X \to Y \) and \( g : Y \to Z \) are separated. Then \( \Delta_f \) and \( \Delta_g \) are closed immersions, and hence \( X \times_Y X \to X \times_Z X \) is a closed immersion (4.3.10 (3)). This implies that \( \Delta_{g \circ f} \) is a closed immersion (4.3.10 (1)), that is, \( g \circ f \) is separated, whence (1).

Suppose \( g \circ f \) is separated. Since \( X \times_Y X \to X \times_Z X \) is an immersion (4.6.1 and 4.4.6 (3)), \( \Delta_f(X) \) coincides with the pull-back of \( \Delta_{g \circ f}(X) \). Since \( \Delta_{g \circ f}(X) \) is closed in \( X \times_Z X \), \( \Delta_f(X) \) is closed in \( X \times_Y X \), whence (2).

Finally, if \( j : U \hookrightarrow X \) is an open immersion, then the diagonal map is an isomorphism, which is clearly a closed immersion, whence (3). \( \square \)

**Proposition 4.6.6.** Let \( f : Y \to X \) be a morphism of adic formal schemes of finite ideal type, and \( \{ V_\alpha \}_{\alpha \in L} \) an open covering of \( X \). Then \( f \) is separated if and only if for any \( \alpha \in L \) the base change \( f^{-1}(V_\alpha) \to V_\alpha \) is separated.

**Proof.** Set \( U_\alpha = f^{-1}(V_\alpha) \). The image \( \Delta_X(X) \) of \( \Delta_X : X \to X \times_X X \) is contained in the open subset \( \bigcup_{\alpha \in L} U_\alpha \times_{V_\alpha} U_\alpha \). By [53], 0. (1.4.8), we have \( \Delta_X^{-1}(U_\alpha \times_{V_\alpha} U_\alpha) = U_\alpha \). Then the assertion follows from 4.6.5 (1), (2), and (3) and 4.3.9. \( \square \)

**Corollary 4.6.7.** Any affine morphism between adic formal schemes of finite ideal type is separated.

**Proof.** In view of 4.6.6, it suffices to show that a morphism

\[
X = \text{Spf } A \longrightarrow Y = \text{Spf } B,
\]

where \( A \) and \( B \) are adic rings of finite ideal type, is separated. Since \( B \widehat{\otimes}_A B \to B \) is clearly surjective, the assertion follows from 1.3.7 and 4.3.7. \( \square \)
Proposition 4.6.8. Let $S$ be an adic formal scheme $S$ of finite ideal type.

(1) For separated $S$-morphisms $f : X \to Y$ and $g : X' \to Y'$ of adic formal schemes of finite ideal type over $S$, the morphism
$$f \times_S g : X \times_S Y \to X' \times_S Y'$$
is separated.

(2) For a separated $S$-morphism $f : X \to Y$ of adic formal schemes of finite ideal type over $S$ and a morphism $S' \to S$ of adic formal schemes of finite ideal type, the induced morphism
$$f_{S'} : X \times_S S' \to Y \times_S S'$$
is separated.

Proof. As we have seen in §1.5.(c), (1) and (2) follow from the special case of (2) with $S = Y$. We use diagram (**) in the proof of 1.6.7. By 4.3.10 (3), if $\Delta_f$ is a closed immersion, then so is $\Delta_{f'}$.

Proposition 4.6.9. Let $f : X \to Y$ be an adic morphism of adic formal schemes of finite ideal type, and suppose $Y$ has an ideal of definition $I$ of finite type. Set
$$X_k = (X, \mathcal{O}_X / I^{k+1}) \quad \text{and} \quad Y_k = (Y, \mathcal{O}_Y / I^{k+1}),$$
and denote by
$$f_k : X_k \to Y_k$$
the induced morphism of schemes. Then the following conditions are equivalent.

(a) $f$ is separated.

(b) $f_k$ is separated for any $k \geq 0$.

(c) $f_0$ is separated.

Proof. This follows from 1.3.5 and the definition of separatedness.

By 1.5.2, 4.6.9, and [53], (5.3.1), we have the following corollary.

Corollary 4.6.10. Let $f : X \to Y$ be a morphism of schemes, and $Z$ a closed subscheme of $Y$ of finite presentation. If $f$ is separated, then the formal completion $\hat{f} : \hat{X} |_{f^{-1}(Z)} \to \hat{Y} |_Z$ is separated.

Proposition 4.6.11. Let $B$ be an adic ring of finite ideal type, and
$$f : X \to Y = \text{Spf } B$$
a morphism between adic formal schemes of finite ideal type. Let $X = \bigcup_{\alpha \in L} U_\alpha$ be an affine open covering of $X$, where $U_\alpha = \text{Spf } A_\alpha$ for an adic ring $A_\alpha$ of finite ideal type for each $\alpha \in L$. Then $f$ is separated if and only if the following conditions are satisfied for any $\alpha, \beta \in L$. 

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(a) The intersection $U_\alpha \cap U_\beta$ is affine and $U_\alpha \cap U_\beta = \text{Spf } A_{\alpha\beta}$ for an adic ring $A_{\alpha\beta}$ of finite ideal type.

(b) The induced map $A_\alpha \hat{\otimes}_B A_\beta \to A_{\alpha\beta}$ is surjective.

Proof. First note that the diagram

\[
\begin{array}{c}
X \\ \downarrow \Delta_X \downarrow \Delta_X \\
U_\lambda \cap U_\mu \\
U_\lambda \times_Y U_\mu
\end{array}
\]

is Cartesian (1.2.7 and 1.6.3). Suppose $f$ is separated, that is, $\Delta_X$ is a closed immersion (4.6.4). Then by 4.3.10 (3) the map $U_\alpha \cap U_\beta \to U_\alpha \times_Y U_\beta$ is a closed immersion. Since $U_\alpha \times_Y U_\beta = \text{Spf } A_\alpha \hat{\otimes}_B A_\beta$ is affine, $U_\alpha \cap U_\beta$ is affine (4.3.5 (1)) and the map $A_\alpha \hat{\otimes}_B A_\beta \to \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$ is surjective (4.3.7).

Conversely, if any $U_\alpha \cap U_\beta = \text{Spf } A_{\alpha\beta}$ is affine and $A_\alpha \hat{\otimes}_B A_\beta \to A_{\alpha\beta}$ is surjective, then $U_\alpha \cap U_\beta \to U_\alpha \times_Y U_\beta$ is a closed immersion (4.3.7). Since the open subsets of $X \times_Y X$ of the form $U_\alpha \times_Y U_\beta$ cover $X \times_Y X$, $\Delta_X$ is a closed immersion due to 4.3.9.

4.6. (b) Separatedness and properties of morphisms

Proposition 4.6.12. Let $f: X \to Y$ be a morphism of adic formal schemes of finite ideal type, and $g: Y \to Z$ a separated morphism of adic formal schemes of finite ideal type. Then the graph $\Gamma_f: X \to X \times_Z Y$ is a closed immersion. If, moreover, $X, Y, Z$ are locally universally rigid-Noetherian and $g$ is of finite type, then $\Gamma_f$ is of finite presentation.

Proof. Since the diagram

\[
\begin{array}{c}
X \\ \downarrow \Gamma_f \downarrow f \\
Y \\
\downarrow \Delta_g \\
Y \times_Z Y
\end{array}
\]

is Cartesian, the first assertion follows from 4.6.4. The last assertion follows from 4.3.12, 2.2.2 (4), and the following lemma.

Lemma 4.6.13. Let $A$ be an adic ring of finite ideal type, and $B$ an $A$-algebra topologically of finite type. Then the kernel of the morphism

\[B \hat{\otimes}_A B \to B, \quad x \otimes y \mapsto xy,\]

is generated by $\{1 \otimes s - s \otimes 1: s \in S\}$, where $S \subseteq B$ is a topological generator of $B$ over $A$. 

Proof. It suffices to show the assertion in the case where $B$ is the restricted power series ring $B = A\langle X_1, \ldots, X_n \rangle$. Then we have to show that the kernel of the morphism

$$A\langle X_1, \ldots, X_n, Y_1, \ldots, Y_n \rangle \rightarrow A\langle Z_1, \ldots, Z_n \rangle$$

sending $X_i$ and $Y_i$ to $Z_i$ for $i = 1, \ldots, n$ is the ideal $a$ generated by the elements $X_i - Y_i$ ($1 \leq i \leq n$). Since the similar assertion is known to hold for polynomial rings, this is equivalent to the ideal $a$ being closed. We show this by induction with respect to $n$.

The case $n = 0$ is trivial.

Consider $A' = A\langle X_n, Y_n \rangle/(X_n - Y_n)$, which is $I$-adically complete (where $I \subseteq A$ is an ideal of definition) due to Exercise 0.7.2; in particular, $A' \cong A\langle Z_n \rangle$. The morphism in question factors through

$$A'\langle X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1} \rangle \rightarrow A'\langle Z_1, \ldots, Z_{n-1} \rangle \cong A\langle Z_1, \ldots, Z_n \rangle.$$

By induction,

$$A'\langle X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1} \rangle/(X_i - Y_i: i = 1, \ldots, n - 1) \cong A\langle Z_1, \ldots, Z_n \rangle.$$

Now again by Exercise 0.7.2 we have

$$A\langle X_1, \ldots, X_n, Y_1, \ldots, Y_n \rangle/(X_n - Y_n) \cong A'\langle X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1} \rangle,$$

whence

$$A\langle X_1, \ldots, X_n, Y_1, \ldots, Y_n \rangle/a \cong A\langle Z_1, \ldots, Z_n \rangle,$$

as desired.

Proposition 4.6.14. Let $P$ be a property of arrows in the category $\text{AcFs}$ satisfying (I) and (C) in 0, §1.5.(b) and the mutually equivalent conditions $(B_i)$ for $i = 1, 2, 3$ (with $DQ = \text{AcFs}$) in 0, §1.5.(c). Suppose that any closed immersion satisfies $P$. Then if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are morphisms of adic formal schemes of finite ideal type such that $g \circ f$ satisfies $P$ and $g$ is separated, $f$ satisfies $P$.

Proof. The morphism $f$ coincides with the composition

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{(g \circ f) \times_Z \text{id}_Y} Y.$$

Since $g$ is separated, the first arrow is a closed immersion (4.6.12), whence the claim.

Corollary 4.6.15. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be morphisms of adic formal schemes of finite ideal type, and suppose $g$ is separated. If $g \circ f$ satisfies one of the following conditions, then so does $f$:
(a) adic,
(b) locally of finite type (resp. of finite type),
(c) quasi-compact (resp. quasi-separated, resp. coherent),
(d) affine adic,
(e) finite,
(f) closed immersion (resp. immersion),
(g) universally closed.

4.7 Proper morphisms

Definition 4.7.1. A morphism \( f : X \to Y \) of adic formal schemes of finite ideal type is said to be proper if it is separated of finite type and universally closed.

By 4.5.7, a proper morphism of schemes \( f : X \to Y \) is also proper as a morphism of \((0\text{-adic})\) formal schemes. The following proposition follows immediately from 4.6.6, the definition of \(\text{‘of finite type’ (1.7.1)}\), and 4.5.6.

Proposition 4.7.2. Let \( f : Y \to X \) be a morphism of adic formal schemes of finite ideal type, and \( \{V_\alpha\}_{\alpha \in L} \) an open covering of \( X \). Then \( f \) is proper if and only if for any \( \alpha \in L \) the base change \( f^{-1}(V_\alpha) \to V_\alpha \) is proper.

By 4.6.9, 1.7.3, and 4.5.9, we have the following result.

Proposition 4.7.3. Let \( f : X \to Y \) be an adic morphism of adic formal schemes of finite ideal type, and suppose \( Y \) has an ideal of definition \( \mathcal{I} \) of finite type. Set

\[
X_k = (X, \mathcal{O}_X / \mathcal{I}^{k+1} \mathcal{O}_X) \quad \text{and} \quad Y_k = (Y, \mathcal{O}_Y / \mathcal{I}^{k+1})
\]

for \( k \geq 0 \). Let

\[
f_k : X_k \to Y_k
\]

be the induced morphism of scheme. The following conditions are equivalent.

(a) \( f \) is proper.
(b) \( f_k \) is proper for any \( k \geq 0 \).
(c) \( f_0 \) is proper.

Proposition 4.7.4. Any finite morphism between adic formal schemes of finite ideal type is proper.

Proof. To check if an adic morphism \( f : X \to Y \) of adic formal schemes is proper, by 4.7.2 we may assume that \( Y \) has an ideal of definition of finite type. Then the assertion follows immediately from 4.2.3 and 4.7.3. \( \square \)
Proposition 4.7.5. (1) If \( f: Z \to Y \) and \( g: Y \to X \) are proper, then so is the composition \( g \circ f \).

(2) For any proper \( S \)-morphisms \( f: X \to Y \) and \( g: X' \to Y' \) of adic formal schemes of finite ideal type over an adic formal scheme \( S \) of finite ideal type, the induced morphism
\[
f \times_S g: X \times_S Y \to X' \times_S Y'
\]
is proper.

(3) For any proper \( S \)-morphism \( f: X \to Y \) of adic formal schemes of finite ideal type over an adic formal scheme \( S \) of finite ideal type, and for any morphism \( S' \to S \) of adic formal schemes of finite ideal type, the induced morphism
\[
f_{S'}: X \times_S S' \to Y \times_S S'
\]
is proper.

(4) Suppose the composition \( g \circ f \) of two adic morphisms of adic formal schemes of finite ideal type is proper. If \( g \) is separated, \( f \) is proper. If \( g \) is separated of finite type and \( f \) is surjective, then \( g \) is proper.

Proof. (1), (2), and (3) follow from 4.6.5, 4.6.8, 1.7.2, and 4.5.8. The first assertion of (4) follows from 4.6.14 and 4.7.4. For the other assertion, it suffices to show that \( g \) is closed. But this is easy to see, since \( f \) is surjective.

In view of 1.5.2, 4.7.3 and [54], II, (5.4.2), yield the following corollary.

Corollary 4.7.6. Let \( f: X \to Y \) be a morphism of schemes, and \( Z \) a closed subscheme of \( Y \) of finite presentation. If \( f \) is proper, then the formal completion
\[
\hat{f}: \hat{X}_{|f^{-1}(Z)} \to \hat{Y}_Z
\]
is proper.

### 4.8 Flat and faithfully flat morphisms

Let \( f: X \to Y \) be a morphism of formal schemes, and \( \mathcal{F} \) an \( \mathcal{O}_X \)-module. As usual (cf. 0.4.1.5), we say that \( \mathcal{F} \) is \( f \)-flat (or \( Y \)-flat) at \( x \in X \) if \( \mathcal{F}_x \) is flat as a module over \( \mathcal{O}_{Y,f(x)} \); likewise, by saying that \( \mathcal{F} \) is \( f \)-flat we mean that \( \mathcal{F} \) is flat at every point of \( X \). Similarly, a morphism of formal schemes \( f: X \to Y \) is said to be flat if it is flat as a morphism of locally ringed spaces, that is, for any \( x \in X \) the induced morphism \( \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \) is flat.
4. Several properties of morphisms

4.8. (a) First properties of flatness

Proposition 4.8.1. Let \( A \to B \) be an adic map of t.u. rigid-Noetherian rings and \( I \subseteq A \) a finitely generated ideal of definition. Consider the associated adic morphism
\[
f : X = \text{Spf } B \to Y = \text{Spf } A.
\]

Let \( M \) be a finitely generated \( A \)-module and \( F = M^A \) the associated a.q.c. sheaf on \( X \) of finite type. For \( k \geq 0 \) we set
\[
X_k = \text{Spec } B_k \quad \text{and} \quad Y_k = \text{Spec } A_k,
\]
where
\[
B_k = B/I^{k+1}B \quad \text{and} \quad A_k = A/I^{k+1},
\]
and denote by
\[
f_k : X_k \to Y_k
\]
the induced morphism of schemes. We likewise set
\[
F_k = F/I^{k+1}F,
\]
for \( k \geq 0 \). The following conditions are equivalent.

(a) \( F \) is \( f \)-flat.

(b) \( F_k \) is \( f_k \)-flat for any \( k \geq 0 \).

(c) \( M \) is \( A \)-flat.

Proof. Implication (a) \( \implies \) (b) is easy; indeed, \( f_k \) is the base change of \( f \) by the closed immersion \( Y_k \to Y \) defined by \( I^{k+1} \). If (b) holds, then \( M_k = M/I^{k+1}M \) is \( A_k \)-flat for any \( k \geq 0 \). Then (c) follows by 0.8.3.8.

Let us show (c) \( \implies \) (a). Let \( M \) be \( A \)-flat. Then \( M_g \) for any \( g \in A \) is \( A_g \)-flat (where we denote the image of \( g \) in \( B \) again by \( g \)). But then, by 0.8.3.8, the completed localization \( M_{(g)} \) is flat over \( A_{(g)} \), since \( A_{(g)} \) is t.u. rigid-Noetherian. Similarly, whenever there exists a canonical morphism \( B_g \to B_h, M_{(h)} \) is flat over \( A_{(g)} \). Indeed, we have
\[
M_{(h)} = M_{(g)} \hat{\otimes}_{B_{(g)}} B_{(h)} = M_{(g)} \otimes_{B_{(g)}} B_{(h)};
\]
on the other hand, since \( B_g \to B_h \) is flat, \( B_{(g)} \to B_{(h)} \) is flat by 0.8.3.8, thus \( M_{(h)} \) is flat over \( A_{(g)} \).

Now let \( y = q \) be a point of \( Y \) (\( q \) is an open prime ideal of \( A \)), and set \( x = f(y) = p \). For any \( g \not\in p, \lim_{h \not\in q} M_{(h)} = \mathcal{F}_y \) is flat over \( A_{(g)} \). Varying \( g \) and taking the inductive limit, we deduce that \( \mathcal{F}_y \) is flat over \( \mathcal{O}_{X,x} = \lim_{\rightarrow g \not\in p} A_{(g)} \), whence (a). \( \square \)
Corollary 4.8.2. Let \( f : X \to Y \) be an adic morphism between locally universally rigid-Noetherian formal schemes, and \( \mathcal{F} \) an a.q.c. sheaf on \( X \) of finite type. Suppose \( Y \) has an ideal of definition \( I \) of finite type, and set
\[
X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X), \quad Y_k = (Y, \mathcal{O}_Y/I^{k+1}),
\]
and
\[
\mathcal{F}_k = \mathcal{F}/I^{k+1}\mathcal{F}
\]
for any \( k \geq 0 \). Let
\[
f_k : X_k \to Y_k
\]
be the induced morphism of schemes. Then the following conditions are equivalent.

(a) \( \mathcal{F} \) is flat over \( Y \).

(b) \( \mathcal{F}_k \) is flat over \( Y_k \) for any \( k \geq 0 \).

Corollary 4.8.3. Let \( f : X \to Y \) be an adic morphism between locally universally rigid-Noetherian formal schemes, and \( \mathcal{F} \) an a.q.c. sheaf of finite type on \( X \). Then the following conditions are equivalent.

(a) \( \mathcal{F} \) is \( f \)-flat.

(b) the functor \( \mathcal{G} \mapsto \mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G} \) from the category of a.q.c. sheaves on \( Y \) of finite type to the category of a.q.c. sheaves on \( X \) of finite type (cf. 3.5.9) is exact.

Corollary 4.8.4. (1) Let \( f : X \to Y \) and \( g : X' \to Y' \) be flat \( S \)-morphisms of locally universally rigid-Noetherian formal schemes that are adic over an adic formal scheme \( S \) of finite ideal type. Suppose \( X \times_S Y \) and \( X' \times_S Y' \) are locally universally rigid-Noetherian. Then the induced morphism
\[
f \times_S g : X \times_S Y \to X' \times_S Y'
\]
is flat.

(2) Let \( f : X \to Y \) be a flat \( S \)-morphism of locally universally rigid-Noetherian formal schemes that are adic over an adic formal scheme \( S \) of finite ideal type, and \( S' \to S \) an adic morphism of adic formal schemes of finite ideal type. Suppose that \( X \times_S S' \) and \( Y \times_S S' \) are locally universally rigid-Noetherian. Then the induced morphism
\[
f_{S'} : X \times_S S' \to Y \times_S S'
\]
is flat.

Proof. Since the statement is local on \( S \), we may assume that \( S \) has an ideal of definition of finite type. Then the corollary follows from 4.8.1, 1.3.5, and [54], IV, (2.1.7).
4. Several properties of morphisms

4.8. (b) Faithfully flat morphisms

**Definition 4.8.5.** Let \( f : X \to Y \) be an adic morphism between locally universally rigid-Noetherian formal schemes, and \( \mathcal{F} \) an a.q.c. sheaf on \( X \) of finite type.

(1) We say \( \mathcal{F} \) is **faithfully flat** over \( Y \) if the functor \( \mathcal{G} \mapsto \mathcal{F} \otimes_{\mathcal{O}_Y} f^{-1}\mathcal{G} \) from the category of a.q.c. sheaves on \( Y \) of finite type to the category of a.q.c. sheaves on \( X \) of finite type is exact and faithful (cf. 4.8.3).

(2) If \( \mathcal{O}_X \) is faithfully flat over \( Y \), then we say the morphism \( f \) is **faithfully flat** or \( X \) is **faithfully flat** over \( Y \).

The condition in (1) is equivalent to that \( \mathcal{F} \) is \( f \)-flat and for an a.q.c. sheaf \( \mathcal{G} \) of finite type on \( Y \), \( \mathcal{F} \otimes_{\mathcal{O}_Y} f^{-1}\mathcal{G} = 0 \) implies \( \mathcal{G} = 0 \).

**Proposition 4.8.6.** Let \( f : X \to Y \) be an adic morphism between locally universally rigid-Noetherian formal schemes, and \( \mathcal{F} \) an a.q.c. sheaf on \( X \) of finite type. \( \mathcal{F} \) is faithfully flat over \( Y \) if and only if \( f \)-flat and \( f(\text{Supp}(\mathcal{F})) = Y \).

To show the proposition, we first need to prove the following lemma.

**Lemma 4.8.7.** Let \( f : X \to Y \) be an adic morphism of locally universally rigid-Noetherian formal schemes, and suppose that \( Y \) has an ideal of definition of finite type \( I \). Let

\[
f_k : X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X) \to Y_k = (Y, \mathcal{O}_Y/I^{k+1})
\]

be the induced morphism of schemes for \( k \geq 0 \). Then for any a.q.c. sheaf \( \mathcal{F} \) of finite type on \( X \) and \( k \geq 0 \),

\[
f(\text{Supp}(\mathcal{F})) = f_k(\text{Supp}(\mathcal{F}_k)),
\]

where \( \mathcal{F}_k = \mathcal{F}/I^{k+1}\mathcal{F} \), holds.

**Proof.** We first show the claim in the affine case \( X = \text{Spf} A \) and \( Y = \text{Spf} B \), where \( A, B \) are t.u. rigid-Noetherian rings, and \( I \subseteq B \) is a finitely generated ideal of definition. In this case, \( \mathcal{F} = M^\Delta \) for a finitely generated \( A \)-module \( M \). For an open prime ideal \( q \) of \( B \), \( q \in f(\text{Supp}(\mathcal{F})) \) if and only if \( M_{\{g\}} = M \otimes_A A_{\{g\}} \neq 0 \) for some \( g \in B \setminus q \). By Nakayama’s lemma, this is equivalent to \( M_g/\mathfrak{m}_g^{k+1}M_g \neq 0 \), whence the claim in this case. The general case reduces to the affine case, since \( y \in Y \) lies in \( f(\text{Supp}(\mathcal{F})) \) if and only if there exists \( x \in \text{Supp}(\mathcal{F}) \) such that \( f(x) = y \).

**Proof of Proposition 4.8.6.** We may assume that \( Y \) has an ideal of definition of finite type \( I \). Let \( f_k : X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X) \to Y_k = (Y, \mathcal{O}_Y/I^{k+1}) \) be the induced morphism of schemes for \( k \geq 0 \), and set \( \mathcal{F}_k = \mathcal{F}/I^{k+1}\mathcal{F} \). By 4.8.7, \( f(\text{Supp}(\mathcal{F})) = f_k(\text{Supp}(\mathcal{F}_k)) \) for any \( k \geq 0 \). Suppose \( \mathcal{F} \) is faithfully flat over \( Y \).
Then $F$ is $f$-flat due to 4.8.3. To show that $f(\text{Supp}(F)) = Y$, it suffices to show that $f(\text{Supp}(F_k)) = Y$. By 4.8.2, the quasi-coherent $O_{X_k}$-module $F_k$ is $f_k$-flat. If $f(\text{Supp}(F_k)) \neq Y$ for some $k$, there exists a point $y \in Y$ such that $F_k \otimes_{O_{Y_k}} k(y) = 0$, where $k(y)$ is the residue field of $y$ ([54], IV, (2.2.1)). This implies, in particular, that $F \otimes_{O_Y} k(y) = 0$, which contradicts the assumption, since $k(y)$ can be regarded as an a.q.c. sheaf on $Y$ of finite type in an obvious way.

Conversely, suppose $F$ is $f$-flat and $f(\text{Supp}(F)) = Y$. By 4.8.2, we know that $F_k$ is $f_k$-flat for any $k \geq 0$. Suppose $F \otimes_{O_Y} \mathcal{G} = 0$ for an a.q.c. sheaf $\mathcal{G}$ on $Y$. Consider for $k \geq 0$ the equalities $(F \otimes_{O_Y} \mathcal{G})_k = F_k \otimes_{O_{Y_k}} \mathcal{G}_k = 0$ (where $\mathcal{G}_k = \mathcal{G}/I^{k+1}\mathcal{G}$). Since $f_k(\text{Supp}(F_k)) = Y_k$, $F_k$ is faithfully flat over $Y_k$ ([54], IV, (2.2.6)), and hence $\mathcal{G}_k = 0$. Since $\mathcal{G} = \lim_{\leftarrow k \geq 0} \mathcal{G}_k$, we have $\mathcal{G} = 0$, as desired.

**Corollary 4.8.8.** Let $f : X \to Y$ be an adic morphism between locally universally rigid-Noetherian formal schemes. Then $f$ is faithfully flat if and only if $f$ is flat and is surjective.

**Corollary 4.8.9.** Let $X$ be an locally universally rigid-Noetherian formal scheme, and $\bar{X} \to X$ a Zariski covering, that is, $\bar{X} = \bigsqcup_{\alpha \in L} U_\alpha \to X$ for an open covering $\{U_\alpha\}_{\alpha \in L}$ of $X$. Then the map $\bar{X} \to X$ is faithfully flat.

**Corollary 4.8.10** (local criterion of flatness). Let $f : X \to Y$ be an adic morphism between locally universally rigid-Noetherian formal schemes, and $F$ an adically quasi-coherent sheaf on $X$ of finite type. Suppose $Y$ has an ideal of definition $I$ of finite type, and set

$$X_k = (X, O_X/I^{k+1}O_X), \quad Y_k = (Y, O_Y/I^{k+1}),$$

and

$$F_k = F/I^{k+1}F$$

for any $k \geq 0$. Let

$$f_k : X_k \to Y_k$$

be the induced morphism of schemes. The following conditions are equivalent.

(a) $F$ is faithfully flat over $Y$.

(b) $F_k$ is faithfully flat over $Y_k$ for any $k \geq 0$.

If, moreover, $X$ and $Y$ are affine $X = \text{Spf} \ B$ and $Y = \text{Spf} \ A$, and if $F = M^\Delta$ for a finitely generated $B$-module $M$, then the conditions are equivalent to

(c) $M$ is faithfully flat over $A$. 
4. Several properties of morphisms

Proof. The equivalence of (a) and (b) follows immediately from 4.8.2, 4.8.6 and [54], II, (2.2.6). The last assertion follows from 0.8.3.5.

Corollary 4.8.11. (1) Let \( f : X \to Y \) and \( g : X' \to Y' \) be faithfully flat \( S \)-morphisms of locally universally rigid-Noetherian formal schemes that are adic over an adic formal scheme \( S \) of finite ideal type. Suppose \( X \times_S Y \) and \( X' \times_S Y' \) are locally universally rigid-Noetherian. Then the induced morphism

\[
f \times_S g : X \times_S Y \to X' \times_S Y'
\]

is faithfully flat.

(2) Let \( f : X \to Y \) be a faithfully flat \( S \)-morphism of locally universally rigid-Noetherian formal schemes that are adic over an adic formal scheme \( S \) of finite ideal type, and \( S' \to S \) an adic morphism of adic formal schemes of finite ideal type. Suppose that \( X \times_S S' \) and \( Y \times_S S' \) are locally universally rigid-Noetherian. Then the induced morphism

\[
f_{S'} : X \times_S S' \to Y \times_S S'
\]

is faithfully flat.

(3) Let \( f : X \to Y \) and \( g : Y \to Z \) two adic morphisms of locally universally rigid-Noetherian formal schemes, and suppose \( f \) is faithfully flat. Then for \( g \) to be flat (resp. faithfully flat) it is necessary and sufficient that \( g \circ f \) is flat (resp. faithfully flat).

Proof. Since we assume that \( S \) has an ideal of definition, the corollary follows from 4.8.10 and 1.3.5, and [54], IV, (2.1.7).

4.8. (c) Adically flat morphisms. As is implicit in the above observations, we need to assume, whenever discussing flatness, that the formal schemes under consideration are locally universally rigid-Noetherian. In more general situation, the weaker notion of ‘adical flatness’ defined as follows is more reasonable.

Definition 4.8.12. (1) Let \( f : X \to Y \) be an adic morphism of adic formal schemes of finite ideal type, and \( x \in X \) a point. Then \( f \) is said to be adically flat at \( x \) if there exist an affine open neighborhood \( V \) of \( y = f(x) \) and an ideal of definition \( \mathcal{I} \) of finite type of \( V \), such that for any \( k \geq 0 \) the induced morphism of schemes \( U_k \to V_k \) is flat at \( x \), where \( U = f^{-1}(V) \), \( V_k \) is the closed subscheme of \( V \) defined by \( \mathcal{I}^{k+1} \), and \( U_k = V_k \times_V U \).

(2) An adic morphism \( f : X \to Y \) between adic formal schemes is said to be adically flat if \( f \) is adically flat at all points of \( X \) (cf. 1.5.5). If \( f \) is, moreover, surjective, we say that \( f \) is adically faithfully flat.
It is clear that in (1) the definition does not depend on the choice of $\mathcal{I}$. Hence we readily have the following result.

**Proposition 4.8.13.** Let $f : X \to Y$ be an adic morphism of adic formal schemes of finite ideal type. Suppose that $Y$ has an ideal of definition $\mathcal{I}$ of finite type. Set $X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X)$ and $Y_k = (Y, \mathcal{O}_Y/I^{k+1})$ for $k \geq 0$, and denote by $f_k : X_k \to Y_k$ the induced morphism of schemes. Then the following conditions are equivalent.

(a) $f$ is adically flat.
(b) $f_k$ is flat for $k \geq 0$.

**Corollary 4.8.14.** Let $f : X \to Y$ be a morphism of schemes, and $Z$ a closed subscheme of $Y$ of finite presentation. If $f$ is flat, then the formal completion $\hat{f} : \hat{X}|_{f^{-1}(Z)} \to \hat{Y}|_Z$ is adically flat.

In particular, by 4.8.2 and 4.8.8 we have the following result.

**Proposition 4.8.15.** An adically flat morphism between locally universally rigid-Noetherian formal schemes is flat. In particular, an adic morphism between locally universally rigid-Noetherian formal schemes is faithfully flat if and only if it is adically faithfully flat.

**Proposition 4.8.16.** (1) The composition of two adically flat morphisms is adically flat. Let $f : X \to Y$ and $g : Y \to Z$ be adic morphisms of adic formal schemes of finite ideal type. Suppose $f$ is adically faithfully flat. Then if $g \circ f$ is adically flat, so is $g$.

(2) For any adically flat $S$-morphisms $f : X \to Y$ and $g : X' \to Y'$ of adic formal schemes of finite ideal type over an adic formal scheme $S$ of finite ideal type, the induced morphism $f \times_S g : X \times_S Y \to X' \times_S Y'$ is adically flat.

(3) For any adically flat $S$-morphism $f : X \to Y$ of adic formal schemes of finite ideal type over an adic formal scheme $S$ of finite ideal type, and for any morphism $S' \to S$ of adic formal schemes of finite ideal type, the induced morphism $f_{S'} : X \times_S S' \to Y \times_S S'$ is adically flat.
The proof is easy, and is left to the reader. Using 0.8.3.7, one can easily deduce the following useful fact.

**Proposition 4.8.17.** Let $f: X \to Y$ and $g: Y \to Z$ be adic morphisms of adic formal schemes of finite ideal type. Suppose that $Z$ has an ideal of definition $I$ of finite type, and set

$$X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X), \quad \text{and} \quad Y_k = (Y, \mathcal{O}_Y/I^{k+1}\mathcal{O}_Y),$$

and

$$Z_k = (Z, \mathcal{O}_Z/I^{k+1})$$

for $k \geq 0$. Suppose, moreover, that $g$ is adically flat. Then the following conditions are equivalent.

(a) $f$ is adically flat.

(b) $g \circ f$ is adically flat, and $f_0: X_0 \to Y_0$ is flat.

**Proposition 4.8.18.** Suppose that we have the commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{g \circ f} & X \\
\downarrow^f & & \downarrow^g \\
Y & \rightarrow & X \\
\end{array}
$$

consisting of adic formal schemes of finite ideal type and that $f$ is adically faithfully flat and quasi-compact. Then $g$ is adic if and only if $g \circ f$ is adic.

**Proof.** The ‘only if’ part is clear due to 1.3.6 (1). To show the ‘if’ part, we may assume that $X$ and $Y$ are affine, $X = \text{Spf} \, A$ and $Y = \text{Spf} \, B$, where $A$ and $B$ are adic rings of finite ideal type. Since $f$ is quasi-compact, one can replace $Z$ by a disjoint union of affine open subspaces, and thus we may assume that $Z$ is affine, $Z = \text{Spf} \, R$, where $R$ is an adic ring of finite ideal type. Hence we have the following diagram

$$
\begin{array}{ccc}
R & \xrightarrow{q} & B \\
\uparrow & & \uparrow \\
A & \rightarrow & B, \\
\end{array}
$$

where we assume that the map $A \to R$ is adic. We need to show that $A \to B$ is adic. Take a finitely generated ideal of definition $I \subseteq A$ (resp. $J \subseteq B$) of $A$ (resp. $B$). Replacing $I$ by a suitable power, we may assume that $I^m R \subseteq I R \subseteq J^n R$ for some $m, n \geq 1$. Since, by the assumption, the maps $B/J^{k+1} \to R/J^{k+1} R$ are faithfully flat for all $k \geq 0$, we have

$$J^m \subseteq I B + J^{k+1} \subseteq J^n$$
for any sufficiently large \( k \). This shows that the closure \( \overline{IB} \) of \( IB \) in the \( J \)-adic ring \( B \) is an ideal of definition of \( B \). Since \( \overline{IB} / J^{k+1} \cong (IB + J^{k+1}) / J^{k+1} \), the ideal \( \overline{IB} \) is finitely generated, by 0.7.2.4. Since \( IB \) is dense in \( \overline{IB} \) and \( \overline{IB} \) is open, we have \( \overline{IB} = IB + \overline{IB}^2 \). By Nakayama’s lemma, \( IB = \overline{IB} \), and hence \( IB \) is an ideal of definition of \( B \). This shows that \( A \to B \) is adic, as desired. □

Exercises

Exercise I.4.1. Let \( f : X \to Y \) be an affine adic morphism of adic formal schemes of finite ideal type, and \( \mathcal{F} \) (resp. \( \mathcal{R} \)) an a.q.c. sheaf (resp. a.q.c. \( \mathcal{O}_X \)-algebra). Suppose that there exists an ideal of definition \( \mathcal{I} \) of \( Y \) of finite type. Show that

\[
\begin{align*}
 f_* \mathcal{I}^{k+1} \mathcal{F} &= \mathcal{I}^{k+1} f_* \mathcal{F} \quad (\text{resp. } f_* \mathcal{I}^{k+1} \mathcal{R} = \mathcal{I}^{k+1} f_* \mathcal{R})
\end{align*}
\]

holds for \( k \geq 0 \).

Exercise I.4.2. Let \( X = \text{Spf} A \) be an affine universally rigid-Noetherian formal scheme, and \( K \subseteq A \) an ideal. Show that \( \mathcal{K} = K^\Delta \) is an ideal of \( \mathcal{O}_X \) and that \( \mathcal{O}_X / \mathcal{K} \) is a.q.c. In particular, \( (Y, \mathcal{O}_X / \mathcal{K}) \), where \( Y \) is the support of \( \mathcal{O}_X / \mathcal{K} \) is a closed formal subscheme of \( X \).

Exercise I.4.3. Let \( A \to B \) be an adic map of adic rings of finite ideal type, and \( I \subseteq A \) a finitely generated ideal of definition. Suppose that \( \text{Spf} B \to \text{Spf} A \) is adically faithfully flat. Let \( F^* = \{ F^n \}_{n \in \mathbb{Z}} \) be a descending filtration of \( A \), separated and exhaustive (cf. 0, §7.1. (a)), consisting of finitely generated ideals such that for any \( q \geq 0 \) and \( n, m \in \mathbb{Z} \) we have \( I^q F^m \subseteq F^{m+q} \) (cf. 0, §7.4. (a)). Show that the following conditions are equivalent:

(a) the filtration \( F^* \) is \( I \)-good (0.7.4.2);

(b) the induced filtration \( F^* B = \{ F^n B \}_{n \in \mathbb{Z}} \) on \( B \) is \( IB \)-good.

5 Differential calculus on formal schemes

This section aims at establishing some basic result on differential calculus on formal schemes. We define the related notions of morphisms of formal schemes, neat, étale, and smooth morphisms, and discuss fundamental properties of them. In order to do this, we first develop in §5.1 the ‘continuous’ version of the theory of derivations and differentials and discuss completions of differential modules. The theory of continuous derivations and differentials has already been discussed in [54], 0IV, §20, and most of our arguments here will be, therefore, brief rehashes of what have been done in this reference.
In §5.2 we define for a morphism $X \to Y$ of adic formal schemes of finite ideal type the sheaf $\Omega^1_{X/Y}$ of 1-differentials as an a.q.c. sheaf. Based on this, we then proceed to discuss étale and smooth morphisms of formal schemes in §5.3. Here the reader should be warned that, according to our definitions, smooth or even étale morphisms are only adically flat, but not necessarily flat. Needless to say, this defect comes from the non-existence of a local criterion of flatness (cf. 0.8.3.4). Hence, in particular, they are flat if the formal schemes under consideration are locally universally rigid-Noetherian (2.1.7).

5.1 Differential calculus for topological rings

Let us first recall some generalities of differential calculus for linearly topologized rings, which has already been developed in [54], 0IV, §20.

5.1. (a) Continuous derivations. Let $A \to B$ be a homomorphism of rings. Suppose that $B$ is endowed with a descending filtration by ideals (0, §7.1.(a)) $J^\bullet = \{J^\lambda\}_{\lambda \in \Lambda}$, and consider the topology on $B$ defined by $J^\bullet$ (0, §7.1.(b)). In this situation, one can similarly consider the topology on any $B$-module $M$ defined by the descending filtration by $B$-submodules $J^\bullet M = \{J^\lambda M\}_{\lambda \in \Lambda}$.

We denote by

$$\text{Der}_{\text{cont}}(B, M)$$

the set of all continuous $A$-derivations of $B$ with values in $M$, that is, continuous additive mappings $\delta: B \to M$ such that $\delta(xy) = x\delta(y) + y\delta(x)$ for $x, y \in B$ and that $\delta(a) = 0$ for $a \in A$. Note that the continuity of $\delta$ is equivalent to the fact that for any $\lambda \in \Lambda$ there exists $\mu \in \Lambda$ such that $\delta(J^\mu) \subseteq J^\lambda M$. It is then easy to verify that the set $\text{Der}_{\text{cont}}(B, M)$ has a canonical $B$-module structure in such a way that it is a $B$-submodule of the $B$-module $\text{Der}_A(B, M)$ of all $A$-derivations of $B$ with values in $M$.

5.1. (b) Differentials and canonical topology. Consider the differential module $\Omega^1_{B/A}$ (without regarding topologies) together with the $A$-derivation

$$d: B \to \Omega^1_{B/A}.$$ 

In order for a $B$-linear morphism $\varphi: \Omega^1_{B/A} \to M$ be such that the composition $\varphi \circ d$ is continuous, it is necessary and sufficient that for any $\lambda \in \Lambda$ there exist $\mu \in \Lambda$ such that $d(J^\mu) \subseteq \varphi^{-1}(J^\lambda M)$. Hence, it is natural to consider the topology on the $B$-module $\Omega^1_{B/A}$ defined by the descending filtration by $B$-submodules $\langle d(J^\bullet) \rangle_B = \{(d(J^\lambda))_B\}_{\lambda \in \Lambda}$, where $\langle d(J^\lambda) \rangle_B$ denotes the $B$-submodule of $\Omega^1_{B/A}$ generated by the image $d(J^\lambda)$. We call this topology on $\Omega^1_{B/A}$ the canonical topology. Note that the $A$-derivation $d: B \to \Omega^1_{B/A}$ is continuous.
Proposition 5.1.1. The map $\varphi \mapsto \varphi \circ d$ induces an isomorphism

$$\text{Hom}_{\text{cont}}(\Omega_{B/A}^1, M) \sim \text{Der}_{\text{cont}}(B, M)$$

of $B$-modules, where the left-hand side is the $B$-module consisting of continuous $B$-linear maps and $\Omega_{B/A}^1$ is endowed with the canonical topology.

Proposition 5.1.2. The canonical topology on $\Omega_{B/A}^1$ is coarser than the linear topology defined by $J^\bullet \Omega_{B/A}^1$. These two topologies coincide if for any $\lambda \in \Lambda$ there exists $\mu \in \Lambda$ such that $J^\mu \subseteq (J^\lambda)^2$.

Note that the hypothesis of the second assertion is satisfied if the topology on $B$ is adic (0, §7.2.(a)).

Proof. For $a \in J^\lambda$ and $x \in B$ we have $ad(x) = d(ax) - xd(a)$, which belongs to $<d(J^\lambda)>_B$. As $\Omega_{B/A}^1$ is generated by elements of the form $d(x)$ ($x \in B$), this shows the inclusion $J^\lambda \Omega_{B/A}^1 \subseteq <d(J^\lambda)>_B$, whence the first assertion. As for the second, observe that $<d((J^\lambda)^2)>_B \subseteq J^\lambda \Omega_{B/A}^1$, which shows that each $J^\lambda \Omega_{B/A}^1$ is open with respect to the topology defined by the filtration $<d(J^\lambda)>_{\lambda \in \Lambda}$. \hfill \Box

5.1. (c) Completion and differentials. Now we assume that $B$ is endowed with the adic topology defined by a finitely generated ideal $J \subseteq B$ (0, §7.2.(a)). Then, by 5.1.2, the canonical topology on the differential module $\Omega_{B/A}^1$ coincides with the $J$-adic topology. We denote the $J$-adic completion (0.7.2.6, 0.7.2.16) of $\Omega_{B/A}^1$ by $\hat{\Omega}_{B/A}^1$, and call it the complete differential module of $B$ relative to $A$. Using the universality of $J$-adic completions, we deduce from 5.1.1 the following result.

Proposition 5.1.3. For any $J$-adically complete $B$-module $M$ the canonical map $\varphi \mapsto \varphi \circ d$ gives rise to an isomorphism

$$\text{Hom}_{\text{cont}}(\hat{\Omega}_{B/A}^1, M) \sim \text{Der}_{\text{cont}}(B, M)$$

of $B$-modules.

Corollary 5.1.4. Let $A, B$ be rings with adic topologies defined by finitely generated ideals and $A \rightarrow B$ a continuous homomorphism. We have the canonical isomorphisms

$$\hat{\Omega}_{B/A}^1 \sim \hat{\Omega}_{\hat{B}/\hat{A}}^1 \sim \hat{\Omega}_{\hat{B}/\hat{A}}^1.$$

Proof. In view of 5.1.3 it suffices to show that the natural maps

$$\text{Der}_{\text{cont}}(\hat{B}, M) \hookrightarrow \text{Der}_{\text{cont}}(\hat{B}, M) \rightarrow \text{Der}_{\text{cont}}(B, M)$$

are isomorphisms.
are bijective for any $J$-adically complete $B$-module $M$. The bijectivity of the first arrow follows immediately from the fact that $A \to B$ is continuous. Since $\delta(J^{k+1}) \subseteq J^k M$ for $k \geq 0$, any $A$-derivation $\delta: B \to M$ can be uniquely extended to an $A$-derivation from $B$, whence the bijectivity of the second arrow.

Let $A$ (resp. $B$, resp. $C$) be a ring with the adic topology defined by a finitely generated ideal $I \subseteq A$ (resp. $J \subseteq B$, resp. $K \subseteq C$), and $A \to B$ and $A \to C$ continuous homomorphisms such that $IB \subseteq J$ and $IC \subseteq K$. As we have seen in §1.2. (a), $B \otimes_A C$ has the topology defined by $H^{\text{ad}} = \{ H^{m,n} \}_{m,n \geq 0}$, where

$$H^{m,n} = \text{image}(J^m \otimes_A C \to B \otimes_A C) + \text{image}(B \otimes_A K^n \to B \otimes_A C)$$

for $m, n \geq 0$. It is shown in (the proof of) 1.2.2 that this topology is actually $H$-adic, where $H = H^{1,1}$. We consider the $B \otimes_A C$-module $\Omega^1_{B/A} \otimes_A C$. This is endowed with the topology defined similarly to that of $B \otimes_A C$, that is, the topology defined by $\{ H^{m,n} \}_{m,n \geq 0}$, where

$$H^{m,n} = \text{image}(J^m \Omega^1_{B/A} \otimes_A C \to \Omega^1_{B/A} \otimes_A C) + \text{image}(\Omega^1_{B/A} \otimes_A K^n \to \Omega^1_{B/A} \otimes_A C)$$

for $m, n \geq 0$. This topology in turn coincides with the topology defined by $\{ H^{m,n}(\Omega^1_{B/A} \otimes_A C) \}_{m,n \geq 0}$, because $H^{m,n} = H^{m,n}(\Omega^1_{B/A} \otimes_A C)$ for $m, n \geq 0$. It follows that the topology on $\Omega^1_{B/A} \otimes_A C$ coincides with the $H$-adic topology. Hence in view of [54], 0IV, (20.5.5), we have the following result.

**Proposition 5.1.5.** We have the canonical isomorphism

$$\Omega^1_{B/A} \otimes_A C \overset{\sim}{\to} \hat{\Omega}^1_{B \hat{\otimes}_A C/C}$$

of topological $B \hat{\otimes}_A C$-modules.

**Corollary 5.1.6.** In the situation as in 5.1.5, we have

$$\hat{\Omega}^1_{B \hat{\otimes}_A C/C}/H^{k+1} \hat{\Omega}^1_{B \hat{\otimes}_A C/C} \cong \Omega^1_{B_k/A_k} \otimes_{A_k} C_k$$

for $k \geq 0$, where $A_k = A/I^{k+1}$, $B_k = B/J^{k+1}$, and $C_k = C/K^{k+1}$.

**Proof.** By 5.1.5, $H^{k+1} \cdot (\Omega^1_{B/A} \otimes_A C)$ is closed in $\Omega^1_{B/A} \hat{\otimes}_A C$, and hence

$$\Omega^1_{B/A} \hat{\otimes}_A C/H^{k+1} \cdot (\Omega^1_{B/A} \hat{\otimes}_A C) \cong \Omega^1_{B/A} \otimes_A C/H^{k+1} \cdot (\Omega^1_{B/A} \otimes_A C),$$

from which the desired equality follows.
Corollary 5.1.7. Let $A$ (resp. $B$) be a ring with the adic topology defined by a finitely generated ideal $I \subseteq A$ (resp. $J \subseteq B$), and $A \to B$ be a continuous homomorphism such that $IB \subseteq J$. Set

$$A_k = A/I^{k+1} \quad \text{and} \quad B_k = B/J^{k+1}$$

for $k \geq 0$. Then for any $k \geq 0$, up to canonical isomorphisms,

$$\Omega^1_{B/A}/J^{k+1}\Omega^1_{B/A} = \Omega^1_{B/A} \otimes_B B_k = \Omega^1_{B_k/A_k}.$$

In particular, the complete differential module $\hat{\Omega}^1_{B/A}$ is canonically isomorphic to the projective limit, $\varprojlim_{k \geq 0} \Omega^1_{B_k/A_k}$.

Proof. We apply 5.1.6 with $C = A$. Since, in this case, $H = J$, we have the following equalities up to canonical isomorphisms:

$$\hat{\Omega}^1_{B/A}/J^{k+1}\hat{\Omega}^1_{B/A} = \hat{\Omega}^1_{B/A}/J^{k+1}\hat{\Omega}^1_{B/A} = \Omega^1_{B/A}/J^{k+1}\Omega^1_{B/A} = \Omega^1_{B_k/A_k},$$

where the first equality is due to 5.1.4.

Proposition 5.1.8. Let $A \to B$ be a continuous homomorphism between adic rings of finite ideal type (1.1.3, 1.1.6), and $S \subseteq B$ a multiplicative subset. Then we have the canonical isomorphism

$$\hat{\Omega}^1_{B/A} \otimes_B B\{S^{-1}\} \sim \hat{\Omega}^1_{B\{S^{-1}\}/A}$$

of topological $B\{S^{-1}\}$-modules (cf. §1.1.(a) for the definition of $B\{S^{-1}\}$).

Proof. Let $J \subseteq B$ be a finitely generated ideal of definition. By [54], §IV, (20.5.9), $\Omega^1_{B/A} \otimes_B S^{-1}B \cong \Omega^1_{S^{-1}B/A}$. Since the topologies on both sides are $J$-adic, we obtain the isomorphism $\hat{\Omega}^1_{B/A} \otimes_B S^{-1}B \cong \hat{\Omega}^1_{S^{-1}B/A}$ between the $J$-adic completions. The left-hand side is clearly isomorphic to $\hat{\Omega}^1_{B/A} \otimes_B B\{S^{-1}\}$, while the right-hand side is isomorphic to $\hat{\Omega}^1_{B\{S^{-1}\}/A}$ due to 5.1.4.

Proposition 5.1.9. Let $A$ and $B$ be adic rings of finite ideal type, $S \subseteq A$ a multiplicative subset, and $A\{S^{-1}\} \to B$ a continuous homomorphism. Then we have the canonical isomorphism

$$\hat{\Omega}^1_{B/A} \sim \hat{\Omega}^1_{B/A\{S^{-1}\}}$$

of topological $B$-modules.

Proof. By [54], §IV, (20.7.17), (cf. 0.7.4.5) the map in question is surjective and the image of $\hat{\Omega}^1_{A\{S^{-1}\}/A} \otimes_{A\{S^{-1}\}} B$ is dense in the kernel. But this is 0, since $\hat{\Omega}^1_{A\{S^{-1\}/A} = \hat{\Omega}^1_{S^{-1}A/A} = 0$, where the first equality is due to 5.1.4.
5. Differential calculus on formal schemes

5.1. (d) Differentials and finiteness conditions. The following result is a consequence of 5.1.4 applied to $B = A\langle X_1, \ldots, X_n \rangle$ (the completion of $A[X_1, \ldots, X_n]$).

**Proposition 5.1.10.** Let $A$ be an adic ring of finite ideal type, and consider

$$B = A\langle X_1, \ldots, X_n \rangle.$$  

The complete differential module $\hat{\Omega}_{B/A}^1$ is a free $B$-module of rank $n$ with a basis given by $dX_1, \ldots, dX_n$.

**Corollary 5.1.11.** Let $A$ be an adic ring of finite ideal type, and $B$ a topologically finitely generated algebra over $A$ (cf. 0.8.4.1). Then $\hat{\Omega}_{B/A}^1$ is a finitely generated $B$-module.

**Proof.** Write $B = R/\mathfrak{a}$, where $R = A\langle X_1, \ldots, X_n \rangle$ and $\mathfrak{a} \subseteq R$ is a closed ideal. Then by [54], 0IV, (20.7.8), $\Omega_{B/A}^1$ is, as a topological $B$-module, the quotient of $\Omega_{R/A}^1 \otimes_R B$ by the image of $\mathfrak{a}$. Hence $\hat{\Omega}_{B/A}^1$ is the quotient of

$$\Omega_{R/A}^1 \otimes_R B = \Omega_{R/A}^1 \otimes_R B = \bigoplus_{i=1}^n B(dX_i \otimes 1)$$

(cf. 5.1.10) by the closure of the image of $\mathfrak{a}$ (cf. 0.7.4.5). □

**Theorem 5.1.12** (fundamental exact sequences). Let $A$ be a t.u. rigid-Noetherian ring (2.1.1 (1)).

(1) Let $B \to C$ be an $A$-algebra homomorphism between topologically finitely generated $A$-algebras. Then we have the canonical exact sequence of $C$-modules

$$\hat{\Omega}_{B/A}^1 \otimes_B C \longrightarrow \hat{\Omega}_{C/A}^1 \longrightarrow \hat{\Omega}_{C/B}^1 \longrightarrow 0.$$  

(2) Let $B$ be a topologically finitely generated $A$-algebra and $\mathfrak{a}$ a finitely generated ideal of $B$. Set $C = B/\mathfrak{a}$. Then we have the canonical exact sequence of $C$-modules

$$\mathfrak{a}/\mathfrak{a}^2 \longrightarrow \hat{\Omega}_{B/A}^1 \otimes_B C \longrightarrow \hat{\Omega}_{C/A}^1 \longrightarrow 0.$$  

Note that in (1) and (2), since $C$ is a t.u. rigid-Noetherian ring, and $\hat{\Omega}_{B/A}^1$ is a finitely generated $B$-module, we have $\hat{\Omega}_{B/A}^1 \otimes_B C = \hat{\Omega}_{B/A}^1 \otimes_B C$ (cf. 2.1.3).

**Proof.** (1) Let $I \subseteq A$ be a finitely generated ideal of definition, and for any $k \geq 0$ set

$$A_k = A/I^{k+1}, \quad B_k = B/I^{k+1}B, \quad C_k = C/I^{k+1}C.$$  

$B_k$ and $C_k$ are finite type $A_k$-algebras. By [54], 0IV, (20.5.7), we have the exact sequence

$$\Omega_{B_k/A_k}^1 \otimes_{B_k} C_k \longrightarrow \Omega_{C_k/A_k}^1 \longrightarrow \Omega_{C_k/B_k}^1 \longrightarrow 0.$$
By \textbf{0.5.1.7}, the projective systems

$$\{\Omega_{B_k/A_k}^1 \otimes B_k^* C_k\}_{k \geq 0}, \quad \{\Omega_{C_k/A_k}^1\}_{k \geq 0}, \quad \{\Omega_{C_k/B_k}^1\}_{k \geq 0}$$

are strict. Hence, due to \textbf{0.5.1.7} and \textbf{0.3.2.5}, we obtain the desired exact sequence upon applying \(\lim_{k \to 0}\).

(2) Define \(B_k\) and \(C_k\) \((k \geq 0)\) similarly as above. The kernel of \(B_k \to C_k\) is \(a_k = aB_k = a/a \cap I^{k+1}\). Hence we have the exact sequence \([54], \text{IV}, (20.5.12)\)

$$a_k/a_k^2 \to \Omega_{B_k/A_k}^1 \otimes B_k^* C_k \to \Omega_{C_k/A_k}^1 \to 0.$$ 

Since \(a\) is finitely generated,

$$0 \to a^2 \to a \to a/a^2 \to 0$$

is an exact sequence of \(I\)-adically complete modules. Since \(A\) is t.u. rigid-Noetherian, \(a = \lim_{k \to 0} a/I^{k+1} a = \lim_{k \to 0} a_k\) etc., whence \(a/a^2 = \lim_{k \to 0} a_k/a_k^2\). Then the desired exact sequence can be obtained by an argument similar to that in the proof of (1).

\textbf{Corollary 5.1.13.} Let \(A\) be a t.u. rigid-Noetherian ring and \(B\) a topologically finitely presented algebra over \(A\). Then \(\hat{\Omega}_{B/A}^1\) is a finitely presented \(B\)-module. More precisely, if

$$B = A\langle X_1, \ldots, X_n \rangle / \mathfrak{a},$$

where \(\mathfrak{a} = (F_1, \ldots, F_m)\) is a finitely generated ideal, then

$$\hat{\Omega}_{B/A}^1 \cong (\bigoplus_{i=1}^m B dX_i) / (\sum_{j=1}^m B dF_j).$$

\section*{5.2 Differential invariants on formal schemes}

\textbf{5.2. (a) The sheaf of differentials}

\textbf{Theorem 5.2.1.} Let \(f : X \to Y\) be a morphism between adic formal schemes of finite type. Then there exists, uniquely up to isomorphism, an adically quasi-coherent sheaf \(\Omega_{X/Y}^1\) of finite type on \(X\), such that for any affine open subset \(V = \text{Spf} A\) of \(Y\) and any affine open subset \(U = \text{Spf} B\) of \(f^{-1}(V)\) (where \(A\) and \(B\) are adic rings of finite ideal type; cf. \textbf{3.7.13}), we have \(\Gamma(U, \Omega_{X/Y}^1) \cong \hat{\Omega}_{B/A}^1\) or, equivalently (cf. \textbf{3.2.8}), \(\Omega_{X/Y}^1|_U \cong (\hat{\Omega}_{B/A}^1)^A\). If \(f\) is locally of finite type, then \(\Omega_{X/Y}^1\) is a.q.c. of finite type.

\textbf{Proof.} The uniqueness is clear, and the existence follows from \textbf{5.1.8} and \textbf{5.1.9}. The last assertion follows from \textbf{5.1.11} and \textbf{3.2.6}. \(\square\)
We call the a.q.c. sheaf $\Omega^1_{X/Y}$ thus obtained the \emph{(complete) sheaf of differentials over $X$ relative to $Y$}; this sheaf is equipped with a canonical morphism of $\mathcal{O}_Y$-modules
\[ d: \mathcal{O}_X \longrightarrow \Omega^1_{X/Y}, \]
the so-called \emph{canonical derivation}.

**Example 5.2.2.** Let $Y$ be an adic formal scheme, and $\mathcal{E}$ a locally free sheaf on $Y$ of finite type. Consider the vector bundle $X = \hat{V}(\mathcal{E})$ associated to $\mathcal{E}$ (Exercise I.5.2). Then due to 5.1.10 $\Omega^1_{X/Y}$ is a locally free sheaf on $X$ of finite type.

Using 5.1.5 and 3.6.4, we readily obtain the following proposition.

**Proposition 5.2.3.** Consider a Cartesian diagram
\[ X' \xrightarrow{\hat{g}} X \xleftarrow{\hat{g}} Y' \to Y \]
of adic formal schemes of finite ideal type, and suppose that either one of the following conditions are satisfied:

(a) $X$ and $X'$ are locally universally rigid-Noetherian, and $X \to Y$ is locally of finite type;

(b) $Y' \to Y$ is adically flat.

Then we have a canonical isomorphism
\[ \hat{g}^* \Omega^1_{X/Y} \sim \Omega^1_{X'/Y'} \]

The following proposition follows immediately from 5.1.7.

**Proposition 5.2.4.** Let $f: X \to Y$ be a morphism between adic formal schemes of finite ideal type, and suppose that $Y$ has an ideal of definition $\mathcal{I}$ of finite type. Set
\[ X_k = (X, \mathcal{O}_X / \mathcal{I}^{k+1} \mathcal{O}_X) \quad \text{and} \quad Y_k = (Y, \mathcal{O}_Y / \mathcal{I}^{k+1}) \]
for any $k \geq 0$. Then we have natural isomorphisms
\[ \Omega^1_{X/Y} \hat{f}^{k+1} \Omega^1_{X/Y} \cong \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_k} \cong \Omega^1_{X_k/Y_k} \]
for $k \geq 0$, where $\Omega^1_{X_k/Y_k}$ denotes the usual differential module for the map of schemes $f_k: X_k \to Y_k$. 

5.2. (b) Differentials on universally rigid-Noetherian formal schemes. The following proposition is a consequence of 5.1.13 and 3.5.6.

Proposition 5.2.5. Let $Y$ be a locally universally rigid-Noetherian formal scheme, see 2.1.7, and $X$ an $Y$-formal scheme locally of finite presentation. Then $\Omega^1_{X/Y}$ is a finitely presented $\mathcal{O}_X$-module.

By 5.1.12 we have the following theorem.

Theorem 5.2.6 (fundamental exact sequences). Let $Z$ be a locally universally rigid-Noetherian formal scheme.

1. Let $f: X \to Y$ and $Y \to Z$ be locally of finite type morphisms of adic formal schemes of finite ideal type. Then we have the natural exact sequence

$$\hat{f}^* \Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$$

of a.q.c. sheaves on $X$ of finite type, where $\hat{f}^*$ denotes the complete pull-back (3.6.1; cf. 3.6.2).

2. Let $Y$ be an adic formal scheme locally of finite type over $Z$ and $i: X \hookrightarrow Y$ be an immersion of finite presentation. Let $\mathcal{N}_{X/Y}$ be the conormal sheaf of $X$ in $Y$ (Exercise I.5.3). Then we have the canonical exact sequence

$$\mathcal{N}_{X/Y} \to \hat{i}^* \Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to 0$$

of a.q.c. sheaves on $X$ of finite type.

5.3 Étale and smooth morphisms

5.3. (a) Neat morphisms

Definition 5.3.1. (1) Let $f: X \to Y$ be an adic morphism between adic formal schemes of finite ideal type. Then $f$ is said to be adically locally of finite presentation if for any point $x \in X$ there exist an open neighborhood $U$ of $x$ in $X$ and an open neighborhood $V$ of $f(x)$ in $Y$ that has an ideal of definition $\mathcal{I}$ of finite type, such that

(a) $f(U) \subseteq V$, and

(b) if we set $U_k = (U, \mathcal{O}_U/\mathcal{I}^{k+1})$ and $V_k = (V, \mathcal{O}_V/\mathcal{I}^{k+1})$ for $k \geq 0$, then the induced map $U_k \to V_k$ of schemes is of finite presentation for any $k \geq 0$.

(2) We say that $f$ is adically of finite presentation if it is adically locally of finite presentation and quasi-compact (1.6.1).
Note that these are properties of the form ‘adically $P$’ (§1.5. (d)), where $P$ = ‘locally of finite presentation’ or ‘of finite presentation’. By 1.7.3, we know that a morphism $f : X \to Y$ of adically finite presentation (resp. adically locally of finite presentation) is of finite type (resp. locally of finite type). Moreover, if $Y$ is locally universally rigid-Noetherian, then $f$ is of adically finite presentation (resp. adically locally of finite presentation) if and only if it is of finite presentation (resp. locally of finite presentation) (2.2.4).

**Proposition 5.3.2.** Let $f : X \to Y$ be a morphism adically locally of finite presentation between adic formal schemes of finite ideal type, and suppose that $Y$ has an ideal of definition $\mathcal{I}$ of finite type. Set

$$X_k = (X, \mathcal{O}_X / \mathcal{I}^{k+1} \mathcal{O}_X) \quad \text{and} \quad Y_k = (Y, \mathcal{O}_Y / \mathcal{I}^{k+1})$$

for $k \geq 0$, and let

$$f_k : X_k \to Y_k$$

be the induced map of schemes. Then the following conditions are equivalent.

(a) $\Omega_{X/Y}^1 = 0$.
(b) $\Omega_{X_k/Y_k}^1 = 0$ for any $k \geq 0$.
(c) $\Omega_{X_0/Y_0}^1 = 0$.
(d) The diagonal map $\Delta_X : X \to X \times_Y X$ is an open immersion.

**Proof.** Implications (a) $\implies$ (b) $\implies$ (c) are clear due to 5.2.4. To show (c) $\implies$ (a), we may work locally and may assume that $X$ and $Y$ are affine, $X = \text{Spf} \ A$ and $Y = \text{Spf} \ B$, where $A$ and $B$ are adic rings of finite ideal type and $I \subseteq A$ is a finitely generated ideal of definition such that $I = I_\Delta$. Then (c) implies that $\hat{\Omega}_{B/A}^1 / \mathcal{I} \hat{\Omega}_{B/A}^1 = 0$. Since $\hat{\Omega}_{B/A}^1$ is a finitely generated $B$-algebra (5.1.11), we have $\hat{\Omega}_{B/A}^1 = 0$ by Nakayama’s lemma, whence (a). Condition (d) is equivalent to the diagonal map $\Delta_{X_k} : X_k \to X_k \times_Y X_k$ being an open immersion for any $k \geq 0$ (cf. 1.3.5). Hence by [54], IV, (17.4.2), we have the equivalence (b) $\iff$ (d). \qed

**Definition 5.3.3.** (1) An adic morphism $f : X \to Y$ of adic formal schemes of finite ideal type is said to be neat, or unramified, if it is adically locally of finite presentation and $\Omega_{X/Y}^1 = 0$.

(2) Let $f : X \to Y$ be an adic morphism of adic formal schemes of finite ideal type, and $x \in X$ a point. We say that $f$ is neat at $x$, or unramified at $x$, if there exists an open neighborhood $U$ of $x$ in $X$ such that the map $U \to Y$ is neat.
Proposition 5.3.4. Let \( f : X \to Y \) be an adic morphism of adic formal schemes of finite ideal type, and suppose \( Y \) has an ideal of definition \( \mathcal{I} \) of finite type. Set
\[
X_k = (X, \mathcal{O}_X / \mathcal{I}^{k+1} \mathcal{O}_X) \quad \text{and} \quad Y_k = (Y, \mathcal{O}_Y / \mathcal{I}^{k+1})
\]
for \( k \geq 0 \). Let
\[
f_k : X_k \to Y_k
\]
be the induced morphism of schemes. Then the following conditions are equivalent.

(a) \( f \) is neat.

(b) \( f_k \) is neat for any \( k \geq 0 \).

If we assume, moreover, that \( f \) is adically locally of finite presentation, then the conditions are equivalent to

(c) \( f_0 \) is neat.

This follows immediately from 5.3.2. By 4.4.2 and [54], IV, (17.3.3) (i), we have the following result.

Proposition 5.3.5. An immersion is neat if and only if it is adically locally of finite presentation.

Proposition 5.3.6. (1) If \( f : Z \to Y \) and \( g : Y \to X \) are neat, then so is the composition \( g \circ f \).

(2) For any neat \( S \)-morphisms \( f : X \to Y \) and \( g : X' \to Y' \) of adic formal schemes of finite ideal type over an adic formal scheme \( S \) of finite ideal type, the induced morphism
\[
f \times_S g : X \times_S Y \to X' \times_S Y'
\]
is neat.

(3) For any neat \( S \)-morphism \( f : X \to Y \) of adic formal schemes of finite ideal type over an adic formal scheme \( S \) of finite ideal type, and for any morphism \( S' \to S \) of adic formal schemes of finite ideal type, the induced morphism
\[
f_{S'} : X \times_S S' \to Y \times_S S'
\]
is neat.

(4) Suppose that the composition \( g \circ f \) of two adic morphisms of adic formal schemes of finite ideal type is neat and that \( g \) is adically locally of finite presentation. Then \( f \) is neat.

Proof. (1) follows from [54], IV, (17.3.3) (ii). By 1.5.1 and 5.3.4, (2) and (3) follow from [54], IV, (17.3.3) (iii) and (iv). Finally, (4) follows from 5.3.4 and [54], IV, (17.3.3) (v).
Corollary 5.3.7. Let $f : X \to Y$ be a morphism of schemes, and $Z$ a closed subscheme of $Y$ of finite presentation. If $f$ is neat, then the formal completion
\[ \hat{f} : \hat{X} |_{f^{-1}(Z)} \to \hat{Y}_Z \]
is neat.

The following two propositions follow from 5.2.6 (1).

Proposition 5.3.8. Consider morphisms $f : X \to Y$ and $g : Y \to Z$ of locally universally rigid-Noetherian formal schemes. Suppose that $f$ is locally of finite type and that $g$ is neat. Then we have the canonical isomorphism
\[ \Omega^1_{X/Z} \sim \Omega^1_{X/Y} \]
of a.q.c. sheaves on $X$.

Proposition 5.3.9. Let $Z$ be a locally universally rigid-Noetherian formal scheme and $f : X \to Y$ a $Z$-morphism between adic formal schemes that are locally of finite type over $Z$. If $f$ neat, then the canonical map
\[ \hat{f}^* \Omega^1_{Y/Z} \to \Omega^1_{X/Z} \]
is surjective. The converse holds if $f$ is locally of finite presentation.

5.3. (b) Étale morphisms

Definition 5.3.10. (1) An adic morphism $f : X \to Y$ between adic formal schemes of finite ideal type is said to be étale if it is neat and adically flat (4.8.12).

(2) Let $f : X \to Y$ be an adic morphism of adic formal schemes, and $x \in X$ a point. We say that $f$ is étale at $x$ if there exists an open neighborhood $U$ of $x$ in $X$ such that the map $U \to Y$ is étale.

Étale morphisms are adically locally of finitely presentation (5.3.1 (1)), and hence are locally of finite type. Moreover, if $f : X \to Y$ is étale and $Y$ is locally universally rigid-Noetherian, then $f$ is locally of finite presentation (due to 2.2.4) and flat (due to 4.8.15).

Proposition 5.3.11. Let $f : X \to Y$ be an adic morphism of adic formal schemes of finite ideal type. Suppose that $Y$ has an ideal of definition $I$ of finite type. Set
\[ X_k = (X, \mathcal{O}_X / I^{k+1} \mathcal{O}_X) \quad \text{and} \quad Y_k = (Y, \mathcal{O}_Y / I^{k+1}) \]
for $k \geq 0$, and denote by
\[ f_k : X_k \to Y_k \]
the induced morphism of schemes.
The following conditions are equivalent.

(a) $f$ is étale.

(b) $f_k$ is étale for $k \geq 0$.

This follows from 5.3.4 and 4.8.13.

**Corollary 5.3.12.** Let $f: X \to Y$ be a morphism of schemes, and $Z$ a closed subscheme of $Y$ of finite presentation. If $f$ is étale, then the formal completion

$$\hat{f}: \hat{X}|_{f^{-1}(Z)} \to \hat{Y}|_Z$$

is étale.

**Proposition 5.3.13.** (1) Any open immersion of adic formal schemes of finite type is étale.

(2) The composition of two étale morphisms is étale. Let both $f: X \to Y$ and $g: Y \to Z$ be adically locally of finite presentation morphisms. Suppose $f$ is adically faithfully flat. Then if $g \circ f$ is étale, so is $g$.

(3) For any étale $S$-morphisms $f: X \to Y$ and $g: X' \to Y'$ of adic formal schemes of finite ideal type over an adic formal scheme $S$ of finite ideal type, the induced morphism

$$f \times_S g: X \times_S Y \to X' \times_S Y'$$

is étale.

(4) For any étale $S$-morphism $f: X \to Y$ of adic formal schemes of finite ideal type over an adic formal scheme $S$ of finite ideal type, and for any morphism $S' \to S$ of adic formal schemes of finite ideal type, the induced morphism

$$f_{S'}: X \times_S S' \to Y \times_S S'$$

is étale.

**Proof.** (1) and the first part of (2) are clear. The second part of (2) follows from 5.3.11 and [54], IV, (17.7.7). By 1.5.1, (3) and (4) follow from [54], IV, (17.3.3). \[ \square \]

**Proposition 5.3.14.** Consider morphisms $f: X \to Y$ and $g: Y \to Z$ of adic formal schemes of finite ideal type. Suppose that $f$ is étale and that $g$ is locally of finite type. Then we have the canonical isomorphism

$$\hat{f}^*\Omega^1_{Y/Z} \xrightarrow{\sim} \Omega^1_{X/Z}$$

of a.q.c. sheaves on $X$.  

Proof. As the question is local, we may assume that $Y$ and $Z$ have ideals of definition $\mathcal{J}$ and $\mathcal{I}$, respectively, such that $\mathcal{I} \mathcal{O}_Y \subseteq \mathcal{J}$. For $k \geq 0$, set
\[
X_k = (X, \mathcal{O}_X/\mathcal{J}^{k+1}\mathcal{O}_X),
\]
\[
Y_k = (Y, \mathcal{O}_Y/\mathcal{J}^{k+1}),
\]
\[
Z_k = (Z, \mathcal{O}_Z/\mathcal{I}^{k+1})
\]
By 5.2.4 (cf. 5.1.7) and 3.6.5, the differential module $\Omega^1_Y/Z$ (resp. $\Omega^1_X/Z$) coincides with the projective limit $\lim_{\leftarrow k \geq 0} f_k^* \Omega^1_{Y_k/Z_k}$ (resp. $\lim_{\leftarrow k \geq 0} \Omega^1_{X_k/Z_k}$). As $f_k: X_k \to Y_k$ is étale (5.3.11), we have $f_k^* \Omega^1_{Y_k/Z_k} \cong \Omega^1_{X_k/Z_k}$ ([54], IV, (17.2.4)), and hence we get the desired isomorphism.

5.3. (c) Smooth morphisms. We first remark that the following proposition holds due to 5.1.10.

Proposition 5.3.15. Let $Y$ be an adic formal scheme of finite ideal type, and consider the affine $n$-space
\[
\hat{\mathbb{A}}^n_Y = \text{Spec} \mathbb{Z}[X_1, \ldots, X_n] \times_{\text{Spec} \mathbb{Z}} Y
\]
(fiber product taken in the category $\mathbf{Fs}$ of all formal schemes) over $Y$ (see Exercise I.5.1 (1)). Then the differential module $\Omega^1_{\hat{\mathbb{A}}^n_Y/Y}$ is a free $\mathcal{O}_{\hat{\mathbb{A}}^n_Y}$-module with a basis given by $dX_1, \ldots, dX_n$.

Definition 5.3.16. (1) An adic morphism $f: X \to Y$ between adic formal schemes of finite ideal type is said to be smooth if for any $x \in X$ there exist an open neighborhood $U$ of $x$ in $X$ and a commutative diagram
\[
\begin{array}{ccc}
U & \to & \hat{\mathbb{A}}^n_Y \\
\downarrow & & \downarrow \\
Y & & 
\end{array}
\]
(for some $n$ that depends on $x$), where the horizontal arrow is étale.

(2) Let $f: X \to Y$ be an adic morphism of adic formal schemes of finite ideal type, and $x \in X$ a point. We say that $f$ is smooth at $x$ if there exists an open neighborhood $U$ of $x$ in $X$ such that the map $U \to Y$ is smooth.

Here are the immediate consequences of the definition: étale morphisms are smooth, and smooth morphisms are adically locally of finite presentation (hence locally of finite type) and adically flat; the affine $n$-space $\hat{\mathbb{A}}^n_Y$ over an adic formal scheme $Y$ is smooth over $Y$.

The following proposition is a direct consequence of the definition and 5.3.14.
Proposition 5.3.17. If \( f : X \rightarrow Y \) is smooth, then \( \Omega^1_{X/Y} \) is a locally free \( \mathcal{O}_X \)-module of finite rank.

Proposition 5.3.18. Let \( f : X \rightarrow Y \) be an adic morphism of adic formal schemes of finite ideal type. Suppose that \( Y \) has an ideal of definition \( I \) of finite type. Set \( X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X) \) and \( Y_k = (Y, \mathcal{O}_Y/I^{k+1}) \) for \( k \geq 0 \), and denote by \( f_k : X_k \rightarrow Y_k \) the induced morphism of schemes. The following conditions are equivalent.

(a) \( f \) is smooth.

(b) \( f_k \) is smooth for \( k \geq 0 \).

Proof. Implication (a) \( \Rightarrow \) (b) follows from 5.3.11 and [54], IV, (17.11.4). To show the converse, we may work locally; we may assume that \( X \) and \( Y \) are affine; set \( X = \text{Spf} B \) and \( Y = \text{Spf} A \), where \( A \) and \( B \) are adic rings of finite ideal type, and let \( I \subseteq A \) be the finitely generated ideal of definition with \( I = I^\Delta \). Set \( A_k = A/I^{k+1} \) and \( B_k = B/I^{k+1}B \) for \( k \geq 0 \). We already have an étale map \( A_0[X_1, \ldots, X_n] \rightarrow B_0 \) over \( A_0 \). Take \( g_1, \ldots, g_n \in B \) whose images in \( B_0 \) are the images of \( X_1, \ldots, X_n \), respectively; since \( B \) is \( I \)-adically complete, one has the \( A \)-algebra homomorphism \( A\langle X_1, \ldots, X_n \rangle \rightarrow B \), mapping \( X_i \) to \( g_i \) for \( i = 1, \ldots, n \), which extends the map \( A_0[X_1, \ldots, X_n] \rightarrow B_0 \). Since this is an \( A \)-algebra morphism, it is an adic map. Thus we have the \( Y \)-morphism \( g : X \rightarrow \hat{\mathbb{A}}^n_Y \) such that \( g_0 \) is étale. Now since \( f \) is adically locally of finite presentation, we deduce that \( g \) is adically locally of finite presentation; by 5.3.4, we know that \( g \) is neat. Now by 4.8.17 we deduce that \( g \) is adically flat, and hence is étale.

Corollary 5.3.19. Let \( f : X \rightarrow Y \) be a morphism of schemes, and \( Z \) a closed subscheme of \( Y \) of finite presentation. If \( f \) is smooth, then the formal completion

\[
\hat{f} : \hat{X} |_{f^{-1}(Z)} \rightarrow \hat{Y} |_Z
\]
is smooth.

Proposition 5.3.20. (1) The composition of two smooth morphisms is smooth. Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be morphisms adically locally of finite presentation. Suppose \( f \) is adically faithfully flat. Then if \( g \circ f \) is smooth, so is \( g \).

(2) For any smooth \( S \)-morphisms \( f : X \rightarrow Y \) and \( g : X' \rightarrow Y' \) of adic formal schemes of finite ideal type over an adic formal scheme \( S \) of finite ideal type, the induced morphism

\[
f \times_S g : X \times_S Y \rightarrow X' \times_S Y'
\]
is smooth.
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(3) For any smooth $S$-morphism $f : X \to Y$ of adic formal schemes of finite ideal type over an adic formal scheme $S$ of finite ideal type, and for any morphism $S' \to S$ of adic formal schemes of finite ideal type, the induced morphism

$$f_{S'} : X \times_S S' \to Y \times_S S'$$

is smooth.

Proof. The first part of (1) follows from 5.3.18 and [54], IV, (17.3.3) (ii). The second part of (1) follows from 5.3.18 and [54], IV, (17.7.7). By 1.5.1, (2) and (3) follow from [54], IV, (17.3.3).

Proposition 5.3.21. Let $Z$ be an adic formal scheme of finite ideal type, and let $Y \to Z$ be a locally of finite type morphism of adic formal schemes of finite ideal type. Let $f : X \to Y$ be a smooth morphism. Then we have the canonical exact sequence

$$0 \to f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$$

of a.q.c. sheaves on $X$, where $\Omega^1_{X/Y}$ is locally free of finite type. In particular, this exact sequence splits locally.

Proof. As the question is local, we may assume that $Z$ has an ideal of definition of finite type $I$. Set $X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X)$, $Y_k = (Y, \mathcal{O}_Y/I^{k+1}\mathcal{O}_Y)$, and $Z_k = (Z, \mathcal{O}_Z/I^{k+1})$ for $k \geq 0$. Since $f_k : X_k \to Y_k$ is smooth (5.3.18), the sequence

$$0 \to f_k^*\Omega^1_{Y_k/Z_k} \to \Omega^1_{X_k/Z_k} \to \Omega^1_{X_k/Y_k} \to 0$$

is exact for any $k \geq 0$ ([54], IV, (17.2.3) (ii)). By 5.2.4 and 3.6.5, the desired sequence (*) is the one obtained from $(*)_k$ by passage to the projective limits $\lim_{\leftarrow k \geq 0}$. Since $\{f_k^*\Omega^1_{Y_k/Z_k}\}_{k \geq 0}$ is strict, the exactness of (*) follows from 0.3.2.14 (1) (cf. 0.3.2.13 (2)). The last assertion follows from 5.3.17.

Exercises

Exercise I.5.1. Let $S$ be a formal scheme.

1. Define $\mathbb{A}^n_S = \mathbb{A}^n_\mathbb{Z} \times_{\text{Spec } \mathbb{Z}} S$ (fiber product taken in $\mathbf{Fs}$), and call it the affine $n$-space over $S$. Show that $\mathbb{A}^n_S \to S$ is affine and that, if $S$ is adic, then $\mathbb{A}^n_S \to S$ is affine and adic.

2. Define $\mathbb{P}^n_S = \mathbb{P}^n_\mathbb{Z} \times_{\text{Spec } \mathbb{Z}} S$ (fiber product taken in $\mathbf{Fs}$), and call it the projective $n$-space over $S$. Show that, if $S$ is adic, then $\mathbb{P}^n_S \to S$ is proper.
Exercise I.5.2. Let $X$ be an adic formal scheme of finite ideal type.

(1) Show that any locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite type is an adically quasi-coherent sheaf.

(2) Show that the completion (3.1.1 (1)) of the symmetric algebra $\text{Sym}_{\mathcal{O}_X}(\mathcal{E})$ is an adically quasi-coherent $\mathcal{O}_X$-algebra of finite type. (The corresponding $\bar{X}$-affine formal scheme (4.1.9) is denoted by $\bar{V}(\mathcal{E})$ and is called the vector bundle over $X$ associated to $\mathcal{E}$.)

Exercise I.5.3 (conormal sheaf). Let $f: Y \hookrightarrow X$ be an immersion between adic formal schemes of finite ideal type. Take a decomposition $f = j \circ i$, where $i: Y \hookrightarrow U$ is a closed immersion and $j: U \hookrightarrow X$ is an open immersion. Let $\mathcal{K}$ be the defining ideal of the closed formal subscheme $Y$ of $U$, that is, the kernel of the map $\mathcal{O}_U \to i^* \mathcal{O}_Y$. Then the $\mathcal{O}_Y$-module $i^* \mathcal{K}$ is called the conormal sheaf of $Y$ in $X$, and is denoted by $\mathcal{N}_{Y/X}$. Show that, if $X$ is locally universally rigid-Noetherian and if $f$ is of finite presentation, then $\mathcal{N}_{Y/X}$ is an a.q.c. sheaf on $Y$ of finite type.

Exercise I.5.4. Let $A$ be an adic ring of finite ideal type, $X$ and $Y$ $A$-formal schemes adically of finite presentation, and $f: X \to Y$ a morphism adically of finite presentation. Show that $f$ is proper if and only if for any integer $N \geq 0$ the induced morphism $\tilde{f} \times_Y \text{id}_{\hat{A}^N_Y}: X \times_Y \hat{A}^N_Y \to \hat{A}^N_Y$ is closed.

6 Formal algebraic spaces

The aim of this section is to lay the foundations of the theory of formal algebraic spaces. Here, formal algebraic spaces are more precisely understood as quasi-separated adic formal algebraic spaces of finite ideal type, generalizing Noetherian formal algebraic spaces already discussed in the last chapter of [72].

Note that an important and non-trivial thing to do first, when introducing formal algebraic spaces as étale sheaves, is to choose the domain category. Since we only need, for the sake of our applications to rigid geometry, to consider the adic situation, we can obviously restrict to the category of adic formal schemes, but with arbitrary morphisms, not only adic ones, in order for several universalities and functorialities to make reasonable sense. Note that, for instance, by considering such a wide category, one can keep having final objects and, moreover, treat schemes at the same time (by this, in particular, the notion of formal algebraic spaces can include that of algebraic spaces). In view of all this, we adopt as the base category the category $\text{AcFs}_S$ of adic formal schemes of finite ideal type over $S$ with arbitrary morphisms.

In §6.1 we deal with preliminaries on descent theory in formal geometry; as our interest is not only in defining (locally) universally rigid-Noetherian formal algebraic spaces, but more general ones, flat descent is not the reasonable notion to
work with; instead, one should consider adically flat descent. In the first part, we provide the general framework of adically flat descent for adically quasi-coherent sheaves and prove several basic theorems.

Next in §6.2 we discuss étale topology and étale sites on adic formal schemes of finite ideal type. Based on these, the notion of a.q.c. sheaves will be extended on the étale site.

The main part of this section is §6.3, where we give the definition of formal algebraic spaces and discuss basic geometries of them. Several properties for morphisms between formal algebraic spaces are defined and discussed in §6.4. Then this section ends up with §6.5, in which we will focus on locally universally rigid-Noetherian formal algebraic spaces and locally universally adhesive formal algebraic spaces.

6.1 Adically flat descent

6.1. (a) Basic assertions. As usual, a diagram of the form

\[
M_0 \xrightarrow{f} M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3
\]

consisting of sets (or sheaves of sets, abelian groups, etc.) is said to be exact if the map \( f \) coincides with the difference kernel of \( g_1 \) and \( g_2 \), that is, \( f \) is injective, \( g_1 \circ f = g_2 \circ f \), and the image of \( f \) coincides with the locus of coincidence of the maps \( g_1 \) and \( g_2 \). In case of (sheaves of) abelian groups, this amounts to the sequence

\[
0 \longrightarrow M_0 \xrightarrow{f} M_1 \xrightarrow{g_1-g_2} M_2
\]

being exact in the usual sense.

**Situation 6.1.1.** Let \( q: Y \rightarrow X \) be a quasi-compact adically faithfully flat morphism of adic formal schemes of finite ideal type; see 4.8.12 (2). We consider the diagram

\[
X \xleftarrow{q} Y \xrightarrow[p_1]{p} Y \times_X Y,
\]

where \( p_1 \) and \( p_2 \) are the projections, and we set \( p = q \circ p_1 = q \circ p_2 \). If \( X \) has an ideal of definition \( I \) of finite type, then for any non-negative integer \( k \geq 0 \) we have the induced diagram of schemes

\[
X_k \xleftarrow{q_k} Y_k \xrightarrow[p_{1,k}]{p_{2,k}} Y_k \times_{X_k} Y_k,
\]

where \( X_k = (X, \mathcal{O}_X/I^{k+1}) \) and \( Y_k = (Y, \mathcal{O}_Y/I^{k+1}\mathcal{O}_Y) \); note that the map \( q_k \) is a quasi-compact faithfully flat morphism between schemes (4.8.13).
Proposition 6.1.2. Consider the situation as in 6.1.1, and let $\mathcal{F}$ be an a.q.c. sheaf on $X$ (resp. an a.q.c. $\mathcal{O}_X$-algebra). Then the induced diagram

$$\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(Y, \widehat{q}^* \mathcal{F}) \longrightarrow \Gamma(Y \times_X Y, \widehat{p}^* \mathcal{F})$$

is exact.

Recall that, since $q$ and $p$ are adically flat, the complete pull-backs $\widehat{q}^* \mathcal{F}$ and $\widehat{p}^* \mathcal{F}$ are a.q.c. sheaves (resp. adically quasi-coherent $\mathcal{O}_X$-algebras); see 3.6.3.

Proposition 6.1.3. Consider the situation as in 6.1.1, and let $\mathcal{F}$ and $\mathcal{G}$ be either a.q.c. sheaves on $X$ or a.q.c. $\mathcal{O}_X$-algebras. Then the induced diagram

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \text{Hom}_{\mathcal{O}_Y}(\widehat{q}^* \mathcal{F}, \widehat{q}^* \mathcal{G}) \longrightarrow \text{Hom}_{\mathcal{O}_{Y \times_X Y}}(\widehat{p}^* \mathcal{F}, \widehat{p}^* \mathcal{G})$$

is exact.

The proofs of these Propositions 6.1.2 and 6.1.3 use the following lemma.

Lemma 6.1.4. Let $A \rightarrow B$ be an adic homomorphism of adic rings of finite ideal type such that the induced map $q: Y = \text{Spf} B \rightarrow X = \text{Spf} A$ is adically faithfully flat. Let $M$ be an $I$-adically complete $A$-module, where $I \subseteq A$ is a finitely generated ideal of definition. Then the sequence

$$M \longrightarrow M \otimes_A B \longrightarrow M \otimes_A (B \otimes_A B)$$

is exact.

Proof. Let $A_k = A/I^{k+1}, B_k = B/I^{k+1}B,$ and $M_k = M/I^{k+1}M$ for $k \geq 0.$ Since $q_k: Y_k = \text{Spec} B_k \rightarrow X_k = \text{Spec} A_k$ is faithfully flat (4.8.13), the sequence

$$M_k \longrightarrow M_k \otimes_{A_k} B_k \longrightarrow M_k \otimes_{A_k} (B_k \otimes_{A_k} B_k)$$

is exact for any $k \geq 0.$ Since (*) is the projective limit $\lim_{\leftarrow k \geq 0}(*)_k$, the desired exactness follows from the left-exactness of the functor $\lim_{\leftarrow k \geq 0}(\text{.3.2.4}).$ \hfill $\Box$

Proof of Propositions 6.1.2 and 6.1.3. Since $\widehat{q}^* \mathcal{O}_X = \mathcal{O}_Y$ and $\widehat{p}^* \mathcal{O}_X = \mathcal{O}_{Y \times_X Y}$, 6.1.2 follows from 6.1.3. Therefore, it suffices to show 6.1.3.

By a standard argument (cf. e.g., the proof of [21], Chapter 6, Proposition 1) one can easily reduce to the case where $X$ and $Y$ are affine, $X = \text{Spf} A$ and $Y = \text{Spf} B$, where $A$ and $B$ are adic rings of finite ideal type. Let $I \subseteq A$ be a finitely generated ideal of definition. Take $I$-adically complete $A$-modules $M$ and $N$ corresponding to $\mathcal{F}$ and $\mathcal{G}$, that is, $\mathcal{F} = M^\Delta$ and $\mathcal{G} = N^\Delta$ (3.2.8). Then, by 3.2.9 and 3.6.4 (2), what we need to show is the exactness of the sequence

$$\text{Hom}_{A}(M, N) \longrightarrow \text{Hom}_{B}(M \otimes_A B, N \otimes_A B) \longrightarrow \text{Hom}_{B \otimes_A B}(M \otimes_A B \otimes_A B, N \otimes_A B \otimes_A B).$$
This follows, by an easy diagram chasing, from the exactness of the sequence

\[ N \rightarrow N \hat{\otimes}_A B \rightarrow N \hat{\otimes}_A (B \hat{\otimes}_A B), \]

which in turn is a consequence of 6.1.4. \[ \square \]

6.1. (b) **Descent of morphisms.** As defined in §1.5. (a), for an adic formal scheme \( X \) of finite ideal type we denote by \( \text{AcF}_{X} \) the category of adic formal schemes of finite ideal type over \( X \) and (not necessarily adic) morphisms over \( X \).

**Proposition 6.1.5.** Consider the situation as in 6.1.1, and let \( Z \) and \( W \) be adic formal schemes of finite ideal type over \( X \). Set

\[ Z' = Y \times_X Z \quad \text{and} \quad Z'' = (Y \times_X Y) \times_X Z, \]

and similarly for \( W \). Then the following sequence is exact:

\[ \text{Hom}_{\text{AcF}_{X}} (Z, W) \rightarrow \text{Hom}_{\text{AcF}_{Y}} (Z', W') \rightarrow \text{Hom}_{\text{AcF}_{X \times Y}} (Z'', W''). \]

**Proof.** Obviously, we may assume that \( X \) is affine \( X = \text{Spf} \ A \), where \( A \) is an adic ring of finite ideal type, with a finitely generated ideal of definition \( I \subseteq A \); this implies that \( Y \) and \( Y \times_X Y \) are quasi-compact and have ideals of definition \( I \otimes_Y \) and \( I \otimes_{Y \times X} \) of finite type, respectively.

By base change with respect to \( Z \rightarrow X \), we may assume \( X = Z = \text{Spf} \ A \). Let \( W = \bigcup_{\alpha \in L} W_{\alpha} \) be an open covering, and set \( W'_\alpha = Y \times_X W_{\alpha} \) and \( W''_\alpha = (Y \times_X Y) \times_X W_{\alpha} \) for each \( \alpha \in L \). Suppose that the assertion is true with \( W, W', \) and \( W'' \) replaced by \( W_{\alpha}, W'_{\alpha}, \) and \( W''_{\alpha} \), respectively, for any \( \alpha \in L \). Let \( f' \in \text{Hom}_{Y} (Z', W') \) be a morphism with the same image in \( \text{Hom}_{Y \times X} (Z'', W'') \). Set \( Z'_{\alpha} = Z' \times_{W'} W'_{\alpha} \) and \( Z''_{\alpha} = Z'' \times_{W''} W''_{\alpha} \) for each \( \alpha \in L \). By [52], Exposé VIII, (4.4), there exists for each \( \alpha \) an open subset \( Z_{\alpha} \) of \( Z \) such that \( Y \times_X Z_{\alpha} = Z'_{\alpha} \); indeed, since the two maps from \( Z''_{\alpha} \) to \( Z'_{\alpha} \) are adic, one can reduce it modulo powers of \( \mathfrak{g} \) to find the inductive system of schemes supported on the open subset of \( Z \), which gives the desired open formal subscheme \( Z_{\alpha} \). Moreover, since \( Z' \rightarrow Z \) is adically faithfully flat, \( \{ Z_{\alpha} \}_{\alpha \in L} \) gives an open covering of \( Z \). By our assumption we can find the unique morphism \( f_{\alpha} : Z_{\alpha} \rightarrow W_{\alpha} \) such that \( f_{\alpha} \times_X Y = f'|_{Z_{\alpha}} \) for each \( \alpha \in L \). By the uniqueness we have the unique map \( f : Z \rightarrow W \) such that \( f \times_X Y = f' \), as desired.

Thus we may assume that both \( Z \) and \( W \) are affine; let \( \mathcal{K} \) be an ideal of definition of finite type of \( W \). We may assume that \( \mathcal{O}_Z \subseteq \mathfrak{g} \) and \( \mathcal{O}_W \subseteq \mathcal{K} \). For \( k \geq 0 \), let \( \text{Hom}^{(k)}_{X} (Z, W) \) (resp. \( \text{Hom}^{(k)}_{Y} (Z', W') \), resp. \( \text{Hom}^{(k)}_{Y \times X} (Z'', W'') \)) be the subset of \( \text{Hom}_{X} (Z, W) \) (resp. \( \text{Hom}_{Y} (Z', W') \), resp. \( \text{Hom}_{Y \times X} (Z'', W'') \)) consisting of morphisms such that \( \mathcal{K}^{k+1} \mathcal{O}_Z \subseteq \mathfrak{g} \) (resp. \( \mathcal{K}^{k+1} \mathcal{O}_{Z'} \subseteq \mathfrak{g} \mathcal{O}_{W'} \), resp.
Since $\text{Hom}_X(Z, W)$ is the union of $\text{Hom}_X^{(k)}(Z, W)$ for $k \geq 0$, it suffices to show that the sequence

$$\text{Hom}_X^{(k)}(Z, W) \rightarrow \text{Hom}_Y^{(k)}(Z', W') \rightarrow \text{Hom}_{Y \times Y}^{(k)}(Z'', W'')$$

is exact for each $k \geq 0$; replacing $K$ by its powers, we may restrict to the case $k = 0$. Set $Z_k = (Z, \mathcal{O}_Z/I^{k+1})$ and $W_k = (W, \mathcal{O}_W/K^{k+1})$ for $k \geq 0$. By 1.4.1 the canonical map

$$\text{Hom}_X^{(0)}(Z, W) \rightarrow \lim_{k \geq 0} \text{Hom}_{X_k}(Z_k, W_k)$$

is bijective, and similarly for the other hom sets. Hence the desired exactness follows from the scheme case [52], Exposé VIII, §5, and the left-exactness of $\lim_{k \geq 0}$ (0.3.2.4).

**Corollary 6.1.6.** In the situation as in 6.1.1, the functor $\text{AcFs}_X \rightarrow \text{AcFs}_Y$ given by $Z \mapsto Z \times_X Y$ is faithful.

6.1. (c) **Descent of properties of morphisms.** We continue with working in the situation as in 6.1.1. Let $Z, W$ be adic formal schemes of finite ideal type over $X$, and $f : Z \rightarrow W$ a morphism over $X$. We consider the $Y$-morphism

$$f \times_Y Y : Z \times_X Y \rightarrow W \times_X Y$$

obtained by base change. By 4.8.18, we have the following ‘descent of adicness’ under adically faithfully flat quasi-compact morphisms.

**Proposition 6.1.7.** The morphism $f$ is adic if and only if $f \times_Y Y$ is adic.

Let $P$ be a property of morphisms of schemes that satisfies both (I) and (C) in §1.5. (b) and is stable under the Zariski topology (0.1.4.8 (1)). Then as in §1.5. (d), one can consider the property ‘adically $P$’ for morphisms of adic formal schemes of finite ideal type.

**Proposition 6.1.8.** Suppose that the property $P$ of morphisms of schemes descends under faithfully flat quasi-compact morphisms of schemes. Then the property ‘adically $P$’ descends under adically faithfully flat quasi-compact morphisms, that is, the morphism $f$ is adically $P$ if and only if $f \times_Y Y$ is adically $P$.

**Proof.** By 6.1.7, we can assume that $f$ is adic. We can work locally on the target $W$ and thus assume that $X$ has an ideal of definition $I$ of finite type. Let $f_k : Z_k = (Z, \mathcal{O}_Z/I^{k+1}) \rightarrow W_k = (W, \mathcal{O}_W/I^{k+1})$ be the induced morphism of schemes for $k \geq 0$; we use similar notation for other formal schemes and morphisms. Since $(f \times_Y Y)_k = f_k \times_Y Y_k$ and since $Y_k \rightarrow X_k$ is faithfully flat and quasi-compact, $f_k$ is $P$ if and only if $(f \times_Y Y)_k$ is $P$. 

\[\square\]
Corollary 6.1.9. The morphism \( f \) is a closed immersion (resp. an open immersion, resp. a quasi-compact immersion) if and only if so is \( f \times_X Y \).

Proof. By 4.3.6, 1.6.9, and 4.4.2, these properties are of the form ‘adically \( P \)’. Hence the assertion follows from 6.1.8 and [52], Exposé VIII, 5.5. \( \Box \)

Corollary 6.1.10. The morphism \( f \) is affine adic (resp. finite) if and only if so is \( f \times_X Y \).

Proof. Due to 4.1.12, 4.2.3 and [52], Exposé VIII, 5.6 and 5.7, we may apply 6.1.8. \( \Box \)

6.1. (d) Effective descent. Consider the situation as in 6.1.1 and let

\[
p_{23}, p_{31}, p_{12} : Y \times_X Y \times_X Y \xrightarrow{\sim} Y \times_X Y
\]

be the projections, where \( p_{ij} \) projects the \( i \)-th and \( j \)-th components of \( Y \times_X Y \times_X Y \) onto \( Y \times_X Y \). For an a.q.c. sheaf \( \mathcal{F} \) on \( Y \), a descent datum on \( \mathcal{F} \) is an isomorphism

\[
\varphi : \widetilde{p_1^* \mathcal{F}} \xrightarrow{\sim} \widetilde{p_2^* \mathcal{F}}
\]

that satisfies the 1-cocycle condition

\[
p_{31}^* \varphi = p_{23}^* \varphi \circ p_{12}^* \varphi.
\]

Pairs \((\mathcal{F}, \varphi)\) consisting of a.q.c. sheaves on \( Y \) and descent data form a category in an obvious manner (cf. [21], 6.1). Note that for any a.q.c. sheaf \( \mathcal{G} \) on \( X \), \( \hat{q}^* \mathcal{G} \) is canonically equipped with the standard descent datum.

Proposition 6.1.11 (effective descent for a.q.c. sheaves). The functor

\[
\mathcal{G} \mapsto \hat{q}^* \mathcal{G}
\]

induces a categorical equivalence between the category of a.q.c. sheaves (resp. a.q.c. sheaves of finite type, resp. a.q.c. \( \mathcal{O}_X \)-algebras, resp. a.q.c. \( \mathcal{O}_X \)-algebras of finite type) on \( X \) and the category of pairs \((\mathcal{F}, \varphi)\) consisting of a.q.c. sheaves (resp. a.q.c. sheaves of finite type, resp. a.q.c. \( \mathcal{O}_Y \)-algebras, resp. a.q.c. \( \mathcal{O}_Y \)-algebras of finite type) on \( Y \) and descent data.

Proof. By 6.1.3, we only need to show that the functor is essentially surjective. By a reduction process similar to that in the proof of 6.1.3, we may assume that \( X \) has an ideal of definition \( I \) of finite type. Given \((\mathcal{F}, \varphi)\), we set \( \mathcal{F}_k = \mathcal{F} / I^{k+1} \mathcal{F} \) for \( k \geq 0 \). Since for each \( k \) the descent datum \( \varphi \) induces a descent datum \( \varphi_k \) on the quasi-coherent sheaf \( \mathcal{F}_k \) on the scheme \( Y_k \), we have a quasi-coherent sheaf \( \mathcal{G}_k \)

\[
\hat{q}^* \mathcal{G}_k = \mathcal{G}_k \quad \text{on} \quad Y_k.
\]
on $X_k$, unique up to isomorphism, such that $\mathcal{F}_k \cong q_k^* \mathcal{G}_k$ ([52], Exposé VIII, Théorème 1.1). By the uniqueness, the sheaves $\mathcal{G}_k$ form a projective system $\{\mathcal{G}_k\}_{k \geq 0}$ of abelian sheaves on $X$, and for any $i \leq j$ the transition map $\mathcal{G}_j \to \mathcal{G}_i$ is surjective with the kernel equal to $I^{i+1} \mathcal{G}_j$. Hence, by 3.4.1 (or 3.4.2), its limit $\mathcal{G} = \lim_{\leftarrow k \geq 0} \mathcal{G}_k$ is an a.q.c. sheaf on $X$ (resp. a.q.c. sheaf on $X$ of finite type, resp. an a.q.c. $\mathcal{O}_X$-algebra, resp. an a.q.c. $\mathcal{O}_X$-algebra of finite type) such that $\mathcal{G} / I^{k+1} \mathcal{G} \cong \mathcal{G}_k$. The last property implies $\hat{q}^* \mathcal{G} \cong \mathcal{F}$ by 3.6.5, as desired. 

Descent data are defined also for adic formal schemes. In the above situation, for an adic formal scheme $W$ adic over $Y$ a descent datum on $W$ is an isomorphism

$$\varphi: p_1^* W \sim p_2^* W,$$

where $p_i^* W = W \times_{Y, p_i} (Y \times_X Y)$, satisfying the similar 1-cocycle condition.

Let $P$ be a property of morphisms of schemes that satisfies (I) and (C) in §1.5. (b) and is stable under the Zariski topology (0.1.4.8 (1)).

**Proposition 6.1.12.** Suppose that the subcategory of $\textbf{Sch}$ consisting of all morphisms satisfying $P$ is an effective descent class (0.1.4.8) with respect to the fpqc topology. The subcategory of $\textbf{AcFs}$ consisting of all adically $P$ morphisms (§1.5. (d)) is an effective descent class with respect to adically flat descent. In other words, in the situation as in 6.1.1, the functor $Z \mapsto q^* Z = Z \times_X Y$ gives rise to a categorical equivalence between

- the category of adic formal schemes $Z$ of finite ideal type over $X$ such that $Z \to X$ is adically $P$, and
- the category of adic formal schemes $W$ of finite ideal type over $Y$ such that $W \to Y$ is adically $P$ together with descent data.

**Proof.** By 6.1.6, it suffices to show that the functor is essentially surjective. We may assume that $X$ has an ideal of definition of finite type $I$; we work with the notation (such as $X_k$, $Y_k$, etc) as above. An adic $Y$-formal scheme $W$ as above with a descent datum gives rise to a $Y_k$-scheme $W_k$ with a descent datum for each $k \geq 0$. Hence, for each $k \geq 0$ we have an $X_k$-scheme $Z_k$, unique up to canonical isomorphisms, such that $Z_k \to X_k$ satisfies $P$. By the uniqueness of $Z_k$’s we have an inductive system $\{Z_k\}_{k \geq 0}$ of schemes over $X$ that satisfies the conditions in Exercise I.1.8, whence the assertion.

We know that the following properties for morphisms of schemes satisfy effective descent with respect to the fpqc topology:

- quasi-compact immersion (resp. open immersion, resp. closed immersion),
- affine,
- quasi-affine.
Corollary 6.1.13. The following properties for morphisms of adic formal schemes of finite ideal type satisfy effective descent with respect to the adically flat descent:
- quasi-compact immersion (resp. open immersion, resp. closed immersion),
- affine adic,
- adically quasi-affine (1.5.5 (2)).

6.1 (e) Adically flat descent and finiteness conditions

Proposition 6.1.14. Consider the situation as in 6.1.1, suppose that $X$ and $Y$ are locally universally rigid-Noetherian (2.1.7).

(1) Let $\mathcal{F}$ be an a.q.c. sheaf of finite type on $X$. Then $\mathcal{F}$ is of finite presentation (resp. locally free of finite type) if and only if $q^* \mathcal{F} = q^* \mathcal{F}$ is of finite presentation (resp. locally free of finite type) over $Y$.

(2) Let $\mathcal{B}$ be an a.q.c. $\mathcal{O}_X$-algebra. Then $\mathcal{B}$ is of finite presentation (3.5.11) if and only if $q^* \mathcal{B}$ is an a.q.c. $\mathcal{O}_Y$-algebra of finite presentation.

Proof. Since the question is local, we may assume that $X$ and $Y$ are affine, $X = \text{Spf} \ A$ and $Y = \text{Spf} \ B$. By 4.8.15, the morphism $q$ is faithfully flat. Hence by 4.8.10 we deduce that $B$ is faithfully flat over $A$. Now the assertions follow from 3.5.6, 3.5.13, and 3.6.4 (cf. 0.7.4.19, 0.8.4.5).

Proposition 6.1.15. Consider the situation as in 6.1.1, let $f : W \to Z$ be a morphism of adic formal schemes of finite ideal type that are adic over $X$.

(1) The morphism $f$ is surjective if and only if so is $q^* f$. If $q^* f$ is injective (resp. bijective), then so is $f$.

(2) The morphism $f$ is quasi-compact (resp. of finite type) if and only if so is $q^* f$. If $X$ and $Y$ are locally universally rigid-Noetherian, then $f$ is of finite presentation if and only if so is $q^* f$.

Proof. We may assume that $X$ has an ideal of definition. Then (1) follows from [52], Exposée VIII, 3.1 and 3.2.

(2) follows from [52], Exposée VIII, 3.3 and 3.6, combined with 1.6.9, 1.7.3, and 2.2.4.
6.2 Étale topology on adic formal schemes

6.2. (a) Étale sites. We consider the category \( \mathbf{AcfS}_S \) of adic formal schemes of finite ideal type over a fixed adic formal scheme \( S \) of finite ideal type. Let \( \mathcal{E} \) be the subcategory of \( \mathbf{AcfS}_S \) consisting of all étale morphisms; \( \mathcal{E} \) is base-change stable due to 5.3.13.

**Proposition 6.2.1.** A collection \( \{ U_\alpha \to U \}_{\alpha \in L} \) of étale morphisms in \( \mathbf{AcfS}_S \) is universally effectively epimorphic (0.1.4.3) if and only if it is a surjective family, that is, the union of the images of \( U_\alpha \to U \) coincides with \( U \).

**Proof.** The ‘only if’ part is easy to see and is left to the reader. We prove the ‘if’ part. Let \( W \to U \) be a morphism in \( \mathbf{AcfS}_S \), \( V \) an object in \( \mathbf{AcfS}_S \), and consider the sequence

\[
\text{Hom}_S(W, V) \to \prod_{\alpha \in L} \text{Hom}_S(W \times_U U_\alpha, V) \to \prod_{\alpha, \beta \in L} \text{Hom}_S(W \times_U U_{\alpha \beta}, V),
\]

where \( U_{\alpha \beta} = U_\alpha \times_U U_\beta \). We need to show that this sequence is exact. By the usual Zariski descent, we may assume that \( S \) and \( U \) have ideals of definition of finite type. By an argument similar to that in the proof of 6.1.5, we may further assume \( W \) and \( V \) are affine and have ideals of definition of finite ideal type. Define the filtration on the Hom sets by the subsets of the form ‘Hom\(^{(k)}\)’, as in the proof of 6.1.5, and reduce to the case \( k = 0 \). Since \( \{ U_{\alpha,k} \to U_k \}_{\alpha \in L} \) for each \( k \) is an étale covering family of schemes, we have the exact sequence

\[
\text{Hom}_{S_k}(W_k, V_k) \to \prod_{\alpha \in L} \text{Hom}_{S_k}(W_k \times_{U_k} U_{\alpha,k}, V_k) \to \prod_{\alpha, \beta \in L} \text{Hom}_{S_k}(W_k \times_{U_k} U_{\alpha \beta,k}, V_k).
\]

We have, for example, \( \text{Hom}_{S}^{(0)}(W, V) \cong \lim_k \text{Hom}_{S_k}(W_k, V_k) \). We apply \( \lim_k \) to the above exact sequences; since \( \lim_k \) is left-exact and commutes with products, concludes that the first sequence above is exact. \( \square \)

**Proposition 6.2.2.** The subcategory \( \mathcal{E} \) of \( \mathbf{AcfS}_S \) satisfies (S\(_1\)), (S\(_2\)), (S\(_3\)(a)), and (S\(_3\)(b)) in 0, §1.4.(b).

**Proof.** It is easy to see that \( \mathcal{E} \) satisfies (S\(_1\)) and (S\(_2\)). Condition (S\(_3\)(a)) follows from the second assertion of 5.3.13 (2). To show the rest, let \( f : X \to Y \) and \( g : Y \to Z \) be adic \( S \)-morphisms of adic formal schemes, and suppose that \( g \circ f \) is étale. To show that \( f \) is étale, we may assume that \( Z \) has an ideal of definition of finite type. Then by 5.3.11 we can easily reduce to the case of schemes, where the assertion is well known ([54], IV, (17.3.5)). \( \square \)
Definition 6.2.3 (large étale site). The topology on the category $\text{AcFs}_S$ associated to the base-change stable subcategory $\mathcal{E}$ consisting of étale morphisms is called the étale topology on $\text{AcFs}_S$, and the resulting site, denoted by $\text{AcFs}_{S,\text{ét}}$, is called the large étale site over $S$.

Similarly, one can define the large étale site $\text{AcCFs}_{S,\text{ét}}$ (resp. $\text{AfAcFs}_{S,\text{ét}}$) with the underlying category $\text{AcCFs}_S$ (resp. $\text{AfAcFs}_S$), the category of coherent (resp. affine) adic formal schemes of finite ideal type over $S$. One can also define similarly the site $\text{AcFs}_{S,\text{ét}}^*$ with the underlying category $\text{AcFs}_S^*$.

As we saw in 0.1.4.5, the étale topology enjoys property $(A_0)$.}

**Proposition 6.2.4.** Any representable presheaf on $\text{AcFs}_{S,\text{ét}}$ is a sheaf.

**Proposition 6.2.5.** In the category $\text{AcFs}_S$ of adic formal schemes of finite ideal type over $S$, the following properties of morphisms yield effective descent classes (cf. 0.1.4.8) under the étale topology:

(a) affine adic,
(b) finite,
(c) open immersion,
(d) closed immersion,
(e) immersion.

**Proof.** The stability follows from 6.1.9 and 6.1.10. As for the ‘effective descent class’ property, we can reduce to the case of schemes (cf. 6.1.13), where the assertion is classically known (for the fppf topology); here we use 1.4.1. □

**Proposition 6.2.6.** In the category $\text{AcFs}_S^*$ of adic formal schemes over $S$ with adic morphisms, the following properties of morphisms are local on the domain (cf. 0.1.4.8) under the étale topology:

(a) locally of finite type,
(b) adically flat,
(c) smooth,
(d) étale.
Proof. Similarly to the proof of 6.2.5, one can reduce the situation to the case of schemes, where the assertions are known to be true. □

**Corollary 6.2.7.** The properties for morphisms as in 6.2.6 and, moreover,

(e) locally of finite presentation

are local on the domain under the étale topology in the category $\text{RigNoeFs}_S$ of locally universally rigid-Noetherian formal schemes.

**Definition 6.2.8** (small étale site). For an adic formal scheme $X$ of finite ideal type we denote by $X_{\text{ét}}$ the category of étale formal schemes over $X$. We consider the class $\mathcal{E}_X$ of étale morphisms in this category (which obviously satisfies statements analogous to 6.2.1 and 6.2.2), and equip $X_{\text{ét}}$ with the associated topology. We call the resulting site, denoted by $X_{\text{ét}}$, the small étale site of $X$.

We denote by $X_{\sim}$ the associated étale topos. As usual, any map $f : X \to Y$ of adic formal schemes of finite ideal type induces the inverse image functor

$$f^{-1} : Y_{\text{ét}} \longrightarrow X_{\text{ét}}$$

in view of 5.3.13 (4), which in turn induces a morphism

$$f_{\sim} = (f_*, f^{-1}) : X_{\sim} \longrightarrow Y_{\sim}$$

of the étale topoi.

For an adic formal scheme $X$ of finite ideal type we denote the associated Zariski topos by $X_{\text{Zar}}$. We have the morphism of topoi

$$\epsilon = (\epsilon_*, \epsilon^{-1}) : X_{\sim} \longrightarrow X_{\text{Zar}},$$

defined in an obvious manner.

**Proposition 6.2.9.** Let $X$ be an adic formal scheme of finite ideal type, and $\mathcal{F}$ an a.q.c. sheaf on $X$ (resp. an a.q.c. $\mathcal{O}_X$-algebra). Define a presheaf $\mathcal{F}_{\text{ét}}$ on $X_{\text{ét}}$ as follows: for any object $q : Y \to X$ of $X_{\text{ét}}$ we set

$$\mathcal{F}_{\text{ét}}(Y) = \Gamma(Y, q^* \mathcal{F}).$$

Then the presheaf $\mathcal{F}_{\text{ét}}$ is a sheaf.

This follows immediately from 6.1.2. Applying the proposition to the structure sheaf $\mathcal{O}_X$, we get the structure sheaf $\mathcal{O}_{\sim}^X$ of the site $X_{\text{ét}}$. Any adic morphism $f : X \to Y$ between adic formal schemes induces a morphism

$$f_{\sim} = (f_*, f^*) : (X_{\sim}, \mathcal{O}_{\sim}^X) \longrightarrow (Y_{\sim}, \mathcal{O}_{\sim}^Y)$$

of ringed topoi. Moreover, for any adic formal scheme $X$ of finite ideal type we have a morphism of ringed topoi

$$\epsilon = (\epsilon_*, \epsilon^*) : (X_{\sim}, \mathcal{O}_{\sim}^X) \longrightarrow (X_{\text{Zar}}, \mathcal{O}_X).$$
6.2. (b) **A.q.c. sheaves on the étale site.** Let $X$ be an adic formal scheme of finite ideal type, and $\mathcal{I}$ an ideal of definition of finite type. As usual, we set $X_k = (X, \mathcal{O}_X/I_k^{k+1})$ for $k \geq 0$. Let $i_k : X_k \hookrightarrow X$ be the closed immersion. Then we have the induced morphism $i_k = (i_k^*, i_k^*): X_{k,\text{ét}} \to X_{\text{ét}}^\sim$ of ringed topoi. For any $\mathcal{O}_{X,\text{ét}}$-module $\mathcal{F}$ and $k \geq 0$, we define

$$\mathcal{F}_k = i_k^* \mathcal{F}.$$ 

Since we have the canonical morphism $X_{k,\text{ét}} \to X_{l,\text{ét}}$ of ringed topoi for $k \leq l$, we obtain a projective system $\{i_k^* \mathcal{F}_k\}_{k \geq 0}$ of $\mathcal{O}_{X,\text{ét}}$-modules. We set

$$\hat{\mathcal{F}} = \lim_{\leftarrow k} i_k^* \mathcal{F}_k,$$

which is again an $\mathcal{O}_{X,\text{ét}}$-module. As in §3.1.(a), one sees that the definition of $\hat{\mathcal{F}}$ does not depend on the choice of the ideal of definition $\mathcal{I}$, and thus one can define $\hat{\mathcal{F}}$ for any $\mathcal{O}_{X,\text{ét}}$-module even in the case $X$ does not have an ideal of definition. The $\mathcal{O}_{X,\text{ét}}$-module $\hat{\mathcal{F}}$ thus obtained is called the **completion** of $\mathcal{F}$. The completion comes with the canonical morphism $\mathcal{F} \to \hat{\mathcal{F}}$; if this morphism is an isomorphism, we say that $\mathcal{F}$ is **complete**.

**Definition 6.2.10.** (1) Let $X$ be an adic formal scheme of finite ideal type, and $\mathcal{F}$ an $\mathcal{O}_{X,\text{ét}}$-module. We say that $\mathcal{F}$ is an **a.q.c. sheaf (on $X_{\text{ét}}$)** if the following conditions are satisfied.

(a) $\mathcal{F}$ is complete.

(b) For any object $q : U \to X$ of $X_{\text{ét}}$ and any ideal of definition $\mathcal{I}$ of finite type of $U$, the sheaf $q^* \mathcal{F} / q^* \mathcal{I} q^* \mathcal{F}$ is a quasi-coherent sheaf with respect to étale topology on the scheme $(U, \mathcal{O}_U/\mathcal{I})$.

(2) An a.q.c. sheaf $\mathcal{F}$ on $X_{\text{ét}}$ is said to be **of finite type** if it is of finite type as an $\mathcal{O}_{X,\text{ét}}$-module.

(3) A morphism between a.q.c. sheaves is a morphism of $\mathcal{O}_{X,\text{ét}}$-modules.

By 3.1.2 and effective étale descent of quasi-coherent sheaves on schemes, (b) is equivalent to

(b)’ there exist a covering $\{q_\alpha : U_\alpha \to X\}_{\alpha \in L}$ of $X_{\text{ét}}$, and for each $\alpha \in L$ an ideal of definition $\mathcal{I}_\alpha$ of finite type of $U_\alpha$, such that for any $\alpha \in L$ and $k \geq 0$ the sheaf $q_\alpha^* \mathcal{F} / q_\alpha^* \mathcal{I}_\alpha^{k+1} q_\alpha^* \mathcal{F}$ is a quasi-coherent sheaf with respect to the étale topology on the scheme $(U_\alpha, \mathcal{O}_{U_\alpha}/\mathcal{I}_\alpha^{k+1})$.

If, in particular, $X$ itself has an ideal of definition $\mathcal{I}$ of finite type, then the last condition is equivalent to (notation being as usual) $\mathcal{F}_k$ being quasi-coherent on $X_{k,\text{ét}}$ for any $k \geq 0$. 
Lemma 6.2.11. Let $X$ be an adic formal scheme of finite ideal type. Then the étale sheaf $\mathcal{F}_{\text{ét}}$ defined as in 6.2.9 is an a.q.c. sheaf on $X_{\text{ét}}$.

Proof. It follows easily from the definition of $\mathcal{F}_{\text{ét}}$ and the definition of projective limits of sheaves in 0, §3.2. (c) that the sheaf $\mathcal{F}_{\text{ét}}$ is complete. By 3.6.5, one sees that the other condition is satisfied. 

Thus we have the functor

$$\mathcal{F} \mapsto \varepsilon^* \mathcal{F} = \mathcal{F}_{\text{ét}}$$

from the category of a.q.c. sheaves over $X$ (resp. a.q.c. $\mathcal{O}_X$-algebras) to the category of a.q.c. sheaves on $X_{\text{ét}}$ (resp. a.q.c. $\mathcal{O}_{X_{\text{ét}}}$-algebras). It is clear that the direct image functor $\varepsilon_*$ maps a.q.c. sheaves on $X_{\text{ét}}$ (resp. a.q.c. $\mathcal{O}_{X_{\text{ét}}}$-algebras) to a.q.c. sheaves over $X$ (resp. a.q.c. $\mathcal{O}_X$-algebras).

Proposition 6.2.12. The functors $\varepsilon_*$ and $\varepsilon^*$ give a categorical equivalence between the category of a.q.c. sheaves of finite type over $X$ (resp. a.q.c. $\mathcal{O}_X$-algebras) and the category of a.q.c. sheaves of finite type on $X_{\text{ét}}$ (resp. a.q.c. $\mathcal{O}_{X_{\text{ét}}}$-algebras).

Proof. We already know by 6.1.3 that the functor $\varepsilon^*$ is fully faithful. Moreover, it is easy to see that, for an a.q.c. sheaf $\mathcal{F}$ of finite type on $X_{\text{ét}}$, $\varepsilon_* \mathcal{F}$ is an a.q.c. sheaf of finite type on $X$. Hence, it suffices to show that for any a.q.c. sheaf $\mathcal{F}$ on $X_{\text{ét}}$ (resp. a.q.c. $\mathcal{O}_{X_{\text{ét}}}$-algebras) we have a canonical isomorphism $\mathcal{F} \cong \varepsilon^* \varepsilon_* \mathcal{F}$. To this end, since the question is local on $X$, one may assume that $X$ has an ideal of definition $I$ of finite type. Moreover, since $\mathcal{F}$ and $\varepsilon^* \varepsilon_* \mathcal{F}$ are complete, we only need to show that $\mathcal{F}_k$ and $(\varepsilon^* \varepsilon_* \mathcal{F})_k$ are canonically isomorphic for any $k \geq 0$, where $\mathcal{F}_k$ etc. are defined as above. We use the notation as above and denote by $\varepsilon_k = (\varepsilon_k^*, \varepsilon_k^*): X_{k, \text{ét}} \rightarrow X_{k, \text{Zar}}$ the morphism of ringed topoi for any $k \geq 0$. Clearly, we have the commutative diagram of ringed topoi

$$
\begin{array}{ccc}
X_{k, \text{ét}} & \xrightarrow{i_k} & X_{\text{ét}} \\
\downarrow{\varepsilon_k} & & \downarrow{\varepsilon} \\
X_{k, \text{Zar}} & \xrightarrow{i_k} & X_{\text{Zar}}
\end{array}
$$

Hence, $(\varepsilon^* \varepsilon_* \mathcal{F})_k = i_k^* \varepsilon^* \varepsilon_* \mathcal{F} = \varepsilon_k^* i_k^* \varepsilon_* \mathcal{F}$. Now by the definition of $\varepsilon_*$ we see that $i_k^* \varepsilon_* \mathcal{F} = \varepsilon_k^* i_k^* \mathcal{F}$ and hence that $(\varepsilon^* \varepsilon_* \mathcal{F})_k = \varepsilon_k^* \varepsilon_k^* i_k^* \mathcal{F}$. By the theory of étale descent of quasi-coherent sheaves on schemes (cf. [9], Exposé VII, 4), we have $\mathcal{F}_k = i_k^* \mathcal{F} = \varepsilon_k^* \varepsilon_k^* i_k^* \mathcal{F}$, as desired. 

\[\square\]
By 6.1.11 and 6.1.14 we immediately have the following result.

**Proposition 6.2.13.** By the functors $\epsilon_*$ and $\check{\epsilon}^*$ a.q.c. sheaves of finite type on $X_{\text{ét}}$ (resp. a.q.c. $\mathcal{O}^\wedge_{X_{\text{ét}}}$-algebras of finite type) correspond to a.q.c. sheaves of finite type on $X$ (resp. a.q.c. $\mathcal{O}_X$-algebras of finite type). If $X$ is locally universally rigid-Noetherian, then a.q.c. sheaves on $X_{\text{ét}}$ of finite presentation (resp. a.q.c. $\mathcal{O}^\wedge_{X_{\text{ét}}}$-algebras of finite presentation) correspond to a.q.c. sheaves on $X$ of finite presentation (resp. a.q.c. $\mathcal{O}_X$-algebras of finite presentation).

### 6.3 Formal algebraic spaces

#### 6.3. (a) Formal algebraic spaces

We consider the large étale site $\text{AcFs}_{S,\text{ét}}$ consisting of adic formal schemes of finite ideal type over a fixed adic formal scheme $S$ of finite ideal type (cf. 6.2.3). As we have seen in 6.2.4, any representable presheaf on the site $\text{AcFs}_{S,\text{ét}}$ is a sheaf. As in 0.1.4. (d), we say that a map $\mathcal{F} \to \mathcal{G}$ of sheaves on $\text{AcFs}_{S,\text{ét}}$ is representable if for any object $Z$ of $\text{AcFs}_{S,\text{ét}}$ (regarded as a sheaf on $\text{AcFs}_{S,\text{ét}}$) and any map $Z \to \mathcal{G}$ of sheaves, the fiber product $Z \times_{\mathcal{G}} \mathcal{F}$ is represented by an object of $\text{AcFs}_{S,\text{ét}}$. For a base-change stable property $P$ for morphisms between adic formal schemes of finite ideal type, we say that a map $\mathcal{F} \to \mathcal{G}$ of sheaves is ‘representable and $P$’ if it is representable and for any $Z \to \mathcal{G}$ as above the induced morphism $Z \times_{\mathcal{G}} \mathcal{F} \to Z$ satisfies $P$. Especially, we say that $\mathcal{F} \to \mathcal{G}$ is an open immersion if it is representable and an open immersion.

Let $\mathcal{F}$ be a sheaf on $\text{AcFs}_{S,\text{ét}}$, and suppose that there exist an object $Y$ of $\text{AcFs}_{S,\text{ét}}$ and a representable étale and surjective morphism $q: Y \to \mathcal{F}$. We say that a collection of morphisms of sheaves $\{\mathcal{F}_\alpha \to \mathcal{F}\}_{\alpha \in L}$ is a Zariski covering of $\mathcal{F}$ if each $\mathcal{F}_\alpha \to \mathcal{F}$ is an open immersion and $\coprod_{\alpha \in L} \mathcal{F}_\alpha \to \mathcal{F}$ is an epimorphism. Since open immersions satisfy effective descent (6.2.5), the last condition is equivalent to that for each $\alpha \in L$ the morphism $\mathcal{F}_\alpha \times_{\mathcal{F}} Y \to Y$ is an open immersion and $\coprod_{\alpha \in L} \mathcal{F}_\alpha \times_{\mathcal{F}} Y \to Y$ is surjective. Note that, in this situation, each $\mathcal{F}_\alpha \times_{\mathcal{F}} Y$ is representable, and the map $\mathcal{F}_\alpha \times_{\mathcal{F}} Y \to \mathcal{F}_\alpha$ is representable and étale surjective.

**Definition 6.3.1.** A formal algebraic space\(^1\) over $S$ is a set-valued sheaf $X$ on the site $\text{AcFs}_{S,\text{ét}}$ such that

1. the diagonal morphism $\Delta_X: X \to X \times_S X$ is representable and quasi-compact, and
2. there exist an adic formal scheme $Y$ of finite ideal type over $S$ and a representable étale surjective morphism $q: Y \to X$.

---

\(^1\)The formal algebraic spaces defined here should be called, more precisely, quasi-separated adic formal algebraic spaces of finite ideal type.
The morphism \( q: Y \to X \) as in (b) will be simply called a representable étale covering of \( X \). It is obvious that, if \( X \) is a formal algebraic space, then any Zariski open subsheaf (that is, a subsheaf \( Y \subseteq X \) such that the inclusion map is an open immersion) is again a formal algebraic space, called an open subspace.

We denote by \( \text{AcFAS}_S \) the category of formal algebraic spaces over \( S \); here morphisms between such formal algebraic spaces are morphisms of sheaves on the site \( \text{AcFs}_S \). It is clear from the definition that any quasi-separated adic formal scheme of finite ideal type over \( S \) is canonically a formal algebraic space over \( S \).

Let \( X \) be a formal algebraic space over \( S \), and \( q: Y \to X \) a representable étale covering map. Then \( R = Y \times_X Y \) is representable, and the projections \( p_1, p_2: R \to Y \) are étale surjective:

\[
Y \times_X Y = R \xrightarrow{p_1} Y \xrightarrow{q} X.
\]

**Proposition 6.3.2** (cf. [72], II.1.3 (a)). (1) The morphisms \( p_1, p_2: R \to Y \) define an étale equivalence relation in \( \text{AcFs}_S,\text{ét} \).

(2) The morphism \( q: Y \to X \) is the cokernel of the morphisms \( p_1, p_2: R \to Y \) in the category of sheaves on \( \text{AcFs}_S,\text{ét} \).

Here an étale equivalence relation in \( \text{AcFs}_S,\text{ét} \) means an \( \tau \)-equivalence relation (0.1.4.11) with \( \tau \) equal to the étale topology. Note that, since

\[
R \xrightarrow{(p_1, p_2)} Y \times_S Y \xrightarrow{q \times q} X \times_X Y \xrightarrow{\Delta_X} X \times_S X
\]

is Cartesian, it follows that \( (p_1, p_2): R \to Y \times_S Y \) is quasi-compact.

**Proposition 6.3.3** (cf. [72], II.1.4). Let \( X \) and \( X' \) be formal algebraic spaces over \( S \) and \( f: X \to X' \) a morphism over \( S \). Then there exist representable étale covering morphisms \( q: Y \to X \) and \( q': Y' \to X' \) and a commutative diagram

\[
\begin{array}{ccc}
Y \times_X Y & \xrightarrow{p_1} & Y \\
\downarrow{h} & & \downarrow{f} \\
Y' \times_{X'} Y' & \xrightarrow{p_1'} & Y' \\
\end{array}
\]

(the commutativity of the left-hand square means \( g \circ p_i = p_i' \circ h \) for \( i = 1, 2 \)).

**Definition 6.3.4.** A formal algebraic space \( X \) is said to be quasi-compact (or coherent) if it has a representable étale covering \( Y \to X \) with \( Y \) quasi-compact.
6. Formal algebraic spaces

We denote by $\text{AcCFAs}_S$ the full subcategory of $\text{AcFAs}_S$ consisting of quasi-compact formal algebraic spaces over $S$.

6.3. (b) Formal algebraic spaces by quotients. Let

$$R \xrightarrow{p_1} Y \xleftarrow{p_2}$$

be an étale equivalence relation (cf. 0.1.4.11) for adic formal schemes of finite ideal type over $S$ such that

- $Y$ is separated and the induced map

$$R \xrightarrow{(p_1,p_2)} Y \times_S Y$$

is quasi-compact.

**Theorem 6.3.5.** In the situation as above, let $q: Y \to X$ be the categorical cokernel of (*) in the category of sheaves on $\text{AcFs}_S^{\text{ ét}}$. Then $X$ is a formal algebraic space over $S$, and $q: Y \to X$ is a representable étale covering.

In order to show the theorem, we first note that we can work locally on $Y$ (cf. [72], I.5.7), and hence assume that $S$ is coherent and that $Y$ is coherent over $S$.

**Proposition 6.3.6.** In the situation as above, suppose that $S$ is coherent and that $Y$ is coherent over $S$ (1.6.6). Then there exists an ideal of definition $I \subseteq \mathcal{O}_Y$ of finite type such that

$$p_1^{-1} I \mathcal{O}_R = p_2^{-1} I \mathcal{O}_R.$$

To show the proposition we need following lemma.

**Lemma 6.3.7.** Consider a Cartesian diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{p} & V \\
\downarrow f & & \downarrow g \\
X & \xleftarrow{q} & U
\end{array}
$$

of coherent adic formal schemes of finite ideal type and adic morphisms. Suppose that

(a) $p$ and $q$ are étale, and $f$ and $g$ are coherent, and

(b) $Z$ and $V$ are schemes (that is, 0-adic formal schemes).

Then $\mathcal{K} = \ker(\mathcal{O}_X \to f_* \mathcal{O}_Z)$ is an ideal of definition of $X$, and we have

$$q^{-1} \mathcal{K} \mathcal{O}_U = \ker(\mathcal{O}_U \to g_* \mathcal{O}_V).$$
Proof. Let $I$ be an ideal of definition of finite type on $X$ \((3.7.12)\). Since $f$ is adic and $Z$ is a quasi-compact scheme, there exists $n \geq 1$ such that $I^n \mathcal{O}_Z = 0$. Thus, replacing $I$ by $I^n$, we may assume that $I \subseteq \mathcal{K}$.

Let us show that $\mathcal{K}$ is an ideal of definition; to this end, we want to show that $\mathcal{K}$ is a.q.c. Since $\mathcal{K}$ contains $I$, it suffices to show that for any $k \geq 0$ the induced sheaf $\mathcal{K}_k = \mathcal{K} / I^{k+1}$ is a quasi-coherent sheaf on the scheme $X_k = (X, \mathcal{O}_X / I^{k+1})$. But this follows from the fact that the morphism $f_k : Z \to X_k$ induced by $f$ is coherent and that

$$
\mathcal{K}_k = \ker(\mathcal{O}_{X_k} \to f_k^* \mathcal{O}_Z).
$$

To show the other assertion, set $U_k = (U, \mathcal{O}_U / I^{k+1} \mathcal{O}_U)$ for $k \geq 0$, and let $q_k : U_k \to X_k$ and $g_k : V \to U_k$ be the morphisms induced by $q$ and $g$, respectively. Since both $q^{-1} \mathcal{K} \mathcal{O}_U$ and $\ker(\mathcal{O}_U \to g_* \mathcal{O}_V)$ contain $I \mathcal{O}_U$ (and hence are complete), it suffices to check the equality

$$
q_k^{-1} \mathcal{K}_k \mathcal{O}_{U_k} = \ker(\mathcal{O}_{U_k} \to g_k^* \mathcal{O}_V)
$$

for any $k \geq 0$. But this is clear, since $p$ and $q_k$ are étale morphisms of schemes and $V \cong Z \times_{X_k} U_k$. \qed

Proof of Proposition 6.3.6. We first claim what follows.

Claim. It suffices to find an ideal of definition $I \subseteq \mathcal{O}_Y$, not necessarily of finite type, such that $p_1^{-1} I \mathcal{O}_R = p_2^{-1} I \mathcal{O}_R$.

Set $Y_k = (Y, \mathcal{O}_Y / I^{k+1})$ and $R_k = (R, \mathcal{O}_R / I^{k+1} \mathcal{O}_R)$ for $k \geq 0$. We have for each $k \geq 0$ the induced morphisms

$$
R_k \xrightarrow{p_{1,k}} Y_k \xrightarrow{p_{2,k}} R_k
$$

of schemes. By 5.3.11 and Exercise 0.1.3, one easily sees that this diagram gives an étale equivalence relation of schemes. Let $X_k$ be the resulting coherent algebraic space with the quotient map $q_k : Y_k \to X_k$. Set $\mathcal{O}^\text{ét}_{X_k} = \lim_{\leftarrow k \geq 0} \mathcal{O}^\text{ét}_{X_k}$, and let $I_X^{(k)}$ be the kernel of $\mathcal{O}^\text{ét}_{X} \to \mathcal{O}^\text{ét}_{X_k}$ for each $k \geq 0$. By étale descent the sheaf $I_X^{(l)} / I_X^{(k)}$ for any $k \geq l \geq 0$ is a quasi-coherent sheaf on $X_k$ such that $q_k^* I_X^{(l)} / I_X^{(k)} = I_X^{(l)} / I_X^{(k)}$.

By 0.5.5.6, $I_X^{(0)} / I_X^{(1)}$ is a filtered inductive limit of quasi-coherent ideals of finite type on $X_1$:

$$
I_X^{(0)} / I_X^{(1)} = \lim_{\lambda \in \Lambda} \overline{I}_\lambda.
$$

For each $\lambda \in \Lambda$, let $I_\lambda$ be the pull-back of $\overline{I}_\lambda$ by the canonical map $\mathcal{O}^\text{ét}_{X} \to \mathcal{O}^\text{ét}_{X_1}$. Then $I_\lambda \mathcal{O}^\text{ét}_Y$ is an adically quasi-coherent ideal of $\mathcal{O}^\text{ét}_Y$ \((3.7.2)\). Take an ideal of
definition $I' \subseteq \mathcal{O}_Y$ of finite type (3.7.12) contained in $I$. Then there exists $\lambda \in \Lambda$ such that $I'\mathcal{O}^\text{et}_Y \subseteq I_\lambda \mathcal{O}^\text{et}_Y$. For a sufficiently large $k \gg 0$ we have the inclusions

$$I^{(k+1)}_X \mathcal{O}^\text{et}_Y \subseteq I' \mathcal{O}^\text{et}_Y \subseteq I_\lambda \mathcal{O}^\text{et}_Y \subseteq I \mathcal{O}^\text{et}_Y.$$  

Since $I_\lambda \mathcal{O}^\text{et}_Y / I^{(k+1)}_X \mathcal{O}^\text{et}_Y = (I_\lambda \mathcal{O}^\text{et}_Y)$ is of finite type, $I_\lambda \mathcal{O}^\text{et}_Y / I' \mathcal{O}^\text{et}_Y$ is of finite type; but since $I' \mathcal{O}^\text{et}_Y$ is of finite type, we deduce that $I_\lambda \mathcal{O}^\text{et}_Y$ is of finite type. Now the ideal sheaf $I'' = \varepsilon_* I_\lambda \mathcal{O}^\text{et}_Y$ with respect to the Zariski topology is an ideal of definition of finite type on $Y$ such that $p_1^{-1} I'' \mathcal{O}_R = p_2^{-1} I'' \mathcal{O}_R$, whence the claim.

To proceed, we consider the diagram (cf. Exercise 0.1.3)

\[
\begin{array}{ccc}
Y & \xleftarrow{p_1} & R \\
| \downarrow p_2 | & & \downarrow \| \\
Y & \xleftarrow{p_2} & R \\
\end{array}
\]

consisting of étale covering morphisms, where $T = R \times_{p_1, Y, p_2} R$; here we have $p_1 \circ p_{23} = p_2 \circ p_{12}$ and $p_2 \circ p_{23} = p_2 \circ p_{13}$.

Now we are going to construct an ideal of definition $I$ on $Y$ satisfying

$$p_1^{-1} I \mathcal{O}_R = p_2^{-1} I \mathcal{O}_R.$$  

Let $\mathcal{J} \subseteq \mathcal{O}_Y$ be an ideal of definition (3.7.12), and define

$$I = \ker(\mathcal{O}_Y \rightarrow p_2* \mathcal{O}_R/p_1^{-1} \mathcal{J} \mathcal{O}_R).$$

Set $R_0 = (R, \mathcal{O}_R/p_1^{-1} \mathcal{J} \mathcal{O}_R)$ and $T_0 = (T, \mathcal{O}_T/p_1^{-1} \mathcal{J} \mathcal{O}_T)$; note the equality $p_{12}^{-1} p_1^{-1} \mathcal{J} \mathcal{O}_T = p_{13}^{-1} p_1^{-1} \mathcal{J} \mathcal{O}_T$. We have the double Cartesian diagram

\[
\begin{array}{ccc}
R_0 & \xleftarrow{p_{12}} & T_0 \\
| \downarrow p_{13} | & & \downarrow \| \\
Y & \xleftarrow{p_2} & R \\
\end{array}
\]

where all horizontal arrows are étale. Since $I = \ker(\mathcal{O}_Y \rightarrow p_2* \mathcal{O}_{R_0})$, we can apply 6.3.7. It follows that $I$ is an ideal of definition on $Y$ and that

$$p_1^{-1} I \mathcal{O}_R = \ker(\mathcal{O}_R \rightarrow p_{23*} \mathcal{O}_{T_0}) = p_2^{-1} I \mathcal{O}_R,$$

which finishes the proof of the proposition. \qed
Proof of Theorem 6.3.5. We use the notation as in 6.3.6. Set \( Y_k = (Y, \mathcal{O}_Y/I^{k+1}) \) and \( R_k = (R, \mathcal{O}_R/I^{k+1}) \) for \( k \geq 0 \). Suppose \( S \) has an ideal of definition of finite type \( \mathcal{J} \subseteq \mathcal{O}_S \). As we may assume that \( \mathcal{J} \mathcal{O}_Y \subseteq \mathcal{I} \), we can consider that \( Y_k \) and \( R_k \) are \( S_k \)-schemes, where \( S_k = (S, \mathcal{O}_S/\mathcal{J}^{k+1}) \), for \( k \geq 0 \). We have the induced diagram

\[
R_k \xrightarrow{\begin{pmatrix} p_{1,k} \\ p_{2,k} \end{pmatrix}} Y_k \tag{*)}_k
\]

for each \( k \geq 0 \). It follows from 5.3.11 and Exercise 0.1.3 that \( (*)_k \) gives an étale equivalence relation by \( S_k \)-schemes such that the induced map

\[
R_k \xrightarrow{(p_{1,k}, p_{2,k})} Y_k \times_{S_k} Y_k
\]

is quasi-compact. Then by [72], II.1.3 (b), the categorical cokernel \( q_k: Y_k \to X_k \) of \( (*)_k \) is a representable étale covering of an algebraic space \( X_k \) over \( S_k \).

It follows from [72], I.5.12 that the map \( (p_{1,k}, p_{2,k}): R_k \to Y_k \times_{S_k} Y_k \) is quasi-affine for any \( k \geq 0 \), and hence that the map

\[
R \xrightarrow{(p_{1,2})} Y \times_S Y
\]

is adically quasi-affine (1.5.5 (2)). Then the assertion is the formal consequence of 0.1.4.13 in view of the fact that adically quasi-affine morphisms form an effective descent class (6.1.12).

Remark 6.3.8. (1) Note that in the situation as in 6.3.5 the diagonal morphism \( \Delta: X \to X \times_S X \) is representable and adically quasi-affine.

(2) Note also that in the situation as in the proof of 6.3.5 we have the natural isomorphism

\[
X \cong \lim_{\longrightarrow} X_k
\]

of sheaves on \( \text{AcFAs}_{S,\acute{e}} \).

6.3. (c) Fiber products

Proposition 6.3.9. Let \( X, Y, \) and \( Z \) be formal algebraic spaces over \( S \), and let \( X \to Z \leftarrow Y \) be morphisms over \( S \). Then the sheaf fiber product \( X \times_Z Y \) is a formal algebraic space over \( S \), and the diagram

\[
\begin{array}{ccc}
X \times_Z Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Z 
\end{array}
\]

is Cartesian in \( \text{AcFAs}_S \).
Proof. We first construct the fiber products locally and then globalize by patching; note that since $X$, $Y$, and $Z$ are quasi-separated, the diagonal map for the sheaf $X \times_Z Y$ is automatically representable and quasi-compact (indeed, $\Delta_{X \times_Z Y} \cong \Delta_X \times_{\Delta_Z} \Delta_Y$). Hence, in view of 6.3.5 the result follows by an argument similar to the proof of [72], II.1.5.

\[\square\]

6.3. (d) \textbf{Étale topology on formal algebraic spaces}

\textbf{Definition 6.3.10.} We say that an $S$-morphism $f : X \to Y$ of formal algebraic spaces is \textit{étale} if there exist a representable étale covering $V \to Y$ and a representable étale covering $U \to X \times_Y V$ such that the resulting map $U \to V$ of adic formal schemes of finite ideal type is étale.

\textbf{Proposition 6.3.11.} (1) The composition of two étale morphisms is étale.

(2) For any étale $Z$-morphisms $f : X \to Y$ and $g : X' \to Y'$ of formal algebraic spaces over a formal algebraic space $Z$, the induced morphism

$$f \times_Z g : X \times_Z Y \longrightarrow X' \times_Z Y'$$

is étale.

(3) For any étale $Z$-morphism $f : X \to Y$ of formal algebraic spaces over a formal algebraic space $Z$, and for any morphism $Z' \to Z$ of formal algebraic spaces, the induced morphism

$$f_{Z'} : X \times_Z Z' \longrightarrow Y \times_Z Z'$$

is étale.

\textbf{Proof.} (1) is easy to see.

To show (2) and (3), by 0.1.4.1 it suffices to show the special case of (3) with $Z = Y$, which is straightforward. \[\square\]

The following proposition is straightforward (use 6.2.2).

\textbf{Proposition 6.3.12.} The base-change stable subcategory $\mathcal{E}$ of $\textbf{AcFAs}_S$ consisting of all étale morphisms satisfies (S1), (S2), (S3(a)), and (S3(b)) in 0, §1.4. (b).

\textbf{Definition 6.3.13} (large étale site). The topology on the category $\textbf{AcFAs}_S$ associated to the subcategory $\mathcal{E}$ consisting of étale maps is called the \textit{étale topology} on $\textbf{AcFAs}_S$, and the resulting site, denoted by $\textbf{AcFAs}_{S, \text{ét}}$, is called the \textit{large étale site} over $S$. 

Chapter I. Formal geometry

Note that for any formal algebraic space $X$ a representable étale covering $Y \to X$ is a covering map with respect to the étale topology. In particular, in the site $\mathbf{AcFAs_{S,\text{ét}}}$, covering families consisting of morphisms from adic formal schemes (of finite ideal type) are cofinal in the set of all covering families. The following proposition follows from 0.1.4.5.

**Proposition 6.3.14.** Any representable presheaf on $\mathbf{AcFAs_{S,\text{ét}}}$ is a sheaf.

**Definition 6.3.15** (Small étale site). For a formal algebraic space $X$ we denote by $X_{\text{ét}}$ the category of étale formal algebraic spaces over $X$. We consider the class $\mathcal{E}$ of étale maps in this category (which obviously satisfies the conditions as in 6.3.12) and equip $X_{\text{ét}}$ with the associated topology. We call the resulting site, denoted by $X_{\text{ét}}$, the small étale site over $X$.

We denote by $X_{\text{ét}}$ the associated étale topos. Any $S$-morphism $f: X \to Y$ of formal algebraic spaces induces the inverse image functor $f^{-1}: Y_{\text{ét}} \to X_{\text{ét}}$, which gives rise to a morphism

$$f_{\text{ét}}^\sim = (f_*, f^{-1}): X_{\text{ét}} \to Y_{\text{ét}}$$

of the étale topoi.

For a formal algebraic space $X$ we define the structure sheaf of $X$ as follows: $\mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_U)$ for any étale map $U \to X$ from an adic formal scheme $U$; this defines a sheaf due to 6.1.2.

**6.3. (e) Ideals of definition and adic morphisms.** Let $X$ be a formal algebraic space, and $q: Y \to X$ a representable étale covering. For any $\mathcal{O}_X$-module $\mathcal{F}$ one can define the complete pull-back $q^* \mathcal{F}$ by an obvious manner (cf. §3.6).

**Definition 6.3.16.** Let $X$ be a formal algebraic space. An ideal of definition of finite type of $X$ is an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ of finite type such that for any representable étale covering $q: Y \to X$, $\hat{f}^* \mathcal{I} = \hat{I} \mathcal{O}_Y$ (cf. 3.7.10) and is an ideal of definition of finite type on $Y$.

By Exercise I.6.3, this is equivalent to say that for any representable étale covering $q: Y \to X$ we have $\hat{f}^* \mathcal{I} = \hat{I} \mathcal{O}_Y$, which is an adically quasi-coherent sheaf of finite type on $Y$ and, for at least one such $q: Y \to X$, $\hat{I} \mathcal{O}_Y$ is an ideal of definition of finite type. Now 6.3.6 yields the following result.

**Proposition 6.3.17.** There exists a Zariski covering $\{X_\alpha \to X\}_{\alpha \in L}$ such that each $X_\alpha$ has an ideal of definition of finite type.

**Definition 6.3.18.** A morphism $f: X \to Y$ of formal algebraic spaces is said to be adic if there exists a Zariski covering $\{Y_\alpha \to Y\}_{\alpha \in L}$ of $Y$, and for each $\alpha \in L$ an ideal of definition of finite type $\mathcal{I}_\alpha$ on $Y_\alpha$, such that $\mathcal{I}_\alpha \mathcal{O}_{X_\alpha}$ is an ideal of definition of $X_\alpha = X \times_Y Y_\alpha$. 
Proposition 6.3.19. Let \( f : X \to Y \) be a morphism of formal algebraic spaces. Then the following conditions are equivalent.

(a) \( f \) is adic.

(b) For any representable étale covering \( V \to Y \) and any representable étale covering \( U \to X \times_Y V \), the morphism \( U \to V \) is adic.

(c) There exist a representable étale covering \( V \to Y \) and a representable étale covering \( U \to X \times_Y V \) such that the morphism \( U \to V \) is adic.

This follows easily from 4.8.18.

Proposition 6.3.20. (1) Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms of formal algebraic spaces. If \( f \) and \( g \) are adic, then so is the composition \( g \circ f \). If \( g \circ f \) and \( g \) are adic, then so is \( f \).

(2) Let \( Z \) be a formal algebraic space, and \( f : X \to X' \) and \( g : Y \to Y' \) two adic \( Z \)-morphisms of formal algebraic spaces. Then

\[
f \times_Z g : X \times_Z Y \to X' \times_Z Y'
\]

is adic.

(3) Let \( Z \) be a formal algebraic space, and \( f : X \to Y \) an adic \( Z \)-morphism between formal algebraic spaces. Then for any morphism \( Z' \to Z \) of formal algebraic spaces the induced morphism

\[
f_{Z'} : X \times_Z Z' \to Y \times_Z Z'
\]

is adic.

Proof. All assertions except for the second one in (1) follow easily from 1.3.6. To show the second assertion of (1), take a representable étale covering \( U \to Z \), and let \( f' : U \times_Z X \to U \times_Z Y \) and \( g' : U \times_Z Y \to U \) be the induced morphisms of adic formal schemes of finite ideal type. Suppose \( g \circ f \) and \( g \) are adic. Then \( g' \circ f' \) and \( g' \) are adic. For \( f \) to be adic, it is enough by 4.8.18 that \( f' \) is adic, which follows from 1.3.6 (1).

We denote by \( \text{AcFAs}_{S}^{a} \) (resp. \( \text{AcFAs}^{a}_{S} \)) the category of formal algebraic spaces over \( S \) with adic morphisms (resp. the category of formal algebraic spaces adic over \( S \)).

Suppose that a formal algebraic space \( X \) has an ideal of definition \( I \), and take a representable étale covering \( q : Y \to X \); replacing \( Y \) by a disjoint union of affine subsets of \( Y \), we may assume that \( Y \) is separated. Then \( Y \) is an adic formal scheme on which the ideal \( \mathcal{I} \mathcal{O}_Y \) is an ideal of definition. Set \( R = Y \times_X Y \) and consider the resulting diagram

\[
\begin{array}{c}
X & \xleftarrow{q} & Y & \xrightarrow{p_1} & R \\
& \searrow & \downarrow & \nearrow & \\
& & Y & \xleftarrow{p_2} & R
\end{array}
\]
Then we note that we are in a similar situation as in \S 6.3. (b). In particular,

\[ X \cong \lim_{\substack{\longrightarrow \\ k \geq 0}} X_k, \]

where \( X_k \) (\( k \geq 0 \)) is the algebraic space defined as in \S 6.3. (b).

**Proposition 6.3.21.** Let \( X \) be a coherent formal algebraic space having an ideal of definition \( I \) of finite type, and \( X_k \) for \( k \geq 0 \) the algebraic spaces defined as above. Then the following conditions are equivalent.

(a) \( X \) is represented by an adic formal scheme of finite ideal type.

(b) \( X_k \) is represented by a scheme for any \( k \geq 0 \).

(c) \( X_0 \) is represented by a scheme.

**Proof.** (a) \( \Rightarrow \) (b) is trivial. (b) \( \Rightarrow \) (c) follows from 0.5.5.5. Suppose (b) holds. As we have seen above, \( X \) is the inductive limit of \( \{ X_k \}_{k \geq 0} \) taken in the category of formal algebraic spaces; but by 1.4.3, it is an adic formal scheme, whence (a). \( \square \)

**6.3. (f) Formal completion of algebraic spaces.** Let \( X \) be an algebraic space over a scheme \( S \), and \( Z \) a closed subspace of \( X \) of finite presentation with the defining ideal sheaf \( I \) (cf. [72], II.5). Take a representable étale covering \( q: Y \to X \), which yields the cokernel sequence

\[ X \leftarrow Y \leftarrow Y \times_X Y, \]

where \( p_1 \) and \( p_2 \) are the projections. Set \( R = Y \times_X Y \). Consider the formal completions \( \hat{Y}|_Z \) and \( \hat{R}|_Z \) along the closed subschemes \( q^{-1}(Z) \) and \( p_1^{-1}q^{-1}(Z) \), respectively. We get a diagram of adic formal schemes of finite ideal type

\[ \hat{Y}|_Z \leftarrow \hat{R}|_Z \]

consisting of étale surjective morphisms.

**Proposition 6.3.22.** Diagram (*) gives an étale equivalence relation in \( \text{AcFS}_S \), and if we put \( \hat{q}: \hat{Y}|_Z \to \hat{X}|_Z \) to be the sheaf quotient, then \( \hat{X}|_Z \) is a formal algebraic space over \( S \) and \( \hat{q} \) gives a representable étale covering.

**Proof.** The first assertion is clear, and the other follow from 6.3.5. \( \square \)
6. Formal algebraic spaces

6.3. (g) A.q.c. sheaves on formal algebraic spaces. Let $X$ be a formal algebraic space, and $\mathcal{I}$ an ideal of definition of finite type. As in §6.3.(e) we have algebraic spaces $X_k$ for $k \geq 0$, which we often denote loosely by $X_k = (X, \mathcal{O}_X/\mathcal{I}^{k+1})$. For any $\mathcal{O}_X$-module $\mathcal{F}$ we put $\mathcal{F}_k = \mathcal{F}/\mathcal{I}^{k+1}\mathcal{F}$ for $k \geq 0$, regarded as an $\mathcal{O}_{X_k}$-module. We thus obtain a projective system $\{\mathcal{F}_k\}_{k \geq 0}$ of $\mathcal{O}_X$-modules. Define the $\mathcal{O}_X$-module $\mathcal{F}$ by $\mathcal{F} = \lim_{\leftarrow k} \mathcal{F}_k$. As in §3.1.(a), one sees that the definition of $\mathcal{F}$ does not depend on the choice of the ideal of definition $\mathcal{I}$, and thus one can define $\mathcal{F}$ for any $\mathcal{O}_X$-module even in the case where $X$ does not have an ideal of definition. The $\mathcal{O}_X$-module $\mathcal{F}$ thus obtained is called the completion of $\mathcal{F}$. As before, the completion comes with the canonical morphism $\mathcal{F} \to \mathcal{F}$, and if this morphism is an isomorphism, we say that $\mathcal{F}$ is complete.

Definition 6.3.23. (1) We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is an a.q.c. sheaf if

(a) $\mathcal{F}$ is complete, and

(b) for any subspace $U \subseteq X$ and any ideal of definition $\mathcal{I}$ of finite type of $U$, the sheaf $(\mathcal{F}|_U)/\mathcal{I}(\mathcal{F}|_U)$ is a quasi-coherent sheaf on the algebraic space $(U, \mathcal{O}_U/\mathcal{I})$.

(2) An a.q.c. sheaf $\mathcal{F}$ on $X$ is said to be of finite type if it is of finite type as an $\mathcal{O}_X$-module.

(3) A morphism between a.q.c. sheaves is a morphism of $\mathcal{O}_X$-modules.

To check (b), it is enough to check the following statement.

(b)' There exist a Zariski covering $\{X_\alpha \to X\}_{\alpha \in L}$ of $X$, and for each $\alpha \in L$ an ideal of definition $\mathcal{I}_\alpha$ of finite type of $X_\alpha$, such that for any $\alpha \in L$ and $k \geq 0$ the sheaf $(\mathcal{F}|_{X_\alpha})/\mathcal{I}_\alpha^{k+1}(\mathcal{F}|_{X_\alpha})$ is a quasi-coherent sheaf on the algebraic space $(X_\alpha, \mathcal{O}_{X_\alpha}/\mathcal{I}_\alpha^{k+1})$.

If $X$ itself has an ideal of definition $\mathcal{I}$ of finite type, then the last condition is equivalent to $\mathcal{F}_k$ (defined as above) being quasi-coherent on $X_k$ for any $k \geq 0$.

We denote, as usual, the category of a.q.c. sheaves on a formal algebraic space $X$ by $\text{AQCoh}_X$ (or $\text{AQCoh}^\text{ét}_X$). By 6.2.12, if $X$ is a quasi-separated adic formal scheme of finite ideal type, then the notion of a.q.c. sheaves on $X$ as a formal algebraic space coincides with that on $X$ as an adic formal scheme. By 3.6.5 and 6.1.11 we have the following result.

Proposition 6.3.24. Let $X$ be a formal algebraic space and $\mathcal{F}$ a complete $\mathcal{O}_X$-module. Then $\mathcal{F}$ is an a.q.c. sheaf (resp. a.q.c. sheaf of finite type, resp. a.q.c. $\mathcal{O}_X$-algebra, resp. a.q.c. $\mathcal{O}_X$-algebra of finite type) if and only if for any representable étale covering $q:Y \to X$, $q^*\mathcal{F}$ is an a.s.c. sheaf (resp. a.q.c. sheaf of finite type, resp. a.q.c. $\mathcal{O}_Y$-algebra, resp. a.q.c. $\mathcal{O}_Y$-algebra of finite type) on $Y$. 
6.4 Several properties of morphisms

We begin with the following definition, consistent with the one given in the beginning of §6.3.(a).

**Definition 6.4.1.** Let $P$ be one of the following properties:

(a) affine adic,
(b) finite,
(c) open immersion,
(d) closed immersion,
(e) immersion.

We say that a morphism $f : X \to Y$ of formal algebraic spaces is $P$ if it is adic and for any morphism $V \to Y$, where $V$ is an adic formal scheme, the fiber product $X \times_Y V$ is represented by an adic formal scheme and the resulting map $f_V : X \times_Y V \to V$ of adic formal schemes has property $P$.

**Proposition 6.4.2.** Let $P$ be one of the properties for morphisms of formal algebraic spaces listed in 6.4.1.

1. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of formal algebraic spaces. If $f$ and $g$ satisfy $P$, then so does the composition $g \circ f$.

2. Let $Z$ be a formal algebraic space, and $f : X \to X'$ and $g : Y \to Y'$ two $Z$-morphisms of formal algebraic spaces satisfying $P$. Then

$$f \times_Z g : X \times_Z Y \to X' \times_Z Y'$$

satisfies $P$.

3. Let $Z$ be a formal algebraic space, and $f : X \to Y$ a $Z$-morphism between formal algebraic spaces satisfying $P$. Then for any morphism $Z' \to Z$ of formal algebraic spaces the induced morphism

$$f_{Z'} : X \times_Z Z' \to Y \times_Z Z'$$

satisfies $P$.

The proof is straightforward, and we omit it. The following proposition follows from 6.2.5.

**Proposition 6.4.3.** Let $P$ be one of the properties for morphisms of formal algebraic spaces listed in 6.4.1. Let $f : X \to Y$ be a morphism of formal algebraic spaces, and $V \to Y$ a representable étale covering of $Y$. Then $f$ satisfies $P$ if and only if $X \times_Y V$ is representable and $f_V : X \times_Y V \to V$ satisfies $P$. 
**Proposition 6.4.4.** Let $P$ be one of the properties for morphisms of formal algebraic spaces listed in 6.4.1. Let $f: X \to Y$ be an adic morphism of formal algebraic spaces, and suppose $Y$ has an ideal of definition $I$ of finite type. For any integer $k \geq 0$, we denote by $f_k: X_k \to Y_k$ the induced morphism of algebraic spaces defined as in §6.3.(e). Then the following conditions are equivalent.

(a) $f$ satisfies $P$.
(b) $f_k$ is satisfies $P$ for any $k \geq 0$.

If $P$ = ‘affine adic’, ‘finite’, or ‘closed immersion’, then the conditions are furthermore equivalent to

(c) $f_0$ is satisfies $P$.

The proof is straightforward and is omitted here.

**Definition 6.4.5.** Let $P$ be one of the following properties:

(a) locally of finite type,
(b) adically flat,
(c) smooth,
(d) étale.

We say that an $S$-morphism $f: X \to Y$ of formal algebraic spaces has $P$ if it is adic and there exist a representable étale covering $V \to Y$ and a representable étale covering $U \to X \times_Y V$ such that the resulting morphism $U \to V$ of adic formal schemes satisfies $P$.

Note that by 6.3.19 the definition of étale morphisms as in (d) is consistent with the one given in 6.3.10. The following proposition is clear.

**Proposition 6.4.6.** Let $P$ be one of the properties for morphisms of formal algebraic spaces listed in 6.4.5. Let $f: X \to Y$ be an adic morphism of formal algebraic spaces, and suppose $Y$ has an ideal of definition $I$ of finite type. For any integer $k \geq 0$ we denote by $f_k: X_k \to Y_k$ the induced morphism of algebraic spaces defined as in §6.3.(e). Then the following conditions are equivalent.

(a) $f$ satisfies $P$.
(b) $f_k$ is satisfies $P$ for any $k \geq 0$.

If $P$ = ‘locally of finite type’, then the conditions are furthermore equivalent to

(c) $f_0$ is satisfies $P$.

**Proposition 6.4.7.** Let $P$ be one of the properties for morphisms of formal algebraic spaces listed in 6.4.5. Let $f: X \to Y$ be an adic morphism of formal algebraic spaces, and $V \to Y$ a representable étale covering of $Y$. Then $f$ satisfies $P$ if and only if the base change $f_V: X \times_Y V \to V$ satisfies $P$. 
This follows from 6.2.6. The following proposition is easy to see.

**Proposition 6.4.8.** Let \( P \) be one of the properties for morphisms of formal algebraic spaces listed in 6.4.5.

1. Let \( f: X \to Y \) and \( g: Y \to Z \) be morphisms of formal algebraic spaces. If \( f \) and \( g \) satisfy \( P \), then so is the composition \( g \circ f \).

2. Let \( Z \) be a formal algebraic space, and \( f: X \to X' \) and \( g: Y \to Y' \) two \( Z \)-morphisms of formal algebraic spaces satisfying \( P \). Then
   \[
   f \times_Z g: X \times_Z Y \longrightarrow X' \times_Z Y'
   \]
   satisfies \( P \).

3. Let \( Z \) be a formal algebraic space, and \( f: X \to Y \) a \( Z \)-morphism between formal algebraic spaces satisfying \( P \). Then for any morphism \( Z' \to Z \) of formal algebraic spaces the induced morphism
   \[
   f_{Z'}: X \times_Z Z' \longrightarrow Y \times_Z Z'
   \]
   satisfies \( P \).

**Proposition 6.4.9.** The following conditions for a morphism \( f: X \to Y \) of formal algebraic spaces are equivalent.

1. For any étale morphism \( V \to Y \) from a quasi-compact adic formal scheme of finite ideal type, the formal algebraic space \( X \times_Y V \) is quasi-compact; see (6.3.4).

2. For any morphism \( V \to Y \) from a quasi-compact adic formal scheme of finite ideal type, the formal algebraic space \( X \times_Y V \) is quasi-compact.

**Proof.** Implication (b) \( \implies \) (a) is trivial. Suppose (a) holds. We take a representable étale covering \( Y' \to Y \), where \( Y' \) is quasi-compact, and set \( V' = Y' \times_Y V \), which is an adic formal scheme étale and surjective over \( V \). Since \( V' \) is quasi-compact, there exists a quasi-compact open subset \( Y'' \) of \( Y' \) containing the image of \( V' \). Since \( X \times_Y Y'' \) is quasi-compact, we deduce that \( X \times_Y V' \) is quasi-compact (1.6.7 (3)). Thus to show the assertion, it suffices to show the following: let \( f: X \to Y \) be a morphism of formal algebraic spaces, where \( Y \) is a quasi-compact adic formal scheme, and \( Y' \to Y \) an étale surjective map; if \( X \times_Y Y' \) is quasi-compact, then so is \( X \). This is easy to see.

**Definition 6.4.10.** A morphism \( f: X \to Y \) of formal algebraic spaces is said to be quasi-compact if the equivalent conditions in 6.4.9 are satisfied.

Clearly, quasi-compact morphisms are closed under composition and are base-change stable.
Definition 6.4.11. A morphism \( f : X \to Y \) of formal algebraic spaces is said to be of finite type if it is locally of finite type and quasi-compact.

Obviously, morphisms of finite type are closed under composition and base-change stable; moreover, (a), (b), and (c) in 6.4.6 with \( P = \text{‘of finite type’} \) are equivalent.

Definition 6.4.12. A morphism \( f : X \to Y \) of formal algebraic spaces is said to be locally separated (resp. separated) if the diagonal map \( \Delta_X : X \to X \times_Y X \) is a quasi-compact immersion (resp. closed immersion).

Proposition 6.4.13. Let \( f : X \to Y \) be an adic morphism of formal algebraic spaces, and suppose \( Y \) has an ideal of definition \( \mathcal{I} \). For any integer \( k \geq 0 \) we denote by \( f_k : X_k \to Y_k \) the induced morphism of algebraic spaces defined as in §6.3.(e). Then the following conditions are equivalent.

(a) \( f \) is locally separated (resp. separated).

(b) \( f_k \) is locally separated (resp. separated) for any \( k \geq 0 \).

Moreover, \( f \) is separated if and only if \( f_0 \) is separated.

Definition 6.4.14. A morphism \( f : X \to Y \) of formal algebraic spaces is said to be proper if it is adic and there exist a Zariski covering \( f : Y \to Y \) and \( \alpha \in \mathcal{L} \) of \( Y \) and an ideal of definition \( \mathcal{I}_\alpha \) of finite type on each \( Y_\alpha \), such that for each \( \alpha \in \mathcal{L} \) the induced morphism \( X_{\alpha,0} = X \times_Y Y_\alpha \to Y_{\alpha,0} \) of algebraic spaces (cf. §6.3.(e)) is proper.

It is easy to see that the condition does not depend on the choice of the Zariski covering \( \{ Y_\alpha \to Y \}_{\alpha \in \mathcal{L}} \) and the ideals of definitions \( \{ \mathcal{I}_\alpha \}_{\alpha \in \mathcal{L}} \). In particular, if \( Y \) has an ideal of definition \( \mathcal{I} \), then \( f \) is proper if and only if it is adic and the induced map \( f_k : X_k \to Y_k \) for any \( k \geq 0 \) is proper. Note that proper morphisms are separated of finite type. The following proposition is easy to establish.

Proposition 6.4.15. (1) Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms of formal algebraic spaces. If \( f \) and \( g \) are separated (resp. proper), then so is \( g \circ f \).

(2) Let \( Z \) be a formal algebraic space, and \( f : X \to X' \) and \( g : Y \to Y' \) two separated (resp. proper) \( Z \)-morphisms of formal algebraic spaces. Then
\[
f \times_Z g : X \times_Z Y \longrightarrow X' \times_Z Y'
\]
is separated (resp. proper).

(3) Let \( Z \) be a formal algebraic space, and \( f : X \to Y \) a separated (resp. proper) \( Z \)-morphism of formal algebraic spaces. Then for any map \( Z' \to Z \) of formal algebraic spaces the induced morphism
\[
f_{Z'} : X \times_Z Z' \longrightarrow Y \times_Z Z'
\]
is separated (resp. proper).
6.5 Universally adhesive and universally rigid-Noetherian formal algebraic spaces

Definition 6.5.1. A formal algebraic space $X$ is said to be \textit{locally universally adhesive} (resp. \textit{locally universally rigid-Noetherian}) if it has an étale covering $Y \to X$ by a locally universally adhesive (resp. locally universally rigid-Noetherian) formal scheme $Y$ (2.1.7). If $Y$ can be taken to be quasi-compact, we say that $X$ is \textit{universally adhesive} (resp. \textit{universally rigid-Noetherian}).

If a formal algebraic space $X$ is locally universally adhesive (resp. locally universally rigid-Noetherian), then any locally of finite type formal algebraic space over $X$ is locally universally adhesive (resp. locally universally rigid-Noetherian). Note that these definitions are consistent with the definitions of the corresponding notions for formal schemes due to 2.1.6.

Proposition 6.5.2. Let $X \to Z \leftarrow Y$ be a diagram of formal algebraic spaces, where $X$ is locally universally adhesive (resp. locally universally rigid-Noetherian) and the map $Y \to Z$ is locally of finite type. Then the fiber product $X \times_Z Y$ is locally universally adhesive (resp. locally universally rigid-Noetherian).

Definition 6.5.3. A morphism $f : X \to Y$ of locally universally rigid-Noetherian formal algebraic spaces is said to be \textit{locally of finite presentation} if it is adic and there exist a representable étale covering $V \to Y$ and a representable étale covering $U \to X \times_Y V$ such that the resulting map $U \to V$ (between locally universally rigid-Noetherian formal schemes) is locally of finite presentation. If, moreover, $f$ is quasi-compact, then $f$ is said to be \textit{of finite presentation}.

The following propositions are straightforward.

Proposition 6.5.4. Let $f : X \to Y$ be an adic morphism of locally universally rigid-Noetherian formal algebraic spaces, and suppose $Y$ has an ideal of definition $I$ of finite type. For any integer $k \geq 0$ we denote by $f_k : X_k \to Y_k$ the induced morphism of algebraic spaces defined as in §6.3. (e). Then the following conditions are equivalent.

(a) $f$ is locally of finite presentation (resp. of finite presentation).

(b) $f_k$ is locally of finite presentation (resp. of finite presentation) for any $k \geq 0$.

Proposition 6.5.5. Let $f : X \to Y$ be an adic morphism of locally universally rigid-Noetherian formal algebraic spaces, and $V \to Y$ a representable étale covering of $Y$. Then $f$ is locally of finite presentation (resp. of finite presentation) if and only if so is the base change $f_V : X \times_Y V \to V$. 
Proposition 6.5.6. (1) Let \( f : X \to Y \) and \( g : Y \to Z \) be morphisms of locally universally rigid-Noetherian formal algebraic spaces. If \( f \) and \( g \) are locally of finite presentation (resp. of finite presentation), then so is \( g \circ f \).

(2) Let \( Z \) be a formal algebraic space, and \( f : X \to X' \) and \( g : Y \to Y' \) two \( Z \)-morphisms of locally universally rigid-Noetherian formal algebraic spaces that are locally of finite presentation (resp. of finite presentation). Suppose \( X' \times_Z Y' \) is locally universally rigid-Noetherian. Then
\[
f \times_Z g : X \times_Z Y \longrightarrow X' \times_Z Y'
\]
is locally of finite presentation (resp. of finite presentation).

(3) Let \( Z \) be a formal algebraic space and \( f : X \to Y \) a \( Z \)-morphism between locally universally rigid-Noetherian formal algebraic spaces that is locally of finite presentation (resp. of finite presentation). Then for any morphism \( Z' \to Z \) of formal algebraic spaces such that \( Y \times_Z Z' \) is locally universally rigid-Noetherian the induced morphism
\[
f_{Z'} : X \times_Z Z' \longrightarrow Y \times_Z Z'
\]
is locally of finite presentation (resp. of finite presentation).

By 6.2.13, one can deal with a.q.c. sheaves of finite presentation over locally universally rigid-Noetherian formal algebraic spaces; here the details are omitted and left to the reader.

Exercises

Exercise I.6.1. Show that any algebraic space \( X \) over a scheme \( S \) is canonically regarded as a formal algebraic space over \( S \). (See 0, §5.2. (a) for our convention on algebraic spaces.)

Exercise I.6.2. Let \( X \) be a (resp. locally separated, resp. separated) formal algebraic space over an adic formal scheme \( S \) of finite ideal type, \( Z \) and \( W \) adic formal schemes over \( S \), and \( Z \to X \) and \( W \to X \) maps of sheaves on \( \text{AcFs}_{S, \text{ét}} \). Show that the sheaf fiber product \( Z \times_X W \) is representable and that the map \( Z \times_X W \to Z \times_S W \) is quasi-compact (resp. a quasi-compact immersion, resp. a closed immersion).

Exercise I.6.3. Let \( f : X \to Y \) be an adically faithfully flat morphism between adic formal schemes of finite ideal type, and \( \mathcal{J} \) an a.q.c. ideal sheaf on \( Y \). Suppose that \( \widehat{f}^* \mathcal{J} \) is an ideal of definition of \( X \). Show that \( \mathcal{J} \) is an ideal of definition of \( Y \).
**Exercise I.6.4.** Show that for any formal algebraic space $X$ there exists a Zariski covering $\{X_\alpha \rightarrow X\}_{\alpha \in \Lambda}$ (that is, covering by open subspaces) such that each $X_\alpha$ is a quasi-compact formal algebraic space. Show, moreover, that in this case if each $X_\alpha$ is a formal scheme, then $X$ is a formal scheme.

**Exercise I.6.5.** Let $X$ be a formal algebraic space.

1. (Cf. [72], II.6.) A point of $X$ is a morphism of the form $\text{Spec} \, k \rightarrow X$ (where $k$ is a field) that is a sheaf monomorphism. We denote by $|X|$ the set of all isomorphism classes of points of $X$. Suppose that $X$ has an ideal of definition of finite type $I$, and let $X_0 = (X, \mathcal{O}_X/I)$ be the associated algebraic space. Show that $|X_0| = |X|$. In particular, $|X| \neq \emptyset$ whenever $X$ is non-empty.

2. The underlying topological space of $X$ is the set $|X|$ endowed with the following topology: a subset of $|X|$ is closed if it is of the form $|Y|$ for a closed subspace $Y$ of $X$. Show that there exists a canonical one-to-one correspondence between the set of all open subspaces of $X$ and the set of all open subsets of $|X|$.

**Exercise I.6.6.** Let $X$ be a non-empty formal algebraic space. Show that there exists a dense open subspace $Y$ of $X$ that is a formal scheme. (Note that, according to our convention, all formal algebraic spaces are quasi-separated.)

### 7 Cohomology theory

In this section we collect fundamental facts on cohomologies of a.q.c. sheaves on formal algebraic spaces. After giving some general facts, we will discuss in §7.2 coherent sheaves on locally universally adhesive formal algebraic spaces of a certain kind. The last subsection, §7.3, collects some derived categorical calculi that are needed in the later arguments.

Here we would like to mention again that when we say ‘$X$ is a formal algebraic space’, we always mean either one of the following situations (cf. 0, §5.2. (a)).

- $X$ is an adic formal scheme of finite ideal type (but not necessarily quasi-separated); in this case, unless otherwise clearly stated, all a.q.c. sheaves on $X$ and their cohomologies are considered with respect to the Zariski topology.

- $X$ is a quasi-separated adic formal algebraic space of finite ideal type; in this case, all a.q.c. sheaves on $X$ and their cohomologies are considered with respect to the étale topology (§6.3. (d)).
7.1 Cohomologies of a.q.c. sheaves

**Theorem 7.1.1.** (1) Let $A$ be an adic ring of finite ideal type (1.1.3 (2), 1.1.6), and set $X = \text{Spf } A$. Then for any a.q.c. sheaf $\mathcal{F}$ (3.1.3) we have

$$H^q(X, \mathcal{F}) = 0$$

for $q \geq 1$.

(2) Let $f : X \rightarrow Y$ be an affine adic morphism between adic formal schemes of finite ideal type, and $\mathcal{F}$ an a.q.c. sheaf on $X$. Then

$$R^q f_* \mathcal{F} = 0$$

for $q \geq 1$.

**Theorem 7.1.2.** (1) Let $A$ be an adic ring of finite ideal type, and set $X = \text{Spf } A$. Then for any complex $\mathcal{F}^\bullet$ of a.q.c. sheaves on $X$ such that $\mathcal{F}^q = 0$ for $q \ll 0$ (resp. $|q| \gg 0$), we have $H^q(X, \mathcal{F}^\bullet) = 0$ for $q \ll 0$ (resp. $|q| \gg 0$), and the object $R^+ \Gamma_X(\mathcal{F}^\bullet)$ of $\mathcal{D}^+(A)$ is represented by the complex $\Gamma_X(\mathcal{F}^\bullet)$.

(2) Let $f : X \rightarrow Y$ be an affine adic morphism between formal schemes of finite ideal type. Let $\mathcal{F}^\bullet$ be a complex of a.q.c. sheaves on $X$ such that $\mathcal{F}^q = 0$ for $q \ll 0$ (resp. $|q| \gg 0$). Then $R^q f_* \mathcal{F}^\bullet = 0$ for $q \ll 0$ (resp. $|q| \gg 0$), and the object $R^+ f_* \mathcal{F}^\bullet$ of $\mathcal{D}^+(Y)$ is represented by the complex $f_* \mathcal{F}^\bullet$.

Similarly to the proof of 0.5.4.3 one can deduce 7.1.2 from 7.1.1. Hence it suffices to show 7.1.1.

**Proof of Theorem 7.1.1.** As (1) is a special case of 1.1.23 (2), we only need to show (2). Since $R^q f_* \mathcal{F}$ is the sheaf on $Y$ associated to the presheaf given by $U \mapsto H^q(f^{-1}(U), \mathcal{F})$, we may assume that $Y$ is affine of the form $Y = \text{Spf } B$, where $B$ is an adic ring that has a finitely generated ideal of definition $I \subseteq B$. Then it suffices to check that $H^q(X, \mathcal{F})$ vanishes for $q \geq 0$. In this situation, $X$ is also of the form $X = \text{Spf } A$, where $A$ is an adic ring having $IA$ as a finitely generated ideal of definition. Then the claimed vanishing follows from (1).

**Corollary 7.1.3.** Let $f : X \rightarrow Y$ be an adic and separated morphism of adic formal schemes of finite ideal type, and suppose $Y$ is quasi-compact. Then there exists an integer $r > 0$ such that for any a.q.c. sheaf $\mathcal{F}$ on $X$ and any $q \geq r$ we have $R^q f_* \mathcal{F} = 0$. If, moreover, $Y$ is affine, then one can take as $r$ (the minimum of) the number of affine open sets that cover $X$. (Hence such an $f$ always has finite cohomological dimension.)

**Proof.** We may assume that $Y$ is affine of the form $Y = \text{Spf } B$, where $B$ is an adic ring of finite type. The formal scheme $X$ admits a finite cover by affine open subsets; by 4.6.4, all intersections of the members of the covering are affine. Hence in view of 7.1.1 the assertion follows from the Čech calculation of the cohomology using the Leray covering. 

\[\square\]
7.2 Coherent sheaves

Definition 7.2.1. A formal algebraic space $X$ is said to be universally cohesive if it is locally universally rigid-Noetherian (6.5.1) and for any locally of finite presentation morphism $Y \to X$ from a formal scheme $Y$, $\mathcal{O}_Y$ is a coherent $\mathcal{O}_Y$-module.

Proposition 7.2.2. Let $A$ be a t.u. rigid-Noetherian ring (2.1.1 (1)) and $I \subseteq A$ a finitely generated ideal of definition. Set

$$X = \text{Spf } A.$$ 

Then $X$ is universally cohesive if $A$ is topologically universally coherent (0.8.5.22) with respect to $I$.

Proof. Suppose $A$ is topologically universally coherent, and let $Y \to X$ be an $X$-formal scheme locally of finite presentation. In order to show that $\mathcal{O}_Y$ is coherent, we may assume that $Y$ is affine, $Y = \text{Spf } B$. By the assumption, $B$ is a topologically finitely presented $B$-algebra and, in view of 0.8.5.23 (2), topologically universally coherent with respect to the ideal $IB$. Hence it suffices to show that $\mathcal{O}_X$ is coherent (since for $Y$ the situation is the same). Let $U$ be an open subset of $X$, and suppose we are given an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}^{\oplus m}_U \to \mathcal{O}_U.$$ 

We need to show that $\mathcal{K}$ is of finite type. We may further assume that $U$ is an affine open subset, and thus that $U = X$, without loss of generality. By 3.2.9 the last morphism comes from a map $A^{\oplus m} \to A$. Let $K$ be its kernel, which is a finitely generated $A$-module. Then by 3.5.6

$$0 \to K^\Delta \to \mathcal{O}^{\oplus m}_X \to \mathcal{O}_X$$

is exact, that is, we have $\mathcal{K} \cong K^\Delta$, which is an a.q.c. sheaf of finite type. 

Proposition 7.2.3. Let $X$ be a locally universally adhesive formal algebraic space such that there exists a covering (Zariski or étale) $\coprod_{\alpha \in \Lambda} U_\alpha \to X$, where each $U_\alpha$ is affine with an ideal of definition $I_\alpha$ of finite type, such that $\mathcal{O}_{U_\alpha}$ is $I_\alpha$-torsion free. Then $X$ is universally cohesive.

This follows from 7.2.2 and 0.8.5.25 (2). Hence, any formal scheme locally of finite presentation over $X$ as above, even if it may have $I$-torsions, is universally cohesive. For example, if $V$ is an $a$-adically complete valuation ring (of arbitrary height), then the $a$-adic formal scheme $\text{Spf } V$, and hence any formal algebraic space locally of finite presentation over $V$, is universally cohesive (0.9.2.7).
Proposition 7.2.4. Let \( X \) be a universally cohesive formal algebraic space, and suppose \( X \) has an ideal of definition \( \mathcal{I} \) of finite type. Set
\[
X_k = (X, \mathcal{O}_X/\mathcal{I}^{k+1}), \quad k \geq 0
\]
(note that \( X_k \) is a universally cohesive scheme). Let \( \mathcal{F} \) be an \( \mathcal{O}_X \)-module, and set
\[
\mathcal{F}_k = \mathcal{F}/\mathcal{I}^{k+1}\mathcal{F}, \quad k \geq 0.
\]
Then the following conditions are equivalent.

(a) \( \mathcal{F} \) is coherent.

(b) \( \mathcal{F}_k \) is coherent on \( X_k \) for any \( k \geq 0 \) and \( \mathcal{F} = \lim_{\leftarrow k} \mathcal{F}_k \).

Proof. Note that in either situation (a) or (b) the sheaf \( \mathcal{F} \) is a.q.c. of finite type (cf. 3.4.1). We may work in the affine situation; set \( X = \text{Spf} A \) where \( A \) is a t.u. rigid-Noetherian ring with a finitely generated ideal of definition \( I \subseteq A \). In this case, \( \mathcal{F} = M^A \) for a uniquely determined finitely generated \( A \)-module \( M \). Then the assertion follows from \( 0.7.4.19 \). \( \square \)

### 7.3 Calculus in derived categories

Let \( X \) be a formal algebraic space. We denote by \( D^*(X) \) (\( * = \text{“”}, +, -, b \)) the derived category associated to the abelian category \( \text{Mod}_X \) of \( \mathcal{O}_X \)-modules. In the case where \( X \) is universally cohesive, one can consider the full subcategory \( D^*_{\text{coh}}(X) \) of \( D^*(X) \) consisting of objects having coherent cohomologies in all degrees. This is a triangulated category equipped with the canonical cohomology functor \( H^0 \) and the canonical \( t \)-structure (cf. 0, §5.3).

Proposition 7.3.1 (cf. 0.C.4.6). Let \( A \) be a t.u. rigid-Noetherian ring with a finitely generated ideal of definition \( I \subseteq A \), and suppose that \( A \) is topologically universally coherent with respect to \( I \). Set \( X = \text{Spf} A \). Then the exact equivalence
\[
\text{Coh}_A \sim \text{Coh}_X
\]
given by \( M \mapsto M^A \) (3.5.6) induces an exact equivalence
\[
D^*(\text{Coh}_A) \sim D^*(\text{Coh}_X),
\]
where \( \text{Coh}_A \) denotes the category of finitely presented \( A \)-modules (which is an abelian category due to 0.3.3.3).
In the following we denote the composition

\[ D^*(A) \sim \overset{\delta^*}{\longrightarrow} D^*(\text{Coh}_X) \overset{\delta^*}{\longrightarrow} D^*_{\text{coh}}(X) \]

(cf. 0, §C.5) by

\[ M \longmapsto M_X. \]

**Proposition 7.3.2.** Let \( A \) and \( I \) be as in 7.3.1, and set \( X = \text{Spf} \ A \). Then the canonical exact functor

\[ \delta^b: D^b(\text{Coh}_X) \longrightarrow D^b_{\text{coh}}(X) \]

(cf. 0, §C.5) is a categorical equivalence.

**Corollary 7.3.3.** In the situation as in 7.3.1 the canonical functor

\[ D^b(\text{Coh}_A) \longrightarrow D^b_{\text{coh}}(X), \quad M \longmapsto M_X, \]

is an exact equivalence. In other words, any object \( M \) of \( D^b_{\text{coh}}(X) \) can be represented (0.C.4.8) by a complex consisting of coherent sheaves and hence by a complex consisting of finitely presented \( A \)-modules.

All these can be shown similarly to 0.5.3.2 and 0.5.3.3. By an argument similar to 0.5.3.5 one has the following result.

**Proposition 7.3.4.** Let \( X \) be a universally cohesive formal algebraic space.

1. For \( F, G \in \text{obj}(D^-_{\text{coh}}(X)) \), the object \( F \otimes_{\mathcal{O}_X}^L G \) belongs to \( D^-_{\text{coh}}(X) \).

2. For \( F \in \text{obj}(D^-_{\text{coh}}(X)) \) and \( G \in \text{obj}(D^+_{\text{coh}}(X)) \), the object \( R\text{Hom}_{\mathcal{O}_X}(F, G) \) belongs to \( D^+_{\text{coh}}(X) \).

**Proposition 7.3.5.** Let \( f: X \rightarrow Y \) be a morphism of universally cohesive formal algebraic spaces. Then the functor \( Lf^* \) maps \( D^-_{\text{coh}}(Y) \) to \( D^-_{\text{coh}}(X) \).

The proof is similar to that of 0.5.3.6. Using 7.1.2 (2) one can show the following proposition by an argument similar to that in 0.5.4.4.

**Proposition 7.3.6.** Let \( X \) be a universally cohesive formal algebraic space and let \( i: Y \hookrightarrow X \) be a closed immersion of finite presentation (hence \( Y \) is universally cohesive). Then \( R^+i_* \) maps \( D^b_{\text{coh}}(Y) \) to \( D^b_{\text{coh}}(X) \).

**Exercises**

**Exercise I.7.1.** Let \( X \) be a universally cohesive (hence locally universally rigid-Noetherian) formal algebraic space, and \( \varphi: \mathcal{F} \rightarrow \mathcal{G} \) a morphism of a.q.c. sheaves of finite type on \( X \). Show that if \( \mathcal{G} \) is coherent, then \( \ker(\varphi) \) is an a.q.c. sheaf of finite type.
8 Finiteness theorem for proper algebraic spaces

In this section we announce and prove the finiteness theorem for cohomologies of coherent sheaves on universally cohesive (0.5.2.1) algebraic spaces (not necessarily locally Noetherian), which generalizes the classically known finiteness theorem in, e.g., [54], III. This theorem provides an important preliminary for (generalized) GFGA theorems, which will be discussed in the next two sections. We put this entirely scheme-theoretic (not formal-scheme-theoretic) section in this chapter, not only because it gives a preliminary for the GFGA, but also some of the techniques for the proof are common to the proofs of the GFGA theorems. Most notably, a variation of Grothendieck’s dévissage is discussed, which we call the carving method. As in case of dévissage, it consists of a reduction to particular cases by an induction with respect to sequences of closed subspaces, that is, induction with respect to a stratification. Aside from its technical merits (e.g., removing Noetherian hypothesis), the method is best suited for treating algebraic spaces, since an inductive argument with respect to a stratification is a basic and fundamental tool to study these spaces.

In the first subsection, §8.1, we announce the finiteness theorem (8.1.3) and some related statements for algebraic spaces. The rest of this section is mainly devoted to the proof. By the carving method, the finiteness theorem is reduced to the projective case, and the theorem in this particular situation is verified by a generalization of Serre’s theorem [54], III, (2.2.1), which will be announced and proved in §8.2. In §8.3, we formulate the carving method in a derived-categorical setting. This will be contained in Proposition 8.3.1 below, and the proof of this proposition is based on the carving lemma (8.3.2). The proof of the finiteness theorem will be finished in §8.4. In the final subsection §8.5 we give an application of the theorem to $I$-goodness (cf. 0.7.4. (a)) of the induced filtrations of the cohomology groups of coherent sheaves, which we will need in our later discussion.

8.1 Finiteness theorem: formulation

First, let us collect some known finiteness results that are already general enough for our purposes.

**Proposition 8.1.1** (cf. [54], III, (1.4.12)). Let $f : X \to S$ be a quasi-compact and separated morphism of algebraic spaces, and suppose that $S$ is quasi-compact. Then there exists an integer $r > 0$ such that for any quasi-coherent sheaf $\mathcal{F}$ on $X$ and any $q \geq r$, we have $R^q f_* \mathcal{F} = 0$. If, moreover, $S$ is affine and $X$ is a scheme, one can take as $r$ (the minimum of) the number of affine open subsets that cover $X$.

This follows readily from 0.5.4.6 and 0.5.4.7 (cf. [72], II.2.7 and II.3.12). The proposition says, in other words, that the morphism $f$ as above has finite cohomological dimension.
Proposition 8.1.2. Let \( f : X \to Y \) be a separated and quasi-compact morphism of algebraic spaces with \( X \) quasi-compact, and \( \mathcal{L} \) an \( f \)-ample invertible sheaf on \( X \). Let \( F \) be a quasi-coherent sheaf on \( X \) of finite type. Then there exists an integer \( N \) such that for any \( n \geq N \) the canonical morphism \( f^* f_* F(n) \to F(n) \) is surjective.

Proof. One can assume that \( Y \) is affine, and hence that \( X \) is a scheme. Then the assertion follows from [54], II, (4.6.8).

Now we formulate our finiteness theorem.

Theorem 8.1.3 (finiteness theorem). Let \( Y \) be a universally cohesive (0.5.2.1) quasi-compact algebraic space, and \( f : X \to Y \) a proper morphism of finite presentation of algebraic spaces (hence \( X \) is also universally cohesive). Then the functor \( Rf_* \) maps an object of \( D^b_{\text{coh}}(X) \) to an object of \( D^b_{\text{coh}}(Y) \).

Note that the premise ‘universally cohesive’ for \( Y \) is fulfilled, in particular, when \( Y \) is Noetherian or is locally of finite presentation over an \( \alpha \)-adically complete valuation ring \( V \) of arbitrary height (0.9.2.8). Before the proof of the theorem we include some useful corollaries.

Corollary 8.1.4. Consider the situation as in 8.1.3. Then the functor \( Rf_* \) maps \( D^b_{\text{coh}}(X) \) to \( D^b_{\text{coh}}(Y) \) for \( \ast = -, +, b \).

Proof. Let \( K \) be an object of \( D^+_b(X) \), and consider \( Rf_* K \). To detect the cohomology \( R^q f_* K \), we can always find a sufficiently large \( n \) such that \( R^q f_* K = R^q f_* (\tau \geq n K) \); therefore, \( Rf_* \) maps \( D^+_b(X) \) to \( D^+_b(Y) \). The other cases are easy to verify (use \( \tau \leq m \) instead of \( \tau \geq n \)).

Corollary 8.1.5. Let \( B \) be a universally coherent ring (0.8.5.22), and let the map \( f : X \to Y = \text{Spec} B \) be a proper morphism of finite presentation between algebraic spaces. Let \( M \in D^{\text{co}}(X) \) and \( N \in D^+_b(X) \). Then \( R^q \text{Hom}_X(M, N) \) (with the natural \( B \)-module structure; cf. 0, §4.3.(c)) is a coherent \( B \)-module for any \( q \).

Proof. By 0.5.3.5 (2) we know that \( R\text{Hom}_X(M, N) \) belongs to \( D^+_b(X) \). Since \( R\text{Hom}_X(M, N) = R\Gamma X \circ R\text{Hom}_X(M, N) \), the assertion follows from 8.1.4.

8.2 Generalized Serre’s theorem

8.2 (a) Formulation. The following theorem together with 8.1.2 generalizes the result of [54], III, (2.2.1).
Theorem 8.2.1. Let $Y$ be a universally cohesive quasi-compact algebraic space, $f: X \to Y$ a proper morphism of finite presentation, and $\mathcal{F}$ a coherent sheaf on $X$. Suppose there exists an $f$-ample invertible sheaf $\mathcal{L}$ on $X$.

1. The sheaf $R^q f_* \mathcal{F}$ for any $q$ is coherent on $Y$.

2. There exists an integer $N$ such that $R^q f_* \mathcal{F}(n) = 0$ for $n \geq N$ and $q > 0$.

The rest of this subsection is devoted to the proof of this theorem.

8.2. (b) Reduction process. We may assume that $Y$ is affine, $Y = \text{Spec } B$, where $B$ is universally coherent (0.5.1.2). We may further assume that $\mathcal{L}$ is very ample relative to $f$. Since $f$ is proper and quasi-projective, it is in fact projective ([72], II.7.8), and hence there exists a closed $Y$-immersion $i: X \hookrightarrow \mathbb{P}^r_B$ for some $r > 0$ such that $\mathcal{L} \cong i^* \mathcal{O}_{\mathbb{P}_B}(1)$. Let $g$ be the structure morphism $\mathbb{P}^r_B \to Y$. As $i$ is affine, we have $R^q f_* \mathcal{F}(n) \cong Rg_* (i_* \mathcal{F}(n))$; also, by the projection formula (cf. [54], 0III. (12.2.3)), $i_* (\mathcal{F}(n)) \cong (i_* \mathcal{F})(n)$. Thus we may assume that $X = \mathbb{P}^r_B$ without loss of generality. Therefore, to show 8.2.1, it suffices to prove the following proposition.

Proposition 8.2.2. Let $B$ be a universally coherent ring, $X = \mathbb{P}^r_B$, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module.

1. For any $q$ the $B$-module $H^q(X, \mathcal{F})$ is finitely presented.

2. There exists an integer $N$ such that $H^q(X, \mathcal{F}(n)) = 0$ for $n \geq N$ and $q > 0$.

8.2. (c) Proof of Proposition 8.2.2

Lemma 8.2.3. Let $B$ be a universally coherent ring and $X = \mathbb{P}^r_B$. Then for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an exact sequence

$$\cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

with sheaves $\mathcal{E}_i$ ($i \geq 0$) of the form $\mathcal{O}_X(n_i)^{\oplus k_i}$.

Proof. The proof is done by successive application of 0.5.4.10; namely, by 0.5.4.10 there exists a surjective morphism $\mathcal{O}_X(n_0)^{\oplus k_0} \to \mathcal{F}$ for some integers $n_0$ and $k_0 > 0$. Let $\mathcal{K}_0$ be its kernel. Since $\mathcal{K}_0$ is again coherent, we similarly have a surjective map $\mathcal{O}_X(n_1)^{\oplus k_1} \to \mathcal{K}_0$. One can then repeat this procedure. \qed

Lemma 8.2.4. Let $B$ be a coherent ring and $X$ an algebraic space over $B$. Let

$$\mathcal{E}^* = (\cdots \to 0 \to \mathcal{E}^s \to \mathcal{E}^{s+1} \to \cdots \to \mathcal{E}^r \to 0 \to \cdots)$$

be a bounded complex of $\mathcal{O}_X$-modules such that for all $q$ and $k$ the cohomology group $H^q(X, \mathcal{E}^k)$ is a finitely presented $B$-module. Then $H^q(X, \mathcal{E}^*)$ for any $q$ is a finitely presented $B$-module.
Proof. The proof is done by induction with respect to the length of $\mathcal{E}^\bullet$. By a suitable shift we may assume that the complex $\mathcal{E}^\bullet$ is of the form

$$\mathcal{E}^\bullet = (\cdots \to 0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to \cdots \to \mathcal{E}^{l-1} \to 0 \to \cdots).$$

Set $\mathcal{E}'^\bullet = \sigma^{\geq 1} \mathcal{E}^\bullet$ (the stupid truncation), that is,

$$\mathcal{E}'^\bullet = (\cdots \to 0 \to 0 \to \mathcal{E}^1 \to \cdots \to \mathcal{E}^{l-1} \to 0 \to \cdots).$$

Then we have the distinguished triangle

$$\mathcal{E}'^\bullet \to \mathcal{E}^\bullet \to \mathcal{E}^0 \xrightarrow{+1}$$

in $\mathcal{K}^b(\text{Mod}_X)$. By the cohomology exact sequence

$$H^{q-1}(X, \mathcal{E}^0) \to H^q(X, \mathcal{E}'^\bullet) \to H^q(X, \mathcal{E}^\bullet) \to H^q(X, \mathcal{E}'^\bullet) \to H^{q+1}(X, \mathcal{E}'^\bullet),$$

and by 0.3.3.3, we deduce that $H^q(X, \mathcal{E}^\bullet)$ is finitely presented, as desired.

By the proposition and 0.5.4.7 we have the following corollary.

**Corollary 8.2.5.** Let $Y$ be a universally cohesive algebraic space, and $f : X \to Y$ a morphism of algebraic spaces. Let $\mathcal{E}^\bullet$ be a bounded complex of $\mathcal{O}_X$-modules such that $R^q f_* \mathcal{E}^k$ are coherent $\mathcal{O}_Y$-modules for all $q$ and $k$. Then $R^q f_* \mathcal{E}^\bullet$ are coherent $\mathcal{O}_Y$-modules for all $q$.

**Proof of Proposition 8.2.2.** We take a resolution of $\mathcal{F}$ as in 8.2.3. By 8.1.1, one already knows that the cohomology group $H^q(X, \mathcal{F})$ vanishes for $q > r$, since $\mathbb{P}^r_B$ is separated over $B$ and covered by $r + 1$ affine open subsets. Hence it is enough to calculate the cohomology groups for $0 \leq q \leq r$. To this end, one is allowed to replace $\mathcal{F}$ by the bounded complex

$$\cdots \to 0 \to \mathcal{E}_{r+1} \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to 0 \to \cdots.$$ 

By 0.5.4.9 and 8.2.4 we deduce that $H^q(X, \mathcal{F})$ is a finitely presented $B$-module for $0 \leq q \leq r$. Combined with the fact that $H^q(X, \mathcal{F})$ for $q > r$ are trivially finitely presented, we get (1).

Let us show (2). For $q > r$ we already know that $H^q(X, \mathcal{F}(n)) = 0$ for any $n$. Hence one can prove the assertion by descending induction with respect to $q$. Consider the surjection $\mathcal{E} = \mathcal{E}_0 \to \mathcal{F}$ as above, and let $\mathcal{K}$ be its kernel (finitely presented due to 0.3.3.3). By induction, there exists $N$ such that $H^{q+1}(X, \mathcal{K}(n)) = 0$ for $n \geq N$. Moreover, one can take $N$ sufficiently large so that for $n \geq N$ the sheaf $\mathcal{E}(n)$ is of the form $\mathcal{O}_X(m)^{\oplus k}$ for a positive integer $m$. Hence $H^q(X, \mathcal{E}(n)) = 0$ and $H^{q+1}(X, \mathcal{E}(n)) = 0$ for $q > 0$ (0.5.4.9). Then the vanishing of $H^q(X, \mathcal{F}(n))$ follows from the cohomology exact sequence. □
8. Finiteness theorem for proper algebraic spaces

8.3 The carving method

8.3. (a) The main assertion. Let $B$ be a universally coherent ring, and consider the category $\text{PAs}_B$, a full subcategory of the category $\text{As}_B$ of $B$-algebraic spaces consisting of proper and finitely presented $B$-algebraic spaces. Note that every algebraic space in $\text{PAs}_B$ is universally cohesive.

Proposition 8.3.1 (carving method). Suppose for each object $f : X \to \text{Spec } B$ of $\text{PAs}_B$ we are given a full subcategory

$$D_f \subseteq D^\text{b}_{\text{coh}}(X)$$

satisfying the following conditions.

(C0) The zero object $0$ belongs to $D_f$, and $D_f$ is stable under isomorphisms in $D^\text{b}_{\text{coh}}(X)$.

(C1) Let $K \to L \to M \to^1$ be a distinguished triangle in $D^\text{b}_{\text{coh}}(X)$. If two of $K$, $L$, and $M$ are in $D_f$, then so is the rest.

(C2) If $f$ is projective, then $D_f = D^\text{b}_{\text{coh}}(X)$.

(C3) Consider a morphism $\pi$ in $\text{PAs}_B$, which amounts to the same as a commutative diagram of the form

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & X \\
\downarrow{g} & & \downarrow{f} \\
\text{Spec } B & &
\end{array}
$$

and suppose $g$ and $\pi$ are projective. Then $R\pi_* \text{ maps } D_g = D^\text{b}_{\text{coh}}(\tilde{X}) \text{ to } D_f$.

(C4) Consider a closed immersion $\iota$ in $\text{PAs}_B$, that is, a commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\iota} & X \\
\downarrow{g} & & \downarrow{f} \\
\text{Spec } B & &
\end{array}
$$

where $\iota$ is a closed immersion. Then if $D_g = D^\text{b}_{\text{coh}}(Z)$, the functor $R\iota_*$ maps $D_g$ to $D_f$.

Then $D_f = D^\text{b}_{\text{coh}}(X)$ for any object $f : X \to \text{Spec } B$ of $\text{PAs}_B$.

8.3. (b) Preparation for the proof and carving lemma. The rest of this subsection is devoted to showing Proposition 8.3.1. Let us first establish the following fact, which will be tacitly used in the sequel.

Claim 0. If $M \in \text{obj}(D_f)$, then $M[n] \in \text{obj}(D_f)$ for any integer $n$. 
Indeed, we have the distinguished triangle

$$M \rightarrow 0 \rightarrow M[1] \rightarrow^+,$$

whence the claim, due to (C1).

**Claim 1.** Consider the diagram as in (C4), where \( \iota \) is a closed immersion defined by a nilpotent quasi-coherent ideal \( \mathcal{I} \subseteq \mathcal{O}_X \). Then if \( \mathbf{D}_g = \mathbf{D}_{\text{coh}}(\mathcal{Z}) \), we have \( \mathbf{D}_f = \mathbf{D}_{\text{coh}}(X) \).

Since \( \iota \) is finitely presented, there exists an integer \( k \geq 0 \) such that \( \mathcal{I}^{k+1} = 0 \). By induction with respect to \( k \) we may assume without loss of generality that \( \mathcal{I}^2 = 0 \). Let \( M \in \text{obj}(\mathbf{D}_{\text{coh}}(X)) \). Using the distinguished triangle

$$\tau^{\leq n} M \rightarrow M \rightarrow \tau^{\geq n+1} M \rightarrow^+$$

and Claim 0, by an inductive application of (C1) we may assume that \( M \) is concentrated in degree 0, that is, \( \mathcal{H}^q(M) = 0 \) unless \( q = 0 \).

Consider the canonical morphism \( M \rightarrow \mathbf{R}_{\iota*}(\tau^{\geq 0} \mathcal{L}_t^* M) \), and embed it into a distinguished triangle

$$M \rightarrow \mathbf{R}_{\iota*}(\tau^{\geq 0} \mathcal{L}_t^* M) \rightarrow N \rightarrow^+ \ .$$

By 0.5.3.6, \( \tau^{\geq 0} \mathcal{L}_t^* M \in \text{obj}(\mathbf{D}_{\text{coh}}(Z)) = \text{obj}(\mathbf{D}_g) \). By (C4), the middle term of (*) belongs to \( \mathbf{D}_f \). Hence by (C1) it suffices to show that \( N \) belongs to \( \mathbf{D}_f \). To this end, we shall show that the canonical morphism \( N \rightarrow \mathbf{R}_{\iota*}(\tau^{\geq 0} \mathcal{L}_t^* N) \) is an isomorphism. Consider the cohomology exact sequence of (*); since \( \iota \) is affine, by 0.5.4.2 (2) we have

$$\mathcal{H}^q(\mathbf{R}_{\iota*}(\tau^{\geq 0} \mathcal{L}_t^* M)) = \begin{cases} \iota_* \iota^* \mathcal{H}^0(M) & \text{if } q = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Hence the initial portion of the cohomology exact sequence gives

$$0 \rightarrow \mathcal{H}^{-1}(N) \rightarrow \mathcal{H}^0(M) \rightarrow \iota_* \iota^* \mathcal{H}^0(M) \rightarrow \mathcal{H}^0(N) \rightarrow 0.$$  

Since \( \iota_* \iota^* \mathcal{H}^0(M) \cong \mathcal{H}^0(M)/\mathcal{I}\mathcal{H}^0(M) \), we deduce that \( \mathcal{H}^{-1}(N) = \mathcal{I}\mathcal{H}^0(M) \) and \( \mathcal{H}^0(N) = 0 \). Other parts of the cohomology exact sequence imply that \( \mathcal{H}^q(\mathbf{R}_{\iota*}(\tau^{\geq 0} \mathcal{L}_t^* M)) = \mathcal{H}^q(N) = 0 \) for \( q > 0 \). Hence \( N \) is concentrated in degree \(-1\), and \( \mathcal{H}^{-1}(N) = \mathcal{I}\mathcal{H}^0(M) \), which is a coherent sheaf on \( X \) (for \( \iota \) is of finite presentation). Since \( \mathcal{I}^2 = 0 \), we deduce that the canonical morphism \( N \rightarrow \mathbf{R}_{\iota*}(\tau^{\geq 0} \mathcal{L}_t^* N) \) is an isomorphism.

Then again by 0.5.3.6 we have \( \tau^{\geq 0} \mathcal{L}_t^* N \in \text{obj}(\mathbf{D}_{\text{coh}}(Z)) = \text{obj}(\mathbf{D}_g) \) and using (C4) and (C0) that \( N \) belongs to \( \mathbf{D}_f \), as desired.
Claim 2. Consider the diagram as in (C4), where \( \iota \) is a closed immersion. Suppose \( \mathbf{D}_g = \mathbf{D}^b_{\text{coh}}(Z) \). Then for any \( M \in \text{obj}(\mathbf{D}^b_{\text{coh}}(X)) \) such that \( H^q(M) \) is supported on \( Z \) for any \( q \), we have \( M \in \text{obj}(\mathbf{D}_f) \).

By a similar reduction process as above, we may assume that \( M \) is concentrated in degree 0. If \( \mathcal{J} \) is the ideal defining \( Z \) in \( X \), there exists a positive integer \( n \) such that \( \mathcal{J}^n H^0(M) = 0 \). By Claim 1, we may further assume \( n = 1 \), that is, \( \mathcal{J} H^0(M) = 0 \). Then the canonical morphism \( M \twoheadrightarrow \mathbb{R}L^0(M) \) is an isomorphism. Since \( \tau_{\leq 0} L^* M \) belongs to \( \mathbf{D}^b_{\text{coh}}(Z) \), we have \( M \in \text{obj}(\mathbf{D}_f) \) by (C4) and (C0).

To proceed, we need the following lemma.

Lemma 8.3.2 (carving lemma). Let \( B \) be any ring, and \( f : X \to Y = \text{Spec } B \) an algebraic space separated and of finite presentation over \( B \). Then there exists a sequence of closed subspaces

\[
X = X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_r \supsetneq X_{r+1} = \emptyset
\]

such that, for any \( 0 \leq i \leq r \),

(a) the closed immersion \( \iota_{i+1} : X_{i+1} \hookrightarrow X_i \) is of finite presentation,

(b) the complement \( X_i \setminus X_{i+1} \) (cf. [72], II.5.12) is quasi-projective, and

(c) there exists a projective morphism \( \pi_i : \tilde{X}_i \to X_i \) of finite presentation such that \( \pi_i \) is an isomorphism over the open subspace \( X_i \setminus X_{i+1} \) and that the composition \( \tilde{X}_i \to Y \) is quasi-projective.

Proof. Take a subring \( B' \) of \( B \) of finite type over \( Z \) and an algebraic space \( X' \) over \( Y' = \text{Spec } B' \) of finite presentation such that there exists a Cartesian diagram

\[
\begin{array}{ccc}
X & \to & X' \\
\downarrow f & & \downarrow f' \\
Y & \to & Y'.
\end{array}
\]

This is possible by the standard argument well known for schemes ([54], IV, §8). We leave the details to the reader as an exercise (Exercise 1.8.2). The algebraic space \( X' \) is then a Noetherian algebraic space, and hence any descending sequence of closed subspaces terminates (cf. [72], II.5.18).

Suppose \( X' = X'_0 \) is not quasi-projective over \( Y' \). Take a non-empty open subspace \( U'_1 \) of \( X' \) quasi-projective over \( Y' \), and let \( X'_1 \) be its complement. If \( X'_1 \) is not quasi-projective, then we repeat this procedure to take \( X'_2 \) with non-empty quasi-projective complement. Since \( X' \) is Noetherian, this procedure stops after finitely many steps. Thus we get the sequence of closed subspaces

\[
X' = X'_0 \supsetneq X'_1 \supsetneq \cdots \supsetneq X'_r \supsetneq X'_{r+1} = \emptyset.
\]
Now we apply Chow’s lemma (stated below) to each $X'_i$ and $U'_i = X'_i \setminus X'_{i+1}$ to get the $U'_i$-admissible blow-up $\tilde{X}'_i \to X'_i$ such that the composition $X'_i \to Y'$ is quasi-projective. Pulling back all these data by the map $Y \to Y'$ onto $Y$, we get the desired sequence of subspaces of $X$.

**Theorem 8.3.3** (Chow’s lemma [89], Première partie, (5.7.14)). Let $Y$ be a coherent algebraic space, $X \to Y$ a separated $Y$-algebraic space of finite type, and $U$ an open subspace quasi-projective over $Y$. Then there exists a $U$-admissible blow-up $W \to X$ such that $X \to Y$ is quasi-projective over $Y$.

Here a blow-up $\tilde{X} \to X$ is said to be $U$-admissible if it is isomorphic to the blow-up along a closed subspace of $X$ of finite presentation disjoint from $U$ (cf. II.E.1.4).

8.3. (c) Proof of Proposition 8.3.1. Let $f: X \to \Spec B$ be an object of $\mathbf{PAs}_B$, and apply 8.3.2 to $f$ to get a sequence of closed subspaces $(\ast)$. Since $f$ is proper, the morphisms $f_i: X_i \to \Spec B$ and $\tilde{f}_i = f_i \circ \pi_i: \tilde{X}_i \to \Spec B$ are proper for $0 \leq i \leq r$. In particular, $f_i$ are projective, and hence $\tilde{X}_i$ are schemes. Note also that $X_r = \tilde{X}_r$, which is therefore a projective scheme over $B$.

We prove the assertion by induction with respect to $r$. If $r = 0$, then $X = X_0$ is projective, and the result follows from (C2). In general, we may assume by induction that $D_{f_1} = D^{b}_{\coh}(X_1)$. In what follows, for the sake of brevity, we write $Z = X_1, g = f_1, \tilde{X} = \tilde{X}_0, \pi = \pi_0, \text{and } \iota = \iota_1$. Thus we are in the situation depicted as follows:

Here $\iota$ is a closed immersion, $\pi$ and $f \circ \pi$ are projective, and $\pi$ is the identity over $X \setminus \iota(Z)$; moreover, we already know that $D_g = D^{b}_{\coh}(Z)$.

Let $M \in \text{obj}(D^{b}_{\coh}(X))$. We are going to show that $M$ belongs to $D_f$. Similarly to the proof of Claim 1, we may assume that $M$ is concentrated in degree 0. Embed the canonical map $M \to R\pi_\ast(\tau_{\geq 0}L\pi^\ast M)$ in the distinguished triangle

$$M \longrightarrow R\pi_\ast(\tau_{\geq 0}L\pi^\ast M) \longrightarrow N \overset{+1}{\longrightarrow}.$$  

By 0.5.3.6, (C2), and (C3), the middle term belongs to $D_f$. Hence by (C1) it suffices to show that $N$ belongs to $D_f$. But by Claim 2 it is enough to verify that all cohomologies of $N$ are supported on $Z$. 

Since $\tau^{\geq 1} R\pi_* (\tau^{\geq 0} L\pi^* M) \cong \tau^{\geq 1} N$ and since $\pi$ is isomorphic over the complement of $Z$ in $X$, $\tau^{\geq 1} N$ has the cohomologies supported on $Z$. Moreover, looking at the initial portion of the cohomology exact sequence of $(*)$,

$$0 \rightarrow \mathcal{H}^{-1}(N) \rightarrow \mathcal{H}^0(M) \rightarrow \mathcal{H}^0(R\pi_*(\tau^{\geq 0} L\pi^* M)) \rightarrow \mathcal{H}^0(N) \rightarrow 0,$$

and recalling the equality $\mathcal{H}^0(R\pi_*(\tau^{\geq 0} L\pi^* M)) = \pi_* \pi^* \mathcal{H}^0(M)$, we deduce that $\mathcal{H}^{-1}(N)$ and $\mathcal{H}^0(N)$ are supported on $Z$. Hence all cohomologies of $N$ are supported on $Z$, as desired.  

8.4 Proof of Theorem 8.1.3

8.4. (a) Reduction process. We assume without loss of generality that $Y$ is affine, $Y = \text{Spec } B$, where $B$ is universally coherent. For any object $(f : X \to \text{Spec } B) \in \text{obj}(\mathcal{PAs}_B)$ we define

$$D_f \subseteq D^b_{\text{coh}}(X)$$

to be the full subcategory consisting of all objects $M \in \text{obj}(D^b_{\text{coh}}(X))$ such that $Rf_* M \in \text{obj}(D^b_{\text{coh}}(\text{Spec } B))$. We want to show that $D_f = D^b_{\text{coh}}(X)$ for any $f \in \text{obj}(\mathcal{PAs}_B)$, and hence it suffices to check the conditions in 8.3.1.

We postpone the checking of $(C2)$, which we suppose is verified for the time being. Both $(C0)$ and $(C1)$ are obviously satisfied. As for $(C3)$, since we already know $(C2)$, $R\pi_*$ maps any object $M$ of $D^b_{\text{coh}}(X)$ to an object of $D^b_{\text{coh}}(X)$. Then, since $Rf_*(R\pi_* M) = Rg_* M$, we have $R\pi_* M \in D_f$. Thus $(C3)$ is satisfied. Finally, in the diagram in $(C4)$, since $\iota$ is of finite presentation, $R\iota_*$ maps an object of $D^b_{\text{coh}}(Z)$ to an object of $D^b_{\text{coh}}(X)$ $(0.5.4.4)$. Hence $(C4)$ can be verified similarly.

8.4. (b) End of the proof. Now we finish the proof of 8.1.3 by checking $(C2)$. By a similar reduction argument as in §8.2. (b), we may assume $X = \mathbb{P}^r_B$. Indeed, if $\iota : X \hookrightarrow \mathbb{P}^r_B$ is the $B$-closed immersion of finite presentation, then $R\iota_*$ maps an object $M$ of $D^b_{\text{coh}}(X)$ to $D^b_{\text{coh}}(\mathbb{P}^r_B)$ $(0.5.4.4)$; if it is proved that the theorem is true for $\mathbb{P}^r_B \to \text{Spec } B$, then it is also true for $X \to \text{Spec } B$ by the composition formula for right derived functors.

Now if $X = \mathbb{P}^r_B$, then by induction with respect to $\text{amp}(M)$ $(0.4.4.9)$ using the distinguished triangles of the form

$$\tau^{\leq n} M \rightarrow M \rightarrow \tau^{\geq n} M \rightarrow ^{+1}$$

and 8.2.4, we may assume $\text{amp}(M) = 0$; by a suitable shift we may further assume that $M$ is concentrated in degree 0. In this case $M$ is represented by a single sheaf. But the assertion in this case is nothing but 8.2.1 (1), which completes the proof.
8.5 Application to \( I \)-goodness

**Proposition 8.5.1.** Let \( f: X \to Y \) be as in 8.1.3 and \( S = \bigoplus_{k \geq 0} S_k \) a quasi-coherent positively graded \( \mathcal{O}_Y \)-algebra of finite presentation. Set \( S^T = f^* S \). Let \( \mathcal{M} = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k \) be a quasi-coherent graded \( S' \)-module of finite presentation. Then \( \mathcal{R}^q f_* \mathcal{M} = \bigoplus_{k \in \mathbb{Z}} \mathcal{R}^q f_* \mathcal{M}_k \) is a quasi-coherent graded \( S \)-module of finite presentation for any \( q \geq 0 \). Moreover, there exists a positive integer \( r \) such that for any \( q \geq r \) we have \( \mathcal{R}^q f_* \mathcal{M} = 0 \).

The first half of the proposition can be shown by the same reasoning as in [54], III, (3.3.1), and hence as in [54], III, (2.4.1), where we use 8.1.3 instead of [54], III, (2.2.1). Since the proof is quite similar to those results, we are not going to repeat it. The last half follows from the proof of [54], III, (3.3.1), and 8.1.1.

**Proposition 8.5.2.** Let \( (B, I) \) be a universally adhesive pair (0.8.5.4) and suppose \( B \) is universally coherent. Let \( f: X \to Y = \text{Spec} B \) be a proper morphism of finite presentation between algebraic spaces and \( \mathcal{F} \) a coherent \( \mathcal{O}_X \)-module. Then for any \( q \geq 0 \) the filtration \( F^\bullet = \{F^n\}_{n \geq 0} \) on the \( B \)-module \( H^q(X, \mathcal{F}) \) by \( F^n = \text{image}(H^q(X, I^n \mathcal{F}) \to H^q(X, \mathcal{F})) \) is \( I \)-good (0.7.4.2).

**Proof.** It is clear that \( I^m F^n \subseteq F^{n+m} \). Consider the \( I \)-torsion part \( \mathcal{F}_{I \text{-tor}} \), and take \( m \geq 0 \) such that \( I^n \mathcal{F} = 0 \) for any \( n \geq m \). Set \( \mathcal{F}_1 = \mathcal{F}/\mathcal{F}_{I \text{-tor}} \). We first claim that \( I^n \mathcal{F} \cong I^n \mathcal{F}_1 \) for \( n \gg 0 \). To see this, we may assume that \( X \) is affine, \( X = \text{Spec} A \), where \( A \) is a finitely presented \( B \)-algebra; note that \( A \) is \( I \)-adically universally adhesive and universally coherent. In this situation we have \( \mathcal{F} = \bar{N} \) for a coherent \( A \)-module \( N \). The canonical morphism \( I^n N \to I^n (N/N_{I \text{-tor}}) \) is clearly surjective, and its kernel is \( I^n N \cap N_{I \text{-tor}} \). By 0.8.5.16, the filtration \( \{I^n N \cap N_{I \text{-tor}}\}_{n \geq 0} \) on \( N_{I \text{-tor}} \) is equivalent to the \( I \)-adic one, and we conclude that \( I^n N \cap N_{I \text{-tor}} = 0 \) for \( n \gg 0 \), i.e., \( I^n N \cong I^n (N/N_{I \text{-tor}}) \), as claimed. The claim shows that, in order to check the \( I \)-goodness of \( F^\bullet \), we may replace \( \mathcal{F} \) by \( \mathcal{F}_1 \) and hence may assume that \( \mathcal{F} \) is \( I \)-torsion free.

Now, consider the Rees algebra

\[
S = R(B, I) = \bigoplus_{n \geq 0} I^n
\]

(cf. 0, §7.5) and the graded \( S \)-module \( M = \bigoplus_{n \geq 0} H^q(X, I^n \mathcal{F}) \). We claim that \( M \) is a finitely generated \( S \)-module; if this is shown, then \( \bigoplus_{n \geq 0} F^n \) is, as a quotient of \( M \), a finitely generated \( S \)-module, and hence the desired result follows from 0.7.5.1.

Set \( S_1 = S/S_{I \text{-tor}} \). Then \( S_1 \) is a finitely presented \( B \)-algebra; indeed, it is clearly of finite type, and hence we have a surjective map \( B[X_1, \ldots, X_n] \to S_1 \); since \( B[X_1, \ldots, X_n] \) is adhesive and \( S_1 \) is \( I \)-torsion free, the kernel is finitely generated. Set \( S = \bar{S} \) (resp. \( S_1 = \bar{S}_1 \)), and \( S' = f^* S \) (resp. \( S'_1 = f^* S_1 \)).
Then $S_1$ is a quasi-coherent positively graded $\mathcal{O}_Y$-algebra of finite presentation. Let $\mathcal{M} = \bigoplus_{n \geq 0} I^n \mathcal{F}$, which is a quasi-coherent graded $S'$-algebra of finite type. We have $H^q(X, \mathcal{M}) = M$. Since $\mathcal{F}$ is $I$-torsion free, $\mathcal{M}$ is $I$-torsion free and hence carries the canonical $S_1'$-module structure; moreover, by the adhesiveness of $S_1'$, $\mathcal{M}$ is finitely presented as an $S_1'$-module. Hence we deduce that the cohomology $H^q(X, \mathcal{M}) = M$ is a finitely presented $S_1$-module (8.5.1); in particular, it is finitely generated as an $S_1$-module, as desired. \[\square\]

\section*{Exercises}

\textbf{Exercise I.8.1.} Let $B$ be a ring, $f : X \to Y = \text{Spec } B$ a quasi-compact separated morphism of algebraic spaces, and $\mathcal{L}$ an $f$-ample invertible sheaf on $X$. Suppose $\mathcal{O}_X$ is coherent on $X$. Show that the canonical functor

$$D^*(\text{Coh}_X) \to D^*_\text{coh} (\text{QCoh}_X)$$

is an equivalence of triangulated categories for $* = -, b$.

\textbf{Exercise I.8.2.} Let $A$ be a ring and $X$ a locally separated (resp. separated) $A$-algebraic space of finite presentation. Then there exist a subring $A_0$ of $A$ of finite type over $\mathbb{Z}$ and a locally separated (resp. separated) $A_0$-algebraic space $X_0$ of finite presentation such that $X_0 \otimes_{A_0} A \cong X$.

\section{9 GFGA comparison theorem}

In this and the next sections, we discuss GFGA (= géométrie formelle et géométrie algébrique) theorems, which generalize the classical GFGA theorems in [54], III. The generalization will be done in the following two directions. First, with the application to rigid geometry in mind, we drop the Noetherian hypothesis and replace it by weaker ones, like universally adhesive, etc. Second, we will argue entirely in the derived categorical language.

In this section, we state and prove the GFGA comparison theorem. The theorem will be stated in the first subsection, §9.1. In §9.2 we give the classical version, the comparison of the cohomologies without derived categorical setting, but considering topologies, along with the formalism of [54], III, §4.1. Note that even this classical version avoids Noetherian hypotheses, and thus gives a generalization of the corresponding theorem in [54], III, §4.1. §9.3 is devoted to the proof of the theorem, in which we again use the carving method developed in §8.3. The final subsection, §9.4, presents, as a corollary, a subsidiary result on comparison of Ext modules, which will be referred to in the next section.
9.1 Formulation of the theorem

9.1. (a) Formal completion functor. Let \((X, W)\) be a pseudo-adhesive pair of algebraic spaces\(^2\) (0.8.6.5) such that \(X\) is universally cohesive (0.5.1.1). Note that such a situation is attained if \((X, W)\) is universally adhesive and \(\mathcal{O}_X\) is \(I_W\)-torsion free or, more generally, if the pair \((X, W)\) is of finite presentation over another such pair (0.8.6.8). For example, one can consider a pair \((X, W)\), where \(X\) is finitely presented over an \(a\)-adically complete valuation ring of arbitrary height and \(W\) is defined by the ideal \(a\).

Set \(\hat{X} = X|_W\), the formal completion of \(X\) along \(W\) (cf. §6.3. (f)), and let \(j: \hat{X} \to X\) be the canonical morphism of formal algebraic spaces. By \(I\)-adic completion (where \(I = I_W\), the defining ideal of \(W\) in \(X\)), we have the functor

\[
\text{Mod}_X \longrightarrow \text{Mod}_{\hat{X}}, \quad \mathcal{F} \longmapsto \widehat{\mathcal{F}} = \mathcal{F}|_W.
\]

But by 1.4.7 (1) this functor restricted to coherent \(\mathcal{O}_X\)-modules is canonically equivalent to the functor

\[
\mathcal{F} \longmapsto j^*\mathcal{F}.
\]

Guided by this, instead of the completion functor, we consider the functor

\[
\text{for: } \text{Mod}_X \longrightarrow \text{Mod}_{\hat{X}}, \quad \mathcal{F} \longmapsto \mathcal{F}^{\text{for}} = j^*\mathcal{F},
\]

which is exact, since \(j\) is flat, and hence induces an exact functor of triangulated categories

\[
\mathbf{D}^*(X) \longrightarrow \mathbf{D}^*(\hat{X}),
\]

for \(\ast = \text{""}, +, -, \text{b}\). We write the functor thus obtained as

\[
M \longmapsto M^{\text{for}}.
\]

Note that, if \(M \in \text{obj}(\mathbf{D}_{\text{coh}}^*(X))\), then

\[
\mathcal{H}^q(M^{\text{for}}) = j^*\mathcal{H}^q(M) = \mathcal{H}^q(M).
\]

Remark 9.1.1. If, moreover, the formal algebraic space \(\hat{X}\) is universally cohesive, then we have the exact functor

\[
\mathbf{D}_{\text{coh}}^*(X) \longrightarrow \mathbf{D}_{\text{coh}}^*(\hat{X}), \quad M \longmapsto M^{\text{for}}
\]

(for \(\ast = \text{""}, +, -, \text{b}\), the so-called comparison functor. Note that under our hypothesis in the beginning of this subsection, \(\hat{X}\) is universally cohesive if \((X, W)\) is finitely presented over a complete t.u. adhesive pair \((A, I)\) such that \(A\) is \(I\)-torsion free (7.2.2). For example, if \(X\) is finitely presented over an \(a\)-adically complete valuation ring \(V\) of arbitrary height and \(W\) is the closed subspace defined by \((a)\), then \(\hat{X}\) is universally cohesive.

\(^2\)See 0, §5.2. (a) and the beginning of §7 for our general conventions for algebraic spaces and formal algebraic spaces.
9. (b) The statement. In the sequel we work in the following situation.

**Situation 9.1.2.** Let \( (Y, Z) \) be a universally pseudo-adhesive pair of algebraic spaces with \( Y \) universally cohesive, and \( f: X \to Y \) a proper and finitely presented morphism. Set \( W = f^{-1}(Z) \). Set \( \hat{X} = X|_W \) and \( \hat{Y} = Y|_Z \), and consider the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{j} & \hat{X} \\
\downarrow{f} & & \downarrow{\hat{f}} \\
Y & \xrightarrow{i} & \hat{Y}
\end{array}
\]

which is Cartesian in the category of formal algebraic spaces (cf. 1.4.5). For any \( k \geq 0 \) we denote by \( X_k \) (resp. \( Y_k \)) the closed subspace of \( X \) (resp. \( Y \)) defined by the ideal \( I^{k+1}\mathcal{O}_X \) (resp. \( I^{k+1} \)), where \( I \) is the defining ideal of \( Z \) on \( Y \).

By 8.1.3 and 8.1.4 we already know that the functor \( Rf_* \) maps \( D^*_\text{coh}(X) \) to \( D^*_\text{coh}(Y) \) for \( * = \text{"} \), \( + \), \( - \), \( b \). Let \( D^*_\text{aqcoh}(\hat{X}) \) be the full subcategory of \( D^*(\hat{X}) \) consisting of the objects having adically quasi-coherent cohomologies in all degrees (\( D^*_\text{aqcoh}(\hat{X}) \) may not be a triangulated category). We have the diagram

\[
\begin{array}{ccc}
D^*_\text{coh}(X) & \xrightarrow{\text{for}} & D^*_\text{aqcoh}(\hat{X}) \\
Rf_* \downarrow & & \downarrow R\hat{f}_* \\
D^*_\text{coh}(Y) & \xrightarrow{\text{for}} & D^*(\hat{Y})
\end{array}
\]

for \( * = \text{"} \), \( + \), \( b \). Note that we can extend the domain of the functor \( Rf_* \) to the whole \( D(\hat{X}) \) due to the fact that \( R\hat{f}_* \) has finite cohomology dimension (7.1.3); cf. [34], C.D. Chapter 2, §2, n° 2, Corollary 2. Hence one can consider diagram (*) also for \( * = \text{"} \), \( - \).

We are going to construct a natural transformation (comparison map)

\[
\rho = \rho_f: \circ Rf_* \longrightarrow R\hat{f}_* \circ \text{for}.
\]

As is well known, there exists a canonical natural transformation

\[
i^{-1} \circ Rf_* \longrightarrow R\hat{f}_* \circ j^{-1}.
\]

Indeed, for any object \( M \) of \( D(X) \) we represent \( M \) by a complex \( \mathcal{I}^* \) consisting of injective \( \mathcal{O}_X \)-modules. Then we have the chain of canonical morphisms

\[
i^{-1}Rf_*M \sim i^{-1}f_*\mathcal{I}^* \longrightarrow \hat{f}_*j^{-1}\mathcal{I}^* \sim R\hat{f}_*j^{-1}M.
\]
where the first and the last arrows are quasi-isomorphisms; note that the last arrow is a quasi-isomorphism because $j^{-1} \mathcal{F}^\bullet$ gives a flasque resolution of $j^{-1} M$. Since $i$ and $j$ are flat as maps of locally ringed spaces, one can extend the above morphism to

$$\text{(R} f_* M)^{\text{for}} \longrightarrow \text{R} \hat{f}_* M^{\text{for}}.$$  

Moreover, since the formation of this morphism is canonical, we get the desired natural transformation $\rho$. Therefore, we finally get the diagram

$$\begin{array}{ccc}
\text{D}^*_{\text{coh}}(X) & \overset{\text{for}}{\longrightarrow} & \text{D}^*_{\text{acoh}}(\hat{X}) \\
R f_* & & R \hat{f}_* \\
\text{D}^*_{\text{coh}}(Y) & \overset{\text{for}}{\longrightarrow} & \text{D}^*(\hat{Y}).
\end{array} \quad (**)$$

Now our main theorem of this section is stated as follows.

**Theorem 9.1.3** (GFGA comparison theorem). Diagram $(**)$ is 2-commutative, that is, $\rho$ gives a natural equivalence for $*=\text{"+"}, \text{+, −, b.}$

### 9.2 The classical comparison theorem

Before proceeding to the proof of 9.1.3, let us mention a classical version of the comparison theorem (the generalized classical comparison theorem) along with the formalism of [54], III, §4.1.

We continue working in the situation of 9.1.2. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, and consider the completion $\hat{\mathcal{F}} = \mathcal{F}|_W$. We set

$$\mathcal{F}_k = \mathcal{F} / I^{k+1} \mathcal{F}, \quad k \geq 0.$$  

Then we have $\hat{\mathcal{F}} = j^{-1} \lim \leftarrow_k \mathcal{F}_k$ and the commutative diagram of cohomologies

$$\begin{array}{ccc}
\text{R}^q f_* \mathcal{F} & \overset{\rho_q}{\longrightarrow} & \text{R}^q \hat{f}_* \hat{\mathcal{F}} \\
\varphi_q & & \psi_q \\
\lim \leftarrow_k \text{R}^q f_* \mathcal{F}_k & & \lim \leftarrow_k \text{R}^q \hat{f}_* \mathcal{F}_k
\end{array} \quad (*)$$

for $q \geq 0$, constructed as follows.

**Construction of $\varphi^q$.** The canonical map $\text{R}^q f_* \mathcal{F} \to \text{R}^q f_* \mathcal{F}_k$ factors through

$$(\text{R}^q f_* \mathcal{F}) \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y / I^{k+1} \mathcal{O}_Y) \longrightarrow \text{R}^q f_* \mathcal{F}_k,$$

from which we obtain $\varphi^q$ by passage to the projective limits $\lim \leftarrow_k$. 

Construction of $\psi^q$. We look at the commutative diagram

$$
\begin{array}{ccc}
X_k & \xleftarrow{\iota_k} & X \\
\downarrow^{h_k} & & \downarrow^j \\
\hat{X} & \xrightarrow{\iota_k} & \hat{X}
\end{array}
$$

where $h_k$ and $\iota_k$ are the closed immersions. Since we have the canonical isomorphism $\mathcal{F}_k \cong (\iota_k)_*(\iota_k)^* \mathcal{F}_k$, we have

$$H^q(X_k, (\iota_k)^* \mathcal{F}_k) \cong H^q(X, \mathcal{F}_k).$$

On the other hand, by 1.4.7 (1) we know that $\hat{\mathcal{F}} = j^* \mathcal{F}$. Hence we have the map

$$H^q(\hat{X}, \hat{\mathcal{F}}) \to H^q(X, \mathcal{F}_k),$$

and hence the map

$$H^q(\hat{X}, \hat{\mathcal{F}}) \to \lim_{\leftarrow k} H^q(X, \mathcal{F}_k).$$

Do the same for all schemes of the form $f^{-1}(V)$ with $V$ étale over $Y$ and apply 0.5.4.7 to obtain the desired map $\psi^q$.

Construction of $\rho^q$. This is essentially done in §9.1.(b). By 8.1.3, we already know that $R^q f_* \mathcal{F}$ is a coherent $\mathcal{O}_Y$-module. By virtue of 1.4.7 (1), we have

$$R^q f_* \mathcal{F} \cong i^* R^q f_* \mathcal{F},$$

and hence we get $\rho^q$ as the composition

$$R^q f_* \mathcal{F} \cong i^* R^q f_* \mathcal{F} \to R^q f_*(j^* \mathcal{F}) \cong R^q f_* \hat{\mathcal{F}}$$

Diagram (*) thus obtained is commutative; to see this, we replace $Y$ by open subsets to reduce to the commutativity of

$$
\begin{array}{ccc}
H^q(X, \mathcal{F}) & \xrightarrow{\rho^q} & H^q(\hat{X}, \hat{\mathcal{F}}) \\
\downarrow^{\psi^q} & & \downarrow^{\psi^q} \\
\lim_{\leftarrow k} H^q(X, \mathcal{F}_k). & & 
\end{array}
$$

But this follows from the commutativity of

$$
\begin{array}{ccc}
H^q(X, \mathcal{F}) & \xrightarrow{\rho^q} & H^q(\hat{X}, j^* \mathcal{F}) \\
\downarrow^{\psi^q} & & \downarrow^{\psi^q} \\
H^q(X_k, \iota_k^* \mathcal{F}) & & 
\end{array}
$$

which is obvious thanks to the fact that $\iota_k = j \circ h_k$. 
Now the classical (but generalized to our situation) comparison theorem is stated as follows.

**Theorem 9.2.1.** In the above situation, suppose, moreover, that \((Y, Z)\) is universally adhesive. Then the morphisms \(\rho^q, \varphi^q\) and \(\psi^q\) in diagram (\(\ast\)) are bicontinuous isomorphisms, where the topology on \(R^q f_* \mathcal{F}\) is the \(I\)-adic topology, and the topologies on \(R^q f_* \widehat{\mathcal{F}}\) and \(\lim_k \leftarrow R^q f_* \mathcal{F}_k\) are the ones given by the filtration by the kernels of the canonical maps to \(R^q f_* \mathcal{F}_k\) for \(k \geq 0\).

**Proof of Theorem 9.1.3 \(\iff\) Theorem 9.2.1.** Assume that \(Y\) is an affine scheme, \(Y = \text{Spec } B\), and let \(I\) be the ideal of \(B\) corresponding to \(\mathcal{F}\). It suffices to show that the maps \(\rho^q, \varphi^q\) and \(\psi^q\) in diagram (\(\ast\ast\)) are bicontinuous isomorphisms.

By 9.1.3 applied to the sheaf \(\mathcal{F}\) (regarded as an object of \(D^b_{\text{coh}}(X)\)), the map

\[
\rho^q : H^q(X, \mathcal{F}) \longrightarrow H^q(\hat{X}, \hat{\mathcal{F}})
\]

is an isomorphism. Since \(\hat{\mathcal{F}}\) is the projective limit of \(\{\mathcal{F}_k\}\), by 0.3.2.16 the map \(\psi^q\) is surjective. On the other hand, the filtration \(F^\bullet = \{F^n\}_{n \geq 0}\) on the \(B\)-module \(H^q(X, \mathcal{F})\) given by

\[
F^n = \text{image}(H^q(X, I^n \mathcal{F}) \to H^q(X, \mathcal{F})) = \ker(H^q(X, \mathcal{F}) \to H^q(X, \mathcal{F}_{n-1}))
\]

is \(I\)-good by 8.5.2. Hence the map \(\varphi^q\) is injective; indeed, \(\overline{H^q(X, \mathcal{F})}\) is topologically isomorphic to the projective limit of \(H^q(X, \mathcal{F})/F^{k+1}\) that are mapped injectively into \(H^q(X, \mathcal{F}_k)\). The injectivity of \(\varphi^q\) now follows from the left-exactness of \(\lim_k \leftarrow\).

Therefore, since (\(\dagger\)) is an isomorphism, we deduce that \(\varphi^q\) and \(\psi^q\) are isomorphisms. Once \(\varphi^q\) is known to be isomorphic, it is automatically topologically isomorphic, since the filtration \(\{F^\bullet\}\) as above is \(I\)-good (and hence both \(\overline{H^q(X, \mathcal{F})}\) and \(\lim_k H^q(X, \mathcal{F}_k)\) are complete with respect to the same filtration). Also, since \(\psi^q\) is now known to be an isomorphism, it is automatically a bicontinuous isomorphism due to the definition of the topologies. This yields the other assertions of the theorem.

**Remark 9.2.2.** Note that our proof of 9.2.1 differs in structure from the proof in [54], III, (4.1.5). In loc. cit. the proof was given by showing that the morphisms \(\varphi_q\) and \(\psi_q\) in diagram (\(\ast\)) are isomorphisms; the isomorphy of the former was established by the \(I\)-goodness of the filtration \(F^\bullet\) as above, and of the latter by (ML) (cf. 0, §3.2.(a)) for the projective system of the cohomologies. But here we first prove that \(\rho_q\) is an isomorphism (disregarding the topologies); then it turns out, as we saw above, that the isomorphy of the remaining maps is almost automatic.
9.3 Proof of Theorem 9.1.3

9.3. (a) Reduction process. By a similar argument as in 8.1.4, it suffices to show the theorem only in the case $* = b$. We may assume that $Y$ is affine, $Y = \text{Spec } B$, and let $I \subseteq B$ the ideal corresponding to $I$. The pair $(B, I)$ is universally adhesive, and the ring $B$ is universally coherent. We apply 8.3.1 to the following situation. For each object $f : X \to \text{Spec } B$ of $\text{PAs}_B$, we define

$$D_f \subseteq D^b_{\text{coh}}(X)$$

to be the full subcategory consisting of objects $M$ such that the comparison morphism

$$\rho_f(M) : (Rf_*M)^{\text{for}} \longrightarrow R\hat{f}_*M^{\text{for}}$$

is an isomorphism in $D(\hat{Y})$.

We claim that, provided that $(C2)$ of 8.3.1 is verified, the other conditions are fulfilled, thereby finishing the proof of 9.1.3. $(C0)$ is trivially satisfied, while $(C1)$ is immediately verified by considering the morphism of exact triangles

$$(Rf_*K)^{\text{for}} \longrightarrow (Rf_*L)^{\text{for}} \longrightarrow (Rf_*M)^{\text{for}} \overset{+1}{\longrightarrow} \approx$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$$R\hat{f}_*K^{\text{for}} \longrightarrow R\hat{f}_*L^{\text{for}} \longrightarrow R\hat{f}_*M^{\text{for}} \overset{+1}{\longrightarrow} \approx$$

Next, let us check $(C3)$: Let $M$ be an object of $D^b_{\text{coh}}(\tilde{X}) = D_g$. Then $R\pi_*M$ belongs to $D^b_{\text{coh}}(X)$ (by 8.1.3). Since $\pi$ is projective, by our assumption that $(C2)$ is true, we have $(R\pi_*M)^{\text{for}} \approx R\hat{\pi}_*M^{\text{for}}$. Then

$$(Rf_*R\pi_*M)^{\text{for}} = (R\hat{\pi}_*M)^{\text{for}} \approx R\hat{\pi}_*M^{\text{for}} = R\hat{f}_*R\hat{\pi}_*M^{\text{for}} = R\hat{f}_*(R\pi_*M)^{\text{for}}.$$

and so $(C3)$ holds.

Finally, if $i$ is a closed immersion as in the diagram of $(C4)$, the comparison map $(Ri_*M)^{\text{for}} \rightarrow R\hat{i}_*M^{\text{for}}$ is shown to be a quasi-isomorphism as follows. By induction with respect to amp$(M)$, and using shifts and distinguished triangles of the form

$$\tau^{\leq n} M \longrightarrow M \longrightarrow \tau^{\leq n+1} M \overset{+1}{\longrightarrow}$$

(using $(C1)$ verified above), we may assume that $M$ is concentrated in degree 0. Then $M$ is represented by a coherent sheaf $F$ on $X$. By 0.5.4.2 (2) and 7.1.1 (2), the $q$-th cohomologies of the both sides are $\tau_*\mathcal{H}^q(F)$ (the completion of $\tau_*\mathcal{H}^q(F)$) and $\hat{i}_*\mathcal{H}^q(F^{\text{for}})$, which are isomorphic to each other, since the completion functor is exact for coherent sheaves. Hence $(C4)$ is verified.
**9.3 (b) Projective case.** Thus we may restrict to the projective case; clearly, by a similar reduction process as in 8.2 (b), using 0.5.4.2 (2) and 7.1.1 (2) in much the same way as above, we may assume \( X = \mathbb{P}^r_B \). Let \( M \) be an object of \( D^b_{\text{coh}}(X) \). As we saw in checking (C4) above, we may assume that \( M \) is concentrated in degree 0 and hence is represented by a coherent sheaf \( \mathcal{F} \) on \( X \).

By 8.2.3, we have an exact sequence

\[ \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0 \]

of sheaves \( \mathcal{E}_i \) of the form \( \mathcal{O}_X(n_i) \oplus k_i \). To compute \( q \)-cohomologies for a fixed \( q \), we can truncate the complex \( \{ \mathcal{E}_* \} \) to a bounded complex

\[ \cdots \to 0 \to \mathcal{E}_{q+1} \to \cdots \to \mathcal{E}_0 \to 0 \to \cdots, \]

and \( \mathcal{F} \) can be replaced by this complex. Hence, it suffices to show the theorem in the case \( \mathcal{F} \) is a line bundle \( \mathcal{O}_X(n) \).

Set \( \mathcal{F} = \mathcal{O}_X(n) \), and let \( X_k \) and \( \mathcal{F}_k = \mathcal{F} / I^{k+1} \mathcal{F} \) be defined as in 9.1.2. The cohomology \( H = H^q(X, \mathcal{F}) \) (resp. \( H_k = H^q(X_k, \mathcal{F}_k) \)) is a free \( B \)-module of finite rank (resp. a free \( B_k \)-module of finite rank); obviously, \( H_k = H / I^{k+1} H \) due to the explicit description of the cohomology group 0.5.4.8. Consequently,

\[ H^q(X, \mathcal{F}) = \lim_{\leftarrow k} H^q(X_k, \mathcal{F}_k). \]

Now since diagram (**) in §9.2 is commutative, it suffices to show that

\[ H^q(\tilde{X}, \lim_{\leftarrow k} \mathcal{F}_k) \to \lim_{\leftarrow k} H^q(X_k, \mathcal{F}_k) \]

is an isomorphism. But since the projective system \( \{ H^q(X_k, \mathcal{F}_k) \}_{k \geq 0} \) is strict for any \( q \), the desired isomorphy follows from 0.3.2.16. Therefore, the theorem in the projective case is proved and hence, by virtue of 8.3.1, the proof of Theorem 9.1.3 is finished.

**9.4 Comparison of Ext modules**

**Proposition 9.4.1.** Let \((B, I)\) be a universally adhesive pair with \( B \) universally coherent, and \( f : X \to Y = \text{Spec} \ B \) a proper morphism of finite presentation of algebraic spaces. Then for \( M \in \text{obj}(D^-_{\text{coh}}(X)) \) and \( N \in \text{obj}(D^+_{\text{coh}}(X)) \) we have the canonical isomorphism

\[ R \text{Hom}_{\mathcal{O}_X}(M, N) \cong R \text{Hom}_{\mathcal{O}_X}(M^\text{for}, N^\text{for}) \]

in \( D^+(B) \) (cf. 0, §4.3. (c) for the \( B \)-module structures on both sides).
**Proof.** First note that by 0.5.3.5 (2) we have $R\mathcal{H}om_X(M, N) \in D^+(X)$. Hence, by 9.1.3, we have

$$(Rf_*R\mathcal{H}om_X(M, N))^\text{for} \cong R\hat{f}_*(R\mathcal{H}om_X(M, N))^\text{for}. $$

On the other hand, since $j: \hat{X} \to X$ is flat,

$$(R\mathcal{H}om_X(M, N))^\text{for} \cong R\mathcal{H}om_{\hat{X}}(M^\text{for}, N^\text{for}) $$

([54], 0III. (12.3.5)). The result follows from this. □

**Corollary 9.4.2.** Let $(B, I)$ be a complete universally adhesive pair with $B$ universally coherent, and $X$ a proper $B$-algebraic space of finite presentation. Then the comparison functor

$$D^b_{\text{coh}}(X) \xrightarrow{\text{for}} D^b(\hat{X})$$

is fully faithful.

**Proof.** We need to prove that the canonical map

$$\text{Hom}_{D(X)}(M, N) \rightarrow \text{Hom}_{D(\hat{X})}(M^\text{for}, N^\text{for})$$

is bijective for $M, N \in D^b(X)$. First note that the left-hand side is isomorphic to $H^0(R\mathcal{H}om_{\theta_X}(M, N))$, which is a finitely presented $B$-module by 8.1.4. Since $B$ is complete, it is complete by 0.8.2.18 (1). Then by 9.4.1 we have

$$H^0(R\mathcal{H}om_{\theta_X}(M, N)) \cong H^0(R\mathcal{H}om_{\theta_{\hat{X}}}(M^\text{for}, N^\text{for}))$$

(as abelian groups); but the latter module is nothing but $\text{Hom}_{D(\hat{X})}(M^\text{for}, N^\text{for})$. □

**10 GFGA existence theorem**

**10.1 Statement of the theorem**

**Situation 10.1.1.** Let $B$ be a t.u. adhesive ring (2.1.1 (2)) with a finitely generated ideal of definition $I \subseteq B$, and suppose that $B$ is topologically universally coherent (0.8.5.22) with respect to $I$. Let $f: X \to Y = \text{Spec } B$ be a proper morphism of algebraic spaces of finite presentation. We will use the notation as in 9.1.2. Note that the algebraic space $X$ is universally cohesive (0.5.2.1), and the formal algebraic space $\hat{X}$ is universally adhesive and universally cohesive (7.2.2).

An important special case of the above situation is as follows: $B$ and $I$ are as above, and $B$ is $I$-torsion free or, more generally, $B$ is topologically finitely presented over a t.u. adhesive ring of this kind (0.8.5.25 (2)). For example, $B$ can be a topologically finitely presented $V$-algebra with $I = aB$, where $V$ is an $a$-adically complete valuation ring of arbitrary height.

In this section we are going to prove the following theorem.
**Theorem 10.1.2** (GFGA existence theorem). *In the situation of 10.1.1, the comparison functor
\[ D^b_{\text{coh}}(X) \xrightarrow{\text{for}} D^b_{\text{coh}}(\hat{X}) \] is an exact equivalence of triangulated categories.*

Note that we have already shown in 9.4.2 that the functor (*) is fully faithful. Hence what to prove here is that the functor is essentially surjective, in other words, every object of \( D^b_{\text{coh}}(\hat{X}) \) is algebraizable.

**10.2 Proof of Theorem 10.1.2**

10.2. (a) Modification of the carving method. Theorem 10.1.2 will be proved similarly to 8.1.3 and 9.1.3 by means of the carving method introduced in 8.3, with a slight modification as follows.

We consider the category \( \text{PAs}_B \) (full subcategory of the category \( \text{As}_B \) of \( B \)-algebraic spaces) consisting of proper and finitely presented \( B \)-algebraic spaces. For any object \( f: X \to \text{Spec} B \) we denote by \( \hat{f}: \hat{X} \to \text{Spf} B \) its \( I \)-adic completion.

**Proposition 10.2.1** (carving method (formal version)). *Suppose for each object \( f: X \to \text{Spec} B \) of \( \text{PAs}_B \) we are given a full subcategory \( D_f \subseteq D^b_{\text{coh}}(\hat{X}) \) such that the following conditions are satisfied.

\begin{enumerate}
\item[(C0)] The zero object 0 belongs to \( D_f \), and \( D_f \) is stable under isomorphisms in \( D^b_{\text{coh}}(\hat{X}) \).
\item[(C1)] Let \( K \to L \to M \xrightarrow{+1} \) be an exact triangle in \( D^b_{\text{coh}}(\hat{X}) \). If two of \( K, L, \) and \( M \) are in \( D_f \), then so are the rest.
\item[(C2)] If \( f \) is projective, then \( D_f = D^b_{\text{coh}}(\hat{X}) \).
\item[(C3)] Consider a morphism \( \pi \) in \( \text{PAs}_B \), which amounts to the same as a commutative diagram of the form
\[ \begin{array}{ccc}
\hat{X} & \xrightarrow{\pi} & X \\
\downarrow g & & \downarrow f \\
\text{Spec} B & & \\
\end{array} \]

and suppose \( g \) and \( \pi \) are projective. Then \( R\hat{\pi}_* \) maps \( D_g = D^b_{\text{coh}}(\hat{X}) \) to \( D_f \).
\end{enumerate}
(C4) Consider a closed immersion \( \iota \) in \( \text{PAs}_B \), that is, a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\iota} & X \\
\downarrow{g} & & \downarrow{f} \\
\text{Spec } B & & \\
\end{array}
\]

where \( \iota \) is a closed immersion. Then if \( D_g = D_{\text{coh}}^b(\tilde{Z}) \), the functor \( R\iota_* \) maps \( D_g \) to \( D_f \).

Then we have \( D_f = D_{\text{coh}}^b(X) \) for any object \( f : X \to \text{Spec } B \) of \( \text{PAs}_B \).

The proof can be done parallel to that of 8.3.1; we only indicate the necessary changes and leave the checking of details to the reader.

On Claim 1. Since \( \iota \) is (automatically) finitely presented, the ideal \( \mathcal{J} \) is of finite type, and hence \( \mathcal{J}\mathcal{O}_{\tilde{X}} \) (closed by 0.7.4.17 and 0.7.4.18) is the defining ideal of the induced closed immersion \( \tilde{\iota} : \tilde{Z} \hookrightarrow \tilde{X} \) (cf. 4.3.11). In particular, it is a square zero ideal. Then all the rest of the proof can be done parallel; we use 7.3.5 instead of 0.5.3.6, and 7.1.1 (2) instead of 0.5.4.2 (2).

On Claim 2. As above, \( \mathcal{J}\mathcal{O}_{\tilde{X}} \) is the defining ideal of \( \tilde{Z} \). Using 7.3.5 instead of 0.5.3.6, one can prove the claim in our version in much the same way; here we need to show the following lemma, which we can show by an argument similar to that in the proof of [54], III, (5.3.4), using our already proven comparison Theorem 9.2.1.

**Lemma 10.2.2.** Let \( f : Z \to X \) be a proper morphism in the category \( \text{PAs}_B \), and \( \mathcal{J} \subseteq \mathcal{O}_X \) a coherent ideal. Set \( U = X \setminus V(\mathcal{J}) \), and suppose that \( f : f^{-1}(U) \to U \) is an isomorphism. Then for any coherent \( \mathcal{O}_{\tilde{X}} \)-module \( \mathcal{F} \) there exist an integer \( n > 0 \) such that the kernel and the cokernel of the map \( \mathcal{F} \to \hat{f}_*\hat{f}^*\mathcal{F} \) is annihilated by \( \mathcal{J}^n \).

The rest of the proof goes just in parallel, involving the carving lemma (8.3.2) in a similar manner as before.

10.2. (b) Reduction process. Thus it suffices to show that the conditions in 10.2.1 are satisfied, when we take \( D_f \) to be the full subcategory of \( D_{\text{coh}}^b(X) \) consisting of objects \( M \) that are algebraizable, that is, there exists \( M_0 \in \text{obj}(D_{\text{coh}}^b(X)) \) such that \( M \cong M_0^{\text{for}} \).

Condition (C0) is trivially satisfied. The following proposition verifies (C1).
Proposition 10.2.3. Consider the situation as in 10.1.1, and let

$$K \rightarrow L \rightarrow M \xrightarrow{+1}$$

be an exact triangle in $\mathcal{D}^b_{\text{coh}}(\hat{X})$. If two of $K$, $L$, and $M$ are algebraizable, then so is the rest.

Proof. Suppose $K$ and $L$ are algebraizable, and take $K_0$ and $L_0$ in $\mathcal{D}^b_{\text{coh}}(X)$ such that

$$K \cong K_0^\text{for} \quad \text{and} \quad L \cong L_0^\text{for}.$$ 

Since $\text{Hom}_{\mathcal{D}^b_{\text{coh}}(X)}(K_0, L_0) \cong \text{Hom}_{\mathcal{D}^b_{\text{coh}}(\hat{X})}(K, L)$ (9.4.2), we have a map $K_0 \rightarrow L_0$ that induces $K \rightarrow L$. Embed $K_0 \rightarrow L_0$ into an exact triangle

$$K_0 \rightarrow L_0 \rightarrow M_0 \xrightarrow{+1}$$

in $\mathcal{D}^b_{\text{coh}}(X)$. Then since the functor for is exact, it induces the exact triangle

$$K \rightarrow L \rightarrow M_0^\text{for} \xrightarrow{+1},$$

whence $M_0^\text{for} \cong M$ ([101], Chapter II, 1.2.4), that is, $M$ is algebraizable. The other cases can be reduced to the above case by a shift. $\square$

Assuming (C2), let us verify (C3). Since we already know that $\mathcal{D}_f = \mathcal{D}^b_{\text{coh}}(\hat{X})$, we may begin with an object of the form $M^\text{for}$ with $M \in \mathcal{D}^b_{\text{coh}}(\hat{X})$. Then by 9.1.3 we have $R\pi_*M^\text{for} \cong (R\pi_*M)^\text{for}$, which is obviously an object of $\mathcal{D}^b_{\text{coh}}(\hat{X})$ (since $R\pi_*M$ belongs to $\mathcal{D}^b_{\text{coh}}(X)$ by 8.1.3; cf. 9.1.1) and hence belongs to $\mathcal{D}_f$.

Condition (C4) can be checked similarly.

10.2. (c) Projective case. Now let us prove 10.1.2 in the case where $f$ is a projective morphism (hence $X$ is a scheme); by what we have established above, this finishes the proof of 10.1.2. Clearly, by a similar reduction process as in 9.3. (b) (using 7.1.1 (2)), we may assume $X = \mathbb{P}^r_B$. To proceed to the main routine of the proof, we need to show some preparatory results.

Proposition 10.2.4. Consider the situation as in 10.1.1, and suppose there exists an $f$-ample line bundle $\mathcal{L}$ on $X$. Then for any coherent $\mathcal{O}\hat{X}$-module $\mathcal{F}$ there exists an integer $N \geq 0$ such that

(a) for $n \geq N$ we have $R^q \hat{f}_*\mathcal{F}(n) = 0$ for any $q > 0$, and $\hat{f}_*\mathcal{F}(n)$ is a coherent sheaf on $\hat{Y}$, and

(b) for $n \geq N$ the canonical morphism $\hat{f}^*\hat{f}_*\mathcal{F}(n) \rightarrow \mathcal{F}(n)$ is surjective.

For the proof we need the following lemma.
Lemma 10.2.5. In the situation as in 10.2.4, suppose \( B \) is \( I \)-torsion free. Let \( \mathcal{F} \) be a coherent \( \mathcal{O}_X \)-module. There exists an integer \( N \geq 0 \) such that

\[
H^q(X_0, I^k \mathcal{F}(n)/I^{k+1} \mathcal{F}(n)) = 0
\]

for any \( n \geq N, k \geq 0 \), and \( q > 0 \) and that \( \bigoplus_{k \geq 0} H^0(X_0, I^k \mathcal{F}(n)/I^{k+1} \mathcal{F}(n)) \) for any \( n \geq N \) is a module of finite presentation over the graded ring \( \text{gr}_I^*(B) = \bigoplus_{k \geq 0} I^k/I^{k+1} \).

Proof. Set \( R = \text{gr}_I^*(B) = \bigoplus_{k \geq 0} I^k/I^{k+1} \). We first claim that \( R \) is a finitely presented \( B \)-algebra. Indeed, if \( R(B, I) \) denotes the Rees algebra (0, §7.5), we have \( R = R(B, I)/IR(B, I) \). Since \( R(B, I) \) is \( I \)-torsion free and of finite type over \( B \), it is of finite presentation over \( B \). Since \( I \) is finitely generated, \( R \) is of finite presentation over \( B \), as desired.

Now set \( X' = X \otimes_B R \), and let \( j: X' \to X \) be the canonical morphism. Since the ring \( R \) is annihilated by \( I \), the formal completion \( \hat{X} \) coincides with \( X' \). Hence the sheaf \( j^* \mathcal{F}(n) = \bigoplus_{k \geq 0} I^k \mathcal{F}(n)/I^{k+1} \mathcal{F}(n) \) is a coherent sheaf on \( X' \), and thus we may apply 8.2.1. Set \( M^q = \bigoplus_{k \geq 0} H^0(X_0, I^k \mathcal{F}(n)/I^{k+1} \mathcal{F}(n)) \). Then \( M^q = H^q(X', j^* \mathcal{F}(n)) \), and it follows from 8.2.1 that there exists \( N \) such that for \( n \geq N \) we have \( M^q = 0 \) for \( q > 0 \) and \( M^0 \) is a finitely presented \( R \)-module. This is exactly what we wanted to show. \( \square \)

Proof of Proposition 10.2.4. First, we claim that it is enough to prove the proposition in the case where \( \mathcal{F} \) is \( I \)-torsion free. Indeed, consider the exact sequence

\[
0 \longrightarrow \mathcal{F}_{I\text{-tor}}(n) \longrightarrow \mathcal{F}(n) \longrightarrow (\mathcal{F}/\mathcal{F}_{I\text{-tor}})(n) \longrightarrow 0 \quad (\ast)
\]

for any \( n \). Since \( \hat{X} \) is universally adhesive, the \( I \)-torsion free \( (\mathcal{F}/\mathcal{F}_{I\text{-tor}})(n) \) is of finite presentation, hence is coherent. Therefore, \( \mathcal{F}_{I\text{-tor}} \) is a coherent sheaf on \( \hat{X} \). Since \( \hat{X} \) is quasi-compact, one can take \( k \geq 0 \) such that \( I^{k+1} \mathcal{F}_{I\text{-tor}} = 0 \). Hence if we put \( X_k = (X, \mathcal{O}_X/I^{k+1} \mathcal{O}_X) \) and \( B_k = B/I^{k+1} \), then \( \mathcal{F}_{I\text{-tor}} \) is isomorphic to the direct image of a coherent sheaf \( \mathcal{G} \) on \( X_k \) (here we used 0.5.4.5 (1)); since \( B_k \) is a finitely presented \( B \)-algebra, we can apply 8.2.1 and 8.1.2 to \( \mathcal{G} \) and \( X_k \to \text{Spec} \, B_k \) to deduce that the properties (a) and (b) are valid for \( \mathcal{F}_{I\text{-tor}} \). So, considering the cohomology exact sequence for \( (\ast) \) for a sufficiently large \( n \), we deduce easily that (a) is valid also for \( \mathcal{F} \); as for (b), by the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{F}_{I\text{-tor}}(n) & \longrightarrow & \mathcal{F}(n) & \longrightarrow & (\mathcal{F}/\mathcal{F}_{I\text{-tor}})(n) & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
& \hat{f}^* \hat{f}_* \mathcal{F}_{I\text{-tor}}(n) & \longrightarrow & \hat{f}^* \hat{f}_* \mathcal{F}(n) & \longrightarrow & \hat{f}^* \hat{f}_*(\mathcal{F}/\mathcal{F}_{I\text{-tor}})(n) & \longrightarrow & 0,
\end{array}
\]

the result follows from the snake lemma.
Thus, we can assume $\mathcal{F}$ is $I$-torsion free. But then one can further assume that $B$ is $I$-torsion free. Indeed, set $B' = B/B_{I\text{-tor}}$; since $B_{I\text{-tor}}$ is finitely generated and $B$ satisfies (APf), $B'$ is $I$-adically complete (0.7.4.17). Since $B'$ is finitely presented and topologically finitely presented $B$-algebra, $B'$ is topologically universally coherent (0.8.5.23 (2)). Consider the Cartesian diagram

\[
\begin{array}{ccc}
\widehat{X}' & \longrightarrow & \widehat{X} \\
\downarrow & & \downarrow \\
\text{Spf } B' & \longrightarrow & \text{Spf } B
\end{array}
\]

where the two horizontal arrows are closed immersions (4.3.7, 4.3.10 (2)). Since $\mathcal{F}$ is $I$-torsion free, there exists a coherent sheaf $\mathcal{G}$ on $\widehat{X}'$ such that $\mathcal{F}$ coincides up to isomorphism with the direct image of $\mathcal{G}$. Hence, if the proposition is proved for $\mathcal{G}$ (and for $\widehat{X}' \rightarrow \text{Spf } B'$), it is also true for $\mathcal{F}$; here we use 7.1.2 (2) and the fact that any finitely presented $B'$-module is finitely presented as $B$-module.

Therefore, we may assume that $B$ is $I$-torsion free and apply 10.2.5. Take $N \geq 0$ as in 10.2.5, and fix $n \geq N$. Starting from Lemma 10.2.5, one can show recursively that

\[ H^q(X_m, I^k \mathcal{F}(n) / I^{k+m+1} \mathcal{F}(n)) = 0 \]

for $m \geq 0$ and $q > 0$. This shows that the map

\[ H^0(X_{m+1}, I^k \mathcal{F}(n) / I^{k+m+2} \mathcal{F}(n)) \longrightarrow H^0(X_m, I^k \mathcal{F}(n) / I^{k+m+1} \mathcal{F}(n)) \]

is surjective. Since

\[ H^q(\widehat{X}, I^k \mathcal{F}(n)) = \lim_{m \geq 0} H^q(X_m, I^k \mathcal{F}(n) / I^{k+m+1} \mathcal{F}(n)), \]

we deduce by 0.3.2.16 that

\[ H^q(\widehat{X}, I^k \mathcal{F}(n)) = 0 \]  

for $q > 0$ and $k \geq 0$, which already proves the first half of (a).

On the other hand, by (*), 10.2.5, and 0.7.5.2, the filtration $\{\hat{f}_* I^k \mathcal{F}(n)\}_{k \geq 0}$ on $\hat{f}_* I^k \mathcal{F}(n)$ is $I$-good. This means that

\[ \hat{f}_* \mathcal{F}(n) = \lim_{\leftarrow k} \hat{f}_* (\mathcal{F}(n) / I^k \mathcal{F}(n)) = \lim_{\leftarrow k} \hat{f}_* (\mathcal{F}(n) / I^k \hat{f}_* \mathcal{F}(n)), \]

where the first equality follows from the left exactness of $\hat{f}_*$ and (*). Since

\[ I^k \hat{f}_* \mathcal{F}(n) \subseteq \hat{f} I^k \mathcal{F}(n), \]
we have the exact sequence
\[ 0 \rightarrow \hat{f}_* \mathcal{F}(n)/I^k \hat{f}_* \mathcal{F}(n) \rightarrow \hat{f}_* \mathcal{F}(n)/I^k \hat{f}_* \mathcal{F}(n) \rightarrow 0, \]
where the last term, equal to \( \hat{f}_*(\mathcal{F}(n)/I^k \mathcal{F}(n)) \), is known to be coherent due to 8.1.3. Take a sufficiently large \( l > 0 \) such that \( \hat{f}_* I^l \mathcal{F}(n) \subseteq I^k \hat{f}_* \mathcal{F}(n) \). We have the exact sequence
\[ 0 \rightarrow I^k (\hat{f}_* \mathcal{F}(n)/\hat{f}_* I^l \mathcal{F}(n)) \rightarrow \hat{f}_* I^k \mathcal{F}(n)/\hat{f}_* I^l \mathcal{F}(n) \rightarrow 0, \]
where, by a similar reasoning as above, the first two terms are coherent. Hence the last term is coherent, and consequently \( \hat{f}_* \mathcal{F}(n) \) is coherent for any \( k \). Hence by 7.2.4, \( \hat{f}_* \mathcal{F}(n) \) is coherent, whence the rest of (a).

To show (b), we take an integer \( N' \geq 0 \) such that, in view of 0.5.4.10, the sheaf \( \mathcal{F}/I \mathcal{F}(n) \) on \( X_0 \) is generated by global sections for \( n \geq N' \). If \( n \geq \max(N, N') \), then (b) holds, since the map \( H^0(\tilde{X}, \mathcal{F}(n)) \rightarrow H^0(\tilde{X}, \mathcal{F}/I \mathcal{F}(n)) \) is surjective. \( \square \)

**Corollary 10.2.6.** In the above situation with \( X = \mathbb{P}_B^n \), let \( \mathcal{F} \) be a coherent sheaf on \( \tilde{X} \). Then we have the exact sequence of the form
\[ \cdots \rightarrow \mathcal{E}_m \rightarrow \mathcal{E}_{m-1} \rightarrow \cdots \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0, \]
where each \( \mathcal{E}_i \) is a free \( \mathcal{O}_{\tilde{X}} \)-module of the form \( \mathcal{O}_{\tilde{X}}(n_i)^{\oplus k_i} \).

**Proof.** By 10.2.4, for a large \( n \) the sheaf \( \mathcal{F}(n) \) is generated by global sections, that is, the morphism \( f^* \hat{f}_* \mathcal{F}(n) \rightarrow \mathcal{F}(n) \) is surjective. Now \( \hat{f}_* \mathcal{F}(n) \) is a coherent sheaf on \( \tilde{Y} = \text{Spf} \, B \), and hence is of the form \( M^\Lambda \) for a \( B \)-module \( M \) of finite presentation (3.5.6). There exists \( n_0 > 0 \) such that we have a surjective map
\[ B^{n_0} \rightarrow M. \]
Pulling it back to \( \tilde{X} \) and composing with the surjection
\[ f^* \hat{f}_* \mathcal{F}(n) \rightarrow \mathcal{F}(n), \]
we get a surjection \( \mathcal{O}_{\tilde{X}}^{\oplus k_0} \rightarrow \mathcal{F}(n) \), whence \( \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0 \) as above. The kernel of this map is again coherent, and hence one can repeat the procedure. \( \square \)
Now we are going to finish the proof of the theorem in the case $X = \mathbb{P}^r_B$, which will establish 10.1.2. Let $M \in \text{obj}(\mathcal{D}^b_{\text{coh}}(X))$. By induction with respect to \text{amp}(M) and using shifts and distinguished triangles of the form
\[ \tau^{\leq n} M \to M \to \tau^{\geq n+1} M \to \]
(and applying 10.2.3), we may assume that $M$ is concentrated in degree 0 and hence is represented by a coherent sheaf $\mathcal{F}$. Then by 10.2.6 and 10.2.3 (since 10.2.3 implies that for any morphism $\mathcal{F} \to \mathcal{G}$ of algebraizable coherent sheaves the kernel and the cokernel are algebraizable), it suffices to show that the sheaves on $\hat{X}$ of the form $\mathcal{O}_X(n)$ are algebraizable, but this is trivial.

### 10.3 Applications

**Proposition 10.3.1.** Let $(B, I)$ be a complete t.u. adhesive pair with $B$ topologically universally coherent with respect to $I$. Then the functor given by $I$-adic formal completion
\[ \text{PSch}_B \to \text{Fs}_B, \quad X \mapsto \hat{X}, \]
from the category of proper $B$-schemes of finite presentation to the category of formal $B$-schemes is fully faithful.

**Proof.** We need to show that for two objects $X$ and $Y$ of $\text{PSch}_B$ the map
\[ \text{Hom}_{\text{PSch}_B}(X, Y) \to \text{Hom}_{\text{Fs}_B}(\hat{X}, \hat{Y}) \]
is bijective. By considering graphs one easily sees that this map is injective; indeed, any graph in $X \times_B Y$ is a closed subspace of finite presentation, and hence one can apply 9.4.2. To show the surjectivity, take a morphism $\Gamma : \hat{X} \to \hat{Y}$ in the right-hand side. This amounts to the same as taking the graph $\Gamma$; the morphism $\Gamma : \hat{X} \times_B \hat{Y}$ is a closed immersion of finite presentation due to 4.6.12. Let $\mathcal{J}$ be the ideal defining $\Gamma$. Then $\mathcal{J}$ is a coherent ideal on $\hat{X} \times_B \hat{Y}$. By 1.4.6 and 10.1.2, we have a coherent ideal $\mathcal{J}_0$ on $X \times_B Y$ (unique up to isomorphism) such that $\mathcal{J}_0^{\text{for}} = \mathcal{J}$. The ideal $\mathcal{J}_0$ defines a closed subspace $\Gamma_0$ of $X \times_B Y$ such that $\hat{\Gamma}_0 \cong \Gamma$. The projection $\Gamma_0 \to X$ is an isomorphism, as so is its completion (Exercise I.10.1). Hence there exists a composite map $X \to Y$ followed by the other projection, whose formal completion coincides with the morphism $\hat{X} \to \hat{Y}$ we started with. \[ \Box \]

**Proposition 10.3.2.** Let $(B, I)$ be a complete t.u. adhesive pair with $B$ topologically universally coherent with respect to $I$, and $X$ a proper $B$-formal algebraic space of finite presentation. Suppose there exists an invertible sheaf $\mathcal{L}$ on $X$ such that $\mathcal{L}_0$ is ample on $X_0$, where $X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X)$ and $\mathcal{L}_k = \mathcal{L}/I^{k+1}\mathcal{L}$ for $k \geq 0$. Then $(X, \mathcal{L})$ is algebraizable, that is, there exist a proper $B$-scheme $\hat{Y}$ of finite presentation and an ample invertible sheaf $\mathcal{M}$ on $Y$ such that $\hat{Y} \cong X$ and $\hat{\mathcal{M}} \cong \mathcal{L}$. (In particular, the scheme $Y$ is projective.)
This can be shown in a similar way to the proof of [54], III, (5.4.5) using 10.2.4 instead of [54], III, (5.2.3). We are not going to repeat the argument here and leave checking details to the reader.

Exercises

Exercise I.10.1 (cf. [54], III, (4.6.8)). Consider the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
g \downarrow & & \downarrow h \\
Z & \xleftarrow{g} & h \\
\end{array}
\]

of schemes, where \( g \) and \( h \) are proper of finite presentation. Let \( W \subseteq Z \) be a closed subscheme of finite presentation. Suppose that the pair \((Z, W)\) is universally adhesive. Set \( \hat{Z} = Z|_W \), \( \hat{Y} = Y|_{h^{-1}(W)} \), and \( \hat{X} = X|_{g^{-1}(W)} \), and consider the resulting diagram

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{f}} & \hat{Y} \\
\hat{g} \downarrow & & \downarrow \hat{h} \\
\hat{Z} & \xleftarrow{\hat{g}} & \hat{h} \\
\end{array}
\]

of formal schemes. Show that \( \hat{f} \) is an isomorphism (resp. a closed immersion) if and only if there exists an open neighborhood \( U \subseteq Z \) of \( W \) such that the morphism \( g^{-1}(U) \rightarrow h^{-1}(U) \) induced by \( f \) is an isomorphism (resp. a closed immersion).

11 Finiteness theorem and Stein factorization

The first part, §11.1, of this section is devoted to the finiteness theorem for proper morphisms between universally adhesive formal schemes. The proof consists of three steps. First, we prove the theorem for a formal scheme with invertible ideal of definition. The theorem in this particular case has been essentially established in [99]; we provide the full proof of this case for the reader’s convenience. The second step deals with the case where the map is a so-called admissible blow-up (cf. II, §1). The final step combines these results to show the general case.

The second part, §11.3, discusses the Stein factorizations for universally adhesive formal schemes. This part refers to the general theory of the Stein factorization for schemes, presented in Appendix §A of this chapter.

11.1 Finiteness theorem for proper morphisms

**Theorem 11.1.1.** Let \( f : X \rightarrow Y \) be a proper morphism of finite presentation of universally adhesive formal schemes, and suppose \( Y \) is universally cohesive. Then the functor \( Rf_* \) maps \( D^*_\text{coh}(X) \) to \( D^*_\text{coh}(Y) \) for \( * = \text{“}, +, -, b \).
Note that by 7.1.3 the functor $Rf_*$ is defined on $D_{coh}(X)$ (cf. [34], C.D. Chapter 2, §2, n° 2, Corollary 2). Clearly, it suffices to show the theorem in the case $*=b$. But then, by a standard reduction process as in §8.4. (b) (by induction with respect to amplitudes), one reduces the theorem to the following one.

**Theorem 11.1.2.** Let $f: X \rightarrow Y$ be a proper morphism of finite presentation of universally adhesive formal schemes, and suppose $Y$ is universally cohesive. Then for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf $R^q f_* \mathcal{F}$ is coherent for any $q \geq 0$.

### 11.2 Proof of Theorem 11.1.1

#### 11.2. (a) Invertible ideal case.
First, we show the theorem in the case where $Y$ has an invertible ideal of definition $I$ and $\mathcal{O}_Y$ is $I$-torsion free. Since one may work locally on $Y$, we can assume, without loss of generality, that $Y$ is affine, $Y = \text{Spf} \ A$, and that $A$ has a principal ideal $I = (a)$ of definition and $A$ is $a$-torsion free. Thus we need to establish the following result.

**Proposition 11.2.1.** Let $(A, I)$ with $I = (a)$ be a complete t.u. adhesive pair such that $A$ is $a$-torsion free and $f: X \rightarrow Y = \text{Spf} \ A$ a proper morphism of finite presentation. (In this situation, $X$ and $Y$ are automatically universally adhesive, and $Y$ is universally cohesive.) Then for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, $R^q f_* \mathcal{F}$ is a coherent $\mathcal{O}_Y$-module for any $q \geq 0$.

The theorem in this case was proved by P. Ullrich [99]; the following argument is essentially the same as the one therein.

**Lemma 11.2.2.** In order to show 11.2.1 it suffices that for $q \geq 0$,

(a) the $A$-module $H^q(X, \mathcal{F})$ is coherent (cf. 0.3.3.1 (1)), and

(b) for any $g \in A$ the canonical map

$$H^q(X, \mathcal{F}) \otimes_A A_{(g)} \rightarrow H^q(f^{-1}(V), \mathcal{F}),$$

where $V = \text{Spf} \ A_{(g)} \hookrightarrow Y = \text{Spf} \ A$, is an isomorphism.

Note that (b) is necessary, for we do not know, a priori, that the sheaf $R^q f_* \mathcal{F}$ is $a$-adically quasi-coherent of finite type.

**Proof.** Set $M = H^q(X, \mathcal{F})$. Then by (a) $M$ is $a$-adically complete (0.8.2.18 (1)), and $M^\Delta$ is a coherent $\mathcal{O}_Y$-module. Since $R^q f_* \mathcal{F}$ is the sheafification of the presheaf given by $V \mapsto H^q(f^{-1}(V), \mathcal{F})$, there exists a canonical map

$$H^q(X, \mathcal{F}) \rightarrow \Gamma(X, R^q f_* \mathcal{F}).$$

By what we have shown in §3.2, we have a canonical morphism $M^\Delta \rightarrow R^q f_* \mathcal{F}$ of $\mathcal{O}_Y$-modules. By (b), the induced map between stalks at each point of $Y$ is an isomorphism, and thus we have $M^\Delta \cong R^q f_* \mathcal{F}$, whence the assertion. \qed
Proof of Proposition 11.2.1. We are going to check (a) and (b) in 11.2.2. We fix a finite open covering \( \mathcal{U} = \{ U_\lambda \}_{\lambda \in \Lambda} \) of \( X \) by affine subsets, and consider the Čech complex \( K^* = C^*(\mathcal{U}, \mathcal{F}) \). For any \( q \geq 0 \) we have \( H^q(K^*) = H^q(X, \mathcal{F}) \). If \( V = \text{Spf} \, A_{(g)} \) as in (b) above, set \( \mathcal{U}|_V = \{ U_\lambda \cap f^{-1}(V) \}_{\lambda \in \Lambda} \). Since \( U_\lambda \to Y \) is an affine map for each \( \lambda \in \Lambda \), \( \mathcal{U}|_V \) is a finite open covering of \( f^{-1}(V) \) by affine open subsets. Thus the Čech complex \( L^* = C^*(\mathcal{U}|_V, \mathcal{F}|_{f^{-1}(V)}) \) gives rise to the cohomologies \( H^q(L^*) = H^q(f^{-1}(V), \mathcal{F}) \) for \( q \geq 0 \). By 7.1.3, there exists \( q_0 \geq 0 \) such that for \( q > q_0 \) we have \( H^q(K^*) = H^q(X, \mathcal{F}) = 0 \) and \( H^q(L^*) = H^q(f^{-1}(V), \mathcal{F}) = 0 \). Thus both (a) and (b) are trivially satisfied for \( q > q_0 \). Hence one can check the conditions by descending induction with respect to \( q \). In the following, we assume that both (a) and (b) are true with \( q \) replaced by \( q + 1 \).

Let us verify (a). Before doing it, note that each member \( K^q \) of the complex \( K^* \) is \( a \)-adically complete, for it is finitely generated over a topologically finitely presented \( A \)-algebra (0.8.2.18). By the adhesiveness, the \( a \)-torsion part \( K_{a-tor}^q \) is again finitely generated over a topologically finitely presented \( A \)-algebra. Hence there exists \( n \geq 0 \) such that \( a^n K_{a-tor}^q = 0 \). Moreover, \( C^*(\mathcal{U}, a^{k+1} \mathcal{F}) = a^{k+1} K^* \) and

\[
0 \longrightarrow C^*(\mathcal{U}, a^{k+1} \mathcal{F}) \longrightarrow C^*(\mathcal{U}, \mathcal{F}) \longrightarrow C^*(\mathcal{U}, \mathcal{F}_k) \longrightarrow 0
\]

is exact for any \( k \geq 0 \), where \( \mathcal{F}_k = \mathcal{F}/a^{k+1} \mathcal{F} \), since each \( U_\lambda \) is affine (cf. 7.1.1). Hence for \( k \geq 0 \) we have \( C^*(\mathcal{U}, \mathcal{F}_k) = K_k^* (= K^*/a^{k+1} K^*) \) (cf. 0, §8.8. (a)).

Since \( \mathcal{F} \) and \( a^{k+1} \mathcal{F} \) are coherent, we know by induction that \( H^{q+1}(K^*) \) and \( H^q(a^{k+1} K^*) \) are coherent \( A \)-modules. Since the sheaf \( \mathcal{F}_k \) is coherent on the scheme \( X_k = (X, \mathcal{O}_X/a^{k+1} \mathcal{O}_X) \) (7.2.4) and the map \( f_k: X_k \to Y_k = \text{Spec} \, A_k \) (where \( A_k = A/a^{k+1} A \)) is proper (4.7.3), the cohomology group \( H^q(K_k^*) \) is a coherent \( A_k \)-module by 8.1.3 and hence is also coherent as an \( A \)-module. Now all the hypotheses of 0.8.8.8 are satisfied, and thus we conclude that \( H^q(K^*) \) is coherent.

Next we verify (b). With the notation as above, what to show is that the map

\[
H^q(K^*) \otimes_A B \longrightarrow H^q(L^*),
\]

where \( B = A_{(g)} \), is an isomorphism. Note that since \( (A, a) \) is t.u. adhesive, \( B \) is flat over \( A \). The checking is done by applying 0.8.8.9. To this end, we need to verify that the maps

\[
H^{q+1}(a^{k+1} K^*) \otimes_A B \longrightarrow H^{q+1}(a^{k+1} L^*)
\]

and

\[
H^q(K_k^*) \otimes_A B \longrightarrow H^q(L_k^*)
\]

are isomorphisms for any \( k \geq n \), where \( n \geq 0 \) is an integer such that

\[
a^n K_{a-tor}^{q+1} = 0 \quad \text{and} \quad a^n L_{a-tor}^{q+1} = 0.
\]
The first map coincides with the canonical map
\[ H^{q+1}(X, a^{k+1} F) \otimes_A B \longrightarrow H^{q+1}(f^{-1}(V), a^{k+1} F), \]
which is an isomorphism by induction. The second map coincides with the canonical map
\[ H^q(X, F_k) \otimes_{A_k} B_k \longrightarrow H^q(f^{-1}(V), F_k), \]
where \( B_k = B/a^{k+1}B = (A_k)_g \), which is an isomorphism, since \( R^q f_\ast kF_k \) is coherent by 8.1.3. Hence (b) is verified, and thus the proof of 11.2.1 is complete.

11.2. (b) Blow-up case. We suppose that \( Y \) is affine, \( Y = \text{Spf} \ A \), where \( A \) is a t.u. adhesive ring with a finitely generated ideal of definition \( I \subseteq A \). Consider the blow-up \( g: \text{Proj} \bigoplus_{k \geq 0} I^k \rightarrow \text{Spec} \ A \), which is proper, since \( I \) is finitely generated. Moreover, it is of finite presentation, since \( A \) is universally adhesive and the structure sheaf of \( \text{Proj} \bigoplus_{k \geq 0} I^k \) is \( I \)-torsion free. Consider the case \( f: X = \text{Proj} \bigoplus_{k \geq 0} I^k \rightarrow Y \), that is, \( X \) is the \( I \)-adic completion of \( \text{Proj} \bigoplus_{k \geq 0} I^k \). In this case, by 10.1.2, there exists a coherent sheaf \( \mathcal{G} \) such that \( \mathcal{G} = \mathcal{F} \). Due to 8.1.3, \( R^q g_\ast \mathcal{G} \) is coherent on \( \text{Spec} \ A \). Then by 9.2.1 \( R^q f_\ast \mathcal{F} = R^q g_\ast \mathcal{G} \), which is coherent by 7.2.4. Hence the theorem is true in this case.

Remark 11.2.3. The above map \( f: X = \text{Proj} \bigoplus_{k \geq 0} I^k \rightarrow Y \) is an example of so-called admissible blow-up, which will be discussed thoroughly in II, §1.

11.2. (c) General case. We may assume that \( Y \) is affine, \( Y = \text{Spf} \ A \), where \( A \) is a t.u. adhesive ring with a finitely generated ideal of definition \( I \) such that \( \mathcal{I} = I^\Delta \). Let \( \pi_Y: Y' \rightarrow Y \) be the admissible blow-up along \( I \) as in §11.2. (b), and \( X' \hookrightarrow X \times_Y Y' \) be the closed formal subscheme defined by dividing out \( I \)-torsions (the so-called strict transform of \( X \)). Since \( X \times_Y Y' \) is again universally adhesive, \( X' \hookrightarrow X \times_Y Y' \) is of finite presentation, and hence the induced map \( f': X' \rightarrow Y' \) is finitely presented and proper. Denote the map \( X' \rightarrow Y \) by \( \pi_X' \); it follows that this map is also an admissible blow-up (cf. II.1.2.9). Thus we have the following commutative square:

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\pi_X & \downarrow & \downarrow \pi_Y \\
X & \xrightarrow{f} & Y.
\end{array}
\]

Note that since the ideal of definition \( I \bigotimes_Y \) is invertible, \( f': X' \rightarrow Y' \) satisfies the hypothesis of §11.2. (a), and hence the theorem is true for \( f' \).
11. Finiteness theorem and Stein factorization

Let $M \in \text{obj}(\mathcal{D}_{\text{coh}}(X))$. By induction with respect to the amplitudes (as in §8.4.(b)), we may assume that $M$ is concentrated in degree 0. We want to show that $Rf_\ast M$ belongs to $\mathcal{D}_{\text{coh}}(Y)$ (by 7.1.3 we already know that $Rf_\ast M$ belongs to $\mathcal{D}(Y)$). Consider the object $R\pi_{X_\ast}(\tau_{\geq 0}L\pi_X^* M)$ and the distinguished triangle

$$M \longrightarrow R\pi_{X_\ast}(\tau_{\geq 0}L\pi_X^* M) \longrightarrow N \longrightarrow .$$

Note that by 7.3.5, §11.2.(b) and by the fact that $\pi_X$ is an admissible blow-up, the object in the middle, and hence $N$ also, is an object of $\mathcal{D}_{\text{coh}}(X)$.

Now since $\pi_X$ is also an admissible blow-up along $I$, all the cohomologies $\mathcal{H}^q(N)$ for $q \in \mathbb{Z}$ are $I$-torsion sheaves, and hence there exists $k \geq 0$ such that $I^{k+1}\mathcal{H}^q(N) = 0$ for any $q$. Consider the map of schemes

$$f_k: X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X) \longrightarrow Y_k = \text{Spf } A_k,$$

where $A_k = A/I^{k+1}$, and the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\iota_X} & & \downarrow{\iota_Y} \\
X_k & \xrightarrow{f_k} & Y_k
\end{array}
$$

where the vertical maps are closed immersions. Since $I^{k+1}\mathcal{H}^q(N) = 0$ for any $q$, the canonical morphism $N \rightarrow R\iota_{X_\ast}(\tau_{\geq 0}L\iota_X^* N)$ is an isomorphism in $\mathcal{D}_{\text{coh}}(X)$.

Now, in order to show the theorem, in view of the distinguished triangle

$$Rf_\ast M \longrightarrow Rf_\ast R\pi_{X_\ast}(\tau_{\geq 0}L\pi_X^* M) \longrightarrow Rf_\ast N \longrightarrow ,$$

we need to show that the second and the third terms belong to $\mathcal{D}_{\text{coh}}(Y)$. As for the second term, since

$$Rf_\ast R\pi_{X_\ast}(\tau_{\geq 0}L\pi_X^* M) \cong R\pi_{Y_\ast}Rf_\ast(\tau_{\geq 0}L\pi_X^* M)$$

and $\tau_{\geq 0}L\pi_X^* M$ belongs to $\mathcal{D}_{\text{coh}}(X')$, we deduce from what we have proved in §11.2.(a) and §11.2.(b) that $Rf_\ast R\pi_{X_\ast}(\tau_{\geq 0}L\pi_X^* M)$ belongs to $\mathcal{D}_{\text{coh}}(Y)$. As for the third, since

$$Rf_\ast N \cong Rf_\ast R\iota_{X_\ast}(\tau_{\geq 0}L\iota^* N) \cong R\iota_{k_\ast}Rf_\ast(\tau_{\geq 0}L\iota^* N),$$

and since $\tau_{\geq 0}L\iota^* N$ belongs to $\mathcal{D}_{\text{coh}}(X_k)$, we deduce the desired assertion from 8.1.3 and 7.3.6.

Thus the proof of 11.1.1 is completed.
11.3 Stein factorization

11.3. (a) Statement of the theorem. In this subsection we show the following theorem.

**Theorem 11.3.1** (Stein factorization). Let $A$ be a t.u. adhesive (resp. t.u. rigid-Noetherian) ring with an invertible ideal of definition of the form $I = (a)$, and $f : X \to Y = \text{Spf} A$ a morphism of finite presentation. Let

$$f_k : X_k = (X, \mathcal{O}_X/I^{k+1}\mathcal{O}_X) \to Y_k = \text{Spec} A/I^{k+1},$$

be the induced morphism of schemes for $k \geq 0$. Suppose that $Y$ is universally cohesive, that $f_0$ is pseudo-affine (A.1.1 (2)), and that $\mathcal{O}_X$ is $I$-torsion free. Then the ring $B = \Gamma(X, \mathcal{O}_X)$ is a t.u. adhesive (resp. t.u. rigid-Noetherian) ring with an ideal of definition $IB$, and in the factorization

$$X \xrightarrow{\pi} Z \xrightarrow{\gamma} Y,$$

where $Z = \text{Spf} B$, the map $\pi$ is proper.

Note that the assumption ‘$Y$ is universally cohesive’ is automatic in the t.u. adhesive case due to 0.8.5.25 (2).

The rest of this subsection is devoted to the proof of the theorem. To this end, we need the following proposition, from which the first assertion of 11.3.1 follows.

**Proposition 11.3.2.** Let $f : X \to Y$ be a morphism of finite presentation between coherent universally rigid-Noetherian formal schemes, and let $I$ be an invertible ideal of definition of $Y$ of finite type. Let

$$f_k : X_k = (X, \mathcal{O}_X/I^k\mathcal{O}_X) \to Y_k = (Y, \mathcal{O}_Y/I^k)$$

be the induced morphism of schemes for each $k \geq 0$. Suppose that $Y$ is universally cohesive and that $f_0$ is pseudo-affine. Then $f_*\mathcal{O}_X$ is an a.q.c. $\mathcal{O}_Y$-algebra of finite type.

To show the proposition, since the question is local with respect to $Y$, one can assume that $Y$ is affine, $Y = \text{Spf} A$, where $A$ is a t.u. rigid-Noetherian ring with an invertible principal ideal of definition $I = (a)$. Let $I$ be the ideal of definition of $Y$ such that $I = I^\Delta$.

**Proposition 11.3.3.** Under the assumptions of Theorem 11.3.1, let $\mathcal{F}$ be a coherent sheaf on $X$, and set

$$\mathcal{F}_k = \mathcal{F}/I^{k+1}\mathcal{F}, \quad k \geq 0.$$

Then for any $q \geq 1$ the following holds.
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(a) The filtration \( \{ F^nH^q(X, \mathcal{F}) \}_{n \geq 0} \) on the cohomology \( H^q(X, \mathcal{F}) \) defined by

\[
F^nH^q(X, \mathcal{F}) = \text{image}(H^q(X, I^n\mathcal{F}) \to H^q(X, \mathcal{F}))
\]

for \( n \geq 0 \) gives the \( I \)-adic topology.

(b) The canonical map

\[
H^q(X, \mathcal{F}) \longrightarrow \lim_{\kappa \geq 0} H^q(X, \mathcal{F}_k)
\]

is an isomorphism.

(c) \( H^q(X, \mathcal{F}) \) is coherent as an \( A \)-module.

Proof. The proof is done by descending induction with respect to \( q \geq 1 \) (cf. 7.1.3). We henceforth assume that (a), (b), and (c) are true for any \( \mathcal{F} \) with \( q \) replaced by \( p \) with \( p \geq q + 1 \).

First, since \( f_0 \) is pseudo-affine, \( f_k \) is pseudo-affine for any \( k \geq 0 \) (A.2.2), and that each \( Y_k = \text{Spec } A/I^{k+1} \) is universally cohesive. Note also that for any \( I \)-torsion coherent sheaf \( \mathcal{G} \) on \( X \) there exists \( m \geq 0 \) such that \( I^mH^q(X, \mathcal{G}) = 0 \) for all \( q \geq 0 \), since they are \( A/I^m \)-modules for some \( m \).

To proceed, we fix a finite open covering \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in \Lambda} \) of \( X \) by affine subsets, and consider the Čech complex \( K^\bullet = C^\bullet(\mathcal{U}, \mathcal{F}) \); note that, since \( f_0 \) is pseudo-affine, \( X \) is separated (cf. 4.6.9). For any \( q \geq 0 \) we have \( H^q(K^\bullet) = H^q(X, \mathcal{F}) \). Note also that for each \( p \geq 0 \) the \( A \)-module \( K^p \) is \( a \)-adically complete and we have \( a^nK^p_{\alpha \text{-tor}} = 0 \) for a sufficiently large \( n \). We consider the exact sequence

\[
0 \longrightarrow a^{k+1}K^\bullet \longrightarrow K^\bullet \longrightarrow K^\bullet_k \longrightarrow 0
\]

of complexes for each \( k \geq 0 \) (cf. 0, §8.8. (a)) and the induced filtration given by \( \{ F^nH^q(K^\bullet) \}_{n \geq 0} \) on each cohomology of \( K^\bullet \). Clearly, the this filtration coincides with the one in (a).

Consider the following portion of the cohomology exact sequence

\[
\cdots \longrightarrow H^q(K^\bullet) \longrightarrow H^q(K^\bullet_k) \longrightarrow H^{q+1}(a^{k+1}K^\bullet) \longrightarrow H^{q+1}(K^\bullet) \longrightarrow \cdots .
\]

By the equality \( H^q(K^\bullet_k) = H^q(X, \mathcal{F}_k) \), where \( \mathcal{F}_k \) is an \( a \)-torsion coherent sheaf, and by the induction hypothesis (applied to the cohomologies \( H^{q+1}(a^{k+1}K^\bullet) = H^{q+1}(X, I^{k+1}\mathcal{F}) \) and \( H^{q+1}(K^\bullet) = H^{q+1}(X, \mathcal{F}) \)), the last three of the above exact sequence are coherent \( A \)-modules. Hence, we conclude that the image of \( H^q(K^\bullet) \to H^q(K^\bullet_k) \) is finitely generated. By 0.8.8.6 (2) we deduce that the cohomology \( H^q(K^\bullet) = H^q(X, \mathcal{F}) \) is finitely generated as an \( A \)-module and, moreover, the induced filtration defines the \( I \)-adic topology. Thus we obtain (a).

Next we show (b). The above argument, applied to each coherent sheaf \( I^{k+1}\mathcal{F} \), shows that the hypotheses of 0.8.8.5 (2) (where \( q \) is replaced by \( q - 1 \)) are satisfied.
We deduce that the projective system \( \{H^q(X, \mathcal{F}_k)\}_{k \geq 0} \) satisfies (ML). Hence, by 0.3.2.16, the map in question is an isomorphism.

Finally, let us show (c). By (a) and (b), which we have already proved, \( H^q(X, \mathcal{F}) \) is \( I \)-adically complete and finitely generated (as we have seen above). Since \( H^q(X, \mathcal{F}_k) \) for each \( k \geq 0 \) is coherent, it follows from (a) that the \( A \)-module \( H^q(X, \mathcal{F})/I^{k+1}H^q(X, \mathcal{F}) \) is coherent for sufficiently large \( k \), and hence we deduce that \( H^q(X, \mathcal{F}) \) is coherent (cf. 7.2.4).

11.3. (b) Proof of Proposition 11.3.2. We may assume that \( Y \) is affine, \( Y = \text{Spf} \ A \), where \( A \) is a t.u. rigid-Noetherian ring and that \( I \) comes from an invertible ideal of definition \( I = (a) \) of \( A \) generated by a non-zero-divisor \( a \in A \). Moreover, we may assume that \( \mathcal{O}_X \) is \( I \)-torsion free, since the \( I \)-torsion part is a coherent ideal of \( \mathcal{O}_X \). We look at the exact sequence

\[
0 \longrightarrow \Gamma(X, I^{k+1}\mathcal{O}_X) \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X, \mathcal{O}_{X_k}) \longrightarrow H^1(X, I^{k+1}\mathcal{O}_X),
\]

and set

\[
B_k = \frac{\Gamma(X, \mathcal{O}_X)}{\Gamma(X, I^{k+1}\mathcal{O}_X)} = \frac{\Gamma(X, \mathcal{O}_X)}{I^{k+1}\Gamma(X, \mathcal{O}_X)} \subseteq \Gamma(X, \mathcal{O}_{X_k})
\]

for \( k \geq 0 \). (Note that the equality \( \Gamma(X, I^{k+1}\mathcal{O}_X) = I^{k+1}\Gamma(X, \mathcal{O}_X) \) follows from our assumption that \( I = (a) \) is invertible.) Since \( H^1(X, I^{k+1}\mathcal{O}_X) \) is a coherent \( A \)-module (11.3.3), there exists \( m \geq 0 \) such that \( I \)-torsions of \( H^1(X, I^{k+1}\mathcal{O}_X) \) are annihilated by \( I^m \). Then by an easy diagram chasing we deduce that \( B_k \) coincides with the image of the map \( \Gamma(X, \mathcal{O}_{X_k+m}) \rightarrow \Gamma(X, \mathcal{O}_{X_k}) \). In particular, \( B_k \) is an \( A_k \)-algebra of finite type (since \( f_{k+m} \) is pseudo-affine).

Since

\[
B = \Gamma(X, \mathcal{O}_X) = \varprojlim_{k \geq 0} \Gamma(X, \mathcal{O}_{X_k})
\]

and the \( I \)-adic completion \( \widehat{B} \) is given by \( \varprojlim_{k \geq 0} B_k \), we deduce that there exists an injective morphism \( \widehat{B} \hookrightarrow B \) (cf. 0.3.2.4); since this morphism factorizes the identity map \( \text{id}_B \), it is also surjective. Hence we have \( B \cong \widehat{B} \). Since \( B_0 \) is finitely generated as an \( A_0 \)-module, one has a map \( A \langle X_1, \ldots, X_d \rangle \rightarrow B \) that induces the surjective map \( A_0[X_1, \ldots, X_d] \rightarrow B_0 \). By [27], Chapter III, §2.11, Proposition 14, the ring \( B \) is topologically finitely generated over \( A \).

To show that \( f_*\mathcal{O}_X \) is a.q.c. of finite type, it remains to show that the canonical map

\[
\Gamma(X, \mathcal{O}_X) \otimes_A A_{\{g\}} \rightarrow \Gamma(f^{-1}(V), \mathcal{O}_X),
\]

where \( V = \text{Spf} \ A_{\{g\}} \) with \( g \in A \), is an isomorphism. This is shown by an argument similar to that in the proof of (b) part in the proof of 11.2.1, which uses 0.8.8.9. Thus the proposition is proved. \( \square \)
11.3. (c) Proof of Theorem 11.3.1. It remains to show that \( \pi: X \to Z = \text{Spf} B \) is proper. Let \( X_k \xrightarrow{\pi_k} Z_k = \text{Spec} A'_k \xrightarrow{g_k} Y_k = \text{Spec} A_k \) be the Stein factorization of \( f_k \) for each \( k \geq 0 \). Consider the exact sequence

\[
0 \to I^{k+1} \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_{X_k}) \to H^1(X, I^{k+1} \mathcal{O}_X)
\]

as before and the image of \( \Gamma(X, \mathcal{O}_X)/I^{k+1} \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_{X_k}) \). Since the \( I \)-torsion part of the coherent \( A \)-module \( H^1(X, I^{k+1} \mathcal{O}_X) \) is annihilated by \( I^m \) for some \( m \geq 0 \), the image of \( \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_{X_k}) \) for sufficiently large \( k \) coincides with the image of \( \Gamma(X, \mathcal{O}_{X_{k+m}}) \to \Gamma(X, \mathcal{O}_{X_k}) \). Hence we have the commutative diagram

\[
\begin{array}{ccc}
X_k & \xrightarrow{\pi_k} & Z_k \\
\downarrow & & \downarrow \\
X_{k+m} & \xrightarrow{\pi_{k+m}} & Z_{k+m}
\end{array}
\]

with \( B_k = \Gamma(X, \mathcal{O}_X)/I^{k+1} \Gamma(X, \mathcal{O}_X) \), where the left vertical map is a closed immersion. Since \( \gamma \) is a closed immersion, and since the map and \( \pi_{k+m} \) are proper, we deduce that \( X_k \to \text{Spec} B_k \) is proper. Then, by 4.7.3, \( \pi: X \to Z = \text{Spf} B \) is proper, and thus the proof of 11.3.1 is finished. \( \square \)

A Appendix: Stein factorization for schemes

A.1 Pseudo-affine morphisms of schemes

A.1. (a) Definition and the first properties. Let \( f: X \to Z \) be a locally of finite type morphism of schemes. The subset \( B_f \) of \( Z \) defined by

\[
B_f = \{ z \in Z : \dim_z X \times_Z k(z) \geq 1 \}
\]

is called the center of the morphism \( f \). By [54], IV, (13.1.5), the center \( B_f \) of \( f \) is a closed subset of \( Z \) if \( f \) is a closed map.

Definition A.1.1. Let \( Y \) be a coherent scheme, and \( f: X \to Y \) a morphism of finite presentation between schemes.

1. A pre-Stein factorization is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & Z \\
\downarrow & & \downarrow \\
& \xrightarrow{g} & Y
\end{array}
\]

such that
(a) \( \pi \) is proper and \( g \) is affine of finite presentation (and hence \( \pi \) is of finite presentation), and

(b) the center \( B_\pi \) of \( \pi \) is finite over \( Y \).

(2) If \( f \) has a pre-Stein factorization, \( f \) is said to be \textit{pseudo-affine}.

We say that two pre-Stein factorizations \( f = g \circ \pi = g' \circ \pi' \), where

\[
X \xrightarrow{\pi} Z \xrightarrow{g} Y \quad \text{and} \quad X \xrightarrow{\pi'} Z' \xrightarrow{g'} Y,
\]

are \textit{isomorphic} if there exists an isomorphism \( h: Z \xrightarrow{\sim} Z' \) of schemes such that the following diagram commutes:

\[
\begin{array}{ccc}
X \xrightarrow{\pi} Z & \xrightarrow{g} & Y \\
\downarrow{h} & \equiv & \downarrow{g'} \\
X \xrightarrow{\pi'} Z' & \xrightarrow{g'} & Y
\end{array}
\]

**Definition A.1.2.** A pre-Stein factorization \( f = g \circ \pi \) of \( f: X \to Y \) as in A.1.1 (1) is called a \textit{Stein factorization} if

(c) the canonical morphism \( O_Z \to \pi_* O_X \) is an isomorphism.

The Stein factorization of \( f \) is, if it exists, unique up to isomorphism.

**Proposition A.1.3.** Let \( Y \) be a coherent scheme and \( f: X \to Y \) a morphism of finite presentation. Suppose that \( Y \) is universally cohesive (0.5.1.1) (e.g. Noetherian) and that \( f \) is pseudo-affine. Then \( f \) has a Stein factorization.

**Proof.** Let \( X \xrightarrow{\pi} Z \xrightarrow{g} Y \) be a pre-Stein factorization of \( f \). Since \( g \) is of finite presentation, \( Z \) is universally cohesive. By 8.1.3, \( \pi_* O_X \) is a coherent \( O_Z \)-module, and hence \( Z' = \text{Spec} \pi_* O_X \to Z \) is affine of finite presentation. Then the factorization \( X \to Z' \to Y \) gives a Stein factorization of \( f \). \( \square \)

**Proposition A.1.4.** (1) Proper (resp. affine) morphisms of finite presentation to a coherent scheme are pseudo-affine.

(2) Let \( f: X \to Y \) be pseudo-affine, and \( Y' \to Y \) a morphism of coherent schemes. Then the induced map \( f_{Y'}: X \times_Y Y' \to Y' \) is pseudo-affine.

**Proposition A.1.5.** Let \( Y \) be a coherent and universally cohesive scheme and \( f: X \to Y \) a morphism of finite presentation. Suppose \( f \) is pseudo-affine. Then for any coherent sheaf \( \mathcal{F} \) on \( X \), \( R^q f_* \mathcal{F} \) for \( q \geq 1 \) is a coherent sheaf on \( Y \), and \( f_* \mathcal{F} \) is a coherent module over a finitely presented \( O_Y \)-algebra.
Proof. Let $X \to Z \to Y$ be the Stein factorization. Since $g$ is affine, we have $R^q f_* F = g_* R^q \pi_* F$ for $q \geq 0$. Since $\pi$ is proper of finite presentation, $R^q \pi_* F$ is a coherent $\mathcal{O}_Z$-module for $q \geq 0$ (8.1.3). If $q > 0$, then the support of the sheaf $R^q f_* F$ is contained in the center $B_\pi$, which is finite over $Y$, and hence $R^q f_* F = g_* R^q \pi_* F$ is a coherent $\mathcal{O}_Y$-module. The sheaf $f_* F = g_* \pi_* F$ is coherent over $g_* \mathcal{O}_Z$, whence the result.

A.1. (b) Pseudo-affineness and compactifications. In this paragraph we need the notion of $U$-admissible blow-ups (already appeared in §8.3. (b)).

Definition A.1.6. Let $X$ be a coherent scheme, and $U \subseteq X$ a quasi-compact open subscheme of $X$. A $U$-admissible blow-up is a blow-up $Y \to X$ of $X$ along a quasi-coherent ideal $\mathcal{J}$ of $\mathcal{O}_X$ of finite type such that the closed subscheme $V(\mathcal{J})$ of $X$ corresponding to $\mathcal{J}$ is disjoint from $U$.

Later in II, §E.1. (b) we will discuss $U$-admissible blow-ups in detail.

Proposition A.1.7. Let $Y$ be a coherent scheme and $f : X \to Y$ a morphism of finite presentation. The following conditions are equivalent.

(a) The morphism $f$ is pseudo-affine.

(b) There exist an open immersion $X \hookrightarrow \overline{X}$ over $Y$ into a proper $Y$-scheme $\overline{X}$ and an effective Cartier divisor $D$ of $\overline{X}$, such that the support of $D$ is $\overline{X} \setminus X$, $D$ is semiample over $Y$, and the normal sheaf $\mathcal{O}_{\overline{X}}(D)|_D$ on $D$ is ample over $Y$.

(c) For any open immersion $X \to X'$ over $Y$ into a proper $Y$-scheme $X'$, there exist an $X$-admissible blow-up $X'' \to X'$ and an effective Cartier divisor $D$ on the closure $\overline{X}$ of $X$ in $X''$, such that the support of $D$ is $\overline{X} \setminus X$, $D$ is semiample over $Y$, and the normal sheaf $\mathcal{O}_{\overline{X}}(D)|_D$ on $D$ is ample over $Y$.

The proof is based on a generalized version of Nagata’s embedding theorem (II.F.1.1).

Proof. First, let us show (a) $\implies$ (b). Let

$$
\begin{array}{ccc}
X & \xrightarrow{\pi} & Z \\
\downarrow & & \downarrow \delta \\
Y
\end{array}
$$

be a pre-Stein factorization of $f$. Take a projective compactification $\overline{Z}$ of $Z$, which is projective of finite presentation over $Y$, and let $\Delta$ be an effective ample Cartier divisor whose support is $\overline{Z} \setminus Z$. One can extend the morphism $f$ to a proper morphism $\overline{X} \to \overline{Z}$ in such a way that $\overline{X} \times_{\overline{Z}} Z \cong X$; indeed, by Nagata’s embedding
theorem, one has a proper map \( \tilde{X} \to \tilde{Z} \) such that \( \tilde{X} \) contains \( X \) as a Zariski open subset; then replace \( \tilde{X} \) by the scheme-theoretic closure of \( X \) in \( \tilde{X} \). We thus arrive at the commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{\pi} & \tilde{X} \\
\downarrow & & \downarrow \tilde{\pi} \\
Z & \xleftarrow{\tilde{\pi}} & \tilde{Z} \\
\downarrow & & \downarrow \tilde{g} \\
Y & & \\
\end{array}
\]

where the horizontal arrows are open immersions and the square is Cartesian. Now, consider the pull-back \( D \) of the divisor \( \Delta \) to \( \tilde{X} \). Clearly, \( D \) is supported on \( \tilde{X} \setminus X \) and is \( \tilde{Y} \)-semiample (since \( \mathcal{O}_Z(\Delta) \) is generated by global sections). Since the map \( \tilde{\pi} \) is an isomorphism around \( \tilde{X} \setminus X, \mathcal{O}(D)|_D \) on \( D \) is \( \tilde{Y} \)-ample.

Next we show \( (b) \implies (c) \). Let \( X \xleftarrow{\iota} X' \) be an open immersion over \( Y \) into a scheme \( X' \) proper over \( Y \), and take \( X \xleftarrow{\iota} \tilde{X} \) as in \( (b) \). Then there exists a diagram of proper \( \tilde{Y} \)-schemes

\[
\begin{array}{ccc}
X'' & \xrightarrow{\tilde{X}} & \tilde{X} \\
\downarrow & & \downarrow \tilde{\pi} \\
X' & \xrightarrow{\iota} & X \\
\downarrow & & \downarrow \iota \\
\end{array}
\]

consisting of \( X \)-admissible blow-ups (cf. \textsc{II}.E.12 (2)); let \( \tilde{X} \) be the closure of \( X \) in \( X'' \). Let \( I \) be the blow-up center of \( X'' \to \tilde{X} \). Then \( I \mathcal{O}_{\tilde{X}} \) is invertible and \( \tilde{X} \)-ample. Moreover, the support of the corresponding divisor \( E \) lies in \( \tilde{X} \setminus X \). Let \( D \) be the effective Cartier divisor of \( \tilde{X} \) as in \( (b) \), and take the pull-back \( \tilde{D} \) on \( \tilde{X} \). Then by \([54], \text{II}, (4.6.13) (ii)\), there exists an integer \( n > 0 \) such that the divisor \( E + n\tilde{D} \) satisfies the conditions as in \( (c) \).

The converse implication \( (c) \implies (b) \) follows from Nagata’s embedding theorem.

It remains to show \( (b) \implies (a) \). Assume \( (b) \) holds. Since \( D \) is \( Y \)-semiample, replacing \( D \) by a multiple of itself if necessary, we may assume that \( D \) induces a morphism \( \tilde{\pi}: \tilde{X} \to P \) over \( Y \), where \( P \) is projective of finite presentation over \( Y \), such that there exists a \( Y \)-ample divisor \( \Delta \) the pull-back of which to \( \tilde{X} \) coincides with \( D \). Let \( Z \) be the open complement of \( \Delta \) in \( P \), which is affine and of finite presentation over \( Y \). Since the support of \( D \) is \( \tilde{X} \setminus X \), we have \( \tilde{X} \times_P Z = X \), and thus we get the proper morphism \( \pi: X \to Z \). The center of \( \tilde{\pi} \) is a closed subset of \( P \). Since \( \mathcal{O}_{\tilde{X}}(D)|_D \) is \( \tilde{Y} \)-ample, \( \tilde{\pi} \) is finite over the boundary \( P \setminus Z \). Hence the center \( B_{\tilde{\pi}} \) lies in \( Z \) and is finite over \( Y \), since it is affine and proper over \( Y \) (cf. \textsc{II}.F.4.1). Thus we have the pre-Stein factorization \( X \xrightarrow{\tilde{\pi}} Z \xrightarrow{\tilde{g}} Y \) of \( f \) and, therefore, the proposition is proved. □
A. Appendix: Stein factorization for schemes

For the proof of the following theorem, see [78], 4.6, (cf. [48]).

Theorem A.2.1 (cohomological criterion). Let $Y$ be a Noetherian scheme, and $f : X \to Y$ a separated morphism of finite type. Then the following conditions are equivalent.

(a) The morphism $f$ is pseudo-affine.

(b) $\mathcal{R}^1 f_* \mathcal{F}$ is coherent on $Y$ for any coherent sheaf $\mathcal{F}$ on $X$.

(c) $\mathcal{R}^q f_* \mathcal{F}$ are coherent for $q \geq 1$ for any coherent sheaf $\mathcal{F}$ on $X$.

Proposition A.2.2. Let $Y = \text{Spec} A$ be an affine scheme and $Y_0 = \text{Spec} A_0$ a closed subscheme of $Y$ defined by a finitely generated nilpotent ideal $I$ of $A$. Let $f : X \to Y$ be a morphism of finite presentation and $f_0 : X_0 = X \times_Y Y_0 \to Y_0$ the induced map. Then $f$ is pseudo-affine if and only if $f_0$ is pseudo-affine.

Proof. The ‘only if’ part follows from A.1.4 (2). We are to show the ‘if’ part. Suppose $f_0$ is pseudo-affine, and let $X_0 \xrightarrow{\pi_0} Z_0 \xrightarrow{g_0} Y_0$ be a pre-Stein factorization of $f_0$. Let us take a filtered inductive system \( \{ A_\alpha, I_\alpha \}_{\alpha \in L} \) indexed by a directed set and consisting of Noetherian rings and nilpotent ideals, such that $A = \lim_{\alpha \in L} A_\alpha$ and $I = \lim_{\alpha \in L} I_\alpha$. Set $Y_\alpha = \text{Spec} A_\alpha$ and $Y_{0_\alpha} = \text{Spec} A_{0_\alpha}$, where $A_{0_\alpha} = A_\alpha/I_\alpha$ for each $\alpha \in L$. Replacing $L$ by a cofinal subset if necessary, we may assume that for each $\alpha \in L$ there exists a $Y_\alpha$-scheme $f_\alpha : X_\alpha \to Y_\alpha$ of finite type such that $X \cong X_\alpha \times_{Y_\alpha} Y$. Let $f_{0_\alpha} : X_{0_\alpha} = X_\alpha \times_{Y_\alpha} Y_{0_\alpha} \to Y_{0_\alpha}$ be the induced map for each $\alpha \in L$. Again replacing $L$ by a cofinal subset if necessary, we may assume that for each $\alpha \in L$ there exists an affine $Y_{0_\alpha}$-scheme $g_{0_\alpha} : Z_{0_\alpha} \to Y_{0_\alpha}$ of finite type such that $Z_0 \cong Z_{0_\alpha} \times_{Y_{0_\alpha}} Y_0$. We may also assume that each $f_{0_\alpha}$ factors through $Z_{0_\alpha}$ by a morphism $\pi_{0_\alpha} : X_{0_\alpha} \to Z_{0_\alpha}$. By [54], IV, (8.10.5), we may assume that the morphisms $\pi_{0_\alpha}$ are proper and that the center $B_{\pi_{0_\alpha}}$ is finite over $Y_{0_\alpha}$ (since $B_{\pi_0} = B_{\pi_{0_\alpha}} \times_{Y_{0_\alpha}} Y_0$). In this way, we arrive at the situation where the map $f$ is realized as the filtered projective limit of the maps $f_\alpha$ between Noetherian schemes such that all $f_{0_\alpha}$ are pseudo-affine.

By A.2.1, we know that for any coherent sheaf $\mathcal{F}$ on $X_{0_\alpha}$ the sheaf $\mathcal{R}^q f_{0_\alpha*} \mathcal{F}$ for $q \geq 1$ is coherent on $Y_{0_\alpha}$. To show the analogous statement for each $f_\alpha$, let $\mathcal{F}$ be a coherent sheaf on $X_\alpha$, and set $\mathcal{F}_0 = \mathcal{F}/I_\alpha \mathcal{F}$; by induction with respect to $d \geq 0$ such that $I^d_\alpha = 0$, we may assume $I^2_\alpha = 0$. Consider the cohomology exact sequence induced by the sequence

$$0 \longrightarrow I_\alpha \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow 0,$$

where the first and the third sheaves are coherent sheaves on $Y_{0_\alpha}$. By 8.1.1 and
in view of the fact that quasi-coherent sheaves of finite type are coherent ([54], I, (1.5.1)), the cohomologies $R^q f_* F$ are coherent on $Y_\alpha$ for $q \geq 1$. Hence again by A.2.1 we deduce that $f_\alpha$ is pseudo-affine. By A.1.4 (2) it follows that $f$ is pseudo-affine, as desired.

**B Appendix: Zariskian schemes**

**B.1 Zariskian schemes**

**B.1. (a) Zariskian rings and Zariskian spectra.** The notion of Zariskian scheme has already appeared in 0, §8.6. In this subsection we give a more systematic and precise account on Zariskian schemes.

Let $A$ be a commutative ring endowed with the $I$-adic topology (0, §7.2.(a)) by an ideal $I \subset A$. The topological ring $A$ is called a Zariskian ring if $1 + I \subseteq A^\times$ (cf. 0, §7.3.(b)). The definition does not depend on the choice of the ideal of definition $I$; indeed, if $J$ is another ideal of definition, that is, $I^m \subseteq J^n \subseteq I$ for some $m, n > 0$, then since $1 + J^n \subseteq A^\times$, $J^n$ is contained in the Jacobson radical of $A$, and hence $J$ is contained in the Jacobson radical, that is to say, $1 + J \subseteq A^\times$ (cf. 0.7.3.2). For any ring $A$ with $I$-adic topology the associated Zariskian ring, denoted by $A^\text{Zar}$, is defined to be $S^{-1}A$, where $S$ is the multiplicative set given by $S = 1 + I$ (cf. 0, §7.3.(b)).

**Definition B.1.1.** Let $A$ be a Zariskian ring and $I$ an ideal of definition. Then the Zariskian spectrum of $A$, denoted by $X = \text{Spz } A$, is the topologically locally ringed space defined as follows.

- $X$ is, as a set, the subset $V(I)$ of $\text{Spec } A$.
- The topology is the induced topology as a subset of $\text{Spec } A$.
- The structure sheaf is the one given by $i^{-1}\mathcal{O}_{\text{Spec } A}$ considered with the $I$-adic topology, where $i: V(I) \hookrightarrow \text{Spec } A$ is the inclusion map.

The topology on $X = \text{Spz } A$ has the open basis $\{\mathcal{D}(f)\}_{f \in A}$ consisting of quasi-compact open subsets, where we set $\mathcal{D}(f) = D(f) \cap X$ for $f \in A$. The open subset $\mathcal{D}(f)$, considered as a topologically locally ringed space with the induced structure sheaf, is isomorphic to the Zariskian spectrum $\text{Spz } A_f^{\text{Zar}}$, where $A_f^{\text{Zar}}$ is the associated Zariskian ring of $A_f$ with respect to the $IA_f$-adic topology. Note that the underlying topological space of $\text{Spz } A$ is coherent (0.2.2.1).

The proofs of the following lemmas are entirely similar to those of 1.1.11 and 1.1.12, respectively.

**Lemma B.1.2.** For a collection of finitely many elements $f_1, \ldots, f_r \in A$ the following conditions are equivalent.
B. Appendix: Zariskian schemes

(a) The open sets $D(f_i), i = 1, \ldots, r$, cover $X = \text{Spz } A$, that is

$$X = \bigcup_{i=1}^{r} D(f_i).$$

(b) For any ideal of definition $I$ of $A$ the open sets $D(\tilde{f}_i), i = 1, \ldots, r$, where $\tilde{f}_i = (f_i \mod I)$, cover $\text{Spec } A/I$, that is,

$$\text{Spec } A/I = \bigcup_{i=1}^{r} D(\tilde{f}_i).$$

(c) The open sets $D(f_i), i = 1, \ldots, r$, cover $\text{Spec } A$, that is,

$$\text{Spec } A = \bigcup_{i=1}^{r} D(f_i).$$

(d) The ideal generated by $f_1, \ldots, f_r$ coincides with $A$.

**Lemma B.1.3.** An open subset $U$ of $X = \text{Spz } A$ is quasi-compact if and only if $U$ is of the form $U = X \setminus V(J)$ in $\text{Spec } A$ for a finitely generated ideal $J \subseteq A$.

Let $A$ be a ring and $I \subseteq A$ an ideal. As above, let $i: V(I) \hookrightarrow \text{Spec } A$ be the inclusion map. For an $A$-module $M$, we define the sheaf $M\hat{}$ on the topological space $V(I)$ by

$$M\hat{} = i^{-1} \tilde{M}.$$ 

**Proposition B.1.4.** For any $f \in A$ we have

$$\Gamma(V(I) \cap D(f), M\hat{}) = M \otimes_A S^{-1}Af,$$

where $S = 1 + IA_f$.

**Proof.** Since $D(f) = \text{Spec } A_f$ and $V(I) \cap D(f) = V(IA_f)$, we may assume without loss of generality that $A = A_f$. First we show that, if $\mathcal{M}$ denotes the presheaf pull-back of $\tilde{M}$ by $i$, then $\Gamma(V(I), \mathcal{M}) = M \otimes_A S^{-1}A_f$. The left-hand side is the inductive limit of $\Gamma(U, \tilde{M})$, where $U$ runs over the set of all open subsets containing $V(I)$. If an open subset $U = \text{Spec } A \setminus V(J)$ contains $V(I)$, then $V(J) \cap V(I) = \emptyset$, that is, $J + I = A$. This implies that there exists $a \in I$ such that $1 + a \in J$. Since $V(I) \subseteq D(1 + a) \subseteq U$, we deduce that the open subsets of the form $D(1 + a)$ with $a \in I$ form a cofinal subset in the set of all open subsets containing $V(I)$. Hence the module in question is the inductive limit of the modules of the form $M \otimes_A A_{(1+a)}$ with $a \in I$, which is nothing but $M \otimes_A (1 + I)^{-1}A$, as desired.

It remains to show that $\Gamma(V(I), \mathcal{M}) = \Gamma(V(I), M\hat{})$. This can be shown by the standard argument as in [54], I, (1.3.7); the detail is left to the reader. □
Corollary B.1.5. If $A$ is a Zariskian ring, then $M\hat{\otimes}$ is an $\mathcal{O}_X$-module, where $X = \text{Spz } A$, and we have $\Gamma(\mathfrak{D}(f), M\hat{\otimes}) = M \otimes_A A^\text{Zar}_f$ for any $f \in A$.

Proposition B.1.6. Let $A$ be a Zariskian ring, and set $X = \text{Spz } A$. Let $M$ be an $A$-module. Then for any point $x \in X$, that is, an open prime ideal $x = \mathfrak{p}$, we have $M_x\otimes = M \otimes_A A_p$.

To show this, we need the following obvious lemma.

Lemma B.1.7. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a filtered inductive system of rings with adic topology such that for any $\lambda \leq \mu$ the map $A_\lambda \to A_\mu$ is adic, that is, for any ideal of definition $I$ of $A_\lambda$, $IA_\mu$ is an ideal of definition of $A_\mu$. Then we have the canonical isomorphism $\lim_{\lambda \in \Lambda} A_\lambda^\text{Zar} \cong \lim_{\lambda \in \Lambda} A_\lambda^\text{Zar}$ of topological rings.

Proof of Proposition B.1.6. By B.1.7, we have
\[
M_x\otimes = \lim_{x \in D(f)} \Gamma(\mathfrak{D}(f), M\hat{\otimes}) = M \otimes_A \lim_{x \in D(f)} A^\text{Zar}_f = M \otimes_A A^\text{Zar}_p.
\]

But due to 0.7.3.3, $A^\text{Zar}_p = A_p$. \hfill $\square$

B.1. (b) Zariskian schemes

Definition B.1.8. (1) A topologically locally ringed space $(X, \mathcal{O}_X)$ is called an affine Zariskian scheme if it is isomorphic to $\text{Spz } A$ for a Zariskian ring $A$.

(2) A topologically locally ringed space $(X, \mathcal{O}_X)$ is called a Zariskian scheme if it has an open covering $X = \bigcup_{\alpha \in L} U_\alpha$ by affine Zariskian schemes.

The morphisms of Zariskian schemes are the morphisms of topologically locally ringed spaces. A Zariskian scheme $X$ is said to be quasi-compact (resp. quasi-separated, resp. coherent) if the underlying topological space of $X$ is quasi-compact (resp. quasi-separated, resp. coherent) (0.2.1.4, 0.2.1.8, and 0.2.2.1). Note that any Zariskian scheme is locally coherent and sober (0.2.2.21 and 0, §2.1.(b)). We denote by $\mathbf{Zs}$ (resp. $\mathbf{CZs}$, resp. $\mathbf{AZs}$) the category of Zariskian (resp. coherent Zariskian, resp. affine Zariskian) schemes. These are full subcategories of the category of topologically locally ringed spaces. Note that schemes are canonically regarded as Zariskian schemes considered with the 0-adic topology. The following theorem is straightforward in view of B.1.5 and B.1.6.

Theorem B.1.9. The functor $A \mapsto \text{Spz } A$ gives rise to a categorical equivalence between the opposite category of the category of Zariskian rings and the category $\mathbf{AZs}$ of affine Zariskian schemes.
For a Zariskian scheme $X$ and an open subset $U$ of the underlying topological space of $X$, $U$ has the induced structure as a Zariskian scheme in such a way that the inclusion map extends to a morphism $j: U \to X$ of Zariskian schemes. The Zariskian scheme obtained in this way is called an \textit{open Zariskian subscheme} of $X$. Similarly, one can define the notion of \textit{open immersions} of Zariskian schemes.

\textbf{Definition B.1.10.} A morphism $f: X \to Y$ of Zariskian schemes is said to be \textit{quasi-compact} if the map of underlying topological spaces is quasi-compact; see (0.2.1.4 (2)).

We have the so-called \textit{Zariskian completion}: let $X$ be a scheme and $Y \subseteq X$ a closed subscheme; then $(Y, i^{-1}\mathcal{O}_X)$, where $i: Y \to X$ is the closed immersion, is a Zariskian scheme, which we denote by $X^\text{Zar}|_Y$.

\section*{B.2 Fiber products}

\textbf{Theorem B.2.1.} \textit{The category of Zariskian schemes has fiber products.}

As usual, the construction is reduced to the affine case. Let

\[
\begin{array}{ccc}
\text{Spz } B & \longrightarrow & \text{Spz } A \\
\downarrow & & \downarrow \\
\text{Spz } C & \leftarrow & \text{Spz } C
\end{array}
\]  

be a diagram of affine Zariskian schemes, where $A$ (resp. $B$, resp. $C$) is a Zariskian ring with an ideals of definition $I$ (resp. $J$, resp. $K$). The fiber product of diagram ($\ast$) is given by a diagram of \textit{affine} Zariskian schemes corresponding to

\[
\begin{array}{ccc}
B \otimes_A C & \longrightarrow & (B \otimes_A C)^{\text{Zar}} \\
\downarrow & & \downarrow \\
B & \leftarrow & C
\end{array}
\]

Here the associated Zariskian ring $(B \otimes_A C)^{\text{Zar}}$ is taken with respect to the ideal $H$ of $B \otimes_A C$ given by

\[ H = \text{image}(J \otimes_A C \to B \otimes_A C) + \text{image}(B \otimes_A K \to B \otimes_A C). \]

\section*{B.3 Ideals of definition and adic morphisms}

\textbf{Definition B.3.1.} Let $X$ be a Zariskian scheme. An \textit{ideal of definition} of $X$ is a quasi-coherent ideal $\mathcal{I}$ of $\mathcal{O}_X$ such that for an affine open covering $X = \bigcup_{\alpha \in L} U_\alpha$ with $U_\alpha \cong \text{Spz } A_\alpha$ for a Zariskian ring $A_\alpha$ for each $\alpha \in L$, $\mathcal{I}|_{U_\alpha}$ is isomorphic to the ideal of the form $I_\alpha^{\text{op}}$ for an ideal of definition $I_\alpha$ of $A_\alpha$. 
Clearly, ideals of definition always exist locally. If $I$ is an ideal of definition of a Zariskian scheme $X$ and $U = \text{Spz} A$ is an affine open subset of $X$, where $A$ is a Zariskian ring, then $I|_U \cong I^\circ$ for an ideal of definition $I$ of $A$.

**Definition B.3.2.** A morphism $f: X \to Y$ of Zariskian schemes is said to be *adic* if for any open subset $V$ of $Y$ having an ideal of definition $I$, $I\mathcal{O}_X|_{f^{-1}(V)}$ is an ideal of definition of the open Zariskian subscheme $f^{-1}(V)$.

We denote by $\text{Zs}^*$ (resp. $\text{CZs}^*$, resp. $\text{AZs}^*$) the category of Zariskian (resp. coherent Zariskian, resp. affine Zariskian) schemes and adic morphisms.

**Proposition B.3.3.** (1) Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of Zariskian schemes. If $f$ and $g$ are adic, then so is the composition $g \circ f$. If $g \circ f$ and $g$ are adic, then so is $f$.

(2) Let $S$ be a Zariskian scheme and $f: X \to X'$ and $g: Y \to Y'$ two adic $S$-morphisms of Zariskian schemes (not necessarily adic over $S$). Then

$$f \times_S g: X \times_S Y \longrightarrow X' \times_S Y'$$

is adic.

(3) Let $S$ be a Zariskian scheme, and $f: X \to Y$ an adic $S$-morphism between Zariskian schemes (not necessarily adic) over $S$. Then for any (not necessarily adic) morphism $S' \to S$ of Zariskian schemes, the induced morphism

$$f_{S'}: X \times_S S' \longrightarrow Y \times_S S'$$

is adic.

The proof is similar to that of 1.3.6.

**B.4 Morphism of finite type and morphism of finite presentation**

**Definition B.4.1.** A morphism $f: X \to Y$ of Zariskian schemes is said to be *locally of finite type* (resp. *locally of finite presentation*) if

(a) the morphism $f$ is adic (B.3.2), and

(b) there exist an affine open covering $\{V_i\}$ of $Y$ with $V_i = \text{Spz} B_i$, where $B_i$ is a Zariskian ring with an ideal of definition $J_i$, and for each $i$ an affine open covering $\{U_{ij}\}$ of $f^{-1}(V_i)$ with $U_{ij} = \text{Spz} A_{ij}$, where $A_{ij}$ is a Zariskian ring with the ideal of definition $J_i A_{ij}$ due to (a), such that each $A_{ij}$ is isomorphic to the associated Zariskian of a finitely generated (resp. finitely presented) algebra over $B_i$.

The morphism $f$ is said to be *of finite type* (resp. *of finite presentation*) if it is locally of finite type and $f$ is quasi-compact.
Note that, if $B$ is the associated Zariskian $(A[X_1, \ldots, X_n]/\mathfrak{a})^{\text{Zar}}$ of a finitely presented $A$-algebra (where $\mathfrak{a}$ is a finitely generated ideal), then

$$B = A[X_1, \ldots, X_n]^{\text{Zar}}/\mathfrak{a}A[X_1, \ldots, X_n]^{\text{Zar}},$$

since, in general, quotient rings (by not necessarily closed ideals) of a Zariskian ring are always Zariskian with respect to the induced ideal of definition. In particular, for any affine Zariskian scheme $\text{Spz} \ A$ and any $f \in A$, the open immersion $\text{Spz} \ A_f^{\text{Zar}} \hookrightarrow \text{Spz} \ A$ is of finite presentation. This fact justifies our definition of ‘(locally) of finite presentation’ as above. By an argument similar to the proof of 1.7.2, we have the following result.

**Proposition B.4.2.** (1) Any open immersion is locally of finite presentation.

(2) The composition of two morphisms locally of finite type (resp. of finite type, resp. locally of finite presentation, resp. of finite presentation) is again locally of finite type (resp. of finite type, resp. locally of finite presentation, resp. of finite presentation). If the composition $g \circ f$ of morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ of formal schemes is locally of finite type and $g$ is adic, then $f$ is locally of finite type. If $g \circ f$ is locally of finite presentation and $g$ is locally of finite type, then $f$ is locally of finite presentation.

(3) Let $S$ be a Zariskian scheme and $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ two adic $S$-morphisms of Zariskian schemes. Suppose $f$ and $g$ are locally of finite type (resp. of finite type, resp. locally of finite presentation, resp. of finite presentation). Then

$$f \times_S g: X \times_S Y \longrightarrow X' \times_S Y'$$

is locally of finite type (resp. of finite type, resp. locally of finite presentation, resp. of finite presentation).

(4) Let $S$ be a Zariskian scheme and $f: X \rightarrow Y$ an adic $S$-morphism between Zariskian schemes. Suppose $f$ is locally of finite type (resp. of finite type, resp. locally of finite presentation, resp. of finite presentation). Then for any morphism $S' \rightarrow S$ of Zariskian schemes the induced morphism

$$f_{S'}: X \times_S S' \longrightarrow Y \times_S S'$$

is locally of finite type (resp. of finite type, resp. locally of finite presentation, resp. of finite presentation).

**C Appendix: FP-approximated sheaves and GFGA theorems**

**C.1 Finiteness up to bounded torsion**

**C.1. (a) Weak isomorphisms.** Let $A$ be a ring and $I \subseteq A$ a finitely generated ideal.
Definition C.1.1. A homomorphism $f: M \to N$ of $A$-modules is said to be weakly injective (resp. weakly surjective) if $\ker(f)$ (resp. $\coker(f)$) is bounded $I$-torsion, that is, there exists a positive integer $n > 0$ such that $I^n \ker(f) = 0$ (resp. $I^n \coker(f) = 0$) (cf. 0.8.1.3).

The following properties are easy to verify. Let $f: L \to M$ and $g: M \to N$ be $A$-linear homomorphisms of $A$-modules.

- If $f$ and $g$ are weakly injective (resp. weakly surjective), then so is the composition $g \circ f$.
- If $g \circ f$ is weakly injective, then so is $f$.
- If $g \circ f$ is weakly surjective, then so is $g$.
- If $g \circ f$ is weakly injective and $f$ is weakly surjective, then $g$ is weakly injective.
- If $g \circ f$ is weakly surjective and $g$ is weakly injective, then $f$ is weakly surjective.

Definition C.1.2. A homomorphism $f: M \to N$ of $A$-modules is called a weak isomorphism if it is weakly injective and weakly surjective.

From the discussion above we easily deduce the following fact.

- For two homomorphisms $f: L \to M$ and $g: M \to N$ of $A$-modules such that $g \circ f$ is a weak isomorphism, if one of $f$ and $g$ is a weak isomorphism, then so is the other.

It is straightforward to see that the family of all weak isomorphisms forms a multiplicative system (cf. [66], 1.6.1) in the category $\text{Mod}_A$ of all $A$-modules. We say that two $A$-modules $M$ and $N$ are weakly isomorphic when regarded as objects in the localized category $\text{Mod}_A/\{\text{weak isomorphisms}\}$.

C.1. (b) Weakly finitely presented modules

Definition C.1.3. An $A$-module $M$ is said to be weakly finitely generated (resp. weakly finitely presented) if it is weakly isomorphic to a finitely generated (resp. finitely presented) $A$-module.

Proposition C.1.4. (1) An $A$-module $M$ is weakly finitely generated if and only if it has a finitely generated $A$-submodule $N \subseteq M$ such that $M/N$ is bounded $I$-torsion.

(2) An $A$-module $M$ is weakly finitely presented if and only if there exists a weak isomorphism $N \to M$ from a finitely presented $A$-module.

Proof. To prove (1), it suffices to show the following conditions.
(a) Let $g: N_1 \to N_2$ be a weak isomorphism and $L_2 \subseteq N_2$ a finitely generated $A$-submodule such that $N_2/L_2$ is bounded $I$-torsion; then there exists a finitely generated $A$-submodule $L_1 \subseteq N_1$ such that $N_1/L_1$ is bounded $I$-torsion.

(b) If $h: L \to N$ be a weak isomorphism with $L$ finitely generated, then $N/\ker h$ is bounded $I$-torsion.

The property (b) is clear. To show (a), let $x_1, \ldots, x_r$ be a set of generators of $L_2$. There exists $n > 0$ such that $ax_j$ belongs of the image of $g (j = 1, \ldots, r)$ for any $a \in I^n$. Let $a_1, \ldots, a_s \in I^n$ generate $I^n$, and take $y_{ij} \in N_1$ such that $g(y_{ij}) = a_i x_j$ for $i = 1, \ldots, s$ and $j = 1, \ldots, r$. Let $L_1$ be the $A$-submodule of $N_1$ generated by all $y_{ij}$’s. Then since the composition $L_1 \to N_1 \to N_2$ is weakly injective, the map $L_1 \to L_2$ is weakly injective; moreover, since $L_1 \to L_2$ is obviously weakly surjective, we deduce that it is a weak isomorphism. Hence the inclusion $L_1 \hookrightarrow N_1$ is a weak isomorphism, as desired. To show (2), one only needs the following condition.

(c) Let $g: N_1 \to N_2$ be a weak isomorphism and $h_2: L_2 \to N_2$ a weak isomorphism from a finitely presented $A$-module $L_2$; then there exist a finitely presented $A$-module $L_1$ and weak isomorphisms $h_1: L_1 \to N_1$ and $L_1 \to L_2$.

The checking is straightforward and is left to the reader.

C.2 Global approximation by finitely presented sheaves

C.2. (a) FP-approximation of sheaves on schemes. Let $X$ be a coherent scheme, $Z \hookrightarrow X$ a closed subscheme of finite presentation, and $I = I_Z$ the defining ideal of $Z$. The notions of ‘weakly injective’, ‘weakly surjective’, and ‘weak isomorphism’ have obvious analogues for morphisms of $\mathcal{O}_X$-modules; for example, a morphism $\varphi: \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a weak isomorphism if there exists a positive integer $n > 0$ such that $I^n \ker(\varphi) = 0$ and $I^n \coker(\varphi) = 0$, that is, $\ker(\varphi)$ and $\coker(\varphi)$ are bounded $I$-torsion. It is straightforward to see that the family of all weak isomorphisms (resp. between quasi-coherent $\mathcal{O}_X$-modules) is a multiplicative system in the category $\text{Mod}_X$ (resp. $\text{Qcoh}_X$) of all $\mathcal{O}_X$-modules (resp. quasi-coherent $\mathcal{O}_X$-modules).

In this subsection we assume that

- $X$ is Noetherian outside $Z$, that is, the open subscheme $X \setminus Z$ is Noetherian, and
- $(X, Z)$ satisfies (UBT), that is, for any finite type map $f: \text{Spec } A \to X$ the induced pair $(A, I)$ (where $I = f^* I$) satisfies (BT) in 0, §8.2. (a).
Definition C.2.1. Let \( \mathcal{F} \) be a quasi-coherent sheaf on \( X \) of finite type.

(1) By an FP-approximation of \( \mathcal{F} \) we mean a weak isomorphism \( \varphi: \mathcal{G} \to \mathcal{F} \) from a finitely presented \( \mathcal{O}_X \)-module.

(2) An FP-approximation \( \varphi: \mathcal{G} \to \mathcal{F} \) of \( \mathcal{F} \) is called an Fp-thickening if it is surjective.

If \( \mathcal{F} \) admits an FP-approximation, then we sometimes say that \( \mathcal{F} \) is FP-approximated. Clearly, a quasi-coherent sheaf that admits an Fp-thickening is of finite type. Conversely, we have the following proposition.

Proposition C.2.2. In the situation as above, any quasi-coherent sheaf of finite type on \( X \) admits an FP-thickening.

This proposition is easy to see in case \( X \) is affine; this follows from what we have seen in the end of the previous subsection. To show the proposition in general, we first introduce the category of FP-thickenings by means of the following notion of morphisms: given two FP-thickenings \( \mathcal{G}_1 \to \mathcal{F} \) and \( \mathcal{G}_2 \to \mathcal{F} \) of \( \mathcal{F} \), a morphism from the former to the latter is a surjective morphism \( \mathcal{G}_1 \to \mathcal{G}_2 \) that makes the triangle \( \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{F} \) commutative.

Lemma C.2.3. Let \( \mathcal{F} \) be a quasi-coherent sheaf of finite type on \( X \). Then the category of all FP-thickenings of \( \mathcal{F} \) is a filtered category (cf. 0, §1.3. (c)).

Proof. Let \( \mathcal{G}_1 \to \mathcal{F} \) and \( \mathcal{G}_2 \to \mathcal{F} \) be FP-thickenings. We need to construct another FP-thickening \( \mathcal{H} \to \mathcal{F} \) that is dominated by \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \). Consider first the fiber product \( \mathcal{K} \) of the maps \( \mathcal{G}_1 \to \mathcal{F} \) and \( \mathcal{G}_2 \to \mathcal{F} \). It is easy to see that \( \mathcal{K} \) is a quasi-coherent subsheaf of the direct product \( \mathcal{G}_1 \oplus \mathcal{G}_2 \), and that the canonical morphisms \( \mathcal{G}_i \to \mathcal{G}_1 \oplus \mathcal{G}_2 / \mathcal{K} \) \((i = 1, 2)\) are surjective. Write \( \mathcal{K} \) as an inductive limit \( \lim_{\lambda \in \Lambda} \mathcal{K}_\lambda \) of quasi-coherent subsheaves of finite type ([54], I, (9.4.9), and IV, (1.7.7)). We need to show that for some \( \lambda \in \Lambda \) the morphisms \( \mathcal{G}_i \to \mathcal{G} \oplus \mathcal{H} / \mathcal{K}_\lambda \) \((i = 1, 2)\) are surjective. To check this, we may assume that \( X \) is affine, \( X = \text{Spec} \, A \). Set \( \mathcal{F} = \mathcal{M} \) and \( \mathcal{G}_i = \mathcal{N}_i \) for \( i = 1, 2 \); \( \mathcal{M} \) is a finitely generated \( \mathcal{A} \)-module, and \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) are finitely presented \( \mathcal{A} \)-modules.

Let \( \{y_{ij}\}_{j \in J_i} \) be a finite set of generators of \( \mathcal{N}_i \) \((i = 1, 2)\). Denote by \( \tilde{y}_{ij} \) the image of \( y_{ij} \) in \( \mathcal{M} \). Take a lift \( x_{1j} \) (resp. \( x_{2j} \)) of \( \tilde{y}_{1j} \) (resp. \( \tilde{y}_{2j} \)) in \( \mathcal{N}_2 \) (resp. \( \mathcal{N}_1 \)). For some \( \lambda \in \Lambda \), \( K_\lambda \) (where \( K_\lambda^\Delta = K_\lambda \)) contains the elements \((y_{1j}, x_{1j})\) \((j \in J_1)\) and \((x_{2j}, y_{2j})\) \((j \in J_2)\). Then the maps \( \mathcal{N}_i \to \mathcal{N}_1 \oplus \mathcal{N}_2 / K_\lambda \) are surjective. \( \Box \)
Proof of Proposition C.2.2. As we have already seen above, the assertion is true if $X$ is affine. Consider a finite open covering $X = \bigcup_{i=1}^r U_i$ such that the assertion is true on each $U_i$ (e.g., a finite affine covering). By induction with respect to $r$, we may work in the case $r = 2$. Let $\mathcal{F}$ be a quasi-coherent sheaf of finite type on $X$, and $\mathcal{G}_i \to \mathcal{F}|_{U_i}$ ($i = 1, 2$) an FP-thickening on $U_i$. Take, on $U_1 \cap U_2$, an FP-thickening $\mathcal{H} \to \mathcal{F}|_{U_1 \cap U_2}$ dominated by $\mathcal{G}_i|_{U_1 \cap U_2}$ for $i = 1, 2$ (C.2.3). Consider $\mathcal{K}_i = \ker(\mathcal{G}_i|_{U_1 \cap U_2} \to \mathcal{H})$ ($i = 1, 2$), which is a quasi-coherent sheaf of finite type, and is bounded $\mathcal{I}$-torsion. Hence, there exists a quasi-coherent sheaf $\mathcal{K}_i$ ($\mathcal{I}$-torsion subsheaf of $\mathcal{G}_i$) of finite type on $U_i$ that extends $\mathcal{K}_i$ (for $i = 1, 2$). Now the quotient sheaves $\mathcal{G}_1/\mathcal{K}_1$ and $\mathcal{G}_2/\mathcal{K}_2$ patch together to a finitely presented sheaf $\mathcal{H}$, which gives an FP-thickening of $\mathcal{F}$.

Corollary C.2.4. A quasi-coherent sheaf $\mathcal{F}$ is FP-approximated if and only if there exists a quasi-coherent subsheaf $\mathcal{G} \subseteq \mathcal{F}$ of finite type such that $\mathcal{F}/\mathcal{G}$ is bounded $\mathcal{I}$-torsion.

Proof. The ‘only if’ part is clear; one only has to take the image of an FP-approximation. Suppose $\mathcal{F}$ has a subsheaf $\mathcal{G}$ as above. By C.2.3 we have an FP-thickening $\mathcal{H} \to \mathcal{G}$, where $\mathcal{H}$ is finitely presented. Then the composite $\mathcal{H} \to \mathcal{F}$ gives an FP-approximation.

Theorem C.2.5. The full subcategory $\text{FPA}_{(X, Z)}$ of the category $\text{Mod}_X$ of $\mathcal{O}_X$-modules consisting of FP-approximated sheaves is a thick (0, §C.5) abelian full subcategory.

As a first step of the proof, we show the following lemma.

Lemma C.2.6. Let $\varphi: \mathcal{F} \to \mathcal{G}$ be a morphism between FP-approximated sheaves on $X$. Then $\ker(\varphi)$ and $\coker(\varphi)$ are FP-approximated.

Proof. First note that $\ker(\varphi)$ and $\coker(\varphi)$ are quasi-coherent sheaves on $X$. To show that $\ker(\varphi)$ is FP-approximated, first take an FP-approximation $\alpha: \mathcal{F}' \to \mathcal{F}$ of $\mathcal{F}$, and consider the composition $\varphi \circ \alpha: \mathcal{F}' \to \mathcal{G}$. Since the image of $\varphi \circ \alpha$ is finitely presented outside $Z$, $\ker(\varphi \circ \alpha)$ is finitely generated outside $Z$. By [54], I, (9.4.7), and IV, (1.7.7), we have a quasi-coherent subsheaf $\mathcal{K}$ of $\ker(\varphi \circ \alpha)$ of finite type such that $\ker(\varphi \circ \alpha)/\mathcal{K}$ is $\mathcal{I}$-torsion; since this is a subsheaf of the finite type $\mathcal{F}'/\mathcal{K}$, the $\mathcal{I}$-torsion is bounded. Set $\mathcal{G}' = \mathcal{F}'/\mathcal{K}$, which is a finitely presented sheaf sitting the commutative diagram 

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\mathcal{F}' & \xrightarrow{\varphi'} & \mathcal{G}'
\end{array}
$$
where $\alpha$ has bounded $I$-torsion kernel and cokernel, and $\beta$ has $I$-torsion kernel. By the snake lemma, the induced morphism $\ker(\varphi') \to \ker(\varphi)$ has bounded $I$-torsion cokernel. Now by C.2.4 we deduce that $\ker(\varphi)$ is FP-approximated.

Next, let us show that $\text{coker}(\varphi)$ is FP-approximated. To this end, take an FP-approximation $\beta': \mathcal{G} \to \mathcal{G}$ of $\mathcal{G}$; since the composition $\mathcal{G}' \to \text{coker}(\varphi)$ obviously has bounded $I$-torsion cokernel, we immediately deduce the desired result, again due to C.2.4.

**Proof of Theorem C.2.5.** In view of C.2.6, the essential point to show is that if

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is an exact sequence of $\mathcal{O}_X$-modules, where $\mathcal{F}$ and $\mathcal{H}$ are FP-approximated, then $\mathcal{G}$ is FP-approximated. Note first that $\mathcal{G}$ is quasi-coherent ([53], (1.4.7)). Recalling C.2.4, we are going to construct a quasi-coherent subsheaf $\mathcal{G}' \subseteq \mathcal{G}$ of finite type such that $\mathcal{G}/\mathcal{G}'$ is bounded $I$-torsion.

Take a quasi-coherent subsheaf $\mathcal{H}'$ (resp. $\mathcal{F}'$) of $\mathcal{H}$ (resp. $\mathcal{F}$) of finite type such that $\mathcal{H}/\mathcal{H}'$ (resp. $\mathcal{F}/\mathcal{F}'$) is bounded $I$-torsion. Let $\{\mathcal{G}_\lambda\}_{\lambda \in \Lambda}$ be a directed set of quasi-coherent subsheaves of $\mathcal{G}$ of finite type such that $\mathcal{G} = \varprojlim_{\lambda \in \Lambda} \mathcal{G}_\lambda$ ([54], I, (9.4.9), and IV, (1.7.7)). Then there exists $\lambda \in \Lambda$ such that the image of $\mathcal{G}_\lambda$ in $\mathcal{H}$ contains $\mathcal{H}'$ and that the preimage $\mathcal{G}_\lambda \cap \mathcal{F}$ contains $\mathcal{F}'$. Now we use the snake lemma to conclude that $\mathcal{G}/\mathcal{G}_\lambda$ is bounded $I$-torsion, as desired.

**C.2. (b) FP-approximation of sheaves on formal schemes.** Similarly to the scheme case, one can define the notion of weak isomorphism on formal schemes: Let $X$ be a coherent adic formal scheme of finite ideal type (1.1.16). Then a morphism of $\mathcal{O}_X$-modules $\varphi: \mathcal{F} \to \mathcal{G}$ is a weak isomorphism if there exists a positive integer $n > 0$ such that $I^n \ker(\varphi) = 0$ and $I^n \text{coker}(\varphi) = 0$, where $I$ is an ideal of definition of finite type of $X$.

Now let $X$ be a coherent universally rigid-Noetherian formal scheme (2.1.7). By 2.1.9, for any affine open $\text{Spf} \ A \subseteq X$ the ring $A$ is a t.u. rigid-Noetherian ring (2.1.1 (1)); in particular, it satisfies (together with a finitely generated ideal of definition) (UBT).

Similarly to the scheme case, we introduce the following definitions.

**Definition C.2.7.** Let $X$ be a coherent universally rigid-Noetherian formal scheme, and $\mathcal{F}$ an $\mathcal{O}_X$-module.

1. By an **FP-approximation** of $\mathcal{F}$ we mean a weak isomorphism $\varphi: \mathcal{G} \to \mathcal{F}$ from a finitely presented $\mathcal{O}_X$-module $\mathcal{G}$ such that the sheaves $\ker(\varphi)$ and $\text{coker}(\varphi)$ are a.q.c. (3.1.3).

2. An FP-approximation $\varphi: \mathcal{G} \to \mathcal{F}$ of $\mathcal{F}$ is called an **FP-thickening** if it is surjective.
The terminology ‘FP-approximated’ will be used similarly. Note that the sheaf \( \mathcal{F} \) is not assumed to be a.q.c. The following proposition is easy to see.

**Proposition C.2.8.** Let \( X \) be a coherent scheme and \( Z \hookrightarrow X \) a closed subscheme of finite presentation such that \( X \) is Noetherian outside \( Z \) and that \( (X, Z) \) satisfies (UBT). Consider the formal completion \( \hat{X} = \hat{X}|_{Z} \), which we assume to be universally rigid-Noetherian. Let \( \mathcal{F} \) be a quasi-coherent sheaf on \( X \) that admits an FP-approximation (resp. FP-thickening). Then the sheaf \( \mathcal{F}^{\text{for}} \) on \( \hat{X} \) obtained by the functor for: \( \text{Mod}_{\mathcal{X}} \to \text{Mod}_{\hat{X}} \) defined in 9.1. (a) admits an FP-approximation (resp. FP-thickening).

If \( \mathcal{F} \) is a.q.c. of finite type and if \( X \) is affine, then, similarly to the scheme case, \( \mathcal{F} \) admits an FP-thickening. More generally we have the following result.

**Proposition C.2.9.** Let \( X \) be a coherent rigid-Noetherian formal scheme. Then any a.q.c. sheaf of finite type admits an FP-thickening.

This proposition can be shown by an argument similar to the scheme case by resorting to the following lemma.

**Lemma C.2.10.** Let \( \mathcal{F} \) be a quasi-coherent sheaf of finite type on \( X \). Then the category of all FP-thickenings of \( \mathcal{F} \) is a filtered category.

The proof of the lemma is quite similar to that of C.2.3 (use Exercise I.3.6 and Exercise I.3.4).

### C.3 Finiteness theorem and GFGA theorems

The proofs of the results in this subsection will be given in [43].

**C.3. (a) Finiteness theorem for FP-approximated sheaves.** Let \( Y \) be a quasi-compact scheme and \( Z \hookrightarrow Y \) a closed subspace of finite presentation. We suppose that \( Y \) is Noetherian outside \( Z \) and that the pair \( (Y, Z) \) satisfies (UBT) (cf. §C.2. (a)). Consider a morphism \( f: X \to Y \) of finite type; \( X \) is Noetherian outside \( W = f^{-1}(Z) \) and the pair \( (X, W) \) satisfies (UBT). Let us denote by \( D^{\bullet}_{\text{FPA}}(X, W) \) (resp. \( D^{\bullet}_{\text{FPA}}(Y, Z) \)) for \( * = \text{“}, +, -, \text{b} \) the full subcategory of the derived category \( D^{\bullet}(X) \) (resp. \( D^{\bullet}(Y) \)) consisting of objects with FP-approximated cohomologies; by C.2.5, this is a triangulated full subcategory.

**Theorem C.3.1.** In the situation as above, we assume that \( f \) is proper. Then \( Rf^{\bullet} \) maps \( D^{\bullet}_{\text{FPA}}(X, W) \) to \( D^{\bullet}_{\text{FPA}}(Y, Z) \) for \( * = \text{“}, +, -, \text{b} \).
C.3. (b) GFGA comparison theorem in rigid-Noetherian situation. Similarly to the previous subsection, let $Y$ be a quasi-compact algebraic space and $Z \hookrightarrow Y$ a closed subspace of finite presentation, and suppose that $Y$ is Noetherian outside $Z$ and that the pair $(Y, Z)$ satisfies \textbf{(UBT)}. For a finite type morphism $f : X \to Y$ we set $W = f^{-1}(Z)$. We consider the formal completions $\hat{X} = \hat{X}|_W$ and $\hat{Y} = \hat{Y}|_Z$ and the functors for: $\text{Mod}_X \to \text{Mod}_{\hat{X}}$ (and similarly on $Y$) defined by $\mathcal{F} \mapsto j^* \mathcal{F}$, where $j : \hat{X} \to X$ is the canonical morphism (cf. §9.1. (a)). If $f$ is proper, we can construct the following diagram in the similar way as in §9.1. (b) for $\mathcal{F}$, $\mathcal{G}$,

$$
\begin{array}{ccc}
D^*_\text{FPA} (X, W) & \xrightarrow{\text{for}} & D^*(\hat{X}) \\
\downarrow Rf_* & & \downarrow R\hat{f}_* \\
D^*_\text{FPA} (Y, Z) & \xrightarrow{\rho \text{ for}} & D^*(\hat{Y}).
\end{array}
$$

\textbf{Theorem C.3.2} (GFGA comparison theorem). The above diagram $(* \text{ is 2-commutative, that is, the natural transformation } \rho \text{ gives a natural equivalence for } \ast = "-", +, -, b.)$

The theorem essentially follows from the following special case.

\textbf{Theorem C.3.3.} Let $A$ be a ring and $I \subseteq A$ a finitely generated ideal. Suppose that the pair $(A, I)$ satisfies \textbf{(UBT)} and that the ring $A$ is Noetherian outside $I$. Let $f : X \to \text{Spec } A$ be a proper morphism. Then for any $\text{FP}$-approximated sheaf $\mathcal{F}$ on $X$ the canonical map

$$
(R^q f_* \mathcal{F})^\text{for} \longrightarrow R^q \hat{f}_* \hat{\mathcal{F}}
$$

is an isomorphism for any $q \geq 0$, where $\hat{\mathcal{F}}$ denotes the $I$-adic completion.

We will need the following implication from the theorem.

\textbf{Corollary C.3.4.} Let $X$ be as in C.3.3, and $\mathcal{F}$ and $\mathcal{G}$ two $\mathcal{O}_X$-modules. Suppose that $\mathcal{F}$ is finitely presented and that $\mathcal{G}$ is $\text{FP}$-approximated. Then the canonical map

$$
\text{Hom}_{\mathcal{O}_X} (\mathcal{F}, \mathcal{G}) \longrightarrow \text{Hom}_{\hat{\mathcal{O}_X}} (\mathcal{F}^\text{for}, \mathcal{G}^\text{for})
$$

is an isomorphism.

C.3. (c) GFGA existence theorem in the rigid-Noetherian case. Let $A$ be a t.u. rigid-Noetherian ring (2.1.1), and $f : X \to \text{Spec } A$ a proper map of schemes. By the $I$-adic completion (where $I \subseteq A$ is a finitely generated ideal of definition), we have the universally rigid-Noetherian formal scheme (2.1.7) $\hat{f} : \hat{X} \to \text{Spf } A$ over $A$.

As in §10.2. (b), we say that an $\mathcal{O}_{\hat{X}}$-module $\mathcal{F}$ is algebraizable if there exists an $\mathcal{O}_X$-module $\mathcal{G}$ such that $\mathcal{F} \cong \mathcal{G}^\text{for}$. 

...
Theorem C.3.5. Suppose $f$ is projective. Then any finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ is algebraizable.

Exercises

Exercise I.C.1. Let $A$ be a ring, and $I \subseteq A$ a finitely generated ideal. We assume that the ring $A$ with the $I$-adic topology satisfies (AP) in §7.4.(c). Let $f : N \to M$ be a weak isomorphism of $A$-modules, where $N$ is assumed to be finitely generated. Show that if either one of $N$ and $M$ is $I$-adically complete, then so is the other.

Exercise I.C.2. Let $X$ be either

- a coherent scheme considered with a closed subscheme $Z \hookrightarrow X$ of finite presentation such that $X \setminus Z$ is Noetherian and $(X, Z)$ satisfies (UBT), or
- a coherent universally rigid-Noetherian formal scheme.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves on $X$ that admits an FP-approximation (resp. FP-thickening). Show that there exist a commutative diagram

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\mathcal{F}' & \xrightarrow{\varphi'} & \mathcal{G}'
\end{array}
$$

where $\alpha$ and $\beta$ are FP-approximations (resp. FP-thickenings).
Chapter II

Rigid spaces

This chapter is the main part of this volume, where we define rigid spaces and develop their geometry. In the first section, §1, we discuss generalities of admissible blow-ups. We then give the definition of rigid spaces in §2, according to Raynaud’s viewpoint. We first define the category of coherent rigid spaces as the quotient of the category of coherent adic formal schemes of finite ideal type modulo the admissible blow-ups. Thus, any coherent rigid space $X$ is, by definition, of the form

$$X^\text{rig}$$

for a coherent adic formal scheme $X$ of finite ideal type, and then $X$ is called a formal model of $X$. We then define general (not necessarily coherent) rigid spaces by ‘patching’. Corresponding to universally adhesive and universally rigid-Noetherian formal schemes (I, §2.1), we have respectively universally adhesive and universally Noetherian rigid spaces:

$$\{\text{univ. adhesive rigid spaces}\} \subseteq \{\text{univ. Noetherian rigid spaces}\} \subseteq \{\text{rigid spaces}\}.$$  

‘Classical’ rigid spaces (called rigid spaces of type (V) in this book), that is, locally of finite type rigid spaces over $(\text{Spf } V)^{\text{rig}}$, where $V$ is an $a$-adically complete valuation ring, are examples of universally adhesive rigid spaces.

In §3 we introduce the visualization, the Zariski–Riemann triple

$$\text{ZR}(X) = ((X), \mathcal{O}_{X}^{\text{int}}, \mathcal{O}_{X})$$

of a rigid space $X$, which ‘visualizes’ the rigid space in the sense that the space $X$, introduced first by an abstract categorical argument, is interpreted as a concrete topological space with a ‘doubly-ringed structure’ which we call a triple. The triples thus obtained are, in important cases, Huber’s adic spaces (cf. §A in the Appendix). We also discuss points of Zariski–Riemann spaces by means of rigid points, which are, similarly to the situation of Zariski’s classical birational geometry mentioned in Introduction, described in terms of valuations. Having thus the notion of visualization of rigid spaces, one is then able to consider several ‘topological properties’ of rigid spaces, some of which we introduce and develop in §4.
We will show, most importantly, that Zariski–Riemann spaces are valuative (Corollary 4.1.8), hence admit the so-called separated quotient

$$\text{sep}_X : \langle X \rangle \longrightarrow [X].$$

Roughly speaking, when one regards the Zariski–Riemann space \(\langle X \rangle\) as the ‘space of arbitrary valuations’, the separated quotient \([X]\) is the subset (endowed with, however, the quotient topology by \(\text{sep}_X\)) of \(\langle X \rangle\) consisting of valuations of height one, and the map \(\text{sep}_X\) is given by ‘maximal generization’. Note that, accordingly, the space \([X]\) can be regarded as a ‘space of seminorms’, and hence is a space of the same kind as those appearing in Berkovich’s analytic geometry; in fact, as we will explain in §C.6, the separated quotient \([X]\) coincides with the underlying topological space of the Berkovich analytic space associated to \(X\), at least when \(X\) is locally of finite type over \(\text{Spf} V^{\text{rig}}\), where \(V\) is an \(a\)-adically complete valuation ring of height one (called rigid space of type \((V_R)\) in this book). Moreover, several related classical features in Tate’s rigid analytic geometry or in Berkovich’s analytic geometry, such as overconvergent structure, can also be ‘visualized’ entirely by means of usual point-set topology techniques. It is perhaps one of the most powerful aspects of our visualization that, in this way, many useful concepts in rigid geometry can be simply boiled down to (often elementary) general topology.

In §5 we begin the study of the ‘analytic geometry’ of our rigid spaces. After discussing coherent sheaves on universally Noetherian rigid spaces in §5, we proceed to the theory of affinoids in §6. In this book, affinoids are defined as rigid spaces of the form \((\text{Spf} A)^{\text{rig}}\), that is, coherent rigid spaces having an affine formal model. Among them, especially important are the universally Noetherian affinoids of this form, where \(\text{Spec} A \setminus V(I)\) is affine. Affinoids of this type are called Stein affinoids, the name coming from the fact that these affinoids enjoy Theorem A and Theorem B for coherent sheaves and thus can be viewed as an analogue of Stein domains in complex analysis. The last-mentioned fact is based on the comparison theorem (Theorem 6.4.1) for affinoids, which roughly asserts that the cohomology of coherent sheaves on a universally Noetherian affinoid \((\text{Spf} A)^{\text{rig}}\) is isomorphic to the cohomology of the corresponding coherent sheaves on the Noetherian scheme \(\text{Spec} A \setminus V(I)\).

In §7 we collect basic properties of morphisms between rigid spaces, such as finite morphisms, immersions, separated morphisms, etc. In §8 we develop some useful tools to investigate points of (the visualizations of) rigid spaces, and apply them to the study of classical points on rigid spaces of type \((V_R)\) or on rigid spaces locally having Noetherian formal models (called rigid spaces of type \((N)\)). The notion of classical points will play an important role in establishing the bridge between our rigid geometry and Tate’s rigid analytic geometry, which will be explained in Appendix §B. Another useful application of our study of classical points is the Noetherness theorem (Theorem 8.3.6), which asserts that the local ring at each point of a rigid space of type \((V)\) or of type \((N)\) is Noetherian.
In §9 we discuss GAGA. Our GAGA functor is a functor

\[ X \mapsto X^{\text{an}} \]

from the category of quasi-separated finite type schemes over \( U = \text{Spec} \ A \setminus V(I) \), where \( A \) is an adic ring with a finitely generated ideal of definition \( I \subseteq A \), to the category of rigid spaces of finite type over \( S = (\text{Spf} \ A)^{\text{rig}} \). The construction of the GAGA functor relies on a generalization of Nagata’s embedding theorem, which we prove in the appendix (Theorem F.1.1), and thus will be carried out by resorting to formal geometry. Hence the GAGA theorems (comparison and existence) discussed here, are consequences of the GFGA theorems that we have already established in Chapter I; note that, like the GFGA theorems, our GAGA theorems will be presented in the derived category language.

In §10 we treat briefly the dimension theory in rigid geometry, and in the final section, §11, we discuss the maximal modulus principle.

1 Admissible blow-ups

In this section we discuss generalities and related topics concerning the so-called admissible blow-ups of formal schemes. As rigid spaces (more precisely, coherent rigid spaces) are defined roughly as ‘limits’ of admissible blow-ups of formal schemes, the notion of admissible blow-ups is, so to speak, a main pillar of the bridge from formal geometry to rigid geometry.

After discussing basic properties of admissible blow-ups, we define in §1.2 the so-called strict transforms of admissible blow-ups and give a collection of basic properties of them. In the final subsection, §1.3, we will see that admissible blow-ups of a fixed coherent formal scheme constitute a cofiltered category that admits a small cofinal set. This fact will be used later when we define coherent rigid spaces.

1.1 Admissible blow-ups

1.1. (a) Admissible blow-ups. Let \( X \) be an adic formal scheme of finite ideal type (I.1.1.16). We assume for the moment that there exists an ideal of definition of finite type \( I \subseteq \mathcal{O}_X \). For \( k \geq 0 \) we set \( X_k = (X, \mathcal{O}_X/I^{k+1}) \), which is a closed subscheme of \( X \). Let \( \mathcal{I} \subseteq \mathcal{O}_X \) be an admissible ideal (I.3.7.4). Consider for each \( k \geq 0 \) the projective \( X_k \)-scheme

\[ X'_k = \text{Proj}(\bigoplus_{n\geq 0} \mathcal{I}^n \otimes \mathcal{O}_X) \longrightarrow X_k. \]

For \( k \leq l \), we have obvious closed immersions \( X'_k \hookrightarrow X'_l \), compatible with the immersion \( X_k \hookrightarrow X_l \), and thus get an inductive system of schemes \( \{X'_k\}_{k \geq 0} \). It is easy to see that this inductive system satisfies the conditions in I.1.4.3, and hence
the inductive limit \( X' = \lim_{k \geq 0} X_k \) is an adic formal scheme of finite ideal type endowed with the structural adic map

\[
X' = \lim_{k \geq 0} \text{Proj} \left( \bigoplus_{n \geq 0} \mathcal{J}^n \otimes \mathcal{O}_{X_k} \right) \xrightarrow{\pi} X,
\]

which is proper due to I.4.7.3. Note that the \( X \)-isomorphism class of the map (*) does not depend on the choice of the ideal of definition \( I \). Hence, the construction of \( \pi: X' \to X \) can be done without a globally defined ideal of definition of finite type on \( X \) as follows. We consider an open covering \( X = \bigcup_{\alpha \in L} X_\alpha \), where each \( X_\alpha \) has an ideal of definition \( I_\alpha \) of finite type. Then, for an admissible ideal \( \mathcal{J} \) on \( X \), define \( \pi: X' \to X \) to be the gluing of \( \pi_\alpha: X'_\alpha \to X_\alpha \) constructed as above by means of the admissible ideal \( \mathcal{J}|_{X_\alpha} \).

**Definition 1.1.1.** Let \( X \) be an adic formal scheme of finite ideal type, and \( \mathcal{J} \subseteq \mathcal{O}_X \) an admissible ideal. An adic morphism \( \pi: X' \to X \) of adic formal schemes of finite ideal type is said to be an *admissible (formal) blow-up along \( \mathcal{J} \)* if it is locally isomorphic to a morphism of the form (*).

The admissible blow-ups are uniquely determined by admissible ideals, up to canonical isomorphisms. Note that if \( X \) is quasi-compact (resp. quasi-separated, resp. coherent), then so is \( X' \). In the sequel, when we want to specify the blow-up center \( \mathcal{J} \), we will write

\[
\pi_\mathcal{J}: X_\mathcal{J} \to X.
\]

The following proposition follows immediately from the fact that admissible blow-ups are of finite type.

**Proposition 1.1.2.** Let \( X \) be an adic formal scheme of finite ideal type and \( \mathcal{J} \subseteq \mathcal{O}_X \) an admissible ideal. If \( X \) is locally universally rigid-Noetherian (resp. locally universally adhesive), then so is the admissible blow-up \( X_\mathcal{J} \) along \( \mathcal{J} \).

1.1. (b) Explicit local description. The formation of admissible blow-ups is an effective local construction with respect to the Zariski topology (cf. 0.1.4.9), and hence most of the properties of admissible blow-ups can be verified by reducing to the affine situation.

Let \( A \) be an adic ring of finite ideal type (I.1.6), \( I \subseteq A \) a finitely generated ideal of definition, and \( J \) an \( I \)-admissible ideal of \( A \). Then the admissible blow-up of \( X = \text{Spf} A \) along \( J^\Delta \) is the formal completion (cf. I, §1.4. (b)) of the usual blow-up of the affine scheme \( \text{Spec} A \),

\[
\text{Proj} R(A, J) \to \text{Spec} A
\]

([54], II, (8.1.3)), where \( R(A, J) = \bigoplus_{n \geq 0} J^n \) is the Rees algebra (cf. 0, §7.5).
Suppose \( f_0, \ldots, f_r \in J \) generate \( J \), and consider the exact sequence

\[
0 \longrightarrow K \longrightarrow A[X_0, \ldots, X_r] \overset{\varphi}{\longrightarrow} R(A, J) \longrightarrow 0,
\]

where \( \varphi \) maps each \( X_i \) to \( f_i \) in degree 1. Let \( \alpha \subseteq A[X_0, \ldots, X_r] \) be the ideal generated by all elements of the form

\[
f_i X_j - f_j X_i
\]

for \( 0 \leq i, j \leq r \). By \([54], \text{II}, (2.9.2) \text{(i)}\), the map \( \varphi \) gives rise to an \( A \)-closed immersion

\[
\text{Proj} \ R(A, J) \hookrightarrow \mathbb{P}^r_A,
\]

which induces, by \([4.3.11]\), the closed immersion

\[
\text{Proj} \ R(A, J) \hookrightarrow \widehat{\mathbb{P}^r_A}
\]

into the formal projective \( r \)-space \( \widehat{\mathbb{P}^r_A} \) over \( \text{Spf} \ A \) (cf. Exercise \([5.1 \text{(2)}]\)). As \( \widehat{\mathbb{P}^r_A} \) is covered by the affine open subsets \( \{U_i = \text{Spf} \ A((X_0, \ldots, X_r))\} \), we want to describe explicitly the closed immersion

\[
\text{Proj} \ R(A, J) \times \widehat{\mathbb{P}^r_A} U_i \hookrightarrow U_i
\]

(\( \dagger \)) for each \( 0 \leq i \leq r \). By \([1.4.6]\), the morphism (\( \dagger \)) is the formal completion of the closed immersion of schemes induced by the morphism of rings

\[
A\left[\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i}\right] \longrightarrow R(A, J)_{(f_i)} = A\left[\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i}\right]/\alpha_i^{f_i-\text{sat}},
\]

where \( \alpha_i = (f_i \frac{X_0}{X_i} - f_0, \ldots, f_i \frac{X_r}{X_i} - f_r) \), and \( \alpha_i^{f_i-\text{sat}} \) is the \( f_i \)-saturation of \( \alpha \); note that in the ring \( R(A, J)_{(f_i)} \) the ideal \( JR(A, J)_{(f_i)} = (f_i) \) is invertible (cf. \([54], \text{II}, (8.1.11)\)). Hence the morphism (\( \dagger \)) is isomorphic to

\[
\text{Spf} \ B \hookrightarrow \text{Spf} A((\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i})),
\]

where

\[
B = A((\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i}))/\alpha_i^{f_i-\text{sat}} A((\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i})).
\]

(Recall \([0], \S 8.4\) for the definition of rings of restricted formal power series.)

**Proposition 1.1.3.** Suppose \( A \) is a t.u. rigid-Noetherian ring \((1.2.1.1) \text{(1)}\), and consider the ideal

\[
b_i = (f_i \frac{X_0}{X_i} - f_0, \ldots, f_i \frac{X_r}{X_i} - f_r) \subseteq A((\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i})).
\]

1. We have \( b_i^{f_i-\text{sat}} = \alpha_i^{f_i-\text{sat}} A((\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i})) \) and \( B = A((\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i}))/b_i^{f_i-\text{sat}} \).
2. If \( A \) is an \( I \)-torsion free t.u. adhesive ring \((1.2.1.1) \text{(2)}\), then \( b_i^{f_i-\text{sat}} \) is finitely generated and the ring \( B \) is \( IB \)-torsion free.
Proof. (1) First note that

\[ b_i \subseteq a_i^{f_i-\text{sat}} A(\left\langle \frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i} \right\rangle) \subseteq b_i^{f_i-\text{sat}}. \]

Hence, to show the equality \( b_i^{f_i-\text{sat}} = a_i^{f_i-\text{sat}} A(\left\langle \frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i} \right\rangle) \), it suffices to show that \( a_i^{f_i-\text{sat}} A(\left\langle \frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i} \right\rangle) \) is \( f_i \)-saturated. Since \( A[\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i}] \) is \( I \)-adically universally pseudo-adhesive, the map

\[ A[\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i}] \longrightarrow A(\left\langle \frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i} \right\rangle) \]

is flat (0.8.2.18), and hence so is its base change

\[ A[\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i}] / a_i^{f_i-\text{sat}} \longrightarrow A(\left\langle \frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i} \right\rangle) / a_i^{f_i-\text{sat}} A(\left\langle \frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i} \right\rangle). \]

Since the left-hand side is \( f_i \)-torsion free, so is the right-hand side. Hence we have that \( a_i^{f_i-\text{sat}} A(\left\langle \frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i} \right\rangle) \) is \( f_i \)-saturated, as desired. Now by 0.7.4.18 we know that the ideal \( b_i^{f_i-\text{sat}} \) is closed, whence the second equality.

(2) If \( A \) is \( I \)-torsion free, then so are the rings \( R(A, J) \) and \( R(A, J)(f_i) \). This implies that \( a_i^{f_i-\text{sat}} A(\left\langle \frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i} \right\rangle) \) is \( f_i \)-saturated. Since \( A[\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i}] \) is \( I \)-adically adhesive, \( a_i^{f_i-\text{sat}} \) is finitely generated (0.8.5.3). Finally, since the completion map

\[ A[\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i}] / a_i^{f_i-\text{sat}} \longrightarrow B = A(\left\langle \frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i} \right\rangle) / b_i^{f_i-\text{sat}} \]

is flat and the ring \( A[\frac{X_0}{X_i}, \ldots, \frac{X_r}{X_i}] / a_i^{f_i-\text{sat}} \) is \( I \)-torsion free, we conclude that \( B \) is \( IB \)-torsion free. \( \square \)

It follows from (2) that if \( X \) is a locally universally adhesive formal scheme (see (I.2.1.7)) such that \( \mathcal{O}_X \) is \( I \)-torsion free, where \( I \) is an ideal of definition of \( X \), then any admissible blow-up of \( X \) is finitely presented.

1.1. (c) Universal mapping property

Proposition 1.1.4. Let \( X \) be an adic formal scheme of finite ideal type, \( J \subseteq \mathcal{O}_X \) an admissible ideal, and \( \pi : X' \rightarrow X \) the admissible blow-up of \( X \) along \( J \).

(1) The morphism \( \pi \) is proper.

(2) The ideal \( J \mathcal{O}_{X'} = (\pi^{-1}J)\mathcal{O}_{X'} \) is invertible. In particular, \( \mathcal{O}_{X'} \) is \( J \)-torsion free.
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(3) Universal mapping property. Given an adic morphism \( \theta : Z \to X \) of adic formal schemes of finite ideal type such that \((\theta^{-1}\mathcal{F})\mathcal{O}_Z\) is invertible, there exists a unique morphism \( Z \to X' \) such that the resulting triangle

\[
\begin{array}{ccc}
Z & \to & X' \\
\downarrow_{\theta} & & \downarrow_{\pi} \\
X & & \\
\end{array}
\]

commutes.

Proof. (1) is clear by definition. To show (2), we may assume that \( X \) is affine, \( X = \text{Spf} \, A \), where \( A \) is an adic ring of finite ideal type. We need to verify that the ring \( B \) as in (\( \xi \)) in §1.1.(b) is \( f \)-torsion free. We will show this in 1.1.5 below.

To show (3), we may reduce to the affine situation \( X = \text{Spf} \, A \) and \( Z = \text{Spf} \, B \); let \( J \subseteq A \) be an admissible ideal such that \( J^\Delta = \emptyset \). We may moreover assume that \( JB \) is a principal ideal generated by a non-zero-divisor \( a \in B \). By the universality of blow-ups of schemes, we have the unique morphism from \( \text{Spec} \, B \) to the blow-up of \( \text{Spec} \, A \) along \( J \), from which the desired morphism \( Z \to X' \) is obtained by completion. The uniqueness is easy to see. \( \square \)

**Lemma 1.1.5.** Let \( A \) be a ring, \( I \subseteq A \) a finitely generated ideal and let \( f \in A \). Suppose that the ideal \( J = (f) \) is \( I \)-admissible (0.8.1.2) and that \( A \) is \( f \)-torsion free. Then the \( I \)-adic completion \( \hat{A} \) is \( f \)-torsion free.

Proof. We may assume that \( I \subseteq J \). We want to show that the map \( \hat{A} \to \hat{A} \) given by \( x \mapsto f^n x \) is injective for any \( n \geq 1 \). Since the image \( J^n \) of \( A \to A \) by \( x \mapsto f^n x \) is an open ideal of \( A \), the subspace topology on \( J^n \) induced by the \( I \)-adic topology on \( A \) is the \( I \)-adic topology. Hence, by 0.3.2.4, the injectivity of \( \hat{A} \to \hat{A} \) follows from the injectivity of the map \( A \to A \) given by \( x \mapsto f^n x \). \( \square \)

As a corollary of 1.1.5 we have the following result.

**Corollary 1.1.6.** Let \( X \) be an adic formal scheme with an ideal of definition of finite type \( I \) and \( X' \to X \) an admissible blow-up. If \( \mathcal{O}_X \) is \( I \)-torsion free, then \( \mathcal{O}_{X'} \) is \( I \hat{\mathcal{O}}_{X'} \)-torsion free. In particular, if \( I \) is invertible on \( X \), then \( I \hat{\mathcal{O}}_{X'} \) is invertible on \( X' \).

Proof. We may assume \( X \) is affine, \( X = \text{Spf} \, A \), and \( X' \to \text{Spf} \, A \) is the admissible blow-up along an admissible ideal \( J \subseteq A \). Let \( I \subseteq A \) be a finitely generated ideal of definition such that \( I^\Delta = \emptyset \); we may assume that \( I \subseteq J \). Using the notation as in §1.1.(b), we have that \( B \) is \( f \)-torsion free, where \( JB = (f) \). Since \( \hat{B} \) is \( f \)-torsion free by 1.1.5, it is \( I \hat{\mathcal{B}} \)-torsion free, because \( I \hat{\mathcal{B}} \subseteq J \hat{\mathcal{B}} \). \( \square \)
1.1. (d) Some basic properties. The following proposition follows immediately from the universality (1.1.4 (3)) and the local description (§1.1. (b)) of admissible blow-ups.

Proposition 1.1.7. Let $X \to Y$ be an adic morphism (resp. of finite type) between adic formal schemes of finite ideal type, and $Y' \to Y$ an admissible blow-up. Then there exist an admissible blow-up $X' \to X$ and an adic morphism $X' \to Y'$ (resp. of finite type) such that the diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

commutes. Explicitly, if $\mathcal{J} \subseteq \mathcal{O}_Y$ is an admissible ideal that gives the admissible blow-up $Y' \to Y$, then one can take $X' \to X$ to be the admissible blow-up along the admissible ideal $\mathcal{J}\mathcal{O}_X$ (cf. 1.3.7.6).

Since the formation of admissible blow-ups is an effective local construction, we have the following result.

Proposition 1.1.8. Let $X$ be an adic formal scheme of finite ideal type, and $U \subseteq X$ an open subset. Let $\mathcal{J}$ be an admissible ideal on $X$, and $X' \to X$ and $U' \to U$ the admissible blow-ups along $\mathcal{J}$ and $\mathcal{J}|_U$, respectively. Then $U' \cong X' \times_X U$. In particular, the induced morphism $U' \to X'$ is an open immersion.

Proposition 1.1.9 (extension of admissible blow-up). Let $X$ be a coherent adic formal scheme of finite ideal type, $U \subseteq X$ a quasi-compact open subset, and let $U' \to U$ be an admissible blow-up. Then there exists an admissible blow-up $X' \to X$ that admits an open immersion $U' \hookrightarrow X'$ such that the resulting square

$$
\begin{array}{ccc}
U' & \subseteq & X' \\
\downarrow & & \downarrow \\
U & \subseteq & X
\end{array}
$$

is Cartesian.

Proof. Let $\mathcal{J}$ be an admissible ideal on $U$ that gives the admissible blow-up $U' \to U$. By 1.3.7.15, there exists an admissible ideal $\mathcal{J}'$ on $X$ that extends $\mathcal{J}$. By 1.1.8, the admissible blow-up $X' \to X$ along $\mathcal{J}'$ has the desired property. □

Proposition 1.1.10. The composition of two admissible blow-ups between coherent adic formal schemes of finite ideal type is again an admissible blow-up.
Note that the analogous fact is known in the scheme case ([89], Première partie, (5.1.4), which we present below in (E.1.7)). Before the proof of the proposition we show the following lemma.

**Lemma 1.1.11.** Let $Y$ be a quasi-compact scheme and $I \subseteq \mathcal{O}_Y$ a quasi-coherent ideal of finite type. Let $X = \hat{Y}$ be the formal completion of $Y$ along $I$. Then for any admissible ideal $J \subseteq \mathcal{O}_X$ there exists a quasi-coherent ideal $K \subseteq \mathcal{O}_Y$ of finite type such that

(a) there exists $n \geq 1$ such that $I^n \subseteq K$, and

(b) $K \mathcal{O}_X = J$.

**Proof.** Take $n \geq 1$ such that $I^n \mathcal{O}_X \subseteq J$. The sheaf $J/I^n$ is a quasi-coherent sheaf of finite type on the scheme $(X, \mathcal{O}_X/\mathcal{I}^n \mathcal{O}_X) = (Y, \mathcal{O}_Y/I^n)$. Let $K$ be the pullback of $J/I^n$ by the canonical projection $\mathcal{O}_Y \to \mathcal{O}_Y/I^n$; this is a quasi-coherent ideal of finite type such that $I^n \subseteq K$ and $K/I^n = J/I^n$. By I.3.7.8 (2), we have $K \mathcal{O}_X = J$. □

**Proof of Proposition 1.1.10.** Consider the sequence of morphisms of coherent adic formal schemes of finite ideal type

$$X'' \overset{\pi'}{\longrightarrow} X' \overset{\pi}{\longrightarrow} X,$$

where $\pi$ is an admissible blow-up along $J \subseteq \mathcal{O}_X$ and $\pi'$ is an admissible blow-up along $J' \subseteq \mathcal{O}_{X'}$. We are going to show that the composition $\pi \circ \pi'$ is an admissible blow-up. Let $I \subseteq \mathcal{O}_X$ be an ideal of definition of finite type (I.3.7.12), and assume without loss of generality that $I \subseteq J$. We want to construct an a.q.c. ideal $J'' \subseteq \mathcal{O}_X$ of finite type such that

(a) $I^{k+1} \subseteq J''$ for some $k \geq 0$, and

(b) $J'' \mathcal{O}_{X'} = J' \mathcal{O}_{X'}$ for some $n \geq 1$.

If this is done, then by the universality of admissible blow-ups (1.1.4 (3)), $\pi \circ \pi'$ is easily seen to be isomorphic to the admissible blow-up of $X$ along $J J''$.

Step 1. We first construct an a.q.c. ideal $J''$ that satisfies (a) and (b), but is not necessarily of finite type. We first deal with the affine case $X = \text{Spf} \ A$, where $A$ is an adic ring with the finitely generated ideal of definition $I \subseteq A$ such that $I = I^\Delta = I \mathcal{O}_X$. Let $J \subseteq A$ be the $I$-admissible ideal such that $J = J^\Delta$; we may assume without loss of generality that $I \subseteq J$. Let $q: Y' \to Y = \text{Spec} \ A$ be the blow-up along the ideal $J$. Then $\pi: X' \to X$ is the $I$-adic formal completion of $q$, and hence by 1.1.11 we have a quasi-coherent ideal $\mathcal{K}' \subseteq \mathcal{O}_{Y'}$ of finite type such that $I^n \mathcal{O}_{Y'} \subseteq \mathcal{K}'$ for some $n \geq 1$ and $\mathcal{K} \mathcal{O}_{X'} = J'$. Let $q': Y'' \to Y'$ be the blow-up along $\mathcal{K}'$. Then $\pi': X'' \to X'$ is the $I$-adic formal completion of $q'$. 

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Since $J\mathcal{O}_{Y'}$ is an ample invertible ideal, by [54], II, (4.6.8), there exists $N_0 \geq 1$ such that $\mathcal{K}' \otimes \mathcal{O}_{Y'} J^n \mathcal{O}_{Y'} = \mathcal{K}' J^n \mathcal{O}_{Y'}$ is generated by global sections for any $n \geq N_0$. Moreover, since $q_*\mathcal{O}_{Y'}/\mathcal{O}_Y$ is $J$-torsion, there exists $N_1 (\geq N_0)$ such that for any $n \geq N_1$, the sheaf of ideals $\mathcal{K}' J^n \mathcal{O}_{Y'}$ is generated by global sections from $\mathcal{O}_Y$. Hence, if we define $\mathcal{K}''$ to be the kernel of

$$\mathcal{O}_Y \longrightarrow q_*\mathcal{O}_{Y'}/q_*\mathcal{K}' J^n \mathcal{O}_{Y'}$$

for $n \geq N_1$, it satisfies

(a)' $I^{k+1} \mathcal{O}_Y \subseteq \mathcal{K}''$ for some $k \geq 0$, and

(b)' $\mathcal{K}'' \mathcal{O}_{Y'} = \mathcal{K}' J^n \mathcal{O}_{Y'}$.

Then $\mathcal{K}'' = \mathcal{K}'' \mathcal{O}_X$ is an a.q.c. ideal satisfying (a) and (b).

Step 2. Let $B$ be another adic ring that admits an open immersion $U = \text{Spf} B \longrightarrow X = \text{Spf} A$.

Note that open immersions are adically flat (I.4.8.12). Set $V = \text{Spec} B$, and let $q: V' \rightarrow V$ be the blow-up along $JB$, which admits a canonical map $V' \rightarrow Y'$, and $q': V'' \rightarrow V'$ the blow-up along $\mathcal{K}' \mathcal{O}_{V'}$. Then we have the commutative diagram

$$\begin{array}{ccc}
V'' & \xrightarrow{q'} & V' & \xrightarrow{q} & V \\
\downarrow & & \downarrow & & \downarrow \\
Y'' & \xrightarrow{q'} & Y' & \xrightarrow{q} & Y
\end{array}$$

which induces by the $I$-adic completion the commutative diagram

$$\begin{array}{ccc}
U'' & \xrightarrow{\pi'} & U' & \xrightarrow{\pi} & U \\
\downarrow & & \downarrow & & \downarrow \\
X'' & \xrightarrow{\pi'} & X' & \xrightarrow{\pi} & X
\end{array}$$

in which all vertical arrows are open immersions. Similarly to Step 1, we define for a sufficiently large $n$ the quasi-coherent ideal $\mathcal{M}'' \subseteq \mathcal{O}_V$ to be the kernel of

$$\mathcal{O}_V \longrightarrow q_*\mathcal{O}_{V'}/q_*\mathcal{K}' J^n \mathcal{O}_{V'}.$$

We claim that $\mathcal{J}'' = \mathcal{K}'' \mathcal{O}_X$, as defined in Step 1 (for the same $n$), enjoys the property that $\mathcal{J}'' \mathcal{O}_U = \mathcal{M}'' \mathcal{O}_U$. To show this, since both sides are complete sheaves, it is enough to check that $\mathcal{J}'' \mathcal{O}_U / I^{l+1} \mathcal{O}_U = \mathcal{M}'' \mathcal{O}_U / I^{l+1} \mathcal{O}_U$ for any $l \geq k$, where $k$
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is chosen so that \( I^{k+1} \mathcal{O}_Y \subseteq \mathcal{K} \) and \( I^{k+1} \mathcal{O}_U \subseteq \mathcal{M} \). Since \( \mathcal{G}' / I^{l+1} \mathcal{O}_X = \mathcal{K}' / I^{l+1} \mathcal{O}_Y \) is the kernel of \( \mathcal{G}' / I^{l+1} \mathcal{O}_Y \to q_* \mathcal{O}_{Y'/q_* \mathcal{K}' \mathcal{J}^n \mathcal{O}_{Y'}, \) and since \( U_l = (U, \mathcal{O}_V / I^{l+1} \mathcal{O}_Y) \leftarrow X_l = (X, \mathcal{O}_Y / I^{l+1} \mathcal{O}_Y) \) is flat, \( \mathcal{G}' \mathcal{O}_{U_l} \) coincides with the kernel of \( \mathcal{G}' / I^{l+1} \mathcal{O}_V \to q_* \mathcal{O}_{V'/q_* \mathcal{K}' \mathcal{J}^n \mathcal{O}_{V'}, \) that is, we have \( \mathcal{G}' \mathcal{O}_{U_l} = \mathcal{M}' / \mathcal{O}_{U_l} \), as desired.

Step 3. Now we discuss the general case. Take a finite affine open covering \( X = \bigcup_{\alpha \in L} X_\alpha \), and consider for each \( \alpha, \beta \in L \) a finite covering of \( X_{\alpha \beta} = X_\alpha \cap X_\beta \) by affine open subsets. If one takes \( n \) and \( k \) to be sufficiently large, one has for each \( \alpha \in L \) an a.q.c. ideal \( \mathcal{J}_\alpha \) such that \( I^{k+1} \mathcal{O}_{X_\alpha} \subseteq \mathcal{J}_\alpha \) and \( \mathcal{J}_\alpha \mathcal{O}_{X_\alpha'} = \mathcal{J}_\alpha^\prime \mathcal{O}_{X_\alpha'} \), where \( X_\alpha' \) is the open subset of \( X' \) that is the admissible blow-up of \( X_\alpha \) along \( \mathcal{J}_\alpha \mathcal{O}_{X_\alpha} \) (cf. 1.1.8). By what we have seen in Step 2, replacing \( n \) by a larger one if necessary, we have \( \mathcal{J}_\alpha^\prime \mathcal{O}_{X_{\alpha \beta}} = \mathcal{J}_\alpha^\prime \mathcal{O}_{X_{\alpha \beta}} \) for all \( \alpha, \beta \in L \). Hence the ideals \( \mathcal{J}_\alpha^\prime \) glue to an a.q.c. ideal \( \mathcal{J}_\alpha^\prime \) of \( \mathcal{O}_X \), which obviously satisfies (a) and (b).

Step 4. In view of [54], I, (9.4.9), and IV, (1.7.7), we know that the quasi-coherent ideal \( \mathcal{G}' / I^{k+1} \) on the scheme \( X_k = (X, \mathcal{O}_X / I^{k+1}) \) is the inductive limit \( \lim_{\longrightarrow \lambda \in \Lambda} \mathcal{J}_\lambda^\prime \) of quasi-coherent ideals of finite type. Since \( \mathcal{J}_\lambda^\prime \mathcal{O}_{X_\lambda} \) is of finite type, there exists \( \lambda \in \Lambda \) such that the pull-back \( \mathcal{J}_\lambda'' \) of \( \mathcal{J}_\lambda^\prime \) by the canonical projection \( \mathcal{O}_X \to \mathcal{O}_{X_k} \) satisfies (b); it also satisfies (a) by the construction. Hence, replacing \( \mathcal{J}_\lambda^\prime \) by this \( \mathcal{J}_\lambda'' \), we finally get the desired admissible ideal, and thus the proof of the proposition is finished.

\[ \square \]

1.2 Strict transform

**Definition 1.2.1.** Let \( X \) be an adic formal scheme of finite ideal type, \( \mathcal{J} \) an admissible ideal, and \( \pi : X' \to X \) the admissible blow-up of \( X \) along \( \mathcal{J} \). For an \( \mathcal{O}_X \)-module \( \mathcal{F} \), the strict transform of \( \mathcal{F} \) by \( \pi \) is the \( \mathcal{O}_{X'} \)-module given by

\[
\pi' \mathcal{F} = \text{the completion of } \pi^* \mathcal{F} / (\pi^* \mathcal{F})_{\mathcal{J}, \mathfrak{tor}}
\]

(cf. I.3.1.1 for the definition of the completion of sheaves).

**Proposition 1.2.2.** If \( \mathcal{F} \) is an a.q.c. sheaf (I.3.1.3) (resp. a.q.c. \( \mathcal{O}_X \)-algebra) on \( X \), then the strict transform \( \pi' \mathcal{F} \) is an a.q.c. sheaf (resp. a.q.c. \( \mathcal{O}_X \)-algebra) on \( X' \). If, moreover, \( \mathcal{F} \) is of finite type, then so is \( \pi' \mathcal{F} \).

To show the proposition, we need to prepare a few lemmas. The following lemma is a generalization of 1.1.5, and the proof is similar.

**Lemma 1.2.3.** Let \( A \) be a ring, \( I \subseteq A \) a finitely generated ideal, and \( f \in A \) an element such that the ideal \( (f) \) is open with respect to the \( I \)-adic topology. If \( \mathcal{M} \) is an \( f \)-torsion free \( A \)-module, then its \( I \)-adic completion \( \hat{\mathcal{M}} \) is \( f \)-torsion free.
Corollary 1.2.4. Let \( A \) be a ring, \( I \subseteq A \) a finitely generated ideal, and \( f \in A \) an element such that the ideal \((f)\) is open with respect to the \( I \)-adic topology. Then for any \( A \)-module \( M \) we have

\[
\overline{M}_{f,\text{tor}} = M_{f,\text{tor}},
\]

where the left-hand side denotes the closure in \( \hat{M} \) of its \( f \)-torsion part, and the right-hand side is the closure of the image of \( M_{f,\text{tor}} \) in \( \hat{M} \).

Proof. The exact sequence \( 0 \to M_{f,\text{tor}} \to M \to M/M_{f,\text{tor}} \to 0 \) induces the exact sequence

\[
0 \longrightarrow \overline{M}_{f,\text{tor}} \longrightarrow \hat{M} \longrightarrow \overline{M}/M_{f,\text{tor}} \longrightarrow 0.
\]

The last module is \( f \)-torsion free by 1.2.3, and hence \( \overline{M}_{f,\text{tor}} \subseteq \overline{M}_{f,\text{tor}} \). On the other hand, since the image of \( M_{f,\text{tor}} \) in \( \hat{M} \) is contained in \( \overline{M}_{f,\text{tor}} \), we have \( \overline{M}_{f,\text{tor}} \supseteq \overline{M}_{f,\text{tor}} \), whence the claim.

Corollary 1.2.5. Let \( A \) be a ring, \( I \subseteq A \) a finitely generated ideal, and \( B \) an \( A \)-algebra. Consider the respective \( I \)-adic completions \( \hat{A} \) and \( \hat{B} \) of \( A \) and \( B \). Let \( f \in B \) be an element such that the ideal \((f)\) is open with respect to the \( IB \)-adic topology. Then for any finitely generated \( A \)-module \( M \) the \( I \)-adic completion of \( \hat{M} \otimes_{\hat{A}} \hat{B} / (\hat{M} \otimes_{\hat{A}} \hat{B})_{f,\text{tor}} \) coincides up to a canonical isomorphism with the \( I \)-adic completion of \( M \otimes_A B / (M \otimes_A B)_{f,\text{tor}} \).

Proof. It follows from 1.2.4 that the closure of the image of \( (\hat{M} \otimes_{\hat{A}} \hat{B})_{f,\text{tor}} \) in \( M \otimes_A B \) coincides with the closure of \( (M \otimes_A B)_{f,\text{tor}} \), which further coincides, again by 1.2.4, with the closure of the image of \( (M \otimes_A B)_{f,\text{tor}} \).

Lemma 1.2.6. Let \( A \) be an adic ring with a finitely generated ideal of definition \( I \subseteq A \), and \( J \subseteq A \) an admissible ideal. Set \( X = \text{Spf} A \) and \( Y = \text{Spec} A \), and consider the commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{i} & X' \\
\downarrow p & & \_downarrow \pi \\
Y & \xleftrightsquigarrow & X
\end{array}
\]

where \( p \) is the blow-up along \( J \), and \( \pi \) is the admissible blow-up along \( J \), the formal completion of \( p \). Then for any \( A \)-module (resp. \( A \)-algebra) \( M \) we have

\[
\pi'M^\Delta = p'\hat{M},
\]

where \( p'\hat{M} \) denotes the scheme-theoretic strict transform of \( \hat{M} \). In particular, \( \pi'M^\Delta \) is an a.q.c. sheaf (resp. a.q.c. \( \Theta_{X'} \)-algebra). If, moreover, \( M \) is finitely generated, then \( \pi'M^\Delta \) is of finite type.
Proof. Consider any affine open subset $U = \text{Spec} \ B$ of $Y'$ such that $JB = (f)$ for a non-zero-divisor $f \in B$. Then $\pi'M^\wedge$ is the sheafification of the presheaf

$$\hat{\mathcal{U}} \longmapsto \text{the completion of } \hat{M} \otimes_A \hat{B} / (\hat{M} \otimes_A \hat{B})_{f\text{-tor}},$$

and $\hat{\mathcal{P}}^\wedge$ is the sheafification of

$$\hat{\mathcal{U}} \longmapsto \text{the completion of } M \otimes_A B / (M \otimes_A B)_{f\text{-tor}}.$$

These two sheaves coincide with each other due to 1.2.5. Since the $f$-torsion part of the quasi-coherent sheaf $\mathcal{P}^\wedge$ is quasi-coherent, $\hat{\mathcal{P}}^\wedge$ is a completion of a quasi-coherent sheaf, and hence is a.q.c.

Proof of Proposition 1.2.2. We may assume that $X$ is affine. Then the admissible blow-up $\pi: X' \to X$ is the formal completion of a blow-up of an affine scheme as in 1.2.6, hence the assertion follows from I.3.2.8 and 1.2.6.

Proposition 1.2.7. Let $X$ be a locally universally rigid-Noetherian formal scheme (I.2.1.7), $\mathcal{J}$ an admissible ideal, and $\pi: X' \to X$ the admissible blow-up along $\mathcal{J}$. Then for any a.q.c. sheaf of finite type $\mathcal{F}$ we have

$$\pi' \mathcal{F} = \pi^* \mathcal{F} / (\pi^* \mathcal{F})_{\mathcal{J}\text{-tor}}.$$

Proof. We may assume that $X$ is affine of the form $X = \text{Spf} \ A$, where $A$ is a t.u. rigid-Noetherian ring (I.2.1.1); we use the notation as in 1.2.6. Since $p^\wedge \hat{M}$ is a quasi-coherent sheaf of finite type on $Y'$, its completion coincides with $i^* p^\wedge \hat{M}$ due to I.3.5.1. Then the assertion follows from the fact that the morphism $i$ is flat.

We will indicate below in Exercise II.1.5 another proof of 1.2.7. It can be shown that, in the situation as in 1.2.7, the strict transform of $\mathcal{J}$ coincides with the pull-back ideal $\mathcal{J}\mathcal{O}_{X'} = (\pi^{-1} \mathcal{J})\mathcal{O}_{X'}$ (cf. Exercise II.1.6).

Let $f: Y \to X$ be an adic morphism of adic formal schemes of finite ideal type and $\pi: X' \to X$ the admissible blow-up of $X$ along an admissible ideal $\mathcal{J} \subseteq \mathcal{O}_X$. Consider the base change $Y \times_X X' \to X'$. By 1.2.2, the completion of $\mathcal{O}_{Y \times_X X'} / \mathcal{O}_{Y \times_X X'}_{\mathcal{J}\text{-tor}}$ is an a.q.c. $\mathcal{O}_{Y \times_X X'}$-algebra, which is a quotient of the structure sheaf $\mathcal{O}_{Y \times_X X'}$ (due to I.1.1.23 (1)). Hence one can consider the closed formal subspace $Y' \hookrightarrow Y \times_X X'$ defined by the ideal $\mathcal{K} = \mathcal{O}_{Y \times_X X'}_{\mathcal{J}\text{-tor}}$ (cf. I.4.3.3).

Definition 1.2.8. In the above situation, the composite map

$$Y' \longmapsto Y \times_X X' \to X'$$

is called the strict transform of $f$ by the admissible blow-up $\pi$. 
**Proposition 1.2.9.** Let \( \pi : X' \to X \) be an admissible blow-up along an admissible ideal \( \mathcal{J} \) of an adic formal scheme \( X \) of finite ideal type and \( f : Y \to X \) an adic morphism of adic formal schemes of finite ideal type. Let \( f' : Y' \to X' \) be the strict transform of \( f \) by \( \pi \). Then the morphism \( Y' \to Y \) is the admissible blow-up along the admissible ideal \( \mathcal{J}\mathcal{O}_Y \) (cf. I.3.7.6).

**Proof.** Let \( Y'' \to Y \) be the admissible blow-up along \( \mathcal{J}\mathcal{O}_Y \). By the universality of admissible blow-ups applied to \( X' \to X \), we get an \( X \)-morphism \( Y'' \to X' \). Hence we get a morphism \( Y'' \to Y \times_X X' \), which induces, in view of 1.1.4 (2), a \( Y \)-morphism \( Y'' \to Y' \). On the other hand, since \( \mathcal{J}\mathcal{O}_{Y'} \) is invertible, the universality of admissible blow-ups applied to \( Y'' \to Y \) implies that there exists a unique \( Y \)-morphism \( Y' \to Y'' \), which is easily seen to be the inverse to \( Y'' \to Y' \). \( \square \)

**Proposition 1.2.10.** Let \( i : Y \to X \) be a closed immersion of finite presentation of locally universally rigid-Noetherian formal schemes and \( Y' \to Y \) an admissible blow-up of \( Y \). Then there exist an admissible blow-up \( X' \to X \) and a closed immersion \( Y' \to X' \) that coincides with the strict transform of \( i \).

**Proof.** Let \( \mathcal{K} \) be the admissible ideal of \( \mathcal{O}_Y \) that gives \( Y' \to Y \). Let \( \mathcal{J} \) be the pull-back of \( \mathcal{K} \) by the map \( \mathcal{O}_X \to i_* \mathcal{O}_Y \), which is an admissible ideal (I.4.3.15). Since \( \mathcal{K} = \mathcal{J}\mathcal{O}_Y \), the assertion follows from 1.2.9. \( \square \)

**Proposition 1.2.11.** Let \( X \) be a locally universally rigid-Noetherian formal scheme with an ideal of definition \( \mathcal{I} \) such that \( \mathcal{O}_X \) is \( \mathcal{I} \)-torsion free and \( \mathcal{F} \) an \( \mathcal{I} \)-torsion free a.q.c. \( \mathcal{O}_X \)-module of finite type. Let \( \mathcal{J} \) be an admissible ideal of \( X \) and \( \pi : X' \to X \) the admissible blow-up along \( \mathcal{J} \). Then

\[
(\pi^* \mathcal{F})_{\mathcal{J}\text{-tor}} = (\pi^* \mathcal{F})_{\mathcal{I}\text{-tor}}.
\]

**Proof.** We may assume that \( X \) is affine, \( X = \text{Spf} A \), with \( \mathcal{I} = \mathcal{I}^A \), where \( A \) is an \( \mathcal{I} \)-torsion free t.u. rigid-Noetherian ring. Let \( \mathcal{J} = J^A \) with \( J = (f_0, \ldots, f_r) \), and \( B \) the ring as in 1.1.3 (1). Let \( M \) be an \( \mathcal{I} \)-torsion free finitely generated \( A \)-module such that \( \mathcal{F} = M^A \). We need to show that \( (M \otimes_A B)_{f_i\text{-tor}} = (M \otimes_A B)_{\mathcal{I}\text{-tor}} \); note that, since \( M \) is finitely generated, we have \( M \otimes_A B = M \otimes_A B (0.8.2.18 \ (1)) \). Since \( J \) is admissible, the inclusion \( (M \otimes_A B)_{f_i\text{-tor}} \subseteq (M \otimes_A B)_{\mathcal{I}\text{-tor}} \) is clear. The other inclusion is shown by an argument similar to that in the proof of 1.1.3. Let \( P \in R(A, J)_{(f_i)} \otimes_A M \) be an \( \mathcal{I} \)-torsion element, and take \( N > 0 \) large enough so that \( f_i^N P \in M \). Since \( M \) is \( \mathcal{I} \)-torsion free, we have \( f_i^N P = 0 \), which implies that \( P \) is a \( f_i \)-torsion element. Hence we have \( (R(A, J)_{(f_i)} \otimes_A M)_{f_i\text{-tor}} = (R(A, J)_{(f_i)} \otimes_A M)_{\mathcal{I}\text{-tor}} \). Since \( R(A, J)_{(f_i)} \to B \) is flat (0.8.2.18 (2)), we have the desired equality by base change. \( \square \)
Corollary 1.2.12. Let $X$ be a locally universally rigid-Noetherian formal scheme, and $\mathcal{I}$ an ideal of definition of finite type of $X$. Suppose $\mathcal{O}_X$ is $\mathcal{I}$-torsion free. Let $\mathcal{F}$ be an $\mathcal{I}$-torsion free a.q.c. sheaf of finite type on $X$. Then for an admissible blow-up $\pi: X' \to X$ the strict transform $\pi^* \mathcal{F}$ is an $\mathcal{I}\mathcal{O}_{X'}$-torsion free a.q.c. sheaf of finite type on $X'$.

Note that, if $X$ is locally universally adhesive, then $\pi^* \mathcal{F}$ is an a.q.c. sheaf of finite presentation on $X'$ (cf. I.3.5.10).

1.3 The cofiltered category of admissible blow-ups

Let $X$ be a coherent adic formal scheme of finite ideal type. We define $\text{BL}_X$ to be the category of all admissible blow-ups of $X$; more precisely

- objects of $\text{BL}_X$ are the admissible blow-ups $\pi: X' \to X$;
- an arrow $\pi' \to \pi$ between two objects $\pi: X' \to X$ and $\pi': X'' \to X$ is a morphism $X'' \to X'$ over $X$:

$$
\begin{array}{ccc}
X'' & \longrightarrow & X' \\
\downarrow \pi' & & \downarrow \pi \\
X & \to & \\
\end{array}
$$

Proposition 1.3.1. (1) The category $\text{BL}_X$ is cofiltered (cf. 0, §1.3.(c)), and $\text{id}_X$ gives the final object.

(2) Define the ordering on the set $\text{AId}_X$ (= the set of all admissible ideals of $X$) as follows: $\mathcal{J} \preceq \mathcal{J}'$ if there exists an admissible ideal $\mathcal{J}''$ such that $\mathcal{J} = \mathcal{J}' \mathcal{J}''$ (cf. I.3.7.9). Then $\text{AId}_X^{\text{opp}}$ is a directed set, and the functor

$$
\text{AId}_X \longrightarrow \text{BL}_X
$$

(where $\text{AId}_X$ is regarded as a category; cf. 0, §1.2.(e)) that maps $\mathcal{J}$ to the admissible blow-up along $\mathcal{J}$ is cofinal.

Proof. To show (1), we need to check the following facts.

(a) For two admissible blow-ups $X' \to X$ and $X'' \to X$, there exist an admissible blow-up $X''' \to X$ and $X$-morphisms $X''' \to X'$ and $X''' \to X''$.

(b) For two admissible blow-ups $X' \to X$ and $X'' \to X$ and two $X$-morphisms $f_0, f_1: X'' \to X'$, there exists an admissible blow-up $X''' \to X$ with an $X$-morphism $g: X''' \to X''$ such that $f_0 \circ g = f_1 \circ g$. 
Let $X' \to X$ and $X'' \to X$ be admissible blow-ups along $\mathcal{J}$ and $\mathcal{J}'$, respectively. Let $X''' \to X$ be the admissible blow-up along $\mathcal{J}\mathcal{J}'$ (cf. I.3.7.9). Then by 1.1.4 (3) one has the morphisms $X''' \to X'$ and $X''' \to X''$ as in (a). If $f_0$ and $f_1$ are as in (b), we have $f_0 \circ g = f_1 \circ g$ by the uniqueness in 1.1.4 (3), whence (b). We have thus shown (1).

(2) is clear.

Corollary 1.3.2. The category $\text{BL}_X$ is cofiltered and essentially small; cf. 0, §1.3. (c).

Exercises

Exercise II.1.1. Let $X$ be an adic formal scheme of finite ideal type, and $\mathcal{J}, \mathcal{J}' \subseteq \mathcal{O}_X$ admissible ideals. Let $\pi: X' \to X$ be the admissible blow-up along $\mathcal{J}$, and $\pi': X'' \to X'$ the admissible blow-up along $(\pi^{-1}\mathcal{J})\mathcal{O}_{X'}$. Show that the composition $\pi \circ \pi': X'' \to X$ coincides up to canonical isomorphism with the admissible blow-up of $X$ along $\mathcal{J} \cdot \mathcal{J}'$.

Exercise II.1.2. Let $\pi: X' \to X$ be an admissible blow-up along an admissible ideal $\mathcal{J}$ on $X$, and $f: Y \to X$ an adic morphism of adic formal schemes of finite ideal type. Then there exist admissible blow-ups $Z \to Y' = X' \times_X Y$ and $Z \to Y$ such that the resulting diagram commutes:

$$
\begin{array}{ccc}
Z & \rightarrow & Y' \\
\downarrow & & \downarrow \pi \\
Y' & \rightarrow & Y
\end{array}
$$

Exercise II.1.3. Let $A$ be a t.u. rigid-Noetherian ring (I.2.1.1 (1)), $I \subseteq A$ a finitely generated ideal of definition, and $J \subseteq A$ an $I$-admissible ideal. Let $M$ be a finitely generated $A$-module. Set $\mathcal{F} = M^\Delta$ and $\mathcal{J} = J^\Delta$. Show that $\mathcal{F}_{\mathcal{J}\text{-tor}} = (M_{J\text{-tor}})^\Delta$.

Exercise II.1.4. Let $X$ be a locally universally rigid-Noetherian formal scheme, $\mathcal{F}$ an a.q.c. sheaf of finite type on $X$, and $\mathcal{J} \subseteq \mathcal{O}_X$ an admissible ideal. Show that $\mathcal{F}/\mathcal{F}_{\mathcal{J}\text{-tor}}$ is an a.q.c. sheaf of finite type.

Exercise II.1.5. Prove 1.2.7 by using Exercise II.1.4.

Exercise II.1.6. Let $A \to B$ be an adic morphism of t.u. rigid-Noetherian rings, $I \subseteq A$ a finitely generated ideal of definition of $A$, and $J \subseteq A$ an $I$-admissible ideal. Suppose $B$ is $J$-torsion free. Set $X = \text{Spf} A$, $Y = \text{Spf} B$, and $\mathcal{J} = J^\Delta$. Let $\mathcal{K}$ be the kernel of $\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \mathcal{J}\mathcal{O}_Y$, and $K$ the kernel of $J \otimes_A B \to JB$.

1. Show that $\mathcal{K} = K^\Delta$.

2. Show that $\mathcal{K} = (\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_Y)_{\mathcal{J}\text{-tor}}$. 
2 Rigid spaces

In this section, we give the definition of rigid spaces and discuss their basic properties. We first define in §2.1 what we call coherent rigid spaces as the objects in the quotient category of coherent adic formal schemes of finite ideal type modulo by all admissible blow-ups. The coherent rigid spaces are, therefore, always of the form ‘\(X^{\text{rig}}\)’, induced from a coherent adic of finite ideal type formal scheme \(X\), which we call a formal model of the coherent rigid space. There are of course plenty of formal models attached to a single coherent rigid space, but they are always connected by admissible blow-ups and blow-downs.

A coherent rigid space thus defined comes with a natural topology, the so-called admissible topology, which will be discussed in §2.2. Roughly speaking, this is the topology most naturally induced by the Zariski topology on the formal schemes. Since we are, at first, only able to speak about ‘coherent’ rigid spaces, open subsets (or equivalently, open immersions) are temporarily restricted to only quasi-compact ones. Endowed with this topology, the category of coherent rigid spaces gives rise to the so-called coherent admissible site. Here it turns out that the adjective ‘coherent’ is justified by the fact that the topos associated to the coherent admissible site is in fact coherent in the sense of [9], Exposé VI, (cf. 0.2.7.5). It also turns out that the terminologies ‘coherent’ rigid space and ‘coherent’ open immersion (as defined in 2.2.2) can be justified in the sense that they are exactly the coherent objects in the topos, the so-called admissible topos. Since this justification is most fluently done in terms of visualization, we postpone it to the next section.

General rigid spaces are defined to be sheaves on the coherent admissible site that satisfy a certain ‘local representability’ property. The already defined coherent rigid spaces are in fact rigid spaces in this generalized sense, since on the coherent admissible site the representable presheaves are sheaves (2.2.10). This fact is closely linked with one of the most important aspects of our birational approach to rigid geometry (cf. Introduction). The (coherent) rigid spaces, which are at first defined formally as objects in the above-mentioned quotient category, admit ‘patching’, modeled on ‘birational’ (= up to admissible blow-ups) patching of formal schemes.

The definition of general rigid spaces given in §2.2.(c) allows one to enhance the admissible topology to a slightly stronger and more consistent one, also called the admissible topology. It is helpful, especially for the reader who is familiar to Tate’s rigid analytic geometry, to remark that the admissible topology in the former sense is the one somewhat similar to the so-called ‘weak topology’, and the latter...
one to the ‘strong topology’ in Tate’s rigid analytic geometry (cf. [18]); see §B.2 for a more precise comparison of the topologies.

In §2.3 we will define the notion of (locally) of finite type morphisms. The final subsection, §2.4, is devoted to fiber products of rigid spaces.

2.1 Coherent rigid spaces and their formal models

2.1. (a) Coherent rigid spaces

**Definition 2.1.1.** We define the category $\text{CRf}$ as follows.

- The objects of $\text{CRf}$ are the same as those of $\text{AcCFs}^*$, the category of coherent adic formal schemes of finite ideal type and adic morphisms:

\[
\text{obj}(\text{CRf}) = \text{obj}(\text{AcCFs}^*);
\]

for an object $X$ of $\text{AcCFs}^*$ we denote by $X^{\text{rig}}$ the same object regarded as an object of $\text{CRf}$.

- For $X, Y \in \text{obj}(\text{AcCFs}^*)$ we define

\[
\text{Hom}_{\text{CRf}}(X^{\text{rig}}, Y^{\text{rig}}) = \lim_{\text{ind} \to NUL!} \text{Hom}_{\text{AcCFs}^*}(\cdot, Y),
\]

where $\text{Hom}_{\text{AcCFs}^*}(\cdot, Y)$ is the functor $\text{BL}_X^{\text{opp}} \to \text{Sets}$ that maps $\pi : X' \to X$ to the set $\text{Hom}_{\text{AcCFs}^*}(X', Y)$ (cf. 0, §1.3. (a)).

We sometimes describe the quotient functor as

\[
Q : \text{AcCFs}^* \longrightarrow \text{CRf}, \quad X \longmapsto Q(X) = X^{\text{rig}}.
\]

Note that by 1.3.1 the inductive limit in the above definition can be replaced by a filtered inductive limit along the directed set $\text{Al}_{\text{opp}} X$ (cf. 0.1.3.1).

In the category $\text{CRf}$, the composition law

\[
\text{Hom}_{\text{CRf}}(X^{\text{rig}}, Y^{\text{rig}}) \times \text{Hom}_{\text{CRf}}(Y^{\text{rig}}, Z^{\text{rig}}) \longrightarrow \text{Hom}_{\text{CRf}}(X^{\text{rig}}, Z^{\text{rig}})
\]

is described as follows. A morphism $\varphi$ in $\text{Hom}_{\text{CRf}}(X^{\text{rig}}, Y^{\text{rig}})$ is given by a diagram of the form

\[
X' \leftarrow X \rightarrow Y
\]

where $X' \to X$ is an admissible blow-up (1.1.1). Similarly, a morphism $\psi$ in $\text{Hom}_{\text{CRf}}(Y^{\text{rig}}, Z^{\text{rig}})$ is given by $Y \leftarrow Y' \to Z$, where the first arrow is an admissible blow-up. By 1.1.7, we have an admissible blow-up $X'' \to X'$ and an adic
morphism \( X'' \rightarrow Y' \) such that the square in the following diagram commutes:

\[
\begin{array}{ccc}
X'' & \xleftarrow{f} & Y' \\
\downarrow & & \downarrow \\
X' & \xleftarrow{g} & Y \\
\end{array}
\]

By 1.1.10 one sees that the composition \( X'' \rightarrow X \) is an admissible blow-up, and hence the diagram \( X \leftarrow X'' \rightarrow Z \) gives an element in \( \text{Hom}_{\text{CRf}}(X^{\text{rig}}, Z^{\text{rig}}) \). One can verify that the resulting element in \( \text{Hom}_{\text{CRf}}(X^{\text{rig}}, Z^{\text{rig}}) \) does not depend on the choice of \( X' \) and \( Y' \) and thus gives the desired composition \( \psi \circ \varphi \).

**Definition 2.1.2.** (1) Objects of \( \text{CRf} \) are called *coherent rigid (formal) spaces*.

(2) For an object \( X \) of \( \text{AcCFs}^* \), the coherent rigid space \( X^{\text{rig}} \) is called the associated (coherent) rigid space. Similarly, for a morphism \( f : X \rightarrow Y \) of \( \text{AcCFs}^* \), the associated morphism of rigid spaces is denoted by \( f^{\text{rig}} : X^{\text{rig}} \rightarrow Y^{\text{rig}} \).

**Remark 2.1.3.** The adjective ‘formal’ in the parentheses in (1) and also the letter ‘\( f \)’ in the notation \( \text{CRf} \) are used for specifying that the above-defined rigid spaces come from formal schemes, indicating future variants including, for example, rigid Henselian spaces and rigid Zariskian spaces (cf. §D in the appendix), which are the similarly defined spaces associated respectively to Henselian schemes and Zariskian schemes.

By an easy but deft use of 1.1.4 (3), one can show the following proposition.

**Proposition 2.1.4.** Let \( X \) and \( Y \) be coherent adic formal schemes of finite ideal type, and consider the rigid spaces \( X^{\text{rig}} \) and \( Y^{\text{rig}} \). Then there exists an isomorphism \( X^{\text{rig}} \cong Y^{\text{rig}} \) in \( \text{CRf} \) if and only if there exists a diagram \( X \leftarrow Z \rightarrow Y \) consisting of admissible blow-ups.

**Corollary 2.1.5.** Let \( f : X \rightarrow Y \) be a morphism in \( \text{AcCFs}^* \). Then

\[
f^{\text{rig}} : X^{\text{rig}} \longrightarrow Y^{\text{rig}}
\]

is an isomorphism if and only there exists a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X & \xleftarrow{g} & Y \\
\end{array}
\]

where both \( Z \rightarrow X \) and \( Z \rightarrow Y \) are admissible blow-ups.

By 2.1.5 and Exercise II.1.2, we have the following corollary.
**Corollary 2.1.6.** Consider the diagram

\[
\begin{array}{c}
X & \xleftarrow{f} & X \times Z Y \\
\downarrow & & \downarrow h \\
Z & \xleftarrow{\phi} & Y.
\end{array}
\]

If \( f \rig \) is an isomorphism in \( \text{CRf} \), then so is \( h \rig \).

### 2.1. (b) Formal models

**Definition 2.1.7.** (1) Let \( \mathcal{X} \) be a coherent rigid space. A formal model of \( \mathcal{X} \) is a couple \( (X, \phi) \) consisting of \( X \in \text{obj}(\text{AcCFs}^*) \) and an isomorphism \( \phi: X \rig \sim \mathcal{X} \) in \( \text{CRf} \). In this way one obtains \( M_\mathcal{X} \), the category of formal models of \( \mathcal{X} \), in which an arrow \( (X, \phi) \rightarrow (X', \phi') \) is defined to be a morphism \( f: X \rightarrow X' \) in \( \text{AcCFs}^* \) such that \( \phi' \circ f \rig = \phi \).

(2) Let \( \varphi: \mathcal{X} \rightarrow \mathcal{Y} \) be a morphism of coherent rigid spaces. A formal model of \( \varphi \) is a triple \( (f, \phi, \psi) \) consisting of a morphism \( f: X \rightarrow Y \) of \( \text{AcCFs}^* \) and isomorphisms \( \phi: X \rig \sim \mathcal{X} \) and \( \psi: Y \rig \sim \mathcal{Y} \) such that the resulting square

\[
\begin{array}{ccc}
X \rig & \xrightarrow{f \rig} & Y \rig \\
\phi & \downarrow & \psi \\
\mathcal{X} & \xrightarrow{\varphi} & \mathcal{Y}
\end{array}
\]

commutes. They constitute the category \( M_\varphi \) of formal models of \( \varphi \), in which an arrow \( (f: X \rightarrow Y, \phi, \psi) \rightarrow (f': X' \rightarrow Y', \phi', \psi') \) is defined to be a couple of morphisms \( (u, v) \) consisting of \( u: X \rightarrow X' \) and \( v: Y \rightarrow Y' \) such that \( v \circ f = f' \circ u \) and that the following diagram is commutative:

\[
\begin{array}{ccc}
X \rig & \xrightarrow{f \rig} & Y \rig \\
\phi & \downarrow & \psi \\
\mathcal{X} & \xrightarrow{\varphi} & \mathcal{Y}
\end{array}
\]

It is readily seen that these categories are cofiltered (due to 1.3.2).
Definition 2.1.8. (1) Let $X$ be a coherent rigid space. A formal model $(X, \phi)$ of $X$ is said to be distinguished if $\mathcal{O}_X$ is $I$-torsion free for some (hence any) ideal of definition $I$ of $X$.

(2) Let $\varphi: X \to Y$ be a morphism of coherent rigid spaces. A formal model $(f: X \to Y, \phi, \psi)$ of $\varphi$ is said to be distinguished if $X$ and $Y$ are distinguished formal models of $X$ and $Y$, respectively.

Proposition 2.1.9 (cf. 1.1.6). Let $(X, \phi)$ be a distinguished formal model of a coherent rigid space $X$, and $\pi: X' \to X$ an admissible blow-up. Then $(X', \phi \circ \pi_{\text{rig}})$ is a distinguished formal model of $X$. If, moreover, $X$ has an invertible ideal of definition $I$, then $I\mathcal{O}_{X'}$ is invertible.

We denote by $M^\text{dist}_{X}$ (resp. $M^\text{dist}_\varphi$) the full subcategory of $M_X$ (resp. $M_\varphi$) consisting of distinguished formal models.

Proposition 2.1.10. The categories $M^\text{dist}_{X}$ and $M^\text{dist}_\varphi$ are cofiltered, and the inclusions $M^\text{dist}_{X} \hookrightarrow M_X$ and $M^\text{dist}_\varphi \hookrightarrow M_\varphi$ are cofinal. (Hence, in particular, any object and any morphism of $\text{CRf}$ have distinguished formal models.)

Proof. The proposition follows from the following observation. Let $X$ be an object of $\text{AcCFS}^*$, and $I$ an ideal of definition of finite type; then the structure sheaf of the admissible blow-up $X'$ of $X$ along $I$ is $I\mathcal{O}_{X'}$-torsion free by 1.1.4 (2). Note the following fact: for a morphism $\varphi: X \to Y$ of coherent rigid spaces with a formal model $f: X \to Y$, let $Y' \to Y$ and $X' \to X$ be the admissible blow-ups along an ideal of definition $I$ of finite type on $Y$ and $I\mathcal{O}_X$, respectively; then by 1.1.4 (3) we have the unique morphism $f': X' \to Y'$ (cf. 1.1.7), which gives a distinguished formal model of $\varphi$.

Definition 2.1.11. A coherent rigid space $X$ is said to be empty if it has an empty formal model.

An empty rigid space will be denoted by $\emptyset$. For instance, if $X$ has a formal model that is a scheme (= 0-adic formal scheme), then $X$ is empty. Later in 3.1.6 we will see that a coherent rigid space $X$ is non-empty if and only if it has a non-empty distinguished formal model. The following proposition follows easily from the existence of formal models for morphisms of coherent rigid spaces.

Proposition 2.1.12. The empty rigid space $\emptyset$ is a strict initial object ([8] Exposé II, 4.5) of the category $\text{CRf}$.

2.1. (c) Comma category $\text{CRf}_S$

Definition 2.1.13. Let $S$ be a coherent rigid space. We define the category $\text{CRf}_S$ as follows.
Objects of \( \text{CRf}_S \) are morphisms \( X \to S \) in \( \text{CRf} \) with the target \( S \).

An arrow from \( X \to S \) to \( Y \to S \) is a morphism \( X \to Y \) in \( \text{CRf} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
S & & S
\end{array}
\]

Let \( S \) be a coherent adic formal scheme of finite ideal type, and set \( S = S^{\text{rig}} \).
Let \( \text{AcCFS}_{/S}^{*} \) be the category of coherent adic formal schemes adic over \( S \) (cf. I, §2.1. (c)). Then we have the canonical functor

\[
\text{AcCFS}_{/S}^{*} \longrightarrow \text{CRf}_S, \quad X/S \longmapsto X^{\text{rig}}/S^{\text{rig}}.
\]

On the other hand, one can mimic the definition of \( \text{CRf} \) to define the quotient category \( \text{AcCFS}_{/S}^{*}/\sim \), that is, a category consisting of objects of \( \text{AcCFS}_{/S}^{*} \) with the set of arrows defined by

\[
\text{Hom}_{\text{AcCFS}_{/S}^{*}/\sim}(X, Y) = \lim_{X' \to X} \text{Hom}_S(X', Y),
\]

where the right-hand inductive limit is taken over all admissible blow-ups \( X' \to X \).
Then the above functor obviously factors through the canonical functor

\[
\text{AcCFS}_{/S}^{*}/\sim \longrightarrow \text{CRf}_S. \quad (*)
\]

Proposition 2.1.14. The functor \((*)\) is a categorical equivalence.

Proof. It suffices to show the following fact: let \( Y \to S \) be a morphism of coherent adic formal schemes of finite ideal type, and set \( Y = Y^{\text{rig}} \); let \( X \to Y \) be a morphism of coherent rigid spaces; then there exists a formal model of the form \( X \to Y \) and, moreover, such a formal model is unique up to admissible blow-ups of \( X \). To see this, take an arbitrary formal model \( X' \to Y' \) of \( X \rightarrow Y \). Replacing \( Y' \) and \( X' \) by admissible blow-ups, we may assume that there exists an admissible blow-up \( Y' \to Y \) (cf. 1.1.7 and 1.3.1). Hence \( X = X' \to Y \) gives the desired formal model. The uniqueness is clear. \( \square \)

2.1. (d) Coherent universally Noetherian and universally adhesive rigid spaces

Definition 2.1.15. A coherent rigid space \( X \) is said to be universally Noetherian (resp. universally adhesive) if it has a formal model \((X, \phi)\) for a coherent universally rigid-Noetherian (resp. coherent universally adhesive) formal scheme \( X \) (I.2.1.7).
2. Rigid spaces

By 1.1.2 we know that, whenever $X$ has a universally rigid-Noetherian (resp. universally adhesive) formal model, any admissible blow-up of it is again universally rigid-Noetherian (resp. universally adhesive). Hence, as in the proof of 2.1.10, one can show that a coherent universally Noetherian (resp. universally adhesive) rigid space has a distinguished universally rigid-Noetherian (resp. universally adhesive) formal model. Note also that, similarly to the case treated in 2.1.14, one can equivalently define universally Noetherian (resp. universally adhesive) rigid spaces as objects in the quotient category of the form $\text{RigNoeCFs}^*/\sim$ (resp. $\text{AdhCFs}^*/\sim$) (cf. 1.2.1. (c) for the notation), constructed similarly from the category of universally rigid-Noetherian (resp. universally adhesive) formal schemes.

2.2 Admissible topology and general rigid spaces

2.2. (a) Coherent admissible sites

**Proposition 2.2.1.** The following conditions for a morphism $U \to X$ of coherent rigid spaces are equivalent.

(a) There exists a formal model $(j, \phi, \psi)$ of $U \to X$ in $\text{AcCFs}^*$ such that the morphism $j : U \to X$ in $\text{AcCFs}^*$ is an open immersion.

(b) There exists a distinguished formal model $(j, \phi, \psi)$ of $U \to X$ such that the morphism $j : U \to X$ in $\text{AcCFs}^*$ is an open immersion.

**Proof.** We only need to show (a) $\implies$ (b). Let $j : U \to X$ be an open immersion of coherent adic formal schemes of finite ideal type, and $I$ an ideal of definition of finite type. Let $X' \to X$ be the admissible blow-up along $I$. Then, as we saw in the proof of 2.1.10, $X'$ gives a distinguished formal model of $X$. Consider the induced open immersion $j_X : U \times_X X' \to X'$. By 1.1.8, $U \times_X X' \to U$ coincides up to isomorphism with the admissible blow-up along $I|_U$. Hence $U \times_X X'$ is a distinguished formal model of $U$, and the open immersion $j_X$ gives a distinguished formal model of $U \to X$. \hfill $\Box$

**Definition 2.2.2.** A morphism $\iota : U \to X$ of coherent rigid spaces is called a coherent open immersion if it fulfills the equivalent conditions in 2.2.1.

We will justify ‘coherent’ in this terminology later in 3.5.4.

**Proposition 2.2.3.** (1) Let $U \leftrightarrow V$ and $V \leftrightarrow X$ be two coherent open immersions of coherent rigid spaces. Then the composition $U \to X$ is a coherent open immersion.

(2) Let $Y \to X$ be a morphism of coherent rigid spaces, and $U \leftrightarrow X$ a coherent open immersion. Then the fiber product $U \times_X Y$ is representable in the category $\text{CRf}$ and the morphism $U \times_X Y \to Y$ is a coherent open immersion. (The general fiber products will be discussed later in 2.4.1.)
Proof. (1) Take open immersions \( U \hookrightarrow V \) and \( V' \hookrightarrow X \) that provides respective formal models for \( U \hookrightarrow V \) and \( V' \hookrightarrow X \). Take admissible blow-ups \( V'' \to V \) and \( V'' \to V' \) (cf. 2.1.4). By 1.1.9, one can take an admissible blow-up \( X' \to X \) such that \( V'' = V' \times_X X' \) and \( V'' \to X' \) is an open immersion. By I.1.2.8 (3), \( U' = U \times_V V'' \to V'' \) is an open immersion and gives a formal model of \( U \hookrightarrow V \) due to 1.1.8. The composition \( U' \to X' \), which is an open immersion due to I.1.2.8 (1), gives a formal model of \( U \to \mathcal{X} \).

(2) Let \( U \subset X \subset Y \) be a diagram in \textbf{AcCFs*}, where \( j \) is an open immersion, that gives rise to \( U \hookrightarrow \mathcal{X} \hookrightarrow Y \) by passage to the associated coherent rigid spaces. Take the fiber product \( U \times_X Y \) in the category \textbf{AcCFs*}. Then one can easily check that the desired fiber product \( U \times_X Y \) is given by \( (U \times_X Y)_{\text{rig}} \) and hence that the morphism \( U \times_X Y \to Y \) is a coherent open immersion. \( \square \)

Proposition 2.2.4. Let \( \mathcal{X} \) be a coherent rigid space and \( \{ \mathcal{U}_\alpha \hookrightarrow \mathcal{X} \}_{\alpha \in L} \) a finite family of coherent open immersions of coherent rigid spaces. Then there exists a (distinguished) formal model \( X \) of \( \mathcal{X} \) and a finite family \( \{ \mathcal{U}_\alpha \hookrightarrow X \}_{\alpha \in L} \) of open immersions of coherent adic formal schemes that induces by passage to the associated coherent rigid spaces the given family \( \{ \mathcal{U}_\alpha \hookrightarrow \mathcal{X} \}_{\alpha \in L} \).

Proof. By induction with respect to the cardinality of \( L \), we reduce to the situation where \( L = \{ 0, 1 \} \). Take a formal model \( \mathcal{U}_0 \hookrightarrow \mathcal{X} \) (resp. \( \mathcal{U}_1 \hookrightarrow \mathcal{X} \)) of \( \mathcal{U}_0 \hookrightarrow \mathcal{X} \) (resp. \( \mathcal{U}_1 \hookrightarrow \mathcal{X} \)). There exist admissible blow-ups \( X'' \to X \) and \( X'' \to X' \) (cf. 2.1.4); moreover, these admissible blow-ups can be taken so that \( X'' \) is a distinguished formal model of \( X \) (2.1.10). Then \( \mathcal{U}_0' = \mathcal{U}_0 \times_X X'' \to X'' \) and \( \mathcal{U}_1' = \mathcal{U}_1 \times_X X'' \to X'' \), which are open immersions by I.1.2.8 (3)), give the desired formal models by 1.1.8. \( \square \)

Definition 2.2.5. Let \( \mathcal{X} \) be a coherent rigid space and \( \{ \mathcal{U}_\alpha \hookrightarrow \mathcal{X} \}_{\alpha \in L} \) a finite family of coherent open immersions of coherent rigid spaces. This family is said to be a covering of \( \mathcal{X} \) if there exist a formal model \( X \) of \( \mathcal{X} \) and a finite Zariski open covering \( \{ \mathcal{U}_\alpha \hookrightarrow X \}_{\alpha \in L} \) that induces \( \{ \mathcal{U}_\alpha \hookrightarrow \mathcal{X} \}_{\alpha \in L} \) by passage to the associated coherent rigid spaces.

By an argument similar to that in the proof of 2.2.4 and by I.4.5.3 (3), one easily sees the following fact: if \( \{ \mathcal{U}_\alpha \hookrightarrow X \}_{\alpha \in L} \) is a finite Zariski covering as above, then for any admissible blow-up \( X' \) one has the finite Zariski covering \( \{ \mathcal{U}_\alpha' = \mathcal{U}_\alpha \times_X X' \hookrightarrow X' \}_{\alpha \in L} \) that also induces \( \{ \mathcal{U}_\alpha \hookrightarrow \mathcal{X} \}_{\alpha \in L} \). This shows, in particular, that the formal model \( X \) in 2.2.5 can be taken to be distinguished.

Definition 2.2.6 (coherent small admissible site for coherent rigid space). Let \( \mathcal{X} \) be a coherent rigid space. Define the site \( \mathcal{X}_{\text{ad}} \) as follows.

- Objects of the category \( \mathcal{X}_{\text{ad}} \) are coherent open immersions \( \mathcal{U} \hookrightarrow \mathcal{X} \) between coherent rigid spaces.
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- Arrows from $U \hookrightarrow X$ to $V \hookrightarrow X$ are morphisms of coherent rigid spaces over $X$.
- For any object $U \hookrightarrow X$ the set of coverings $\text{Cov}(U)$ consists of finite families of morphisms $\{U_\alpha \hookrightarrow U\}_{\alpha \in L}$ that gives a covering of the coherent rigid space $U$ in the sense of 2.2.5.

The last-mentioned notion of coverings defines, by 2.2.3 and I.4.5.3, a pretopology on the category $\mathcal{X}_{ad}$. The site $\mathcal{X}_{ad}$ thus obtained is said to be the **coherent small admissible site** associated to the coherent rigid space $X$. We denote by $\mathcal{X}_\sim$ the topos induced from the site $\mathcal{X}_{ad}$, called the **admissible topos** associated to $X$.

**Definition 2.2.7** (large admissible site of coherent rigid spaces). We endow $\text{CRf}$ with the following topology. For any object $X$ of $\text{CRf}$ the set of coverings $\text{Cov}(X)$ consists of finite families of coherent open immersions $\{U_\alpha \hookrightarrow X\}_{\alpha \in L}$ that gives a covering of $X$ in the sense of 2.2.5. We denote the site thus obtained by $\text{CRf}_{ad}$, and the associated topos by $\text{CRf}_{ad}$.

For a coherent rigid space $S$ the large admissible site $\text{CRf}_{S,ad}$, defined on the comma category $\text{CRf}_S$, and its associated topos $\text{CRf}_{S,ad}$ are defined similarly.

2.2. (b) Properties of coherent admissible sites. The following proposition is clear by the definition of the admissible topology.

**Proposition 2.2.8.** Any object of the site $\mathcal{X}_{ad}$ (resp. $\text{CRf}_{ad}$, resp. $\text{CRf}_{S,ad}$) is quasi-compact as an object of $\mathcal{X}_\sim$ (resp. $\mathcal{X}_\sim$, resp. $\mathcal{X}_\sim$). In particular, the topos $\mathcal{X}_\sim$ (resp. $\text{CRf}_{ad}$, resp. $\text{CRf}_{S,ad}$) has a generating full subcategory consisting of quasi-compact objects.

**Proposition 2.2.9.** Let $\mathcal{X}$ be a coherent rigid space. Then the topos $\mathcal{X}_\sim$ is coherent (0.2.7.5).

*Proof.* By 2.2.8 and 2.2.3 (2) any object of $\mathcal{X}_{ad}$ is coherent. Since the final object $X$ is a coherent objects, 2.2.3 (2) also implies that the category $\mathcal{X}_{ad}$ is stable under finite projective limits.

**Proposition 2.2.10.** On the site $\text{CRf}_{ad}$ any representable presheaf is a sheaf.

To prove this, and for later purpose, here we introduce the notion of *patching* of coherent rigid spaces:
**Proposition 2.2.11** (birational patching of coherent rigid spaces). (1) Let $\mathcal{X}$ be a coherent rigid space, and $U_0 \hookrightarrow \mathcal{X}$ and $U_1 \hookrightarrow \mathcal{X}$ coherent open immersions. Then the Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{X} & \to & \mathcal{X} \\
\downarrow \quad & \quad \downarrow \\
U_0 & \to & U_1
\end{array}
\]

where $U_{01} = U \times X U_1$, is co-Cartesian in $\text{CRf}$.

(2) Consider the diagram

\[
\begin{array}{ccc}
\mathcal{X} & \leftarrow & \mathcal{U} \\
\alpha \quad & \quad \beta \\
& \downarrow \quad & \downarrow \\
\mathcal{Y} & \quad \quad & \mathcal{Z}
\end{array}
\]

in $\text{CRf}$ where $\alpha$ and $\beta$ are coherent open immersions. Then there exists a co-Cartesian and Cartesian square in $\text{CRf}$

\[
\begin{array}{ccc}
\mathcal{X} & \leftarrow & \mathcal{U} \\
\alpha \quad & \quad \beta \\
& \downarrow \quad & \downarrow \\
\mathcal{Y} & \quad \quad & \mathcal{Z}
\end{array}
\]

where $\varphi$ and $\psi$ are coherent open immersions.

**Proof.** (1) Let $\mathcal{W}$ be a coherent rigid space, and $\gamma_i: U_i \to \mathcal{W}$ ($i = 0, 1$) morphisms of coherent rigid spaces. Suppose $\gamma_0 = \gamma_1$ on $U_{01}$. Take two formal models $g_0: U_0 \to W_0$ and $g_1: U_1 \to W_1$ of $\gamma_0$ and $\gamma_1$, respectively. One can replace $W_0$ and $W_1$ by a common admissible blow-up of them (due to 2.1.4), and $U_0$ and $U_1$ by the strict transforms, so that we may suppose $W_0 = W_1$, which we denote by $W$. We can moreover replace each $U_i$ ($i = 0, 1$) by an admissible blow-up, which is a coherent open subset of a formal model $X_i$ ($i = 0, 1$) of $\mathcal{X}$; replacing $X_0$ and $X_1$ by a common admissible blow-ups, we may assume $X_0 = X_1$, which we denote by $X$. By the assumption, there exists an admissible blow-up $U'_{01}$ of $U_{01} = U_0 \cap U_1$ on which $g_0$ and $g_1$ are equal. By 1.1.9, there exists an admissible blow-up $X' \to X$ that extends the admissible blow-up $U'_{01} \to U_{01}$ such that $U'_{01} \cong U_{01} \times_X X'$. Hence, replacing $X$ with $X'$ and $U_i$ ($i = 0, 1$) with the pull-back by the map $X' \to X$, we may assume that $g_0 = g_1$ on $U_{01}$. Then we have the unique morphism $g: X \to W$ such that $g|_{U_i} = g_i$ for $i = 0, 1$, and $\gamma = g^{\text{rig}}: \mathcal{X} \to \mathcal{W}$ such that $\gamma|_{U_i} = \gamma_i$ for $i = 0, 1$. The uniqueness of $\gamma$ is straightforward.
(2) Take formal models \( U \hookrightarrow X \) and \( U' \hookrightarrow Y \) of \( \alpha \) and \( \beta \), respectively, which are open immersions. Since \( U^{\text{rig}} \cong U'^{\text{rig}} \), there exist two admissible blow-ups \( U'' \to U \) and \( U'' \to U' \) (2.1.4). By 1.1.9 there exists an admissible blow-up \( X' \to X \) of \( X \) such that \( U'' = U \times_X X' \). Hence one can replace the formal model \( U \hookrightarrow X \) by \( U'' \hookrightarrow X' \). Doing the same for \( U \hookrightarrow Y \), one sees that we may assume \( U = U' \), that is, we may start with a diagram \( X \hookrightarrow U \hookrightarrow Y \) of open immersions that induces the diagram (\(*\)) by passage to the associated coherent rigid spaces. Now consider the push-out \( Z = X \sqcup_U Y \), the patching of \( X \) and \( Y \) along \( U \), in \( \text{AcCFs}^* \), and set \( Z = Z^{\text{rig}} \). Thus we get the desired commutative diagram, which is Cartesian, for we have \( U = X \times_Z Y \). The diagram thus obtained is co-Cartesian as well, due to (1).

In the situation as in 2.2.11, the coherent rigid space \( Z \) is denoted by \( X \sqcup_U Y \) and called the coherent rigid space obtained by patching of \( X \) and \( Y \) along \( U \). In case \( U \) is empty (2.1.11), we write \( X \sqcup Y \) (disjoint sum) for \( X \sqcup_U Y \). Since the square diagram in 2.2.11 is Cartesian, we have the following corollary.

**Corollary 2.2.12.** The disjoint sum \( X \sqcup Y \) is universally disjoint ([8], Exposé II, 4.5).

**Corollary 2.2.13.** In the category \( \text{CRf} \) any finite colimit consisting of coherent open immersions is representable.

To prove 2.2.10, we still need a few more lemmas.

**Lemma 2.2.14.** Let \( j: U \hookrightarrow X \) be a coherent open immersion of coherent rigid spaces. If the singleton set \( \{ j: U \hookrightarrow X \} \) is a covering (in the sense of 2.2.5), then \( j \) is an isomorphism.

This is clear from the definition of coverings (2.2.5).

**Lemma 2.2.15.** Let \( \varphi: X \hookrightarrow Z \) and \( \psi: Y \hookrightarrow Z \) be coherent open immersions of coherent rigid spaces, and set \( U = X \times_Z Y \). Then the canonical morphism \( X \sqcup_U Y \to Z \) (cf. 2.2.11) is a coherent open immersion.

**Proof.** Take a formal model \( Z \) of \( Z \) and quasi-compact open immersions \( X \hookrightarrow Z \) and \( Y \hookrightarrow Z \) that induce \( \varphi \) and \( \psi \), respectively, by the passage to the associated rigid spaces. Set \( U = X \times_Z Y \). Then we have \( U = U^{\text{rig}} \). As in the proof of 2.2.11, the morphism \( X \sqcup_U Y \to Z \) of coherent rigid spaces is represented by the open immersion \( X \sqcup_U Y \hookrightarrow Z \), whence the result.

By 2.2.15 and 2.2.14 we have the following lemma.

**Lemma 2.2.16.** Let \( U_0 \hookrightarrow X \) and \( U_1 \hookrightarrow X \) be coherent open immersions of coherent rigid spaces. Set \( U_{01} = U_0 \times_X U_1 \). Suppose that the induced morphism \( U_0 \sqcup U_1 \to X \) is a covering of the site \( \text{CRf}^\text{ad} \). Then \( U_0 \sqcup_{U_{01}} U_1 \to X \) is an isomorphism.
Proof of Proposition 2.2.10. Let \( X \) and \( \mathcal{U} \) be coherent rigid spaces, and suppose a covering \( \bigsqcup_{\alpha \in L} \mathcal{U}_\alpha \to \mathcal{U} \) is given. We need to show that the sequence

\[
\text{Hom}_{\text{CRf}}(\mathcal{U}, X) \longrightarrow \prod_{\alpha \in L} \text{Hom}_{\text{CRf}}(\mathcal{U}_\alpha, X) \longrightarrow \prod_{\alpha, \beta \in L} \text{Hom}_{\text{CRf}}(\mathcal{U}_{\alpha\beta}, X)
\]

is exact, where \( \mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \times_\mathcal{U} \mathcal{U}_\beta \) (cf. 2.2.3 (2)). By induction with respect to the cardinality of the index set \( I \), it suffices to show the assertion in the case \( I = \{0, 1\} \). But this case follows promptly from 2.2.16 and the fact that the square diagram as in 2.2.11 (1) is co-Cartesian. \( \square \)

2.2. (c) General rigid space

Definition 2.2.17. (1) A sheaf \( \mathcal{F} \) of sets on \( \text{CRf}_{\text{ad}} \) is said to be a stretch of coherent rigid spaces if there exists an inductive system \( \{\mathcal{U}_i\}_{i \in J} \) of coherent rigid spaces indexed by a directed set such that

(a) for any \( i, j \in J \) with \( i \leq j \) the transitions map \( \mathcal{U}_i \to \mathcal{U}_j \) is a coherent open immersion and

(b) for any coherent rigid space \( X \) we have

\[
\mathcal{F}(X) = \varprojlim_{i \in J} \text{Hom}(X, \mathcal{U}_i)
\]

(that is, \( \mathcal{F} \) is the inductive limit of the sheaves represented by \( \mathcal{U}_i \)).

In this situation, we also say that \( \mathcal{F} \) is represented by the stretch of the coherent rigid spaces \( \{\mathcal{U}_i\}_{i \in J} \).

(2) A morphism \( \mathcal{F} \to \mathcal{G} \) of sheaves of sets on \( \text{CRf}_{\text{ad}} \) is said to be (represented by) a stretch of coherent open immersions if for any representable sheaf \( \mathcal{X} \) and any morphism \( \mathcal{X} \to \mathcal{G} \) of sheaves, \( \mathcal{F} \times_\mathcal{G} \mathcal{X} \) is represented by a stretch of coherent rigid spaces \( \{\mathcal{U}_i\}_{i \in J} \) and the map \( \mathcal{F} \times_\mathcal{G} \mathcal{X} \to \mathcal{X} \) coincides with the inductive limit of \( \{\mathcal{U}_i \hookrightarrow \mathcal{X}\}_{i \in J} \).

Definition 2.2.18 (general rigid space). A sheaf \( \mathcal{F} \) of sets on \( \text{CRf}_{\text{ad}} \) is called a (general) rigid space if

(a) there exists a surjective morphism of sheaves

\[
\mathcal{Y} = \bigsqcup_{\alpha \in L} \mathcal{Y}_\alpha \longrightarrow \mathcal{F}
\]

(with the cardinality of the index set \( L \) being arbitrary), where each \( \mathcal{Y}_\alpha \) is represented by a coherent rigid space, and
(b) for any $\alpha, \beta \in L$ the projection
\[
Y_\alpha \times_\mathcal{F} Y_\beta \longrightarrow Y_\alpha
\]
is a stretch of coherent open immersions (2.2.17 (2)).

A morphism $\varphi: \mathcal{F} \to \mathcal{G}$ of rigid spaces is, by definition, a morphism of sheaves. We denote by $\mathbf{Rf}$ the category of rigid spaces.

**Proposition 2.2.19.** Any sheaf representable by a coherent rigid space is a rigid space. More generally, any stretch of coherent rigid spaces is a rigid space.

The proof is straightforward. Obviously, we have the fully faithful functor
\[
\mathbf{CRf} \longrightarrow \mathbf{Rf}
\]
that maps a coherent rigid space $\mathcal{X}$ to the sheaf represented by $\mathcal{X}$.

**Definition 2.2.20** (open immersion). Let $\varphi: \mathcal{F} \to \mathcal{G}$ be a morphism of rigid spaces. We say that $\varphi$ is an open immersion if it is a stretch of coherent open immersions (2.2.17 (2)). A $\mathcal{G}$-isomorphism class of open immersions is called an open rigid subspace of $\mathcal{G}$.

By the definition of open immersions and 2.2.3 (2), we immediately deduce the following result.

**Proposition 2.2.21.** Let $\mathcal{U} \to \mathcal{X}$ be a coherent open immersion between coherent rigid spaces. Then, regarded as a morphism between rigid spaces, it is an open immersion.

Later in 3.5.4 we will show that, conversely, any open immersion between coherent rigid spaces are coherent open immersions.

By 2.2.3 and 2.2.15, we have the following proposition.

**Proposition 2.2.22.** (1) Let $\mathcal{U} \hookrightarrow \mathcal{V}$ and $\mathcal{V} \hookrightarrow \mathcal{F}$ be two open immersions of rigid spaces. Then the composite $\mathcal{U} \to \mathcal{F}$ is an open immersion.

(2) Let $\mathcal{G} \to \mathcal{F}$ be a morphism of rigid spaces, and $\mathcal{U} \hookrightarrow \mathcal{F}$ an open immersion. Then the fiber product $\mathcal{U} \times_\mathcal{F} \mathcal{G}$ is representable in the category $\mathbf{Rf}$ and the morphism $\mathcal{U} \times_\mathcal{F} \mathcal{G} \to \mathcal{G}$ is an open immersion.

2.2. (d) Universally Noetherian and universally adhesive rigid spaces

**Definition 2.2.23.** A (general) rigid space $\mathcal{F}$ is called a locally universally Noetherian (resp. locally universally adhesive) rigid space if it has a covering as in 2.2.18 (a) such that each $Y_\alpha$ ($\alpha \in L$) is a coherent universally Noetherian (resp. universally adhesive) rigid space (2.1.15). If the covering can be chosen such that the index set $L$ is finite, then $\mathcal{F}$ is said to be a universally Noetherian (resp. universally adhesive) rigid space.
According to our later terminology in §3.5.1 below, a locally universally Noetherian (resp. locally universally adhesive) rigid space is universally Noetherian (resp. universally adhesive) if it is quasi-compact.

Using some of the results in §3 below, one can show without vicious circle the following fact: if \( \mathcal{F} \) is represented by a coherent rigid space, then \( \mathcal{F} \) is locally universally Noetherian (resp. locally universally adhesive) if and only if it is universally Noetherian (resp. universally adhesive) in the sense as in 2.1.15. Indeed, if \( \mathcal{F} \), assumed to be representable by a coherent rigid space, is locally universally Noetherian (resp. locally universally adhesive) in the sense of 2.2.23, then by 3.5.2 we may assume that the covering \( \{ \mathcal{Y}_\alpha \}_{\alpha \in L} \) is finite. By 3.5.4 we see that each map \( \mathcal{Y}_\alpha \to \mathcal{F} \) is represented by a coherent open immersion. Let \( Y_\alpha \) be a universally rigid-Noetherian (resp. universally adhesive) formal model of \( \mathcal{Y}_\alpha \) for each \( \alpha \), and \( X \) a formal model of the coherent rigid space that represents \( \mathcal{F} \). By 3.1.3, replacing \( X \) by an admissible blow-up if necessary, we may assume that \( X \) has a Zariski covering \( \{ U_\alpha \}_{\alpha \in L} \) consisting of quasi-compact open subsets \( U_\alpha \) together with an admissible blow-up \( U_\alpha \to Y_\alpha \) for each \( \alpha \in L \) (here we used 3.4.1). Since each \( U_\alpha \) is universally rigid-Noetherian (resp. universally adhesive), we deduce that \( X \) is universally rigid-Noetherian (resp. universally adhesive).

2.2. (e) Admissible sites

**Definition 2.2.24** (small admissible site). For a rigid space \( \mathcal{F} \) we denote by \( \mathcal{F}_{\text{ad}} \) the site defined as follows. As a category, it is the category of all open immersions \( \mathcal{U} \to \mathcal{F} \) and morphisms over \( \mathcal{F} \). For an object \( \mathcal{U} \to \mathcal{F} \) the collection of coverings \( \text{Cov}(\mathcal{U}) \) consists of families \( \{ \mathcal{U}_\alpha \to \mathcal{U} \}_{\alpha \in L} \) (indexed by an arbitrary set) of morphisms such that \( \coprod \mathcal{U}_\alpha \to \mathcal{U} \) is an epimorphism of sheaves on \( \text{CRf}_{\text{ad}} \). Note that this defines a pretopology due to 2.2.3. We denote by \( \mathcal{F}_{\text{ad}} \) the topos associated to the site \( \mathcal{F}_{\text{ad}} \).

Note that, if \( \mathcal{X} \) is a coherent rigid space and \( \mathcal{F} \) is the general rigid space represented by \( \mathcal{X} \) (that is, the image of \( \mathcal{X} \) by the functor \( \text{CRf} \leftrightarrow \text{Rf} \)), then the site \( \mathcal{X}_{\text{ad}} \) defined in 2.2.6 and the site \( \mathcal{F}_{\text{ad}} \) defined in 2.2.24 are different. We will see in 3.4.4, however, there exists a canonical morphism of sites \( \mathcal{X}_{\text{ad}} \to \mathcal{F}_{\text{ad}} \) inducing an equivalence of topoi \( \mathcal{X}_{\text{ad}} \to \mathcal{F}_{\text{ad}} \).

**Definition 2.2.25** (large admissible site). We endow the category \( \text{Rf} \) with the following topology: for any object \( \mathcal{F} \) of \( \text{Rf} \) the collection of coverings \( \text{Cov}(\mathcal{F}) \) consists of families of open immersions \( \{ \mathcal{U}_\alpha \leftrightarrow \mathcal{F} \}_{\alpha \in L} \) (indexed by an arbitrary set) such that the map of sheaves \( \coprod \mathcal{U}_\alpha \to \mathcal{F} \) on \( \text{CRf}_{\text{ad}} \) is an epimorphism. Thanks to 2.2.22, this gives a pretopology on the category \( \text{Rf} \). We denote this site by \( \text{Rf}_{\text{ad}} \) and the associated topos by \( \text{Rf}_{\text{ad}} \).

For a rigid space \( \mathcal{G} \) the large admissible site \( \text{Rf}_{\mathcal{G}, \text{ad}} \), defined on the obvious comma category \( \text{Rf}_{\mathcal{G}} \), and the associated topos \( \text{Rf}_{\mathcal{G}, \text{ad}} \) are defined similarly.
2.3 Morphism of finite type

**Definition 2.3.1.** (1) A morphism $\varphi: \mathcal{X} \to \mathcal{Y}$ of coherent rigid spaces is said to be of finite type if it has a formal model $f: X \to Y$ of finite type (I.1.7.1).

(2) Let $\varphi: \mathcal{X} \to \mathcal{Y}$ be a morphism of rigid spaces where $\mathcal{Y}$ is a coherent rigid space. Then $\varphi$ is said to be locally of finite type if there exists a covering family $\{\mathcal{U}_\alpha \hookrightarrow \mathcal{X}\}_{\alpha \in L}$ of the site $\mathcal{X}_{ad}$, where each $\mathcal{U}_\alpha$ is a coherent rigid space, such that for any $\alpha \in L$ the morphism $\mathcal{U}_\alpha \to \mathcal{Y}$ is of finite type in the sense of (1). If, moreover, such a covering family can be taken to be finite, we say that $\varphi$ is of finite type.

(3) For a morphism of general rigid spaces $\varphi: \mathcal{X} \to \mathcal{Y}$, we say $\varphi$ is locally of finite type (resp. of finite type) if there exists a covering family $\{\mathcal{V}_\alpha \hookrightarrow \mathcal{Y}\}_{\alpha \in L}$ of the site $\mathcal{Y}_{ad}$, where each $\mathcal{V}_\alpha$ is a coherent rigid space, such that for any $\alpha \in L$ the induced morphism $\mathcal{X} \times_\mathcal{Y} \mathcal{V}_\alpha \to \mathcal{V}_\alpha$ is locally of finite type (resp. of finite type) in the sense of (2). (Note that $\mathcal{X} \times_\mathcal{Y} \mathcal{V}_\alpha$ is represented by a rigid space.)

One can show that these definitions are consistent with each other; cf. 3.5.2.

**Proposition 2.3.2.** (1) An open immersion is locally of finite type.

(2) The composite of two morphisms locally of finite type (resp. of finite type) is again locally of finite type (resp. of finite type). If the composite $\psi \circ \varphi$ of morphisms $\varphi: \mathcal{X} \to \mathcal{Y}$ and $\psi: \mathcal{Y} \to \mathcal{Z}$ is locally of finite type, then $\varphi$ is locally of finite type.

**Proof.** (1) is clear. We show (2). Let $\varphi: \mathcal{X} \to \mathcal{Y}$ and $\psi: \mathcal{Y} \to \mathcal{Z}$ be morphisms of finite type. We may assume without loss of generality that $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are coherent rigid spaces. Take respective formal models $f: X \to Y$ and $g: Y' \to Z$. Since $Y^{rig} \cong Y'^{rig}$, by 2.1.4 we have the following diagram

$$
\begin{array}{ccc}
\pi & \pi' \\
X & \rightarrow & Y' \\
\downarrow & & \downarrow \\
Y & \rightarrow & Z,
\end{array}
$$

where the two oblique arrows are admissible blow-ups. Consider

$$
f': X' = X \times_\mathcal{Y} Y'' \longrightarrow Y''.
$$

By 2.1.6 the morphism $f'$ gives another formal model of $\varphi$, and the composite $g \circ \pi' \circ f'$, which is of finite type (I.1.7.2 (2) and (4)), gives a formal model of $\psi \circ \varphi$.

To show the other assertion, take a formal model $h: X'' \to Z'$ of finite type of the composite $\psi \circ \varphi$. Since there exist admissible blow-ups $Z'' \to Z$ and $Z'' \to Z'$, one can assume by the base change to $Z''$ that $Z' = Z$. Replacing $X''$ by a suitable admissible blow-up, we may assume that there exists a map $\pi'': X'' \to X'$ such that $h = g \circ \pi' \circ f' \circ \pi''$. Since $h$ is of finite type, we deduce that $f' \circ \pi''$, which is a formal model of $\varphi$, is of finite type (I.1.7.2 (2)).
**Proposition 2.3.3.** Let \( \varphi: \mathcal{X} \to \mathcal{Y} \) be a locally of finite type morphism between rigid spaces. If \( \mathcal{Y} \) is locally universally Noetherian (resp. locally universally adhesive), then so is \( \mathcal{X} \).

*Proof.* We may assume without loss of generality that \( \mathcal{X} \) and \( \mathcal{Y} \) are coherent and that \( \varphi \) has a formal model \( f: X \to Y \) of finite type; here, since admissible blow-ups are of finite type, we may furthermore assume that \( Y \) is universally rigid-Noetherian (resp. universally adhesive). Then \( X \) is universally rigid-Noetherian (resp. universally adhesive) \((I.2.1.8)\).

**Proposition 2.3.4.** Let \( \varphi: \mathcal{X} \to \mathcal{Y} \) be a finite type morphism of coherent rigid spaces. Then there exists a distinguished formal model \( f: X \to Y \) of \( \varphi \) of finite type. Moreover, such formal models are cofinal in the category \( M_\varphi \). If \( \mathcal{Y} \) is universally adhesive, then such formal models are always of finite presentation.

*Proof.* The first and the second assertions follow from 1.1.7 applied to the admissible blow-up along an ideal of definition of finite type (cf. 2.1.10). Suppose \( \mathcal{Y} \) is universally adhesive (and hence so is \( \mathcal{X} \) due to 2.3.3). If there exists a distinguished formal model as above of finite type consisting of universally adhesive formal schemes, it is automatically of finite presentation due to \( I.2.3.1 \).

### 2.4 Fiber products of rigid spaces

**Proposition 2.4.1.** For any diagram in \( \text{Rf} \) of the form \( \mathcal{X} \overset{\varphi}{\rightarrow} S \overset{\psi}{\leftarrow} \mathcal{Y} \) the fiber product \( \mathcal{X} \times_S \mathcal{Y} \) is representable in \( \text{Rf} \).

*Proof.* We first deal with the case where the rigid spaces \( \mathcal{X} \), \( S \), and \( \mathcal{Y} \) are coherent. Take a diagram \( X \overset{f}{\rightarrow} S \overset{g}{\leftarrow} Y \) in \( \text{AcCFs}^* \) such that \( f^{\text{rig}} = \varphi \) and \( g^{\text{rig}} = \psi \). Then by \( I.1.2.5 \) there exists a fiber product \( X \times_S Y \) in \( \text{AcCFs}^* \). The desired coherent rigid space \( \mathcal{X} \times_S \mathcal{Y} \) is then defined to be \( (X \times_S Y)^{\text{rig}} \) together with the projections induced from the projections \( X \times_S Y \to X \) and \( X \times_S Y \to Y \). By 2.1.6 one sees easily that this gives a well-defined coherent rigid space, and \( \mathcal{X} \times_S \mathcal{Y} \) thus obtained indeed gives a fiber product of the diagram in question.

In general, first construct the fiber product \( \mathcal{X} \times_S \mathcal{Y} \) in the category of sheaves on the site \( \text{CRf}_{\text{ad}} \). Then it is easy to see by the standard argument, with the aid of 2.2.3 (2), that, by what we have just seen for coherent rigid spaces, the last sheaf is a rigid space.

**Corollary 2.4.2.** The quotient functor \( Q: \text{AcCFs}^* \to \text{CRf} \) preserves fiber products.
Proposition 2.4.3. (1) If \( \varphi: X \to X' \) and \( \psi: Y \to Y' \) are two open immersions (resp. morphisms locally of finite type, resp. morphisms of finite type) over a rigid space \( S \), then the induced morphism \( \varphi_S \times_S \psi: X \times_S Y \to X' \times_S Y' \) is an open immersion (resp. a morphism locally of finite type, resp. a morphism of finite type).

(2) If \( \varphi: X \to Y \) is an open immersion (resp. a morphism locally of finite type, resp. a morphism of finite type) over a rigid space \( S \) and \( S' \to S \) is a morphism of rigid spaces, then the induced morphism \( \varphi_{S'}: X \times_S S' \to Y \times_S S' \) is an open immersion (resp. a morphism locally of finite type, resp. a morphism of finite type).

Proof. By 0.1.4.1, we only need to show that for a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\downarrow & & \downarrow \\
Z & \xleftarrow{\psi} & Z
\end{array}
\]

of rigid spaces, where \( \varphi \) is an open immersion (resp. a morphism locally of finite type, resp. a morphism of finite type), the induced morphism

\[
\varphi_Z: X \times Y Z \to Z
\]

is an open immersion (resp. a morphism locally of finite type, resp. a morphism of finite type). We may assume that \( X, \ Z, \) and \( Y \) are coherent rigid spaces and that \( \varphi \) is a coherent open immersion (resp. a morphism of finite type, resp. a morphism of finite type). Then the claim follows from I.1.2.8 (2) and I.1.7.2 (3).

\[\square\]

2.5 Examples of rigid spaces

2.5. (a) Rigid spaces of type (V)

Definition 2.5.1. A rigid space of type (V) is a rigid space \( X \) locally of finite type over \( \text{Spf} \ V^{\text{rig}} \), where \( V \) is an \( a \)-adically complete valuation ring with \( a \in m_V \setminus \{0\} \) (cf. 0.\$9.1). If, in particular, the valuation ring \( V \) is of height one, then \( X \) is called a rigid space of type (V\(_{\text{R}}\)).

Since the topological ring \( V \) is t.u. adhesive (0.9.2.7), the coherent rigid space \( \text{Spf} \ V^{\text{rig}} \) is universally adhesive, and hence by 2.3.3 any rigid space of type (V) is locally universally adhesive (2.2.23).

2.5. (b) Rigid spaces of type (N)

Definition 2.5.2. A rigid space of type (N) is a rigid space \( X \) that admits an open covering \( \{U_a\}_{a \in L} \) consisting of coherent rigid spaces having Noetherian formal models.

Since any Noetherian adic ring is t.u. adhesive, it follows that any rigid space of type (N) is locally universally adhesive (2.2.23). It can be shown, with the aid of 3.1.3 below, that if a coherent rigid space \( X \) is of type (N), then \( X \) has a Noetherian formal model (Exercise II.3.4).
2.5. (c) Unit disk over a rigid space. Let $\mathcal{X}$ be a rigid space. The so-called (closed) unit disk over $\mathcal{X}$, denoted by $D^n_{\mathcal{X}}$, is the rigid space defined as follows. When $\mathcal{X}$ is of the form $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$, where $A$ is an adic ring of finite ideal type, then $D^n_{\mathcal{X}} = (\text{Spf } A((X_1, \ldots, X_n)))^{\text{rig}}$ (cf. 0, §8.4) or, what amounts to the same, the coherent rigid space associated to the formal affine $n$-space $\hat{A}_A^n$ over $A$ (Exercise I.5.1 (1)). Since any rigid space has an admissible covering by open subspaces of the above form, one can define $D^n_{\mathcal{X}}$ for general $\mathcal{X}$ by patching.

2.5. (d) Projective space over a rigid space. The projective space $\mathbb{P}^{n,\text{an}}_{\mathcal{X}}$ over a rigid space $\mathcal{X}$ is constructed as follows. When $\mathcal{X}$ is of the form $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$, one considers the formal projective space $\hat{\mathbb{P}}^n_{\mathcal{X}} = \hat{\mathbb{P}}^n_A$ (Exercise I.5.1 (2)); then set $\mathbb{P}^{n,\text{an}}_{\mathcal{X}} = (\hat{\mathbb{P}}^n_{\mathcal{X}})^{\text{rig}}$. For a general rigid space $\mathcal{X}$ one can define $\mathbb{P}^{n,\text{an}}_{\mathcal{X}}$ by gluing.

Exercises

Exercise II.2.1. Show that in the category $\text{Rf}$ of rigid spaces any colimit consisting of open immersions is representable.

Exercise II.2.2. Consider the category $\text{AcFs}^*$ of all adic formal schemes of finite ideal type with adic morphisms. Then define the natural functor

$$\text{AcFs}^* \longrightarrow \text{Rf}$$

extending $Q: \text{AcCFs}^* \rightarrow \text{CRf} (X \mapsto X^{\text{rig}})$.

3 Visualization

The introduction of rigid spaces in the previous section is, so to speak, one of the two most fundamental starting points of rigid geometry. The other one is the so-called visualization, which we will discuss in this section. Our definition of rigid spaces as ‘generic fibers’ of formal schemes, which we have given in the previous section, is based on the creed that rigid geometry is like a ‘birational geometry’ of formal schemes. Because of this, one can say that the visualization of rigid spaces is the way to enhance the birational geometric aspect of the rigid geometry. It does this job by adopting Zariski’s old idea of birational geometry, by means of
the so-called Zariski–Riemann spaces, which we will introduce in \S 3.1. There, the Zariski–Riemann spaces are first constructed for coherent rigid spaces. Coherent rigid spaces have formal models by definition, and the Zariski–Riemann space associated to a given coherent rigid space is defined to be the projective limit of all formal models. As the admissible blow-ups of an arbitrary formal model comprise a cofinal part of the totality of all formal models of the coherent rigid space, the Zariski–Riemann space is equivalently defined as the filtered projective limit of admissible blow-ups of a fixed formal model.

One of the most significant topological features of the Zariski–Riemann spaces comes promptly after the definition in Theorem 3.1.2, which asserts that the Zariski–Riemann space associated to a coherent rigid space is in fact a coherent sober topological space (0.2.2.1). As stated in the introduction of 0, \S 2, the coherence or, especially, the quasi-compactness of the Zariski–Riemann spaces plays a very important role in our theory of rigid geometry, similarly to how the quasi-compactness of Zariski’s generalized Riemann spaces, proved by Zariski in 1944, played one of the most important roles in his works of resolution of singularities of algebraic varieties. It is also worthwhile remarking that introducing the Zariski–Riemann spaces as above allows one to have a more lucid picture of what we have called the ‘birational patching’ in the previous section (cf. 2.2.11). Indeed, the ‘birational patching’ gives the usual topological patching of the Zariski–Riemann spaces along quasi-compact open subsets. Using this, in particular, the general construction of Zariski–Riemann spaces associated to a general rigid space can be given quite naturally.

In the next subsection, \S 3.2, we discuss the structure sheaves on the Zariski–Riemann spaces. As the Zariski–Riemann spaces are locally given by projective limits of formal schemes, they have the natural sheaves of rings, simply given by the inductive limits of the structure sheaves of the formal schemes. This sheaf, called the integral structure sheaf and denoted by $\mathcal{O}^{\text{int}}_X$ is, needless to say, an important object to consider. But, in view of the central tenet of rigid geometry that rigid geometry is the geometry of ‘generic fibers’ of formal schemes, one has to ‘invert’ the ideal of definition in the sheaf $\mathcal{O}^{\text{int}}_X$ to obtain the ‘correct’ structure sheaf of the rigid spaces. This ‘inversion’ of ideals of definition is possible thanks to the fact that the stalk at each point of the integral structure sheaf $\mathcal{O}^{\text{int}}_X$ is a valuative ring (0.8.7.1) with respect to the ideal of definition (3.2.6). The resulting sheaf of rings, denoted by $\mathcal{O}_X$, is the one we take up as the structure sheaf of the rigid space $\mathcal{X}$, called the rigid structure sheaf or just structure sheaf of $\mathcal{X}$. Since the integral structure sheaf is, from the viewpoint of rigid geometry, regarded as the canonical formal model of the structure sheaf, it retains the importance as a natural object associated to the rigid space. For this reason, when considering visualization of rigid spaces, one should consider the triple $\text{ZR}(\mathcal{X}) = (\mathcal{X}, \mathcal{O}^{\text{int}}_\mathcal{X}, \mathcal{O}_\mathcal{X})$, the so-called Zariski–Riemann triple, consisting of the Zariski–Riemann space $\mathcal{X}$, the
integral structure sheaf, and the rigid structure sheaf, rather than the locally ringed space \((\mathcal{X}, \mathcal{O}_\mathcal{X})\) alone.

In §3.3 we will study the points of Zariski–Riemann spaces. There, we will see that the notion of points coincides with the one usually called *rigid points* in classical rigid geometry. This is the place where valuation rings come into play in rigid geometry; remember that, as stated in the introduction of 0, §6, valuation rings are the most natural ‘value rings’ at points in rigid spaces, and thus play the role of fields in algebraic geometry.

Subsection §3.4 is devoted to a comparison between admissible topology and the topology (in the usual sense) of the Zariski–Riemann spaces. In the final Subsection, §3.5, we consider quasi-compactness and quasi-separateness and show that these notions for rigid spaces and those for corresponding Zariski–Riemann spaces coincide.

### 3.1 Zariski–Riemann spaces

#### 3.1. (a) Construction in the coherent case.

Let \(\mathcal{X}\) be a coherent rigid space, and take a formal model \((X, \phi)\) of \(\mathcal{X}\). Consider the functor \(S_X : \text{BL}_X \to \text{LRsp}\) from the category of admissible blow-ups of \(X\) (§1.3) to the category of locally ringed spaces that maps each admissible blow-up \(X' \to X\) to the underlying locally ringed space of \(X'\). Consider the corresponding limit

\[
\lim \leftarrow S_X
\]

(cf. 0, §1.3 for the notation). Since the functor \(\text{AId}_X \to \text{BL}_X\) (cf. 1.3.1 (2)) is cofinal, one can replace the above limit by the projective limit (0.1.3.2)

\[
\lim_{\mathcal{J} \in \text{AId}_X} X_{\mathcal{J}};
\]

note that such a limit exists in the category \(\text{LRsp}\) thanks to 0.4.1.10. Note also that the limit (*) does not depend on the choice of the formal model \(X\); indeed, for example, if we change \(X\) by an admissible blow-up \(X'\) of \(X\), then we have a canonical functor \(\text{BL}_{X'} \to \text{BL}_X\) (simply by composition; cf. 1.1.10), which is clearly cofinal, and hence the limits taken along \(\text{BL}_X\) and \(\text{BL}_{X'}\) coincide up to canonical isomorphisms in the category \(\text{LRsp}\). Hence the limit (*) depends only on \(\mathcal{X}\) up to canonical isomorphisms.

**Definition 3.1.1.** The underlying topological space of the limit (*) is called the *Zariski–Riemann space associated to \(\mathcal{X}\)* and denoted by \((\mathcal{X})\).
Let $\mathcal{X}$ be a coherent rigid space, and $\langle \mathcal{X} \rangle$ the associated Zariski–Riemann space. Let $X$ be any formal model of $\mathcal{X}$. Since $\langle \mathcal{X} \rangle$ is the projective limit of all admissible blow-ups of $X$, we have the projection map $\langle \mathcal{X} \rangle \rightarrow X$ of locally ringed spaces, called the *specialization map* and denoted by

$$\text{sp}_X : \langle \mathcal{X} \rangle \rightarrow X.$$  

Clearly, we have $\langle \emptyset \rangle = \emptyset$ (cf. 2.1.11 for the definition of empty rigid space $\emptyset$). More generally, for a *scheme* $X$ (that is, a 0-adic formal scheme; cf. I.1.1.15) we have $\langle X^{\text{rig}} \rangle = \emptyset$.

3.1. (b) **Functoriality.** Let $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism in $\mathbf{CRf}$. Then one can take a formal model $f : X \rightarrow Y$ of $\varphi$, and by 1.1.7 we get a canonical continuous map $\langle \varphi \rangle : \langle \mathcal{X} \rangle \rightarrow \langle \mathcal{Y} \rangle$ of topological spaces. Thus the formation of the associated Zariski–Riemann spaces gives rise to a functor

$$\langle \cdot \rangle : \mathbf{CRf} \rightarrow \mathbf{Top},$$

where $\mathbf{Top}$ denotes the category of topological spaces.

3.1. (c) **Topological feature**

**Theorem 3.1.2.** Let $\mathcal{X}$ be a coherent rigid space, and $\langle \mathcal{X} \rangle$ the associated Zariski–Riemann space.

1. The topological space $\langle \mathcal{X} \rangle$ is coherent (0.2.7.5) and sober (0, §2.1.(b)).

2. For any formal model $X$ of $\mathcal{X}$, the specialization map $\text{sp}_X$ is quasi-compact (0.2.1.4 (2)), and closed.

**Proof.** Since the topological space $\langle \mathcal{X} \rangle$ is, by definition, the projective limit of coherent sober topological spaces with all the transition maps quasi-compact, (1) follows from 0.2.2.10 (1). Since admissible blow-ups are proper, $\text{sp}_X$ is a closed map. Hence (2) follows from 0.2.2.14. \(\square\)

By the definition of the projective limit topology, the collection

$$\left\{ \text{sp}_X^{-1}(U) \middle| X \text{ is a formal model of } \mathcal{X} \text{ and } U \text{ is a quasi-compact open subset of } X \right\}$$

gives an open basis of $\langle \mathcal{X} \rangle$ consisting of quasi-compact open subsets. The following proposition shows that this collection is actually the set of all quasi-compact open subsets of $\langle \mathcal{X} \rangle$. 

---

*Note:* The above text is a reproduction of a section from a mathematical document, focusing on the concepts of coherent rigid spaces, Zariski–Riemann spaces, specialization maps, and their functorial properties. The text is structured to maintain the flow and logical progression typical of mathematical expositions, ensuring clarity and coherence in the presentation of complex ideas.
Chapter II. Rigid spaces

Proposition 3.1.3. (1) For any quasi-compact open subset \( \mathcal{U} \) of \( \langle \mathcal{X} \rangle \) there exist a formal model \( X \) and a quasi-compact open subset \( U \subseteq X \) such that \( \mathcal{U} = \text{sp}^{-1}_X(U) \). Moreover, the formal model \( X \) here can be taken to be distinguished.

(2) Let \( X \) be a coherent adic formal scheme of finite ideal type, and \( U \subseteq X \) a quasi-compact open subset. Set \( \mathcal{X} = X^\text{rig} \) and \( \mathcal{U} = U^\text{rig} \). Then the induced map \( \langle \mathcal{U} \rangle \to \langle \mathcal{X} \rangle \) maps \( \langle \mathcal{U} \rangle \) homeomorphically onto the quasi-compact open subset \( \text{sp}^{-1}_X(U) \) of \( \langle \mathcal{X} \rangle \).

The proposition says that the coherent small admissible site \( \mathcal{X}_\text{ad} \) is isomorphic to the projective limit of the Zariski sites \( X_\text{Zar} \) of the formal models.

Proof. Obviously, it is enough to consider the case where \( \langle \mathcal{X} \rangle \) is non-empty.

(1) The first assertion follows from \ref{02.2.9}. Once such a model \( X \) and a quasi-compact open subset \( U \) are found, we can similarly find a quasi-compact open subset in any admissible blow-up \( X' \) of \( X \); indeed, we just take \( U' = X' \times_X U \) so that \( \mathcal{U} = \text{sp}^{-1}_X(U') \). By \ref{2.1.10}, in particular, we can assume that \( X' \) is distinguished.

(2) Consider the ordered sets \( \text{AId}_X \) and \( \text{AId}_U \) as in \ref{1.3.1}. The map

\[ \text{AId}_X \longrightarrow \text{AId}_U \]

given simply by restriction is an ordered map and is surjective by \ref{1.3.7.15}. Hence the map \( \langle \mathcal{U} \rangle \to \langle \mathcal{X} \rangle \) in question is the projective limit of the system of morphisms \( \{ U|_{U_j} \to X|_{X_j} \}_{j \in \text{AId}_X} \). By \ref{1.1.8}, these morphisms are open immersions, and we have \( U|_{U_j} \cong X|_{X_j} \times_X U \). Since the fiber product by an open immersion of (formal) schemes and the fiber product of the underlying sets coincide (\ref{1.1.2.7}), the desired result follows from \ref{04.1.11}.

Proposition 3.1.4. The functor \( \langle \cdot \rangle : \text{CRf} \to \text{Top} \) maps coherent open immersions to open immersions and preserves finite colimits consisting of coherent open immersions.

Proof. The first assertion follows from \ref{3.1.3} (2). To show the other assertion, it suffices to show that the functor \( \langle \cdot \rangle \) preserves cofiber products by coherent open immersions. Consider the patching diagram as in \ref{2.2.11}, and take, as in the proof there, the patching \( Z = X \amalg_U Y \) of coherent adic formal schemes of finite ideal type by open immersions that represents the patching diagram. Since the restriction maps \( \text{AId}_Z \to \text{AId}_X \), \( \text{AId}_Z \to \text{AId}_Y \), and \( \text{AId}_Z \to \text{AId}_U \) are surjective (\ref{1.3.7.15}), we have

\[ \lim_{\leftarrow} Z|_{U_j} = \lim_{\leftarrow} X|_{U_j} \coprod_{\lim_{\leftarrow} U_j} \lim_{\leftarrow} Y|_{U_j}, \]

that is, \( \langle X \amalg_U Y \rangle = \langle X \rangle \amalg_{\langle U \rangle} \langle Y \rangle \).

\( \square \)
3. (d) Non-emptiness

**Proposition 3.1.5.** Let $X$ be a coherent rigid space and $X$ a distinguished formal model of $X$. Then the specialization map $\text{sp}_X : \langle X \rangle \to X$ is surjective.

**Proof.** We may assume in view of 3.1.3 that $X$ is affine, $X = \text{Spf} A$. By the assumption, the adic ring $A$ is $I$-torsion free, where $I \subseteq A$ is an ideal of definition. Note that, in this situation, $Y \setminus V(I)$ is dense in $Y$. Set $Y = \text{Spec} A$, and consider the canonical map $X = \text{Spf} A \to Y = \text{Spec} A$. Take any admissible ideal $J \subseteq A$, and let $Y' \to Y$ be the blow-up along $J$. The formal completion $X' \to X$ is an admissible blow-up of $X$, and any admissible blow-up is of this form. Since $Y' \to Y$ is proper and the identity outside $V(I)$, it is surjective. Hence $X' \to X$ is surjective. Now the assertion follows from 0.2.2.13 (2).

**Corollary 3.1.6.** The following conditions for a coherent rigid space $X$ are equivalent.

(a) The associated Zariski–Riemann space $\langle X \rangle$ is non-empty.

(b) There exists a non-empty distinguished formal model of $X$.

(c) There exists a non-empty formal model $X$ of $X$ with an ideal of definition of finite type $I$ such that $X$ is not $I$-torsion.

(d) $X$ is not an empty rigid space (2.1.11).

**Proof.** Implication (a) $\Rightarrow$ (b) is clear. (b) $\Rightarrow$ (a) follows from 3.1.5. (b) $\Rightarrow$ (c) is trivial. Let us show that (c) $\Rightarrow$ (b). Let $X' \to X$ be the admissible blow-up along $I$. Since $X$ is not $I$-torsion, $X'$ is non-empty. Thus $X'$ gives a non-empty distinguished formal model of $X$. As (d) $\Rightarrow$ (b) is trivial, it only remains to show that (b) $\Rightarrow$ (d). Suppose that (b) holds, and let $X$ be a formal model of $X$. Then there exists a non-empty distinguished formal model $X'$ that dominates $X$ (2.1.9 and 2.1.10). Hence $X$ is non-empty, thereby (b) $\Rightarrow$ (d).

3. (e) General construction

**Definition 3.1.7 (Zariski–Riemann space for general rigid spaces).** (1) Let $\mathcal{F}$ be a sheaf represented by a stretch of coherent rigid spaces $\{\mathcal{U}_i\}_{i \in I}$ (2.2.17 (1)). Then the induced inductive system $\{\langle \mathcal{U}_i \rangle\}_{i \in I}$ is a filtered inductive system of topological spaces such that each transition map is an open immersion (3.1.3 (2)). We set

$$\langle \mathcal{F} \rangle = \lim_{i \in I} \langle \mathcal{U}_i \rangle,$$

which is a sober topological space (0.2.1.2). It is clear that $\langle \mathcal{F} \rangle$ does not depend, up to canonical isomorphisms, on the choice of an inductive system $\{\mathcal{U}_i\}_{i \in I}$ representing $\mathcal{F}$. The space $\langle \mathcal{F} \rangle$ is called the associated Zariski–Riemann space of $\mathcal{F}$.
For an open immersion $\varphi: \mathcal{F} \hookrightarrow \mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are rigid spaces represented by
a stretch of coherent rigid spaces, one defines

$$\langle \varphi \rangle: \langle \mathcal{F} \rangle \rightarrow \langle \mathcal{G} \rangle$$

in a similar way.

(2) Let $\mathcal{F}$ be a rigid space, and take $\mathcal{Y} = \bigsqcup_{\alpha \in L} \mathcal{Y}_\alpha \to \mathcal{F}$ as in 2.2.18. Set

$$\langle \mathcal{Y} \rangle = \bigsqcup_{\alpha \in L} \langle \mathcal{Y}_\alpha \rangle.$$ 

For each $\alpha, \beta \in L$ the projections $\text{pr}: \mathcal{Y}_\alpha \times_{\mathcal{F}} \mathcal{Y}_\beta \to \mathcal{Y}_\alpha$ are open immersions, and hence one defines $\langle \mathcal{Y}_\alpha \times_{\mathcal{F}} \mathcal{Y}_\beta \rangle$ and $\langle \text{pr} \rangle$ as in (1). Set

$$\langle \mathcal{Y} \times_{\mathcal{F}} \mathcal{Y} \rangle = \bigsqcup_{\alpha, \beta \in L} \langle \mathcal{Y}_\alpha \times_{\mathcal{F}} \mathcal{Y}_\beta \rangle,$$ 

and similarly for $\langle \text{pr} \rangle$. The maps $\langle \text{pr} \rangle$ are local isomorphisms of topological spaces. Define

$$\langle \mathcal{F} \rangle = \langle \mathcal{Y} \rangle \amalg \langle \mathcal{Y} \times_{\mathcal{F}} \mathcal{Y} \rangle \langle \mathcal{Y} \rangle$$

as a topological space. It is easy to see that $\langle \mathcal{F} \rangle$ does not depend on the choice of the presentation $\mathcal{Y} = \bigsqcup_{\alpha \in L} \mathcal{Y}_i \to \mathcal{F}$. We call $\langle \mathcal{F} \rangle$ the associated Zariski–Riemann space of $\mathcal{F}$.

Note that (1) and (2) are consistent with each other due to 3.1.4.

The mapping $\mathcal{F} \mapsto \langle \mathcal{F} \rangle$ is functorial: for a morphism $\varphi: \mathcal{F} \to \mathcal{G}$ of rigid spaces we have the canonically induced map $\langle \varphi \rangle: \langle \mathcal{F} \rangle \to \langle \mathcal{G} \rangle$ of topological spaces. Thus the functor $(\dagger)$ in §3.1. (b) extends to

$$\langle \cdot \rangle: \mathcal{Rf} \longrightarrow \text{Top}.$$ 

By the construction of the Zariski–Riemann spaces and 3.1.2 (1), we have the following result.

**Proposition 3.1.8.** The topological space $\langle \mathcal{F} \rangle$ for a rigid space $\mathcal{F}$ is locally coherent (0.2.2.21) and sober (0, §2.1.(b)).

**Proposition 3.1.9.** The functor $\langle \cdot \rangle: \mathcal{Rf} \to \text{Top}$ maps open immersions to open immersions and preserves colimits consisting of open immersions.

**Proof.** By 3.1.4 one sees that the functor $\langle \cdot \rangle: \mathcal{C^0Rf} \to \text{Top}$ preserves base change of coherent open immersions by coherent open immersions. Hence, to show that $\langle \cdot \rangle: \mathcal{Rf} \to \text{Top}$ preserves open immersions, it suffices to check that, if $\varphi: \mathcal{F} \to \mathcal{G}$ is an open immersion, where $\mathcal{F}$ and $\mathcal{G}$ are rigid space represented by a stretch
of coherent rigid spaces, \( \langle \varphi \rangle : \langle \mathcal{F} \rangle \to \langle \mathcal{G} \rangle \) is an open immersion of topological space; but this is clear from the definition. In order to check that \( \langle \cdot \rangle \) preserves colimits for open immersions, it is enough to show that it preserves finite colimits for open immersion, since \( \langle \cdot \rangle \) obviously preserves filtered inductive limits for open immersions. But this easily reduces to the case of finite colimits of coherent rigid spaces, which we have already dealt with in 3.1.4.

3.1. (f) Connectedness

**Definition 3.1.10.** We say that a rigid space \( X \) is *connected* if the associated Zariski–Riemann space \( \langle X \rangle \) is connected.

**Proposition 3.1.11.** A coherent rigid space \( X \) is connected if and only if any distinguished formal model is connected.

*Proof.* The ‘only if’ part follows from 3.1.5. By 1.1.6, \( \langle X \rangle \) is the projective limit of a projective system consisting of distinguished formal models. Hence the ‘if’ part follows from Exercise 0.2.9.

3.1. (g) Notation  Let us use the following simplified notation for cohomologies in the sequel.

**Notation 3.1.12.** (1) Let \( X \) be a rigid space and let \( \langle X \rangle \) be the associated Zariski–Riemann space. For an abelian sheaf \( \mathcal{F} \) on \( \langle X \rangle \) we write

\[
H^q(\langle X \rangle, \mathcal{F}) = H^q(\langle \mathcal{X} \rangle, \mathcal{F})
\]

for any \( q \geq 0 \). When \( q = 0 \), we often denote it by \( \Gamma(\langle X \rangle, \mathcal{F}) \) or by \( \Gamma_{\mathcal{X}}(\mathcal{F}) \). Note that \( H^q(\langle X \rangle, -) \) is the \( q \)-th right derived functor of \( \Gamma_{\mathcal{X}} \).

(2) Let \( \varphi: \mathcal{X} \to \mathcal{Y} \) be a morphism of rigid spaces. Then for any abelian sheaf \( \mathcal{F} \) on \( \langle X \rangle \) we write

\[
R^q \varphi_* \mathcal{F} = R^q \langle \varphi \rangle_* \mathcal{F}
\]

for any \( q \geq 0 \). When \( q = 0 \), we often denote it by \( \varphi_* \mathcal{F} \). Note that \( R^q \varphi_* \) is the \( q \)-th right derived functor of \( \varphi_* \).

3.2 Structure sheaves and local rings

3.2. (a) Integral structure sheaf.  Let \( \mathcal{X} \) be a coherent rigid space, and \( X \) a formal model of \( \mathcal{X} \). Recall that the Zariski–Riemann space \( \langle X \rangle \) has been defined as the underlying topological space of the limit (\( \ast \)) in §3.1. (a), taken in the category of locally ringed spaces. Hence it has the canonical sheaf of rings, denoted by \( \mathcal{O}\text{\textsc{int}}_\mathcal{X} \), such that

\[
(\langle \mathcal{X} \rangle, \mathcal{O}\text{\textsc{int}}_\mathcal{X}) = \lim S_X.
\]
For any point \( x \in \langle \mathcal{X} \rangle \) we have

\[
\mathcal{O}^{\text{int}}_{\mathcal{X}, x} = \lim_{\mathcal{g} \in \text{Alg}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}_{\mathcal{g}}, \text{sp}_{\mathcal{g}}}(x),
\]

where the inductive limit is taken in the category of local rings (with local homomorphisms) (cf. 0.4.1.10). If \( \mathcal{U} \hookrightarrow \mathcal{X} \) is a coherent open immersion of coherent rigid spaces, then by 3.1.3 and 0.4.1.11 we have \( \mathcal{O}^{\text{int}}_{\mathcal{X}, \langle \mathcal{U} \rangle} = \mathcal{O}^{\text{int}}_{\mathcal{U}} \). Hence one can extend the definition of \( \mathcal{O}^{\text{int}}_{\mathcal{X}} \) to general rigid spaces \( \mathcal{X} \) as follows.

**Definition 3.2.1** (integral structure sheaf). Let \( \mathcal{X} \) be a rigid space, and \( \langle \mathcal{X} \rangle \) the associated Zariski–Riemann space. The **integral structure sheaf**, denoted by \( \mathcal{O}^{\text{int}}_{\mathcal{X}} \), is the sheaf of local rings on the topological space \( \langle \mathcal{X} \rangle \) such that for any open immersion \( j: \mathcal{U} \hookrightarrow \mathcal{X} \) from a coherent rigid space \( \mathcal{U} \) we have \( j^{-1} \mathcal{O}^{\text{int}}_{\mathcal{X}} = \mathcal{O}^{\text{int}}_{\mathcal{U}} \), where \( \mathcal{O}^{\text{int}}_{\mathcal{F}} \) is the one defined as above.

The following proposition is easy to see.

**Proposition 3.2.2.** (1) If \( \mathcal{U} \hookrightarrow \mathcal{X} \) is an open immersion of rigid spaces, the induced morphism \( \langle \langle \mathcal{U}, \mathcal{O}^{\text{int}}_{\mathcal{U}} \rangle \rangle \to \langle \langle \mathcal{X}, \mathcal{O}^{\text{int}}_{\mathcal{X}} \rangle \rangle \) is an open immersion of locally ringed spaces.

(2) Let \( \mathcal{X} \) be a coherent rigid space, and \( \mathcal{U} \) a quasi-compact open subset of \( \langle \mathcal{X} \rangle \). Then there exist an open immersion \( \mathcal{U} \hookrightarrow \mathcal{X} \) from a coherent rigid space such that the open immersion \( \langle \langle \mathcal{U}, \mathcal{O}^{\text{int}}_{\mathcal{X}} \rangle \rangle \to \langle \langle \mathcal{X}, \mathcal{O}^{\text{int}}_{\mathcal{X}} \rangle \rangle \) is isomorphic to the induced open immersion \( \langle \langle \mathcal{U}, \mathcal{O}^{\text{int}}_{\mathcal{U}} \rangle \rangle \to \langle \langle \mathcal{X}, \mathcal{O}^{\text{int}}_{\mathcal{X}} \rangle \rangle \).

**Definition 3.2.3** (ideal of definition). (1) Let \( \mathcal{X} \) be a coherent rigid space, and \( \langle \mathcal{X} \rangle \) the associated Zariski–Riemann space. An ideal \( \mathcal{I} \subseteq \mathcal{O}^{\text{int}}_{\mathcal{X}} \) is said to be an **ideal of definition of finite type** if there exist a formal model \( \mathcal{X} \) and an ideal of definition of finite type \( \mathcal{I} \mathcal{X} \) such that \( \mathcal{I} = (\text{sp}_{\mathcal{X}}^{-1} \mathcal{I}_{\mathcal{X}}) \mathcal{O}^{\text{int}}_{\mathcal{X}} \).

(2) Let \( \mathcal{X} \) be a rigid space and \( \langle \mathcal{X} \rangle \) the associated Zariski–Riemann space. An ideal \( \mathcal{I} \subseteq \mathcal{O}^{\text{int}}_{\mathcal{X}} \) is said to be an **ideal of definition of finite type** if for any open immersion \( j: \mathcal{U} \hookrightarrow \mathcal{X} \) from a coherent rigid space, \( \mathcal{I} \mathcal{X}_{\mathcal{U}} \) is an ideal of definition of finite type in the above sense.

The above two definitions (1) and (2) of ideals of definition coincide if \( \mathcal{X} \) is a coherent rigid space, that is, if \( \mathcal{I} \) is an ideal of definition of finite type in the sense of (2); then there exists a formal model \( \mathcal{X} \) of \( \mathcal{X} \) and an ideal of definition of finite type \( \mathcal{I} \mathcal{X} \) of \( \mathcal{X} \) such that \( \mathcal{I} = (\text{sp}_{\mathcal{X}}^{-1} \mathcal{I}_{\mathcal{X}}) \mathcal{O}^{\text{int}}_{\mathcal{X}} \). Recall that, by I.3.7.12, any coherent rigid space has an ideal of definition of finite type.

**Proposition 3.2.4.** Let \( \mathcal{X} \) be a coherent rigid space, and \( \mathcal{I} \) an ideal of definition of finite type. Then there exists a distinguished formal model \( \mathcal{X} \) of \( \mathcal{X} \) and an invertible ideal of definition \( \mathcal{I} \mathcal{X} \) such that \( \mathcal{I} = (\text{sp}_{\mathcal{X}}^{-1} \mathcal{I}_{\mathcal{X}}) \mathcal{O}^{\text{int}}_{\mathcal{X}} \). Moreover, such formal models are cofinal in the category of all formal models of \( \mathcal{X} \).
Proof. Let $X$ be a formal model of $\mathcal{X}$, and $I_X$ an ideal of definition of finite type on $X$ such that $I = (\text{sp}_X^{-1} I_X)\mathcal{O}_{\mathcal{X}}^\text{int}$. Replacing $X$ by the admissible blow-up along $I_X$, we can make $I_X$ invertible, and hence $X$ is distinguished. \hfill $\square$

**Corollary 3.2.5.** Let $\mathcal{X}$ be a rigid space and $I$ an ideal of definition of finite type. Then $I$ is an invertible ideal of $\mathcal{O}_{\mathcal{X}}^\text{int}$, and hence $\mathcal{O}_{\mathcal{X}}^\text{int}$ is $I$-torsion free.

*Proof.* We may assume that $\mathcal{X}$ is coherent. By 3.2.4 there exists a formal model $X$ with an invertible ideal of definition $I_X$ such that $I = (\text{sp}_X^{-1} I_X)\mathcal{O}_{\mathcal{X}}^\text{int}$. Let $\pi: X' \to X$ be an admissible blow-up. Then $I_{X'} = I_X \mathcal{O}_{X'}$ is an invertible ideal of definition (1.1.6). Since $\lim_{\longrightarrow} X' \to X$, $\text{sp}_X^{-1} I_{X'}$ is an ideal of $\mathcal{O}_{\mathcal{X}}^\text{int}$, we have

$$I = \lim_{\longrightarrow} \text{sp}_X^{-1} I_{X'}.$$ 

Using this we easily deduce that $\mathcal{O}_{\mathcal{X}}^\text{int}$ is $I$-torsion free and that $I$ is invertible. \hfill $\square$

3.2. (b) Rigid structure sheaf

**Proposition 3.2.6.** Let $\mathcal{X}$ be a coherent rigid space, $\langle \mathcal{X} \rangle$ the associated Zariski–Riemann space, and $x \in \langle \mathcal{X} \rangle$ a point. Let $I$ be an ideal of definition of finite type, and set $I_x = I_X$. Then the stalk $A_x = \mathcal{O}_{\mathcal{X},x}^\text{int}$ of the integral structure sheaf is $I_x$-valutative (0.8.7.1) and $I_x$-adically Henselian (0, §7.3. (a)).

*Proof.* Let $\mathcal{X} = X^\text{rig}$ and $I = I_X \mathcal{O}_{\mathcal{X}}^\text{int}$, where $I_X$ is an ideal of definition of $X$ of finite type. Let $\widetilde{I} = I_{X,\text{sp}_X(x)}$. Then $A_x$ is the filtered inductive limit of $\widetilde{I}$-adically complete local rings and hence is Henselian with respect to $I_x$ (0.7.3.1).

Let $J$ be an $I_x$-admissible ideal (0.8.1.2) of $A_x$. We need to show that $J$ is an invertible ideal of $A_x$. Replacing $X$ by an admissible blow-up if necessary, we may assume that there exists a finitely generated ideal $J \subseteq \mathcal{O}_{X,\text{sp}_X(x)}$ such that $\widetilde{J} A_x = J$. Now, since we may work locally around $x \in \langle \mathcal{X} \rangle$, we can replace $X$ by any coherent open subspace $U \subseteq \mathcal{X}$ such that $x \in \langle U \rangle$. In view of 3.2.2 (2), this allows us to replace $X$ by any quasi-compact open neighborhood $U \subseteq X$ of $\text{sp}_X(x)$. We may therefore assume that $X$ is affine, $X = \text{Spf} A$, and, moreover, that $A$ has a finitely generated ideal of definition $I \subseteq A$ and an $I$-admissible ideal $J_X \subseteq A$ such that $I^\Delta = I_X$ and $J_X \mathcal{O}_{X,\text{sp}_X(x)} = \widetilde{J}$ (hence $J_X A_x = J$). Now consider the admissible blow-up $X' \to X$ along $J_X$. Then $J_X \mathcal{O}_{X'}$ is invertible. Since $\mathcal{O}_{\mathcal{X}}^\text{int}$ is $I_X$-torsion free due to 3.2.5, it is $J_X$-torsion free, and hence $J = J_X \mathcal{O}_{\mathcal{X},x}^\text{int}$ is an invertible ideal of $A_x = \mathcal{O}_{\mathcal{X},x}^\text{int}$, as desired. \hfill $\square$

**Corollary 3.2.7.** Let $\mathcal{X}$ be a coherent rigid space, $\langle \mathcal{X} \rangle$ the associated Zariski–Riemann space, and $x \in \langle \mathcal{X} \rangle$ a point. Then for any ideal of definition $I$ of finite type of $\langle \mathcal{X} \rangle$, we have $I_x = (a)$ for a non-zero-divisor $a \in \mathcal{O}_{\mathcal{X},x}^\text{int}$.
By 0.8.7.8 and 0.8.7.13 we have the following corollary.

**Corollary 3.2.8.** Let $\mathcal{X}$ be a coherent rigid space, $\langle \mathcal{X} \rangle$ the associated Zariski–Riemann space, and $x \in \langle \mathcal{X} \rangle$ a point. Let $I$ be an ideal of definition of finite type of $\langle \mathcal{X} \rangle$, and set $(A_x, I_x) = (\mathcal{O}_{\mathcal{X},x}^{\text{int}}, I_x)$. Set $I_x = (a)$ (cf. 3.2.7) and $J_x = \bigcap_{n \geq 1} I_x^n$.

(a) $B_x = \lim_{n \to \infty} \text{Hom}(I_x^n, A_x) = A_x[\frac{1}{a}]$ is a local ring, and $V_x = A_x/J_x$ is a valuation ring, $\tilde{a}$-adically separated, for the residue field $K_x$ of $B_x$, where $I_x V_x = (\tilde{a})$.

(b) $A_x = \{ f \in B_x : (f \mod m_{B_x}) \in V_x \}$.

(c) $J_x = m_{B_x}$.

Moreover, $B_x$ is a Henselian local ring, and $V_x$ is Henselian with respect to $\tilde{a}$-adic topology.

**Definition 3.2.9** (Rigid structure sheaf). Let $\mathcal{X}$ be a rigid space. We define a sheaf of rings $\mathcal{O}_\mathcal{X}$ on its associated Zariski–Riemann space $\langle \mathcal{X} \rangle$ as follows.

(1) If $\mathcal{X}$ is a coherent rigid space, then

$$\mathcal{O}_\mathcal{X} = \lim_{x \to \infty} \text{Hom}_{\mathcal{O}_{\mathcal{X},x}^{\text{int}}}(I_x^n, \mathcal{O}_{\mathcal{X},x}^{\text{int}}),$$

where $I$ is an ideal of definition of finite type (cf. Deligne’s formula [53], (6.9.17)). (Exercise II.3.2 verifies that this does not depend on the choice of $I$.)

(2) In general, we construct $\mathcal{O}_\mathcal{X}$ by patching; that is, $\mathcal{O}_\mathcal{X}$ is the sheaf such that for any open immersion $U \hookrightarrow \mathcal{X}$ from a coherent rigid space, $\mathcal{O}_\mathcal{X}|_U$ is the one defined as in (1).

We call the sheaf $\mathcal{O}_\mathcal{X}$ the (rigid) structure sheaf of $\langle \mathcal{X} \rangle$. By the local description given in 3.2.8 we have the following proposition.

**Proposition 3.2.10.** The sheaf $\mathcal{O}_\mathcal{X}$ is a sheaf of Henselian local rings, that is, for any $x \in \langle \mathcal{X} \rangle$ the stalk $\mathcal{O}_{\mathcal{X},x}$ is a Henselian local ring.

3.2. (c) Zariski–Riemann triple. We have so far obtained two sheaves $\mathcal{O}_{\mathcal{X},x}^{\text{int}}$ and $\mathcal{O}_{\mathcal{X}}$, by which the space $\langle \mathcal{X} \rangle$ is endowed with locally ringed structures in two ways.

**Definition 3.2.11.** Let $\mathcal{X}$ be a rigid space. Then the triple

$$\textbf{ZR}(\mathcal{X}) = (\langle \mathcal{X} \rangle, \mathcal{O}_{\mathcal{X},x}^{\text{int}}, \mathcal{O}_{\mathcal{X}})$$

is called the Zariski–Riemann triple associated to the rigid space $\mathcal{X}$.

See §A.1 for a general theory of triples.

**Convention.** By the structure sheaf of a rigid space $\mathcal{X}$ we always mean the rigid structure sheaf $\mathcal{O}_{\mathcal{X}}$ on $\langle \mathcal{X} \rangle$, unless otherwise clearly stated.
3. (d) Reducedness

Definition 3.2.12. We say that a rigid space $\mathcal{X}$ is reduced if the ringed space $((\mathcal{X}), \mathcal{O}_\mathcal{X})$ is reduced in the sense as in 0, §4.1.

It will be shown in 6.4.2 that a coherent universally Noetherian rigid space $\mathcal{X}$ is reduced if and only if any distinguished formal model of $\mathcal{X}$ is reduced.

3.2. (e) Description of the local rings

Notation 3.2.13. Let $\mathcal{X}$ be a rigid space and $x \in \langle \mathcal{X} \rangle$. We will often use the following notations (cf. 3.2.8):

- $A_x = \mathcal{O}_{\mathcal{X},x}^{\text{int}}$.

One can always find a coherent open neighborhood of $x$ of the form $\langle \text{Spf } A \rangle^{\text{rig}}$ such that the adic ring $A$ has a principal invertible ideal of definition $I = (a)$. Considering the invertible ideal

- $I_x = IA_x = aA_x$

of $A_x$, one has

- $B_x = A_x[\frac{1}{a}] = \mathcal{O}_{\mathcal{X},x}, K_x = B_x/m_{B_x}$;
- $J_x = m_{B_x} = \bigcap_{n \geq 1} I_{x}^{n};$
- $V_x = A_x/J_x, k_x = V_x/m_{V_x}$.

Let $\tilde{a}$ be the image of $a$ in $V_x$. Then $V_x$ is an $\tilde{a}$-adically separated and Henselian valuation ring with the residue field $k_x$ such that $\text{Frac}(V_x) = K_x$, and we have $A_x = \{ f \in B_x : (f \mod m_{B_x}) \in V_x \}$.

Note that the objects $A_x, B_x, K_x, J_x, V_x$, and $k_x$ do not depend on the choice of $I = (a) \subseteq A$. When we want to describe these local data, the following notion will often prove useful.

Definition 3.2.14. Let $\mathcal{X}$ be a rigid space and $x \in \langle \mathcal{X} \rangle$ a point.

1. A formal neighborhood of $x$ in $\mathcal{X}$ is a pair $(U, \iota)$ consisting of a coherent adic formal scheme of finite ideal type $U$ and an open immersion

$$\iota: U = U^{\text{rig}} \hookrightarrow \mathcal{X}$$

such that $x \in \langle \iota \rangle((\langle U \rangle))$. 
(2) By a morphism between formal neighborhoods \((U, i)\) and \((U', i')\) we mean an adic morphism \(h: U \to U'\) such that the resulting diagram

\[
\begin{array}{ccc}
U_{\text{rig}} & \xrightarrow{h_{\text{rig}}} & X \\
\downarrow i_{\text{rig}} & & \downarrow \quad \\
U'_{\text{rig}} & \xrightarrow{i'} & X
\end{array}
\]

commutes.

We denote by \(\text{FN}_{X, x}\) or simply by \(\text{FN}_x\) the category of formal neighborhoods of \(x\) in \(X\). Formal neighborhoods are usually considered up to isomorphism in this category.

A formal neighborhood \(U = (U, i)\) is said to be affine if the formal scheme \(U\) is affine. Note here that the category \(\text{FN}_{X, x}\) is canonically cofiltered, and affine formal neighborhoods give a cofinal family.

Suppose that \(X\) is coherent, and let \(X\) be a formal model of \(X\). Then any quasi-compact Zariski open neighborhood \(U \subseteq X\) of \(\text{sp}_X(x)\) gives rise to a formal neighborhood \((U, i_{\text{rig}})\) of \(x\), where \(i: U \hookrightarrow X\) is the inclusion map. By 3.1.3, considering all quasi-compact Zariski (affine) open neighborhoods of \(\text{sp}_{X'}(x)\) in any admissible blow-up \(X'\) of \(X\), one has a system of formal neighborhoods that gives a cofinal system of quasi-compact open neighborhoods of \(x\) in \(\langle X \rangle\).

Now suppose we are in the situation as in 3.2.13 with \(X = X_{\text{rig}}, X = \text{Spf}\ A\). We fix a system of affine formal neighborhoods considered together with embedded formal models

\[
\{ (\mathcal{U}_\alpha = (\text{Spf}\ A_\alpha)_{\text{rig}}, j_\alpha: U_\alpha = \text{Spf}\ A_\alpha \hookrightarrow X_\alpha) \}_{\alpha \in L}
\]

indexed by a directed set \(L\); each \(U_\alpha = \text{Spf}\ A_\alpha\) is an affine open subset of an admissible blow-up \(X_\alpha\) of \(X = \text{Spf}\ A\) that contains the point \(\text{sp}_\alpha(x)\), where we denote the specialization map \(\text{sp}_{X_\alpha}: \langle X \rangle \to X_\alpha\) simply by \(\text{sp}_\alpha\). (Here we may assume that the directed set \(L\) has a minimum, say \(0 \in L\), and \(A_0 = A\).) For \(\alpha \leq \beta\) we assume that there exists a commutative diagram

\[
\begin{array}{ccc}
U_\beta & \xrightarrow{j_\beta} & X_\beta \\
\downarrow \quad & & \downarrow \pi_{\beta\alpha} \\
U_\alpha & \xrightarrow{j_\alpha} & X_\alpha
\end{array}
\]

where \(\pi_{\beta\alpha}\) is an admissible blow-up, such that for \(\alpha \leq \beta \leq \gamma\) we have \(\pi_{\gamma\alpha} = \pi_{\beta\alpha} \circ \pi_{\gamma\beta}\). In this situation we have the inductive system of rings \(\{A_\alpha\}_{\alpha \in L}\). Note that, by 1.1.6, each \(A_\alpha\) has the invertible ideal of definition \(IA_\alpha = (a)\).
Taking \( L \) to be the set of all pairs \((X', U')\) consisting of admissible blow-ups \( X' \) of \( X_0 \) and an affine neighborhood \( U' \subseteq X' \) of \( \text{sp}_X(x) \) with the ordering defined as above, we can construct a system as above such that the image of the forgetful map \((\mathcal{U}_\alpha, j_\alpha: U_\alpha \leftarrow X_\alpha) \mapsto (\mathcal{U}_\alpha, j_\alpha^\text{rig}) \) is cofinal in \( \mathbf{FN}_{X,x} \).

**Proposition 3.2.15.** (1) We have the canonical identifications

\[
A_x = \mathcal{O}_{\mathcal{X},x}^\text{int} = \lim_{\alpha \in L} A_\alpha \quad \text{and} \quad B_x = \mathcal{O}_{X,x} = \lim_{\alpha \in L} A_\alpha [\frac{1}{a}].
\]

(2) Let \( J_\alpha \) for each \( \alpha \in L \) be the ideal of \( A_\alpha \) that is the pull-back of \( J_x \) by the map \( A_\alpha \to A_x = \mathcal{O}_{\mathcal{X},x}^\text{int} \). Then \( J_x = \lim_{\alpha \in L} J_\alpha \) and

\[
V_x = \lim_{\alpha} A_\alpha / J_\alpha.
\]

**Proof.** By the definition of the sheaf \( \mathcal{O}_{\mathcal{X}}^\text{int} (\S 3.2. (a)) \),

\[
A_x = \lim_{\alpha \in L} \Gamma(\mathcal{U}_\alpha, \mathcal{O}_{\mathcal{X}}^\text{int}) = \lim_{\alpha \in L} \lim_{X' \subseteq \mathcal{X}} \Gamma(X', \mathcal{O}_{X'}),
\]

where \( X' \) in the second limit runs through the totality of all admissible blow-ups of \( \text{Spf} A_\alpha \). One deduces by a standard argument (cf. 0.1.3.1), using the extension of admissible blow-ups (1.1.9), that the last limit is canonically isomorphic to \( \lim_{\alpha \in L} A_\alpha \). Then the second equality follows immediately. (2) follows from the first equality of (1) and the exactness of filtered inductive limits (cf. 0.3.1.3). \( \square \)

### 3.2. (f) Generalization maps

Let \( \mathcal{X} \) be a rigid space, and \( x, x' \in (\mathcal{X}) \), and suppose \( x' \) is a generalization of \( x \) (cf. 0, §2.1. (a)). We have the generalization map (0, §4.1. (b))

\[
A_x \longrightarrow A_{x'}.
\]

**Proposition 3.2.16.** The generalization map \((*)\) maps \( J_x \) to \( J_{x'} \), and induces a local homomorphism \( B_x \to B_{x'} \). In particular, it induces an injective morphism \( K_x \to K_{x'} \), which maps \( V_x \) injectively into \( V_{x'} \).

**Proof.** Since any open neighborhood of \( x \) in \((\mathcal{X})\) contains \( x' \), we may work in the setting of 3.2.13, where \( \mathcal{X} = X^\text{rig} \) with \( X = \text{Spf} A \). Then it is clear that the generalization map \( A_x \to A_{x'} \), which is obviously \( a \)-adic, maps \( J_x \) to \( J_{x'} \) and induces \( B_x \to B_{x'} \) and \( V_x \to V_{x'} \). Since \( J_x = m_{B_x} \) and \( J_{x'} = m_{B_{x'}} \), the map \( B_x \to B_{x'} \) is a local homomorphism and hence induces the injective morphism \( K_x \to K_{x'} \) between the residue fields. Since \( K_x = \text{Frac}(V_x) \) and \( K_{x'} = \text{Frac}(V_{x'}) \), the morphism \( V_x \to V_{x'} \) is injective. \( \square \)
Theorem 3.2.17. Suppose \( \mathcal{X} \) is locally universally Noetherian (2.2.23).

(1) Let \( \mathfrak{p}' \) be the prime ideal of \( A_x = \mathcal{O}^\text{int}_{\mathcal{X},x} \) that is the pull-back of the maximal ideal of \( A_{x'} = \mathcal{O}^\text{int}_{\mathcal{X},x'} \) by the generization map \( A_x \to A_{x'} \). Then the induced \( \mathfrak{a} \)-adic map \( (A_x)_{\mathfrak{p}} \to A_{x'} \) is faithfully flat, and \( (A_x)_{\mathfrak{p}}/(a) \to A_{x'}/(a) \) is an isomorphism.

(2) Set \( \mathfrak{q}' = \mathfrak{p}'/J_x \). Then the \( \mathfrak{a} \)-adic map \( (V_x)_{\mathfrak{q}} \to V_{x'} \) induced by the generization map \( V_x \to V_{x'} \) (3.2.16) is faithfully flat, and the induced map

\[
(V_x)_{\mathfrak{q}}/(a) \to V_{x'}/(a)
\]

is an isomorphism. In particular, we have the natural isomorphism

\[
(V_x)_{\mathfrak{q}} \sim \hat{V}_{x'}
\]

between the \( \mathfrak{a} \)-adic completions.

(3) The generization map \( B_x \to B_{x'} \) (3.2.16) is faithfully flat.

Proof. Take a cofinal system of affine formal neighborhoods \( \{ \mathcal{U}_\alpha = (\text{Spf} A_\alpha)^{\text{rig}} \}_{\alpha \in L} \) of \( x \) as in \( \S 3.2. \) (e) and the filtered inductive system \( \{ J_\alpha \}_{\alpha \in L} \) of ideals as in 3.2.15 (2). Since each affinoid neighborhood \( \langle \mathcal{U}_\alpha \rangle \) of \( x \) contains \( x' \), replacing the index set \( L \) by a larger directed set, one can form the analogous cofinal system of affine neighborhoods \( \{ \mathcal{U}'_\alpha = (\text{Spf} A'_\alpha)^{\text{rig}} \}_{\alpha \in L} \) of \( x' \) and a filtered inductive system \( \{ J'_\alpha \}_{\alpha \in L} \) of ideals such that \( U'_\alpha = \text{Spf} A'_\alpha \) is an affine open subset of \( U_\alpha = \text{Spf} A_\alpha \) for each \( \alpha \in L \). Indeed, for each \( \alpha \in L \) one can form a cofinal system of open neighborhoods of \( x' \) of the form \( \{ U'_{\alpha,\lambda} = (\text{Spf} A'_{\alpha,\lambda})^{\text{rig}} \}_{\lambda \in A_\alpha} \) in \( U_\alpha \); hence, one can replace \( L \) by the set \( \{ (\alpha, \lambda) : \lambda \in A_\alpha \} \), by defining \( U_{\alpha,\lambda} = U_\alpha \).

Let \( p'_\alpha \) be the open prime ideal of \( A_\alpha \) that is the image of \( x' \) under the specialization map \( \text{sp}_{A_\alpha} : \langle U_\alpha \rangle \to U_\alpha = \text{Spf} A_\alpha \). We can assume that each \( A'_{\alpha} \) is a complete localization of \( A_\alpha \) of the form \( A'_{\alpha} = (A_\alpha)_{(f_\alpha)} \), where \( f_\alpha \in A_\alpha \) is an element not contained in \( p'_\alpha \). The sequence of open prime ideals \( \{ p'_\alpha \}_{\alpha} \) gives by passage to the inductive limit the open prime ideal \( p' \) of \( \mathcal{O}^\text{int}_{\mathcal{X},x} = \lim_{\longrightarrow \alpha \in L} A_\alpha \) corresponding to the generization \( x' \).

(1) Since \( A_\alpha \) is a t.u. rigid-Noetherian ring, the canonical map \( A_\alpha \to (A_\alpha)_{(f_\alpha)} \) is flat (0.8.2.18). Hence, by 3.2.15 (1), it follows that the morphism

\[
A_x = \lim_{\longrightarrow \alpha \in L} A_\alpha \longrightarrow \lim_{\longrightarrow \alpha \in L} A'_\alpha = A_{x'}
\]

is flat. To show that \( (A_x)_{p'}/(a) \to A_{x'}/(a) \) is faithfully flat, it suffices to show that \( (A_x)(p')/(a) \to A_{x'}/(a) \) is an isomorphism (0.7.3.6). By the exactness of the functor for filtered inductive limits, we know that the ring \( (A_x)(p')/(a) \) is isomorphic to the inductive limit

\[
\lim_{\longrightarrow \alpha \in L} (A_\alpha/(a))_{p'_{\alpha}}.
\]
where \( \tilde{p}'_\alpha = p'_\alpha / (a) \). Since this limit is obviously isomorphic to \( \lim_{\alpha \in L} A'_\alpha / (a) \), we have the desired assertion.

(2) By 3.2.16 the map \( V_x \to V_{x'} \) is injective, and hence is flat. The pull-back of the maximal ideal \( m_{V_{x'}} \) is the open prime ideal \( q' \). To show that \( (V_x)_{q'} \to V_{x'}/(a) \) is faithfully flat, it suffices to show that \( (A_x)_{p'}/(a) \to A_{x'}/(a) \) is an isomorphism, as shown in (1).

(3) follows immediately from (1) due to the equalities \( B_x = A_x[\frac{1}{a}] \) and \( B_{x'} = A_{x'}[\frac{1}{a}] \).

3.3 Points on Zariski–Riemann spaces

3.3. (a) Rigid points

**Definition 3.3.1.** (1) A rigid point of a rigid space \( \mathcal{X} \) is a morphism of rigid spaces of the form
\[
\alpha: \mathcal{T} = (\text{Spf } V)^{\text{rig}} \longrightarrow \mathcal{X},
\]
where \( V \) is an \( a \)-adically complete valuation ring \( (a \in m_V \setminus \{0\}) \).

(2) Let \( X \) be a coherent adic formal scheme of finite ideal type. A rigid point of \( X \) is an adic morphism of the form
\[
\alpha: \text{Spf } V \longrightarrow X,
\]
where \( V \) is an \( a \)-adically complete valuation ring \( (a \in V \setminus \{0\}) \).

Two rigid points, \( \alpha: \mathcal{T} = (\text{Spf } V)^{\text{rig}} \longrightarrow \mathcal{X} \) and \( \beta: \mathcal{S} = (\text{Spf } W)^{\text{rig}} \longrightarrow \mathcal{X} \), are said to be isomorphic if there exists an isomorphism of rigid spaces \( \mathcal{T} \sim \mathcal{S} \) over \( \mathcal{X} \); the isomorphism of rigid points of \( X \) is defined similarly.

**Definition 3.3.2.** Let \( \mathcal{X} \) be a rigid space and \( Y \) an adic formal scheme of finite ideal type (I.1.1.16). Let \( \alpha: Y \to (\mathcal{X}, \mathcal{O}_\mathcal{X}^{\text{int}}) \) be a morphism of locally ringed spaces. We say that the map \( \alpha \) is adic if for any open immersion \( \mathcal{U} \hookrightarrow \mathcal{X} \) and an ideal of definition of finite type \( I \) of \( (\mathcal{U}) \), the ideal \( (\alpha^{-1} I) \mathcal{O}_{\alpha^{-1}(\mathcal{U})} \) is an ideal of definition of the open formal subscheme \( \alpha^{-1}(\mathcal{U}) \) of \( Y \).

**Proposition 3.3.3.** There exists a canonical bijection between the set of all isomorphism classes of rigid points of a rigid space \( \mathcal{X} \) and the set of all isomorphism classes of adic morphisms \( \alpha: \text{Spf } V \to (\mathcal{X}, \mathcal{O}_\mathcal{X}^{\text{int}}) \) of locally ringed spaces, where \( V \) is an \( a \)-adically complete valuation ring, where two such morphisms, \( \alpha: \text{Spf } V \to (\mathcal{X}) \) and \( \beta: \text{Spf } W \to (\mathcal{X}) \), are said to be isomorphic if there exists an isomorphism of formal schemes \( \text{Spf } V \sim \text{Spf } W \) such that the resulting triangle diagram commutes.
The bijection maps a rigid point \( \alpha: \mathcal{T} = (\text{Spf} V)^{\text{rig}} \to \mathcal{X} \) to the map \( \langle \alpha \rangle \); note that we have \( \langle (\text{Spf} V)^{\text{rig}} \rangle = \text{Spf} V \) (Exercise II.3.3).

**Proof.** We give the inverse map. Let \( \alpha: \text{Spf} V \to \langle \mathcal{X} \rangle, \mathcal{O}_{\mathcal{X},x}^\text{int} \) be an adic morphism, and \( x \) the image of the closed point. Take a coherent open subspace \( U \subseteq \mathcal{X} \) such that \( x \in \langle U \rangle \). We have \( \alpha(\text{Spf} V) \subseteq \langle U \rangle \). Let \( U \) be a formal model of \( U \), and consider the specialization map \( \text{sp}_U: \langle U \rangle \to U \). Then the composition \( \text{sp}_U \circ \alpha \) gives a rigid point of \( U \). Taking \( \text{rk} \), we get a rigid point

\[
\mathcal{T} = (\text{Spf} V)^{\text{rig}} \to U \hookrightarrow \mathcal{X}.
\]

In the sequel, by a rigid point of a rigid space \( \mathcal{X} \) we sometimes mean a morphism of locally ringed spaces as in 3.3.3.

**Proposition 3.3.4.** Let \( \mathcal{X} \) be a rigid space.

1. For \( x \in \langle \mathcal{X} \rangle \) there exists a rigid point of the form

\[
\alpha_x: \text{Spf} \overline{V}_x \to \langle \mathcal{X} \rangle, \quad \mathcal{m}_{\overline{V}_x} \mapsto x.
\]

where \( \overline{V}_x \) is the \( \acute{a} \)-adic completion of \( V_x \) (in the notation as in 3.2.13), such that the induced map of stalks at \( x \) is the canonical map

\[
A_x = \mathcal{O}_{\mathcal{X},x}^\text{int} \to \overline{V}_x.
\]

2. Let \( \alpha: \text{Spf} V \to \langle \mathcal{X} \rangle \) be a rigid point such that \( \alpha(\mathcal{m}_V) = x \). Then there exists uniquely an injective homomorphism \( \overline{V}_x \to V \) such that \( V \) dominates \( \overline{V}_x \) and the diagram

\[
\begin{array}{ccc}
\text{Spf} V & \xrightarrow{\alpha} & \langle \mathcal{X} \rangle \\
\downarrow & & \downarrow \\
\text{Spf} \overline{V}_x & \xrightarrow{\alpha_x} & \mathcal{X}
\end{array}
\]

commutes.

**Proof.** We may assume that \( \mathcal{X} \) is coherent.

1. Take the \( \acute{a} \)-adic completion \( \overline{V}_x \) of the \( \acute{a} \)-adically separated valuation ring \( V_x \). The map \( A_x = \mathcal{O}_{\mathcal{X},x}^\text{int} \to \overline{V}_x \) induces the inductive system of homomorphisms \( \{ \mathcal{O}_{X',\text{sp}X'(x)} \to \overline{V}_x \} \), where \( X' \) runs through all admissible blow-ups \( X' \to X \) of \( X \), and hence the projective system of adic morphisms \( \{ \text{Spf} \overline{V}_x \to X' \} \). The desired map \( \alpha_x \) is the projective limit of this system of morphisms.

2. The morphism \( \alpha \) gives an \( \acute{a} \)-adic homomorphism \( \mathcal{O}_{\mathcal{X},x}^\text{int} \to V \). Since \( V \) is \( \acute{a} \)-adically complete, there exists a unique factorizing map \( \overline{V}_x \to V \). Since the last map is \( \acute{a} \)-adic, we deduce by 0.6.7.6 that the map \( \overline{V}_x \to V \) is injective. Since \( \alpha(\mathcal{m}_V) = \alpha_x(\mathcal{m}_{\overline{V}_x}) = x \), \( V \) dominates \( \overline{V}_x \), as desired. \( \square \)
Definition 3.3.5. Let \( x \in \langle \mathcal{X} \rangle \) be a point. Then the rigid point \( \alpha_x : \text{Spf} \tilde{V}_x \to \langle \mathcal{X} \rangle \) as in 3.3.4 (1) is called the associated rigid point of \( x \).

In the rest of this subsection, \( \mathcal{X} \) denotes a coherent rigid space. Let

\[
\alpha : \text{Spf} V \longrightarrow \langle \mathcal{X} \rangle
\]

be a rigid point of \( \mathcal{X} \). It defines the point \( \alpha(m_V) \) in \( \langle \mathcal{X} \rangle \). Thus we get a mapping

\[
\left\{ \text{isomorphism classes of rigid points of } \langle \mathcal{X} \rangle \right\} \longrightarrow \langle \mathcal{X} \rangle, \quad \alpha \mapsto \alpha(m_V) . \tag{*}
\]

If \( X \) is a formal model of \( \mathcal{X} \), then we have another mapping

\[
\left\{ \text{isomorphism classes of rigid points of } \langle \mathcal{X} \rangle \right\} \longrightarrow \left\{ \text{isomorphism classes of rigid points of } X \right\} , \quad \alpha \mapsto \text{sp}_X \circ \alpha . \tag{**}
\]

Note that \( \text{sp}_X \circ \alpha \) is a morphism of formal schemes, for it is a morphism of locally ringed spaces, and is adic (and hence continuous).

Proposition 3.3.6. (1) The mapping \((*)\) is surjective. Define an equivalence relation \( \approx \) on the left-hand set generated by the relation \( \sim \) given as follows: for rigid points \( \alpha : \text{Spf} V \to \langle \mathcal{X} \rangle \) and \( \beta : \text{Spf} W \to \langle \mathcal{X} \rangle \), \( \alpha \sim \beta \) if there exists an injective map \( f : V \leftarrow W \) such that \( \alpha \circ \text{Spf} f = \beta \) and \( W \) dominates \( V \). Then \((*)\) induces a bijection

\[
\left\{ \text{isomorphism classes of rigid points of } \langle \mathcal{X} \rangle \right\} / \approx \longrightarrow \langle \mathcal{X} \rangle . \tag{*}'
\]

(2) The mapping \((**)\) is bijective.

Note that in (1) the map \( \text{Spf} f : \text{Spf} W \to \text{Spf} V \) is automatically adic and that \( V \subset \text{Frac}(V) \cap W \subset \text{Frac}(W) \) (cf. (a) in 0.6.2.1). Note also that if \( \alpha \sim \beta \), then \( \alpha(m_V) = \alpha(m_W) \); thus the map \((*)'\) as above is well defined.

For the proof of 3.3.6 we need the following lemma.

Lemma 3.3.7. Let \( \pi : X' \to X \) be an admissible blow-up and \( \alpha : \text{Spf} V \to X \) a rigid point. Then there exists a unique rigid point \( \alpha' : \text{Spf} V \to X' \) such that the resulting diagram commutes:

\[
\begin{array}{ccc}
\text{Spf} V & \xrightarrow{\alpha} & X \\
\downarrow \alpha' & & \downarrow \pi \\
X' & & \\
\end{array}
\]
Proof. Let \( \mathcal{J} \) be an admissible ideal of \( X \) that gives the admissible blow-up \( \pi \). Since \( \mathcal{J} \) is of finite type, the ideal \( \mathcal{J} V \) of \( V \) is invertible (0.6.2.2 (2)). Hence, there exists a unique factoring map \( \alpha' \) as above. By I.1.3.6 (1), the morphism \( \alpha' \) is adic. \( \square \)

Proof of Proposition 3.3.6. (1) Let \( x \in \langle \mathcal{X} \rangle \) be a point. We use the notation of 3.2.13. By 3.3.4 (1), we have a mapping \( x \mapsto \alpha_x \) from the set \( \langle \mathcal{X} \rangle \) to the set of all isomorphism classes of rigid points of \( \mathcal{X} \). Clearly, for \( x \in \langle \mathcal{X} \rangle \) we have \( \alpha_x(m_{\mathcal{V}_x}) = x \). Hence \((*)\) is surjective. To show the other half of (1), it suffices to check that for a rigid point \( \alpha: \text{Spf} V \to \langle \mathcal{X} \rangle \) such that \( \alpha(m_{\mathcal{V}}) = x \), we have \( \alpha_x \sim \alpha \). But this follows from 3.3.4 (2).

(2) Let \( \alpha: \text{Spf} V \to X \) be a rigid point. Since \( \langle \mathcal{X} \rangle \) is the projective limit of all admissible blow-ups of \( X \), by 3.3.7 we have a unique adic map
\[
(\alpha): \text{Spf} V \longrightarrow \langle \mathcal{X} \rangle
\]
such that \( \text{sp}_X \circ (\alpha) = \alpha \). This yields a map from the set of all isomorphism classes of rigid points of \( \mathcal{X} \) to the set of all isomorphism classes of rigid points of \( \langle \mathcal{X} \rangle \). By the relation \( \text{sp}_X \circ (\alpha) = \alpha \) and the uniqueness of the lifting, this map gives the inverse to \( (**) \). \( \square \)

3.3. (b) Seminorms associated to points. Let \( \mathcal{X} \) be a coherent rigid space, and \( \mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}^\text{int} \) an ideal of definition of finite type (3.2.3). We have the valuation ring \( V_x \) at \( x \) for the field \( K_x \) as in 3.2.13. Since \( V_x \) is \( a \)-adically separated (where \( \mathcal{I} A_x = (a) \)), we have the associated height-one prime \( p = \sqrt{(a)} \) (0.6.7.4), and hence we have the corresponding height-one valuation on the field \( K_x \) with the valuation ring \( V_p \). We can then define, choosing a once for all fixed real number \( 0 < c < 1 \), the corresponding (non-Archimedean) norm (0, §6.3. (c)), denoted by
\[
\| \cdot \|_{x,\mathcal{I},c}: K_x \longrightarrow \mathbb{R}_{\geq 0},
\]
uniquely determined by \( \| a \|_{x,\mathcal{I},c} = c \). It is clear that another choice of \( \mathcal{I} \) and \( c \) only leads to an equivalent norm.

For \( f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \) and \( x \in \langle \mathcal{X} \rangle \), we denote by \( f(x) \) the image of \( f \) under the composite map
\[
\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \longrightarrow B_x (= \mathcal{O}_{\mathcal{X},x}) \longrightarrow K_x (= B_x/m_{B_x}),
\]
and write
\[
\| f(x) \|_{\mathcal{I},c} = \| f(x) \|_{x,\mathcal{I},c}.
\]
This construction yields a mapping (denoted also by \( \| \cdot \|_{x,\mathcal{I},c} \))
\[
\| \cdot \|_{x,\mathcal{I},c}: \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \longrightarrow \mathbb{R}_{\geq 0}, \quad f \mapsto \| f(x) \|_{\mathcal{I},c},
\]
which is a *multiplicative seminorm* on the ring $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$, that is
\begin{itemize}
  \item $\|0\|_{x, I, c} = 0$; $\|- f\|_{x, I, c} = \|f\|_{x, I, c}$;
  \item $\|f + g\|_{x, I, c} \leq \max\{\|f\|_{x, I, c}, \|g\|_{x, I, c}\}$;
  \item $\|f \cdot g\|_{x, I, c} = \|f\|_{x, I, c} \cdot \|g\|_{x, I, c}$;
  \item $\|1\|_{x, I, c} = 1$.
\end{itemize}

**Proposition 3.3.8.** Let $x, x' \in (\mathcal{X})$, and suppose that $x'$ is a generization of $x$. Then we have the equality of the seminorms on $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$:
\[
\|\cdot\|_{x, I, c} = \|\cdot\|_{x', I, c}.
\]

**Proof.** In view of 3.2.16, we have the generization map $K_x \hookrightarrow K_{x'}$ that maps $V_x$ into $V_{x'}$. Consider the associated height-one prime $\mathfrak{p}' = \sqrt{aV_{x'}}$ of $V_{x'}$; in view of 0.6.7.6, the prime $\mathfrak{p} = \mathfrak{p'} \cap V_x$ is equal to $\sqrt{aV_x}$ and hence is the associated height-one prime of $V_x$. In particular, we have the local injective morphism $V_{x, \mathfrak{p}} \hookrightarrow V_{x', \mathfrak{p}'}$ and hence the injective mapping $K^\times_x/V_{x, \mathfrak{p}} \hookrightarrow K^\times_{x'/V_{x', \mathfrak{p}'}}$ between the value groups. Since this is an ordered mapping between the totally ordered sets isomorphic to $\mathbb{R}$ mapping the class of $a$ to the class of $a$, it is an ordered isomorphism. The claimed equality follows from this. \[\square\]

3.3 (c) Spectral seminorms. Let $X$ and $\mathcal{I}$ as in §3.3 (b). For $f \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ we set
\[
\|f\|_{\text{sp}, I, c} = \sup_{x \in (\mathcal{X})} \|f(x)\|_{I, c},
\]
which defines the map
\[
\|\cdot\|_{\text{sp}, I, c}: \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) \rightarrow \mathbb{R}_{\geq 0}.
\]
It is easy to see that $\|\cdot\|_{\text{sp}, I, c}$ gives a seminorm on the ring $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$, that is
\begin{itemize}
  \item $\|0\|_{\text{sp}, I, c} = 0$; $\|- f\|_{\text{sp}, I, c} = \|f\|_{\text{sp}, I, c}$;
  \item $\|f + g\|_{\text{sp}, I, c} \leq \max\{\|f\|_{\text{sp}, I, c}, \|g\|_{\text{sp}, I, c}\}$;
  \item $\|f \cdot g\|_{\text{sp}, I, c} \leq \|f\|_{\text{sp}, I, c} \cdot \|g\|_{\text{sp}, I, c}$;
  \item $\|1\|_{\text{sp}, I, c} = 1$.
\end{itemize}
We called this the *spectral seminorm* on $\mathcal{X}$. Clearly, changing $\mathcal{I}$ and $c$ only leads to an equivalent seminorm.
3.4 Comparison of topologies

**Proposition 3.4.1.** Let \( \{ j_\alpha : \mathcal{U}_\alpha \hookrightarrow \mathcal{F} \}_{\alpha \in \Lambda} \) be a collection of open immersions of rigid spaces. The following conditions are equivalent.

(a) \( \{ j_\alpha : \mathcal{U}_\alpha \hookrightarrow \mathcal{F} \}_{\alpha \in \Lambda} \) is a covering family in the site \( \mathcal{F}_{\text{ad}} \) or, equivalently, in \( \mathbf{Rf}_{\text{ad}} \) (that is, \( \bigcup_{\alpha \in \Lambda} \mathcal{U}_\alpha \hookrightarrow \mathcal{F} \) is an epimorphism of sheaves on \( \mathbf{CRf}_{\text{ad}} \)).

(b) \( \{ (j_\alpha) : (\mathcal{U}_\alpha) \hookrightarrow (\mathcal{F}) \}_{\alpha \in \Lambda} \) is a covering of the topological space \( (\mathcal{F}) \) (that is, \( (\mathcal{F}) = \bigcup_{\alpha \in \Lambda} (i_\alpha)((\mathcal{U}_\alpha)) \)).

For the proof we need the following lemma.

**Lemma 3.4.2.** Let \( j : \mathcal{U} \hookrightarrow \mathcal{X} \) be a coherent open immersion of coherent rigid spaces. Then the following conditions are equivalent.

(a) \( j \) is an isomorphism.

(b) The singleton set \( \{ j : \mathcal{U} \hookrightarrow \mathcal{X} \} \) is a covering in the coherent small site \( \mathcal{X}_{\text{ad}} \).

(c) The singleton set \( \{ j : \mathcal{U} \hookrightarrow \mathcal{X} \} \) is a covering in the large admissible site \( \mathbf{Rf}_{\text{ad}} \).

(d) \( (j) \) is an isomorphism.

**Proof.** The equivalence of (a) and (b) is obvious. The implication (a) \( \implies \) (c) is clear. Suppose (c) holds and \( (j) \) is not an isomorphism. Since \( (j) : (\mathcal{U}) \hookrightarrow (\mathcal{X}) \) is an open immersion of locally ringed spaces, this means that \( (j) \) is not surjective, and hence there exists \( x \in (\mathcal{X}) \) not lying in the image of \( (j) \). Consider the associated rigid point \( \alpha_x : (\text{Spf } \widehat{\mathcal{V}}_x)^{\text{rig}} \to \mathcal{X} \). By definition of the coverings in the site \( \mathbf{Rf}_{\text{ad}} \) (2.2.25), this should come from a map \( (\text{Spf } \widehat{\mathcal{V}}_x)^{\text{rig}} \to \mathcal{U} \), which contradicts what we have supposed, since this implies that \( x \) lies in the image of \( (j) \). Finally, we show (d) \( \implies \) (a). Suppose \( (j) \) is an isomorphism. In view of 2.2.1 there exists a distinguished formal model \( U \hookrightarrow X \) of \( j \). It suffices to show that \( U = X \). Suppose \( X \setminus U \) is non-empty, and take a point \( x \in X \setminus U \). By 3.1.5 there exists a rigid point \( \alpha : \text{Spf } \mathcal{V} \to X \) such that \( \alpha(\mathcal{V}) = x \). In view of 3.3.6 (2) there exists a unique lift \( \tilde{\alpha} : \text{Spf } \mathcal{V} \to (\mathcal{X}) \) of \( \alpha \). By 3.1.3 the point \( \alpha(\mathcal{V}) \) lies outside \( (\mathcal{U}) \) in \( (\mathcal{X}) \), which is absurd. \( \square \)

**Proof of Proposition 3.4.1.** That (a) \( \implies \) (b) follows from 3.1.9. Let us prove that (b) \( \implies \) (a). If \( \{ j_\alpha : \mathcal{U}_\alpha \hookrightarrow \mathcal{F} \}_{\alpha \in \Lambda} \) is not a covering, then there exists a coherent open rigid subspace \( \mathcal{V} \) of \( \mathcal{F} \) such that \( \{ j_\alpha : \mathcal{U}_\alpha \times_\mathcal{F} \mathcal{V} \hookrightarrow \mathcal{V} \}_{\alpha \in \Lambda} \) is not a covering. Hence, to show (a), we can assume that \( \mathcal{F} \) is coherent. In this case, since \( (\mathcal{F}) \) is quasi-compact, it is covered by only finitely many of \( (\mathcal{U}_\alpha) \). Hence we can assume that \( \Lambda \) is a finite set. Moreover, since \( (\mathcal{F}) \) has an open basis consisting of quasi-compact open subsets, we can assume that each \( \mathcal{U}_\alpha \) is a coherent rigid space.
Let $\mathcal{U}$ be the rigid space defined by the quotient of sheaves

$$\coprod_{\alpha, \beta \in L} \mathcal{U}_\alpha \times_{\mathcal{F}} \mathcal{U}_\beta \xrightarrow{\sim} \coprod_{\alpha \in L} \mathcal{U}_\alpha \to \mathcal{U}.$$ 

Thanks to 2.2.13 and the exactness of $\text{CRf} \to \text{RF}_{\text{ad}}$, $\mathcal{U}$ is represented by a coherent rigid space, and the canonical morphism $j: \mathcal{U} \to \mathcal{F}$ is a coherent open immersion (2.2.15). Then the assertion follows from 3.1.4 and 3.4.2. \hfill $\Box$

**Corollary 3.4.3.** Let $\mathcal{X}$ be a coherent rigid space, and $\{\mathcal{U}_\alpha \to \mathcal{X}\}_{\alpha \in L}$ a covering family in the small admissible site $\mathcal{X}_{\text{ad}}$ (2.2.24), where $\mathcal{X}$ is regarded as a general rigid space. Then the covering family $\{\mathcal{U}_\alpha \to \mathcal{X}\}_{\alpha \in L}$ can be refined by a covering family in the coherent small admissible site (2.2.5). In particular, there exists a finite subset $L' \subseteq L$ such that $\{\mathcal{U}_\alpha \to \mathcal{X}\}_{\alpha \in L'}$ gives a covering in the site $\mathcal{X}_{\text{ad}}$.

**Proof.** Considering a covering of each $\mathcal{U}_\alpha$ by coherent rigid spaces, we may assume that each $\mathcal{U}_\alpha$ is coherent. Then by 3.4.1 and by the fact that $\{\mathcal{X}\}$ is quasi-compact due to 3.1.2 (1), we have a finite subset $L' \subseteq L$ such that $\{\mathcal{U}_\alpha \to \mathcal{X}\}_{\alpha \in L'}$ already gives a covering. By 3.4.2, this gives a covering in the coherent small admissible site, as desired. \hfill $\Box$

Let $\mathcal{F}$ be a rigid space represented by a coherent rigid space $\mathcal{X}$. Let $\mathcal{F}_{\text{ad}}$ denote the small admissible site as in 2.2.24, and $\mathcal{X}_{\text{ad}}$ the coherent small admissible site as in 2.2.5. There exists a canonical comparison morphism of sites

$$\mathcal{X}_{\text{ad}} \longrightarrow \mathcal{F}_{\text{ad}}$$

obtained as follows. Consider the obvious functor of categories $\mathcal{X}_{\text{ad}} \to \mathcal{F}_{\text{ad}}$ that maps $\mathcal{U} \to \mathcal{X}$ to the associated morphism of the representable functors. To see that the functor in question is a morphism of sites, it suffices to show that it maps coverings to coverings, since it is obvious that coherent open rigid subspaces of $\mathcal{X}$ and coherent open immersions generate the site $\mathcal{F}_{\text{ad}}$. But this follows from 3.4.1, since a covering in the coherent site $\mathcal{X}_{\text{ad}}$ induces an open covering by passage to the associated Zariski–Riemann spaces (3.1.4). Since by 3.4.3 the topology on $\mathcal{F}_{\text{ad}}$ is generated by the topology on $\mathcal{X}_{\text{ad}}$, we have the following theorem.

**Theorem 3.4.4.** The morphism of sites $(\ast)$ gives rise to an equivalence of topoi

$$\mathcal{X}_{\text{ad}} \sim \mathcal{F}_{\text{ad}}.$$ 

Let $\mathcal{F}$ be a rigid space. Then for any open immersion $\mathcal{U} \hookrightarrow \mathcal{F}$ the induced map $\langle \mathcal{U} \rangle \hookrightarrow \langle \mathcal{F} \rangle$ is an open immersion of topological spaces (3.1.9). Hence one has a natural functor

$$\mathcal{F}_{\text{ad}} \longrightarrow \text{Ouv}(\langle \mathcal{F} \rangle),$$

(\ast\ast)
where $\text{Ouv}(\langle F \rangle)$ is the category of open subsets of $\langle F \rangle$. We claim that there exists a canonical morphism of sites

$$\text{Ouv}(\langle F \rangle) \longrightarrow F_{\text{ad}}$$

underlied by the functor $(\ast\ast)$. By 3.1.9, the above functor maps all covering families to open coverings of $\langle F \rangle$. By 3.1.8, the topological space $\langle F \rangle$ has a generator consisting of quasi-compact open subsets coming from coherent open rigid subspaces of $F$. Hence we deduce that $(\ast\ast)$ induces a morphism of sites as above.

**Theorem 3.4.5.** Let $F$ be a rigid space and $\langle F \rangle$ the associated Zariski–Riemann space. Then the morphism $(\dagger)$ of sites induces an equivalence of topoi

$$\text{top}(\langle F \rangle) \sim F_{\text{ad}}$$

where $\text{top}(\cdot)$ is the functor giving the associated topos.

**Proof.** We may assume that $F$ is represented by a coherent rigid space. Then the assertion follows from 3.1.3 and 3.4.1. \qed

**Theorem 3.4.6.** The canonical functor $\text{CRf} \hookrightarrow F$ gives rise to a morphism of sites $\text{CRf}_{\text{ad}} \rightarrow F_{\text{ad}}$. The site $F_{\text{ad}}$ is generated by the objects in the image of $\text{CRf} \hookrightarrow F$. In particular, the topos $F_{\text{ad}}$ (and, similarly, $F^*_{\text{ad}}$ for a rigid space $G$) is generated by quasi-compact objects.

**Proof.** It is clear that the functor $\text{CRf} \hookrightarrow F$ gives rise to a morphism of sites $\text{CRf}_{\text{ad}} \rightarrow F_{\text{ad}}$. By 3.4.3, the site $F_{\text{ad}}$ is generated by the objects in the image of $\text{CRf} \hookrightarrow F$. Finally, by the second assertion of 3.4.3, the image of $\text{CRf} \hookrightarrow F$ consists of quasi-compact objects. \qed

### 3.5 Finiteness conditions and consistency of terminologies

#### 3.5. (a) Finiteness conditions

**Definition 3.5.1.** (1) A rigid space $F$ is said to be *quasi-compact* if it is quasi-compact as an object of the topos $\text{CRf}_{\text{ad}}$ (cf. 0.2.7.3 (1)).

(2) A rigid space $F$ is said to be *quasi-separated* if it is quasi-separated as an object of the topos $\text{CRf}_{\text{ad}}$, that is, the diagonal map $F \rightarrow F \times F$ of sheaves is quasi-compact (cf. 0.2.7.3 (2)).

Note that if a rigid space $F$ is quasi-compact, then one can take the covering $Y = \bigsqcup_{\alpha \in L} Y_\alpha$ as in 2.2.18 (a) with the index set $L$ finite.
**Proposition 3.5.2.** A rigid space $F$ is quasi-compact and quasi-separated if and only if it is represented by a coherent rigid space.

This proposition allows us to say consistently that a rigid space $F$ is coherent if it is quasi-compact and quasi-separated.

**Proof.** Suppose $F$ is quasi-compact and quasi-separated. Take a surjective map $Y = \bigsqcup_{\alpha \in L} Y_{\alpha} \to F$ as in 2.2.18, where $L$ is finite. Since $F$ is quasi-separated, each $Y_{\alpha} \times_{F} Y_{\beta}$ is quasi-compact. Since $F$ is covered by finitely many stretches of coherent rigid spaces, it is covered by finite stretches of coherent rigid spaces (2.2.17 (1)) and hence is coherent. Then one can refine the covering

$$Y = \bigsqcup_{\alpha \in L} Y_{\alpha} \to F$$

so that each projection $Y_{\alpha} \times_{F} Y_{\beta} \to Y_{\alpha}$ is a coherent open immersion. Hence $F$ is isomorphic to the coherent rigid space $\bigsqcup Y_{\alpha} \times_{F} Y_{\beta}$. Note that since the canonical functor from a site to the associated topos is exact, it preserves finite cofiber products.

**Proposition 3.5.3.** A rigid space $F$ is quasi-separated if and only if it is a stretch of coherent rigid spaces (2.2.17 (1)).

**Proof.** The ‘if’ part is clear. Suppose $F$ is quasi-separated, and take

$$Y = \bigsqcup_{\alpha \in L} Y_{\alpha} \to F$$

as in 2.2.18. Let $\mathcal{L}$ be the set of all finite subsets of $L$ considered with the inclusion order. Set $Y_{L'} = \bigsqcup_{\alpha \in L'} Y_{\alpha}$ for any $L' \in \mathcal{L}$, and let $X_{L'}$ be the image of $Y_{L'}$. Then $X_{L'}$ is quasi-compact and quasi-separated and hence is represented by a coherent rigid space by 3.5.2. Since $F = \lim_{\longrightarrow} X_{L'}$, we are done.

---

**3.5. (b) Consistency of open immersions.** The following propositions shows that the terminologies ‘open immersion’ and ‘coherent open immersion’ are consistent.

**Proposition 3.5.4.** Let $U$ and $X$ be coherent rigid spaces and $\iota : U \hookrightarrow X$ an open immersion. Then $\iota$ is a coherent open immersion. Moreover, it is coherent (0.2.7.4 (3)) as an arrow in the topos $\mathcal{R}_{\text{ad}}$.

**Proof.** One can write $U$ as the union of an increasing sequence of coherent open rigid subspaces $\{U_i\}$ such that each $U_i \hookrightarrow X$ is a coherent open immersion. Since $U$ is quasi-compact as an object of $\mathcal{R}_{\text{ad}}$ (3.4.3), there is an $i$ such that $U = U_i$, and hence $\iota$ is a coherent open immersion. The last assertion follows from 2.2.3 and 3.5.2.
3.5. (c) **Rigid space as quotient.** The following proposition follows immediately from the definition of rigid spaces.

**Proposition 3.5.5.** Let $\mathcal{Y}$ be a rigid space, and $\mathcal{R}$ an equivalence relation in the sheaf $\mathcal{Y} \times \mathcal{Y}$ on the site $\text{CRF}_{ad}$ with the projection maps

$$
\mathcal{R} \xrightarrow{q_1} \mathcal{Y} \xrightarrow{q_2} \mathcal{Y}
$$

such that there exists a covering family $\{\mathcal{V}_\alpha \to \mathcal{R}\}$ such that $q_i: \mathcal{V}_\alpha \to \mathcal{Y}$ for each $i = 1, 2$ and any $\alpha$ is an open immersion. Then the quotient $\mathcal{F}$ of $\mathcal{Y}$ by $\mathcal{R}$ is a rigid space.

3.5. (d) **Consistency of finiteness conditions**

**Proposition 3.5.6.** Let $\mathcal{F}$ be a rigid space. Then the topos $\mathcal{F}_{ad}$ is algebraic (0.2.7.5). Moreover, the following conditions are equivalent.

(a) $\mathcal{F}$ is quasi-separated (resp. coherent).

(b) $\mathcal{F}_{ad}$ is quasi-separated (resp. coherent).

**Proof.** Let $C$ be the set of all objects in $\mathcal{F}_{ad}$ represented by coherent rigid spaces. Then $C$ generates the topos $\mathcal{F}_{ad}$ (by 3.4.3). Clearly, $C$ is stable under fiber products and equalizers. Hence $\mathcal{F}_{ad}$ is algebraic. Suppose $\mathcal{F}$ is quasi-separated. Then $C$ is further stable under products (over $\mathcal{F}$). Hence $\mathcal{F}_{ad}$ is quasi-separated. If $\mathcal{F}$ is coherent, then $C$ is stable under all finite colimits, and hence $\mathcal{F}_{ad}$ is coherent. Thus, (a) $\implies$ (b). If, conversely, the topos $\mathcal{F}_{ad}$ is quasi-separated, then $\mathcal{F}$ is a colimit of coherent rigid spaces for which all transition maps are open immersions. Hence by a similar reasoning as in 3.5.3 one sees that $\mathcal{F}$ is a stretch of coherent rigid spaces and hence is quasi-separated. If $\mathcal{F}_{ad}$ is coherent, then such a colimit is taken to be finite, and hence $\mathcal{F}$ is coherent. \qed

**Proposition 3.5.7.** Let $\mathcal{F}$ be a rigid space and let $\langle \mathcal{F} \rangle$ be the associated Zariski–Riemann space. Then the following conditions are equivalent.

(a) $\mathcal{F}$ is quasi-compact (resp. quasi-separated, resp. coherent).

(b) $\langle \mathcal{F} \rangle$ is quasi-compact (resp. quasi-separated, resp. coherent) (cf. 0.2.1.4, 0.2.1.8, and 0.2.2.1).

**Proof.** Implication (a) $\implies$ (b) is clear by the construction of $\langle \mathcal{F} \rangle$ and 3.1.2 (1). If $\langle \mathcal{F} \rangle$ is quasi-compact, then $\mathcal{F}$ is quasi-compact by 3.4.1. If $\langle \mathcal{F} \rangle$ is quasi-separated (resp. coherent), then the topos $\langle \mathcal{F} \rangle_{\sim}$ is quasi-separated (resp. coherent), and hence so is $\mathcal{F}_{ad}$ by 3.4.5. But then by 3.5.6 we deduce that $\mathcal{F}$ is quasi-separated (resp. coherent). \qed
3.5. (e) Rigid spaces associated to adic formal schemes. Let $X$ be an adic formal scheme of finite ideal type, which is not necessarily coherent. We are going to construct the associated rigid space $X^{\text{rig}}$.

(1) Suppose $X$ is coherent. In this case, we just take $X^{\text{rig}}$ as in §2.1. (a); the resulting rigid space $X^{\text{rig}}$ is, therefore, coherent.

(2) Suppose $X$ is quasi-separated. In this case, since $X$ is locally coherent, there exists an increasing family $\{U_i\}_{i \in I}$ (where $I$ is a directed set) of coherent open subsets such that $X = \bigcup U_i$. Then consider $U_i^{\text{rig}}$ for each $i$, and note that the induced morphism $U_i^{\text{rig}} \to U_j^{\text{rig}}$ is a coherent open immersion for any $i \leq j$. Thus we get the desired rigid space $X^{\text{rig}}$ by taking the union of $U_i^{\text{rig}}$; note that the rigid space $X$ in this case is a stretch of coherent rigid spaces and hence is quasi-separated (3.5.3).

(3) In general, let $X = \bigcup_{\alpha \in L} U_\alpha$ be an open covering by coherent open subsets. For each $\alpha \in L$, consider $U_\alpha^{\text{rig}}$ as in (1). Since each intersection $U_\alpha \cap U_\beta$ is quasi-separated, one can consider the rigid space $(U_\alpha \cap U_\beta)^{\text{rig}}$ as in (2). Since $(U_\alpha \cap U_\beta)^{\text{rig}} \to U_\alpha^{\text{rig}}$ is obviously an open immersion, one can define $X^{\text{rig}}$ by patching all $U_\alpha^{\text{rig}}$ along $(U_\alpha \cap U_\beta)^{\text{rig}}$.

Exercises

Exercise II.3.1 (Deligne’s formula). Let $(X, Z)$ be a universally pseudo-adhesive pair of schemes such that $X$ is coherent. Denote the defining ideal of $Z$ by $I_Z$. Set $U = X \setminus Z$, and let $j: U \to X$ be the canonical open immersion. Then for any finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$, the canonical map

$$\lim_{\longrightarrow n \geq 0} \text{Hom}_{\mathcal{O}_X}(I_Z^n, \mathcal{F}) \longrightarrow \text{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

is bijective. In particular, we have $\lim_{\longrightarrow n \geq 0} \text{Hom}(I_Z^n, \mathcal{G}) \sim \Gamma(U, \mathcal{G})$.

Exercise II.3.2. Let $X$ be a quasi-compact rigid space and $I, I' \subseteq \mathcal{O}_X^{\text{int}}$ ideals of definition of finite type. Show that there exist integers $n, m > 0$ such that $I^m \subseteq I' \subseteq I$.

Exercise II.3.3. Let $T = \text{Spf} V$, where $V$ is an $a$-adically complete valuation ring with $a \in m_V \setminus \{0\}$, and consider the associated coherent rigid space $T = T^{\text{rig}}$. Show that the topological space $(T)$ is homeomorphic to the underlying topological space of $T$ and, moreover, that

(a) the integral structure sheaf $\mathcal{O}_T^{\text{int}}$ is isomorphic to $\mathcal{O}_T$, and

(b) the rigid structure sheaf $\mathcal{O}_T$ is isomorphic to $\mathcal{O}_T[\frac{1}{a}]$. 
Exercise II.3.4. Show that the following conditions for a coherent rigid space $\mathcal{X}$ are equivalent.

(a) $\mathcal{X}$ has a Noetherian formal model.

(b) There exists a cofinal family $\{X_\lambda\}_{\lambda \in \Lambda}$ of formal models of $\mathcal{X}$ consisting of Noetherian formal schemes.

(c) Any quasi-compact open subspace $U$ of $\mathcal{X}$ has a Noetherian formal model.

(d) There exists an open covering $\{U_\alpha\}_{\alpha \in L}$ of the coherent small admissible site $\mathcal{X}_{ad}$ such that each $U_\alpha$ has a Noetherian formal model.

4 Topological properties

In this section we study several topological aspects of rigid spaces. Most of the topological properties, which capture several geometric structures of rigid spaces, are defined and discussed in terms of the associated Zariski–Riemann spaces and are often easily grasped simply by point-set topology. Many of the statements in this section will be, therefore, recasts of general topology statements, already discussed in §0, §2.

The main aim of §4.1 is to show that for any rigid space $\mathcal{X}$ the associated Zariski–Riemann space $\langle \mathcal{X} \rangle$, which we have already shown to be locally coherent and sober (3.1.8), is a valuative space (0.2.3.1). The proof of this fundamental result requires a careful study of generizations and specializations of points of adic formal schemes of finite ideal type, especially their behavior under passage to the admissible blow-ups. It turns out that, for a given point of the associated Zariski–Riemann space, the corresponding rigid point carries all generizations of the point.

In §4.3 we study the separated quotient of the associated Zariski–Riemann spaces (cf. §0, §2.3. (c)). It will turn out that the separated quotient is, as a set, the subset of the Zariski–Riemann space consisting of the points of height 1. A good thing about separated quotients is that they have nice topological features; for example, the separated quotient of the Zariski–Riemann space associated to a coherent rigid space is a compact\(^1\) space.

4.1 Generization and specialization

Proposition 4.1.1. Let $\mathcal{X} = X^{rig}$ be a coherent rigid space, and $x, y \in \langle \mathcal{X} \rangle$. Then $y$ is a generization (§0, §2.1. (a)) of $x$ if and only if for any admissible blow-up $X' \to X$, $\text{sp}_{X'}(y)$ is a generization of $\text{sp}_{X'}(x)$ in $X'$.

To show this, we need the following preparatory lemma.

\(^1\)Recall that, as mentioned in the introduction, all compact topological spaces are assumed to be Hausdorff.
Lemma 4.1.2. Let \( X \) be a coherent adic formal scheme of finite ideal type, and \( \mathcal{X} = X^{\text{rig}} \). Let \( F \subseteq \langle \mathcal{X} \rangle \) be a subset.

1. For any \( X' \in \text{obj}(\text{BL}_X) \) we have

\[ \text{sp}_{X'}(\overline{F}) = \text{sp}_{X'}(F), \]

where \( \overline{F} \) is the closure of \( F \) in \( \langle \mathcal{X} \rangle \), and \( \text{sp}_{X'}(F) \) is the closure of \( \text{sp}_{X'}(F) \) in \( X' \).

2. We have

\[ \overline{F} = \bigcap_{X' \in \text{obj}(\text{BL}_X)} \text{sp}_{X'}^{-1}(\text{sp}_{X'}(F)) = \lim_{X' \in \text{obj}(\text{BL}_X)} \text{sp}_{X'}(F). \]

Proof. Since each \( \text{sp}_{X'} \) is a closed map (3.1.2 (2)), (1) holds. Then (2) follows from 0.2.2.19 (2).

Proof of Proposition 4.1.1. Suppose \( y \) is a generalization of \( x \), and set \( C = \{y\} \). By 4.1.2, \( \text{sp}_{X'}(C) = \{\text{sp}_{X'}(y)\} \). Hence \( \text{sp}_{X'}(x) \) is a specialization of \( \text{sp}_{X'}(y) \). Conversely, suppose \( \text{sp}_{X'}(y) \) is a generalization of \( \text{sp}_{X'}(x) \) for any \( X' \), and set \( C = \{y\} \). By 4.1.2 (2),

\[ C = \lim_{X' \to X} \{\text{sp}_{X'}(y)\}, \]

and so \( x \in C \), as desired.

Proposition 4.1.3. Let \( \mathcal{X} \) be a coherent rigid space and \( x \in \langle \mathcal{X} \rangle \). Let

\[ \alpha_x : \text{Spf} \widehat{V}_x \to \langle \mathcal{X} \rangle \]

be the associated rigid point of \( x \) (3.3.5). Then \( \alpha_x \) maps the set \( \text{Spf} \widehat{V}_x \) bijectively onto the set of all generalizations of \( x \).

Proof. Let us first show that any point in the image of \( \alpha_x \) is a generalization of \( x \).
Let \( y = \alpha_x(p) \) for \( p \in \text{Spf} \widehat{V}_x \), and \( X \) a formal model of \( \mathcal{X} \). By 4.1.1, it suffices to show that for any admissible blow-up \( X' \to X \) the point \( \text{sp}_{X'}(y) \) is a generalization of \( \text{sp}_{X'}(x) \). But this is clear, since \( p \) is a generalization (in \( \text{Spf} \widehat{V}_x \)) of \( m_{\widehat{V}_x} \).

Conversely, suppose \( y \) is a generalization of \( x \), and let us show that there exists a unique \( p \in \text{Spf} \widehat{V}_x \) such that \( y = \alpha_x(p) \). For any admissible blow-up \( X' \to X \), find the prime ideal \( p_{X'} \) of \( \mathcal{O}_{X', \text{sp}_{X'}(x)} \) that corresponds to the generalization \( \text{sp}_{X'}(y) \). The system \( \{p_{X'}\} \) defines an ideal (easily seen to be prime) of the inductive limit \( A_x = \mathcal{O}_{(\mathcal{X})} = \lim_{\longrightarrow \mathcal{O}_{X', \text{sp}_{X'}(x)}} \). Denote this by \( p \). Since each \( p_{X'} \) is open, \( p \) is an open ideal with respect to the \( I_x \)-adic topology. Moreover, it is straightforward...
to see that $p$ contains $J_x$ (in the notation as in 3.2.13); indeed, as one can easily verify, the ideal $p_X A_x$ contains $J_x$ for any $X' \to X$. Hence $p$ determines a prime ideal of $V_x$, which we again denote by $p$. Then by 0.9.1.1 (5) we find that $\alpha_x$ maps $\text{Spf } \widehat{V}_x$ bijectively onto the set of all generizations of $x$. □

4.1.3, 3.3.4 (2), and 0.6.7.6 immediately yield the following corollary.

**Corollary 4.1.4.** Let $X$ be a coherent rigid space and $\alpha: \text{Spf } V \to \langle X \rangle$ a rigid point. Then $\alpha$ maps the set $\text{Spf } V$ surjectively onto the set of all generizations of $\alpha(m_V)$.

As in 0, §2.1 (a), for any point $x \in \langle X \rangle$ we denote the set of all generizations of $x$ in $\langle X \rangle$ by $G_x$, and consider the order on $G_x$ defined as follows: $y \leq z$ if $z$ is a generization of $y$.

**Corollary 4.1.5.** The set $G_x$ equipped with the above order is a totally ordered set.

**Proof.** This follows immediately from the fact that $G_x$ is the bijective image under an order preserving map of $\text{Spf } \widehat{V}_x$ with the order by inclusion, and that $\text{Spf } \widehat{V}_x$ is totally ordered (since $\widehat{V}_x$ is a valuation ring; cf. 0.6.2.1 (c)). □

**Definition 4.1.6.** Let $X$ be a rigid space and $x \in \langle X \rangle$ a point of the associated Zariski–Riemann space.

(1) The **height** of the point $x$ is the height of the associated valuation ring $V_x$ (defined as in 3.2.13). Note that this definition coincides with the one already given in 0, §2.3.

(2) A generization $y$ of the point $x$ is called the **maximal generization** if it is the maximal element in the ordered set $G_x$ as in 4.1.5.

The maximal generization of $x$ is, if it exists, uniquely determined, since $G_x$ is totally ordered (4.1.5).

**Proposition 4.1.7.** Let $X$ be a coherent rigid space and $x \in \langle X \rangle$ a point of the associated Zariski–Riemann space. Then the maximal generization of $x$ exists. Moreover, the following conditions for a point $y \in \langle X \rangle$ are equivalent.

(a) $y$ is the maximal generization of $x$.

(b) $y$ is a generization of $x$ and is of height one.

(c) There exists a rigid point $\alpha: \text{Spf } V \to \langle X \rangle$ such that $\alpha(m_V) = x$ and $\alpha(p_V) = y$, where $p_V = \sqrt{(a)}$ is the associated height-one prime (0.6.7.4).

(d) For any rigid point $\alpha: \text{Spf } V \to \langle X \rangle$ such that $\alpha(m_V) = x$, we have $\alpha(p_V) = y$, where $p_V = \sqrt{(a)}$. 
Proof. Take the associated rigid point $\alpha_x : \text{Spf} \tilde{V}_x \to (\mathcal{X})$. By 4.1.3 and 0.6.7.5, the image $\tilde{x}$ of the associated height one prime $p_{\tilde{x}} = \sqrt{(a)}$ is the maximal generalization of $x$, whence the existence. The equivalence of the above conditions follows immediately from 3.3.4 and 0.6.7.6. \qed

Corollary 4.1.8. Let $\mathcal{X}$ be a rigid space and $(\mathcal{X})$ the associated Zariski–Riemann space. Then the topological space $(\mathcal{X})$ is valuative (0.2.3.1). Moreover, for any morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ of rigid spaces the induced map $(\varphi) : (\mathcal{X}) \to (\mathcal{Y})$ between the associated Zariski–Riemann spaces is valuative (0.2.3.21).

Proof. By 3.1.8 the topological space $(\mathcal{X})$ is locally coherent (0.2.2.21) and sober (0. §2.1. (b)) and is valuative due to 4.1.5 and 4.1.7. The last assertion follows from 4.1.7. \qed

4.2 Tubes

4.2 (a) Tubes. The following statements are recasts of already proven ones; see 0.2.2.27, 0.2.3.5, and 0.2.3.7.

Theorem 4.2.1. Let $\mathcal{X}$ be a rigid space and $\mathcal{U} \hookrightarrow (\mathcal{X})$ a retrocompact (0.2.1.7) open subset of $(\mathcal{X})$. Then

$$\overline{\mathcal{U}} = \bigcup_{x \in \mathcal{U}} \{x\}.$$  

In other words, the closure $\overline{\mathcal{U}}$ is the set of all specializations of points of $\mathcal{U}$.

Corollary 4.2.2. Let $\mathcal{X}$ be a rigid space and $\mathcal{U} \hookrightarrow (\mathcal{X})$ a retrocompact open subset of $(\mathcal{X})$. For $x \in (\mathcal{X})$ to belong to $\overline{\mathcal{U}}$ it is necessary and sufficient that the maximal generalization $\tilde{x}$ of $x$ belongs to $\mathcal{U}$.

Proposition 4.2.3. Let $\mathcal{X}$ be a rigid space, and $\{\mathcal{U}_\alpha\}_{\alpha \in L}$ a family of retrocompact open sets of $(\mathcal{X})$. Then

$$\overline{\bigcap_{\alpha \in L} \mathcal{U}_\alpha} = \bigcap_{\alpha \in L} \overline{\mathcal{U}_\alpha}.$$  

Definition 4.2.4 (cf. 0.2.3.4). Let $\mathcal{X}$ be a rigid space. A tube closed subset of $\mathcal{X}$ is a closed subset of $(\mathcal{X})$ of the form $\overline{\mathcal{U}}$ for a retrocompact open subset $\mathcal{U} \subseteq (\mathcal{X})$. The complement of a tube closed subset is called a tube open subset. Tube closed and tube open subsets are collectively called tube subsets.

Let $X$ be a coherent adic formal scheme of finite ideal type, and $Y$ a closed subscheme of $X$ of finite presentation defined by an admissible ideal (cf. I.3.7.7). We set

$$C_{Y|X} = \text{Sp}_X^{-1}(Y)^{\circ}.$$
where \((\cdot)^\circ\) denotes the topological interior kernel. The subset \(C_{Y|X} \subseteq \{X^\text{rig}\}\) is called the tube of \(Y\) in \(X\). Note that \(C_{Y|X}\) depends only on the topological structure of \(Y\). There is a canonical projection map \(C_{Y|X} \to Y\) induced from the specialization map \(\text{sp}_X\), which is clearly continuous.

**Proposition 4.2.5.** Let \(X\) be a coherent adic formal scheme of finite ideal type, and \(Y\) a closed subscheme of \(X\) of finite presentation defined by an admissible ideal. Then the subset \(C_{Y|X}\) is a tube open subset of \(X^\text{rig}\). Conversely, any tube open subset of the coherent rigid space \(X\) is of this form for some \(X\) and \(Y\).

**Proof.** Let \(U = X \setminus Y\), which is a coherent open subset of \(X\), and \(\mathcal{U} = \text{sp}^{-1}_X(U)\). Then \(C_{Y|X}\) is the complement of \(\mathcal{U}\), whence the first assertion. Conversely, let \(\mathcal{U}\) be a quasi-compact open subset of \((X)\). We want to show that its complement \(C = (X) \setminus \mathcal{U}\) is of the form \(C_{Y|X}\). Take a formal model \(X\) and a quasi-compact open subset \(U \subseteq X\) such that \(\text{sp}^{-1}_X(U) = \mathcal{U}\). Since \(U\) is quasi-compact, there exists a closed subscheme \(Y\) of \(X_0\) of finite presentation with the underlying topological space \(X \setminus U\) (Exercise 0.5.1). Then we have \(C = C_{Y|X}\). \(\square\)

The following statements are transcriptions of 0.2.3.6 and 0.2.3.8.

**Proposition 4.2.6.** Let \(X\) be rigid space, and \(C = (X) \setminus \mathcal{U}\) (where \(\mathcal{U} \subseteq (X)\) is retrocompact open) a tube open subset. Then a point \(x \in (X)\) belongs to \(C\) if and only if the maximal generization \(\tilde{x}\) does not belong to \(\mathcal{U}\). In particular, every height-one point of \((X) \setminus \mathcal{U}\) lies in \(C\).

**Proposition 4.2.7.** (1) Any finite union of tube closed (resp. tube open) subsets is a tube closed (resp. tube open) subset.

(2) Any finite intersection of tube closed (resp. tube open) subsets is a tube closed (resp. tube open) subset.

4.2. (b) Explicit description

**Proposition 4.2.8** (closure formula). Let \(X\) be a coherent adic formal scheme of finite ideal type, \(Y \subseteq X\) a closed subscheme of \(X\) defined by an admissible ideal \(\mathcal{J}_Y\), and \(U = X \setminus Y\) the open complement. Let \(X = X^\text{rig}\) be the associated coherent rigid space, and set \(\mathcal{U} = \text{sp}^{-1}_X(U)\), where \(\text{sp}_X : X \to X\) is the specialization map. Let \(I\) be an ideal of definition of \(X\), and \(\pi_n : X_n \to X\) the admissible blow-up along \(\mathcal{J}_n = I + \mathcal{J}_Y^n\). Then

\[
\mathcal{U} = \bigcap_{n \geq 1} \text{sp}^{-1}_X(V_n),
\]

where \(V_n\) is the maximal open subset of \(X_n\) such that \(\mathcal{J}_n \otimes_{X_n} V_n = \mathcal{J}_Y^n \otimes_{X_n} V_n\) for \(n \geq 1\).
Proof. Let $D$ be the right-hand side of the formula. We first show that for $x \in \langle X \rangle$ to belong to $D$ it is necessary and sufficient that the maximal generization $\tilde{x}$ of $x$ belongs to $\mathfrak{U}$. Let $x \in \langle X \rangle$ and take $V = V_x$ (as in 3.2.13). Let $I$ and $J$ be the pull-back ideals of $V$ of $I$ and $J_Y$, respectively, which are principal. By the definition, a point $x$ lies in $D$ if and only if $I$ is divisible by $J^n$ for any $n \geq 1$. But the last condition is equivalent to $JV' = V'$, where $V'$ is the height-one valuation ring associated to $V$, and hence is also equivalent to $\tilde{x} \in \mathfrak{U}$. By 4.2.2, $x \in D$ if and only if $\tilde{x} \in \mathfrak{U}$.

Proposition 4.2.9 (open interior formula). Let $X$ be a coherent adic formal scheme of finite ideal type, and $Y \subseteq X$ a closed subscheme of $X$ defined by an admissible ideal $J_Y$, and consider the associated coherent rigid space $X = X^\text{rig}$. Let $I$ be an ideal of definition of $X$. For $n \geq 1$ we set $J_n = I + J_Y^n$, and let $\pi_n : X_n \to X$ be the admissible blow-up along $J_n$. Then

$$C_Y = \text{sp}^-Y = \lim_{\longrightarrow n \geq 1} \text{sp}^-X_n(U_n),$$

(*)

where $U_n$ for each $n \geq 1$ is the open subset of $X_n$ that is maximal among the open subsets $U$ of $X_n$ such that $J_n \mathfrak{O}_{X_n}|_U = I \mathfrak{O}_{X_n}|_U$.

Note that the open subsets $U_n$ are quasi-compact. To prove the proposition, we need the following lemma.

Lemma 4.2.10. Let $X$ and $Y$ be as in 4.2.9, and denote the right-hand side of (*) by $C'$.

1. We have $C' \subseteq \text{sp}^-X(Y)$.

2. For $x \in \langle X \rangle$ (where $X = X^\text{rig}$) to belong to $C'$ it is necessary and sufficient that the maximal generization $\tilde{x}$ of $x$ belongs to $\text{sp}^-X(Y)$. In particular, every height-one point of $\text{sp}^-X(Y)$ belongs to $C'$.

Proof. (1) Let $x \in C'$, and take $V = V_x$ (as in 3.2.13). Let $I$ and $J$ be the pull-back ideals of $V$ of $I$ and $J_Y$, respectively, which are principal. By the definition of $C'$ there exists $n \geq 1$ such that $J^n$ is divisible by $I$. Hence $\text{sp}^-X$ maps $x$ to a point in $Y$.

(2) If $x \in C'$, then there exists $n \geq 1$ such that $J^n$ is divisible by $I$, as before. This implies, a fortiori, that any point of $\text{Spf} \hat{V}$ is mapped to a point of $Y$ by the composition $\text{sp}^-X \circ \alpha_x$; hence so is, in particular, the height-one point of $\text{Spf} \hat{V}$. Conversely, if $\tilde{x}$ is mapped by $\text{sp}^-X$ to a point in $Y$, then $J^n$ is divisible by $I$ for some $n$; but this means that $x \in C'$.
Proof of Proposition 4.2.9. By 4.2.10 (1), it suffices to show the inclusion $C_Y|X \subseteq C'$. Let $x \in C_Y|X$. Suppose that the maximal generization $\tilde{x}$ belongs to $\mathcal{U} = \text{sp}_X^{-1}(X \setminus Y)$. Then $x \in \tilde{\mathcal{U}}$. On the other hand, since $\mathcal{U}$ and $\text{sp}_X^{-1}(Y)$ are disjoint, we deduce that $\tilde{\mathcal{U}}$ and $C_Y|X$ are disjoint, which contradicts $x \in \tilde{\mathcal{U}}$. Hence $\tilde{x}$ does not belong to $\mathcal{U}$, and thus $\tilde{x} \in \text{sp}_X^{-1}(Y)$. Since $\tilde{x}$ is of height one, we deduce by 4.2.10 (2) that $x \in C'$, as desired. \qed

Proposition 4.2.11 (norm description of tube subsets). Let $X = \text{Spf} A$, where $A$ is an adic ring of finite ideal type with a finitely generated ideal of definition $I \subseteq A$, and set $\mathcal{X} = X^{\text{rig}}$. Let $Y \subseteq X$ be a closed subscheme defined by an admissible ideal $J = (f_1, \ldots, f_n) \subseteq A$. Then we have

$$C_Y|X = \{x \in [\mathcal{X}]: \|f_i(x)\|_{I,c} < 1 \text{ for } i = 1, \ldots, n\},$$

where $I = I \otimes^{\text{int}}_A \mathcal{X}$. Here we consider $f_i$'s as elements of $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ by the map $A \to \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$.

Proof. In view of 4.2.6 and 3.3.8, it suffices to show that

$$\text{sp}_X^{-1}(Y) \cap [\mathcal{X}] = \{x \in [\mathcal{X}]: \|f_i(x)\|_{I,c} < 1 \text{ for } i = 1, \ldots, n\}.$$

For any height-one point $x \in [\mathcal{X}]$, consider the associated rigid point (3.3.5) $\alpha: \text{Spf} \tilde{V}_x \to [\mathcal{X}]$. The composition $\text{sp}_X \circ \alpha: \text{Spf} \tilde{V}_x \to X = \text{Spf} A$ is an adic morphism of formal schemes, hence induces an adic morphism $A \to \tilde{V}_x$. Now for $i = 1, \ldots, n$ the inequality $\|f_i(x)\|_{I,c} < 1$ holds if and only if the image of $f_i$ in $V_x$ lies in the maximal ideal $m_{V_x}$, which is equivalent to, in view of 0.9.1.1 (2), that the image of $f_i$ in $\tilde{V}_x$ lies in its maximal ideal $m_{\tilde{V}_x}$. Since $\text{sp}_X(x)$, as an open prime ideal of $A$, is the inverse image of $m_{\tilde{V}_x}$ under the map $A \to \tilde{V}_x$, this is equivalent to $\text{sp}_X(x) \in Y$. \qed

4.3 Separation map and overconvergent sets

4.3. (a) Separation map. Let $\mathcal{X}$ be a rigid space, and consider the associated Zariski–Riemann space $[\mathcal{X}]$. Consider the separation map (0, §2.3.(c))

$$\text{sep}_\mathcal{X}: [\mathcal{X}] \longrightarrow [\mathcal{X}], \quad x \longmapsto \text{the maximal generization of } x;$$

here $[\mathcal{X}]$ denotes the separated quotient consisting of height-one points of $[\mathcal{X}]$, cf. (4.1.7).
**Proposition 4.3.1** (functoriality; cf. 0.2.3.10). Any morphism $f: \mathcal{X} \to \mathcal{Y}$ of rigid spaces induces a unique continuous map $[f]: [\mathcal{X}] \to [\mathcal{Y}]$ that makes the following diagram commutative:

$$
\begin{array}{ccc}
\langle \mathcal{X} \rangle & \xrightarrow{(f)} & \langle \mathcal{Y} \rangle \\
\text{sep}_{\mathcal{X}} & \downarrow & \text{sep}_{\mathcal{Y}} \\
[\mathcal{X}] & \xrightarrow{[f]} & [\mathcal{Y}].
\end{array}
$$

In other words, the formation of separated quotients is functorial.

### 4.3. (b) Overconvergent sets and tube subsets

**Definition 4.3.2** (cf. 0.2.3.11). Let $\mathcal{X}$ be a rigid space, and $S$ a closed (resp. open) subset of $\langle \mathcal{X} \rangle$. Then $S$ is said to be **overconvergent** if for any $x \in S$, any generalization (resp. specialization) of $x$ belongs to $S$.

Obviously, if $S$ is a overconvergent set, then its complement $\langle \mathcal{X} \rangle \setminus S$ is also overconvergent. The following statements are special cases of 0.2.3.13, 0.2.3.14, and 0.2.3.15.

**Proposition 4.3.3.** Let $\mathcal{X}$ be a rigid space, and $S$ a closed or an open subset of $\langle \mathcal{X} \rangle$. Then $S$ is overconvergent if and only if $S = \text{sep}_{\mathcal{X}}^{-1}(\text{sep}_{\mathcal{X}}(S))$. (Hence there exists a canonical bijection between the set of all overconvergent open (resp. closed) subsets of $\langle \mathcal{X} \rangle$ and the set of all open (resp. closed) subsets of $[\mathcal{X}]$.)

**Corollary 4.3.4.** (1) Any finite intersection of overconvergent open subsets is an overconvergent open subset. Any union of overconvergent open subsets is an overconvergent open subset.

(2) Any finite union of overconvergent closed subsets is an overconvergent closed subset. Any intersection of overconvergent closed subsets is an overconvergent closed subset.

**Proposition 4.3.5.** Any tube closed (resp. tube open) subset of a rigid space is an overconvergent closed (resp. overconvergent open) subset.

**Definition 4.3.6** (cf. 0.2.3.16). Let $\mathcal{X}$ be a rigid space, and consider the separated quotient $[\mathcal{X}]$ of $\langle \mathcal{X} \rangle$. A closed (resp. open) subset $C$ of $[\mathcal{X}]$ is said to be a **tube closed (resp. tube open)** subset if $\text{sep}_{\mathcal{X}}^{-1}(C)$ is a tube closed (resp. tube open) subset of $\mathcal{X}$.

By 4.3.5, there exists a canonical order preserving bijection between the set of all tube closed (resp. tube open) subsets of $\mathcal{X}$ and that of tube closed (resp. tube open) subsets of $[\mathcal{X}]$. Due to 0.2.3.17 we have the following proposition.
Proposition 4.3.7. Let \( \mathcal{X} \) be a coherent rigid space.

1. For any overconvergent closed set \( F \) the set of all tube open subsets containing \( F \) forms a fundamental system of neighborhoods of \( F \).

2. For overconvergent closed subsets \( F_1 \) and \( F_2 \) of \( \langle \mathcal{X} \rangle \) such that \( F_1 \cap F_2 = \emptyset \), there exist tube open subsets \( U_1 \) and \( U_2 \) such that \( F_i \subseteq U_i \) for \( i = 1, 2 \) and that \( U_1 \cap U_2 = \emptyset \).

Corollary 4.3.8. Let \( \mathcal{X} \) be a coherent rigid space.

1. The separated quotient \( [\mathcal{X}] \) is a normal topological space.

2. The separated quotient \( [\mathcal{X}] \) is a compact (hence Hausdorff) space.

Proof. (1) follows from proposition 4.3.7 (1), since \( [\mathcal{X}] \) satisfied the T\(_1\)-axiom by definition.

(2) follows from (1) and the quasi-compactness of \( \langle \mathcal{X} \rangle \). \( \square \)

Corollary 4.3.9. Let \( \mathcal{X} \) be a coherent rigid space. Then the set of all tube open subsets of \( [\mathcal{X}] \) forms an open basis of the topological space \( [\mathcal{X}] \).

Proof. This follows from 4.3.7 (2) and the fact that, since \( [\mathcal{X}] \) satisfies T\(_1\)-axiom, any singleton set \( \{x\} \) for \( x \in [\mathcal{X}] \) is a closed subset. \( \square \)

4.3. (c) Overconvergent interior

Definition 4.3.10 (cf. 0.2.3.28). Let \( \mathcal{X} \) be a rigid space, and \( F \) a subset of \( \langle \mathcal{X} \rangle \). We denote by \( \text{int}_\mathcal{X}(F) \) the maximal overconvergent open subset in \( F \) and call it the overconvergent interior of \( F \) in \( \mathcal{X} \).

The existence of \( \text{int}_\mathcal{X}(F) \) follows from 4.3.4. The following statements were already proved in 0.2.3.29, 0.2.3.30, and 0.2.3.31.

Proposition 4.3.11. Let \( \mathcal{X} \) be a rigid space, and \( \mathcal{U} \subseteq \mathcal{X} \) a quasi-compact open subspace. Suppose that

\[ (*) \quad \text{there is a coherent open subspace } \mathcal{V} \text{ of } \mathcal{X} \text{ such that } \widetilde{\mathcal{U}} \subseteq \mathcal{V}. \]

Then for a height-one point \( y \) to belong to \( \text{int}_\mathcal{X}(\mathcal{U}) \) it is necessary and sufficient that \( \{y\} \subset \mathcal{U} \).

Condition \( (*) \) is automatic if \( \mathcal{X} \) is quasi-separated and locally quasi-compact; see 4.4.1 and 4.4.3 below.
Corollary 4.3.12. Under the assumptions of 4.3.11, the overconvergent interior \( \text{int}_X(\mathcal{U}) \) is described as

\[
\text{int}_X(\mathcal{U}) = \text{sep}_X^{-1}([\mathcal{U}] \setminus \text{sep}(\partial \mathcal{U})),
\]

where \( \partial \mathcal{U} = \overline{\mathcal{U}} \setminus \mathcal{U} \).

Corollary 4.3.13. Let \( X \) be a quasi-separated rigid space, and \( \mathcal{U} \) a quasi-compact open subspace of \( X \). Suppose \( \overline{\mathcal{U}} \) is quasi-compact. Then for a height-one point \( y \) to belong to \( \text{int}_X(\mathcal{U}) \) it is necessary and sufficient that \( \{y\} \subset \mathcal{U} \).

4.4 Locally quasi-compact and paracompact rigid spaces

4.4. (a) Locally quasi-compact rigid spaces

Definition 4.4.1. A rigid space \( X \) is said to be locally quasi-compact if the associated Zariski–Riemann space \( \langle X \rangle \) is locally strongly compact (0.2.5.1).

By 0.2.5.7, 0.2.5.5, 0.2.5.9, and Exercise 0.2.15, the following statements.

Theorem 4.4.2. Let \( X \) be a locally quasi-compact rigid space.

1. The separated quotient \( \overline{[X]} \) is a locally compact space (in particular, it is locally Hausdorff), and the separation map \( \text{sep}_X: \langle X \rangle \rightarrow [X] \) is proper.

2. \( X \) is quasi-compact (resp. quasi-separated) if and only if \( [X] \) is quasi-compact (resp. Hausdorff).

Proposition 4.4.3. Let \( X \) be a quasi-separated rigid space. Then the following conditions are equivalent.

(a) \( X \) is locally quasi-compact.

(b) For any quasi-compact open set \( \mathcal{U} \) the closure \( \overline{\mathcal{U}} \) is quasi-compact.

(c) There is an open covering \( \{\mathcal{U}_\alpha\}_{\alpha \in L} \) of \( X \) such that \( \mathcal{U}_\alpha \) and \( \overline{\mathcal{U}_\alpha} \) are quasi-compact for any \( \alpha \in L \).

4.4. (b) Paracompact rigid spaces

Definition 4.4.4 (cf. 0.2.5.13). A rigid space \( X \) is said to be paracompact if the topological space \( \langle X \rangle \) is paracompact.

For the proofs of the following statements, see 0.2.5.14 and 0.2.5.15, respectively.
Lemma 4.4.5. Let \( \mathcal{X} \) be a rigid space.

(1) \( \mathcal{X} \) is paracompact if it admits a locally finite covering by quasi-compact open sets.

(2) If \( \{ \mathcal{U}_\alpha \}_{\alpha \in L} \) is a locally finite covering such that \( \mathcal{U}_\alpha \) is quasi-compact for any \( \alpha \in L \), then for each \( \alpha \in L \) there are finitely many \( \beta \in L \) such that \( \mathcal{U}_\beta \) intersect \( \mathcal{U}_\alpha \).

Proposition 4.4.6. Let \( \mathcal{X} \) be a paracompact quasi-separated rigid space. Then \( \mathcal{X} \) is locally quasi-compact.

Exercises

Exercise II.4.1 (general closure formula). Let \( X \) be a coherent adic formal scheme of finite ideal type, \( \mathcal{J} \) and \( \mathcal{K} \) admissible ideals, and \( U \) the complement of \( V(\mathcal{J}) \) in \( X \). Let \( X_n \) for \( n \geq 1 \) be the admissible blow-up along the ideal \( \mathcal{K} + \mathcal{J}^n \), \( V_n \) the maximal open subset of \( X_n \), where \( \mathcal{J}^n \) generates the pull-back of \( \mathcal{K} + \mathcal{J}^n \). Set \( \mathcal{U} = \text{sp}_{X}^{-1}(U) \) and \( \mathfrak{V}_n = \text{sp}_{X_n}^{-1}(V_n) \) for \( n \geq 1 \).

1. Show that \( V_n \) is quasi-compact and that the closure \( \overline{\mathcal{U}} \) is given by the formula

\[
\overline{\mathcal{U}} = \bigcap_{n \geq 1} \mathfrak{V}_n .
\]

2. Suppose, moreover, that \( \mathcal{K} \) is an ideal of definition of \( X \). Show that the closure \( C_n \) of \( U \) inside \( X_n \) is contained in \( V_n \) and hence that the map \( C_{n+1} \to C_n \) is proper.

3. Suppose \( \mathcal{K} \) is an ideal of definition. Show that

\[
\overline{\mathcal{U}} = \bigcap_{n \geq 1} \mathfrak{V}_n .
\]

4. Show that for any quasi-compact open subset \( \mathcal{U} \) of a coherent rigid space \( X \) the closure \( \overline{\mathcal{U}} \) admits a countable system of neighborhoods.

Exercise II.4.2. Let \( \varphi: \mathcal{X} \to \mathcal{Y} \) be a morphism between rigid spaces, and \( T \subseteq \langle \mathcal{Y} \rangle \) a tube closed (resp. tube open) subset. Show that \( \langle \varphi \rangle^{-1}(T) \) is a tube closed (resp. tube open) subset of \( \langle \mathcal{X} \rangle \).

Exercise II.4.3. Let \( \mathcal{X} \) be a coherent rigid space, and \( I \subseteq \mathcal{O}^{\text{int}}_{\mathcal{X}} \) an ideal of definition of finite type. Let \( 0 < c < 1 \) be a real number.

1. Show that for any \( f \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) \) the map \( \| f(\cdot) \|_{I,c} : \langle \mathcal{X} \rangle \to \mathbb{R}_{\geq 0} \) factors through the separation map \( \text{sep}_{\mathcal{X}} : \langle \mathcal{X} \rangle \to [\mathcal{X}] \).

2. Show that the map \( \| f(\cdot) \|_{I,c} : \langle \mathcal{X} \rangle \to \mathbb{R}_{\geq 0} \) is continuous.
5 Coherent sheaves

The aim of this section is to discuss coherent sheaves on locally universally Noetherian rigid spaces (2.2.23) and their formal models. Here, by a coherent sheaf on a rigid space \( X \) we mean a coherent sheaf of modules over the rigid structure sheaf \( \mathcal{O}_X \) (cf. 3.2.9) on the associated Zariski–Riemann space \( \langle X \rangle \). In order to lay reasonable foundations for the study of coherent sheaves on rigid spaces, it is of course necessary first to establish that the rigid structure sheaf \( \mathcal{O}_X \) is coherent as a module over itself, that is, the locally ringed space \( \langle X \rangle, \mathcal{O}_X \) is cohesive (0.4.1.7) (the analogue of Oka’s theorem in complex analysis). To show this fundamental result, we first study in §5.1 formal models of an \( \mathcal{O}_X \)-module, that is, when \( X \) is coherent, \( \mathcal{O}_X \)-modules that give rise to the given \( \mathcal{O}_X \)-module by passage to the functor ‘rig’ given by ‘inverting’ the ideal of definition. Then the coherence of the rigid structure sheaf is obtained from the existence theorem (weak form) of finitely presented formal models for finitely presented \( \mathcal{O}_X \)-modules, which we will discuss in §5.2.

In §5.3 a stronger result on the existence of finitely presented formal models will be stated and proved. This result asserts, roughly speaking, that the category of coherent sheaves on a coherent universally Noetherian rigid space \( X \) is equivalent to the quotient category of finitely presented sheaves on a fixed formal model \( X \) of \( X \) modulo the so-called weak isomorphisms (cf. I, §C.2. (a)), that is, isomorphisms up to torsion by ideals of definition.

5.1 Formal models of sheaves

5.1. (a) The ‘rig’ functor for \( \mathcal{O}_X \)-modules

**Notation 5.1.1.** For a rigid space \( X \) we denote by

\[
\text{Mod}^{\text{int}}_X \quad \text{and} \quad \text{Mod}_X
\]

the category of \( \mathcal{O}^{\text{int}}_X \)-modules and the category of \( \mathcal{O}_X \)-modules, respectively.

Let \( X \) be a coherent rigid space and \( X \) a formal model of \( X \). For any \( \mathcal{O}_X \)-module \( F \) we denote by \( F^{\text{rig}} \) the \( \mathcal{O}_X \)-module defined by

\[
F^{\text{rig}} = \text{sp}_X^{-1} F \otimes_{\text{sp}_X^{-1} \mathcal{O}_X} \mathcal{O}_X,
\]

where \( \text{sp}_X: \langle X \rangle \to X \) is the specialization map defined in §3.1. (a). For a morphism \( \varphi: F \to G \) of \( \mathcal{O}_X \)-modules we denote by \( \varphi^{\text{rig}} \) the induced morphism

\[
\varphi^{\text{rig}}: F^{\text{rig}} \to G^{\text{rig}}
\]

of \( \mathcal{O}_X \)-modules. Thus we get the functor

\[
\text{rig}: \text{Mod}_X \to \text{Mod}_X.
\]
Proposition 5.1.2. Let $X$ be an a coherent adic formal scheme with an ideal of definition $\mathcal{I}_X$ of finite type, and $\mathcal{F}_X$ an $\mathcal{O}_X$-module. For any admissible blow-up $\pi : X' \to X$, set $\mathcal{I}_X' = \mathcal{I}_X \mathcal{O}_{X'}$ and $\mathcal{F}_X' = \pi^* \mathcal{F}_X$. Then

$$\mathcal{F}_X^{\text{rig}} = \lim_{\pi : X' \to X} \lim_{n \geq 0} \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{I}_X^n, \mathcal{F}_X'),$$

where the first inductive limit is taken along the cofiltered category $\text{BL}_X$ of all admissible blow-ups of $X$ (§1.3). Moreover, $\mathcal{F}_X'$ in the right-hand side can be replaced by the strict transform $\pi' \mathcal{F}$ (1.2.1).

To show this, we first need to prove the following lemma.

Lemma 5.1.3. Let $X$ be a coherent rigid space, and $\mathcal{F}$ an $\mathcal{O}_X^{\text{int}}$-module. Then we have a canonical isomorphism

$$\mathcal{F} \otimes_{\mathcal{O}_X^{\text{int}}} \mathcal{O}_X \sim \lim_{n > 0} \mathcal{H}om_{\mathcal{O}_X^{\text{int}}}(I^n, \mathcal{F})$$

of $\mathcal{O}_X$-modules, where $I$ is an ideal of definition of $\langle X \rangle$ of finite type (3.2.3).

Proof. The morphism is constructed as follows. By 0.3.1.4, the left-hand side is equal to

$$\mathcal{F} \otimes_{\mathcal{O}_X^{\text{int}}} \lim_{n > 0} \mathcal{H}om_{\mathcal{O}_X^{\text{int}}}(I^n, \mathcal{O}_X^{\text{int}}) = \lim_{n > 0} \mathcal{F} \otimes_{\mathcal{O}_X^{\text{int}}} \mathcal{H}om_{\mathcal{O}_X^{\text{int}}}(I^n, \mathcal{O}_X^{\text{int}}).$$

On the other hand, we have a canonical morphism (cf. [54], 0.1, (5.4.2))

$$\lim_{n > 0} \mathcal{F} \otimes_{\mathcal{O}_X^{\text{int}}} \mathcal{H}om_{\mathcal{O}_X^{\text{int}}}(I^n, \mathcal{O}_X^{\text{int}}) \longrightarrow \lim_{n > 0} \mathcal{H}om_{\mathcal{O}_X^{\text{int}}}(I^n, \mathcal{F}).$$

The desired morphism is the composition of these two morphisms. To show that it is an isomorphism, we may check it stalkwise at any point $x \in \langle X \rangle$. By 3.2.7 we may put $I_x = (a)$ by a non-zero divisor $a \in A_x = \mathcal{O}_X^{\text{int}}_{x, x}$. Then the map between stalks is described as

$$\mathcal{F}_x \otimes_{A_x} A_x[\frac{1}{a}] = \mathcal{F}_x[\frac{1}{a}] \longrightarrow \lim_{n > 0} \mathcal{H}om_{\mathcal{O}_x^{\text{int}}}(I^n, \mathcal{F})_x = \lim_{n > 0} \mathcal{H}om_{A_x}(a^n A_x, \mathcal{F}_x),$$

where the last equality is due to [51], (4.1.1), and the fact that $I$ is invertible (3.2.5). Now the last module is easily seen to be isomorphic to $\mathcal{F}_x[\frac{1}{a}]$, whence the lemma.

Proof of Proposition 5.1.2. Consider the canonical map

$$\lim_{\pi : X' \to X} \lim_{n \geq 0} \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{I}_X^n, \mathcal{F}_X') \longrightarrow \lim_{n > 0} \mathcal{H}om_{\mathcal{O}_X^{\text{int}}}(I^n, \mathcal{F}_X) \otimes_{\mathcal{O}_X^{\text{int}}} \mathcal{F}_X \cong \mathcal{F}_X^{\text{rig}},$$

where the last module is the inductive limit taken along the cofiltered category $\text{BL}_X$ of all admissible blow-ups of $X$. This completes the proof.
where the last isomorphism is due to 5.1.3. To show that this is an isomorphism, one can check stalkwise. Replacing $X$ by the admissible blow-up along $I_X$, one can assume that $I_X$ is invertible. Then for any admissible blow-up $\pi: X' \to X$ the ideal $I_{X'}$ is again invertible (1.1.6). Let $x \in (X)$, and set $I_{X',x} = (a)$. Then $O_X = \mathcal{O}_{X,x}^{\text{int}}[\frac{1}{a}]$, and hence $\mathcal{F}_{X,x}^{\text{rig}} = \mathcal{O}_{X,x}^{\text{int}}[\frac{1}{a}]$. On the other hand,

$$\lim_{\pi: X' \to X} \lim_{n \geq 0} \lim_{x} \mathcal{O}_{X',x} \left( (I_{X'}^{n}, \mathcal{F}_{X'})_{x} \right) \xrightarrow{\pi: X' \to X} \lim_{n \geq 0} \lim_{x} \mathcal{O}_{X',x} \left( (a^n), \mathcal{F}_{X',x}(x) \right) \xrightarrow{\pi: X' \to X} \lim_{\pi: X' \to X} \mathcal{F}_{X',x}(x) \otimes \mathcal{O}_{X',x}(x)[\frac{1}{a}],$$

which coincides with $\mathcal{F}^{\text{rig}}_X$ by 0.3.1.4. The second assertion is clear.

\textbf{Corollary 5.1.4.} Let $X$ be a coherent adic formal scheme of finite ideal type, and set $\mathcal{X} = X^{\text{rig}}$. Then the functor $\cdot^{\text{rig}}: \text{Mod} \to \text{Mod}$ is right-exact. If, moreover, $X$ is universally rigid-Noetherian (I.2.1.7), then it is exact

\textbf{Proof.} The first part is clear. To show the rest, as the proof of 5.1.2 indicates, it suffices to show the following: let $\pi: X' \to X$ be an admissible blow-up of a coherent universally rigid-Noetherian formal schemes (I.2.1.7), where $X'$ is assumed to have an invertible ideal of definition, and $x \in X'$; then the map $\mathcal{O}_{X',x} \to \mathcal{O}_{X',x}[\frac{1}{a}]$ (where $a$ is a local generator of an ideal of definition of $X'$) is flat. To this end, we may assume $X$ is affine, $X = \text{Spf} A$, where $A$ is a t.u. rigid-Noetherian ring (I.2.1.1 (1)); since Zariski localization is flat on universally rigid-Noetherian formal schemes (0.8.2.18), it suffices to verify that, in the notation of 1.1.3, the map $\text{Spec} B \setminus V(IB) \to \text{Spec} A$ is flat. This map can be written as the composition $\text{Spec} B \setminus V(IB) \to \text{Spec} A \setminus V(I) \hookrightarrow \text{Spec} A$, where $B$ is the $I$-adic completion of an affine patch of a blow-up along an admissible ideal. By 0.8.2.18, this means that the morphism $\text{Spec} B \setminus V(IB) \to \text{Spec} A \setminus V(I)$ is flat.

\textbf{Corollary 5.1.5.} Let $X$ be a coherent adic formal scheme of finite ideal type. The following conditions for an $\mathcal{O}_X$-module $\mathcal{F}_X$ of finite type are equivalent.

(a) $\mathcal{F}_X^{\text{rig}} = 0$.

(b) There exists an admissible blow-up $\pi: X' \to X$ such that $\pi^* \mathcal{F}_X$ is an $I_{X'}$-torsion module, where $I_{X'}$ is an ideal of definition of finite type of $X'$.

(c) There exists an admissible blow-up $\pi: X' \to X$ such that the strict transform $\pi^! \mathcal{F}_X$ is zero.
Corollary 5.1.6. Let $X$ be a coherent universally rigid-Noetherian formal scheme, and $I_X$ an ideal of definition of finite type of $X$. Let $\varphi_X : F_X \to G_X$ be a morphism of $\mathcal{O}_X$-modules of finite type. Then the following conditions are equivalent.

(a) $\varphi_X^{\rig} = 0$.

(b) There exists an admissible blow-up $\pi : X' \to X$ such that the induced morphism between strict transforms $\varphi_{X'} : F_{X'} \to G_{X'}$ is the zero map.

Proof. Implication (b) $\implies$ (a) is clear. The converse follows from 5.1.5 applied to the image of $\varphi_X$. $\square$

5.1. (b) Formal models and lattice models

Definition 5.1.7. Let $X$ be a coherent universally Noetherian rigid space (2.2.23).

(1) Let $F$ be an $\mathcal{O}_X$-module. A formal model of $F$ is a datum $((X, \phi), (F_X, \varphi))$ consisting of

- a formal model $(X, \phi)$ of $X$ (2.1.7 (1)),
- an a.q.c. $\mathcal{O}_X$-module $F_X$ of finite type (I.3.1.3),
- an isomorphism $\varphi : F_X^{\rig} \sim F$.

A formal model $((X, \phi), (F_X, \varphi))$ is called a lattice model if $(X, \phi)$ is distinguished, see 2.1.8 (1), and $F_X$ is of finite type and $I_X$-torsion free for some (hence all) ideal of definition $I_X$ of finite type.

(2) Let $\Phi : F \to G$ be a morphism of $\mathcal{O}_X$-modules. A formal model of $\Phi$ is a datum $((X, \phi), (\Phi_X : F_X \to G_X, \varphi, \psi))$ consisting of

- a formal model $(X, \phi)$ of $X$,
- a morphism $\Phi_X : F_X \to G_X$ of $\mathcal{O}_X$-modules, where $F_X$ and $G_X$ are a.q.c. of finite type,
- isomorphisms $\varphi : F_X^{\rig} \sim F$ and $\psi : G_X^{\rig} \sim G$

such that the following square is commutative:

\[
\begin{array}{ccc}
F_X^{\rig} & \xrightarrow{\Phi_X^{\rig}} & G_X^{\rig} \\
\downarrow{\varphi} & & \downarrow{\psi} \\
F & \xrightarrow{\Phi} & G.
\end{array}
\]

If both $((X, \phi), (F_X, \varphi))$ and $((X, \phi), (G_X, \psi))$ are lattice models, then we say that the formal model $((X, \phi), (\Phi_X : F_X \to G_X, \varphi, \psi))$ is a lattice model.
Lattice models are particularly important when $X$ is a coherent universally adhesive rigid space: if $X$ is universally adhesive and if $((X, \phi), (\mathcal{F}_X, \varphi))$ is a lattice model of $\mathcal{F}$, then $\mathcal{F}_X$ is a coherent $\mathcal{O}_X$-module (I.7.2.3).

**Proposition 5.1.8.** If $((X, \phi), (\mathcal{F}_X, \varphi))$ is a formal model of $\mathcal{F}$, where $(X, \phi)$ is a distinguished formal model of $X$, then $((X, \phi), (\mathcal{F}_X', \varphi'))$ with $\mathcal{F}_X' = \mathcal{F}_X / \mathcal{F}_X, I$-tor gives a lattice model of $\mathcal{F}$, where $I$ is an ideal of definition of $X$.

**Proof.** By Exercise II.1.4, $\mathcal{F}_X'$ is a.q.c. of finite type. By 5.1.4 and 5.1.5 we deduce that $\mathcal{F}_X'$ gives a formal model. □

The following proposition is obvious.

**Proposition 5.1.9.** Let $X$ be a coherent universally Noetherian rigid space.

(1) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $((X, \phi), (\mathcal{F}_X, \varphi))$ a formal model of $\mathcal{F}$. Then for any admissible blow-up $\pi: X' \to X$, $((X', \phi \circ \pi_{\text{rig}}), (\pi^* \mathcal{F}_X, \varphi))$ is a formal model of $\mathcal{F}$. If $((X, \phi), (\mathcal{F}_X, \varphi))$ is a lattice model, then $((X', \phi \circ \pi_{\text{rig}}), (\pi^* \mathcal{F}_X, \varphi))$ is again a lattice model, where $\pi^* \mathcal{F}$ denotes the strict transform.

(2) Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules, $\Phi: \mathcal{F} \to \mathcal{G}$ a morphism $\mathcal{O}_X$-modules, and let $((X, \phi), (\Phi_X, \varphi, \psi))$ be a formal model of $\Phi$. Then for any admissible blow-up $\pi: X' \to X$ the pair $((X', \phi \circ \pi_{\text{rig}}), (\pi^* \Phi_X, \varphi, \psi))$ is a formal model of $\Phi$. If $((X, \phi), (\Phi_X, \varphi, \psi))$ is a lattice model, then $((X', \phi \circ \pi_{\text{rig}}), (\pi^* \Phi_X, \varphi, \psi))$ is again a lattice model, where $\pi^* \Phi_X: \pi^* \mathcal{F}_X \to \pi^* \mathcal{G}_X$ is the induced morphism between strict transforms.

**Proposition 5.1.10.** Let $X$ be a coherent universally rigid-Noetherian rigid space, and $X$ a distinguished formal model of $X$. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module, and suppose we have two finitely presented formal models $\mathcal{F}_X$ and $\mathcal{G}_X$ of $\mathcal{F}$ on $X$. Then, replacing $X$ by an admissible blow-up and $\mathcal{F}_X$ and $\mathcal{G}_X$ by their strict transforms if necessary, one can find positive integers $n, m, l$ such that, up to isomorphisms, the following inclusions hold:

$$I^n_X \mathcal{F}_X \subseteq I^m_X \mathcal{G}_X \subseteq I^l_X \mathcal{F}_X,$$

where $I_X$ is an invertible ideal of definition of $X$ of finite type.

**Proof.** First note that, since the $I_X$-torsion parts of $\mathcal{F}_X$ and $\mathcal{G}_X$ are locally bounded, there exists $l > 0$ such that $I^l_X \mathcal{F}_X$ and $I^l_X \mathcal{G}_X$ are $I_X$-torsion free. Moreover, since $I_X$ is invertible, $I^l_X \mathcal{F}_X$ and $I^l_X \mathcal{G}_X$ are finitely presented.

Let $x \in X$. The hypotheses imply that, replacing $X$ by an admissible blow-up and $\mathcal{F}_X$ and $\mathcal{G}_X$ by their strict transforms, we have $\mathcal{F}_{X,x}[\frac{1}{a}] \cong \mathcal{G}_{X,x}[\frac{1}{a}]$, where $I_{X,x} = (a)$. Consequently,

$$a^n \mathcal{F}_{X,x} \subseteq a^m \mathcal{G}_{X,x} \subseteq a^l \mathcal{F}_{X,x}$$
for some $n, m > 0$. Hence, by [51], (4.1.1), there exists an open neighborhood $U$ of $x$ in $X$ such that

$$(I_X|U)^n\mathcal{F}_X|U \subseteq (I_X|U)^m\mathcal{G}_X|U \subseteq (I_X|U)^l\mathcal{F}_X|U.$$ 

Using the fact that $X$ is quasi-compact, one gets the desired inclusions for sufficiently large $n$ and $m$. 

5.1. (c) Weak isomorphisms

**Definition 5.1.11** (cf. I, §C.2.(a)). Let $X$ be a coherent adic formal scheme, and $I_X$ an ideal of definition of finite type.

1. A morphism $\varphi: \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is said to be a *weak isomorphism* if there exists an integer $s \geq 0$ such that $I_X^s \ker(\varphi) = 0$ and $I_X^s \coker(\varphi) = 0$.

2. We say that two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ are *weakly isomorphic* if they are connected by a chain of weak isomorphisms

$$\mathcal{F} \leftarrow \mathcal{H}_1 \rightarrow \mathcal{H}_2 \leftarrow \cdots \leftarrow \mathcal{H}_r \rightarrow \mathcal{G}.$$ 

**Proposition 5.1.12.** Let $\varphi_X: \mathcal{F}_X \to \mathcal{G}_X$ be a morphism between a.q.c. sheaves of finite type on a coherent universally rigid-Noetherian formal scheme $X$. Then the following conditions are equivalent.

(a) There exists an admissible blow-up $\pi: X' \to X$ such that $\pi^*\varphi_X$ is a weak isomorphism.

(b) $\varphi_X^{\text{rig}}$ is an isomorphism.

**Proof.** By 5.1.4 and 5.1.5 we readily see that (a) implies (b). Suppose $\varphi_X^{\text{rig}}$ is an isomorphism. By 5.1.5, we have an admissible blow-up $\pi: X' \to X$ such that the kernel and the cokernel of $\pi^*\varphi_X$ are $I_{X'}$-torsion, where $I_{X'} = I\mathcal{O}_{X'}$. By I.3.2, these are bounded $I_{X'}$-torsion.

**Corollary 5.1.13.** Let $X$ be a coherent universally rigid-Noetherian formal scheme, and $\varphi_X: \mathcal{F}_X \to \mathcal{G}_X$ and $\psi_X: \mathcal{G}_X \to \mathcal{H}_X$ weak isomorphisms between a.q.c. sheaves of finite type on $X$. Then there exists an admissible blow-up $\pi: X' \to X$ such that $\pi^*(\psi_X \circ \varphi_X)$ and $\pi^*(\psi_X \circ \varphi_X)$ are weak isomorphisms.

**Corollary 5.1.14.** Let $X$ be a coherent universally Noetherian rigid space, and $X$ a distinguished formal model of $X$. Let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module, and $\mathcal{F}_X$ a formal model of $\mathcal{F}$ on $X$. Then any a.q.c. sheaf $\mathcal{G}_X$ of finite type on $X$ weakly isomorphic to $\mathcal{F}_X$ is a formal model of $\mathcal{F}$. In particular, for any ideal of definition $I_X$ of finite type of $X$ and any positive integer $n$, $I_X^n\mathcal{F}_X$ is a formal model of $\mathcal{F}$.

The following proposition is a consequence of I.C.2.9.
5. Coherent sheaves

**Proposition 5.1.15.** Let $X$ be coherent universally rigid-Noetherian formal scheme and let $\mathcal{X} = X^{\text{rig}}$. If a finite type $\mathcal{O}_X$-module $\mathcal{F}$ has a formal model on $X$, then it has a formal model given by a finitely presented $\mathcal{O}_X$-module.

5.2 Existence of finitely presented formal models (weak version)

The main goal of this subsection is to prove the following theorem.

**Theorem 5.2.1** (Existence of finitely presented formal models (weak form)). Let $\mathcal{X}$ be a coherent universally Noetherian rigid space and $\varphi: F \to G$ a morphism of finitely presented $\mathcal{O}_\mathcal{X}$-modules. Then there exist a formal model $X$ of $\mathcal{X}$ and a formal model $\varphi_X: F_X \to G_X$ of $\varphi$ by finitely presented $\mathcal{O}_X$-modules.

As a special case where $F = G$ and $\varphi = \text{id}_F$, we have the following corollary.

**Corollary 5.2.2.** Let $\mathcal{X}$ be a coherent universally Noetherian rigid space and $\mathcal{F}$ a finitely presented $\mathcal{O}_\mathcal{X}$-module. Then there exists a finitely presented formal model $\mathcal{F}_X$ of $\mathcal{F}$ on a formal model $X$ of $\mathcal{X}$.

Furthermore, using 2.1.10 and 5.1.8 we deduce the following corollary.

**Corollary 5.2.3.** Let $\mathcal{X}$ and $\varphi$ be as in 5.2.1. Then there exist a distinguished formal model $X$ of $\mathcal{X}$ and a lattice model $\varphi'_X: F'_X \to G'_X$ of $\varphi$ on $X$.

Note that, if $X$ is universally adhesive (hence $\mathcal{X}$ is universally adhesive), then lattice models are automatically finitely presented. To prove the theorem, we first need to show the following lemma.

**Lemma 5.2.4.** Let $X$ be a coherent universally rigid-Noetherian formal scheme with an ideal of definition $I_X$ of finite type, and $\mathcal{F}_X$ and $\mathcal{G}_X$ finitely presented $\mathcal{O}_X$-modules. Suppose there exists a morphism $\varphi: \mathcal{F}^{\text{rig}}_X \to \mathcal{G}^{\text{rig}}_X$ of $\mathcal{O}_\mathcal{X}$-modules, where $\mathcal{X} = X^{\text{rig}}$. Then, replacing $X$ by an admissible blow-up and $\mathcal{F}_X$ and $\mathcal{G}_X$ by their strict transforms, we can find a morphism $\varphi: I^{\text{rig}}_X \mathcal{F}_X \to \mathcal{G}_X$ of $\mathcal{O}_X$-modules, where $s$ is a positive integer, such that $\varphi^{\text{rig}} = \varphi$.

**Proof.** Step 1. Replacing $X$ by the admissible blow-up along $I_X$, we may assume that the ideal of definition $I_X$ is invertible. Let $x \in (\mathcal{X})$. By 5.1.2 (and using the notation therein), we have

$$\mathcal{G}^{\text{rig}}_x = \lim_{\substack{\longrightarrow \mathcal{F}^{\text{rig}}_{X'} \to \mathcal{G}^{\text{rig}}_X}} \mathcal{G}^{\text{rig}}_{X',\text{sp}X'(x)}[\frac{1}{a}],$$

where $\mathcal{G}_{X'}$ is the pull-back (total transform) of $\mathcal{G}_X$ by an admissible blow-up $X' \to X$, and $a$ is the generator of $I_{X,\text{sp}X(x)}$. On the other hand, by the definition of $\mathcal{F}^{\text{rig}}_X$ we can write

$$\mathcal{F}^{\text{rig}}_X = \mathcal{F}_{X,\text{sp}X(x)} \otimes_{\mathcal{O}_{X,\text{sp}X(x)}} \mathcal{O}^{\text{inf}}_{\mathcal{X},x}[\frac{1}{a}].$$
Hence the stalk map $\varphi_x$ gives

$$\varphi_x : \mathcal{F}_{X, \text{sp}_X}(x) \otimes \mathcal{O}_{X, \text{sp}_X}(x) \mathcal{O}_{X, x}^{\text{int}} \longrightarrow \lim_{X' \to X} \mathcal{G}_{X', \text{sp}_{X'}(x)}[\frac{1}{a}].$$

Let

$$\psi_x : \mathcal{F}_{X, \text{sp}_X}(x) \longrightarrow \lim_{X' \to X} \mathcal{G}_{X', \text{sp}_{X'}(x)}[\frac{1}{a}]$$

be the composition of $\varphi_x$ preceded by the canonical map

$$\mathcal{F}_{X, \text{sp}_X}(x) \longrightarrow \mathcal{F}_{X, \text{sp}_X}(x) \otimes \mathcal{O}_{X, \text{sp}_X}(x) \mathcal{O}_{X, x}^{\text{int}}[\frac{1}{a}];$$

$\psi_x$ is a morphism of $\mathcal{O}_{X, \text{sp}_X}(x)$-modules. Since $\mathcal{F}_{X, \text{sp}_X}(x)$ is a finitely presented $\mathcal{O}_{X, \text{sp}_X}(x)$-module, there exist an admissible blow-up $X' \to X$ and a map

$$\phi_x : \mathcal{F}_{X, \text{sp}_X}(x) \longrightarrow \mathcal{G}_{X', \text{sp}_{X'}(x)}[\frac{1}{a}]$$

such that $\psi_x$ is equal to the composition of $\phi_x$ followed by the canonical map. Hence we get a map

$$\phi'_x : \mathcal{F}_{X', \text{sp}_{X'}(x)} \longrightarrow \mathcal{G}_{X', \text{sp}_{X'}(x)}[\frac{1}{a}]$$

by base change. Since $\mathcal{F}_{X', \text{sp}_{X'}(x)}$ is finitely generated, one can find an integer $s \geq 0$ and a map

$$\tilde{\varphi}_x : a^s \mathcal{F}_{X', \text{sp}_{X'}(x)} \longrightarrow \mathcal{G}_{X', \text{sp}_{X'}(x)}$$

that gives the restriction of $\phi'_x$. By construction,

$$\tilde{\varphi}_x \otimes \mathcal{O}_{X', \text{sp}_{X'}(x)} \mathcal{O}_{X, x} = \varphi_x.$$

Step 2. The previous step implies that for any $x \in \mathcal{X}$ one can replace $X$ by a suitable admissible blow-up of $X$ in such a way that there exists a map

$$\tilde{\varphi}_x : a^s \mathcal{F}_{X, \text{sp}_X}(x) \longrightarrow \mathcal{G}_{X, \text{sp}_X}(x)$$

of $\mathcal{O}_{X, \text{sp}_X}(x)$-modules that induces the stalk map $\varphi_x$ by base change. Here $s_x$ is a non-negative integer, which depends on the point $x$. By [51], (4.1.1) (here we use the fact that $\mathcal{F}_X$ is finitely presented), one can take an open neighborhood $U$ of $\text{sp}_X(x)$ in $X$ and a morphism $\tilde{\varphi}_U : I^s \mathcal{F}_X|_U \to \mathcal{G}_X|_U$ such that $\tilde{\varphi}_{U,x} = \tilde{\varphi}_x$. Moreover, one can take $U$ sufficiently small such that $\tilde{\varphi}^{\text{rig}}_U = \text{sp}_X^{-1} \tilde{\varphi}_U \otimes \mathcal{O}_{X|_U}^{\text{int}} \mathcal{O}_X|_U$ coincides with $\varphi|_U$, where $U = \text{sp}_X^{-1}(U)$.
Step 3. Thus we get an open covering \( \mathcal{X} = \bigcup_{\alpha=1}^{n} \mathcal{U}_\alpha \) such that we have for each \( \alpha \)

- an admissible blow-up \( X_\alpha \to X \) and a quasi-compact open subset \( U_\alpha \) of \( X_\alpha \) such that \( \mathcal{U}_\alpha = \text{sp}_X^{-1}(U_\alpha) \),

- a positive integer \( s_\alpha \), and

- a morphism \( \tilde{\varphi}_\alpha : I_X^s \mathcal{F}|_{U'_\alpha} \to \mathcal{G}|_{U'_\alpha} \) such that \( \tilde{\varphi}_\alpha^{\text{rig}} = \varphi|_{\mathcal{U}_\alpha} \).

Since \( \mathcal{X} \) is quasi-compact, the above open covering can be taken to be finite, and hence the positive integer \( s = s_\alpha \) can be chosen independently of \( \alpha \). Moreover, successive use of 1.1.9 yields an admissible blow-up \( X_0 \) that dominates all \( X_\alpha \) on which there exists an open covering \( X_0 = \bigcup_{\alpha=1}^{n} U'_\alpha \) consisting of quasi-compact open subsets for which for each \( \alpha \) there exists a morphism

\[
\tilde{\varphi}_\alpha : I_{X_0}^s \mathcal{F}|_{U'_\alpha} \to \mathcal{G}|_{U'_\alpha}
\]

such that \( \tilde{\varphi}_\alpha^{\text{rig}} = \varphi|_{\mathcal{U}_\alpha} \). Now by 5.1.6 there exists a further admissible blow-up \( X'' \) on which the strict transforms of \( \tilde{\varphi}_\alpha \) glue together to a morphism

\[
\tilde{\varphi} : I_{X''}^s \mathcal{F}|_{U''_\alpha} \to \pi' \mathcal{G}|_{U''_\alpha},
\]

as desired.

**Proof of Theorem 5.2.1.** First we claim that any finitely presented \( \mathcal{O}_X \)-module \( \mathcal{F} \) has a finitely presented formal model \( \mathcal{F}_X \). Once this is shown, the assertion of the theorem immediately follows from 5.2.4 since, replacing \( X \) by a further admissible blow-up, one can assume that \( \mathcal{F}_X \) is \( I_X \)-torsion free (where \( I_X \) is an ideal of definition of finite type) and that \( X \) has an invertible ideal of definition. But for this, by 5.1.15 we only have to show that \( \mathcal{F} \) has a formal model.

Let \( \mathcal{F} \) be a finitely presented \( \mathcal{O}_X \)-module. Then there exist a finite open covering \( \mathcal{X} = \bigcup_{\alpha=1}^{r} \mathcal{U}_\alpha \) consisting of quasi-compact open subsets and, for each \( \alpha \), an exact sequence

\[
\mathcal{O}_X^{\oplus p}|_{\mathcal{U}_\alpha} \xrightarrow{\varphi_\alpha^p} \mathcal{O}_X^{\oplus q}|_{\mathcal{U}_\alpha} \to \mathcal{F}|_{\mathcal{U}_\alpha} \to 0.
\]

(Here the numbers \( p \) and \( q \) depend on \( \alpha \).) By 3.1.3, 2.1.10, and 3.1.5, there exist a distinguished formal model \( X \) of \( X \) and an open covering \( \mathcal{X} = \bigcup_{\alpha=1}^{r} U_\alpha \) such that \( \mathcal{U}_\alpha = \text{sp}_X^{-1}(U_\alpha) \) for each \( \alpha \). Moreover, each \( U_\alpha \) can be identified with \( \mathcal{U}_\alpha \), where \( \mathcal{U}_\alpha = U_\alpha^{\text{rig}} \). Replacing \( X \) by the admissible blow-up along \( I_X \), we may assume that \( I_X \) is invertible.

By 5.2.4, replacing \( X \) by an admissible blow-up, we have for each \( \alpha = 1, \ldots, r \) a morphism

\[
\tilde{\varphi}_\alpha : I_X^s \mathcal{O}_X^{\oplus p}|_{U_\alpha} \to \mathcal{O}_X^{\oplus q}|_{U_\alpha}
\]

such that \( \tilde{\varphi}_\alpha^{\text{rig}} = \varphi_\alpha \). Since \( \alpha \) runs through a finite set, one can choose such an admissible blow-up independently of \( \alpha \). Set \( \mathcal{F}_\alpha = \text{coker}(\tilde{\varphi}_\alpha) \). Then \( \mathcal{F}_\alpha \) is a finitely presented \( \mathcal{O}_X|_{U_\alpha} \)-module such that \( \mathcal{F}_\alpha^{\text{rig}} \cong \mathcal{F}|_{\mathcal{U}_\alpha} \).
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Since $X$ is coherent, $U_{\alpha \beta} = U_{\alpha} \cap U_{\beta}$ are quasi-compact. By 5.1.10 we deduce that, replacing $X$ by an admissible blow-up if necessary, there exist, for each pair $\alpha, \beta$ of indices, positive integers $m, l$ such that

$$I_{X}^{m} \tilde{F}_{\alpha} \subseteq I_{X}^{m} \tilde{F}_{\beta} \subseteq I_{X}^{l} \tilde{F}_{\alpha}$$

on $U_{\alpha \beta}$ (here we used 1.1.9 to extend the possible admissible blow-up of $U_{\alpha \beta}$ to an admissible blow-up of $X$).

We want to show that there exists a finitely presented $\mathcal{O}_{X}$-module $\tilde{F}_{X}$ such that, for any $\alpha = 1, \ldots, r$,

$$I_{X}^{s} \tilde{F}_{\alpha} \subseteq \tilde{F}_{X} \subseteq I_{X}^{t} \tilde{F}_{\alpha}$$

on $U_{\alpha}$ for some $s, t > 0$. This is shown by induction with respect to $n$, and thus we reduce to the case $n = 2$, that is, $X = U_{\alpha} \cup U_{\beta}$. But then the issue is to extend the sheaf $I_{X}^{m} \tilde{F}_{\beta}$ to $X$. It follows from Exercise I.3.3 that one can do this to get the desired extension as an a.q.c. sheaf $\tilde{F}_{X}$ of finite type. By the construction we have $\tilde{F}^{\text{rig}} \cong \tilde{F}_{X}$. □

**Corollary 5.2.5.** Let $\mathcal{X}$ be a locally universally Noetherian rigid space. Then $\mathcal{O}_{\mathcal{X}}$ is a coherent $\mathcal{O}_{\mathcal{X}}$-module. In other words, the locally ringed space $((\mathcal{X}), \mathcal{O}_{\mathcal{X}})$ is cohesive (0.4.1.7).

**Proof.** Consider, for any open subspace $\mathcal{U} \subseteq \mathcal{X}$, a morphism of $\mathcal{O}_{\mathcal{U}}$-modules of the form $\varphi: \mathcal{O}_{\mathcal{U}}^{\oplus p} \to \mathcal{O}_{\mathcal{U}}$. We need to show that $\ker(\varphi)$ is of finite type. As we may work locally on $\mathcal{U}$, we may assume that $\mathcal{U}$ is coherent, which, by a further formal reduction, we may set $\mathcal{X} = \mathcal{U}$. By 5.2.4, we have a formal model $X$ of $\mathcal{X}$ with an invertible ideal of definition $I_{X}$ and a morphism $\varphi_{X}: \mathcal{O}_{X}^{\oplus p} \to \mathcal{O}_{X}$ that gives rise to $\varphi$ by $^{\text{rig}}$. Then, as the question is local on $\mathcal{X}$, we may further assume that $X$ is affine, $X = \text{Spf} \ A$. The map $\varphi_{X}$ is given by a $A$-linear map $A^{\oplus p} \to A$. Let $K$ be the kernel. Since $A$ is t.u. rigid-Noetherian, we have an injective map $K' \hookrightarrow K$ from a finitely generated $A$-module such that $K / K'$ is $I$-torsion (where $I \subseteq A$ is an ideal of definition). Since $K / K' \subseteq A^{\oplus p} / K'$, we know that $K / K'$ is bounded $I$-torsion. If we set $\mathcal{K}' = (K')^{\Delta}$ and $\mathcal{K} = K^{\Delta}$, we have a weak isomorphism $\mathcal{K}' \leftarrow \mathcal{K}$. Now by 5.1.4 the kernel of $\varphi$ is isomorphic to $\mathcal{K}^{\text{rig}}$ and hence to $\mathcal{K}'^{\text{rig}}$, which is of finite type. □

**5.3 Existence of finitely presented formal models (strong version)**

As usual, we denote by $\text{Coh}_{\mathcal{X}}$ the category of coherent $\mathcal{O}_{\mathcal{X}}$-modules. The goal of this subsection is to show the following theorem.
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**Theorem 5.3.1.** Let \(\mathcal{X}\) be a coherent universally Noetherian rigid space (2.2.23), and \(X\) a universally rigid-Noetherian formal model of \(\mathcal{X}\). Then the functor

\[
\text{Mod}_{\mathcal{X}}^{\text{FP}} \rightarrow \text{Coh}_{\mathcal{X}},
\]

where \(\text{Mod}_{\mathcal{X}}^{\text{FP}}\) denotes the full subcategory of \(\text{Mod}_{\mathcal{X}}\) consisting of finitely presented \(\mathcal{O}_{\mathcal{X}}\)-modules, induces an equivalence of the categories

\[
\text{Mod}_{\mathcal{X}}^{\text{FP}} / \{\text{weak isomorphisms}\} \sim \text{Coh}_{\mathcal{X}}.
\]

As a corollary, we get a stronger result on existence of lattice models.

**Corollary 5.3.2** (existence of finitely presented formal models (strong form)). Let \(\mathcal{X}\) be a coherent universally Noetherian rigid space, let \(X\) be a universally rigid-Noetherian formal model of \(\mathcal{X}\), and let \(\varphi: \mathcal{F} \rightarrow \mathcal{G}\) be a morphism of finitely presented \(\mathcal{O}_{\mathcal{X}}\)-modules. Then there exists a formal model \(\varphi_{\mathcal{X}}: \mathcal{F}_{\mathcal{X}} \rightarrow \mathcal{G}_{\mathcal{X}}\) of \(\varphi\) on \(X\) by finitely presented \(\mathcal{O}_{\mathcal{X}}\)-modules.

Similarly to §5.2, we immediately have the following two corollaries.

**Corollary 5.3.3.** Let \(\mathcal{X}, X\), and \(\varphi\) be as in 5.3.2. Then there exists a lattice model \(\varphi'_{\mathcal{X}}: \mathcal{F}'_{\mathcal{X}} \rightarrow \mathcal{G}'_{\mathcal{X}}\) of \(\varphi\) on \(X\).

**Corollary 5.3.4.** Let \(\mathcal{X}, X\), and \(\varphi\) be as in 5.3.2. Then there exists a lattice model \(\varphi'_{\mathcal{X}}: \mathcal{F}'_{\mathcal{X}} \rightarrow \mathcal{G}'_{\mathcal{X}}\) of \(\varphi\) on \(X\).

In order to show the theorem, we need a few preparatory results.

**Proposition 5.3.5.** Let \(X\) be a coherent universally adhesive formal scheme that is \(I_{X}\)-torsion free (resp. coherent universally rigid-Noetherian formal scheme), where \(I_{X}\) is an ideal of definition of finite type on \(X\). Let \(\pi: X' \rightarrow X\) be an admissible blow-up.

1. For any finitely presented \(\mathcal{O}_{X}\)-module \(\mathcal{G}\), \(R^{q}\pi_{*}\pi^{*}\mathcal{G}\) is finitely presented (resp. FP-approximated (I.C.2.7)). If \(q > 0\), it is bounded \(I_{X}\)-torsion. Moreover, the canonical morphism \(\mathcal{G} \rightarrow \pi_{*}\pi^{*}\mathcal{G}\) is a weak isomorphism.

2. For any finitely presented \(\mathcal{O}_{X'}\)-module \(\mathcal{F}\), \(R^{q}\pi_{*}\mathcal{F}\) is finitely presented (resp. FP-approximated). If \(q > 0\), it is bounded \(I_{X}\)-torsion. Moreover, the canonical morphism \(\pi^{*}\pi_{*}\mathcal{F} \rightarrow \mathcal{F}\) is a weak isomorphism.
Proof. We may assume that \( X \) is affine (note that the property ‘FP-approximated’ is local with respect to Zariski topology), \( X = \text{Spf} \, A \), and that \( I_X = I \Delta, \) where \( A \) is a t.u. adhesive ring (resp. t.u. rigid-Noetherian ring), and \( I \subseteq A \) is a finitely generated ideal of definition. In this situation, the admissible blow-up \( \pi \) is the formal completion of the usual blow-up \( p : Y' \to Y = \text{Spec} \, A \) along an admissible ideal \( J \) of \( A \).

(1) Since \( X \) is affine, there exists a finitely presented \( \mathcal{O}_Y \)-module \( \mathcal{H} \) such that \( \mathcal{H}^{\text{for}} = \mathcal{G} \), where \( ^{\text{for}} \) denotes the ‘virtual’ formal completion functor defined in §9.1. (a). By I.1.4.7, \( \pi^* \mathcal{G} = (p^* \mathcal{H})^{\text{for}} \). Now we apply the comparison theorem (I.9.1.3) (resp. (I.C.3.3)) to deduce that \( R^q \pi_* \pi^* \mathcal{G} = (R^q p_* p^* \mathcal{H})^{\text{for}} \) for \( q \geq 0 \). By the finiteness theorem (I.8.1.3) (resp. (I.C.3.1)), \( R^q p_* p^* \mathcal{H} \) is coherent (resp. FP-approximated) and hence \( R^q \pi_* \pi^* \mathcal{G} \) is coherent (resp. FP-approximated by I.C.2.8). Moreover, when \( q > 0 \), \( R^q p_* p^* \mathcal{H} \) is I-torsion, and hence so is \( R^q \pi_* \pi^* \mathcal{G} \). It is now clear that \( \mathcal{G} \to \pi_* \pi^* \mathcal{G} \) is a weak isomorphism, for the kernel and cokernel of the morphism \( \mathcal{H} \to p_* p^* \mathcal{H} \) are bounded I-torsion.

(2) By the existence theorem (I.10.1.2) (resp. (I.C.3.5)) one can take a coherent \( \mathcal{O}_Y \)-module \( \mathcal{H} \) such that \( \mathcal{H}^{\text{for}} = \mathcal{F} \). Then the assertion follows by a similar argument as above. \( \square \)

Corollary 5.3.6. Let \( X \) be a coherent universally adhesive formal scheme that is \( I_X \)-torsion free (resp. coherent universally rigid-Noetherian formal scheme), where \( I_X \) is an ideal of definition of finite type on \( X \). Let \( \pi : X' \to X \) be an admissible blow-up.

(1) The sheaf \( \pi_* \mathcal{O}_{X'} \) is finitely presented (resp. FP-approximated), and the canonical morphism \( \mathcal{O}_X \to \pi_* \mathcal{O}_{X'} \) is a weak isomorphism.

(2) If \( q > 0 \), \( R^q \pi_* \mathcal{O}_{X'} \) is bounded \( I_X \)-torsion.

Proof of Theorem 5.3.1. First we prove that the functor in question is essentially surjective. Let \( \mathcal{F} \) be an object of \( \text{Coh}_X \). By 5.2.1 and 5.1.9 (1), there exist an admissible blow-up \( \pi : X' \to X \) and a finitely presented formal model \( \mathcal{F}_{X'} \) of \( \mathcal{F} \) on \( X' \). By 5.3.5 (2), \( \mathcal{F}_X = \pi_* \mathcal{F}_{X'} \) is finitely presented (resp. FP-approximated, which can be replaced by a finitely presented formal model by an FP-approximation (I.C.2.7 (1))). Hence the essential surjectivity is proven. By a similar argument one can also show that for any morphism \( \varphi \) in \( \text{Coh}_X \) there exists a morphism \( \varphi_X \) of \( \text{Mod}^{\text{FP}}_X \) such that \( \varphi_X^{\text{rig}} = \varphi \) (resp. here we need to use the fact that the category of FP-approximations is filtered; cf. Exercise I.C.2). Hence the functor in question is full and essentially surjective. The faithfullness follows from 5.1.12 and 5.3.5 (2). \( \square \)
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5.4 Integral models

**Notation 5.4.1.** Let \( X \) be a coherent rigid space, and \( X \) a formal model of \( X \). For any \( \mathcal{O}_X \)-module \( \mathcal{F} \) we denote by \( \text{sp}_X^{\ast} \mathcal{F} \) the \( \mathcal{O}_X^{\text{int}} \)-module defined by

\[
\text{sp}_X^{-1} \mathcal{F} \otimes_{\text{sp}_X^{-1} \mathcal{O}_X} \mathcal{O}_X^{\text{int}},
\]

that is, the pull-back by the morphism of locally ringed spaces

\[
((\mathcal{X}), \mathcal{O}_X^{\text{int}}) \rightarrow (X, \mathcal{O}_X)
\]

whose underlying morphism of topological spaces is \( \text{sp}_X \).

**Theorem 5.4.2.** Let \( \mathcal{X} \) be a coherent rigid space.

1. For any finitely presented \( \mathcal{O}_X^{\text{int}} \)-module \( \mathcal{F} \) there exist a distinguished formal model \( X \) of \( \mathcal{X} \) and a finitely presented \( \mathcal{O}_X \)-module \( \mathcal{F}_X \) such that \( \mathcal{F} \cong \text{sp}_X^{\ast} \mathcal{F}_X \).

2. Let \( X \) be a distinguished formal model of \( \mathcal{X} \), and let \( \mathcal{F}_X \) and \( \mathcal{G}_X \) be \( \mathcal{O}_X \)-modules. Suppose \( \mathcal{F}_X \) is finitely presented. Let \( \varphi_X, \psi_X : \mathcal{F}_X \rightarrow \mathcal{G}_X \) be two morphisms of \( \mathcal{O}_X \)-modules such that \( \text{sp}_X^{\ast} \varphi_X = \text{sp}_X^{\ast} \psi_X \). Then there exists an admissible blow-up \( \pi : X' \rightarrow X \) such that \( \pi^{\ast} \varphi_X = \pi^{\ast} \psi_X \).

3. Let \( \mathcal{F} \) and \( \mathcal{G} \) be finitely presented \( \mathcal{O}_X^{\text{int}} \)-modules, and \( \varphi : \mathcal{F} \rightarrow \mathcal{G} \) a morphism of \( \mathcal{O}_X^{\text{int}} \)-modules. Then there exist a distinguished formal model \( X \) of \( \mathcal{X} \), finitely presented \( \mathcal{O}_X \)-modules \( \mathcal{F}_X \) and \( \mathcal{G}_X \), and a morphism \( \varphi_X : \mathcal{F}_X \rightarrow \mathcal{G}_X \) of \( \mathcal{O}_X \)-modules, such that \( \text{sp}_X^{\ast} \varphi_X = \varphi \).

Note that the formal model \( X' \) in (2) is again distinguished (2.1.9). Before proceeding to the proof, let us insert here a useful set-up for the discussion, which we will also use in later sections.

Here, for the proof of the theorem, and also for the later purpose, let us set up a useful situation.

**Situation 5.4.3.** Let \( \mathcal{X} \) be a coherent rigid space, and \( X \) a distinguished formal model. For notational convenience, we display the set \( \text{AId}_X \) of all admissible ideals of \( \mathcal{O}_X \) in the form \( \text{AId}_X = \{ \mathcal{J}_\alpha : \alpha \in L \} \), where \( L \) is an index set. The set \( L \) is considered with the order defined by

\[
\alpha \leq \beta \iff \text{there exists } \gamma \in L \text{ such that } \mathcal{J}_\beta = \mathcal{J}_\alpha \mathcal{J}_\gamma.
\]

Note that the ordered set \( L \) is isomorphic to \( \text{AId}_X^{\text{opp}} \) and hence is a directed set; see 1.3.1 (2). The set \( \text{AId}_X \) has the trivial ideal \( \mathcal{O}_X \), the corresponding element of which in \( L \) is denoted by \( 0 \), that is, \( \mathcal{J}_0 = \mathcal{O}_X \). We write \( X_\alpha = X_{\mathcal{J}_\alpha} \) (the admissible blow-up along the admissible ideal \( \mathcal{J}_\alpha \)) for each \( \alpha \), and the corresponding morphism \( X_\alpha \rightarrow X_0 = X \) is denoted by \( \pi_{0\alpha} \). By 1.3.1 (2), the functor

\[
L^{\text{opp}} \rightarrow \text{BL}_X, \quad \alpha \mapsto (\pi_{0\alpha} : X_\alpha \rightarrow X)
\]
is cofinal. Note that by 2.1.9 each $X_\alpha$ is again a distinguished formal model of $X$. For $\alpha, \beta \in L$ such that $\alpha \leq \beta$ we have the unique commutative diagram

\[
\begin{array}{ccc}
X_\alpha & \xrightarrow{\pi_{\alpha\beta}} & X_\beta \\
\downarrow{\pi_{0\alpha}} & & \downarrow{\pi_{0\beta}} \\
X & & \\
\end{array}
\]

thanks to the universality of admissible blow-ups (1.1.4 (3)). Again by universality, one see easily that the morphism $\pi_{\alpha\beta}$ is also an admissible blow-up; in fact, it is the admissible blow-up along $(\pi_{0\alpha}^{-1} J_\gamma) \mathcal{O}_{X_\alpha}$, where $J_\beta = J_\alpha J_\gamma$. Moreover, for $\alpha \leq \beta \leq \gamma$ we have $\pi_{\alpha\beta} \circ \pi_{\beta\gamma} = \pi_{\alpha\gamma}$. Thus we get a filtered projective system \{$X_\alpha, \pi_{\alpha\beta}$\} consisting of admissible blow-ups among distinguished formal models of $X$ indexed by the directed set $L$. By 1.3.1 (2),

\[
\lim_{\alpha \in L} X_\alpha = (\langle X \rangle, \mathcal{O}_{\langle X \rangle}^{\text{int}}).
\]

For brevity we write $\text{sp}_{\alpha}: \langle X \rangle \to X_\alpha$ in place of $\text{sp}_{X_\alpha}$.

**Proof of Theorem 5.4.2.** In the situation as in 5.4.3, we may assume that $\langle X \rangle$ is the projective limit along the directed set $L$. Then theorem follows from the formal results 0.4.2.1 and 0.4.2.2. \qed

For a (not necessarily coherent) rigid space $X$ we have the functor

\[
\text{Mod}_X^{\text{int}} \to \text{Mod}_X, \quad \mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X^{\text{int}}} \mathcal{O}_X,
\]

where the last $\mathcal{O}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_X^{\text{int}}} \mathcal{O}_X$ is, by a slight abuse of notation, often denoted by $\mathcal{F}^{\text{rig}}$ (resp. $\varphi^{\text{rig}}$) (and similarly for morphisms in $\text{Mod}_X^{\text{int}}$).

**Definition 5.4.4.** Let $X$ be a coherent universally Noetherian rigid space.

1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. An integral model of $\mathcal{F}$ is a pair of the form $(\widetilde{\mathcal{F}}, \varphi)$, where $\widetilde{\mathcal{F}}$ is a finitely presented $\mathcal{O}_X^{\text{int}}$-module and $\varphi$ is an isomorphism $\varphi: \widetilde{\mathcal{F}}^{\text{rig}} \to \mathcal{F}$.

2. Let $\Phi: \mathcal{F} \to \mathcal{G}$ be a morphism of coherent $\mathcal{O}_X$-modules. An integral model of $\Phi$ is a datum of the form $(\widetilde{\Phi}: \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}, \varphi, \psi)$ consisting of

- a morphism $\widetilde{\Phi}: \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ of $\mathcal{O}_X^{\text{int}}$-modules, where $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are finitely presented,

- and isomorphisms $\varphi: \widetilde{\mathcal{F}}^{\text{rig}} \to \mathcal{F}$ and $\psi: \widetilde{\mathcal{G}}^{\text{rig}} \to \mathcal{G}$.
such that the following square is commutative:

\[
\begin{array}{ccc}
\tilde{F} \text{rig} & \overset{\Phi \text{rig}}{\longrightarrow} & \tilde{G} \text{rig} \\
\varphi \downarrow & & \downarrow \psi \\
F & \overset{\Phi}{\longrightarrow} & G.
\end{array}
\]

**Definition 5.4.5.** Let $X$ be a coherent rigid space, and $I$ an ideal of definition of $\langle X \rangle$ of finite type. A morphism $\varphi : F \rightarrow G$ of $\mathcal{O}_X^{\text{int}}$-modules is said to be a weak isomorphism if there exists an integer $s \geq 0$ such that $I^s \ker \varphi = 0$ and $I^s \coker \varphi = 0$.

**Theorem 5.4.6.** Let $X$ be a coherent universally Noetherian rigid space. Then the functor

\[
\text{Mod}_{\mathcal{O}_X^{\text{int}}}^{\text{FP}} \longrightarrow \text{Coh}_X^{\text{rig}},
\]

where $\text{Mod}_{\mathcal{O}_X^{\text{int}}}^{\text{FP}}$ denotes the category of finitely presented $\mathcal{O}_X^{\text{int}}$-modules, induces a categorical equivalence

\[
\text{Mod}_{\mathcal{O}_X^{\text{int}}}^{\text{FP}}/\{\text{weak isomorphisms}\} \sim \text{Coh}_X^{\text{rig}},
\]

where the left-hand category is the localized category by the set of all weak isomorphisms. Moreover, for a locally free $\mathcal{O}_X$-module $F$ one can find a locally free $\mathcal{O}_X^{\text{int}}$-module $\tilde{F}$ of the same rank such that $\tilde{F}^{\text{rig}} \cong F$.

The theorem says that any morphism $\Phi$ of coherent $\mathcal{O}_X$-modules has an integral model, uniquely determined up to weak isomorphism. The proof can be done by combining 5.2.1 and 5.4.2 and using Exercise II.5.2. The details are left to the reader. Notice that, since $\mathcal{O}_X$ is coherent, any finitely presented $\mathcal{O}_X$-module is coherent.

**Exercises**

**Exercise II.5.1.** Under the notation as in 5.1.10, suppose that $X$ is universally adhesive and that there exists an $I_X$-torsion free $\mathcal{O}_X$-module $\mathcal{M}$ that contains $F_X$ and $G_X$ as $\mathcal{O}_X$-submodules. Show that both $F_X + G_X$ and $F_X \cap G_X$ are lattice models of $F$.

**Exercise II.5.2.** Let $X$ be a coherent universally Noetherian rigid space, and $F$ a locally free $\mathcal{O}_X$-module of finite rank. Show that there exists a lattice model $(X, F_X)$ of $F$ such that $F_X$ is a locally free $\mathcal{O}_X$-module of the same rank.
6 Affinoids

In this section we discuss the so-called affinoids, which are, by definition, coherent rigid spaces having an affine formal model. Our approach to the basic geometry of affinoids arises from the following viewpoint. Since an affinoid $\mathcal{X}$ has a formal model of the form $X = \text{Spf} A$, which is the $I$-adic completion of the scheme Spec $A$, the geometries of the rigid space $\mathcal{X}$ should reflect the geometries of the scheme Spec $A \setminus V(I)$, the complement of the closed subset defined by an ideal of definition. In §6.2 we will see this in the context of morphisms between affinoids; roughly speaking, if $A$ and $B$ are t.u. adhesive, then morphisms $\mathcal{X} = (\text{Spf} A)^\text{rig} \to \mathcal{Y} = (\text{Spf} B)^\text{rig}$ between affinoids always come from morphisms of affine formal models of the form $\text{Spf} A' \to \text{Spf} B$, where $A'$ is finite and isomorphic outside $I$ over $A$. In §6.3 we will see that, in the context of coherent sheaves, the category of coherent sheaves on an affinoid $\mathcal{X} = (\text{Spf} A)^\text{rig}$, where $A$ is a t.u. rigid-Noetherian ring, is equivalent to the category of coherent sheaves on the Noetherian scheme Spec $A \setminus V(I)$.

Subsection §6.4 is devoted to the calculation of the cohomologies of coherent sheaves on affinoids, where we find that the cohomologies on an affinoid $\mathcal{X} = (\text{Spf} A)^\text{rig}$ can be calculated by means of the cohomologies of the corresponding coherent sheaf on the scheme Spec $A \setminus V(I)$. This fundamental result for the calculation of cohomology on locally universally Noetherian rigid spaces leads us to the notion of Stein affinoids, which is a genuine analogue of the classical Stein domains in complex analytic geometry or of affine schemes in algebraic geometry. In fact, any locally universally Noetherian rigid spaces can have an open covering consisting of Stein affinoids and, moreover, such coverings, called Stein affinoid coverings, are cofinal in the set of all coverings. Due to the basic Theorem A and Theorem B proved in §6.5, Stein affinoid coverings are Leray coverings for computing the cohomologies of coherent sheaves, and this fact, as in the classical complex analytic geometry, algebraic geometry, etc., provides the foundations for the calculation of cohomology on rigid spaces. Note that, in the classical situation, every affinoid is a Stein affinoid.

In the final subsection, §6.6, we focus on the comparison between universally Noetherian affinoids and their associated schemes, that is, the Noetherian schemes of the form ‘Spec $A \setminus V(I)$’ as above. This will give us a useful bridge between local geometries of rigid spaces and those of schemes.

6.1 Affinoids and affinoid coverings

6.1. (a) Definition and basic properties

Definition 6.1.1. A coherent rigid space $\mathcal{X}$ is called an affinoid if there exists a formal model $(X, \phi)$ of $\mathcal{X}$ with $X$ affine.
We denote by $\mathbf{ARf}$ the full subcategory of $\mathbf{CRf}$ consisting of affinoids.

**Definition 6.1.2.** Let $\mathcal{X}$ be a rigid space.

(1) An **affinoid open subspace** of $\mathcal{X}$ is an isomorphism class over $\mathcal{X}$ of objects $\mathcal{U} \hookrightarrow \mathcal{X}$ in the small site $\mathcal{X}_{\text{ad}}$ (2.2.24) such that $\mathcal{U}$ is an affinoid. For a point $x \in (\mathcal{X})$, an **affinoid neighborhood** of $x$ is an affinoid open subspace $\mathcal{U} \hookrightarrow \mathcal{X}$ such that the image of $\langle \mathcal{U} \rangle$ contains $x$.

(2) An **affinoid covering** of $\mathcal{X}$ is a covering

$$\bigsqcup_{\alpha \in L} \mathcal{U}_\alpha \longrightarrow \mathcal{X}$$

of the site $\mathcal{X}_{\text{ad}}$ such that each $\mathcal{U}_\alpha$ is an affinoid.

As the following proposition shows, any rigid space has an open basis consisting of affinoid open subspaces.

**Proposition 6.1.3.** Let $\mathcal{X}$ be a rigid space, $x \in (\mathcal{X})$, and $\mathcal{V}$ an open neighborhood of $x$ in $(\mathcal{X})$. Then there exists an affinoid neighborhood $\mathcal{U} \hookrightarrow \mathcal{X}$ of $x$ such that the image of $\langle \mathcal{U} \rangle$ is contained in $\mathcal{V}$.

**Proof.** We may assume $\mathcal{X}$ is coherent. Moreover, since the topology on $(\mathcal{X})$ is generated by quasi-compact open subsets, we may assume that $\mathcal{V}$ is a quasi-compact open neighborhood of $x$. Take a formal model $\mathcal{X}$ of $\mathcal{X}$ that admits a quasi-compact open subset $V \subseteq X$ such that $\text{sp}_X^{-1}(V) = \mathcal{V}$, and an affine open neighborhood $U \subseteq V$ of $\text{sp}_X(x)$. Then $\mathcal{U} = U^\text{rig}$ admits an open immersion $\mathcal{U} \hookrightarrow \mathcal{X}$ enjoying the desired properties. \qed

An affinoid $\mathcal{X}$ is said to be **distinguished** if it is of the form $\mathcal{X} = (\text{Spf } A)^\text{rig}$ for an $I$-torsion free $A$, where $I \subseteq A$ is an ideal of definition. If $\mathcal{X} = (\text{Spf } A)^\text{rig}$ is an affinoid, then considering the admissible blow-up along a finitely generated ideal of definition $I \subseteq A$, one finds that $\mathcal{X}$ is covered by distinguished affinoids. Hence any rigid space has an open basis consisting of distinguished affinoid open subspaces.

**Corollary 6.1.4.** Let $\mathcal{X}$ be a rigid space. Then any covering of the (coherent) small admissible site $\mathcal{X}_{\text{ad}}$ is refined by an affinoid covering consisting of distinguished affinoids.

If a rigid space $\mathcal{X}$ is locally universally Noetherian (resp. locally universally adhesive), then, clearly, $\mathcal{X}$ has an open basis consisting of affinoid open subspaces by affinoids of the form $(\text{Spf } A)^\text{rig}$, where $A$ is a t.u. rigid-Noetherian (resp. t.u. adhesive) ring (I.2.1.1). In this situation, in view of the following proposition, the difference between affinoids and distinguished affinoids is not important.
Proposition 6.1.5. Let $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$ be an affinoid, where $A$ is a t.u. rigid-Noetherian (resp. t.u. adhesive) ring, and $I \subseteq A$ an ideal of definition. Then $A' = A/A_{I, \text{tor}}$ is again a t.u. rigid-Noetherian (resp. t.u. adhesive) ring, and we have $\mathcal{X} = (\text{Spf } A')^{\text{rig}}$. In particular, $\mathcal{X}$ is a distinguished affinoid.

Proof. By 0.7.4.18, $A_{I, \text{tor}}$ is closed in $A$ with respect to the $I$-adic topology. Hence, $A' = A/A_{I, \text{tor}}$ is $I$-adically complete and hence is t.u. rigid-Noetherian (resp. t.u. adhesive). Let $Y \to \text{Spf } A$ be the admissible blow-up along $I$, and $Y' \to \text{Spf } A'$ the admissible blow-up along $IA'$. The latter map coincides with the strict transform of the former by the closed immersion $\text{Spf } A' \hookrightarrow \text{Spf } A$. Since $I\mathcal{O}_Y$ is invertible, it follows that $Y' \cong Y$, that is, $\text{Spf } A$ and $\text{Spf } A'$ are dominated by a common admissible blow-up. □

Convention. In the sequel, whenever we discuss an affinoid $\mathcal{X}$ that we know from the context to be universally Noetherian (resp. universally adhesive), it is always supposed, unless otherwise clearly stated, to be of the form $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$ for a t.u. rigid-Noetherian (resp. t.u. adhesive) ring $A$.

The above convention will not be necessary, if one can show the following statement (for which we do not know the proof): If $\mathcal{X}$ is universally Noetherian and $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$, then $A$ is t.u. rigid-Noetherian.

6.1. (b) Affinoid subdomains

Definition 6.1.6. An affinoid open subspace of an affinoid $\mathcal{X}$ is called an 
affinoid subdomain of $\mathcal{X}$.

Examples 6.1.7 (examples of affinoid subdomains; cf. [18], (7.2.3/2) and (7.2.3/5)). Let $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$ be an affinoid, and $I = (a) \subseteq A$ an ideal of definition.

1. (Weierstrass subdomain) Let $f_1, \ldots, f_m \in \Gamma(\text{Spec } A \setminus V(I), \mathcal{O}_{\text{Spec } A}) = A[\frac{1}{a}]$. We can find a positive integer $k$ such that $a^k f_i \in A$ for $i = 1, \ldots, m$. Set $J = (a^k, a^k f_1, \ldots, a^k f_m)$. Then $\mathcal{J} = J^A$ is an admissible ideal of $X = \text{Spf } A$. Consider the admissible blow-up $\pi: X' \to X$ along $\mathcal{J}$, and let $U$ be the affine open part of $X'$, where $\mathcal{J}\mathcal{O}_U$ is generated by $a$:

$$U = \text{Spf } A[(\frac{a^k f_1}{a^k}, \ldots, \frac{a^k f_m}{a^k})]/I\text{-tor}.$$ 

The associated rigid space $U^{\text{rig}}$ is an affinoid subdomain of $\mathcal{X}$, denoted by

$$\mathcal{X}(f) = \mathcal{X}(f_1, \ldots, f_m)$$

and called a Weierstrass subdomain.
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(2) (Laurent subdomain) In the above situation, we moreover take \( g_1, \ldots, g_n \in A[1/d] \) and \( l \geq 1 \) such that \( a^l g_j \in A \) for \( j = 1, \ldots, n \). Set \( \mathcal{K}_j = (a^l, a^l g_j)^\Delta \), which are admissible ideals on \( X \). Define \( U_j (j = 1, \ldots, n) \) inductively as follows: \( U_0 = U \); let \( \pi_j : X'_j \to U_{j-1} \) be the admissible blow-up along the ideal \( \mathcal{K}_j \mathcal{O}_{U_{j-1}} \), and \( U_j (j = 1, \ldots, n) \) the affine open part of \( X'_j \) on which \( a^l g_j \) generates the ideal \( \mathcal{K}_j \mathcal{O}_{X'_j} \). Then the rigid space \( U_n^{rig} \) is an affinoid subdomain of \( X \), denoted by

\[
\mathcal{X}(f, g^{-1}) = \mathcal{X}(f_1, \ldots, f_m, g_1^{-1}, \ldots, g_n^{-1})
\]

and called a Laurent subdomain.

(3) (rational subdomain) Take elements \( f_1, \ldots, f_m, g \in A[1/d] \) such that the ideal \( (f_1, \ldots, f_m, g) \) of the ring \( A[1/d] \) is the unit ideal. As before, fix \( k \geq 1 \) such that \( a^k f_1, \ldots, a^k f_m, a^k g \in A \). Then \( J = (a^k f_1, \ldots, a^k f_m, a^k g) \) is an open ideal, and hence \( \mathcal{J} = J^\Delta \) is an admissible ideal on \( X = \text{Spf} A \). Take the admissible blow-up \( X' \to X \) along \( \mathcal{J} \), and let \( U \) be the affine part of \( X' \), where the ideal \( \mathcal{J} \mathcal{O}_{X'} \) is generated by \( a^k g \):

\[
U = \text{Spf} A((\frac{a^k f_1}{a^k g}, \ldots, \frac{a^k f_m}{a^k g})/I\text{-tor}.
\]

The associated rigid space \( U^{rig} \) is an affinoid subdomain of \( X \), denoted by

\[
\mathcal{X}(\frac{f}{g}) = \mathcal{X}(\frac{f_1}{g}, \ldots, \frac{f_m}{g})
\]

and called a rational subdomain.

6.2 Morphisms between affinoids

**Definition 6.2.1.** Let \( A \to A' \) be an adic morphism of adic rings of finite ideal type, and \( I \subseteq A \) an ideal of definition. We say that the map \( A \to A' \) (or \( \text{Spf } A' \to \text{Spf } A \)) is an isomorphism outside \( I \) if the morphism \( \text{Spec } A' \setminus V(IA') \to \text{Spec } A \setminus V(I) \) of schemes is an isomorphism.

Let \( A \) be an adic ring of finite ideal type with a finitely generated ideal of definition \( I \subseteq A \), and \( \pi : X' \to X = \text{Spf } A \) an admissible blow-up. We have a ring homomorphism \( A \to A' = \Gamma(X', \mathcal{O}_{X'}) \) (cf. Exercise I.1.9). We consider the \( IA'\)-adic topology on \( A' \).

**Proposition 6.2.2.** Suppose that \( A \) is t.u. rigid-Noetherian. Then

1. \( A \to A' \) is a weak isomorphism (I.C.1.2).
2. \( A' \) endowed with the \( IA'\)-adic topology is rigid-Noetherian (I.2.1.1 (1)).
The morphism $X' \to \text{Spf } A'$ is an admissible blow-up. In particular, we have $(\text{Spf } A')^{\text{rig}} \cong (\text{Spf } A')^{\text{rig}}$.

Moreover, if $A$ is t.u. adhesive (I.2.1.1 (2)) and $I$-torsion free, then $A'$ is finite over $A$ (and hence is again t.u. adhesive).

Proof. (1) It is clear (due to GFGA comparison) that $A$ and $A'$ are isomorphic outside $I$ (that is, we have $\text{Spec } A' \setminus V(IA') \cong \text{Spec } A \setminus V(I)$); in particular, $A'$ is Noetherian outside $I$. Hence, the kernel of $A \to A'$ is $I$-torsion; since $A_{I-\text{tor}}$ is bounded $I$-torsion, the kernel is also bounded $I$-torsion. Let $J = (f_0, \ldots, f_r) \subseteq A$ be an admissible ideal that gives the admissible blow-up $\pi$, and $X' = \bigcup_{i=0}^r U_i$ the affine covering as in §1.1.(b). It is clear that the cokernel of each map $A \to U_i$ is bounded $I$-torsion, and hence the cokernel of $A \to A'$ is also bounded $I$-torsion.

(2) Since it is already shown that $A'$ is Noetherian outside $IA'$, it suffices to show that $A'$ is $IA'$-adic complete, which follows from Exercise I.C.1.

(3) By the universal mapping property (1.1.4 (3)), the morphism $X' \to \text{Spf } A'$ coincides with the admissible blow-up along the admissible ideal $JA'$. If $A$ is t.u. adhesive and $I$-torsion free, then by 5.3.6 (1) the sheaf $\mathcal{O}_{X'}$ is a coherent $\mathcal{O}_X$-module, and hence $A' = \Gamma(X, \mathcal{O}_{X'})$ is a coherent $A$-module (I.3.5.6).

Definition 6.2.3. Let $f : A \to A'$ be an adic morphism between adic rings of finite ideal type. We say that $f$ is a strict weak isomorphism if it is a weak isomorphism and induces an isomorphism $(\text{Spf } A')^{\text{rig}} \cong (\text{Spf } A)^{\text{rig}}$ of rigid spaces.

It follows from 6.1.5 that if $A$ is a t.u. rigid-Noetherian ring and $I \subseteq A$ is an ideal of definition, then $A \to A' = A/A_{I-\text{tor}}$ is a strict weak isomorphism. The above proposition shows that if $A$ is t.u. rigid-Noetherian, then any admissible blow-up $X' \to \text{Spf } A$ induces the strict weak isomorphism $A \to A' = \Gamma(X', \mathcal{O}_{X'})$.

Proposition 6.2.4. Let $p : X' = \text{Spf } A' \to X = \text{Spf } A$ be a morphism of affine adic formal schemes of finite ideal type, and suppose that $p$ is finite (I.4.2.2) and an isomorphism outside $I$. Then $A \to A'$ is a strict weak isomorphism. In particular, a finite weak isomorphism between adic rings is a strict weak isomorphism.

Proof. Let $q : Y' = \text{Spec } A' \to Y = \text{Spec } A$ be such that $\tilde{q} = p$. Then by I.4.2.1 the morphism $q$ is finite. Set $U = Y \setminus V(I)$, which is a quasi-compact open subset of $Y$. By [89], Première partie, (5.7.12), (included below in E.1.9), there exists a $q^{-1}(U)$-admissible blow-up $W \to Y'$ such that the composition $W \to Y' \to Y$ is an $U$-admissible blow-up. Taking the $I$-adic completions, we get the admissible blow-ups $Z = \tilde{W} \to \tilde{Y}' = X'$ and $Z \to \tilde{Y} = X$. \qed
**Proposition 6.2.5.** Consider a morphism between affinoids

\[ \varphi: (\text{Spf } A)^{\text{rig}} \to (\text{Spf } B)^{\text{rig}}, \]

where \( A \) and \( B \) are t.u. rigid-Noetherian rings.

(1) There exists a diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{} & A \\
\downarrow & & \downarrow \\
B & \xleftarrow{} & A \\
\end{array}
\]

consisting of adic morphisms of adic rings such that

(a) \( A' \) is a rigid-Noetherian ring,

(b) \( A \to A' \) is a strict weak isomorphism, and

(c) the induced morphism \( (\text{Spf } A)^{\text{rig}} \xrightarrow{\sim} (\text{Spf } A')^{\text{rig}} \to (\text{Spf } B)^{\text{rig}} \) coincides with the morphism \( \varphi \).

If, moreover, \( \varphi \) is an isomorphism, then the above diagram can be taken such that \( B \to A' \) is also a strict weak isomorphism.

(2) If \( A \) is t.u. adhesive, and is \( I \)-torsion free, where \( I \subseteq A \) is an ideal of definition, then the above diagram can be taken such that \( A \to A' \) is a finite weak isomorphism (and hence \( A' \) is again t.u. adhesive). If, moreover, \( \varphi \) is an isomorphism and \( B \) is a \( J \)-torsion free t.u. adhesive ring (where \( J \subseteq B \) is an ideal of definition), then the diagram can be taken such that \( B \to A' \) is also a finite weak isomorphism.

**Proof.** The morphism \( \varphi \) comes from a diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{} & \text{Spf } A \\
\downarrow & & \downarrow \\
\text{Spf } B & \xleftarrow{} & \text{Spf } A \\
\end{array}
\]

where the left-hand arrow is an admissible blow-up. Set \( A' = \Gamma(X', \mathcal{O}_{X'}) \). Then, as we have seen in 6.2.2, the ring \( A' \) with the \( IA' \)-adic topology is a rigid-Noetherian ring, and the map \( A \to A' \) is a strict weak isomorphism, since both \( \text{Spf } A \) and \( \text{Spf } A' \) are dominated by \( X' \). Moreover, we have the canonical map \( B \to A' \) (cf. Exercise I.1.9). If \( \varphi \) is an isomorphism, then one can take a diagram as above such that both \( X' \to \text{Spf } A \) and \( X' \to \text{Spf } B \) are admissible blow-ups (2.1.4). Hence, in this case, \( B \to A' \) is also a strict weak isomorphism. The rest of the assertion follow from 6.2.2. \qed
Proposition 6.2.6. Let $A$ be an $I$-torsion free t.u. rigid-Noetherian, where $I \subseteq A$ is an ideal of definition of $A$ such that $\text{Spec } A \setminus V(I)$ is affine. Set $\mathcal{X} = (\text{Spf } A)^\text{rig}$ and $\text{Spec } B = \text{Spec } A \setminus V(I)$. Then the integral closure $A^\text{int}$ of $A$ in $B$ is canonically isomorphic to $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^\text{int})$.

The proposition follows immediately from 6.2.4 and 6.2.5 if the ring $A$ is t.u. adhesive. The general case follows from GFGA comparison and the following observation. Let $A$ be a ring with a finitely generated ideal $I$. We assume that $A$ is $I$-torsion free, and $U = X \setminus V(I)$, where $X = \text{Spec } A$ is affine; we set $B = \Gamma(U, \mathcal{O}_U)$. We need to compare the integral closure $A^\text{int}$ of $A$ in $B$ with the ring

$$\tilde{A} = \lim_{X' \to X} \Gamma(X', \mathcal{O}_{X'}) ,$$

where $X'$ runs over all $U$-admissible blow-ups (cf. E.1.4 below) of $X$.

Lemma 6.2.7. We have $\tilde{A} = A^\text{int}$.

Proof. The inclusion $A^\text{int} \subseteq \tilde{A}$ can be shown by an argument similar to that in the proof of 6.2.4. For any $f \in \tilde{A}$, take a $U$-admissible blow-up $\pi: X' \to X$ such that $I \mathcal{O}_{X'}$ is invertible and $f \in \Gamma(X', \mathcal{O}_{X'})$. Set $Z = \text{Spec } A[f]$. We want to show that $Z \to X$ is finite. Since $A[f]$ is $I$-torsion free, $U$ is dense in $Z$. We have a factorization $X' \xrightarrow{h} Z \xrightarrow{g} X$ of $\pi$. Now since $h$ is proper and hence $h(X')$ is closed and contains $U$, $h(X') = Z$, that is, $h$ is surjective. Hence $g$ is proper. Since $g$ is proper and affine, it is finite (cf. F.4.1 below), as desired.

Theorem 6.2.8. Let $A$ and $B$ be adic rings with finitely generated ideals of definition $I \subseteq A$ and $J \subseteq B$, respectively. Suppose that $A$ and $B$ are t.u. rigid-Noetherian (resp. t.u. adhesive and that $A$ is $I$-torsion free and $B$ is $J$-torsion free). Then the canonical map

$$\left\{ \begin{array}{c|c}
 p & A' \\
 \downarrow & \downarrow q \\
 A & B
\end{array} \right\} \sim \longrightarrow \text{Hom}_{\text{CRf}}(\mathcal{X}, \mathcal{Y}).$$

is a bijection. Here the equivalence relation $\sim$ on the left-hand side is defined as follows: $(A \to A' \leftarrow B)$ and $(A \to A'' \leftarrow B)$ are equivalent if there exists a third diagram $(A \to A'' \leftarrow B)$ as above sitting in the commutative diagram

\[
\begin{array}{c}
\text{A' } \\
\downarrow \quad \quad \quad \downarrow \\
A \to A'' \leftarrow B \\
\downarrow \quad \quad \quad \downarrow \\
\text{A''}.
\end{array}
\]
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Proof. In view of 6.2.5, we only need to show the injectivity. Suppose we are given a diagram

\[
\begin{array}{ccc}
A' & \rightarrow & B \\
\downarrow & & \downarrow \\
A & \rightarrow & A''
\end{array}
\]

where the left-hand maps are strict (resp. finite) weak isomorphisms. Since we have \((\text{Spf } A')^{\text{rig}} = (\text{Spf } A'')^{\text{rig}}\), there exists admissible blow-ups \(X'' \rightarrow \text{Spf } A'\) and \(X'' \rightarrow \text{Spf } A''\). Moreover, one can choose such an \(X''\) that admits an admissible blow-up \(X'' \rightarrow \text{Spf } A\) compatible with the above two admissible blow-ups. Now, applying 6.2.2, we have the diagram as in the theorem.

6.3 Coherent sheaves on affinoids

Let \(\mathcal{X} = (\text{Spf } A)^{\text{rig}}\) be an affinoid, where \(A\) is a t.u. rigid-Noetherian ring, and let \(I \subseteq A\) be a finitely generated ideal of definition. Set \(X = \text{Spf } A\). Recall that there exists a canonical exact categorical equivalence

\[
\text{Mod}_{\text{FP}}^{\text{rig}} \rightarrow \text{Mod}_{\text{FP}}^X, \quad M \mapsto M^\Delta
\]

(1.3.5.6). On the other hand, if we set \(Y = \text{Spec } A\), then by [53], (1.4.2) and (1.4.3), we have an exact categorical equivalence

\[
\text{Mod}_{\text{FP}}^{\text{rig}} \rightarrow \text{Mod}_{\text{FP}}^Y, \quad M \mapsto \tilde{M}.
\]

Thus we have an exact equivalence of the categories

\[
\text{Mod}_{\text{FP}}^X \rightarrow \text{Mod}_{\text{FP}}^Y, \quad M^\Delta \mapsto \tilde{M}.
\]

Set \(U = Y \setminus V(I)\), and consider the diagram of categories

\[
\begin{array}{cccc}
\text{Coh}_{\mathcal{X}} & \leftarrow & \text{Mod}_{\text{FP}}^X & \rightarrow \text{Mod}_{\text{FP}}^Y & \rightarrow \text{Coh}_U,
\end{array}
\]

where the last functor is given by the restriction to \(U\), \(\mathcal{F} \mapsto \mathcal{F}|_U\). By 5.3.1, for any object \(\mathcal{F}\) (resp. arrow \(\varphi\)) in \(\text{Coh}_{\mathcal{X}}\) there exists up to weak isomorphisms an object \(\mathcal{F}_X\) (resp. an arrow \(\varphi_X\)) in \(\text{Mod}_{\text{FP}}^X\) such that \(\mathcal{F}_X^{\text{rig}} = \mathcal{F}\) (resp. \(\varphi_X^{\text{rig}} = \varphi\)), which gives rise to an object \(\mathcal{F}_U\) (resp. arrow \(\varphi_U\)) in \(\text{Coh}_U\). It is clear that this gives a well-defined functor

\[
\text{Coh}_{\mathcal{X}} \rightarrow \text{Coh}_U.
\]

(*)
Theorem 6.3.1. The functor \((\ast)\) is an exact categorical equivalence.

Proof. By 5.3.1, it suffices to show the equivalence

\[
\text{Mod}^{\text{FP}}_Y / \{\text{weak isomorphisms}\} \sim \text{Coh}_U,
\]

where by a weak isomorphism we mean a morphism \(\varphi: \mathcal{F} \to \mathcal{G}\) in \(\text{Mod}^{\text{FP}}_Y\) whose kernel and cokernel are annihilated by \(I^s\) for some \(s > 0\). But then, only essential surjectivity calls for a proof, which follows from [53], (6.9.11).

\[\square\]

Notation 6.3.2. The proof of 6.3.1 shows that we have a quasi-inverse functor

\[\text{Coh}_U \longrightarrow \text{Coh}_X\]

defined similarly to \((\ast)\), denoted by \(\mathcal{F} \mapsto \mathcal{F}^{\text{rig}}\) by a slight abuse of notation.

In general, for a ringed space \((X, \mathcal{O}_X)\) with coherent structure sheaf, a coherent sheaf \(\mathcal{F}\) is said to be Noetherian if for any point \(x \in X\) there exists an open neighborhood \(U\) such that any increasing sequence \(\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \cdots\) of coherent subsheaves of \(\mathcal{F}|_U\) terminates, that is, there exists a number \(N\) such that \(\mathcal{G}_N = \mathcal{G}_{N+1} = \ldots\)

\(^2\)By 6.3.1, we immediately have the following corollary.

Corollary 6.3.3. Let \(X\) be a locally universally Noetherian rigid space. Then the sheaf \(\mathcal{O}_X\) is Noetherian.

6.4 Comparison theorem for affinoids

Theorem 6.4.1 (comparison theorem for affinoids). Let \(X = (\text{Spf} A)^{\text{rig}}\) be an affinoid, where \(A\) is a t.u. rigid-Noetherian ring (I.2.1.1), and set \(X = \text{Spf} A\). Consider the affine scheme \(Y = \text{Spec} A\), and set \(U = Y \setminus V(I)\), where \(I \subseteq A\) is a finitely generated ideal of definition. Then for any coherent sheaf \(\mathcal{F}\) on the Noetherian scheme \(U\), we have a canonical isomorphism

\[
H^q(X, \mathcal{F}^{\text{rig}}) \cong H^q(U, \mathcal{F})
\]

for each \(q \geq 0\).

Proof. In view of 6.1.5, we may assume without loss of generality that \(A\) is \(I\)-torsion free. Let \(I_X = I^A\) and \(I = (\text{sp}^{-1} X) \mathcal{O}_X^{\text{int}}\). Let \(\mathcal{G}_Y\) be a finitely presented sheaf on \(Y\) such that \(\mathcal{G}_Y|_U = \mathcal{F}\) ([53], (6.9.11)). The formal completion \(\mathcal{F}_X = \mathcal{G}_Y\)

\(^2\)Note that the notion of Noetheriness for sheaves as defined here is different from the one in [66], Definition 11.1.1 (iii). Note also that, if the topological space \(X\) is locally coherent, then the open set \(U\) in the definition can be arbitrary coherent open neighborhood of \(x\).
gives a finitely presented formal model of $\mathcal{F}^\text{rig}$ on $X$. Set $\tilde{\mathcal{F}} = \operatorname{sp}_X^* \mathcal{F}_X$, which is an $\mathcal{O}_X^{\text{int}}$-module on $(X)$. By 5.1.3 we have

$$H^q(\mathcal{X}, \mathcal{F}^\text{rig}) = H^q(\mathcal{X}, \lim_{n \geq 0} \mathcal{H}\text{om}(I^n, \tilde{\mathcal{F}})) = \lim_{n \geq 0} H^q(\mathcal{X}, \mathcal{H}\text{om}(I^n, \tilde{\mathcal{F}})),$$

where the last equality is due to 0.3.1.16.

Set $I_Y = \tilde{I}$, which is a coherent ideal of $\mathcal{O}_Y$. Let $\pi: X' \to X$ be the admissible blow-up along $I_X$, and $p: Y' \to Y$ the blow-up of the scheme $Y$ along $I_Y$. Then $\pi$ is the formal completion of $p$. Set $I_{X'} = (\pi^{-1} I_X)\mathcal{O}_{X'}$, which is an invertible ideal of definition. Starting with $X_0 = X'$, we construct the system $\{X_\alpha\}_{\alpha \in L}$ as in 5.4.3. For any $\alpha$ one can take the corresponding blow-up $Y_\alpha \to Y' = Y_0$, whose formal completion is $X_\alpha \to X' = X_0$. Moreover, for $\alpha \leq \beta$ there exists a morphism $p_{\alpha \beta}: Y_\beta \to Y_\alpha$ of schemes (in fact, a blow-up) whose formal completion coincides with $\pi_{\alpha \beta}$. For any $\alpha \in L$, let $I_\alpha = (\pi_\alpha^{-1} I_{X'})\mathcal{O}_{X_\alpha}$, and $\mathcal{F}_\alpha$ the pull-back of $\mathcal{F}_X$, which is the formal completion of the sheaf $\mathcal{G}_\alpha$, the pull-back of $\mathcal{G}_Y$ by the map $Y_\alpha \to Y$. Then, by 3.2.5, $I = \operatorname{sp}_X^* I_\alpha$ for any $\alpha \in L$. By [53], 0. (5.4.9), and 0.4.4.3 we calculate

$$\lim_{n \geq 0} H^q(\mathcal{X}, \mathcal{H}\text{om}(I^n, \tilde{\mathcal{F}})) = \lim_{n \geq 0} H^q(\mathcal{X}, \operatorname{sp}_0^* \mathcal{H}\text{om}_{\mathcal{O}_{X'}}(I_0^n, \mathcal{F}_0))$$

$$= \lim_{n \geq 0} \lim_{\alpha \in L} H^q(\mathcal{X}_\alpha, \mathcal{H}\text{om}_{\mathcal{O}_{X_\alpha}}(I_\alpha^n, \mathcal{F}_\alpha)).$$

By the comparison theorem (I.C.3.3),

$$\lim_{\alpha \in L} \lim_{n \geq 0} H^q(\mathcal{X}_\alpha, \mathcal{H}\text{om}_{\mathcal{O}_{X_\alpha}}(I_\alpha^n, \mathcal{F}_\alpha)) = \lim_{\alpha \in L} \lim_{n \geq 0} H^q(Y_\alpha, \mathcal{H}\text{om}_{\mathcal{O}_{Y_\alpha}}(\mathcal{I}_\alpha^n, \mathcal{G}_\alpha))$$

$$= \lim_{\alpha \in L} H^q(Y_\alpha, \lim_{n \geq 0} \mathcal{H}\text{om}_{\mathcal{O}_{Y_\alpha}}(\mathcal{I}_\alpha^n, \mathcal{G}_\alpha)),$$

where $\mathcal{I}_\alpha$ is the invertible ideal of $\mathcal{O}_{Y_\alpha}$ corresponding to $I_\alpha$.

Set $U_\alpha = Y_\alpha \setminus V(\mathcal{I}_\alpha)$. Since $Y_\alpha$ is $\mathcal{G}_\alpha$-torsion free, $U_\alpha$ is a non-empty scheme. By Deligne’s formula (cf. Exercise II.3.1) we have

$$\lim_{n \geq 0} \mathcal{H}\text{om}_{\mathcal{O}_{Y_\alpha}}(\mathcal{I}_\alpha^n, \mathcal{G}_\alpha) = j_\alpha^* j_\alpha^* \mathcal{G}_\alpha,$$

where $j_\alpha: U_\alpha \hookrightarrow Y_\alpha$ is the open immersion. Since the blow-up $Y_\alpha \to Y = \text{Spec } A$ is isomorphic on $U = \text{Spec } A \setminus V(I)$, and since $U \hookrightarrow Y_\alpha$ is an affine morphism, we have by I.7.1.1 (2)

$$\lim_{\alpha \in L} H^q(Y_\alpha, \lim_{n \geq 0} \mathcal{H}\text{om}_{\mathcal{O}_{Y_\alpha}}(\mathcal{I}_\alpha^n, \mathcal{G}_\alpha)) = H^q(U, \mathcal{G}_Y|_U).$$

Consequently, we have $H^q(\mathcal{X}, \mathcal{F}^\text{rig}) = H^q(U, \mathcal{F})$, as desired. \qed
Corollary 6.4.2. Let $\mathcal{X}$ be a coherent universally Noetherian rigid space. The following conditions are equivalent.

(a) $\mathcal{X}$ is reduced (3.2.12).

(b) There exists a cofinal collection $\{X_\lambda\}_{\lambda \in \Lambda}$ of formal models of $\mathcal{X}$ consisting of reduced distinguished formal models (2.1.8).

(c) Any distinguished formal model of $\mathcal{X}$ is reduced.

Here a formal scheme is said to be reduced if it is reduced as a ringed space; see 0, §4.1. (a).

Proof. Suppose $\mathcal{X}$ is reduced, and let $X$ be a distinguished formal model. By 6.4.1, for any affine open subset $\text{Spf} \ A$ of $X$, the ring $H^0(\text{Spec} \ A \setminus V(I), \mathcal{O}_{\text{Spec} \ A})$ (where $I$ is an ideal of definition) is reduced. Hence the nilradical of $A$ is an $I$-torsion ideal, which must be equal to zero, since $A$ is $I$-torsion free. Thus we have (a) $\implies$ (c). Implication (c) $\implies$ (b) is trivial. If (b) holds, then by 6.4.1 for any point $x \in (\mathcal{X})$ the stalk $\mathcal{O}_{\mathcal{X},x}$ is a reduced ring (since a filtered inductive limit of reduced rings is reduced), whence (b) $\implies$ (a). \hfill $\square$

6.5 Stein affinoids

6.5. (a) Stein affinoids and Stein affinoid coverings

Proposition 6.5.1. Let $\mathcal{X} = (\text{Spf} \ A)^{\text{rig}}$ be an affinoid, where $A$ is a t.u. rigid-Noetherian ring. Set $X = \text{Spf} \ A$. The following conditions are equivalent.

(a) $H^1(\mathcal{X}, \mathcal{F}) = 0$ for any coherent $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$ (that is, $\mathcal{X}$ is ‘Stein’).

(b) $H^q(\mathcal{X}, \mathcal{F}) = 0$ for $q \geq 1$ and for any coherent $\mathcal{O}_\mathcal{X}$-module $\mathcal{F}$.

(c) $\text{Spec} \ A \setminus V(I)$ is an affine scheme, where $I$ is a finitely generated ideal of definition of $A$.

(d) There exists an $I'$-torsion free t.u. rigid-Noetherian ring $A'$ (where $I' \subseteq A'$ is a finitely generated ideal of definition) such that $\mathcal{X} = (\text{Spf} \ A')^{\text{rig}}$ and that $\text{Spec} \ A' \setminus V(I')$ is affine.

Proof. First we show (c) $\implies$ (d). Set $A' = A/A_{I-\text{tor}}$. Then, by 6.1.5, we have $\mathcal{X} = (\text{Spf} \ A')^{\text{rig}}$. Since $\text{Spec} \ A' \hookrightarrow \text{Spec} \ A$ is affine, $\text{Spec} \ A' \setminus V(IA')$ is affine, whence (d).

Next, let us show (d) $\implies$ (b). We work with the notation as in the proof of 6.4.1. Take a coherent sheaf $\mathcal{H}$ on the Noetherian affine scheme $U' = \text{Spec} \ A' \setminus V(I')$ such that $\mathcal{F} = \mathcal{H}^{\text{rig}}$; this is possible due to 6.3.1. Then, by 6.4.1, we have $H^q(\mathcal{X}, \mathcal{F}) = H^q(U', \mathcal{H})$. If $q > 0$, then the right-hand cohomology is zero due to 0.5.4.2 (1), whence (b).
(b) \implies (a) is clear. It remains to show (a) \implies (c). Set \( U = \text{Spec} \ A \setminus V(I) \).

We are to show that the Noetherian scheme \( U \) is affine. To this end, by Serre’s criterion ([54], II, §5.2), it suffices to show that for any coherent sheaf \( \mathcal{H} \) on \( U \) its first cohomology group on \( U \) vanishes. But this follows from 6.3.1 and 6.4.1. \( \square \)

In 6.5.1, it is important to assume that \( \mathcal{X} \) is an affinoid; in fact, there exists an example of non-affinoid rigid space on which the higher cohomologies of any coherent sheaf vanish [77].

**Definition 6.5.2.** A coherent universally Noetherian rigid space \( \mathcal{X} \) is called a **Stein affinoid** if it is an affinoid of the form \( \text{Spf} \ A/\rig \) for a t.u. rigid-Noetherian ring \( A \) that satisfies the conditions in 6.5.1.

**Example 6.5.3.** Let \( R \) be an integrally closed Noetherian local domain of dimension \( \geq 2 \), and set \( I = \mathfrak{m}_R \). Suppose \( R \) is \( I \)-adically complete. Then \( R \) is a t.u. adhesive ring, and the rigid space \( \mathcal{X} = (\text{Spf} \ R)^\rig \) is an affinoid, but is not a Stein affinoid.

**Proposition 6.5.4.** Let \( \mathcal{X} \) be a locally universally Noetherian rigid space. Then for any point \( x \in (\mathcal{X}) \) there exists an affinoid open neighborhood \( \mathcal{U} \hookrightarrow \mathcal{X} \) given by a Stein affinoid \( \mathcal{U} \). Moreover, such affinoid open neighborhoods are cofinal in the set of all open neighborhoods of \( x \).

**Proof.** In the proof of 6.1.3, replace \( X \) by the admissible blow-up along an ideal of definition. Since the ideal of definition of \( X \) is invertible, the scheme \( \text{Spec} \ A \setminus V(I) \) for any affine open subset \( U = \text{Spf} \ A \) of \( X \), where \( I \subseteq A \) is the ideal of definition, is affine. \( \square \)

Clearly, if the rigid space \( \mathcal{X} \) in the setting 6.5.4 is locally universally adhesive, then one can take \( \mathcal{U} \) as above to be of the form \( (\text{Spf} \ A)^\rig \) for a t.u. adhesive ring \( A \).

**Definition 6.5.5.** Let \( \mathcal{X} \) be a locally universally Noetherian rigid space.

1. A **Stein affinoid open subspace** of \( \mathcal{X} \) is an isomorphism class over \( \mathcal{X} \) of objects \( \mathcal{U} \hookrightarrow \mathcal{X} \) in the small site \( \mathcal{X}_{\text{ad}} \) (2.2.24) such that \( \mathcal{U} \) is a Stein affinoid.

2. A **Stein affinoid covering** of \( \mathcal{X} \) is a covering

\[
\bigsqcup_{\alpha \in L} \mathcal{U}_\alpha \longrightarrow \mathcal{X}
\]

of the site \( \mathcal{X}_{\text{ad}} \) such that each \( \mathcal{U}_\alpha \) is a Stein affinoid.

The second assertion of 6.5.4 readily yields the following result.

**Proposition 6.5.6.** Let \( \mathcal{X} \) be a locally universally Noetherian rigid space. Then any admissible covering of \( \mathcal{X} \) is refined by a Stein affinoid covering.


6.5. (b) Theorem A and Theorem B

Theorem 6.5.7 (Theorem A and Theorem B). Let $\mathcal{X}$ be a Stein affinoid, and $\mathcal{F}$ a coherent $\mathcal{O}_\mathcal{X}$-module.

(1) The sheaf $\mathcal{F}$ is generated by global sections. If $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$, where $A$ is an I-torsion free t.u. rigid-Noetherian ring ($I \subseteq A$ is an ideal of definition) such that $U = \text{Spec } A \setminus V(I)$ is affine, then there exists a finitely presented $A$-module $M$ such that $M^{\text{rig}} \cong \mathcal{F}$ and

$$H^0(\mathcal{X}, \mathcal{F}) = \lim_{n \geq 0} \text{Hom}_{A}(I^n, M).$$

In particular, $\mathcal{F} \mapsto \Gamma(\mathcal{X}, \mathcal{F})$ gives the quasi-inverse to the functor

$$\text{Coh}_U \longrightarrow \text{Coh}_\mathcal{X}$$

as in 6.3.2.

(2) For $q > 0$ we have $H^q(\mathcal{X}, \mathcal{F}) = 0$.

Proof. (2) is already proved in 6.5.1. As in the proof of 6.5.1, we have $H^q(\mathcal{X}, \mathcal{F}) = H^q(U, \mathcal{H})$, where $\mathcal{H}$ is a coherent sheaf on $U$ such that $\mathcal{F} = \mathcal{H}^{\text{rig}}$ (cf. 6.3.1). If $q = 0$, it is isomorphic to $\lim_{n \geq 0} \text{Hom}_{A}(I^n, M)$ by Deligne’s formula. □

6.6 Associated schemes

6.6. (a) Definition and functoriality. Let $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$ be a universally Noetherian affinoid, where $A$ is a t.u. rigid-Noetherian ring with a finitely generated ideal of definition $I \subseteq A$. We set

$$s(\mathcal{X}) = \text{Spec } A \setminus V(I),$$

which is a Noetherian scheme. It follows from 6.2.5 that $s(\mathcal{X})$ does not depend on the choice of $A$. The Noetherian scheme $s(\mathcal{X})$ thus defined is said to be the associated scheme to the universally Noetherian affinoid $\mathcal{X}$.

Let $\mathcal{Y} = (\text{Spf } B)^{\text{rig}}$ be another universally Noetherian affinoid (where $B$ is a t.u. rigid-Noetherian ring), and $\varphi: \mathcal{Y} \rightarrow \mathcal{X}$ a morphism of rigid spaces. Then by 6.2.5 we have the canonically induced morphism

$$s(\varphi): s(\mathcal{Y}) \longrightarrow s(\mathcal{X})$$

of schemes. Thus the mapping $s: \mathcal{X} \mapsto s(\mathcal{X})$ defines a functor $s$ from the category of all universally Noetherian affinoid to the category of Noetherian schemes.
Proposition 6.6.1. (1) Let \( i : \mathcal{Y} \rightarrow \mathcal{X} \) be an open immersion between universally Noetherian affinoids. Then the morphism \( s(i) : s(\mathcal{Y}) \rightarrow s(\mathcal{X}) \) is flat.

(2) Let \( \{ \mathcal{U}_\alpha \}_{\alpha \in L} \) be a finite affinoid covering of a universally Noetherian affinoid \( \mathcal{X} \). Then the induced morphism \( \coprod_{\alpha \in L} s(\mathcal{U}_\alpha) \rightarrow s(\mathcal{X}) \) is faithfully flat.

Note that the affinoids \( \mathcal{U}_\alpha \) in (2) are also assumed to be of the form \( \mathcal{U}_\alpha = (\text{Spf} \, A_\alpha)^{\text{rig}} \), where \( A_\alpha \) is a t.u. rigid-Noetherian ring (cf. our convention in the end of \( \S 6.1. (a) \)).

Proof. Let \( \mathcal{Y} = (\text{Spf} \, B)^{\text{rig}} \hookrightarrow \mathcal{X} = (\text{Spf} \, A)^{\text{rig}} \) be an open immersion between universally Noetherian affinoids (where \( A \) and \( B \) are t.u. rigid-Noetherian rings). By 3.1.3 and 1.1.9, we have a diagram of the form

\[
\begin{array}{ccc}
U & \xrightarrow{\phi} & X \\
\downarrow & & \downarrow \\
\text{Spf} B & & \text{Spf} A
\end{array}
\]

where the vertical arrows are admissible blow-ups, and the horizontal arrow is a quasi-compact open immersion. Since admissible blow-ups over affine formal schemes are algebraizable and since such an admissible blow-up has an open basis consisting of algebraizable affine open subspaces, to show (1) we only need to show the following facts.

(i) For a diagram of the form

\[
\begin{array}{ccc}
\text{Spec} \, R & \xleftarrow{\phi} & X \\
\downarrow & & \downarrow \\
\text{Spec} \, A
\end{array}
\]

where the horizontal arrow is an open immersion and the vertical map is a blow-up along an admissible ideal of \( A \), the map

\[
\text{Spec} \, \hat{R} \setminus V(I \hat{R}) \longrightarrow \text{Spec} \, A \setminus V(I)
\]

is flat (where \( I \subseteq A \) is a finitely generated ideal of definition).

(ii) For a set of finitely many diagrams \( \{ \text{Spec} \, R_\alpha \hookrightarrow X \rightarrow \text{Spec} \, A \}_{\alpha \in L} \) of the form as in (i) (with a fixed \( X \rightarrow \text{Spec} \, A \)) such that \( \{ \text{Spf} \, \hat{R}_\alpha \}_{\alpha \in L} \) covers the formal completion of \( X \), the morphism

\[
\coprod_{\alpha \in L} \text{Spec} \, \hat{R}_\alpha \setminus V(I \hat{R}_\alpha) \longrightarrow \text{Spec} \, A \setminus V(I)
\]

is faithfully flat.
In (i), since $R$ is universally rigid-Noetherian, we know that the completion map $R \to \widehat{R}$ is flat, and hence the assertion is clear. In (ii), the map in question is flat due to (i). Hence we only need to show that the map is surjective, and hence only to show that all closed points are in the image (due to the going-down theorem; cf. [81], Theorem 9.5). For any closed point $x \in \text{Spec} \ A \setminus V(I)$ one has a valuation ring $V$ and a map $\text{Spec} \ V \to \text{Spec} \ A$ such that the image of the generic point is $x$ and the image of the closed point lies in $V(I)$. By the valuative criterion, we have a lift $\text{Spec} \ V \to X$, and hence $\text{Spf} \ V \to \widehat{X}$. There exists $\alpha \in L$ such that this map factors though $\text{Spf} \ V \to \text{Spf} \widehat{R}_\alpha$. Then we have the map $\text{Spec} \ V \to \text{Spec} \widehat{R}_\alpha$, the image of the generic point under which lies outside $V(I \widehat{R}_\alpha)$ and is mapped to $x$. This proves (1). The proof of (2) is similar; one can reduce to the situation as in (ii).

6.6. (b) The comparison map. Let $\mathcal{X} = (\text{Spf} \ A)^\text{rig}$ be a universally Noetherian affinoid. The associated scheme $s(\mathcal{X})$ admits a canonical map, called the comparison map,

$$s: \langle \mathcal{X} \rangle \longrightarrow s(\mathcal{X}),$$

constructed as follows.

For $x \in \langle \mathcal{X} \rangle$, take a rigid point $\alpha: \text{Spf} \ V \to \langle \mathcal{X} \rangle$, where $V$ is an $\alpha$-adically complete valuation ring with $\alpha \in m_V \setminus \{0\}$, that maps the closed point to $x$ (3.3.1 (1)). Consider the composition $s_p X \circ \alpha: \text{Spf} \ V \to X = \text{Spf} \ A$, which is an adic morphism of affine formal schemes. The last morphism induces the morphism

$$\text{Spec} \ V \longrightarrow \text{Spec} \ A$$

of affine schemes and hence the morphism

$$\text{Spec} \ V \left[ \frac{1}{\alpha} \right] \longrightarrow \text{Spec} \ A \setminus V(I) = s(\mathcal{X}).$$

Since $V \left[ \frac{1}{\alpha} \right]$ is a field (0.6.7.2), the map (**) gives rise to a point of $s(\mathcal{X})$, which we denote by $s(x)$, and thus we obtain the desired map (**) set-theoretically. Note that in the above construction the point $s(x)$ does not depend on the choice of the rigid point $\alpha$ due to 0.6.7.6 and 0.6.7.3.

Remark 6.6.2. Note that by the above construction we easily deduce the following fact: for $x, x' \in \langle \mathcal{X} \rangle$ such that $x' \in G_x$ (that is, $x'$ is a generization of $x$) we have $s(x) = s(x')$. Hence the map $s$ factors through the separation map (§4.3. (a))

$$\langle \mathcal{X} \rangle \underset{\text{sep}_\mathcal{X}}{\longrightarrow} [\mathcal{X}] \overset{s}{\longrightarrow} s(\mathcal{X})$$

where, by a slight abuse of notation, we denote by the same $s$ the resulting continuous map $[\mathcal{X}] \to s(\mathcal{X})$. 
When $\mathcal{X}$ is a Stein affinoid (6.5.2), the map (*) admits another description as follows. In this case, by 6.4.1 we have

$$s(\mathcal{X}) = \text{Spec} \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}),$$

which is a Noetherian affine scheme. For $x \in \langle \mathcal{X} \rangle$ the associated rigid point $\alpha: \text{Spf} V \to \langle \mathcal{X} \rangle$ induces the map $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to K_x = V[1/a]$, which is nothing but the one induced by the restriction map $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to B_x = \mathcal{O}_{\mathcal{X}, x}$. Hence the point $s(x)$ coincides with the prime ideal that is the pull-back of the maximal ideal of $B_x$ by the last map.

**Proposition 6.6.3.** The set-theoretic map (*) extends canonically to a flat morphism of locally ringed spaces (denoted by the same symbol)

$$s: (\langle \mathcal{X} \rangle, \mathcal{O}_\mathcal{X}) \longrightarrow (s(\mathcal{X}), \mathcal{O}_{s(\mathcal{X})}),$$

where $\mathcal{O}_{s(\mathcal{X})}$ is the structure sheaf of the scheme $s(\mathcal{X})$.

**Proof.** First note that the issue is local on $\mathcal{X}$; more precisely, if $\mathcal{X} = \bigcup_{\alpha \in L} \mathcal{U}_\alpha$ is an affinoid covering, then it suffices to show that the morphism $\langle \mathcal{U}_\alpha \rangle \to s(\mathcal{X})$ extends canonically to a flat morphism of locally ringed spaces for each $\alpha \in L$. Since this morphism factors through $s(\mathcal{U}_\alpha)$ and the morphism $s(\mathcal{U}_\alpha) \to s(\mathcal{X})$ of schemes is flat (6.6.1 (1)), we only have to show that each $\langle \mathcal{U}_\alpha \rangle \to s(\mathcal{U}_\alpha)$ extends canonically to a flat morphism of locally ringed spaces. Thus we may assume that $\mathcal{X}$ is a Stein affinoid. But in this case, $s$ is nothing but the morphism corresponding to the identity map of $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ by the correspondence in [53], (1.6.3), and hence the assertion is clear. 

The following corollary is easy to see by 3.2.15 (1); the last assertion follows from 6.6.1 (1).

**Corollary 6.6.4.** Let $\mathcal{X} = \langle \text{Spf} A \rangle^{\text{rig}}$ be a universally Noetherian affinoid, $x \in \langle \mathcal{X} \rangle$ a point, and $\{\mathcal{U}_\alpha = \langle \text{Spf} A_\alpha \rangle^{\text{rig}}\}_{\alpha \in L}$ a cofinal system of formal neighborhoods of $x$. For each $\alpha \in L$, let $x_\alpha$ be the image of $x$ under the map $s: \langle \mathcal{U}_\alpha \rangle \to s(\mathcal{U}_\alpha)$. Then

$$\mathcal{O}_{\mathcal{X}, x} = \lim_{\alpha \in L} \mathcal{O}_{s(\mathcal{U}_\alpha), x_\alpha}.$$ 

Moreover, for $\alpha \leq \beta$ the transition map $\mathcal{O}_{s(\mathcal{U}_\alpha), x_\alpha} \to \mathcal{O}_{s(\mathcal{U}_\beta), x_\beta}$ is flat.
Proposition 6.6.5. Let $\mathcal{X}$ be a universally Noetherian affinoid. Then the correspondence $\mathcal{F} \mapsto s^* \mathcal{F}$ establishes the categorical equivalence

$$\text{Coh}_s(\mathcal{X}) \sim \text{Coh}_\mathcal{X}.$$ 

Moreover, for any coherent sheaf $\mathcal{F}$ on $s(\mathcal{X})$ we have a canonical isomorphism between the cohomologies

$$H^q(s(\mathcal{X}), \mathcal{F}) \sim H^q(\mathcal{X}, s^* \mathcal{F})$$

for each $q \geq 0$.

Proof. Since $s^* \mathcal{F}$ is nothing but $\mathcal{F}^{\text{rig}}$ in the sense as in 6.3.2, the assertions are rehashes of 6.3.1 and 6.4.1.

Exercises

Exercise II.6.1. Show that the intersection of two Weierstrass (resp. Laurent, resp. rational) subdomains of an affinoid is again a Weierstrass (resp. Laurent, resp. rational) subdomain.

Exercise II.6.2. Let $X$ be a locally universally rigid-Noetherian formal scheme, see (I.2.1.7), and $\pi: X' \to X$ an admissible blow-up. Show that $\pi_* \mathcal{O}_{X'}$ is an a.q.c. $\mathcal{O}_X$-algebra.

Exercise II.6.3. Let $\mathcal{X}$ be a universally Noetherian affinoid, and consider the comparison map $s: (\mathcal{X}) \to s(\mathcal{X})$. Show that the image of $s$ contains all closed points of $s(\mathcal{X})$.

7 Basic properties of morphisms of rigid spaces

In this section we discuss several properties of morphisms between rigid spaces. We give the definitions of those properties and establish some of the basic results, such as base change stability, interrelation with other properties, etc. In discussing separated morphisms and proper morphisms, we also give the valuative criterion. Note that our properness in rigid geometry presented here is the so-called Raynaud properness, that is, the one defined by means of the properness of the formal models. This differs, a priori, from the one introduced by Kiehl in [69], the so-called Kiehl properness. It will be our objective in one of the later chapters to show that, at least in the adhesive case, these notions of properness are actually equivalent. In §7.5. (d) we present the finiteness theorem for proper maps between universally adhesive rigid spaces.
7. Basic properties of morphisms of rigid spaces

7.1 Quasi-compact and quasi-separated morphisms

**Proposition 7.1.1.** Let \( \varphi: X \to Y \) be a morphism of rigid spaces. Then the following conditions are equivalent.

(a) There exists a covering \( \{V_\alpha \to Y\}_{\alpha \in L} \) in the small site \( Y_{ad} \) with each \( V_\alpha \) a coherent rigid space such that \( V_\alpha \times_Y X \) is a quasi-compact rigid space \((3.5.1 \ (1))\) for each \( \alpha \in L \).

(b) For any morphism \( S \to Y \) of rigid spaces with \( S \) quasi-compact, \( S \times_Y X \) is quasi-compact.

(c) The induced map \( \langle \varphi \rangle: \langle X \rangle \to \langle Y \rangle \) is quasi-compact as a map of topological spaces \((cf. \ 0.2.1.4 \ (2))\).

(d) The morphism of small admissible topoi \( \varphi_{ad}: X_{ad} \to Y_{ad} \) is quasi-compact \((cf. \ 0.2.7.7)\).

**Proof.** As (b) \( \implies \) (a) is trivial, let us first show (a) \( \implies \) (b). One easily reduces to the case where \( Y \) and \( S \) are coherent. In this case, \( X \) is quasi-compact and hence is covered by finitely many coherent open rigid subspaces \( U_\alpha \). Then \( S \times_Y X \) is covered by finitely many coherent open rigid spaces \( U_\alpha \times_Y S \) and hence is quasi-compact. To show the equivalence of (a) and (c), again one can reduce to the case where \( Y \) is coherent. Then the issue is to show that \( X \) is quasi-compact if and only if \( \langle X \rangle \) is quasi-compact. But this has been done in 3.5.7. Finally, the equivalence of (c) and (d) follows from 3.4.5. \( \square \)

**Proposition 7.1.2.** Let \( \varphi: X \to Y \) be a morphism of rigid spaces. Then the following conditions are equivalent.

(a) For any morphism \( S \to Y \) from a quasi-separated rigid space \((3.5.1 \ (2))\) \( S \), \( S \times_Y X \) is quasi-separated.

(b) The induced map \( \langle \varphi \rangle: \langle X \rangle \to \langle Y \rangle \) is quasi-separated as a map of topological spaces.

(c) The morphism of small admissible topoi \( \varphi_{ad}: X_{ad} \to Y_{ad} \) is quasi-separated \((cf. \ 0.2.7.7)\).

(d) The diagonal morphism \( X \to X \times_Y X \) is quasi-compact.

**Proof.** By 3.4.5, (b) and (c) are equivalent. Condition (c) is equivalent to (a) when the morphisms \( S \leftrightarrow Y \) are assumed to be an open immersion with \( S \) coherent. By 7.1.1 (b), (c) is still equivalent to (a) without this assumption.

Suppose that (d) holds. To deduce (c), it is enough to show that for any open immersion \( S \leftrightarrow Y \) from a quasi-separated rigid space, the base change \( X_S \) is quasi-separated. To this end, we may assume that \( Y \) is quasi-separated and that \( Y = S \). For any quasi-compact open subspaces \( U, V \subseteq X \), the intersection \( U \times_X V \) coincides with the pull-back of \( U \times_Y V \) by the diagonal, and hence is quasi-compact. Hence \( X \) is quasi-separated, thereby (c).
Conversely, since $Y_{ad}$ has a generating family consisting of coherent open subspaces, we deduce (c) $\implies$ (d), using 7.1.1.

**Definition 7.1.3.** A morphism $\varphi: X \to Y$ of rigid space is said to be **quasi-compact** (resp. **quasi-separated**) if it satisfies the conditions in 7.1.1 (resp. 7.1.2). If $\varphi$ is quasi-compact and quasi-separated, it is said to be **coherent**.

**Remark 7.1.4.** Note that an open immersion $i: \mathcal{U} \hookrightarrow X$ between coherent rigid spaces is always coherent in the above sense; cf. 3.5.4. This again explains the consistency of our terminology ‘coherent open immersion’ (defined in 2.2.2).

The following proposition is easy to see (cf. 0.1.4.1).

**Proposition 7.1.5.** (1) A locally of finite type morphism is of finite type if and only if it is quasi-compact.

(2) The composition of two quasi-compact (resp. quasi-separated, resp. coherent) morphisms is quasi-compact (resp. quasi-separated, resp. coherent).

(3) If $\varphi: X \to X'$ and $\psi: Y \to Y'$ are two quasi-compact (resp. quasi-separated, resp. coherent) morphisms over a rigid space $S$, then the induced morphism $\varphi \times_S \psi: X \times_S Y \to X' \times_S Y'$ is quasi-compact (resp. quasi-separated, resp. coherent).

(4) If $\varphi: X \to Y$ is a quasi-compact (resp. quasi-separated, resp. coherent) morphism over a rigid space $S$ and $S' \to S$ is a morphism of rigid spaces, then the induced morphism $\varphi_\times S': X \times_S S' \to Y \times_S S'$ is quasi-compact (resp. quasi-separated, resp. coherent).

### 7.2 Finite morphisms

**Definition 7.2.1.** (1) A morphism $\varphi: X \to Y$ of coherent rigid spaces is said to be **finite** if it has a finite (I.4.2.2) formal model $f: X \to Y$.

(2) A morphism $\varphi: X \to Y$ of rigid spaces is said to be **finite** if it is coherent (7.1.3) and for any open immersion $\mathcal{V} \hookrightarrow Y$ from a coherent rigid space, the induced morphism $X \times_Y \mathcal{V} \to \mathcal{V}$ between coherent rigid spaces is finite in the sense of (1).

It follows from I.4.2.4 (4) that for a morphism $\varphi: X \to Y$ of coherent rigid spaces the definition (2) of the finiteness is consistent with (1).
Proposition 7.2.2. Let \( \varphi: \mathcal{X} \rightarrow \mathcal{Y} \) be a morphism of coherent universally Noetherian rigid spaces. Then \( \varphi \) is finite if and only if there exists a finite and distinguished formal model \( f: X \rightarrow Y \) of \( \varphi \).

Proof. The ‘if’ part is trivial. Let \( f: X \rightarrow Y \) be a finite formal model of \( \varphi \), and \( Y' \rightarrow Y \) an admissible blow-up such that \( Y' \) is \( I \)-torsion free, where \( I \) is an ideal of definition of \( Y \). Let \( X' \leftarrow X \) be the strict transform (1.2.8). Then by 1.4.2.4 (2) and (4) and 1.4.3.5, the morphism \( X' \rightarrow Y' \) is finite.

Proposition 7.2.3. Let \( \mathcal{Y} \) be a coherent universally Noetherian rigid space, and \( \mathcal{A} \) an \( \mathcal{O}_\mathcal{Y} \)-algebra that is coherent as an \( \mathcal{O}_\mathcal{Y} \)-module. Then there exists a unique up to \( \mathcal{Y} \)-isomorphisms finite morphism \( \varphi: \mathcal{X} \rightarrow \mathcal{Y} \) such that \( \langle \varphi \rangle_\ast \mathcal{O}_\mathcal{X} \cong \mathcal{A} \).

Proof. Let \( Y \) be a universally rigid-Noetherian formal model of \( \mathcal{Y} \). Then by 5.3.1 there exists an \( \mathcal{O}_\mathcal{Y} \)-algebra \( \mathcal{A}_Y \) that is finitely presented as an \( \mathcal{O}_\mathcal{Y} \)-module. Let \( X = \text{Spf} \mathcal{A}_Y \) (I.4.1.9). Then the structural map \( f: X \rightarrow Y \) is finite (I.4.2.6). Let \( \varphi = f^{\text{rig}}: \mathcal{X} = X^{\text{rig}} \rightarrow \mathcal{Y} \) be the associated morphism of coherent rigid spaces. To show that \( \langle \varphi \rangle_\ast \mathcal{O}_\mathcal{X} \cong \mathcal{A} \), we can assume that \( Y \) (and hence \( X \)) is affine, and it suffices to show that \( \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) = \Gamma(\mathcal{Y}, \mathcal{A}) \). But this follows from 6.4.1.

To show the uniqueness, suppose that there is another such finite morphism \( \varphi': \mathcal{X}' \rightarrow \mathcal{Y} \). Let \( X \rightarrow Y \) and \( X' \rightarrow Y \) be universally rigid-Noetherian formal models of \( \mathcal{X} \rightarrow \mathcal{Y} \) and \( \mathcal{X}' \rightarrow \mathcal{Y} \), respectively. Replacing \( Y \) by an affine open subset, we may assume that \( Y \) is affinoid of the form \( \text{Spf}(B)^{\text{rig}} \), where \( B \) is a t.u. rigid-Noetherian ring, and that \( \mathcal{X} \) and \( \mathcal{X}' \) are affinoids of similar form. Let \( X = \text{Spf} A \) (resp. \( X' = \text{Spf} A' \)) be an affine formal model of \( \mathcal{X} \) (resp. \( \mathcal{X}' \)) such that \( A \) (resp. \( A' \)) is a t.u. rigid-Noetherian ring. By 6.4.1, \( \text{Spec} A \) and \( \text{Spec} A' \) are isomorphic outside the closed loci defined by an ideal of definition. By [89], Première partie, (5.7.12) (recorded below in E.1.9), we have admissible blow-ups \( X'' \rightarrow X \) and \( X'' \rightarrow X' \), and thus \( \mathcal{X} \cong \mathcal{X}' \), as desired.

In the situation as in 7.2.3, the morphism \( \varphi \) is called the finite morphism associated to \( \mathcal{A} \). By the uniqueness, one can also define the finite morphism associated to a coherent \( \mathcal{O}_\mathcal{X} \)-algebra on a general rigid space.

Proposition 7.2.4. Let \( \varphi: \mathcal{X} \rightarrow \mathcal{Y} \) be a morphism of locally universally Noetherian rigid spaces. Then \( \varphi \) is finite if and only if it is the finite morphism associated to an \( \mathcal{O}_\mathcal{Y} \)-algebra that is coherent as an \( \mathcal{O}_\mathcal{X} \)-module.

For the proof we need the following lemma.

Lemma 7.2.5. Let \( \varphi: \mathcal{X} \rightarrow \mathcal{Y} \) be a finite morphism of locally universally Noetherian rigid spaces, and \( \mathcal{F} \) a coherent \( \mathcal{O}_\mathcal{X} \)-module. Then \( \langle \varphi \rangle_\ast \mathcal{F} \) is a coherent \( \mathcal{O}_\mathcal{Y} \)-module.
Proof. Take a finite formal model \( f : X \to Y \) of \( \phi \). Since finite morphisms are stable under base change (I.4.2.4 (4)), we may assume that \( Y \) has an invertible ideal of definition \( I \) (replacing \( Y \) by an admissible blow-up along an ideal of definition of finite type). To show that \( (\phi)_* \mathcal{F} \) is coherent, we may work locally and hence may assume that \( Y \) is affine, \( Y = \text{Spf} B \), with the invertible ideal of definition \( I = (a) \). Accordingly (I.4.2.4 (1)), \( X \) is also affine, \( X = \text{Spf} A \). By 6.3.1, the sheaf \( \mathcal{F} \) corresponds to a coherent \( A[\frac{1}{a}] \)-module \( M \). By 6.4.1,

\[
\Gamma((\mathcal{X}), \mathcal{O}_\mathcal{X}) = A[\frac{1}{a}], \quad \Gamma((\mathcal{X}), \mathcal{F}) = M
\]

which are finite over the Noetherian ring \( \Gamma((\mathcal{Y}), \mathcal{O}_\mathcal{Y}) = B[\frac{1}{a}] \), since \( A \) is finite over \( B \) (I.4.2.1). Since this holds for any sufficiently small affinoid open sets of \( Y \), we deduce that \( (\phi)_* \mathcal{F} \) is a coherent \( \mathcal{O}_Y \)-module. \( \square \)

Proof of Proposition 7.2.4. The ‘if’ part is clear. To show the converse, first note that by 7.2.5 the sheaf \( (\phi)_* \mathcal{O}_\mathcal{X} = \mathcal{A} \) is a coherent \( \mathcal{O}_Y \)-module. Then \( \phi \) is isomorphic to the one associated to \( \mathcal{A} \), as one can verify by an argument similar to that in the proof of (the uniqueness of) 7.2.3. \( \square \)

Proposition 7.2.6. (1) The composition of two finite morphisms between locally universally Noetherian rigid spaces is finite.

(2) If \( \phi : \mathcal{X} \to \mathcal{X}' \) and \( \psi : \mathcal{Y} \to \mathcal{Y}' \) are two finite morphisms over a rigid space \( S \), then the induced morphism \( \phi \times_S \psi : \mathcal{X} \times_S \mathcal{Y} \to \mathcal{X}' \times_S \mathcal{Y}' \) is finite.

(3) If \( \phi : \mathcal{X} \to \mathcal{Y} \) is a finite morphism over a rigid space \( S \) and \( S' \to S \) is a morphism of rigid spaces, then the induced morphism \( \phi_{S'} : \mathcal{X} \times_S S' \to \mathcal{Y} \times_S S' \) is finite.

Proof. Statements (2) and (3) follow from I.4.2.4 (3) and (4). (Note that if (1) holds, then properties (2) and (3) are equivalent due to 0.1.4.1.) To show (1), let \( \phi : \mathcal{X} \to \mathcal{Y} \) and \( \psi : \mathcal{Y} \to \mathcal{Z} \) be finite morphisms between coherent rigid spaces. One can take a diagram

\[
X \xrightarrow{f} Y' \xrightarrow{\pi} Y \xrightarrow{g} Z
\]

consisting of coherent universally rigid-Noetherian formal schemes such that

- \( f \) (resp. \( g \)) is a finite formal model of \( \phi \) (resp. \( \psi \)) and
- \( \pi \) is an admissible blow-up.

Since we can work locally on \( \mathcal{Z} \), we can further assume that \( \mathcal{Z} \) is affine, \( \mathcal{Z} = \text{Spec} \ A \). Then, by GFGA existence theorem (I.C.3.5), one can algebraize the diagram into the diagram of schemes

\[
\tilde{X} \xrightarrow{\tilde{f}} \tilde{Y}' \xrightarrow{\tilde{\pi}} \tilde{Y} \xrightarrow{\tilde{g}} \tilde{Z} = \text{Spec} \ A,
\]
where \( \tilde{f} \) and \( \tilde{g} \) are finite morphisms and \( \tilde{\pi} \) is a blow-up. By the algebraic flattening theorem ([89]), there exists a \( U \)-admissible blow-up \( \tilde{Z}' \to Z \) (with \( U = \text{Spec} \ A \setminus V(I) \), where \( I \subseteq A \) is an ideal of definition) such that the strict transform \( \tilde{X}' \) (resp. \( \tilde{Y}'' \)) of \( \tilde{X} \) (resp. \( \tilde{Y}' \)) is finite over \( \tilde{Z}' \). By passage to the \( I \)-adic completions of the resulting strict transforms, we obtain the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y'' \\
& \searrow & \downarrow \psi \\
& & Z'
\end{array}
\]

consisting of finite morphisms such that \( \varphi = (f')^{\text{rig}} \) and \( \psi = (g')^{\text{rig}} \). Since \( g' \circ f' \) is finite, \( \psi \circ \varphi \) is finite, as desired.

**Proposition 7.2.7.** Let \( \varphi: X \to Y \) be a finite morphism of rigid spaces. Then, for any point \( y \in (Y) \), the fiber \( (\varphi)^{-1}(y) \) is a finite set.

**Proof.** Let \( \alpha: \text{Spf} \, \tilde{V}_y \to (Y) \) be the associated rigid point of \( y \) (3.3.5). We need to show that \( (X \times_Y (\text{Spf} \, \tilde{V}_y))^{\text{rig}} \) is a finite set. Hence, replacing \( Y \) by \( (\text{Spf} \, \tilde{V}_y)^{\text{rig}} \), we may assume that \( Y \) is of the form \( (\text{Spf} \, V)^{\text{rig}} \), where \( V \) is an \( a \)-adically complete valuation ring. Then \( X = (\text{Spf} \, A)^{\text{rig}} \), where \( A \) is finite flat over \( V \). Since any morphism from \( A \) to a valuation ring \( V' \) is induced from a unique morphism from \( A_{\text{red}} \) to \( V' \), we may further assume that \( A = A_{\text{red}} \); note that \( A_{\text{red}} \) for a topologically of finite type \( V \)-algebra \( A \) is \( a \)-adically complete. Since \( A \) is finite over \( V \), we have

\[
A \otimes_{\tilde{V}_y} K \cong L_1 \times \cdots \times L_n,
\]

where \( L_i/K \) is a finite extension field for \( i = 1, \ldots, n \). Now, in view of 3.3.6, the points of \( (X) \) are in one-to-one correspondence with the valuation subrings of \( A \otimes_{\tilde{V}_y} K \) containing \( A \) and dominating \( V \). To see that there are only finitely many such objects, it suffices to invoke the classical fact that the number of valuation subrings of the finite extension \( L_i \) \( (i = 1, \ldots, n) \) of \( K \) dominating \( V \) is finite (e.g., [27], Chapter VI, §8.3, Theorem 1).

**7.3 Closed immersions**

**7.3. (a) Definition and first properties**

**Proposition 7.3.1.** Let \( i: Y \to X \) be a morphism of coherent universally Noetherian rigid spaces. Then the following conditions are equivalent.

(a) There exists a formal model \( i: Y \to X \) of \( \varphi \) that is a closed immersion between coherent universally rigid-Noetherian formal schemes.

(b) There exists a formal model \( i: Y \to X \) of \( \varphi \) that is a closed immersion of finite presentation between coherent universally rigid-Noetherian formal schemes.
(c) There exists a cofinal family of formal models \( \{i_\lambda: Y_\lambda \to X_\lambda\} \) of \( i \) consisting of closed immersions between coherent universally rigid-Noetherian formal schemes.

As a preparation for the proof, we first need the following lemma.

**Lemma 7.3.2.** Let \( i: Y \hookrightarrow X \) be a closed immersion of coherent universally rigid-Noetherian formal schemes. Then \( i \) factorizes as

\[
Y \xrightarrow{i'} Y' \xrightarrow{i''} X,
\]

where \( i'' \) is a closed immersion of finite presentation and \( i' \) is a closed immersion defined by a bounded \( I_{Y'} \)-torsion a.q.c. ideal, where \( I_{Y'} \) is an ideal of definition of \( Y' \) (cf. I.4.3.12).

**Proof.** Consider the surjective morphism \( \mathcal{O}_X \to f_*\mathcal{O}_Y \) (I.4.3.8). By I.4.3.12, the kernel \( \mathcal{K} \) of this morphism is an a.q.c. ideal of \( \mathcal{O}_X \). Then using Exercise I.3.4, one finds an a.q.c. subideal \( \mathcal{K}' \subseteq \mathcal{K} \) of finite type such that \( \mathcal{K}/\mathcal{K}' \) is bounded \( I_X \)-torsion, where \( I_X \) is an ideal of definition of finite type on \( X \). Let \( i'': Y' \hookrightarrow X \) be the closed immersion of finite presentation with the defining ideal \( \mathcal{K}' \) (I.4.3.12). Then we have the closed immersion \( i': Y \hookrightarrow Y' \) defined by \( \mathcal{K}/\mathcal{K}' \). \( \square \)

**Proof of Proposition 7.3.1.** Let us show (a) \( \implies \) (b). For a formal model \( i: Y \to X \) of \( i \) that is a closed immersion of coherent universally rigid-Noetherian formal schemes, we obtain a factorization as in 7.3.2. We need to show that the closed immersion \( i'' \) gives another formal model of \( \varphi \). To this end, we perform the admissible blow-up \( Y'' \to Y' \) of \( Y' \) along an ideal of definition of finite type. Such an admissible blow-up comes as the strict transform of the admissible blow-up of \( X \) along the blow-up center of \( Y'' \to Y' \). The strict transform of \( Y \) is then isomorphic to \( Y' \), thereby the claim.

Implication (b) \( \implies \) (c) follows from 1.2.10, 7.3.2, and the fact that the strict transform of a closed immersion by an admissible blow-up is again a closed immersion. Implication (c) \( \implies \) (a) is trivial. \( \square \)

**Remark 7.3.3.** Note that, if the \( X \) in 7.3.1 (a) is coherent universally adhesive; see (I.2.1.7) (and hence so is \( Y \), and \( X \) and \( Y \) are coherent universally adhesive rigid spaces), then one can always replace \( X \) and \( Y \) by distinguished formal models by taking admissible blow-up along an ideal of definition of finite type. In this situation, the closed immersion \( i: Y \hookrightarrow X \) is automatically finitely presented. In particular, one can further assume in (c) that each \( i_\lambda \) is a closed immersion of finite presentation.

**Definition 7.3.4.** A morphism \( i: Y \to X \) of coherent universally Noetherian rigid spaces (2.1.15) is said to be a **closed immersion** if it satisfies the conditions in 7.3.1.
Proposition 7.3.5. Let $X$ be a coherent universally Noetherian rigid space and $\mathcal{K} \subseteq \mathcal{O}_X$ a coherent ideal. Then there exists uniquely up to canonical isomorphisms a closed immersion $Y \hookrightarrow X$ that induces an isomorphism of locally ringed spaces $((Y), \mathcal{O}_Y) \sim (Z, \mathcal{O}_X/\mathcal{K})$, where $Z$ is the support of the sheaf $\mathcal{O}_X/\mathcal{K}$. Moreover, any closed immersion $Y \hookrightarrow X$ is isomorphic to the one obtained in this way by a uniquely determined coherent ideal of $\mathcal{O}_X$.

The uniquely determined coherent ideal for a closed immersion $Y \hookrightarrow X$ stated in the proposition will be called the defining ideal.

Proof. Let $X$ be a coherent universally rigid-Noetherian formal model of $X$, which we assume without loss of generality to have an invertible ideal of definition $I_X$. Then by 5.3.1 we have a finitely presented ideal $K_X$ of $\mathcal{O}_X$ such that $K_X^{\text{rig}} = \mathcal{K}$; indeed, we have a finitely presented sheaf $K_0 = K_X^{\text{rig}}$ with the map $K_0 \rightarrow \mathcal{O}_X$ such that $K_X^{\text{rig}} = K_X$; then $K_X = I_X^n K_X'$ for a sufficiently large $n > 0$ is a finitely presented ideal of $\mathcal{O}_X$ having the same property. Let $i: Y \hookrightarrow X$ be the closed immersion of finite presentation corresponding to $K_0$ (I.4.3.12). Then we have the morphism $i = i^{\text{rig}}: Y = Y^{\text{rig}} \hookrightarrow X$ of coherent rigid spaces.

As in the proof of 7.3.1, one has a cofinal family of formal models $\{i_\lambda: Y_\lambda \rightarrow X_\lambda\}$ of $i$ consisting of closed immersions between coherent universally rigid-Noetherian formal schemes that dominates $i: Y \hookrightarrow X$. We have $\langle Y \rangle = \lim Y_\lambda$.

For each $\lambda$, let $K_{X_\lambda}$ be the defining ideal of the closed immersion $i_\lambda$. Set $\mathcal{K} = \lim sp^{\text{rig}}_{X_\lambda} K_{X_\lambda}$, which is an ideal of $\mathcal{O}_X^{\text{int}}$ such that $\mathcal{K} \otimes \mathcal{O}_X = \mathcal{K}$. Since $\mathcal{O}_X^{\text{int}}/\mathcal{K}$ is easily seen to be $I$-torsion free (where $I$ is an ideal of definition (3.2.3)), we deduce that

$$\text{Supp } \mathcal{O}_X^{\text{int}}/\mathcal{K} = \text{Supp } \mathcal{O}_X/\mathcal{K}.$$ 

Finally, by 0.2.2.13 (3), one sees that the map $\langle Y \rangle \rightarrow \langle X \rangle$ is closed, and then it is easy to see that it is actually an isomorphism onto the closed subset $\text{Supp } \mathcal{O}_X^{\text{int}}/\mathcal{K}$.

To show the last assertion, let $i: Y \hookrightarrow X$ be a closed immersion, and take a formal model $i: Y \hookrightarrow X$ that is a closed immersion of finite presentation between coherent universally rigid-Noetherian formal schemes. Since the question is local on $X$, we may assume that $X$ is affine of the form $X = \text{Spf } A$, where $A$ is a t.u. rigid-Noetherian ring (I.2.1.1). Then the closed immersion $i: Y \hookrightarrow X$ comes from a surjective map $A \rightarrow B$, where $Y = \text{Spf } B$. Let $K$ be the kernel of $A \rightarrow B$. Then $K^{\Delta}$ is an a.q.c. ideal of $\mathcal{O}_X$ that gives the kernel of $\mathcal{O}_X \rightarrow i_* \mathcal{O}_Y$. It is then easy to check that the coherent ideal $\mathcal{K} = (K^{\Delta})^{\text{rig}}$ of $\mathcal{O}_X$ recovers, up to isomorphism, the closed immersion $Y \hookrightarrow X$. The uniqueness is clear. \qed
Corollary 7.3.6. Let \( \iota: Y \to X \) be a morphism of coherent universally Noetherian rigid spaces and \( \{U_\alpha\}_{\alpha \in L} \) a covering in the site \( X_{\text{ad}} \). Then \( \iota \) is a closed immersion if and only if for each \( \alpha \in L \) the base change \( Y \times_X U_\alpha \to U_\alpha \) is a closed immersion.

The last corollary allows one to define closed immersions between locally universally Noetherian rigid spaces (2.2.23) consistently as follows.

Definition 7.3.7. (1) A morphism \( \phi: Y \to X \) of locally universally Noetherian rigid spaces (2.2.23) is called a closed immersion if it is coherent (7.1.3) and \( X \) has a covering \( \{X_\alpha\}_{\alpha \in L} \) by coherent rigid spaces as in 2.2.18, such that for each \( \alpha \in L \) the base change \( Y \times_X X_\alpha \to X_\alpha \) is a closed immersion in the sense of 7.3.4.

(2) A closed rigid subspace of a rigid space \( X \) is an equivalence class (by \( X \)-isomorphisms) of closed immersions \( Y \hookrightarrow X \).

By 7.3.5, any closed immersion as in (1) comes from a uniquely determined coherent ideal, which we also call the defining ideal.

Proposition 7.3.8. A finite morphism \( \iota: Y \to X \) of locally universally Noetherian rigid spaces is a closed immersion if and only if the map \( \mathcal{O}_X \to \iota_* \mathcal{O}_Y \) (cf. 7.2.5) is surjective.

Proposition 7.3.9. Let \( X \) be a Stein affinoid (6.5.2), and \( Y \hookrightarrow X \) a closed immersion. Then \( Y \) is a Stein affinoid.

Proof. Take a formal model \( X = \text{Spf} \, A \) of \( X \) with a t.u. rigid-Noetherian ring \( A \) such that \( \text{Spec} \, A \setminus V(I) \) is affine, where \( I \subseteq A \) is an ideal of definition. As in the proof of 7.3.5, the defining ideal of \( Y \hookrightarrow X \) comes from a finitely presented ideal of \( \mathcal{O}_X \) and thus from a finitely presented ideal \( K \subseteq A \). Set \( B = A/K \) and \( Y = \text{Spf} \, B \). Then the closed immersion \( Y \hookrightarrow X \) of finite presentation gives a formal model of \( Y \hookrightarrow X \). Clearly, \( \text{Spec} \, B \setminus V(IB) \) is affine.

Proposition 7.3.10. Let \( Y \hookrightarrow X \) be a closed immersion of locally universally Noetherian rigid spaces. Then the induced map \( \langle Y \rangle \to \langle X \rangle \) maps \( \langle Y \rangle \) homeomorphically onto an overconvergent closed subset of \( \langle X \rangle \).

Proof. Let \( x \in \langle Y \rangle \) and \( y \) be a generization of \( x \) in \( \langle X \rangle \). We need to show that \( y \) belongs to \( \langle Y \rangle \). Let \( \alpha: \text{Spf} \, \widehat{V}_x \to \langle X \rangle \) and \( \beta: \text{Spf} \, \widehat{W}_x \to \langle Y \rangle \) be the associated rigid points (3.3.5). The surjective map \( \mathcal{O}_{\langle X \rangle, x} \to \mathcal{O}_{\langle Y \rangle, y} \) (cf. 1.4.3.7) gives rise to a local surjective map \( h: V_x \to W_x \). Since \( h \) is, at the same time, \( a \)-adic, it is an injective map (cf. the proof of 0.6.7.6). Hence, \( h \) is an isomorphism, and thus \( \widehat{V}_x \cong \widehat{W}_x \). Now since \( y \) belongs to the image of \( \alpha \), we deduce that \( y \) belongs to \( \langle Y \rangle \), as desired.
Proposition 7.3.11. (1) Any closed immersion is a finite morphism.

(2) If $\varphi: Z \to Y$ and $\psi: Y \to X$ are closed immersions, then so is the composition $\psi \circ \varphi$.

(3) If $\varphi: X \to X'$ and $\psi: Y \to Y'$ are two closed immersions over a rigid space $S$ such that either $X$ and $X'$, or $X$ and $Y'$ are locally of finite type over $S$, then the induced morphism $\varphi \times_S \psi: X \times_S Y \to X' \times_S Y'$ is a closed immersion.

(4) If $\varphi: X \to Y$ is a closed immersion over a rigid space $S$ and $S' \to S$ is a morphism of rigid spaces such that either $X$ and $Y$ are locally of finite type over $S$ or that $S'$ is locally of finite type over $S$, then the induced morphism $\varphi_S: X \times_S S' \to Y \times_S S'$ is a closed immersion.

Proof. (1) is clear. (2) can be shown by an argument similar to that in 7.2.6 (1). (3) follows easily from I.4.3.10 (2). Finally, (4) follows due to 0.1.4.1. \( \square \)

7.3. (b) Irreducible rigid spaces

Definition 7.3.12. We say that a locally universally Noetherian rigid space $X$ is irreducible (or, more adequately, globally irreducible) if the following condition is satisfied: if $\langle X \rangle = \langle Y \rangle \cup \langle Z \rangle$, where $Y$ and $Z$ are closed rigid subspaces (7.3.7) of $X$, then either $\langle X \rangle = \langle Y \rangle$ or $\langle X \rangle = \langle Z \rangle$ holds.

7.3. (c) Open complement

Definition 7.3.13. Let $X$ be a locally universally Noetherian rigid space, and let $Y \hookrightarrow X$ be a closed subspace. The open complement of $Y$ in $X$ is the open subspace $U$ of $X$, denoted by $X \setminus Y$, such that $\langle U \rangle = \langle X \rangle \setminus \langle Y \rangle$.

The Zariski–Riemann space $\langle U \rangle$ of the open complement $U$ is, therefore, an overconvergent open subset of $X$, due to 7.3.10. The actual construction of $U$ is given as follows.

Construction 7.3.14. It is enough to perform the construction in the case where the space $X$ is coherent. Take a cofinal family of formal models $\{i_\lambda: Y_\lambda \to X_\lambda\}$ of $Y \hookrightarrow X$ consisting of closed immersions between coherent universally rigid-Noetherian formal schemes (7.3.1). For any $\lambda$, let $U_\lambda$ be the open complement of $Y_\lambda$ in $X_\lambda$. Then $\bigcup_{\lambda \in A} \text{sp}^{-1}(U_\lambda)$ gives the complement $\langle X \rangle \setminus \langle Y \rangle$.

Now define for any $\alpha$ the coherent rigid space $\mathcal{U}_\alpha = U_\alpha^{\text{rig}}$. For $\lambda \leq \mu$ we have an obvious coherent open immersion $\mathcal{U}_\lambda \hookrightarrow \mathcal{U}_\mu$. Hence $\{\mathcal{U}_\lambda\}$ is an increasing family of coherent rigid spaces and defines a quasi-separated rigid space (stretch of coherent rigid spaces; cf. 2.2.17 (1)) $\mathcal{U} = \bigcup_{\lambda \in A} \mathcal{U}_\lambda$. There exists an obvious open immersion $\mathcal{U} \hookrightarrow X$. By 3.1.9, $\langle \mathcal{U} \rangle = \langle X \rangle \setminus \langle Y \rangle$, as desired.
7.3. (d) Closed subspaces of an affinoid. Let \( \mathcal{X} = (\text{Spf } A)^{\text{rig}} \) be a universally Noetherian affinoid, where \( A \) is a t.u. rigid-Noetherian ring with a finitely generated ideal of definition \( I \subseteq A \). Consider the associated Noetherian scheme \( s(\mathcal{X}) = \text{Spec } A \setminus V(I) \) (see \( \S 6.6.1 \)) and the comparison map \( s: (\mathcal{X}) \to s(\mathcal{X}) \) (see \( \S 6.6.2 \)).

For any closed subscheme \( Z \subseteq s(\mathcal{X}) \), take a closed subscheme \( \tilde{Z} \subseteq \text{Spec } A \) of finite presentation such that \( Z = \tilde{Z} \cap s(\mathcal{X}) \). Take the \( I \)-adic completion \( \hat{Z} \) of \( \tilde{Z} \), which is a closed formal subscheme of \( \text{Spf } A \) of finite presentation. Then we have the closed subspace \( Z = (\hat{Z})^{\text{rig}} \) of \( \mathcal{X} \). By the construction, if \( \mathcal{I} \) is the defining coherent ideal of \( Z \) on \( s(\mathcal{X}) \), then the coherent ideal \( \mathcal{I}^{\text{rig}} (6.3.2) \) gives the defining ideal of \( Z \) on \( \mathcal{X} \). Note that, since the coherent ideal \( \mathcal{I}^{\text{rig}} \) is nothing but the pull-back sheaf \( s^* \mathcal{I} = \mathcal{I}\mathcal{O}_{\mathcal{X}} \) (cf. \( 6.6.3 \) and \( 6.6.5 \)), the locally ringed space \( ((Z), \mathcal{O}_Z) \) is canonically isomorphic to the fiber product

\[
Z \times_{s(\mathcal{X})} (\mathcal{X})
\]

in the category \( \text{LRsp} \).

**Proposition 7.3.15.** (1) For any closed subset \( Z \) (resp. open subset \( U \)) of \( s(\mathcal{X}) \), the pull-back \( s^{-1}(Z) \) (resp. \( s^{-1}(U) \)) in \( (\mathcal{X}) \) is an overconvergent closed (resp. open) subset \( (4.3.2) \).

(2) For any closed subscheme \( Z \) of \( s(\mathcal{X}) \) there exists a unique closed subspace \( Z \) of \( \mathcal{X} \) such that \( ((Z), \mathcal{O}_Z) \) is isomorphic to the fiber product \( Z \times_{s(\mathcal{X})} (\mathcal{X}) \) in the category of locally ringed spaces. Moreover, the correspondence \( Z \mapsto Z \) establishes the bijection between the set of all closed subschemes (resp. irreducible closed subschemes) of \( s(\mathcal{X}) \) and the set of all closed subspaces (resp. irreducible closed subspaces) of \( \mathcal{X} \).

**Proof.** (1) follows from \( 4.3.3 \) and \( 6.6.2 \). Since closed subspaces of \( \mathcal{X} \) are determined by its coherent defining ideal of \( \mathcal{O}_{\mathcal{X}} \), (2) follows from \( 6.6.5 \). \( \square \)

**Notation 7.3.16.** For a closed subscheme \( Z \) of \( s(\mathcal{X}) \), the corresponding closed subspace \( Z \) of \( \mathcal{X} \) as in 7.3.15 (2) is denoted by

\[ s^*Z. \]

On the other hand, for a closed subspace \( Z \) of \( \mathcal{X} \), the corresponding closed subscheme of \( s(\mathcal{X}) \) (consistently) denoted by

\[ s(Z). \]

Note that we have \( s^*Z = s^{-1}(Z) \) as a topological space.
7.4 Immersions

7.4. (a) Immersions and rigid subspaces

Definition 7.4.1. A morphism \( \varphi: Y \rightarrow X \) of locally universally Noetherian rigid spaces (2.2.23) is said to be an immersion if it is a composition \( \varphi = j \circ i \), where \( i \) is a closed immersion and \( j \) is an open immersion.

Proposition 7.4.2. Let \( \varphi: Y \rightarrow X \) be a morphism of locally universally Noetherian rigid spaces, and \( \{ \mathcal{V}_\alpha \leftarrow X \}_{\alpha \in L} \) a family of open immersions such that \( \varphi \) factors through the open immersion \( \bigcup_{\alpha \in L} \mathcal{V}_\alpha \leftarrow X \). Then \( \varphi \) is an immersion if and only if for any \( \alpha \in L \) the base change \( Y \times_X \mathcal{V}_\alpha \rightarrow \mathcal{V}_\alpha \) is an immersion.

Proof. The ‘only if’ part is clear. Let us show the other part. Take an open immersion \( U_\alpha \leftarrow \mathcal{V}_\alpha \) for each \( \alpha \in L \), such that the immersion \( Y \times_X \mathcal{V}_\alpha \rightarrow \mathcal{V}_\alpha \) factors through the closed immersion \( Y \times_X \mathcal{V}_\alpha \leftarrow \mathcal{U}_\alpha \). Set \( \mathcal{U} = \bigcup_{\alpha \in L} \mathcal{U}_\alpha \), which is an open subspace of \( X \). Then by 7.3.6 the morphism \( Y \rightarrow \mathcal{U} \) is a closed immersion.

Proposition 7.4.3. (1) If \( \varphi: Z \rightarrow Y \) and \( \psi: Y \rightarrow X \) are immersions, then so is the composition \( \psi \circ \varphi \).

(2) If \( \varphi: X \rightarrow X' \) and \( \psi: Y \rightarrow Y' \) are two immersions over a rigid space \( S \) such that either \( X \) and \( X' \) or \( X \) and \( Y' \) are locally of finite type over \( S \), then the induced morphism

\[
\varphi \times_S \psi: X \times_S Y \longrightarrow X' \times_S Y'
\]

is an immersion.

(3) If \( \varphi: X \rightarrow Y \) is an immersion over a rigid space \( S \) and \( S' \rightarrow S \) is a morphism of rigid spaces such that either \( X \) and \( Y \) are locally of finite type over \( S \) or that \( S' \) is locally of finite type over \( S \), then the induced morphism

\[
\varphi_{S'}: X \times_S S' \longrightarrow Y \times_S S'
\]

is an immersion.

To show (1), we need to prove the following lemma.

Lemma 7.4.4. Let \( Z \leftarrow Y \) be an open immersion, and \( Y \leftarrow X \) a closed immersion. Then the composition \( Z \leftarrow X \) is an immersion.

Proof. We identify \( Y \) with a closed subspace of \( X \), and \( Z \) with an open subspace of \( Y \). There exists an open subset \( \mathfrak{U} \) of \( (X) \) that contains \( (Z) \) as a closed subset. Let \( \mathcal{U} \subseteq X \) be the open subspace supported on \( \mathfrak{U} \). Then \( Z \leftarrow \mathcal{U} \) is a closed subset given by the defining ideal of \( Y \leftarrow X \) restricted on \( \mathcal{U} \). \( \square \)
Chapter II. Rigid spaces

**Proof of Proposition 7.4.3.** (1) follows from 7.4.4. (2) follows from 7.3.11 (3) and the corresponding (obvious) assertion for open immersions. Due to 0.1.4.1, assertion (3) follows automatically.

**Proposition 7.4.5.** Let \( \varphi: \mathcal{X} \to \mathcal{Y} \) be a morphism of locally universally Noetherian rigid spaces. Then the diagonal morphism \( \Delta \mathcal{X}: \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \) is an immersion.

**Proof.** As in the proof of I.4.6.1, we may assume, in view of 7.4.2, that \( \varphi \) has a formal model of the form \( \text{Spf} \ A \to \text{Spf} \ B \). The diagonal map \( \Delta \mathcal{X} \) in this case is a closed immersion due to I.4.3.7.

**Definition 7.4.6.** Let \( \mathcal{X} \) be a locally universally Noetherian rigid space. A locally closed rigid subspace of \( \mathcal{X} \) is an \( \mathcal{X} \)-isomorphism class of immersions \( \mathcal{Y} \to \mathcal{X} \).

### 7.5 Separated morphisms and proper morphisms

#### 7.5. (a) Closed morphisms

**Proposition 7.5.1.** Let \( \varphi: \mathcal{X} \to \mathcal{Y} \) be a morphism between coherent rigid spaces. Then the following conditions are equivalent.

- (a) \( \langle \varphi \rangle: \langle \mathcal{X} \rangle \to \langle \mathcal{Y} \rangle \) is a closed map.
- (b) For any point \( x \in \langle \mathcal{X} \rangle \),
  \[ \langle \varphi \rangle(\{x\}) = \{\langle \varphi \rangle(x)\}. \]
- (c) Any distinguished formal model \( f: X \to Y \) of \( \varphi \) is closed.
- (d) There exists a cofinal subset \( \mathcal{C} \) of formal models of \( \varphi \) consisting of closed maps.

**Proof.** Implication \( (a) \Rightarrow (b) \) is a consequence from general topology. To show \( (b) \Rightarrow (c) \), let \( f: X \to Y \) be a distinguished formal model of \( \varphi \), and take an ideal of definition \( I_Y \) of finite type of \( Y \). Set \( I_X = I_Y \mathfrak{O}_X \). Denote by \( X_0 \) and \( Y_0 \) the closed subschemes of \( X \) and \( Y \) defined respectively by \( I_X \) and \( I_Y \), and let \( f_0: X_0 \to Y_0 \) be the induced morphism. We need to show that \( f(Z) \) is closed for any closed subset \( Z \subset X \). By [54], II, (7.2.2), it suffices to show that \( f(Z) \) is closed under specialization.

Take \( z = f(x) \in Z \). By 3.1.5, we can find a point \( u \in \langle \mathcal{X} \rangle \) such that \( \text{sp}_X(u) = x \). Since the specialization map \( \text{sp}_X \) is closed (3.1.2 (2)), \( \text{sp}_Y(\langle \varphi(u) \rangle) = \{z\} \). It then follows from our hypothesis that

\[ \text{sp}_Y(\langle \varphi(u) \rangle) = \text{sp}_Y(\langle \varphi(\{u\}) \rangle) = f(\text{sp}_X(\{u\})). \]

Now since \( Z \) is closed, \( \text{sp}_X(\{u\}) \subseteq Z \). Hence, \( \{z\} \subseteq F(Z) \), as desired.

Implication \( (c) \Rightarrow (d) \) is obvious; \( (d) \Rightarrow (a) \) follows from 0.2.2.13 (3).
Definition 7.5.2. A morphism \( \varphi : \mathcal{X} \to \mathcal{Y} \) of rigid spaces is said to be closed if the associated map \( (\varphi) : (\mathcal{X}) \to (\mathcal{Y}) \) is closed.

The following proposition follows immediately from 7.3.5.

Proposition 7.5.3. An immersion between locally universally Noetherian rigid spaces is a closed immersion if and only if it is closed.

Proof. The ‘only if’ part is clear. Let \( \varphi : \mathcal{Y} \hookrightarrow \mathcal{X} \) be an immersion, which we suppose to be closed. By 7.3.6, we may assume that \( \mathcal{X} \) is coherent. Write \( \varphi = j \circ i \), where \( i : \mathcal{Y} \hookrightarrow \mathcal{U} \) is a closed immersion, and \( j : \mathcal{U} \hookrightarrow \mathcal{X} \) is an open immersion. Since the image of \( (\mathcal{Y}) \) in \( (\mathcal{X}) \) is closed, and hence is quasi-compact, we may assume that \( \mathcal{U} \) is quasi-compact, and hence that it is a quasi-compact open subspace of \( \mathcal{X} \). Then one can take, using 1.1.9 if necessary, the coherent rigid-Noetherian distinguished formal models \( Y \hookrightarrow U \hookrightarrow X \) consisting of a closed immersion followed by a coherent open immersion, so that the composite \( Y \hookrightarrow X \) gives a formal model of \( \phi \). By 7.5.1, \( Y \hookrightarrow X \) is closed, and is an immersion. Then by I.4.5.10, one deduces that \( Y \hookrightarrow X \) is a closed immersion, and hence \( \varphi \) is a closed immersion, as desired. \( \square \)

7.5. (b) Separated morphisms and proper morphisms

Definition 7.5.4. Let \( \varphi : \mathcal{X} \to \mathcal{Y} \) be a morphism of rigid spaces.

(1) The morphism \( \varphi \) is said to be universally closed if for any morphism \( \mathcal{Z} \to \mathcal{Y} \) of rigid spaces the base change \( \varphi_{\mathcal{Z}} : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z} \) is closed.

(2) The morphism \( \varphi \) is said to be separated if the diagonal morphism

\[
\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}
\]

is quasi-compact and closed.

(3) The morphism \( \varphi \) is said to be proper if it is of finite type, separated, and universally closed.

Note that, due to 7.4.5 and 7.5.3, a morphism \( \varphi : \mathcal{X} \to \mathcal{Y} \) of locally universally Noetherian rigid spaces (2.2.23) is separated if and only if the diagonal morphism \( \Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \) is a closed immersion. Proposition 7.5.5 follows immediately from the definition.

Proposition 7.5.5. Any separated morphism is quasi-separated.
**Theorem 7.5.6.** Let \( \varphi : X \to Y \) be a finite type morphism of coherent rigid spaces. The following conditions are equivalent.

(a) \( \varphi \) is separated.

(b) Any distinguished formal model \( f : X \to Y \) of \( \varphi \) is separated.

(c) There exists a separated formal model \( f : X \to Y \).

(d) There exists a cofinal set of formal models of \( \varphi \) consisting of separated morphisms.

**Proof.** First let us prove that (a) \( \implies \) (b). Take a distinguished formal model \( f : X \to Y \) of \( \varphi \), and consider the commutative diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & Z \\
\downarrow & & \downarrow \\
X & \longrightarrow & X \times_Y X
\end{array}
\]

where the right vertical map is the admissible blow-up along an ideal of definition of finite type, and the left vertical map is the strict transform. Since \( X \) is a distinguished formal model of \( X \), the map \( X' \to X \), which is again an admissible blow-up, is surjective. Hence, to show that \( X \to X \times_Y X \) is closed, it suffices to show that \( X' \to X \times_Y X \) is closed. Since the admissible blow-up is proper, we only need to show that \( X' \to Z \) is closed. But since this is a distinguished formal model of \( X \to X \times_Y X \), it is closed due to 7.5.1.

Implication (b) \( \implies \) (c) is obvious. To show that (c) \( \implies \) (d), take a separated formal model \( f : X \to Y \) of \( \varphi \). For any formal model \( g : X' \to X' \) of \( \varphi \) there exists a formal model \( h \) dominating \( g \) of the form

\[
\begin{array}{ccc}
X'' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
X & \longrightarrow & X \times_Y X
\end{array}
\]

where \( \pi \) and \( Y'' \to Y \) are admissible blow-ups. The morphism \( h \) is clearly separated (I.4.6.8 (2)), whence the claim.

Finally, implication (d) \( \implies \) (a) follows from the following observation: by 7.5.1 the diagonal morphism \( X \to X \times_Y X \) is closed. \( \square \)

**Proposition 7.5.7.** Let \( \varphi : X \to Y \) be a separated morphism of locally universally adhesive rigid spaces (2.2.23), where \( Y = (\text{Spf } B) \text{rig} \) is a Stein affinoid with \( B \) a t.u. adhesive ring (I.2.1.1). Then for any Stein affinoid open subspaces \( \mathcal{U} = (\text{Spf } P) \text{rig} \) and \( \mathcal{V} = (\text{Spf } Q) \text{rig} \) of \( X \) with \( P \) and \( Q \) t.u. adhesive, \( \mathcal{U} \times_X \mathcal{V} \) is a Stein affinoid open subspace of the form \((\text{Spf } R) \text{rig} \) with \( R \) t.u. adhesive.
Consider the open subspace $U \subset X \times Y$ of $X \times_y X$, which is a Stein affinoid, since we have $U = \text{Spf} P' \times_{\text{Spf} B} \text{Spf} Q'$, where $P'$ and $Q'$ are as in 6.2.5. Note that the schemes $\text{Spec} B \setminus V(I)$ (where $I \subseteq B$ is an ideal of definition), $\text{Spec} P' \setminus V(IP')$, and $\text{Spec} Q' \setminus V(IQ')$ are affine.

First we show that $U$ is a Stein affinoid. It is an affinoid of the form $\text{Spf} P_0 \otimes B Q_0$; the scheme $\text{Spec} P_0 \otimes B Q_0 \setminus V(IP_0 \otimes B Q_0)$ is the pull-back of the affine scheme $\text{Spec} P_0 \otimes B Q_0 \setminus V(IP_0 \otimes B Q_0)$ by the affine map $\text{Spec} P_0 \otimes B Q_0 \longrightarrow \text{Spec} P' \otimes B Q'$ and hence is affine. This shows that $U$ is a Stein affinoid.

Now the space $U \times_y V$ in question is the pull-back of $U$ by the closed immersion $X ! X \times Y$ and hence is a closed subspace of the Stein affinoid $U \times_y V$. Now the assertion follows from 7.3.9.

**Proposition 7.5.8.** Let $\varphi: X \to Y$ be a morphism of locally universally Noetherian rigid spaces, and $\psi: Y \to Z$ a separated and locally of finite type morphism between locally universally Noetherian rigid spaces. Then the graph map

$$\Gamma_\varphi: X \longrightarrow X \times_Z Y$$

is a closed immersion.

**Proof.** Since $Y \to Z$ is locally of finite type, the fiber product $X \times_Z Y$ is locally universally Noetherian. Then the proof goes similarly to I.4.6.12. □

By an argument similar to that in I.4.6.14, we have the following result.

**Proposition 7.5.9.** Let $P$ be a property of morphisms in the category of morphisms in the category of locally universally Noetherian rigid spaces satisfying (I), (C) in 0, §1.5. (b) and the mutually equivalent conditions (B_i) for $i = 1, 2, 3$ with $Q = \text{‘locally of finite type’ in 0, §1.5. (c).}$ Suppose that any closed immersion satisfies $P$. Then the following holds: if $\varphi: X \to Y$ and $\psi: Y \to Z$ are morphisms of locally universally Noetherian rigid spaces such that $\psi \circ \varphi$ satisfies $P$ and $\psi$ is separated and locally of finite type, then $\varphi$ satisfies $P$.

**Corollary 7.5.10.** Let $\varphi: X \to Y$ and $\psi: Y \to Z$ be morphisms of locally universally Noetherian rigid spaces, and suppose $\psi$ is separated and locally of finite type. If $\psi \circ \varphi$ satisfies one of the following conditions, then so does $\varphi$:

(a) locally of finite type (resp. of finite type),
(b) quasi-compact (resp. quasi-separated, resp. coherent),
(c) finite,
(d) closed immersion (resp. immersion),
(e) closed (universally closed).
**Proposition 7.5.11.** (1) The composition of two separated morphisms of locally universally Noetherian rigid spaces is separated.

(2) An open immersion of rigid spaces is separated. A closed immersion of rigid spaces is separated.

(3) Let $\phi : X \to Y$ be a morphism of rigid spaces, and $\{ V_\alpha \to Y \}_{\alpha \in L}$ an admissible covering. Then $\phi$ is separated if and only if $X \times_Y V_\alpha \to V_\alpha$ is separated for each $\alpha \in L$.

**Proof.** (1) Let $\phi : X \to Y$ and $\psi : Y \to Z$ be separated morphisms of rigid spaces. Since the diagonal morphism $Y \to Y \times_Z Y$ is a closed immersion, so is the morphism $X \times_Y X \to X \times_Z X$ (as it is the base change of the former morphism by the canonical morphism $X \times_Z X \to Y \times_Z Y$). The diagonal $X \to X \times_Z X$ coincides with the composite of $X \to X \times_Y X$ followed by $X \times_Y X \to X \times_Z X$, whence the result.

(2) is clear.

(3) Set $U_\alpha = X \times_Y V_\alpha$ for each $\alpha \in L$. Then the diagonal $U_\alpha \to U_\alpha \times_{V_\alpha} U_\alpha$ is the base change of $X \to X \times_Y X$ by the open immersion $U_\alpha \times_{V_\alpha} U_\alpha \to X \times_Y X$. Then taking the associated maps between the Zariski–Riemann spaces, we deduce the desired equivalence of conditions by 7.5.1. \hfill \Box

**Corollary 7.5.12.** All rigid spaces here are supposed to be locally universally Noetherian.

(1) If $\phi : X \to X'$ and $\psi : Y \to Y'$ are two separated morphisms over a rigid space $S$, then the induced morphism $\phi \times_S \psi : X \times_S Y \to X' \times_S Y'$ is separated.

(2) If $\phi : X \to Y$ is a separated morphism over a rigid space $S$ and $S' \to S$ is a morphism of rigid spaces, then the induced morphism $\phi_S : X \times_S S' \to Y \times_S S'$ is separated.

(3) If the composition $\psi \circ \phi$ of two morphisms of rigid spaces is separated, then $\phi$ is separated.

**Proof.** By 7.5.11 (3), we may assume that the rigid spaces in question are all coherent. Then the assertions follow easily from 7.5.6, I.4.6.8, and I.4.6.5. To show (3), we use the fact that an admissible blow-up is separated. Note that (1) and (2) are equivalent due to 0.1.4.1. \hfill \Box
Theorem 7.5.13. Let $\varphi: X \to Y$ be a separated morphism of finite type of coherent rigid spaces. The following conditions are equivalent.

(a) $\varphi$ is proper (7.5.4 (3)).

(b) Any distinguished formal model $f: X \to Y$ of $\varphi$ is proper.

(c) There exists a proper formal model $f: X \to Y$.

(d) There exists a cofinal set of formal models of $\varphi$ consisting of proper morphisms.

Proof. First we show (a) $\implies$ (b). Take a distinguished formal model $f: X \to Y$ of $\varphi$. To see that $f$ is proper, it suffices by Exercise 7.5.4 to show that

$$f \times_Y \text{id}_{\mathbb{A}_{Y}^{N}}: X \times_{Y} \mathbb{A}_{Y}^{N} \to \mathbb{A}_{Y}^{N}$$

is closed for any $N \geq 0$. But the morphism $f \times_Y \text{id}_{\mathbb{A}_{Y}^{N}}$ is nothing but the formal model of $\varphi \times_Y \text{id}_{\mathbb{A}_{Y}^{N}}$ (cf. §2.5.(c)), and so the claim follows from 7.5.1.

Implication (b) $\implies$ (c) is obvious. To show (c) $\implies$ (d), take a proper formal model $f: X \to Y$ of $\varphi$. For any formal model $g: X' \to Y'$ of $\varphi$, there exists a model $h$ dominating $g$ of the form

$$X'' \to X \times_{Y} Y'' \to Y'',$$

where $\pi$ and $Y'' \to Y$ are admissible blow-ups. The morphism $h$ is clearly proper, see I.4.7.5 (3), whence the claim.

Finally, let us show (d) $\implies$ (a). By 7.5.1, the morphism $\varphi$ is closed. It suffices then to see that (d) is preserved after base change. But since (d) implies (c), and since (c) is preserved after base change, the claim follows.

Proposition 7.5.14. (1) A finite morphism is proper.

(2) The composition of two proper morphism between locally universally Noetherian rigid spaces is proper.

(3) Let $\varphi: X \to Y$ be a morphism of rigid spaces, and $\{V_\alpha \leftarrow Y\}_{\alpha \in L}$ a covering in the small admissible site $Y_{ad}$. Then $\varphi$ is proper if and only if $X \times_Y V_\alpha \to V_\alpha$ is proper for each $\alpha \in L$.

Proof. (2) and (3) are easy to see. We show (1). Let $\varphi: X \to Y$ be a finite morphism of rigid spaces. Since properness is a local condition due to (3), we may assume that both $Y$ and $X$ are coherent. Then it is proper by 7.5.13 and I.4.7.4. ✷
Corollary 7.5.15. (1) If \( \varphi: \mathcal{X} \to \mathcal{X}' \) and \( \psi: \mathcal{Y} \to \mathcal{Y}' \) are two proper morphisms over a rigid space \( \mathcal{S} \), then the induced morphism \( \varphi \times_{\mathcal{S}} \psi: \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{X}' \times_{\mathcal{S}} \mathcal{Y}' \) is proper.

(2) If \( \varphi: \mathcal{X} \to \mathcal{Y} \) is a proper morphism over a rigid space \( \mathcal{S} \) and \( \mathcal{S}' \to \mathcal{S} \) is a morphism of rigid spaces, then the induced morphism \( \varphi_{\mathcal{S}'}: \mathcal{X} \times_{\mathcal{S}} \mathcal{S}' \to \mathcal{Y} \times_{\mathcal{S}} \mathcal{S}' \) is proper.

(3) Suppose that the composition \( \psi \circ \varphi \) of two morphisms between rigid spaces is proper. If \( \psi \) is separated, \( \varphi \) is proper.

Proof. We first claim the following fact: if \( \mathcal{X} \to \mathcal{Y} \) is a separated morphism and \( \mathcal{Y} \) is quasi-separated, then \( \mathcal{X} \) is quasi-separated. This indeed follows from 7.5.5. Hence, in particular, if it is proper and \( \mathcal{Y} \) is coherent, then \( \mathcal{X} \) is again coherent. Combining this with 7.5.14 (3), we may assume that the rigid spaces in question are all coherent; then the claim follows easily from 7.5.13 and I.4.7.5. In view of 0.1.4.1 (2) also holds. Finally, (3) can be shown by an argument similar to that in the proof of 7.5.12 (3). \( \square \)

Corollary 7.5.16. Let \( \varphi: \mathcal{X} \to \mathcal{Y} \) be a separated morphism of finite type of rigid spaces. Then \( \varphi \) is proper if and only if \( \varphi \times_{\mathcal{Y}} \text{id}_{\mathcal{D}^N_{\mathcal{Y}}}: \mathcal{X} \times_{\mathcal{Y}} \mathcal{D}^N_{\mathcal{Y}} \to \mathcal{D}^N_{\mathcal{Y}} \) is closed for any \( N \geq 0 \).

Proof. One can assume that \( \mathcal{Y} \) (and hence \( \mathcal{X} \) also) is coherent. Then the corollary follows from the argument in the proof of (a) \( \implies \) (b) of 7.5.13. \( \square \)

7.5. (c) Valuative criterion

Theorem 7.5.17. Let \( \varphi: \mathcal{X} \to \mathcal{Y} \) be a morphism of finite type between locally universally Noetherian rigid spaces.

(1) The morphism \( \varphi \) is separated if and only if the following condition is satisfied: Let \( \alpha: \mathcal{T} \to \mathcal{Y} \) be a rigid point (cf. 3.3.1 (1)) with a generization \( \mathcal{T}' \to \mathcal{T} \), and suppose we are given a commutative diagram

\[
\begin{array}{ccc}
\mathcal{T}' & \xrightarrow{\beta} & \mathcal{X} \\
\downarrow & & \downarrow \varphi \\
\mathcal{T} & \xrightarrow{\alpha} & \mathcal{Y}
\end{array}
\]

Then there exists at most one morphism \( \mathcal{T} \to \mathcal{X} \) making the resulting diagram commutative.

(2) The morphism \( \varphi \) is proper if and only if the following condition is satisfied: Let \( \alpha: \mathcal{T} \to \mathcal{Y} \) be a rigid point with a generization \( \mathcal{T}' \to \mathcal{T} \), and suppose we are given a commutative diagram as above. Then there exists a unique morphism \( \mathcal{T} \to \mathcal{X} \) making the resulting diagram commutative.
Here by a generization of a rigid point $\alpha : \mathcal{T} = (\text{Spf } V)^{\text{rig}} \to \mathcal{Y}$ we mean the rigid point of the form $\mathcal{T}' = (\text{Spf } V')^{\text{rig}} \to \mathcal{T} \to \mathcal{Y}$, where $V' = \hat{\mathcal{V}}_p$ by a prime ideal $p \subseteq V$.

**Proof.** (1) Suppose that $\varphi$ is separated and that a rigid point $\alpha : \mathcal{T} = (\text{Spf } V)^{\text{rig}} \to \mathcal{Y}$ is given. Consider the base change $\mathcal{X} \times_{\mathcal{Y}} \mathcal{T} \to \mathcal{T}$, which is again separated by 7.5.12 (2). We need to show that if $\beta_1$ and $\beta_2$ are two sections having the common generization $\bar{\beta}$ (as in the diagram below), then $\beta_1 = \beta_2$:

\[
\begin{array}{ccc}
\mathcal{T}' & \xrightarrow{\beta} & \mathcal{X} \times_{\mathcal{Y}} \mathcal{T} \\
\downarrow \beta_1, \beta_2 & \searrow \bar{\beta} \\
\mathcal{T} & \xrightarrow{\alpha} & \mathcal{Y}
\end{array}
\]

To show this, we may assume that $\mathcal{Y} = \mathcal{T}$ and thus that $\mathcal{X} = \mathcal{X} \times_{\mathcal{T}} \mathcal{T}$. The two sections define the morphism $\mathcal{T} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{X}$, which, restricted to $\mathcal{T}'$, factors through the diagonal morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{X}$. Since the diagonal morphism is a closed immersion, it gives rise to a closed immersion $\langle \mathcal{X} \rangle \hookrightarrow \langle \mathcal{X} \times_{\mathcal{T}} \mathcal{X} \rangle$. Hence the map $\langle \mathcal{T} \rangle \to \langle \mathcal{X} \times_{\mathcal{T}} \mathcal{X} \rangle$ factors through $\langle \mathcal{X} \rangle$. By 3.3.3, we get a rigid point $\mathcal{T} \to \mathcal{X}$, which coincides with the sections $\beta_1$ and $\beta_2$. Hence $\beta_1 = \beta_2$, as desired.

Suppose, conversely, that the condition in (1) is satisfied. We want to show that the diagonal morphism $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{X}$ is closed. By 7.5.1, this is the case if and only if

$$\langle \Delta_{\mathcal{X}}(\{x\}) \rangle = \langle \Delta_{\mathcal{X}}(x) \rangle$$

for any $x \in \langle \mathcal{X} \rangle$. As the left-hand side is clearly contained in the right-hand side, we need to show that for any $x \in \langle \mathcal{X} \rangle$ and a specialization $y'$ of $y = \langle \Delta_{\mathcal{X}} \rangle(x)$, there exists a specialization $x'$ of $x$ that is mapped to $y'$.

Let $W = \hat{\mathcal{V}}_x$, and let $\beta : \mathcal{T}' = (\text{Spf } \hat{\mathcal{V}}_x)^{\text{rig}} \to \mathcal{X}$ be the rigid point associated to $x$ (3.3.5). Consider the inclusion of valuation rings $\hat{\mathcal{V}}_{y'} \hookrightarrow \hat{\mathcal{V}}_y \hookrightarrow W$; the last injectivity is due to the fact that the map is an adic homomorphism (0.6.7.6). Let $V$ be the integral closure of $\hat{\mathcal{V}}_{y'}$ in $W$. Then $V$ is again an $a$-adically complete valuation ring, and $W$ is a localization of $V$. We have a rigid point

$$\tilde{\beta} : \mathcal{T} = (\text{Spf } V)^{\text{rig}} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}.$$ 

The rigid point $\tilde{\beta}$ gives rise to two rigid points $\tilde{\beta}_1, \tilde{\beta}_2 : \mathcal{T} \to \mathcal{X}$ by projections. Let $\alpha : \mathcal{T} \to \mathcal{Y}$ be the composition $\varphi \circ \tilde{\beta}_1 = \varphi \circ \tilde{\beta}_2$. Then we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{T}' & \xrightarrow{\beta} & \mathcal{X} \\
\downarrow \varphi & \swarrow \alpha \\
\mathcal{T} & \xrightarrow{\varphi} & \mathcal{Y}
\end{array}
\]
together with two $\mathcal{S}$-sections $\tilde{\beta}_1$ and $\tilde{\beta}_2$. Since $\mathcal{S}$ is a generalization of $\mathcal{S}$, $\tilde{\beta}_1 = \tilde{\beta}_2$.

Let $x'$ be the image of the closed point under the map

$$\langle \tilde{\beta}_1 \rangle \rightarrow \langle \tilde{\beta}_2 \rangle: \text{Spf } V \rightarrow \langle \mathcal{X} \rangle.$$ 

Then by the construction the point $x'$ is a specialization of $x$ and is mapped to $y'$ by the map $\langle \Delta \mathcal{X} \rangle$.

(2) We first show the ‘only if’ part. Suppose that $\varphi$ is proper and that the diagram as above is given. Since the properness is stable under base change (7.5.15 (2)), we may assume $\mathcal{T} = \mathcal{Y}$. Then we have the diagram

$$\begin{array}{ccc}
\mathcal{T}' & \xrightarrow{\beta} & \mathcal{X} \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
\mathcal{T} & & 
\end{array}$$

where $\beta$ is a $\mathcal{T}'$-section of $\varphi$, and we need to show that there exists a section $\tilde{\beta}$ of $\varphi$ that extends $\beta$ (the uniqueness follows from separatedness and what we have already shown above). Set $\mathcal{T} = (\text{Spf } V)^{\text{rig}}$ and $\mathcal{T}' = (\text{Spf } W)^{\text{rig}}$, where $W$ is a localization of $V$. Let $x$ be the image of the closed point under the map $\langle \beta \rangle: \text{Spf } W \rightarrow \langle \mathcal{X} \rangle$. Since $\varphi$ is closed, $\langle \varphi \rangle(\{x\}) = \{\langle \varphi \rangle(x)\} = \text{Spf } V$. Hence there exists a specialization $x'$ of $x$ that is mapped to the closed point of $\text{Spf } V$ by $\langle \varphi \rangle$. By 4.1.3 and 3.3.4 (2), the map $\langle \beta \rangle$ factors through the associated rigid point $\text{Spf } \widehat{V}_{x'} \rightarrow \langle \mathcal{X} \rangle$. Hence we have a chain of maps

$$\text{Spf } W \rightarrow \text{Spf } \widehat{V}_{x'} \rightarrow \text{Spf } V;$$

accordingly, we have

$$V \rightarrow \widehat{V}_{x'} \rightarrow W.$$ 

All these maps are injective, since they are adic homomorphisms (0.6.7.6). Moreover, the first map is local; that is, $\widehat{V}_{x'}$ dominates $V$. Since $W$ is a localization of $V$, we have $\text{Frac}(V) = \text{Frac}(\widehat{V}_{x'})$. Hence, $V = \widehat{V}_{x'}$. Thus we get the rigid point $\text{Spf } \mathcal{V} \rightarrow \langle \mathcal{X} \rangle$ and a section $\mathcal{T} \rightarrow \mathcal{X}$, as desired.

Suppose, conversely, that the condition in (2) is satisfied. Due to the uniqueness, $\varphi$ is separated; since the condition in question is stable under base change, it suffices to show that the morphism $\varphi$ is closed. By 7.5.1 this is equivalent to that for any $x \in \langle \mathcal{X} \rangle$ we have the equality

$$\langle \varphi \rangle(\{x\}) = \{\langle \varphi \rangle(x)\}. $$

As the left-hand side is clearly contained in the right-hand side, we need to show that for any $x \in \langle \mathcal{X} \rangle$ and a specialization $y'$ of $y = \langle \varphi \rangle(x)$, there exists a specialization $x'$ of $x$ that is mapped to $y'$. 


Set $W = \widehat{V}_x$, and let

$$\beta: \mathcal{T} = (\text{Spf } W)^{\text{rig}} \longrightarrow \mathcal{X}$$

be the associated rigid point. Similarly to the proof of (1) as above, we have inclusions of valuation rings

$$\widehat{V}_y' \hookrightarrow \widehat{V}_y \hookrightarrow W.$$

Let $V$ be the integral closure of $\widehat{V}_y'$ in $W$. Then we have a rigid point

$$\alpha: \mathcal{T} = (\text{Spf } V)^{\text{rig}} \longrightarrow \mathcal{Y}.$$

We have, by the hypothesis, a $\mathcal{T}$-section $\mathcal{T} \to \mathcal{X}$ of $\varphi$. Let $x'$ be the image of the closed point under the corresponding map $\text{Spf } V \to \langle \mathcal{X} \rangle$. Then $x'$ is a specialization of $x$ that is mapped to $y'$.

\[ \square \]

**7.5. (d) Finiteness theorem.** Let us include here a finiteness theorem of cohomologies of coherent sheaves for proper morphisms of universally adhesive rigid spaces. First we note the following ‘boundedness’ result for cohomologies of separated (7.5.4 (2)) and quasi-compact (7.1.3) morphisms of rigid spaces.

**Proposition 7.5.18.** Let $\varphi: \mathcal{X} \to \mathcal{Y}$ be a separated and quasi-compact morphism between universally adhesive rigid spaces (2.2.23). There exists an integer $r > 0$ such that for any coherent $\mathcal{O}_\mathcal{X}$-module $F$ and any $q \geq r$, we have $R^q \varphi_* F = 0$. If, moreover, $\mathcal{Y}$ is a Stein affinoid, then one can take as $r$ the minimum number of Stein affinoids in a Stein affinoid covering of $\mathcal{X}$.

**Proof:** This follows immediately from that fact that, by 7.5.7 and 6.5.7, a Stein affinoid covering is a Leray covering for coherent sheaves. \[ \square \]

For a rigid space $\mathcal{X}$ we denote by $D^*(\mathcal{X}) (\ast = \text{"""""""\""""""""\""""""""\""""""""\""""""""\""""""""\""""""""\""""""""\""""""""\""""""""\""""""""\""""""""\""""""""

\[ \square \]

**Theorem 7.5.19** (finiteness theorem for proper morphisms). Let $\varphi: \mathcal{X} \to \mathcal{Y}$ be a proper morphism between universally adhesive rigid spaces (2.2.23). Then the functor $R\varphi_*$ maps $D^*_{\text{coh}}(\mathcal{X})$ to $D^*_{\text{coh}}(\mathcal{Y})$ for $\ast = \text{""""""""\""""""""\""""""""\""""""""\""""""""\""""""""\""""""""\""""""""\""""""""\""""""""

To show the theorem, we need the following lemma.
Lemma 7.5.20. Let \( \{X_i, p_{ij}\}_{i \in I} \) and \( \{Y_i, q_{ij}\}_{i \in I} \) be projective systems of ringed spaces indexed by a common directed set \( I \), and \( \{f_i : X_i \rightarrow Y_i\}_{i \in I} \) a morphism of these systems (that is, \( q_{ij} \circ f_j = f_i \circ p_{ij} \) for any \( i \geq j \)) such that

- for any \( i \in I \) the underlying topological spaces of \( X_i \) and \( Y_i \) are coherent (0.2.2.1) and sober (0.2.1.1(b)),
- for any \( i \leq j \) the underlying continuous mapping of the transition maps \( p_{ij} : X_j \rightarrow X_i \) and \( q_{ij} : Y_j \rightarrow Y_i \) are quasi-compact (0.2.1.4(2)), and
- for any \( i \in I \) the underlying continuous mapping of \( f_i : X_i \rightarrow Y_i \) is quasi-compact.

Let \( 0 \in I \) be an index and \( F_0 \) an \( O_{X_0} \)-module. Then the canonical morphism

\[
\lim_{i \geq 0} q_i^* R^q f_i_* p_0^* F_0 \rightarrow R^q f_* p_0^* F_0
\]

is an isomorphism for any \( q \geq 0 \), where \( q_i \) for \( i \in I \) is the canonical projection \( Y = \lim_{\leftarrow j \in I} Y_j \rightarrow Y \).

Proof. Since \( \lim_{i \in I} p_i^{-1} p_0^* F_0 \cong p_0^* F_0 \) by 0.4.2.4, the desired result for \( q = 0 \) follows from 0.3.1.15 and 0.4.2.7. The general assertion can be shown by an argument similar to that in 0, §3.1.(g). \( \square \)

Proof of Theorem 7.5.19. By a standard reduction process similar to that in I, §8.4.(b), it suffices to show that for any coherent \( O_X \)-module \( F \) the sheaf \( R^q \phi_* F \) is a coherent \( O_Y \)-module. For this we may assume that \( Y \) is coherent (and then so is \( X \)). By 5.3.1, there exist a distinguished formal model \( f : X \rightarrow Y \) of \( \phi \) and a coherent \( O_X \)-module \( F_X \) such that \( F_X^\text{rig} = F \). By 7.5.13, the morphism \( f \) is proper. Now applying I.11.1.1, we know that \( R^q f_* F_X \) is a coherent \( O_Y \)-module. Since any distinguished model of \( \phi \) is proper and distinguished models are cofinal among all the formal models, we get the assertion by 7.5.20 and Deligne’s formula (Exercise II.3.1).

\( \square \)

7.6 Projective morphisms

Let \( \mathcal{Y} \) be a coherent universally Noetherian rigid space, and \( \phi : \mathcal{X} \rightarrow \mathcal{Y} \) a proper morphism (hence \( \mathcal{X} \) is coherent). Let \( \mathcal{L} \) be an invertible \( O_\mathcal{X} \)-module. By 5.4.6, one can take an invertible \( O_\mathcal{X}^{\text{int}} \)-module \( \mathcal{L}^+ \) that gives an integral model of \( \mathcal{L} \).

Definition 7.6.1. The sheaf \( \mathcal{L}^+ \) is said to be \( \phi \)-positive, or positive relative to \( \phi \), if there exist a formal model \( f : X \rightarrow Y \) of \( \phi \) and an invertible sheaf \( \mathcal{L}_X \) such that \( \text{sp}_X^* \mathcal{L}_X = \mathcal{L}^+ \) and that \( \mathcal{L}_X \) is ample, that is, \( \mathcal{L}_0 = \mathcal{L}/\mathcal{I}_X \mathcal{L} \) is \( f_0 \)-ample, where \( \mathcal{I}_X \) is an ideal of definition of finite type of \( X \) and \( f_0 : X_0 \rightarrow Y_0 \) is the induced morphism of schemes obtained by mod \( \mathcal{I}_X \)-reduction.
**Definition 7.6.2.** A proper morphism \( \varphi: \mathcal{X} \to \mathcal{Y} \) of coherent universally Noetherian rigid spaces is said to be **projective** if there exists a pair \((\mathcal{L}, \mathcal{L}^+)\) consisting of an invertible \(\mathcal{O}_{\mathcal{X}}\)-module \(\mathcal{L}\) and an invertible integral model of it such that \(\mathcal{L}^+\) is \(\varphi\)-positive.

**Proposition 7.6.3.** Let \( \varphi: \mathcal{X} \to \mathcal{Y} \) be a morphism of coherent universally Noetherian rigid spaces, where \( \mathcal{Y} = (\text{Spf } B)^{\text{rig}} \) is an affinoid with \( B \) t.u. rigid-Noetherian. Then \( \varphi \) is projective if and only if there exist a finitely generated \( B \)-module \( E \) and a \( \mathcal{Y} \)-closed immersion \( \mathcal{X} \hookrightarrow \hat{\mathcal{P}(E)}^{\text{rig}} \).

**Proof.** The ‘if’ part is easy to see. To show the ‘only if’ part, we may assume that \( B \) is \( I \)-torsion free, where \( I \) is an ideal of definition of \( B \). Let \((\mathcal{L}, \mathcal{L}^+)\), where \( \mathcal{L} \) is of an invertible \(\mathcal{O}_{\mathcal{X}}\)-module and \(\mathcal{L}^+\) is an invertible integral model that is \(\varphi\)-positive. Take a formal model \( f: X \to Y' \) of \(\varphi\) and an \( f \)-ample \(\mathcal{L}_X \) such that \(\text{sp}^*_X \mathcal{L}_X = \mathcal{L}^+ \). By [54], II, (4.6.13) (iii), we may assume that between \( Y' \) and \( Y = \text{Spf } B \) there is an admissible blow-up \( \pi: Y' \to Y \). We may also assume that \( \mathcal{O}_X \) is \( I \)-torsion free. Let \( J \) be the admissible ideal of \( Y \) such that \( \pi \) is the admissible blow-up along \( J \). Then \( J\mathcal{O}_{Y'} \) is \( \pi \)-ample, and hence \( \mathcal{L}_X \otimes_{\mathcal{O}_Y} f^*(J\mathcal{O}_{Y'})^{\otimes n} \) for some \( n > 0 \) is \( \pi \circ f \)-ample ([54], II, (4.6.13) (ii)). By an argument similar to that in the proof of [54], III, (5.4.3), applied to the proper map \( \pi \circ f \), we get a \( \mathcal{Y} \)-closed immersion \( X \hookrightarrow \hat{\mathcal{P}(E)} \) for some \( B \)-module \( E \) of finite type, whence the desired result. \( \square \)

**Proposition 7.6.4.** (1) Any closed immersion is projective.

(2) If \( \varphi: Z \to \mathcal{Y} \) and \( \psi: \mathcal{Y} \to \mathcal{X} \) are projective, then so is the composition \( \psi \circ \varphi \). If \( \psi \circ \varphi \) is projective and \( \psi \) is separated, then \( \varphi \) is projective.

(3) If \( \varphi: \mathcal{X} \to \mathcal{X}' \) and \( \psi: \mathcal{Y} \to \mathcal{Y}' \) are two projective morphisms over a rigid space \( S \) such that either \( \mathcal{X} \) and \( \mathcal{X}' \) or \( \mathcal{X} \) and \( \mathcal{Y}' \) are locally of finite type over \( S \), then the induced morphism \( \varphi \times_S \psi: \mathcal{X} \times_S \mathcal{Y} \to \mathcal{X}' \times_S \mathcal{Y}' \) is projective.

(4) If \( \varphi: \mathcal{X} \to \mathcal{Y} \) is a projective morphism over a rigid space \( S \) and \( S' \to S \) is a morphism of rigid spaces such that either \( \mathcal{X} \) and \( \mathcal{Y} \) are locally of finite type over \( S \) or that \( S' \) is locally of finite type over \( S \), then the induced morphism

\[
\varphi_{S'}: \mathcal{X} \times_S S' \to \mathcal{Y} \times_S S'
\]

is projective.

**Proof.** (1) is clear. To show (2), let \((\mathcal{L}, \mathcal{L}^+)\) (resp. \((\mathcal{M}, \mathcal{M}^+)\)) be the pair as in 7.6.2 for the morphism \( \varphi \) (resp. \( \psi \)). Let \( f: X \to Y \) (resp. \( g: Y' \to Z \)) be a formal model of \( \varphi \) (resp. \( \psi \)) with the relatively ample sheaf \( \mathcal{L}_X \) (resp. \( \mathcal{M}_{Y'} \)) as in 7.6.1. By [54], II, (4.6.13) (iii), we may assume that \( Y \) is an admissible blow-up of \( Y' \).
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Write the admissible blow-up as $\pi : Y \rightarrow Y'$, and let $\mathcal{L}$ be the blow-up center. Then, by [54], II. (4.6.13) (ii), the sheaf $\mathcal{L} \otimes f^*(\mathcal{J} \otimes \pi^* \mathcal{M}_{Y'}^m)^{\otimes n}$ for sufficiently large $n$ and $m$ is ample relative to the composite map $g \circ \pi \circ f$. Let this sheaf be $\mathcal{N}_X$. Then the pair $((\text{sp}_X^* \mathcal{N}_X)^\text{rig}, \text{sp}_Y^* \mathcal{N}_Y)$ guarantees that the composition $\psi \circ \varphi$ is projective. Thus the first half of (2) is proved. (3) and (4) are proved by similar arguments with the aid of [54], II. (4.6.13) (iii) and (iv). Finally, in view of 7.5.9, the second half of (2) follows.

Exercises

Exercise II.7.1. Let $\mathcal{X} \to \mathcal{Z} \leftarrow \mathcal{Y}$ be a diagram consisting of universally Noetherian affinoids, where the second morphism is a closed immersion. Show that the canonical morphism of the associated schemes (cf. 6.6. (a))

$$s(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}) \longrightarrow s(\mathcal{X}) \times_{s(\mathcal{Z})} s(\mathcal{Y})$$

is an isomorphism.

Exercise II.7.2. Let $\mathcal{Y}$ be a rigid space and $\alpha : \mathcal{T} \to \mathcal{Y}$ a rigid point. We say that the rigid point $\alpha$ is essentially of finite type if $\mathcal{T}$ is $\mathcal{Y}$-isomorphic to the associated rigid point (3.3.5) to a point $x$ of the disk $\mathbb{D}_Y^N$ for some $N \geq 0$. Show that in 7.5.17 (2) it is enough to use a rigid point of $\mathcal{Y}$ essentially of finite type.

8 Classical points

The main aim of this section is to discuss the so-called classical points. Classical points are, roughly speaking, the points of rigid spaces of type (V) (2.5.1) or of type (N) (2.5.2) that are, so to speak, already dealt with in classical rigid geometry. In §8.1 we first introduce a useful concept, the spectral functors, which provides a general framework for dealing with several kinds of points on rigid spaces. Spectral functors are useful not only in this section, but in later sections in the appendix, where we discuss the classical rigid geometry (§B) and Berkovich analytic geometry (§C.6).

In §8.2 we define and discuss classical points. The basic facts discussed in this subsection provide a platform for comparing later our rigid geometry with the classical rigid geometry and with Berkovich analytic geometry.

In the final subsection, §8.3, we prove the Noetherness theorem (8.3.6), which asserts that if $\mathcal{X}$ is a rigid space of type (V) or of type (N), then the local ring $B_x = \mathcal{O}_{\mathcal{X},x}$ at any point $x \in (\mathcal{X})$ is a Noetherian ring.
8. Classical points

8.1 Spectral functors

8.1. (a) Definitions

**Definition 8.1.1.** Let $\mathcal{C}$ be a subcategory of the category of rigid spaces $\mathbf{Rf}$.

1. We say that $\mathcal{C}$ is $O$-stable if for any $X \in \text{obj}(\mathcal{C})$ any open immersion $U \hookrightarrow X$ belongs to $\mathcal{C}$.

2. We say that $\mathcal{C}$ is $\text{QCO}$-stable if for any $X \in \text{obj}(\mathcal{C})$ any quasi-compact open immersion $U \hookrightarrow X$ belongs to $\mathcal{C}$.

**Definition 8.1.2.** Let $\mathcal{C}$ be a subcategory of $\mathbf{Rf}$, and $S: \mathcal{C} \to \text{Top}$ a functor. Consider the following conditions.

(a) There exists a natural transformation from $S$ to $\langle \cdot \rangle$ that induces for any $X \in \text{obj}(\mathcal{C})$ an inclusion of sets $S(X) \subset \langle X \rangle$.

(b) $S(X) = \emptyset$ if and only if $X = \emptyset$.

(c) For any quasi-compact open immersion $U \hookrightarrow V$ in $\mathcal{C}$ the induced map of topological spaces $S(U) \to S(V)$ maps $S(U)$ homeomorphically onto $S(V)$ with the subspace topology induced by the topology on $S(V)$.

(d) For a quasi-compact open immersion $U \hookrightarrow V$ in $\mathcal{C}$ the equality $S(U) = S(V)$ holds if and only if $U = V$.

The functor $S$ is called a prespectral functor if it satisfies (a), (b), and (c). If it furthermore satisfies (d), then it is called a spectral functor.

Note that we do not assume in (a) that the topology on $S(X)$ is the topology induced by $\langle X \rangle$. Note also that if $S$ is prespectral (resp. spectral), then $S|_{\mathcal{C}'}$ is prespectral (resp. spectral) for any subcategory $\mathcal{C}'$ of $\mathcal{C}$.

**Proposition 8.1.3.** Let $\mathcal{C}$ be a $\text{QCO}$-stable subcategory of $\mathbf{Rf}$ and $S: \mathcal{C} \to \text{Top}$ a spectral functor. Then for any $X$ in $\mathcal{C}$ and any quasi-compact open immersions $U \hookrightarrow X$ and $V \hookrightarrow X$, $S(U) = S(V)$ implies $U = V$.

*Proof.* The open immersion $U \cap V \hookrightarrow U$ is quasi-compact and hence belongs to $\mathcal{C}$ due to the QCO-stability. By 8.1.2 (c), we have $S(U) = S(X) \cap \langle U \rangle$ and $S(V) = S(X) \cap \langle V \rangle$. Further, $S(U) = S(U) \cap S(V) = S(X) \cap \langle U \rangle \cap (V) = S(X) \cap (\langle U \cap V \rangle)$ by our assumption. Since the composition $U \cap V \hookrightarrow U \hookrightarrow X$ is a quasi-compact immersion, again by 8.1.2 (c) we have $S(U \cap V) = S(X) \cap (\langle U \cap V \rangle)$ and hence $S(U \cap V) = S(U)$. Then $U \cap V = U$ follows from 8.1.2 (d). By switching the roles of $U$ and $V$, we have $V = U \cap V$. Thus we get $U = V$, as desired. \qed
8.1. (b) **Continuity.** Let $S: \mathcal{C} \to \text{Top}$ be a prespectral functor, and consider an inductive system $\{U_i\}_{i \in I}$ in $\mathcal{C}$ indexed by a directed set $I$ and consisting of quasi-compact open immersions $U_i \hookrightarrow U_j$ for $i \leq j$. Then by 8.1.2 (c) one can define $S(U)$, where $U = \lim_{\longrightarrow} \bigcup_{i \in I} U_i$, by the formula

$$S(U) = \lim_{\longrightarrow} S(U_i).$$

In other words, any prespectral functor behaves *continuously* under filtered limits by quasi-compact open immersions.

Let us discuss more general continuity properties of prespectral functors.

**Definition 8.1.4.** Let $\mathcal{C}$ be a subcategory of $\text{Rf}$. A prespectral functor $S$ is said to be *continuous* if for any small category $\mathcal{D}$ and any functor $F: \mathcal{D} \to \mathcal{C}$ such that $F(f)$ is an open immersion for any morphism $f$ in $\mathcal{D}$, we have

$$S(\lim \triangleright F) = \lim \triangleright S \circ F$$

in the category of topological spaces.

The following proposition is easy to verify; we leave the checking to the reader.

**Proposition 8.1.5.** Let $\mathcal{C}$ be an $\mathcal{O}$-stable subcategory of $\text{Rf}$. Suppose $\mathcal{C}$ is stable under the taking of disjoint sums. Let $S: \mathcal{C} \to \text{Top}$ be a continuous prespectral functor.

1. $S$ commutes with disjoint sums, that is, for $X_\alpha \in \mathcal{C}$ indexed by a set $L$, we have

$$S\left( \bigsqcup_{\alpha \in L} X_\alpha \right) = \bigsqcup_{\alpha \in L} S(X_\alpha).$$

2. The functor $S$ preserves the equivalence relation, that is, for $X \in \text{obj}(\mathcal{C})$ and an open covering $\{U_\alpha\}_{\alpha \in L}$ of $X$, let $R$ be the equivalence relation defining $X$, that is,

$$R \bigsqsupseteq \bigsqcup_{\alpha \in L} U_\alpha \longrightarrow X$$

is exact; then

$$S(R) \bigsqsupseteq \bigsqcup_{\alpha \in L} S(U_\alpha) \longrightarrow S(X)$$

is exact.

**Remark 8.1.6.** If $S: \mathcal{C} \to \text{Top}$ is a continuous spectral functor in the setting of 8.1.5, then if $U \hookrightarrow V$ is an open immersion, $S(U) = S(V)$ if and only if $U = V$. The proof is given by an argument similar to that of 8.1.3.
Proposition 8.1.7.  The functor

\[ \cdot : \text{Rf} \longrightarrow \text{Top} \]

(the separated quotient; cf. §4.3. (a)) is a continuous prespectral functor.

Proof.  It is easy to see that the functor \( \cdot \) is prespectral.  The continuity follows from 0.2.3.25.

Proposition 8.1.8.  Suppose \( \mathcal{C} \) is QCO-stable.  Let \( S : \mathcal{C} \rightarrow \text{Top} \) be a prespectral functor.  Then the following conditions are equivalent.

(a) Let \( \mathcal{X} \) be a coherent space in \( \mathcal{C} \), and \( \mathcal{U} \subseteq \mathcal{X} \) a quasi-compact open subspace.  Then \( \langle \mathcal{U} \rangle \cap S(\mathcal{X}) = S(\mathcal{X}) \) implies \( \mathcal{U} = \mathcal{X} \).

(b) Let \( \mathcal{X} \) be a non-empty coherent space in \( \mathcal{C} \), and \( \mathcal{X} \) a distinguished formal model of \( \mathcal{X} \).  Then for any non-empty closed formal subscheme \( \mathcal{Y} \) of \( \mathcal{X} \) of finite presentation, we have \( \text{sp}_{\mathcal{X}}^{-1}(\mathcal{Y}) \cap S(\mathcal{X}) \neq \emptyset \).

Proof.  Let us first show that (a) \( \Rightarrow \) (b).  Let \( \mathcal{X} \), \( \mathcal{X} \), and \( \mathcal{Y} \) be as in (b).  Set \( U = X \setminus Y \).  Then \( U \) is a quasi-compact open subset of \( X \), and hence \( \text{sp}_{\mathcal{X}}^{-1}(U) \) is of the form \( \langle \mathcal{U} \rangle \), where \( \mathcal{U} = U^{\text{rig}} \) is a quasi-compact open subspace of \( \mathcal{X} \) (3.1.3 (2)).  Suppose \( S(\mathcal{X}) \subseteq \langle \mathcal{U} \rangle \).  Then \( \mathcal{U} = \mathcal{X} \).  Since \( X \) is distinguished, \( U = X \) (3.1.5), which contradicts the assumption \( \mathcal{Y} \neq \emptyset \).

Next we show the converse.  Suppose \( S(\mathcal{X}) \subseteq \langle \mathcal{U} \rangle \).  Take a distinguished formal model \( X \) of \( \mathcal{X} \) containing a quasi-compact open subset \( U \) corresponding to \( \mathcal{U} \).  If \( \mathcal{U} \neq \mathcal{X} \), then by 3.1.5 the complement \( Y = X \setminus U \) is non-empty.  Since \( U \) is quasi-compact, \( Y \) carries a structure of a formal scheme in such a way that we have a closed immersion \( Y \hookrightarrow X \) of finite presentation.  But \( \text{sp}_{X}^{-1}(Y) \cap S(\mathcal{X}) \neq \emptyset \) contradicts \( S(\mathcal{X}) \subseteq \langle \mathcal{U} \rangle \).  \( \square \)

8.1. (c) Regularity.  Let \( X \) be a topological space.  Then an open subset \( U \subseteq X \) is said to be regular if \( U \) = \( U \).  Similarly, a closed subset \( \mathcal{C} \) of \( X \) is said to be regular if \( \mathcal{C} \).  Note that, if \( S \) is a regular open (resp. closed) subset of \( X \), the complement \( X \setminus S \) is a regular closed (resp. open) subset.

Definition 8.1.9.  A prespectral functor \( S \) is said to be real valued if \( S(\mathcal{X}) \) is a subset of \( [\mathcal{X}] \) for any \( \mathcal{X} \in \text{obj}(\mathcal{C}) \).

Proposition 8.1.10.  Let \( \mathcal{C} \) be a QCO-stable subcategory of \( \text{Rf} \), and \( S : \mathcal{C} \rightarrow \text{Top} \) a real valued spectral functor.

(1) For a quasi-compact open immersion \( \mathcal{U} \hookrightarrow \mathcal{X} \) in \( \mathcal{C} \), the open subset \( \langle \mathcal{U} \rangle \) and its closure \( \langle \overline{\mathcal{U}} \rangle \) are regular in \( \langle \mathcal{X} \rangle \).

(2) For a rigid space \( \mathcal{X} \) in \( \mathcal{C} \), any tube open subset (4.2.4) of \( \langle \mathcal{X} \rangle \) is regular.
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Proof. First we prove (1). By 8.1.2, $S(\mathcal{U}) = S(\mathcal{X}) \cap \langle \mathcal{U} \rangle$. Since $S$ is real valued, it is also equal to $S(\mathcal{X}) \cap [\mathcal{U}]$. Hence, by 8.1.3, for two quasi-compact open immersions $\mathcal{U} \hookrightarrow \mathcal{X}$ and $\mathcal{V} \hookrightarrow \mathcal{X}$, $[\mathcal{U}] = [\mathcal{V}]$ implies $\mathcal{U} = \mathcal{V}$. Then the regularity of $\langle \mathcal{U} \rangle$ follows by an argument similar to that in the proof of 0.2.4.4. The regularity of $\langle \mathcal{U} \rangle$ follows as a consequence.

We prove (2). Since $\mathcal{C}$ is QCO-stable, any retrocompact open set $\mathcal{U}$ of $\mathcal{X}$ belongs to $\mathcal{C}$. By (1), $(\mathcal{X}) \setminus \langle \mathcal{U} \rangle$ is a regular open subset. Any tube open subset of $\mathcal{X}$ is of this form by 4.2.5, and the claim follows. \[\square\]

8.1. (d) Density argument

Theorem 8.1.11. Let $\mathcal{X}$ be a coherent rigid space, and $S$ a spectral functor (8.1.2) defined on the category of quasi-compact open subspaces of $\mathcal{X}$. Then

$$\Gamma((\mathcal{X}), \mathcal{O}_{\mathcal{X}}^{\text{int}}) = \{ f \in \Gamma((\mathcal{X}), \mathcal{O}_{\mathcal{X}}): f_x \in \mathcal{O}_{\mathcal{X},x}^{\text{int}} \text{ for any } x \in S(\mathcal{X}) \}.$$ 

We first prove the following lemma.

Lemma 8.1.12. Let $\mathcal{X}$ be a coherent rigid space. Define a subset $\mathcal{W}_f \subseteq (\mathcal{X})$ for $f \in \Gamma((\mathcal{X}), \mathcal{O}_{\mathcal{X}})$ by

$$\mathcal{W}_f = \{ x \in (\mathcal{X}): f_x \in \mathcal{O}_{\mathcal{X},x}^{\text{int}} \}.$$ 

Then $\mathcal{W}_f$ is a quasi-compact open subset of $\langle \mathcal{X} \rangle$.

Proof. Take $x \in (\mathcal{X})$. If $x \in \mathcal{W}_f$, then it is clear that $\mathcal{W}_f$ contains a quasi-compact open neighborhood of $x$. Suppose $x \notin \mathcal{W}_f$. With the notation as in 3.2.13, since $(f \mod m_{B_x}) \in V_x$ is non-zero, $f$ is invertible in $B_x = \mathcal{O}_{\mathcal{X},x}$. This implies that $g = 1/f$ belongs to $A_x = \mathcal{O}_{\mathcal{X},x}^{\text{int}}$. Thus there exist a quasi-compact open neighborhood of $x$ of the form $\mathcal{U} = U^{\text{rig}}$, where $U$ is a quasi-compact open subset of a formal model $X$ of $\mathcal{X}$, and an element $h \in \Gamma(U, \mathcal{O}_U)$ that gives $g$ via the inductive limit. Replacing $U$ by a smaller set if necessary, we may assume that $U$ is affine, $U = \text{Spf} B$, that $hB$ is an open ideal of $B$, and that $fh = 1$. Then we have $\mathcal{W}_f \cap \langle \mathcal{U} \rangle = (\text{Spf } B(h))^{\text{rig}}$. This implies that the inclusion map $\mathcal{W}_f \hookrightarrow \langle \mathcal{X} \rangle$ is quasi-compact (cf. 0.2.1.4 (2)) and hence that $\mathcal{W}_f$ is quasi-compact. \[\square\]

Proof of Theorem 8.1.11. Suppose $f \in \Gamma((\mathcal{X}), \mathcal{O}_{\mathcal{X}})$ belongs to the right-hand side of the equality. Then $S(\mathcal{X}) = S(\mathcal{W}_f)$, where $\mathcal{W}_f$ is the quasi-compact open subspace of $\mathcal{X}$ such that $\langle \mathcal{W}_f \rangle = \mathcal{W}_f$ as in 8.1.12. Since $S$ is a spectral functor, we have $\mathcal{W}_f = \mathcal{X}$, and hence the assertion follows. \[\square\]
8.2 Classical points

8.2. (a) Point-like rigid spaces

Definition 8.2.1. A rigid space $Z$ is said to be *point-like* if it is coherent and reduced and there exists a unique minimal point in $\langle Z \rangle$.

Proposition 8.2.2. Let $Z$ be a universally Noetherian point-like rigid space.

1. The set $\langle Z \rangle$ coincides with the set $G_x$ of all generizations of the minimal point $x$. In particular, the set $\langle Z \rangle$ is totally ordered with respect to the ordering by generization.

2. Any distinguished formal model $\hat{Z}$ of $Z$ is reduced and has the underlying topological space totally ordered by the ordering by generization with the unique minimal point.

Proof. (1) is clear by definition. Let $Z$ be a distinguished formal model of $Z$. Then $Z$ is reduced by 6.4.2. By 3.1.5, the specialization map $\text{sp}_Z : \langle Z \rangle \to Z$ is surjective, which yields (2).

Example 8.2.3. If $V$ is an $a$-adically complete valuation ring, then $(\text{Spf} V)^{\text{rig}}$ is a point-like rigid space.

8.2. (b) Structure of point-like rigid spaces

Proposition 8.2.4. Any universally Noetherian point-like rigid space is a Stein affinoid.

Proof. Let $Z$ be a point-like rigid space, and $x \in \langle Z \rangle$ the minimal point. By 6.5.4, one can take a Stein affinoid neighborhood $V$ of $x$. Then by 8.2.2 (1) one sees that $Z = V$.

Proposition 8.2.5. Let $Z = (\text{Spf} A)^{\text{rig}}$ be a universally Noetherian Stein affinoid, and suppose that $A$ is an $I$-torsion free t.u. rigid-Noetherian ring, where $I \subseteq A$ is an ideal of definition of $A$. Set $\text{Spec } B = s(Z) = \text{Spec } A \setminus V(I)$ (that is, $B = \Gamma(Z, \mathcal{O}_Z)$), and suppose that $\text{Spec } B$ is Jacobson. Then $Z$ is point-like if and only if the following conditions are satisfied.

(a) $A$ is a local integral domain and $B = \text{Frac}(A)$.

(b) the integral closure $A^{\text{int}}$ of $A$ in $B$ is an $I$-adically separated Henselian valuation ring.

Moreover, in this situation, we have $A^{\text{int}} = \mathcal{O}_{Z,z}^{\text{int}}$, where $z$ is the unique minimal point of $\langle Z \rangle$. 
Proof. Suppose that $\mathcal{Z}$ is point-like, and let $z$ be the unique minimal point of $\langle \mathcal{Z} \rangle$. Since $s(\mathcal{Z}) = \text{Spec} A \setminus V(I) = \text{Spec} B$ is reduced and $A$ is $I$-torsion free, $A$ is a reduced ring. Let $\alpha: (\text{Spf} \, \hat{V}_z)^{\text{rig}} \to \mathcal{Z}$ be the associated rigid point at $z$. Since $\langle \mathcal{Z} \rangle$ coincides with the set of all generalizations of $\mathcal{Z}$, and since $\text{Spf} A$ is a distinguished formal model of $\mathcal{Z}$, the composite map $\text{Spf} \, \hat{V}_z \to \langle \mathcal{Z} \rangle \to \text{Spf} A$ is surjective (cf. 3.1.5).

We claim that the underlying topological space of $s(\mathcal{Z}) = \text{Spec} B$ consists of one point. Let $x \in s(\mathcal{Z})$ be a closed point, and suppose $s(\mathcal{Z}) \neq \{x\}$. Since $\text{Spec} B$ is Jacobson, there exists another closed point $y \neq x$ of $s(\mathcal{Z})$. But, as we have seen in 7.3.15 (2), non-empty closed subschemes of $s(\mathcal{Z})$ correspond bijectively to non-empty closed subspaces of $\mathcal{Z}$, which are, however, always supported on the whole topological space $\langle \mathcal{Z} \rangle$ due to 7.3.15 (1), since $\mathcal{Z}$ is point-like. But this is absurd, and hence we deduce by contradiction that $s(\mathcal{Z}) = \text{Spec} B$ consists of one point. This shows that the map $\text{Spec} \, \hat{V}_z \to \text{Spec} A$ is surjective, and hence $\text{Spec} A$ is irreducible. Therefore, $A$ is a local integral domain such that $B = \text{Frac}(A)$; in particular, we have a local injective homomorphism $A \hookrightarrow \hat{V}_z$.

By 6.2.6 we know that $A^{\text{int}} = \Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}^{\text{int}}) = \mathcal{O}_{\mathcal{Z}, z}^{\text{int}}$. The composite morphism $A \hookrightarrow A^{\text{int}} = \mathcal{O}_{\mathcal{Z}, z}^{\text{int}} \to \hat{V}_z$ is, since $A \hookrightarrow \hat{V}_z$ is injective, a local injective homomorphism. Now since $A$ is a subring of $V_z$, and since $V_z$ is integrally closed, we have $A^{\text{int}} \subseteq V_z$ and hence $A^{\text{int}} = V_z$, which is an $I$-adically separated Henselian valuation ring (cf. 3.2.13).

Conversely, suppose that $\mathcal{Z}$ satisfies (a) and (b). Set $Y = \text{Spec} A$ and $U = \text{Spec} B$, and consider the $U$-admissible blow-ups (which are, by passage to the formal completions, exactly corresponding to admissible blow-ups of $X = \text{Spf} A$) of $Y$ (cf. E.1.4). We denote by $\langle Y \rangle$ the classical Zariski–Riemann space, that is, the projective limit of all $U$-admissible blow-ups (cf. E.2.2). By the valuative criterion, one has a morphism $\text{Spec} A^{\text{int}} \to \langle Y \rangle$; let $\hat{z}$ be the image of the closed point. Now we claim that any blow-up $Y' \to Y = \text{Spec} A$ along an admissible ideal is affine. Take an affine open subset $U' = \text{Spec} A' \subseteq Y'$ that contains the image of $\hat{z}$ under the specialization map $\langle Y \rangle \to Y'$. We have an inclusion $A' \hookrightarrow A^{\text{int}}$ that factorizes $A \hookrightarrow A^{\text{int}}$. Suppose there exists a closed point $y \in Y' \setminus U'$. There exist a valuation ring $V$ and a morphism $\text{Spec} V \to Y'$ that dominates $y$. Since $y$ is also mapped to $z$, which is the unique closed point of $\text{Spec} A$, we have a map $A^{\text{int}} \hookrightarrow V$ that factorizes $A \hookrightarrow V$. Hence in $\langle Y \rangle$ the point $\hat{z}$ is a specialization of the image of the closed point of $\text{Spec} V \to \langle Y \rangle$. But this is absurd, since the specialization map $\langle Y \rangle \to Y'$ is a closed map (cf. E.2.5).

In particular, we deduce that any admissible blow-up $X' \to \text{Spf} A$ is affine, say $X' = \text{Spf} A'$, where $A'$ is contained in $A^{\text{int}}$. Hence we have the surjective map $\text{Spf} \, A^{\text{int}} \to X'$. Since this holds for any admissible blow-up $X'$, by 0.2.2.13 (2) we have the surjective map $\text{Spf} A^{\text{int}} \to \langle \mathcal{Z} \rangle$. Since $A^{\text{int}}$ is an $I$-adically complete
valuation ring \((0.9.1.1)\), we deduce that \((Z)\) has a unique minimal point. By 6.2.6, we have \(O_{Z,z}^\text{int} = A^\text{int} = V_z\). Since the local ring at the maximal point is given by \(B\), the fractional field of \(O_{Z,z}^\text{int} = A^\text{int} = V_z\), it follows by 3.2.17 that the local ring at each point of \((Z)\) is a subring of \(B\), and hence is an integral domain. Hence \(Z\) is reduced, and thus we have that \(Z\) is point-like.

**Proposition 8.2.6.** (1) Let \(V\) be an \(a\)-adically complete valuation ring and \(Z\) a point-like rigid space of finite type over \(S = (\text{Spf } V)^\text{rig}\). Then \(Z\) is of the form \(Z^D_{\text{Spf }, W} / \text{rig}^D\), where \(W\) is quasi-finite, flat, and finitely presented over \(V\). If \(V\) is of height one, then \(W\) is finite over \(V\).

(2) Let \(Z\) be a point-like rigid space of type \((N)\) (§2.5. (b)). Then \(Z\) is of the form \(Z^D_{\text{Spf }, W} / \text{rig}^D\), where \(W\) is an \(a\)-adically complete discrete valuation ring.

We show the following lemma before the proof.

**Lemma 8.2.7.** Let \(A\) be a Noetherian \(I\)-adically Zariskian ring. Then the scheme \(\text{Spec } A \setminus V(I)\) is Jacobson ([54], IV, (10.4.1)), and the closure \(\overline{\{p\}}\) in \(\text{Spec } A\) of any closed point \(p \in \text{Spec } A \setminus V(I)\) is of the form \(\text{Spec } B\), where \(B\) is a 1-dimensional semi-local ring. Moreover, we have \(\overline{\{p\}} \setminus V(I) = \{p\}\).

**Proof.** Since \(1 + I \subseteq A^\times\), one easily sees that any non-empty closed subset of \(\text{Spec } A\) meets \(V(I)\). Hence it follows from [54], IV, (10.5.7), that \(\text{Spec } A \setminus V(I)\) is a Jacobson scheme. Let \(p\) be an ideal maximal in \(\text{Spec } A \setminus V(I)\), and set \(Y = \overline{\{p\}}\), where the closure is taken in \(\text{Spec } A\). Since \(p\) is not an open prime, it is not maximal in \(A\) (0.7.3.3). Let \(q\) be any prime ideal that strictly contains \(p\). Then \(q\) is an open prime ideal, and hence \(A_q\) is \(IA_q\)-adically Zariskian (0.7.3.3). By [109], Chapter VIII, Theorem 10, \(A/q\) is an Artinian local ring. Hence \(q\) is actually a maximal ideal of \(A\), and thus we deduce that the ring \(B\) as above is of dimension 1. By [54], IV, (10.5.3), we deduce that \(B\) is a semi-local ring.

**Proof of Proposition 8.2.6.** By 8.2.4, \(Z = (\text{Spf } W)^\text{rig}\), where \(W\) is topologically finitely generated \(V\)-algebra in case (1), or is a Noetherian adic ring in case (2). In both cases the scheme \(\text{Spec } W \setminus V(I)\) (\(I \subseteq W\) is an ideal of definition) is Jacobson due to 0.9.3.10 and 8.2.7. In view of 8.2.5, we may assume that \(W\) is an \(I\)-adically complete local integral domain and that its integral closure \(W^\text{int}\) in \(\operatorname{Frac}(W)\) is an \(I\)-adically separated Henselian valuation ring.

(1) By 6.2.8, we may assume that the map \(Z \rightarrow S\) has a distinguished formal model of the form \(\text{Spf } W \rightarrow \text{Spf } V\). Since we have the surjective map \(\text{Spec } W^\text{int} \rightarrow \text{Spec } W\), \(W\) is a local ring. By the assumption, \(W\) is topologically of finite presentation over \(V\) (2.3.4), and \(a\)-torsion free (hence \(W\) is flat over \(V\)).
We claim that the map \( \text{Spec } W \to \text{Spec } V \) is finite in case \( V \) is of height one. The map \( \text{Spf } W \to \text{Spf } V \) is surjective, and the closed fiber of \( \text{Spec } W \to \text{Spec } V \) is of dimension 0, since no scheme of finite type over a field of positive dimension can be totally ordered with respect to the order by generization. By Noether normalization (0.9.2.10), we deduce that \( W \) is finite over \( V \).

For a general \( V \) we use the technique alluded to in 0.9.2.5. Let \( q = \sqrt{(a)} \) be a height-one prime (cf. 0.6.7.3). Since \( W/qW \) is flat over \( V/q \), and since the generic fiber of the map \( \text{Spec } W/qW \to \text{Spec } V/q \) is of dimension 0 (by the above-mentioned reasoning), we deduce that \( W \) is of relative dimension 0 over \( V \). Hence \( W \) is quasi-finite over \( V \), as desired.

(2) By 8.2.7, \( W \) is a local integral domain of dimension 1. Since \( W \) is Noetherian, it follows that \( I \) contains \( m^n_W \) for some \( n > 0 \); that is, the \( I \)-adic topology coincides with the topology defined by the maximal ideal. By [84], (32.2), we deduce that its normalization \( W^{\text{int}} \) is finite over \( W \). Hence in view of 6.2.4 we may assume that \( W \) is normal. Then \( W \) is a Noetherian 1-dimensional integrally closed local domain, hence is a discrete valuation ring, as desired.

8.2. (c) Classical points

**Definition 8.2.8.** Let \( X \) be a rigid space of type (V) (2.5.1) or of type (N) (2.5.2).

1. A classical point of \( X \) is a point-like locally closed rigid subspace \( Z \subseteq X \) that is retrocompact, i.e., the immersion \( Z \hookrightarrow X \) is quasi-compact.

2. A classical point \( Z \subseteq X \) is said to be closed if it is a closed subspace.

Recall that a rigid space \( X \) is said to be of type (\( V_R \)) if it is locally of finite type over a rigid space of the form \( (\text{Spf } V)^{\text{rig}} \), where \( V \) is an \( a \)-adically complete valuation ring \((a \in m_V \setminus \{0\}) \) of height one (2.5.1).

**Proposition 8.2.9.** (1) Any classical point of a rigid space of type (\( V_R \)) is closed.

(2) Let \( X \) be a coherent rigid space of type (N). Suppose \( X \) has a Noetherian distinguished formal model \( \mathcal{X} \) and an ideal of definition \( \mathcal{I} \) such that the scheme \( X_0 = (X, \mathcal{O}_X/I) \) is Jacobson. Then any classical point of \( X \) is closed.

**Proof.** (1) follows immediately from 8.2.6 (1). To show (2), let \( Z \hookrightarrow X \) be a classical point; \( Z \) is a closed rigid subspace of a coherent open subspace \( U \subseteq X \), and is of the form \( Z = (\text{Spf } W)^{\text{rig}} \) for an \( a \)-adically complete discrete valuation ring \( W \); see 8.2.6 (2). Passing to formal models, we have the sequence of morphisms

\[
\text{Spf } W \hookrightarrow i \quad U \hookrightarrow X' \xrightarrow{\pi} X,
\]

where \( U^{\text{rig}} = U \), \( i \) is a closed immersion, \( j \) is an open immersion, and \( \pi \) is an admissible blow-up. Dividing out by the ideal of definition \( I \), we get the sequence...
of morphisms of schemes

\[ \text{Spec } W/aW \overset{i_0}{\hookrightarrow} U_0 \overset{j}{\hookrightarrow} X'_0 \overset{\pi}{\longrightarrow} X_0. \]

Since \( X'_0 \) is Jacobson, the image of \( \text{Spec } W/aW \) in \( X'_0 \) is a closed point ([54], IV, (10.4.7)). Hence, by 4.5.10, \( \text{Spec } W \hookrightarrow X' \) is a closed immersion, and hence \( Z \hookrightarrow X \) is a closed immersion.

**Proposition 8.2.10.** Let \( \mathcal{X} = (\text{Spf } A)^{\text{rig}} \) be an affinoid of type \( (V_R) \) or of type \( (N) \), where \( A \) is \( I \)-torsion free for a finitely generated ideal of definition \( I \subseteq A \).

1. For any closed point \( x \) of the Noetherian scheme \( s(\mathcal{X}) \), there exists a unique closed classical point \( Z = Z^{\text{rig}} \hookrightarrow \mathcal{X} \) of \( \mathcal{X} \) such that the image of \( s(Z) \to s(\mathcal{X}) \) is \( x \).

2. Suppose (in type \( (N) \) case) that \( \text{Spec } A/I \) is Jacobson. Then, for any classical point \( Z \hookrightarrow \mathcal{X} \) of \( \mathcal{X} \), \( s(Z) \) is a point, and the image of the map \( s(Z) \to s(\mathcal{X}) \) is a closed point of \( s(\mathcal{X}) \).

**Proof.** (1) Let \( x \in s(\mathcal{X}) \) be a closed point. The construction as in §7.3. (d) gives the closed subspace \( Z = \{ x \} \times_{s(\mathcal{X})} (\mathcal{X}) \hookrightarrow \mathcal{X} \) corresponding to \( x \). We claim that this indeed gives a closed classical point of \( \mathcal{X} \). To describe \( Z \), let us take the reduced closure \( Y = \text{Spec } W \) of \( \{ x \} \) in \( \text{Spec } A \). Then we have \( Z = (\text{Spf } W)^{\text{rig}} \). In type \( (V_R) \) case, by 0.9.2.12, \( W \) is finite over \( V \), and hence, by 6.2.6 and 8.2.5, \( Z = (\text{Spf } W)^{\text{rig}} \) is a closed classical point of \( \mathcal{X} \). In type \( (N) \) case, \( W \) is a Noetherian 1-dimensional semi-local domain (8.2.7); by an argument similar to that in the proof of 8.2.6 (2), we deduce that \( W^{\text{int}} \), the integral closure of \( W \), is finite over \( W \), and is a complete discrete valuation ring. Thus, similarly, \( Z = (\text{Spf } W)^{\text{rig}} \) is a closed classical point. The uniqueness is easy to see, and is left to the reader.

(2) Let \( Z = (\text{Spf } W)^{\text{rig}} \hookrightarrow \mathcal{X} = (\text{Spf } A)^{\text{rig}} \) be a classical point. One can replace, due to 6.2.8, \( W \) by a finite algebra isomorphic outside \( I \), so that we have an immersion \( \text{Spf } W \hookrightarrow \text{Spf } A \). By 8.2.9, one can even do this in such a way that \( \text{Spf } W \hookrightarrow \text{Spf } A \) is a closed immersion. Then one obtains the closed immersion \( \text{Spec } W \to \text{Spec } A \), and hence, \( s(Z) = \text{Spec } W[\frac{1}{\pi}] \) is a point mapped to a closed point of \( s(\mathcal{X}) = \text{Spec } A[\frac{1}{\pi}] \).

**Corollary 8.2.11.** Let \( \mathcal{X} = (\text{Spf } A)^{\text{rig}} \) be either an affinoid of type \( (V_R) \), or an affinoid of type \( (N) \) having a distinguished Noetherian formal model \( X \), together with an ideal of definition \( I \), such that the scheme \( X_0 = (X, \mathcal{O}_X/I) \) is Jacobson. Then there exists a natural one to one correspondence between the set of all classical points of \( \mathcal{X} \) and the set \( s(\mathcal{X})_{\text{cl}} \) of all closed points of the scheme \( s(\mathcal{X}) \).

**Corollary 8.2.12.** Let \( \mathcal{X} \) be a rigid space of type \( (V_R) \) or of type \( (N) \), and \( \mathcal{F} \) a coherent sheaf on \( \mathcal{X} \). Then \( \mathcal{F} = 0 \) if and only if \( \mathcal{F}|_Z = 0 \) for any classical point \( Z \hookrightarrow \mathcal{X} \).
Chapter II. Rigid spaces

Proof. We may assume that $X$ is an affinoid, say $X = (\text{Spf } A)^{\text{rig}}$. By 6.3.1, the sheaf $\mathcal{F}$ corresponds to a coherent sheaf $\mathcal{G}$ on the associated Noetherian scheme $s(X) = \text{Spec } A \setminus V(I)$. Clearly, $\mathcal{F} = 0$ if and only if $\mathcal{G} = 0$, which is further equivalent to that $\mathcal{G}_x = 0$ for any closed point of $s(X)$ (due to [54], IV, (5.1.11)). In view of the correspondence established in 8.2.11, this is equivalent to $\mathcal{F}|_Z = 0$ for any classical point $Z \hookrightarrow X$ (here we have again used 6.3.1).

Corollary 8.2.13. Let $X$ be a non-empty rigid space of either type $(\mathbb{V}_\mathbb{R})$ or type $(\mathbb{N})$. Then $X$ has a classical point. Moreover, any non-empty rigid subspace $Y$ of $X$ has a classical point of $X$.

Proof. We may assume that $X$ is affinoid. Then the first assertion follows from Corollary 8.2.11. For the second one, we can further assume that $Y$ is an affinoid, and consider the immersion $Y \hookrightarrow X$. If $Z \hookrightarrow Y$ is a classical point, then the composition $Z \hookrightarrow X$ defines a classical point of $X$ (7.4.3 (1)).

In the sequel, we denote by

$$\langle X \rangle_{\text{cl}}$$

the set of all (isomorphism classes of) classical points of $X$. Note that, if $X$ is of type $(\mathbb{V}_\mathbb{R})$ or $(\mathbb{N})$, then due to 8.2.10 and 7.3.15 (2) the set $\langle X \rangle_{\text{cl}}$ can be naturally regarded as a subset of $\langle X \rangle$. Moreover, if $Y \subseteq X$ is a rigid subspace, then

$$\langle Y \rangle_{\text{cl}} = \langle X \rangle_{\text{cl}} \cap \langle Y \rangle.$$

8.2. (d) Functoriality

Proposition 8.2.14. Let $Y$ be a rigid space of type $(\mathbb{V}_\mathbb{R})$, and $\varphi: X \to Y$ a morphism locally of finite type. Let $\iota: Z \hookrightarrow X$ be a classical point of $X$. Then there exists a unique classical point $\iota': Z' \hookrightarrow Y$ of $Y$ for which the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\iota} & X \\
\downarrow{\phi} & & \downarrow{\varphi} \\
Z' & \xrightarrow{\iota'} & Y
\end{array}
$$

where $\phi$ is a finite morphism, is commutative.

Proof. We may assume $Z$, $X$, and $Y$ are affinoid, say $Z = (\text{Spf } W)^{\text{rig}}$, $X = (\text{Spf } A)^{\text{rig}}$, and $Y = (\text{Spf } B)^{\text{rig}}$, where $W$, $A$, and $B$ are $\alpha$-torsion free, and that $\iota$ is a closed classical point. We may also assume that there exist a map $\text{Spf } A \to \text{Spf } B$ of finite presentation and a closed immersion $\text{Spf } W \hookrightarrow \text{Spf } A$ of finite presentation that give formal models of $\varphi$ and $\iota$, respectively. By 8.2.10, the closed immersion $\text{Spec } W \hookrightarrow \text{Spec } A$ defines a closed point $x$ of $s(X) = \text{Spec } A[\frac{1}{\alpha}]$. Since $A[\frac{1}{\alpha}]$
and \(B\left[\frac{1}{a}\right]\) are classical affinoid algebras over \(K = \text{Frac}(V)\) (0, §9.3. (a)), the image \(y\) of \(x\) by the map \(\text{Spec} A \rightarrow \text{Spec} B\) is a closed point of \(s(y)\) by 0.9.3.7 and 0.9.3.6. Hence the point \(y\) gives rise to a classical point \(\iota' : Z' \hookrightarrow Y\), which comes from the formal model of the form \(\text{Spf} W' \rightarrow \text{Spf} B\) as in 8.2.10. By the construction, we have the map \(\text{Spf} W' \rightarrow \text{Spf} W\) that gives the map \(\phi\) as above. The morphism \(\text{Spec} W \rightarrow \text{Spec} W'\) is finite due to 0.9.2.12. The uniqueness follows immediately from the construction.

It follows from the proposition that we have the functor

\[ X \mapsto \langle X \rangle^{\text{cl}} \]

defined on the category of rigid spaces of type \((V_\mathbb{R})\). By what we have seen in §8.2. (c), this is a prespectral functor (8.1.1). We will see in the next subsection that it is in fact a spectral functor.

8.2. (e) Spectrality

**Proposition 8.2.15.** Let \(X\) be either a coherent formal scheme of finite type over an \(a\)-adically complete valuation ring \(V\) of height one (where \(a \in \mathfrak{m}_V \setminus \{0\}\)) or a coherent Noetherian formal scheme. We assume that \(X\) is \(I\)-torsion free, where \(I\) is an ideal of definition of finite type on \(X\). Then the map

\[ \text{sp}_X | [X] : [X] \rightarrow X \]

given by the specialization map (§3.1. (a)), where \(X = X^{\text{rig}}\), is surjective.

Let us first prove the proposition in the \((V_\mathbb{R})\) case.

**Lemma 8.2.16.** Let \(V\) be an \(a\)-adically complete valuation ring of height one (where \(a \in \mathfrak{m}_V \setminus \{0\}\)), \(K = \text{Frac}(V)\), and \(A\) a topologically of finite type \(V\)-algebra. Suppose \(A\) is integral and \(V\)-flat and admits a finite injection

\[ R = V \langle T_1, \ldots, T_n \rangle \hookrightarrow A. \]

Let \(L = \text{Frac}(R)\), and denote the localization of \(R\) at \(\sqrt{a}R\) by \(W\) (note that \(W\) is a valuation ring). Then the integral closure \(A^{\text{int}}\) of \(A\) in \(B = A[\frac{1}{a}]\) is described as

\[ A^{\text{int}} = \{ f \in A[\frac{1}{a}] : f \text{ is integral over } A \otimes_R W \}. \]

**Proof.** Only the inclusion ‘\(\supseteq\)’ calls for the proof. Let \(f \in A[\frac{1}{a}]\) be an element of the right-hand side, and \(P = P(T)\) the characteristic polynomial of the multiplication map by \(f\) in \(\text{End}_L(A \otimes_R L)\). Since \(f\) is integral over the Tate algebra \(R[\frac{1}{a}] = R \otimes_V K\), all coefficients of \(P(T)\) are integral over \(R[\frac{1}{a}]\) ([27], Chapter V, §1.6, Proposition 17). Since \(R[\frac{1}{a}]\) is integrally closed (cf. 0.9.3.12),
we have \( P(T) \in R\left\lbrack \frac{1}{f} \right\rbrack[T] \). Similarly, since \( f \) is integral over \( A \otimes_R W \), we have \( P(T) \in W[T] \). Consequently, \( P(T) \in R[T] \). Now since \( P(f) = 0 \) in \( A \otimes_R L \), and since \( A \) is integral, we deduce that \( P(f) = 0 \) in \( A \). This implies that \( f \) is integral over \( R \) and hence integral over \( A \).

**Proof of Proposition 8.2.15 in the type \((V_R)\) case.** First note that we may assume that \( X \) is reduced and thus that \( X \) is reduced (here we take the reduced model of \( X \), which will be discussed later in §8.3.(b)). Take \( y \in X \), and let \( Y \) be a closed subscheme of \( X \) defined by an admissible ideal such that the underlying topological space of \( Y \) coincides with \( \{y\} \), the closure of the singleton set \( \{y\} \) in \( X \); such a \( Y \) exists, for the scheme \( X_{\text{red}} \) is Noetherian.

First we assume that \( y \) is a closed point of \( X \). We may replace \( X \) by the admissible blow-up along the defining ideal of \( Y \). Moreover, we may replace \( X \) by an affine neighborhood of \( y \). In this way, we may assume that \( X \) is affine, and \( Y \) is defined by an element \( f \in A \). Since \( A \) is an integral domain and \( (f) \subset A \) is an admissible ideal, we have \( 1/f \in A \left\lbrack \frac{1}{f} \right\rbrack \). By the Noether normalization theorem (0.9.2.10), we have a finite injection \( R = V \langle T_1, \ldots, T_n \rangle \hookrightarrow A \). Let \( L = \text{Frac}(R) \), and \( W \) the localization of \( R \) at \( \sqrt{\alpha} R \). Consider the finite \( W \)-algebra \( D = A \otimes_R W \). Since \( D \) is \( \alpha \)-torsion free, \( D \) is finite flat over \( W \), and \( fD \) is an admissible ideal of \( D \). By 8.2.16, the equality \( fD = D \) would imply that \( 1/f \) is integral over \( A \), which is absurd, since \( Y \) is non-empty. Hence, \( fD \neq D \), and we are reduced to the case \( X = \text{Spf} D \otimes_W \hat{W} \) with \( V \) replaced by \( \hat{W} \); the assertion in this case is obvious, for \( D \otimes_W \hat{W} \) is finite over \( \hat{W} \).

In the general case, we take an extension \( V \to V' \) of \( a \)-adically complete valuation rings, and perform the base-change by \( \text{Spf} V' \to \text{Spf} V \); let us denote by \( X_{V'} \) and \( Y_{V'} \) the \( V' \)-formal schemes obtained by the base-change of \( X \) and \( Y \), respectively. Let \( k' \) be the residue field of \( V' \). If one can take such an extension \( V \to V' \) so that there exists an \( k' \)-rational point of \( Y_{k'} \) mapped by the canonical map to the point \( y \), the situation is reduced to the above-treated case. This is indeed possible as follows. The generic point \( y = \text{Spec} K' \) of \( Y \) can be seen as a \( K' \)-rational point \( y' \) of \( Y \times_{\text{Spec} k} \text{Spec} K' \), where \( k = V/m_V \) is the residue field of \( V \). Take a local flat extension \( V \to U \) such that the closed fiber of \( \text{Spec} U \to \text{Spec} V \) is isomorphic to \( \text{Spec} K' \to \text{Spec} k \). Taking a valuation ring dominating \( U \) and the \( a \)-adic completion, we have the desired extension of valuation rings.

The following lemma, which we will use in the proof in type (N) case, is easy to see, and the proof is left to the reader.

**Lemma 8.2.17.** Let \( X \) be a coherent universally Noetherian rigid space and \( X \) a distinguished formal model with an ideal of definition \( I_X \) of finite type. Suppose that for any point \( x \in X \) there exist a valuation ring \( V \) of finite one and a local homomorphism \( \mathcal{O}_{X,x} \to V \) such that \( I_{X,x} V \neq V, 0 \). Then \( \text{sp}_X \lbrack \mathcal{X} \rbrack : \lbrack X \rbrack \to X \) is surjective.
Proof of Proposition 8.2.15 in type (N) case. By 8.2.17 it suffices to prove the following fact: for a Noetherian local ring $A$ with the maximal ideal $m_A$ and an ideal $I \subseteq A$ such that $A$ is $I$-torsion free, there exists a height-one valuation ring $V$ and a local homomorphism $A \to V$ such that $IV \neq V, 0$. We may assume that $A$ is complete. Since $A$ is $I$-torsion free, $U = \text{Spec} \ A \setminus V(m_A)$ is non-empty. Since $U$ is Jacobson ([54], IV, (10.5.9)), there is a closed point $x$ of $\text{Spec} \ A \setminus V(f)$ that is also closed in $U$. By replacing $\text{Spec} \ A$ by the closure of $x$ in $\text{Spec} \ A$, we may assume that $A$ is integral and one-dimensional ([54], IV, (10.5.9)). The claim is clear in this case. \hfill $\square$

Now let us state an immediate but important corollary of 8.2.15.

Corollary 8.2.18. Let $X$ be as in 8.2.15, $Y$ a closed subscheme of $X$ defined by an admissible ideal of $X$, and $U = X \setminus Y$. Set $\mathcal{X} = X^\text{rig}$ and $U = U^\text{rig}$. Then the following conditions are equivalent.

(a) $Y$ is non-empty.

(b) $[U] \neq [\mathcal{X}]$.

(c) The tube open subset $C_Y|_X$ (cf. 4.2.5) is non-empty.

In particular, we deduce the following theorem.

Theorem 8.2.19. The functor $\mathcal{X} \mapsto [\mathcal{X}]$ given by going to the separated quotients (§4.3. (a)) is a spectral functor on the category of rigid spaces of type $(V_\mathbb{R})$ or of type (N).

Note that the theorem implies, in particular, that the valuative space $\langle \mathcal{X} \rangle$ is reflexive (0.2.4.1).

Proposition 8.2.20. Let $\mathcal{C}$ be a category of rigid spaces. Suppose that $\mathcal{X} \mapsto [\mathcal{X}]$ is a spectral functor on $\mathcal{C}$. Then any prespectral functor $S$ on $\mathcal{C}$ is a spectral functor.

Proof. Let $X$ be a distinguished formal model of a coherent rigid space $\mathcal{X}$ in $\mathcal{C}$, $Y$ a non-empty closed subscheme of $X$ defined by an admissible ideal of $X$, and $U = X \setminus Y$. We need to show that $S(U) \neq S(\mathcal{X})$, where $U = U^\text{rig}$. Note that (b) and (c) in 8.2.18 are equivalent to each other also in our situation. Since $[U] \neq [\mathcal{X}]$ by the assumption, the tube open subset $T = C_Y|_X$ corresponding to $Y$ is non-empty. This means that $S(T)$ is non-empty, since $S$ is a prespectral functor, and hence that $S(U) \neq S(\mathcal{X})$ as $S(U) \cap S(T) = S(U \cap T) = \emptyset$. \hfill $\square$

By 8.2.19, Proposition 8.2.20 has the following corollary.

Corollary 8.2.21. The functor $\mathcal{X} \mapsto \langle \mathcal{X} \rangle^\text{cl}$ defined on the category of rigid spaces of type $(V_\mathbb{R})$ is a spectral functor.
The following statement, which for the type \((V_R)\) case is contained in 8.2.19, is an immediate corollary of 8.2.18.

**Corollary 8.2.22.** Let \(X\) be a rigid space of type \((V_R)\) or of type \((N)\), and \(U\) a quasi-compact open subspace of \(X\). If \(U \subsetneq X\), then \(\langle U \rangle^{\text{cl}} \subsetneq \langle X \rangle^{\text{cl}}\).

**Corollary 8.2.23.** Let \(X\) be a rigid space of type \((V_R)\) or of type \((N)\), and \(U, V\) quasi-compact open subspaces of \(X\). Then \(\langle U \rangle^{\text{cl}} = \langle V \rangle^{\text{cl}}\) implies \(U = V\).

**Proof.** Apply 8.2.22 to \(U \cup V\).


8.3 **Noetherness theorem**

8.3. (a) **Comparison of complete local rings**

**Proposition 8.3.1.** Let \(X = (\text{Spf } A)^{\text{rig}}\) be an affinoid of type \((V)\) or \((N)\), and \(x \in \langle X \rangle\) a closed classical point. Then for any \(n \geq 1\) we have the canonical isomorphism

\[
\Theta_{X,x}/m_{X,x}^n \cong \Theta_{s(x),s(x)}/m_{s(x),s(x)}^n.
\]

In particular, we have

\[
\hat{\Theta}_{X,x} \cong \hat{\Theta}_{s(x),s(x)},
\]

where \(\hat{\cdot}\) denotes the completions with respect to the maximal ideals.

For the proof, we refer to [18], (7.3.2/3). Here we offer a non-elementary proof, based on GAGA.

**Proof.** We may assume that \(A\) is a-torsion free. The closed classical point \(x\) is a closed subspace \(Z \hookrightarrow X\), and let \(J_Z \leq \Theta_X\) be the coherent ideal that defines \(Z\). Take the corresponding coherent ideal \(J_{s(Z)} \leq \Theta_{s(X)}\) by the correspondence as in 6.6.5, which is the defining ideal of the closed point \(s(Z) = s(x)\) on \(s(X)\); see the proof of 8.2.10 (1). Then for any \(n \geq 1\), we have

\[
\Theta_{s(X),s(x)}/m_{s(X),s(x)}^n \cong \Gamma(s(X), \Theta_{s(X)}/J_{s(Z)}^n) \\
\cong \Gamma(X, \Theta_{X}/J_{Z}^n) \cong \Theta_{X,x}/m_{X,x}^n,
\]

as desired.

**Corollary 8.3.2.** Let \(V\) be an \(a\)-adically complete valuation ring of height one and \(Y \hookrightarrow X\) an open immersion of affinoids of finite type over \((\text{Spf } V)^{\text{rig}}\). Then the associated morphism \(s(Y) \rightarrow s(X)\) of Noetherian schemes is flat, and maps the set of all closed points of \(s(Y)\) injectively to the set of all closed points of \(s(X)\). Moreover, for any closed point \(y \in s(Y)\) we have

\[
\hat{\Theta}_{s(Y),y} \cong \hat{\Theta}_{s(X),x},
\]

where \(x\) is the image of \(y\).
Proof. The flatness was already shown in 6.6.1 (1). Closed points of \( s(\mathcal{X}) \) are in one to one correspondence with classical points of \( \mathcal{X} \) due to 8.2.11; the restrictions of them to \( \mathcal{Y} \) are either empty or classical points of \( \mathcal{Y} \), and hence the fiber of each closed point of \( s(\mathcal{X}) \) under the map \( s(\mathcal{Y}) \to s(\mathcal{X}) \) consists of at most one closed point (cf. 8.2.10 (2)). The other assertion follows immediately from 8.3.1. \( \square \)

### 8.3. (b) Reducedness and irreducibility

**Proposition 8.3.3.** Let \( \mathcal{X} \) be a rigid space of type (V), and \( \mathcal{N}_\mathcal{X} \) the subsheaf of \( \mathcal{O}_\mathcal{X} \) consisting of locally nilpotent sections. Then \( \mathcal{N}_\mathcal{X} \) is a coherent ideal of \( \mathcal{O}_\mathcal{X} \).

**Proof.** Considering affinoid coverings of \( \mathcal{X} \), we may assume that \( \mathcal{X} \) is an affinoid \( \mathcal{X} = (\text{Spf } A)^\text{rig} \), where \( A \) is \( a \)-torsion free and topologically of finite type algebra over an \( a \) - adically complete valuation ring \( V \). Considering the base change by the height one localization \( V_\mathfrak{p} \) of \( V \), where \( \mathfrak{p} = \sqrt{(a)} \), one can assume without loss of generality that \( V \) is of height one. Let \( N \subseteq A[\frac{1}{a}] \) be the nilpotent radical of the Noetherian ring \( A[\frac{1}{a}] \), and consider the corresponding coherent ideal \( \mathcal{N} \subseteq \mathcal{O}_\mathcal{X} \) on \( \mathcal{X} = (\text{Spf } A)^\text{rig} \); see §6.3. Clearly we have \( \mathcal{N} \subseteq \mathcal{N}_\mathcal{X} \). We want to show \( \mathcal{N} = \mathcal{N}_\mathcal{X} \).

Considering \( A/A \cap N \), which is again an \( a \)-torsion free topologically of finite type \( V \)-algebra, one deduces the other inclusion from the following statement: if \( A[\frac{1}{a}] \) is reduced (that is, \( N = 0 \)), then \( \mathcal{X} \) is reduced, that is, \( \mathcal{O}_{\mathcal{X},x} \) is reduced for any \( x \in \langle \mathcal{X} \rangle \). To show this, due to 3.2.15, it suffices to show that, for any affinoid subdomain \( \mathcal{U} = (\text{Spf } B)^\text{rig} \subseteq \mathcal{X} \), the ring \( B[\frac{1}{a}] \) is reduced. But for this, we only need to show that the local ring of \( U = s(\mathcal{U}) = \text{Spec } B[\frac{1}{a}] \) at every closed point is reduced. For any \( y \in U^{\text{cl}} \), the morphism \( \mathcal{O}_{\mathcal{U},y} \to \widehat{\mathcal{O}}_{\mathcal{U},y} \) is faithfully flat, and hence it suffices to show that \( \widehat{\mathcal{O}}_{\mathcal{U},y} \) is reduced. Then, in view of 8.3.1, after all, what we need to show is: if \( A[\frac{1}{a}] \) is reduced, then, for any classical point \( x \in \langle \mathcal{X} \rangle^{\text{cl}} \) the complete local ring \( \widehat{\mathcal{O}}_{\mathcal{X},x} \) is reduced. But, again due to 8.3.1, this follows from the reducedness of \( \widehat{\mathcal{O}}_{\mathcal{X},s(x)} \), where \( X = s(\mathcal{X}) = \text{Spec } A[\frac{1}{a}] \), which is guaranteed by the fact that the classical affinoid algebra \( A[\frac{1}{a}] \) is excellent (0.9.3.13). \( \square \)

By 7.3.5 we have the closed immersion

\[ \mathcal{X}_{\text{red}} \hookrightarrow \mathcal{X} \]

with the defining ideal \( \mathcal{N}_\mathcal{X} \). The rigid space \( \mathcal{X}_{\text{red}} \) is determined up to canonical isomorphisms. We call \( \mathcal{X}_{\text{red}} \) the *reduced model* of \( \mathcal{X} \). By 3.3.6 (1) we deduce that the topological spaces \( \langle \mathcal{X} \rangle \) and \( \langle \mathcal{X}_{\text{red}} \rangle \) coincide.

Let \( \mathcal{X} \) be a rigid space of type (V), \( \mathcal{J} \) a coherent ideal of \( \mathcal{O}_\mathcal{X} \), and \( \mathcal{Z} \subseteq \mathcal{X} \) the closed subspace defined by \( \mathcal{J} \). Applying 8.3.3 to \( \mathcal{Z} \), one obtains a coherent ideal of \( \mathcal{O}_\mathcal{X} \) containing \( \mathcal{J} \) of which the stalk at any \( x \in \langle \mathcal{X} \rangle \) coincides with \( \sqrt{\mathcal{J}}_x \). We denote this sheaf by \( \sqrt{\mathcal{J}} \). This is a coherent ideal sheaf of \( \mathcal{O}_\mathcal{X} \), of which the
corresponding closed subspace is \( Z_{\text{red}} \). In case \( X \) is an affinoid, say \( X = (\text{Spf} \, A)^{\text{rig}} \), then, if \( I \) corresponds to an ideal \( I \subseteq A[\frac{1}{a}] \) as in §6.3 (that is, \( I = I \mathcal{O}_X \)), then \( \sqrt{I} \) corresponds to \( \sqrt{I} \).

**Corollary 8.3.4.** Let \( X = (\text{Spf} \, A)^{\text{rig}} \) be an affinoid of type \((V, \mathbb{R})\), and \( Z_1, Z_2 \) closed subspaces of \( X \). Let \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) be the defining ideal of \( Z_1 \) and \( Z_2 \), respectively. Then \( \langle Z_1 \rangle = \langle Z_2 \rangle \) if and only if \( \sqrt{\mathcal{I}_1} = \sqrt{\mathcal{I}_2} \).

**Proof.** The ‘if’ part is easy. To show the ‘only if’ part, let \( Z_1 \) and \( Z_2 \) be the closed subschemes of \( s(X) = \text{Spec} \, A[\frac{1}{a}] \) corresponding to \( Z_1 \) and \( Z_2 \), respectively, by the correspondence as in 7.3.15. If \( \langle Z_1 \rangle = \langle Z_2 \rangle \), we have \( \langle Z_1 \rangle^{\text{cl}} = \langle Z_2 \rangle^{\text{cl}} \), and hence the closed points in \( Z_1 \) coincides with those of \( Z_2 \) (8.2.11). Since the classical affinoid algebra \( A[\frac{1}{a}] \) is Jacobson (0.9.3.10), this means, if \( I_1 \) and \( I_2 \) are the ideals of \( A[\frac{1}{a}] \) defining \( Z_1 \) and \( Z_2 \), respectively, we have \( \sqrt{I_1} = \sqrt{I_2} \). Since \( \mathcal{I}_i = I_i \mathcal{O}_X \) for \( i = 1, 2 \), we have \( \sqrt{\mathcal{I}_1} = \sqrt{\mathcal{I}_2} \). \( \square \)

**Proposition 8.3.5.** Let \( V \) be an \( a \)-adically complete valuation ring of height one, and \( A \) a topologically finitely generated \( V \)-algebra.

1. The affinoid \( X = (\text{Spf} \, A)^{\text{rig}} \) is reduced if and only if so is the Noetherian scheme \( s(X) = \text{Spec} \, A[\frac{1}{a}] \).

2. The affinoid \( X = (\text{Spf} \, A)^{\text{rig}} \) is irreducible (7.3.12) if and only if so is the Noetherian scheme \( s(X) = \text{Spec} \, A[\frac{1}{a}] \).

**Proof.** (1) Set \( B = A[\frac{1}{a}] = \Gamma(X, \mathcal{O}_X) \), and let \( N \) be the nilpotent radical of the ring \( B \). As the proof of 8.3.3 indicates, \( X \) is reduced if and only if \( N = 0 \), whence (1).

(2) We may assume that \( s(X) = \text{Spec} \, A[\frac{1}{a}] \) is reduced, or what amounts to the same, that the ring \( A[\frac{1}{a}] \) is reduced. Consider reduced closed rigid subspaces \( Y, Z \subseteq X \), and the corresponding reduced closed subschemes \( s(Y), s(Z) \) of \( s(X) \) (cf. 7.3.15). It follows from 8.3.4 that \( \langle X \rangle = \langle Y \rangle \cup \langle Z \rangle \) if and only if \( s(X) = s(Y) \cup s(Z) \), and similarly that, for example, \( \langle Z \rangle = \langle X \rangle \) holds if and only if \( s(X) = s(Z) \). Hence we deduce that \( X \) is irreducible if and only if the scheme \( s(X) \) is irreducible. \( \square \)

Let \( A \) and \( X \) be as in 8.3.5, and

\[
s(X) = \bigcup_{i=1}^{r} X_i
\]

be the irreducible decomposition of the Noetherian scheme \( s(X) = \text{Spec} \, A[\frac{1}{a}] \).
For each \( i = 1, \ldots, r \), we have the uniquely determined closed rigid subspace \( \mathcal{X}_i \) of \( \mathcal{X} \) such that \( s(\mathcal{X}_i) = X_i \) (cf. 6.6.5); in fact, if \( q_i \) is the ideal of \( B = A[\frac{1}{\alpha}] \) corresponding to \( X_i \), then \( \mathcal{X}_i = (\text{Spf} \ A/q_i \cap A)^{\text{rig}} \). By 8.3.5, each \( Z_i \) \((i = 1, \ldots, r)\) is irreducible, and we have the irreducible decomposition

\[
\langle \mathcal{X} \rangle = \bigcup_{i=1}^{r} \langle \mathcal{X}_i \rangle.
\]

8.3. (c) Noetherness theorem

**Theorem 8.3.6.** Let \( \mathcal{X} \) be a rigid space of type (V) or of type (N), and \( x \in \langle \mathcal{X} \rangle \) a point. Then the local ring \( \mathcal{O}_{\mathcal{X},x} \) is Noetherian.

**Proof of Theorem 8.3.6 in the type (V) case.** We may assume that \( \mathcal{X} \) is an affinoid. First we show the assertion in the following case: \( \mathcal{X} \) is of finite type over \((\text{Spf} \ V)^{\text{rig}}\), where \( V \) is an \( a \)-adically complete valuation ring of height one, and \( x \in \langle \mathcal{X} \rangle \) is a classical point. We use the notation as in the proof of 8.3.1, with \( R_\alpha = \mathcal{O}_{s(\mathcal{U}_\alpha),x_\alpha} \) for \( \alpha \in L \). By 6.6.4, we only need to check (c) in 0.3.1.5. As we have shown in the proof of 8.3.1, we have \( R_\alpha/m_\alpha^n \cong R_\beta/m_\beta^n \) for any \( \alpha \leq \beta \) and \( n \geq 1 \). Hence we have \( m_\alpha R_\beta + m_\beta^n = m_\beta \) for any \( n \geq 1 \). But since \( R_\beta \) is Noetherian, every ideal is closed, and hence we have \( m_\alpha R_\beta = m_\beta \), as desired.

To show the assertion in the type (V) case in general, note first that by 3.2.17 (3), 0.9.1.10, and 0.9.2.4, we may assume that \( V \) is of height one. Consider for any point \( x \in \langle \mathcal{X} \rangle \) the associated rigid point \( \text{Spf} \ V^\text{rig} \rightarrow \mathcal{X} \) (3.3.5). By 3.2.17 (3), we may assume that \( x \) is of height one and hence that \( V' \) is of height one. Consider the base-change \( \mathcal{X}' = \mathcal{X} \times_{(\text{Spf} \ V)^{\text{rig}}} (\text{Spf} \ V')^{\text{rig}} \) and the induced diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{x} & \mathcal{X}' \\
\downarrow & & \downarrow \\
(\text{Spf} \ V)^{\text{rig}} & \leftarrow & (\text{Spf} \ V')^{\text{rig}}
\end{array}
\]

The rigid point \((\text{Spf} \ V')^{\text{rig}} \rightarrow \mathcal{X}'\) determines a classical point \( x' \). By the special case treated above, we know that \( \mathcal{O}_{\mathcal{X}',x'} \) is Noetherian. Since \( V' \) is faithfully flat over \( V \), the map \( \mathcal{O}_{\mathcal{X},x} \rightarrow \mathcal{O}_{\mathcal{X}',x'} \) is faithfully flat, and hence \( \mathcal{O}_{\mathcal{X},x} \) is Noetherian, as desired. \( \square \)

**Proof of Theorem 8.3.6 in the type (N) case.** We may assume that \( \mathcal{X} \) is an affinoid, \( \mathcal{X} = (\text{Spf} \ A)^{\text{rig}} \), where \( A \) is an \( I \)-adically complete Noetherian ring for an ideal \( I \subseteq A \). Take a rigid point \((\text{Spf} \ V)^{\text{rig}} \rightarrow \mathcal{X} \), where \( V \) is an \( a \)-adically complete valuation ring \((a \in m_V \setminus \{0\})\), that maps the closed point to the point \( x \in \langle \mathcal{X} \rangle \). Take a cofinal system of formal neighborhoods \( \{ U_\alpha = (\text{Spf} \ A_\alpha)^{\text{rig}} \}_{\alpha \in L} \) of \( x \) in
such a way that each Spf $A_\alpha$ lies in an admissible blow-up of Spf $A$ as an affine open subspace; note that each $A_\alpha$ is Noetherian. Let $x_\alpha$ for each $\alpha \in L$ be the image of $x$ under the specialization map $\langle X \rangle \to \text{Spf } A_\alpha$, that is, the image of the closed point under Spec $V \to \text{Spec } A_\alpha$. Furthermore, let $\xi_\alpha \in \text{Spec } A_\alpha$ be the image of the generic point under the morphism Spec $V \to \text{Spec } A_\alpha$, that is, the image of $x_\alpha$ under the map $\text{Spec } V \to \text{Spec } A_\alpha$. We have $\mathcal{O}_{X,x} = \lim_{\alpha \in L} \mathcal{O}_{\text{Spec } A_\alpha, \xi_\alpha}$, which we need to show to be Noetherian. To this end, set $R_\alpha = \mathcal{O}_{\text{Spec } A_\alpha, x_\alpha}$; we denote also by $\xi_\alpha$ the image of $x_\alpha$ in Spec $R_\alpha$.

Consider the completion $\widehat{R_\alpha}$ with respect to the maximal ideal; each $R_\alpha \to \widehat{R_\alpha}$ is faithfully flat, and for each $\alpha \leq \beta$ we have the induced local map $\widehat{R_\alpha} \to \widehat{R_\beta}$. Note that each $\widehat{R_\alpha}$ is quasi-excellent, since it is a Noetherian complete local ring. Set $B_\alpha = \mathcal{O}_{\text{Spec } \widehat{R_\alpha}, \hat{\xi_\alpha}}$ (where $\hat{\xi_\alpha} \in \text{Spec } \widehat{R_\alpha}$ is the point above $\xi_\alpha \in \text{Spec } R_\alpha$); for $\alpha \leq \beta$, we have the induced local morphisms $B_\alpha \to B_\beta$.

Now for $\alpha \leq \beta$ the transition map $A_\alpha \to A_\beta$ comes from a Zariski open part of an admissible blow-up, hence so is $R_\alpha \to R_\beta$. Since $R_\alpha \to \widehat{R_\alpha}$ is flat, $\widehat{R_\alpha} \to \widehat{R_\beta}$ is a composition of an admissible blow-up (with respect to the $I$-adic topology) followed by completion with respect to a maximal ideal. Since $\widehat{R_\alpha}$ is quasi-excellent, the formal fiber $B_\alpha \to B_\beta$ is regular. Now we fix an $\alpha_0 \in L$ and replace $L$ by the cofinal subset $\{\alpha \in L: \alpha \geq \alpha_0\}$. Then $\{\dim B_\alpha\}_{\alpha \in L}$ is upper-bounded by $\dim A_{\alpha_0}$. Hence, by 0.3.1.6, we deduce that $B = \lim_{\alpha \in L} B_\alpha$ is Noetherian. Now since any $\mathcal{O}_{\text{Spec } A_\alpha, \xi_\alpha} \to B_\alpha$ is faithfully flat, we deduce that $\mathcal{O}_{X,x} \to B$ is faithfully flat; since $B$ is known to be Noetherian, so is $\mathcal{O}_{X,x}$, as desired.  

9 GAGA

In this section we discuss GAGA theorems in rigid geometry. The first subsection, §9.1, is devoted to the definition of GAGA functor, which associates to any separated of finite type scheme $X$ over $U = \text{Spec } A \setminus V(I)$, where $A$ is an adic ring with a finitely generated ideal of definition $I \subseteq A$, the ‘analytification’ $X^\text{an}$, a separated rigid space over the affinoid (Spf $A)^\text{rig}$. Some of the basic properties of the GAGA functor will be discussed in §9.1.(c). We also give a generalization of the GAGA functor to non-separated schemes (§9.1.(e)).

After discussing affinoid valued points (§9.2), we introduce the so-called comparison maps and comparison functors in §9.3, by means of which the GAGA theorems, the GAGA comparison theorem and GAGA existence theorem, are formulated and proved in §9.4 and §9.5 from GFGA theorems. Similarly to the GFGA theorems (I, §9 and §10), our GAGA theorems are stated in terms of derived categorical language.
9.1 Construction of GAGA functor

9.1. (a) The category $\text{Emb}_{X|S}$. Let $A$ be an adic ring of finite ideal type, and $I \subseteq A$ a finitely generated ideal of definition. Set

$$S = \text{Spec } A \leftarrow U = S \setminus D,$$

where $D = V(I)$ is the closed subset corresponding to $I$, and

$$S = (\text{Spf } A)^{\text{rig}}.$$

Let $f : X \to U$ be a separated $U$-scheme of finite type. We define the category $\text{Emb}_{X|S}$ as follows.

- The objects are the commutative diagrams

$$
\begin{array}{ccc}
X & \xleftarrow{f} & \bar{X} \\
\downarrow \kern{1cm} \downarrow & & \downarrow \kern{1cm} \downarrow \\
U & \xleftarrow{\bar{f}} & S
\end{array}
$$

where $\bar{f} : \bar{X} \to S$ is a proper $S$-scheme, and $X \xleftarrow{f} \bar{X}$ is a birational open immersion, that is, an open immersion onto a dense open subspace of $X$ (cf. §E.1.2 (3) below).

- A morphism $(X \xleftarrow{f} \bar{X}) \to (X \xleftarrow{f'} \bar{X}')$ is an $X$-admissible $S$-modification $\bar{X} \to \bar{X}'$ (cf. §E.1.2).

Note that the category $\text{Emb}_{X|S}$ is non-empty due to Nagata’s embedding theorem (F.1.1).

**Lemma 9.1.1.** (1) The category $\text{Emb}_{X|S}$ is cofiltered.

(2) Let $\bar{X} = (X \xleftarrow{f} \bar{X}, f : \bar{X} \to S)$ be an object of $\text{Emb}_{X|S}$, and $\text{Ald}^*_\text{op}(\bar{X}, X)$ be the set of all quasi-coherent ideals of $\mathcal{O}_{\bar{X}}$ of finite type such that the corresponding closed subscheme is disjoint from $X$ (cf. §E.1. (b)). We introduce an order on the set $\text{Ald}^*_\text{op}(\bar{X}, X)$ as follows: $\mathcal{J} \preceq \mathcal{J}'$ if there exists $\mathcal{J}'' \in \text{Ald}^*_\text{op}(\bar{X}, X)$ such that $\mathcal{J} = \mathcal{J}'' \cdot \mathcal{J}'$. Then $\text{Ald}^*_\text{op}(\bar{X}, X)$ is a directed set, and the functor

$$\text{Ald}^*_\text{op}(\bar{X}, X) \longrightarrow \text{Emb}_{X|S}$$

that maps $\mathcal{J}$ into $\bar{X}_g = (X \xleftarrow{f_g} \bar{X}_g, f_g : \bar{X}_g \to S)$, where $X_g \to X$ is the blow-up along $\mathcal{J}$, is cofinal.
Proof. (1) We have to show that the following conditions are fulfilled.

(a) For \( \overline{X}_1 = (X \hookrightarrow \overline{X}_1, f_1: \overline{X}_1 \to S) \) and \( \overline{X}_2 = (X \hookrightarrow \overline{X}_2, f_2: \overline{X}_2 \to S) \), there exist an object \( \overline{X}_3 = (X \hookrightarrow \overline{X}_3, f_3: \overline{X}_3 \to S) \) and \( X \)-admissible \( S \)-modifications \( \overline{X}_3 \to \overline{X}_1 \) and \( \overline{X}_3 \to \overline{X}_2 \).

(b) For \( \overline{X}_1 = (X \hookrightarrow \overline{X}_1, f_1: \overline{X}_1 \to S) \) and \( \overline{X}_2 = (X \hookrightarrow \overline{X}_2, f_2: \overline{X}_2 \to S) \) and two \( X \)-admissible \( S \)-modifications \( q_0, q_1: \overline{X}_2 \to \overline{X}_1 \), there exist an object \( \overline{X}_3 = (X \hookrightarrow \overline{X}_3, f_3: \overline{X}_3 \to S) \) and an \( X \)-admissible \( S \)-modification \( p: \overline{X}_3 \to \overline{X}_2 \) such that \( q_0 \circ p = q_1 \circ p \).

To show (a), let \( \overline{X}_3 = \overline{X}_1 \star \overline{X}_2 \), the join of \( \overline{X}_1 \) and \( \overline{X}_2 \) (cf. E.1.10); that is, the closure of \( X \) in the product \( \overline{X}_1 \times_S \overline{X}_2 \). Then \( \overline{X}_3 \to S \) is proper. Clearly, the morphisms by projections \( \overline{X}_3 \to \overline{X}_1 \) and \( \overline{X}_3 \to \overline{X}_2 \) are \( X \)-admissible modifications.

To show (b), consider the Cartesian diagram of \( S \)-schemes

\[
\begin{array}{ccc}
\overline{X}_2 & \xrightarrow{(q_0,q_1)} & \overline{X}_1 \\
\downarrow & & \downarrow \Delta \overline{X}_1 \\
Z & \longrightarrow & \overline{X}_1
\end{array}
\]

Since \( \overline{X}_1 \) is separated over \( S \), the right-hand vertical map is a closed immersion, and hence so is the left-hand one. The scheme \( Z \) contains a copy of \( X \). Let \( \overline{X}_3 \) be the scheme-theoretic closure of \( X \) in \( Z \). Then \( \overline{X}_3 \to S \) defines an object of \( \text{Emb}_{X|S} \), and the map \( p: \overline{X}_3 \to \overline{X}_2 \) is clearly an \( X \)-admissible \( S \)-modification such that \( q_0 \circ p = q_1 \circ p \).

(2) follows immediately from [89], Première partie, (5.7.12), (see E.1.9).

9.1. (b) Construction of \( X^{\text{an}} \)

Construction 9.1.2. We continue working in the setting of §9.1 (a). Let \( f: X \to U \) be a separated \( U \)-scheme of finite type, and

\[
\begin{array}{ccc}
X & \xrightarrow{\overline{f}} & \overline{X} \\
\downarrow f & & \downarrow \overline{f} \\
U & \xrightarrow{\text{incl.}} & S
\end{array}
\]

be an object of \( \text{Emb}_{X|S} \). Set

\[
\begin{align*}
Z &= (\overline{X} \times_S U) \setminus X, \\
\overline{Z} &= \text{the closure of } Z \text{ in } \overline{X}, \\
\widetilde{X} &= \overline{X} \setminus \overline{Z}.
\end{align*}
\]
Let \( \hat{X} \hookrightarrow \tilde{X} \) be the \( I \)-adic completion of the open immersion \( \tilde{X} \hookrightarrow \tilde{X} \). We have the open immersion

\[
(\hat{X})^{\text{rig}} \hookrightarrow (\tilde{X})^{\text{rig}}
\]

of coherent rigid spaces.

**Definition 9.1.3.** Let the notation be as in 9.1.2. We define a sheaf \( X^{\text{an}} \) on the site \( \mathbf{Rf}_{S, \text{ad}} \) by

\[
X^{\text{an}} = \lim_{\to} (\hat{X})^{\text{rig}},
\]

where the inductive limit is taken along the filtered category \( \text{Emb}_{X|S}^{\text{opp}} \) or, equivalently, along the set \( \text{AId}^*_{(\tilde{X}, X)} \).

**Proposition 9.1.4.** The sheaf \( X^{\text{an}} \) on \( \mathbf{Rf}_{S, \text{ad}} \) is a quasi-separated rigid space.

**Proof.** Let \( \tilde{X} = (X \hookrightarrow \tilde{X}, \tilde{f}: \tilde{X} \to S) \) be an object of \( \text{Emb}_{X|S} \), and consider an \( X \)-admissible blow-up \( \tilde{X}_1 \to \tilde{X} \), that is, a blow-up along an ideal in \( \text{AId}^*_{(\tilde{X}, X)} \).

Then the induced morphism \( \hat{X}_1 \to \hat{X} \) is an admissible blow-up of coherent adic formal schemes of finite ideal type, and thus we have the canonical isomorphism \( (\hat{X}_1)^{\text{rig}} \cong (\hat{X})^{\text{rig}} \). It follows that the corresponding \( (\hat{X}_1)^{\text{rig}} \) sits in the following commutative diagram consisting of open immersions:

\[
\begin{array}{ccc}
(\hat{X}_1)^{\text{rig}} & \to & (\hat{X})^{\text{rig}} \\
\downarrow & & \downarrow \\
(\hat{X})^{\text{rig}} & \to & (\tilde{X})^{\text{rig}}.
\end{array}
\]

As the set \( \text{AId}^*_{(\tilde{X}, X)}^{\text{opp}} \) is cofinal in \( \text{Emb}_{X|S} \), we deduce that \( X^{\text{an}} \) is a stretch of coherent rigid spaces and hence is quasi-separated (3.5.3). \( \Box \)

Note that by the construction we always have a canonical open immersion

\[
(\hat{X})^{\text{rig}} \hookrightarrow X^{\text{an}}
\]

for any object \( (X \hookrightarrow \tilde{X}) \) of \( \text{Emb}_{X|S} \).

**Proposition 9.1.5.** If \( f: X \to U \) is proper, then \( X^{\text{an}} = (\hat{X})^{\text{rig}} \). In particular, \( X^{\text{an}} \) is a coherent rigid space.

**Proof.** Take an arbitrary \( (X \hookrightarrow \tilde{X}) \in \text{obj}(\text{Emb}_{X|S}) \), and let \( \tilde{X}' \) be the closure of \( X \) in \( \tilde{X} \). Then by [89], Première partie, (5.7.12), (see E.1.9 below) one sees that \( (\hat{X})^{\text{rig}} \cong (\tilde{X}')^{\text{rig}} \). When we replace \( \tilde{X} \) by \( \tilde{X}' \), then we have \( \tilde{Z} = \emptyset \), thereby the result. \( \Box \)
In particular, we have $U^{\text{an}} = S$ ($= (\text{Spf } A)^{\text{rig}}$).

**Construction 9.1.6.** Next we construct, for a given $U$-morphism $h: Y \to X$ of $U$-schemes of finite type, the morphism of rigid spaces

$$h^{\text{an}}, Y^{\text{an}} \to X^{\text{an}}.$$ 

Let us take arbitrary $(X \hookrightarrow \bar{X}) \in \text{obj}(\text{Emb}_{X|S})$ and $(Y \hookrightarrow \bar{Y}) \in \text{obj}(\text{Emb}_{Y|S})$. Let $\bar{Y}_1$ be the closure of the graph of $h$ in the product $\bar{Y} \times_S \bar{X}$; $\bar{Y}_1$ is proper and admits a birational open immersion $Y \hookrightarrow \bar{Y}_1$, that is, $(Y \hookrightarrow \bar{Y}_1)$ defines an object of $\text{Emb}_{Y|S}$ that sits in the commutative square

$$
\begin{array}{ccc}
Y & \to & X \\
\downarrow & & \downarrow \\
\bar{Y}_1 & \to & \bar{X}
\end{array}
$$

Thus we get $\bar{Y} \to \bar{X}$. As both $(X \hookrightarrow \bar{X}) \in \text{obj}(\text{Emb}_{X|S})$ and $(Y \hookrightarrow \bar{Y}) \in \text{obj}(\text{Emb}_{Y|S})$ are taken arbitrary, we get the desired morphism $h^{\text{an}}$.

Thus we get a functor

$$X \mapsto X^{\text{an}}$$

from the category of separated $U$-schemes of finite type to the category of rigid spaces over $S$, called the GAGA functor.

**Remark 9.1.7.** Our construction of the GAGA functor relies on Nagata’s embedding theorem F.1.1. The first author has proven a strong version of the Nagata’s embedding theorem (F.5.1) by which one can define the GAGA functor for algebraic spaces as follows. In the setting of §9.1.(a), we consider a separated $U$-algebraic space $f: X \to U$ of finite type. We define the category $\text{Emb}_{X|S}$ as before, with the additional condition that the boundary $\bar{X} \setminus X$ is a scheme (cf. F.5.1 (a)). Now in 9.1.2, since $\bar{X} \times_S D$ and $\bar{X} \times S D$ are schemes, the formal completions $\hat{X}$ and $\hat{X}$ are formal schemes and hence define the rigid spaces $(\hat{X})^{\text{rig}}$ and $(\hat{X})^{\text{rig}}$. Then one can just carry out the same construction to get the desired GAGA functor $X \mapsto X^{\text{an}}$ from the category of separated $U$-algebraic spaces of finite type to the category of rigid spaces over $S$. See [31] for another approach to define GAGA functors for algebraic spaces.

Let us finally remark that, if the adic ring $A$ in §9.1.(a) is t.u. rigid-Noetherian (resp. t.u. adhesive) (I.2.1.1), then the rigid space $X^{\text{an}}$ is locally universally Noetherian (resp. locally universally adhesive) (2.2.23).
9.1. (c) Some basic properties of the GAGA functor

Proposition 9.1.8. The GAGA functor

\[ X \mapsto X^{\text{an}} \]

maps a U-open immersion \( V \hookrightarrow X \) to an S-open immersion \( V^{\text{an}} \hookrightarrow X^{\text{an}} \) and, in case the adic ring \( A \) in §9.1. (a) is t.u. rigid-Noetherian, a U-closed immersion \( Y \hookrightarrow X \) to an S-closed immersion \( Y^{\text{an}} \hookrightarrow X^{\text{an}} \).

Proof. First we prove that the functor \( X \mapsto X^{\text{an}} \) maps an open immersion to an open immersion. As in 9.1.6, one can consider morphisms of the form \( \overline{V} \to \overline{X} \) starting from objects in \( \text{Emb}_{X|S} \) and \( \text{Emb}_{V|S} \). By [89], Première partie, (5.7.11), (included as E.1.8 below), the morphism \( \overline{V} \to \overline{X} \) can be replaced, by means of an \( X \)-admissible blow-up and the strict transform, by an open immersion. Hence the \( I \)-adic completion \( \overline{V} \to \overline{X} \) is an open immersion, and thus \( (\overline{V})^{\text{rig}} \to (\overline{X})^{\text{rig}} \) is an open immersion of coherent rigid spaces. As the rigid spaces \( V^{\text{an}} \) and \( X^{\text{an}} \) are unions of them, we deduce that the morphism \( V^{\text{an}} \hookrightarrow X^{\text{an}} \) is an open immersion by definition 2.2.20.

Next we deal with the assertion for a closed immersion \( h: Y \hookrightarrow X \). To show that \( h^{\text{an}} \) is a closed immersion, it suffices to show the following facts.

(a) For any object \( (X \hookrightarrow \overline{X}) \) of \( \text{Emb}_{X|S} \), the closure \( \overline{Y} \) of \( Y \) in \( \overline{X} \) gives an object of \( \text{Emb}_{Y|S} \).

(b) For any object \( (Y \hookrightarrow \overline{Y}) \) of \( \text{Emb}_{Y|S} \), there exists an object \( (X \hookrightarrow \overline{X}) \) of \( \text{Emb}_{X|S} \) such that the closure of \( Y \) in \( \overline{X} \) dominates \( \overline{Y} \).

If these properties are satisfied, we get a cofinal family \( \{ h_{\lambda}: \overline{Y}_{\lambda} \hookrightarrow \overline{X}_{\lambda} \} \) of closed immersions over \( S \) indexed by a directed set such that \( \lim_{\rightarrow \lambda} (h_{\lambda})^{\text{rig}} = h^{\text{an}} \), thereby the assertion.

Assertion (a) is clear. To see (b), take any \( (X \hookrightarrow \overline{X}) \) and take the closure of the graph of the map \( Y \hookrightarrow \overline{X} \) in \( \overline{Y} \times_S \overline{X} \), which clearly dominates \( \overline{Y} \).

\[
\text{Proposition 9.1.9. Suppose in the setting of §9.1. (a) that } A \text{ is t.u. rigid-Noetherian. Let } X \text{ be a separated } U \text{-scheme of finite type, and } Y \text{ a closed subscheme of } X. \text{ Then }
\]

\[
(X \setminus Y)^{\text{an}} = X^{\text{an}} \setminus Y^{\text{an}}
\]

(cf. 7.3.13 for the notion of open complements of rigid spaces).

Proof. As in the proof of the second part of 9.1.8, we have a cofinal family of \( S \) closed immersions \( \{ h_{\lambda}: \overline{Y}_{\lambda} \hookrightarrow \overline{X}_{\lambda} \} \) such that \( \lim_{\rightarrow \lambda} (h_{\lambda})^{\text{rig}} = h^{\text{an}} \). Then \( \overline{V}_{\lambda} = \overline{X}_{\lambda} \setminus \overline{Y}_{\lambda} \) gives the cofinal family of embeddings \( V = X \setminus Y \hookrightarrow \overline{V} \) as in 9.1.2. Therefore, \( V^{\text{an}} = \lim_{\rightarrow \lambda} (\overline{V}_{\lambda})^{\text{rig}} = X^{\text{an}} \setminus Y^{\text{an}} \).
Proposition 9.1.10. The GAGA functor $X \mapsto X^{\text{an}}$ is left-exact. Moreover, it is compatible with base change, that is, for any adic morphism $A \to A'$ with $S' = \text{Spec } A'$ and $U' = S' \setminus V(IA')$, where $I \subseteq A$ is a finitely generated ideal of definition, and for any separated of finite type $U$-scheme $X$ (resp. morphism $h$ between separated of finite type $U$-schemes), we have $(X \times_U U')^{\text{an}} \cong X^{\text{an}} \times_S S'$ (resp. $(h \times_U U')^{\text{an}} \cong h^{\text{an}} \times_S S'$), where $S' = (\text{Spf } A')^{\text{rig}}$.

Proof. To show that the GAGA functor is left-exact, it suffices to show that it preserves fiber products. Consider a diagram $X \to Z \leftarrow Y$ of separated $U$-schemes of finite type. We first suppose that $X, Y, Z$ are proper over $U$. In this case, $(X \times_Z Y)^{\text{an}}$ is the associated rigid space of the formal completion of $\hat{X} \times_{\hat{Z}} \hat{Y}$, where $\hat{X} \to \hat{Z} \leftarrow \hat{Y}$ is a diagram consisting of Nagata compactifications. Hence, in this case, the assertion is clear. In general, we first take Nagata compactifications $X, Y,$ and $Z$ to embed them into a proper $U$-schemes and then apply 9.2.2 below to compare rigid points of $(X \times_Z Y)^{\text{an}}$ and those of $X^{\text{an}} \times_{Z^{\text{an}}} Y^{\text{an}}$. The compatibility with base change can be proved similarly. □

Corollary 9.1.11. Suppose that we are in the same situation as in §9.1. (a) and that $A$ is t.u. rigid-Noetherian. Then for any separated $U$-scheme $X$ of finite type the rigid space $X^{\text{an}}$ is separated.

Proof. By 9.1.10 and 9.1.8, the diagonal morphism

$$X^{\text{an}} \to X^{\text{an}} \times_S X^{\text{an}} = (X \times_U X)^{\text{an}}$$

is a closed immersion. □

9.1. (d) Some examples

Examples 9.1.12. (1) The additive group $\mathbb{G}^{\text{an}}_a$. Consider the closed subscheme $D$ of $\mathbb{P}^1_U$ corresponding to the $\infty$-section, that is, $\mathbb{P}^1_U \setminus D = \mathbb{G}_{a,U}$. Clearly, we have $D^{\text{an}} \cong U^{\text{an}} = \hat{S}^{\text{rig}}$. Then by 9.1.9, we have

$$\mathbb{G}^{\text{an}}_a = \mathbb{P}^{1,\text{an}}_U \setminus D^{\text{an}};$$

note that by 9.1.5 the projective space $\mathbb{P}^{1,\text{an}}_U$ coincides with $\hat{\mathbb{P}}^{1,\text{an}}_S = (\hat{\mathbb{P}}^{1}_S)^{\text{rig}}$ defined in §2.5. (d). Let $\mathcal{J}_D$ be the ideal defining $\hat{D}$ (the $\infty$-section over $S$) in $\hat{\mathbb{P}}^{1}_S$, and consider the admissible blow-up of $\hat{\mathbb{P}}^{1}_S$ by the admissible ideal $\mathcal{J}_D + I^n\mathcal{O}_{\hat{\mathbb{P}}^{1}_S}$ for each $n$. Define $U_n$ to be the open part of the admissible blow-up where $\mathcal{J}_D$ generates the strict transform of $\mathcal{J}_D + I^n\mathcal{O}_{\hat{\mathbb{P}}^{1}_S}$. Then we have

$$\mathbb{G}^{\text{an}}_a = \bigcup_{n \geq 0} U^{\text{rig}}_n$$

(cf. Exercise II.9.1).
(2) The multiplicative group $\mathbb{G}_m^\text{an}$. This has a similar description as above where $D$ is replaced by the union of the $\infty$-section and the 0-section.

**9.1. (e) GAGA functor for non-separated schemes.** Let $U$ and $S$ be as in §9.1. (a), and $X \to U$ a quasi-separated $U$-scheme locally of finite type. (Note that, in the case where the ring $A$ is t.u. rigid-Noetherian, any $U$-scheme of finite type $X$ is quasi-separated, since it is Noetherian.) Let $X = \bigcup_{\alpha \in L} U_\alpha$ be a finite open covering by quasi-compact and separated schemes. Then we define $X^\text{an}$ to be the rigid space representing the sheaf on $\text{CRF}_{S^\text{an}, \text{ad}}$ that sits in the following exact sequence

$$\bigsqcup_{\alpha, \beta \in L} (U_\alpha \cap U_\beta)^\text{an} \longrightarrow \bigsqcup_{\alpha \in L} U_\alpha^\text{an} \longrightarrow X^\text{an}.$$

By 9.1.8, the rigid space $X^\text{an}$ does not depend on the choice of the open covering. By 9.1.8, in case $X$ is separated, the above $X^\text{an}$ coincides with the one defined in 9.1.3.

A similar construction can be used to define $h^\text{an}$ for a morphism $h : X \to Y$ over $U$ between quasi-separated $U$-schemes of finite type. This yields the extension of the GAGA functor to the category of all quasi-separated $U$-schemes of finite type.

### 9.2 Affinoid valued points

We continue working in the setting of §9.1. (a). Let $\mathcal{T}$ be an affinoid, and suppose a morphism

$$\alpha : \mathcal{T} \longrightarrow X^\text{an}$$

of rigid spaces is given.

**Lemma 9.2.1.** There exists $\widetilde{X}$ as in 9.1.2 such that the morphism $\alpha$ factors through a morphism $\mathcal{T} \to (\widetilde{X})^\text{rig}$.

**Proof.** Since $\mathcal{T}$ is quasi-compact, there exists a quasi-compact open subspace $\mathcal{U}$ of $X^\text{an}$ such that $\alpha$ maps $\mathcal{T}$ to $\mathcal{U}$. Since $X^\text{an}$ is quasi-separated, $\mathcal{U}$ is coherent and hence is coherent in the small admissible site $(X^\text{an})_{\text{ad}}$. By [9], Exposé VI, 1.23, there exists an $\widetilde{X}$ as in 9.1.2 such that $\mathcal{U} \subseteq (\widetilde{X})^\text{rig}$, thereby the lemma.

We henceforth denote the morphisms $\mathcal{T} \to (\widetilde{X})^\text{rig}$ and $\mathcal{T} \to (\widetilde{X})^\text{rig}$ also by $\alpha$. We now assume that $\mathcal{T}$ is a universally adhesive affinoid of the form $\mathcal{T} = \text{Spf } B$; we may furthermore assume without loss of generality that $B$ is $J$-torsion free, where $J \subseteq B$ is an ideal of definition.
Consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\alpha} & (\widehat{X})_{\text{rig}} \\
 & \downarrow & \downarrow (f)_{\text{rig}} \\
S_{\text{rig}} & \xrightarrow{\beta} & S
\end{array}
\]

There exists an admissible blow-up \( T' \rightarrow T \) such that this diagram is induced from a commutative diagram of formal schemes of the form

\[
\begin{array}{ccc}
T' & \xrightarrow{\alpha'} & \widehat{X} \\
\downarrow & & \downarrow j \\
T & \xrightarrow{\beta'} & \widehat{S}
\end{array}
\]

which in turn induces the diagram

\[
\begin{array}{ccc}
T' & \xrightarrow{\alpha'} & \widehat{X} \times_{\widehat{S}} T \\
\downarrow & & \downarrow \\
T & \rightarrow & T.
\end{array}
\]

Since the morphisms in the last diagram are all proper (and of finite presentation, since \( B \) is \( J \)-torsion free), by \ref{1.10.3.1} we have

\[
\begin{array}{ccc}
Y' & \xrightarrow{\alpha'} & \widehat{X} \times_S Y \\
\downarrow & & \downarrow \\
Y & \rightarrow & Y
\end{array}
\]

where \( Y = \text{Spec} \, B \) and \( Y' \rightarrow Y \) is the blow-up that gives the admissible blow-up \( T' \rightarrow T \) by passage to the formal completions.

Since the morphism \( T' \rightarrow \widehat{X} \) as above maps \( T' \) to the open formal subscheme \( \widehat{X} \), we see that the morphism \( Y' \rightarrow \widehat{X} \) maps \( Y' \) to \( \widehat{X} \). Therefore,

\[
\tilde{\alpha}: Y_U = \text{Spec} \, B \setminus V(J) = s(\mathcal{T}) \rightarrow X, \quad (\star)
\]

by the base change; cf. §6.6. (a) for the notion of the associated schemes \( s(\mathcal{T}) \).
Theorem 9.2.2. Let \( \mathcal{T} = (\text{Spf } B)^{\text{rig}} \) be an affinoid, where \( B \) is a t.u. adhesive ring. Then the map

\[
\{ \text{morphisms } \alpha: \mathcal{T} \to X^\text{an} \text{ of rigid spaces} \} \to \left\{ \text{pairs } (\beta, h) \text{ consisting of } \beta: \mathcal{T} \to S^\text{an} \text{ and } h: s(\mathcal{T}) \to X \text{ such that the diagram } s(\mathcal{T}) \to X \to U \text{ commutes, where the arrow } s(\mathcal{T}) \to s(S^\text{an}) = U \text{ is the one obtained from } \beta \right\}
\]

given by

\[
\alpha \mapsto (f^\text{an} \circ \alpha, \tilde{\alpha})
\]

(\text{where } \tilde{\alpha} \text{ is the one as in } (\ast)) \text{ is a bijection.}

Proof. We are going to construct the inverse map of the map (\ast\ast). Take a pair \((\beta, h)\) as above. As before, set \( \mathcal{T} = T^{\text{rig}} \), where \( T = \text{Spf } B \), and set \( Y = \text{Spec } B \); we suppose, moreover, that \( B \) is \( J \)-torsion free, where \( J \subseteq B \) is an ideal of definition. For a Nagata compactification \( \tilde{X} \) as in 9.1.2, there exist a \( Y_U \)-admissible blow-up \( Y' \to Y \) and a morphism \( \tilde{h}: Y' \to \tilde{X} \) that gives rise to the map \( h \) by base change.

So far, we get a morphism \( \alpha: \mathcal{T} \to \tilde{X}^\text{an} \) of rigid spaces. We need to show that the image of this map lies in \( X^\text{an} \subseteq \tilde{X}^\text{an} \). Let \( \tilde{Z} \) be as in 9.1.2, and consider the base change

\[
\tilde{X} \times_S Y' \to Y'.
\]

The morphism \( \tilde{h} \) gives a section of this morphism and hence gives a closed immersion

\[
i_1: Y' \hookrightarrow \tilde{X} \times_S Y'.
\]

On the other hand, we have the closed immersion

\[
i_2: \tilde{Z} \times_S Y' \hookrightarrow \tilde{X} \times_S Y'.
\]

Let \( \mathcal{I}_j \) be the defining ideal of the closed immersion \( i_j \) for \( j = 1, 2 \). These ideals are of finite type, since the schemes we are working with are all \( J \)-torsion free. Since \( i_1 \times_S U \) and \( i_2 \times_S U \) have the disjoint images, the ideal \( \mathcal{I}_1 + \mathcal{I}_2 \) is an open ideal with respect to the \( J \)-adic topology. Let \( \mathcal{I} \) be the push-out of the ideal \( \mathcal{I}_1 + \mathcal{I}_2 \) by the projection map \( \tilde{X} \times_S Y' \to \tilde{X} \). Then the blow-up along the ideal \( \mathcal{I} \) is an \( \tilde{X} \)-admissible blow-up. Replacing \( \tilde{X} \) by the one resulting via the blow-up (and \( Y' \) and \( \tilde{Z} \) by the strict transforms), we see that the image of \( \tilde{h} \) is disjoint from \( \tilde{Z} \). Hence we have the morphism \( \tilde{h}: Y' \to \tilde{X} \), as desired.
By passage to the associated rigid spaces, we get
\[ \alpha = (h)_{\text{rig}}: \mathcal{T} \rightarrow (\hat{X})_{\text{rig}} \hookrightarrow X_{\text{an}}. \]

Then one can show that this \( \alpha \) depends only on the data \((\beta, h)\) and that the map \((\beta, h) \mapsto \alpha\) gives the inverse map of (**) \(\square\).

**Example 9.2.3.** Let \( Z \) be a locally universally adhesive rigid space over \( S_{\text{an}} \). As usual, we denote by \( \mathbb{G}^\text{an}_a(Z) \) the set of \( S_{\text{an}} \)-morphisms \( Z \rightarrow \mathbb{G}^\text{an}_a \). We have
\[ \mathbb{G}^\text{an}_a(Z) = \Gamma(Z, \mathscr{O}_Z). \]

Indeed, since \( \mathbb{G}^\text{an}_a \) is a sheaf on the site \( \text{CRf}_{S_{\text{an}}, \text{ad}} \), we may assume that \( Z \) is a Stein affinoid (cf. 6.5.6), and then the equality follows from 9.2.2.

Similarly, we have
\[ \mathbb{G}^\text{an}_m(Z) = \Gamma(Z, \mathscr{O}_Z^\times) \]
for any rigid space \( Z \).

### 9.3 Comparison map and comparison functor

**9.3. (a) Comparison map.** We continue working in the setting of §9.1. (a) where we assume that the adic ring \( A \) is t.u. rigid-Noetherian. Let \( f: X \rightarrow U \) be a separated \( U \)-scheme of finite type. For any object \((X \leftarrow \hat{X})\) of \( \text{Emb}_X|S \), consider the \( S \)-scheme \( \hat{X} \) as in 9.1.2. For any affine open subset \( \text{Spec} B \leftarrow \hat{X} \), its open part \( \text{Spec} B \setminus V(IB) \) is an open subset of \( X \). In this situation, we have the comparison map
\[ s: ((\text{Spf} B)_{\text{rig}}) \rightarrow s((\text{Spf} \hat{B})_{\text{rig}}) = \text{Spec} \hat{B} \setminus V(I\hat{B}) \]
(§6.6. (b)) and the composite morphism
\[ ((\text{Spf} \hat{B})_{\text{rig}}) \rightarrow \text{Spec} \hat{B} \setminus V(I\hat{B}) \rightarrow \text{Spec} B \setminus V(IB) \hookrightarrow X \]  

\( \ast \)
of locally ringed spaces.

One can glue the morphisms \( \ast \) to a morphism
\[ \rho_X: (\langle X_{\text{an}} \rangle, \mathscr{O}_X) \rightarrow (X, \mathscr{O}_X) \]
of locally ringed spaces. Indeed, the set-theoretic map \( \rho_X: \langle X_{\text{an}} \rangle \rightarrow X \) is actually obtained as follows. Let \( x \in \langle X_{\text{an}} \rangle \). Consider the associated rigid point
\[ \alpha: \mathcal{T} \rightarrow X_{\text{an}}, \]
where \( \mathcal{T} = (\text{Spf} \hat{V}_x)_{\text{rig}} \) (3.3.5). Then, by 9.2.2, one has the morphism of schemes \( \text{Spec} \hat{V}_x \left[ \frac{1}{a} \right] \rightarrow X \); since \( \hat{V}_x \left[ \frac{1}{a} \right] \) is a field (0.6.7.2), it defines a point \( y \). Thus we
get the map \( \{X^{an}\} \to X \) given by \( x \mapsto y \). It is clear that, compared with the construction of the comparison map in the affinoid case (§6.6.(b)), this last map coincides with the one as in (\( \ast \)) on each affinoid (Spf \( \hat{B} \))\^{\text{rig}}. Hence we deduce that the maps (\( \ast \)) for \( \alpha \in \hat{L} \) and \( i \in J_\alpha \) glue together to a map \( \rho_X \) of locally ringed spaces as above. Note that, simply by gluing, one can also define the morphism \( \rho_X \) in the case where \( X \) is not necessarily separated.

The morphism \( \rho_X \) thus obtained for any \( U \)-scheme \( X \) of finite type is called the \textit{comparison map}.

\textbf{Proposition 9.3.1.} The comparison map \( \rho_X \) is flat.

\textbf{Proof.} Due to 6.6.3 it suffices to show that the map

\[ \text{Spec } \hat{B} \setminus V(I \hat{B}) \to \text{Spec } B \setminus V(IB) \]

in (\( \ast \)) is flat. Since \( B \) is \( IB \)-adically t.u. rigid-Noetherian (I.2.1.1), we know that the map \( B \to \hat{B} \) is flat (0.8.2.18). The assertion follows immediately from this. \( \square \)

9.3. (b) \textit{Comparison functor}

\textbf{Proposition 9.3.2.} The comparison map \( \rho_X \) gives rise to an exact functor

\[ \rho_X^*: \text{Mod}_X \to \text{Mod}_{X^{an}}. \]

Moreover, it maps the full subcategory \( \text{Coh}_X \) to \( \text{Coh}_{X^{an}} \).

We call the functor \( \rho_X^* \) the \textit{comparison functor}.

\textbf{Proof.} The first half is clear by 9.3.1. Since \( X \) is a Noetherian scheme, coherent \( \mathcal{O}_X \)-modules are exactly the \( \mathcal{O}_X \)-modules of finite presentation. Similarly, coherent \( \mathcal{O}_{X^{an}} \)-modules are nothing but finitely presented \( \mathcal{O}_X \)-modules (5.2.5). Hence the second assertion follows. \( \square \)

\textbf{Remark 9.3.3.} Here is another description of the comparison functor for coherent sheaves; here we assume that the scheme \( X \) is separated over \( U \). Let \( \mathcal{F} \) be a coherent sheaf on the scheme \( X \). Consider the quasi-compact open immersion \( X \hookrightarrow \widehat{X} \) as in 9.1.2, and extend \( \mathcal{F} \) to a finitely presented \( \mathcal{O}_{\widehat{X}} \)-module \( \widehat{\mathcal{F}} \) ([53], 6.9.11). Then the formal completion \( \widehat{\mathcal{F}} \) on \( \widehat{X} \) is finitely presented, and thus we get \( (\widehat{\mathcal{F}})^{\text{rig}} \) (§5.1.(a)), which is a coherent sheaf on \( (\widehat{X}_\alpha)^{\text{rig}} \). By gluing we get the desired coherent sheaf, which is nothing but \( \rho_X^* \mathcal{F} \).
9.4 GAGA comparison theorem

In this and next subsections, we work in the situation as in §9.1.(a) with the additional assumption that

(*) the adic ring $A$ is t.u. rigid-Noetherian.

The theorems in these subsections are announced in this case, but the proofs will be done under the stronger assumption that $A$ is t.u. adhesive. The general case can be proven with the aid of what we discussed in I, §C; we present the details found in [43].

Let

$$f : X \longrightarrow Y$$

be a proper $U$-morphisms of separated and of finite type $U$-schemes. We have the commutative diagram

$$
\begin{array}{ccc}
X & \xleftarrow{\mathbf{\rho}_X} & \left(\mathcal{O}_{X^{\text{an}}}, (X^{\text{an}},\mathcal{O}_{X^{\text{an}}})
\end{array}
$$

$$
\begin{array}{ccc}
f & & f^{\text{an}}
\end{array}
$$

$$
\begin{array}{ccc}
Y & \xleftarrow{\mathbf{\rho}_Y} & \left(\mathcal{O}_{Y^{\text{an}}}, (Y^{\text{an}},\mathcal{O}_{Y^{\text{an}}})
\end{array}
$$

of locally ringed spaces, where $\mathbf{\rho}_X$ and $\mathbf{\rho}_Y$ are the comparison maps (§9.3.(a)).

By I.8.1.3, $Rf_*$ maps $\mathbf{D}^\bullet_{\text{coh}}(X)$ to $\mathbf{D}^\bullet_{\text{coh}}(Y)$ for $\bullet = "", +, -, b$. On the other hand, since the comparison functor $\mathbf{\rho}_X^*$ is exact (9.3.2), it induces an exact functor

$$\mathbf{D}^\bullet(X) \longrightarrow \mathbf{D}^\bullet(X^{\text{an}})$$

(cf. 0.C.4.6), where $\mathbf{D}^\bullet(X^{\text{an}}) = \mathbf{D}^\bullet((X^{\text{an}},\mathcal{O}_{X^{\text{an}}})$. We write this functor as

$$M \longmapsto M^{\text{rig}}.$$  

Since $\mathbf{\rho}_X$ is flat, one easily sees that this functor maps $\mathbf{D}^\bullet_{\text{coh}}(X)$ to $\mathbf{D}^\bullet_{\text{coh}}(X^{\text{an}})$. Similarly, we get a canonical functor from $\mathbf{D}^\bullet_{\text{coh}}(Y)$ to $\mathbf{D}^\bullet_{\text{coh}}(Y^{\text{an}})$.

Thus we get the diagram of triangulated categories

$$
\begin{array}{ccc}
\mathbf{D}^\bullet_{\text{coh}}(X) & \xrightarrow{\text{rig}} & \mathbf{D}^\bullet_{\text{coh}}(X^{\text{an}})
\end{array}
$$

$$
\begin{array}{ccc}
Rf_* & & Rf^{\text{an}}
\end{array}
$$

$$
\begin{array}{ccc}
\mathbf{D}^\bullet_{\text{coh}}(Y) & \xrightarrow{\text{rig}} & \mathbf{D}^\bullet(Y^{\text{an}})
\end{array}
$$

Note that by 7.5.18 the functor $Rf^{\text{an}}_*$ has finite cohomology dimension, whence the right-hand vertical arrow.
By an argument similar to that in I, §9.1, one has the comparison map

\[ \rho = \rho_f : \text{rig} \circ Rf_* \longrightarrow Rf^\text{an}_* \circ \text{rig}, \]

and thus we obtain the diagram with a 2-arrow

\[
\begin{array}{ccc}
D^\ast_{\text{coh}}(X) & \xrightarrow{\text{rig}} & D^\ast_{\text{coh}}(X^\text{an}) \\
Rf_* & \downarrow & \downarrow Rf^\text{an}_* \\
D^\ast_{\text{coh}}(Y) & \xrightarrow{\text{rig}} & D^\ast(Y^\text{an})
\end{array}
\]

(\ast \ast)

**Theorem 9.4.1** (GAGA comparison theorem). Suppose \( f : X \to Y \) is proper. Then the natural transformation \( \rho \) gives a natural equivalence; hence diagram (\ast \ast) is 2-commutative.

We give here the proof only in case where the ring \( A \) in §9.1. (a) is t.u. adhesive; for the proof of the general case, see [43].

**Proof.** First note that to show the theorem, we may restrict ourselves to considering only objects \( M \) of \( D^b_{\text{coh}}(X) \) that are concentrated in degree 0 by using a reduction process similar to that in I, §9.3. (b) in the following way:

(1) first we may assume \( * = b \) (standard);

(2) by induction with respect to the amplitude of \( M \), taking the distinguished triangle

\[ \tau^{\leq n} M \longrightarrow M \longrightarrow \tau^{\geq n+1} M \xrightarrow{+1} \]

into account, we may assume that \( \text{amp}(M) = 0 \);

(3) finally, by a shift, we arrive at the hypothesis as above.

The theorem in this case will be shown in 9.4.2 below.

\[ \square \]

**Proposition 9.4.2.** Working under the assumption of 9.4.1, let \( \mathcal{F} \) be a coherent \( \mathcal{O}_X \)-module. Then the canonical morphism

\[ (R^q f_* \mathcal{F})^\text{rig} \longrightarrow Rf^\text{an}_* \mathcal{F}^\text{rig} \]

is an isomorphism for all \( q \geq 0 \).
**Proof.** We first deal with the case where $Y$ is proper over $U$. In this case, one can take suitable objects $X \hookrightarrow \bar{X}$ in $\text{Emb}_{X|S}$ and $Y \hookrightarrow \bar{Y}$ in $\text{Emb}_{Y|S}$ sitting in the Cartesian diagram

$$
\begin{array}{ccc}
X & \xleftarrow{\subset} & \bar{X} \\
\downarrow^{f} & & \downarrow^{\bar{f}} \\
Y & \xleftarrow{\subset} & \bar{Y}
\end{array}
$$

Since $X^\text{an} = (\widehat{X})^\text{rig}$ and $Y^\text{an} = (\widehat{Y})^\text{rig}$, one can show the claim by the similar reasoning as in 6.4.1.

In general, $Y^\text{an}$ is the open complement (7.3.13) of a closed subspace $Z$ in $(\widehat{Y})^\text{rig}$ (due to 9.1.9). Since $f$ is proper, $X^\text{an}$ is, similarly, the open complement of $W$, the pull-back of $Z$, in $(\widehat{X})^\text{rig}$. Note that $\bar{X}^\text{an}_U = (\widehat{X})^\text{rig}$ and $\bar{Y}^\text{an}_U = (\widehat{Y})^\text{rig}$. As the assertion is local on $Y$, one can reduce to the above case in the following way. First extend the coherent sheaf $\mathcal{F}$ on $X$ to a coherent sheaf $\mathcal{G}$ on the proper scheme $\bar{X}_U$; this is possible because $\bar{X}_U$ is a Noetherian scheme. Then due to the first step, the assertion is true for the morphism $\bar{X}_U \to \bar{Y}_U$.

Now since $f$ is proper, the commutative diagram

$$
\begin{array}{ccc}
X & \xleftarrow{\subset} & \bar{X}_U \\
\downarrow^{f} & & \downarrow^{\bar{f}_U} \\
Y & \xleftarrow{\subset} & \bar{Y}_U
\end{array}
$$

is Cartesian, and by 9.1.10 we have the Cartesian diagram

$$
\begin{array}{ccc}
X^\text{an} & \xleftarrow{\subset} & \bar{X}^\text{an}_U \\
\downarrow^{f^\text{an}} & & \downarrow^{\bar{f}^\text{an}_U} \\
Y^\text{an} & \xleftarrow{\subset} & \bar{Y}^\text{an}_U
\end{array}
$$

of rigid spaces. Hence we get the desired result by base change. □

The following corollaries can be shown similarly, at least in the t.u. adhesive situation, to I, §9.4.

**Proposition 9.4.3.** In the setting of 9.4.1, let $f : X \to U$ be a proper $U$-scheme. Then for $M \in \text{obj}(D^{-}_\text{coh}(X))$ and $N \in D^+_\text{coh}(X)$ we have the canonical isomorphism

$$
R\text{Hom}_{\mathcal{O}_X}(M, N) \cong R\text{Hom}_{\mathcal{O}_X}^{\text{rig}}(M^\text{rig}, N^\text{rig})
$$

in $D^+(\text{Ab})$. 

Corollary 9.4.4. In the setting of 9.4.1, let \( f : X \to U \) be a proper \( U \)-scheme. Then the comparison functor
\[
D^b_{\text{coh}}(X) \to \text{rig} \to D^b_{\text{coh}}(X^{\text{an}})
\]
is fully faithful.

9.5 GAGA existence theorem

We continue to work in the setting described in the beginning of §9.4.

Theorem 9.5.1 (GAGA existence theorem). Let \( f : X \to U \) be a proper \( U \)-scheme. Then the comparison functor
\[
D^b_{\text{coh}}(X) \to \text{rig} \to D^b_{\text{coh}}(X^{\text{an}})
\]
is an exact equivalence of triangulated categories.

Similarly to the comparison theorem, here we give the proof only in the case where the ring \( A \) is t.u. adhesive; for the proof of the general case, see [43].

Proof. As we have already seen that the functor in question is exact and fully faithful (9.4.4), we only need to show that it is essentially surjective. But then, by the reduction process carried out by induction with respect to the amplitudes of objects of \( D^b_{\text{coh}}(X^{\text{an}}) \) involving the distinguished triangle
\[
\tau_{\leq n} M \to M \to \tau_{\geq n+1} M \to M^{+1}
\]
(similarly to that in I, §10.2.(c)), it suffices to show that the functor
\[
\text{rig}: \text{Coh}_X \to \text{Coh}_{X^{\text{an}}}
\]
is essentially surjective.

Take a diagram
\[
\begin{array}{ccc}
X & \to & \tilde{X} \\
\downarrow f & & \downarrow \tilde{f} \\
U & \to & S
\end{array}
\]
which is an object of \( \text{Emb}_{X|S} \); as we saw in the proof of 9.1.5, we may, moreover, assume that this diagram is Cartesian. We have \( X^{\text{an}} = (\tilde{X})^{\text{rig}} \) (9.1.5). Moreover, we may assume that \( \tilde{X} \) is \( I \)-torsion free.

Let \( \mathcal{F} \) be a coherent \( \mathcal{O}_{X^{\text{an}}} \)-module. Then by 5.3.1 there exists a coherent \( \mathcal{O}_{\tilde{X}} \)-module \( \mathcal{G} \) such that \((\mathcal{G})^{\text{rig}} = \mathcal{F}\). By I.10.1.2, we get a coherent \( \mathcal{O}_{\tilde{X}} \)-module \( \mathcal{G} \) whose formal completion coincides with \( \mathcal{G} \). Now the sheaf \( \mathcal{G}|_X \) is the desired one, for we have \((\mathcal{G}|_X)^{\text{rig}} = \mathcal{F}\), as we have seen in 9.3.3. \( \square \)
By an argument similar to that in I.10.3.1, one obtains the following corollary.

**Corollary 9.5.2.** The GAGA-functor

\[ {^{\text{an.}} \text{PSch}_U} \rightarrow \text{Rf}_S, \]

where \( \text{PSch}_U \) denotes the category of proper \( U \)-schemes, is fully faithful.

### 9.6 Adic part for non-adic morphisms

Finally, let us observe that one can apply Berthelot’s construction of tubes [14] to construct canonical rigid spaces associated to non-adic morphisms of formal schemes. The relation with GAGA will be shown in Exercise II.9.2.

**9.6. (a) Adic part.** Let \( f: X \rightarrow Y \) be a morphism between adic formal schemes of finite ideal type. We do not assume that \( f \) is adic. Let \( \mathcal{X} = X^{\text{rig}} \) and \( \mathcal{Y} = Y^{\text{rig}} \) be the associated rigid spaces (§3.5. (e)). If \( X \) is not adic over \( Y \), \( \mathcal{X} \) does not admit the structural map \( \mathcal{X} \rightarrow \mathcal{Y} \). However, following [14], §0.2, one can construct a canonical open rigid subspace \( \mathcal{X}^{\text{adic}}_{/\mathcal{Y}} \subseteq \mathcal{X} \), called the *adic part over \( \mathcal{Y} \), that admits the \( \mathcal{Y} \)-structure, as follows.

Consider the subset \( \mathcal{U} \) of \( \mathcal{X} \) defined as follows:

\[
\mathcal{U} = \left\{ x \in \mathcal{X} \left| f \circ \text{sp}_X \circ \alpha_x: \text{Spf} \overset{\sim}{V}_x \rightarrow Y \text{ is adic, where } \alpha_x: \text{Spf} \overset{\sim}{V}_x \rightarrow \mathcal{X} \text{ is the associated rigid point of } x \text{ (3.3.5)} \right. \right\}.
\]

**Proposition 9.6.1.** (1) \( \mathcal{U} \) is open in \( \langle \mathcal{X} \rangle \).

(2) There exists a canonical open rigid subspace \( \mathcal{X}^{\text{adic}}_{/\mathcal{Y}} \) of \( \mathcal{X} \) with the canonical morphism \( \mathcal{X}^{\text{adic}}_{/\mathcal{Y}} \rightarrow \mathcal{Y} \) such that \( \langle \mathcal{X}^{\text{adic}}_{/\mathcal{Y}} \rangle = \mathcal{U} \).

**Proof.** To show (1), we can restrict ourselves to affine situation. Consider \( f: X = \text{Spf} A \rightarrow Y = \text{Spf} B \),

and let \( I \) (resp. \( J \)) be a finitely generated ideal of definition of \( A \) (resp. \( B \)) such that \( JA \subseteq I \). We may moreover suppose, replacing \( Y \) by the admissible blow-up along \( J \) if necessary, that \( J \) is invertible and principal, \( J = aB \). Note that \( A \) is \( a \)-adically complete due to 0.7.2.5 (b). Suppose \( x \in \mathcal{U} \). Since the composition \( B \rightarrow A \rightarrow \overset{\sim}{V}_x \) is adic, \( a \overset{\sim}{V}_x \) is an ideal of definition of \( \overset{\sim}{V}_x \), and hence \( I^n \overset{\sim}{V}_x \subseteq a \overset{\sim}{V}_x \) for some \( n > 0 \). Set \( I^n = (f_1, \ldots, f_r) \) and consider the admissible blow-up \( \overset{\sim}{X} \) of \( X = \text{Spf} A \) along the admissible ideal \( JA + I^n = (a, f_1, \ldots, f_r) \); \( \overset{\sim}{X} \) has the affine open part given by \( \text{Spf} A_{I^n} \), where

\[
A_{I^n} = A(\langle \frac{f_1}{a}, \ldots, \frac{f_r}{a} \rangle),
\]
that is, the part where \((JA + I^n)\mathcal{O}_X\) is generated by \(J\mathcal{O}_X\). The map 

\[ \text{sp}_X \circ \alpha_X : \text{Spf} \, \widehat{V}_X \longrightarrow X = \text{Spf} \, A \]

factors through \(\text{Spf} \, A_{fn}\). Since \((\text{Spf} \, A_{fn})^{\text{rig}}\) is an open subspace of \(\mathcal{X} = X^{\text{rig}}\) and since any point \(y \in ((\text{Spf} \, A_{fn})^{\text{rig}})\) satisfies \(I \, \widehat{V}_y \subseteq a \, \widehat{V}_x\), we have \((\text{Spf} \, A_{fn})^{\text{rig}} \subseteq \mathfrak{U}\), and thus the openness of \(\mathfrak{U}\) follows.

As the above argument shows, for any \(x \in \mathfrak{U}\) there exists a positive integer \(n > 0\) such that \(I^n \, \widehat{V}_x \subseteq a \, \widehat{V}_x\), and hence the open subset \(\mathfrak{U} \subseteq \langle \mathcal{X} \rangle\) is described as

\[ \mathfrak{U} = \bigcup_{n>0} ((\text{Spf} \, A_{fn})^{\text{rig}}), \]

where each \((\text{Spf} \, A_{fn})\) is naturally identified with a coherent open subset of \(\langle \mathcal{X} \rangle\). Hence one can construct the desired rigid subspace in this case by

\[ \mathcal{X}_{/\mathfrak{U}}^{\text{adic}} = \lim_{\longrightarrow \atop{n>0}} (\text{Spf} \, A_{fn})^{\text{rig}}, \]

which clearly does not depend on the choice of the ideal of definition \(I\). The construction in the general case is given by gluing, whence (2). \(\square\)

**Remark 9.6.2.** In the situation as in the proof of 9.6.1, consider \(Z = \text{Spf} \, A\), where \(A\) is now considered as an adic ring for the \(J\)-adic topology, and \(W = \text{Spec} \, A/I\). Then by the open interior formula (4.2.9), we have \(\mathfrak{U} = C_{W/Z}\); cf. Exercise II.9.2.

**Remark 9.6.3** (cf. [14], (0.2.6)). In the affine case the rigid space \(\mathcal{X}_{/\mathfrak{U}}^{\text{adic}}\) admits, as in the proof of 9.6.1, another open covering as follows. Set \(I = (f_1, \ldots, f_r)\). Define, for any positive integer \(n > 0\),

\[ A(n) = A\langle \left\{ \frac{f_1^n}{a}, \ldots, \frac{f_r^n}{a} \right\} \rangle = A\langle \{T_1, \ldots, T_r\} \rangle/(f_1^n - aT_1, \ldots, f_r^n - aT_r). \]

Then the rigid space \(\mathcal{U}(n) = (\text{Spf} \, A(n))^{\text{rig}}\) is an open subspace of \(\mathcal{X}\) such that

\[ (\text{Spf} \, A_{fn})^{\text{rig}} \subseteq \mathcal{U}(n) \subseteq (\text{Spf} \, A_{fnr})^{\text{rig}} \]

for any \(n > 0\). Hence,

\[ \mathcal{X}_{/\mathfrak{U}}^{\text{adic}} = \lim_{\longrightarrow \atop{n>0}} \mathcal{U}(n). \]

Note that the open immersion \(\mathcal{U}(n) \hookrightarrow \mathcal{U}(m)\) for \(m > n\) is given by the map \(A(m) \to A(n)\), which maps \(f_i^m/a\) to \(f_i^m/f_i^n a\) for \(i = 1, \ldots, r\).
9.6. (b) **Functoriality.** In the situation as above, suppose $X$ is coherent. Let $Z$ be a coherent adic formal scheme of finite ideal type, and $g: Z \to Y$ an adic morphism. Suppose we are given a morphism $Z = Z^\text{rig} \to \mathcal{X}_Y^\text{adic}$ of rigid spaces over $Y$. Then the composition with the open immersion $\mathcal{X}_Y^\text{adic} \to \mathcal{X}$ is induced by the rig-functor from a $Y$-morphism $Z' \to X$ of formal schemes, where $Z'$ is an admissible blow-up of $Z$. Conversely, suppose we are given a $Y$-morphism of the form $Z' \to X$ from an admissible blow-up $Z'$ of $Z$. For any admissible ideal $\mathcal{I}$ of $X$, since $Z'$ is adic over $Y$, $\mathcal{I} \mathcal{O}_{Z'}$ is an admissible ideal of $Z'$. In other words, for any admissible blow-up $X' \to X$, there exist an admissible blow-up $Z'' \to Z'$ and a morphism $Z'' \to X'$ such that the diagram

\[
\begin{array}{ccc}
Z'' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Z' & \longrightarrow & X
\end{array}
\]

commutes. Hence we have a morphism $Z \to \mathcal{X}$ of rigid spaces. For any point $z \in \langle Z \rangle$, the map $\text{Spf} \widehat{\mathcal{V}}_z \to Y$ is adic, and so the composition $\text{Spf} \widehat{\mathcal{V}}_z \to \langle Z \rangle \to \langle \mathcal{X} \rangle$ has its image in the open part $\mathcal{U}$ as in §9.6.(a). We therefore obtain a morphism $Z \to \mathcal{X}_Y^\text{adic}$ of rigid spaces over $Y$.

Thus we have established the existence of a canonical bijection

\[
\text{lim}_{Z'} \text{Hom}_Y(Z', X) \xrightarrow{\sim} \text{Hom}_{\text{Rf}_Y}(Z, \mathcal{X}_Y^\text{adic}),
\]

where $Z'$ runs over all admissible blow-ups of $Z$.

**Proposition 9.6.4.** Let $Y$ be an adic formal scheme of finite ideal type, and $Y = Y^\text{rig}$. Then the mapping $X \mapsto (X^\text{rig})_Y^\text{adic}$ yields a natural functor $\text{AcFs}_Y \to \text{Rf}_Y$ from the category of adic formal schemes over $Y$ to the category of rigid spaces over $Y$.

**Proof.** We need to show that any morphism $W \to X$ in $\text{AcFs}_Y$ naturally induces a morphism $(W^\text{rig})_Y^\text{adic} \to (X^\text{rig})_Y^\text{adic}$. For any open subspace $\mathcal{U} \subseteq \mathcal{X} = X^\text{rig}$, we have $\mathcal{U}_Y^\text{adic} = \mathcal{U} \cap \mathcal{X}_Y^\text{adic}$. Hence, by a standard patching argument, we may assume that $X$ is coherent.

The proof of 9.6.1 shows that every point of $(W^\text{rig})_Y^\text{adic}$ has a coherent open neighborhood of the form $Z^\text{rig}$ for a formal subscheme $Z \subseteq W'$, adic over $Y$, of an admissible blow-up $W'$ of $W$. By what we have seen above, we have canonically a morphism $Z^\text{rig} \to \mathcal{X}_Y^\text{adic}$. Then by gluing we obtain the desired morphism $(W^\text{rig})_Y^\text{adic} \to \mathcal{X}_Y^\text{adic}$. \qed
9.6. (c) Adic part for formally locally of finite type morphisms

**Definition 9.6.5.** Let \( f : X \to Y \) be a (not necessarily adic) morphism between adic formal schemes of finite ideal type. We say that \( f \) is **formally locally of finite type** if, for any \( x \in X \), there exist an open neighborhood \( V \subseteq Y \) of \( y = f(x) \) and an open neighborhood \( U \subseteq X \) of \( x \) such that \( f(U) \subseteq V \), and that, if \( \mathcal{I} \) (resp. \( \mathcal{V} \)) is an ideal of definition of \( U \) (resp. \( V \)) of finite type such that \( \mathcal{I} \mathcal{O}_X \subseteq \mathcal{I} \), the induced morphism \( (X, \mathcal{O}_X/\mathcal{I}) \to (Y, \mathcal{O}_Y/\mathcal{J}) \) of schemes is locally of finite type. If, in addition, \( f \) is quasi-compact, it is said to be **formally of finite type**.

Note that, in view of I.1.7.3, if \( f : X \to Y \) is adic, then \( f \) is formally locally of finite type (resp. formally of finite type) if and only if it is locally of finite type (resp. of finite type).

**Proposition 9.6.6.** Let \( f : X \to Y \) be a formally locally of finite type morphism between adic formal schemes of finite ideal type, and set \( X = X^\text{rig} \) and \( Y = Y^\text{rig} \). Then the rigid space \( X^\text{adic}/Y \) is locally of finite type over \( Y \).

**Proof.** It suffices to treat the affine case. In the notation as in the proof of 9.6.1, the assumption implies that \( B/J \to A/I \) is of finite type. We have \( A/I = A_I/IA_I \), which is, then, topologically of finite type over \( B \). Hence the assertion follows. \( \square \)

9.6. (d) Examples

**Example 9.6.7 (open unit disk).** Let \( B \) be an adic ring of finite ideal type, with a finitely generated ideal of definition \( J \), and consider the formal power series ring \( A = B[[X_1, \ldots, X_r]] \) with an ideal of definition \( I = JA + (X_1, \ldots, X_r) \). In this case the rigid space \( X^\text{adic}_{/Y} \) (where \( X = (\text{Spf } A)^{\text{rig}} \) and \( Y = (\text{Spf } B)^{\text{rig}} \)) is called the **open unit disk over \( Y \)**. To describe \( X^\text{adic}_{/Y} \) locally over \( Y \), we assume that \( J \) is invertible and \( J = aB \). As in 9.6.3, we have

\[
X^\text{adic}_{/Y} = \bigcup_{n > 0} \left( \text{Spf } A\left(\frac{X^n_1}{a}, \ldots, \frac{X^n_r}{a}\right)\right)^{\text{rig}}.
\]

Here, note that we have

\[
A\left(\frac{X^n_1}{a}, \ldots, \frac{X^n_r}{a}\right) = B\langle X_1, \ldots, X_r, Y_1, \ldots, Y_r \rangle/(X^n_1 - aY_1, \ldots, X^n_r - aY_r),
\]

and hence

\[
\left( \text{Spf } A\left(\frac{X^n_1}{a}, \ldots, \frac{X^n_r}{a}\right)\right)^{\text{rig}}
\]

is naturally viewed as a rational subdomain (6.1.7)

\[
\mathbb{D}^r_y\left(\frac{X^n_1}{a}, \ldots, \frac{X^n_r}{a}\right)
\]

of the unit disk \( \mathbb{D}^r_y \) over \( Y \) (§2.5 (c)). Thus, \( X^\text{adic}_{/Y} \) is isomorphic to the open subspace of \( \mathbb{D}^r_y \) that is the union of all \( \mathbb{D}^r_y\left(\frac{X^n_1}{a}, \ldots, \frac{X^n_r}{a}\right) \).
By the last description of the open unit disk $X^\text{adic}_Y$, if, for example, $B = V$ is an $a$-adically complete valuation ring, then it has the following set-theoretic description:

$$\langle X^\text{adic}_Y \rangle = \{ x \in \langle D^r_y \rangle: \| X_i(x) \|_{a,c} < 1, \ i = 1, \ldots, r \},$$

where $\| \cdot \|_{x,a,c}$ is the seminorm defined as in §3.3.(b).

**Example 9.6.8** (open annulus). Let $V$ be an $a$-adically complete valuation ring, and set $A = V[X, Y]/(XY - a)$. Then one sees that the adic part $X^\text{adic}_Y$ (where $X = (\text{Spf } A)^{\text{rig}}$ and $Y = (\text{Spf } V)^{\text{rig}}$) is isomorphic to the open part of the unit disk $D^r_Y = (\text{Spf } V(\{X\}))^{\text{rig}}$ characterized by

$$\langle X^\text{adic}_Y \rangle = \{ x \in \langle D^r_y \rangle: c < \| X(x) \|_{a,c} < 1 \}.$$

**Example 9.6.9** (Berthelot’s tube (cf. [14], §1)). Let $P$ be a locally of finite type scheme over $S = \text{Spec } V$, where $V$ is an $a$-adically complete valuation ring of height one, and $X$ a closed subscheme of the scheme $P_0 = (P, \mathcal{O}_P/a\mathcal{O}_P)$. Consider the formal completion $\hat{P}|_X$ of $P$ along $X$. Then

$$((\hat{P}|_X)^{\text{rig}})^{\text{adic}}_S = ]X[P,$$

where $S = (\text{Spf } V)^{\text{rig}}$.

**Exercises**

**Exercise II.9.1.** In the setting of §9.1.(a), let $X$ be a separated $U$-scheme of finite type. Show that $X^\text{an}$ is a union of countably many affinoids.

**Exercise II.9.2.** Let $A$ be an adic ring of finite ideal type, $I \subseteq A$ an ideal of definition, and $J \subseteq A$ a finitely generated ideal contained in $I$. Consider $D = V(J) \subseteq X = \text{Spec } A$ and the formal completions $\hat{X} = \text{Spf } A$ and $\hat{X}|_D$ of $X$ along $I$ and $J$, respectively. Set $X = (\hat{X})^{\text{rig}}$ and $S = (\hat{X}|_D)^{\text{rig}}$. Show that $X^\text{adic}_S$ is canonically isomorphic to $(X \setminus D)^{\text{an}}$.

**10 Dimension of rigid spaces**

This section gives generalities on dimension and codimension in rigid geometry. Like in complex analytic geometry, the dimension of rigid spaces is defined to be the supremum of local dimension at each point (§10.1.(a)), which is defined as the Krull dimension of the local ring. In §10.1.(f) we will briefly discuss the so-called
10. Dimension of rigid spaces

10.1 Dimension of rigid spaces

10.1 (a) Dimension

Definition 10.1.1. Let $\mathcal{X}$ be a rigid space.

1. Let $x \in \mathcal{X}$ be a point. The dimension of $\mathcal{X}$ at $x$, denoted by

$$\dim_x(\mathcal{X}),$$

is the Krull dimension $\dim(\mathcal{O}_{\mathcal{X},x})$ of the local ring $\mathcal{O}_{\mathcal{X},x}$ at $x$.

2. The dimension of $\mathcal{X}$, denoted by $\dim(\mathcal{X})$, is defined by

$$\dim(\mathcal{X}) = \sup_{x \in \mathcal{X}} \dim_x(\mathcal{X}).$$

We set $\dim(\mathcal{X}) = -\infty$ if $\mathcal{X}$ is empty. Notice that, if $\mathcal{X}$ is locally universally Noetherian, and if $y$ is a generization of $x$, then we have $\dim_y(\mathcal{X}) \geq \dim_x(\mathcal{X})$, since $\mathcal{O}_{\mathcal{X},y}$ is faithfully flat over $\mathcal{O}_{\mathcal{X},x}$ (3.2.17 (3)).

Example 10.1.2. Let $V$ be an $a$-adically complete valuation ring of arbitrary (positive) height, and set $\mathcal{X} = (\text{Spf } V)^{\text{rig}}$. Then

$$\dim(\mathcal{X}) = 0,$$

since, in this case, $\mathcal{X}$ consists only of one point with the local ring being isomorphic to the fractional field of $V$.
**Proposition 10.1.3.** Let $\mathcal{X}$ be a universally Noetherian rigid space, and $\mathcal{Y}$ a rigid subspace (7.4.6) of $\mathcal{X}$.

(1) For any point $x \in \mathcal{Y}$ we have $\dim_x(\mathcal{Y}) \leq \dim_x(\mathcal{X})$.

(2) We have $\dim(\mathcal{Y}) \leq \dim(\mathcal{X})$.

**Proof.** (1) follows immediately from the fact that $\mathcal{O}_{\mathcal{Y},x}$ is a quotient of $\mathcal{O}_{\mathcal{X},x}$; see 7.3.5.

(2) follows immediately from the definition of the dimensions. $\square$

**Proposition 10.1.4.** Let $\mathcal{X}$ be a rigid space, and $\{\mathcal{U}_\alpha\}_{\alpha \in L}$ a covering of $\mathcal{X}_{\text{ad}}$ (2.2.24). Then we have

$$\dim(\mathcal{X}) = \sup_{\alpha \in L} \dim(\mathcal{U}_\alpha).$$

**Proof.** By 10.1.3 we have $\sup_{\alpha \in L} \dim(\mathcal{U}_\alpha) \leq \dim(\mathcal{X})$. On the other hand, for any $x \in (\mathcal{X})$ there exists $\mathcal{U}_\alpha$ containing $x$. Then $\dim_x(\mathcal{X}) = \dim_x(\mathcal{U}_\alpha) \leq \dim(\mathcal{U}_\alpha)$. Hence we have $\dim(\mathcal{X}) \leq \sup_{\alpha \in L} \dim(\mathcal{U}_\alpha)$. $\square$

**Proposition 10.1.5.** Let $\mathcal{X}$ be a universally Noetherian rigid space, and $x \in (\mathcal{X})$ a point. Then we have

$$\dim_x(\mathcal{X}) \leq \sup_{x \in \mathcal{U}} \dim(\mathcal{U}, \mathcal{O}_\mathcal{U}),$$

where $\mathcal{U}$ in the right-hand side runs through all Stein affinoid neighborhoods of $x$ in $\mathcal{X}$, and the dimension $\dim(\mathcal{U}, \mathcal{O}_\mathcal{U})$ is the Krull dimension of the ring $\Gamma(\mathcal{U}, \mathcal{O}_\mathcal{U})$.

**Proof.** Let $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$ be a strictly increasing chain of prime ideals of $\mathcal{O}_{\mathcal{X},x}$. Let $\mathcal{U}$ be an arbitrary Stein affinoid open neighborhood of $x$. Set $\mathcal{U} = (\text{Spf } A)^\text{rig}$, where $A$ is a t.u. rigid-Noetherian ring, and define $B$ by $\text{Spec } A \setminus V(I) = \text{Spec } B$, where $I \subseteq A$ is an ideal of definition (that is, $B = \Gamma(\mathcal{U}, \mathcal{O}_\mathcal{U})$). Consider $P_i, \mathcal{U} = \ker(B \to \mathcal{O}_{\mathcal{X},x}/p_i)$ for $i = 0, \ldots, n$, which is a prime ideal of $B$. Since

$$\lim_{x \in \mathcal{U}} P_i, \mathcal{U} = p_i,$$

where the left-hand limit is taken along all Stein affinoid open neighborhoods of $x$, there exists a sufficiently small $\mathcal{U}$ such that

$$P_0, \mathcal{U} \subsetneq P_1, \mathcal{U} \subsetneq \cdots \subsetneq P_n, \mathcal{U},$$

from which the assertion follows. $\square$
10.1. (b) Germs of rigid subspaces. Let \( \mathcal{X} \) be a locally universally Noetherian rigid space (2.2.23), and \( x \in (\mathcal{X}) \) a point. We define an equivalence relation on the set of all rigid subspaces of \( \mathcal{X} \) as follows: two rigid subspaces \( Z_1 \) and \( Z_2 \) are equivalent if and only if there exists a quasi-compact open neighborhood \( U \) of \( x \) such that \( Z_1 \cap U = Z_2 \cap U \) as rigid subspaces of \( U \). An equivalence class with respect to this equivalence relation is called a germ of a rigid subspace at \( x \). Given a rigid subspace \( Z \subseteq \mathcal{X} \), the associated germ at \( x \) will be denoted by \( Z \times x \).

Clearly, for a rigid subspace \( Z \) and a quasi-compact open neighborhood \( U \) of \( x \), we have \( Z \times x = (Z \cap U) \times x \). Hence, when considering a germ \( Z \times x \), one can always replace \( Z \) by a rigid subspace of the form \( Z \cap U \), which, moreover, can be assumed to be a closed rigid subspace of \( U \).

A germ \( Z \times x \) is said to be reduced if it comes from a reduced rigid subspace \( Z \) (cf. 3.2.12).

Let \( Z_{1,x} \) and \( Z_{2,x} \) be two germs of rigid subspaces at \( x \). We say that a germ \( Z_{1,x} \) is contained in \( Z_{2,x} \), written \( Z_{1,x} \subseteq Z_{2,x} \), if there exists a quasi-compact open neighborhood \( U \) of \( x \) such that \( Z_1 \cap U \) is a rigid subspace of \( Z_2 \cap U \). In type (V) case, by \( \S 8.3 \)(b), any germ contains the uniquely determined reduced model.

Let \( Z_x \) be a germ of a rigid subspace at \( x \), and suppose \( Z \) is a closed subspace of a quasi-compact open neighborhood \( U \) of \( x \). Then we have the defining coherent ideal \( \mathcal{I} \) of \( \mathcal{O}_U \). Taking the stalk at \( x \), we get a finitely generated ideal \( \mathcal{I}_x \) of \( \mathcal{O}_{\mathcal{X},x} \). We denote this ideal by \( I(Z_x) \) and call the ideal of \( Z_x \).

**Proposition 10.1.6.** Let \( \mathcal{X} \) be a locally universally Noetherian rigid space. Then the correspondence

\[
Z_x \leftrightarrow I(Z_x)
\]

establishes a bijection between the set of all germs of rigid subspaces at \( x \) and the set of all finitely generated ideals of \( \mathcal{O}_{\mathcal{X},x} \). Moreover, if \( Z_{1,x} \) and \( Z_{2,x} \) are two germs at \( x \), then \( Z_{1,x} \subseteq Z_{2,x} \) if and only if \( I(Z_{1,x}) \supseteq I(Z_{2,x}) \). In type (V) case, reduced germs correspond to finitely generated radical ideals.

**Proof.** The inverse to the above-mentioned correspondence is constructed as follows (cf. \( \S 7.3 \)(d)). Let \( J \) be a finitely generated ideal of \( \mathcal{O}_{\mathcal{X},x} \). Then there exist an affinoid open neighborhood \( \mathcal{U} = (\text{Spf} \ A)^{\text{rig}} \) of \( x \) (where \( A \) is a t.u. rigid-Noetherian ring) and a finitely generated ideal \( J' \) of \( A[\frac{1}{a}] \) such that \( J = J' \mathcal{O}_{\mathcal{X},x} \) (cf. 3.2.15 (1)). Take a finitely generated ideal \( \tilde{J} \) of \( A \) such that \( \tilde{J} A[\frac{1}{a}] = J' \). Let \( Z \hookrightarrow \text{Spf} \ A \) be the closed immersion of finite presentation corresponding to \( \tilde{J} \), and consider the associated rigid space \( \mathcal{Z} = Z_r \), which is the closed subspace of \( \mathcal{U} \). The germ \( Z_x \) of \( Z \) is easily seen to be independent of the choice of \( \mathcal{U}, A, \) and \( \tilde{J} \), and thus we have the desired inverse to the correspondence \( Z_x \leftrightarrow I(Z_x) \).
In type (V) case, we know, thanks to 8.3.6, that the local ring $\mathcal{O}_{X,x}$ is Noetherian, and hence every ideal corresponds to a germ. Then it follows from 8.3.3 that a germ $Z_x$ is reduced if and only if the corresponding ideal $I(Z_x)$ is a radical ideal.

Let $Z_{1,x}$ and $Z_{2,x}$ be two germs of rigid subspaces at $x$. We may assume that these germs come from closed rigid subspaces $Z_1$ and $Z_2$ of a quasi-compact open neighborhood $U$ of $x$. Then the intersection $Z_{1,x} \cap Z_{2,x}$ is defined to be the germ of $Z_1 \times_U Z_2$ at $x$ (cf. 7.3.11 (4)). It is clear that we have the equality

$$I(Z_{1,x} \cap Z_{2,x}) = I(Z_{1,x}) + I(Z_{2,x})$$

of finitely generated ideals of $\mathcal{O}_{X,x}$. One can define the union $Z_{1,x} \cup Z_{2,x}$ as the germ corresponding to the product $I(Z_{1,x}) \cdot I(Z_{2,x})$.

A germ $Z_x$ at $x$ is said to be prime if the corresponding ideal $I(Z_x)$ is a prime ideal of $\mathcal{O}_{X,x}$. In type (V) case, this is equivalent to that $Z_x$ is reduced and irreducible (that is, non-empty and whenever $Z_x = Z_{1,x} \cup Z_{2,x}$ by reduced germs $Z_{1,x}, Z_{2,x}$, we have either $Z_x = Z_{1,x}$ or $Z_x = Z_{2,x}$).

Let $X$ be a locally universally Noetherian rigid space, and $x \in (X)$ a point such that the local ring $\mathcal{O}_{X,x}$ is Noetherian; the last assumption is always valid if $X$ is of type (V) or of (N); see 8.3.6. A chain of prime germs of rigid subspaces at $x$ is the diagram

$$(*) \quad Z_{0,x} \hookrightarrow Z_{1,x} \hookrightarrow \cdots \hookrightarrow Z_{n,x}$$

consisting of prime germs at $x$ such that for $i = 0, \ldots, n-1$ we have $Z_{i,x} \neq Z_{i+1,x}$. The number $n$ in $(*)$ is called the length of the chain $(*)$.

**Proposition 10.1.7.** Let $X$ be a rigid space, and $x \in (X)$ a point such that the local ring $\mathcal{O}_{X,x}$ is Noetherian (e.g. $X$ is of type (V) or of (N)). Then the local ring $\mathcal{O}_{X,x}$ is Noetherian, and we have $\dim_X (X) < +\infty$. Moreover, $\dim_X (X)$ coincides with the supremum of the lengths of all chains of prime germs of rigid subspaces at $x$.

**Proof.** The local ring $\mathcal{O}_{X,x}$ is Noetherian due to 8.3.6, and hence is of finite Krull dimension. The other assertion is clear by the definition of prime germs. □
10.1. (c) Dimension of rigid spaces of type (V) or of type (N)

**Theorem 10.1.8** (comparison of the dimensions for affinoids). Let $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$ be an affinoid of type (V) or of type (N). Then for any closed classical point $x \in (\mathcal{X})^{\text{cl}}$ (8.2.8) we have

$$\dim_x(\mathcal{X}) = \dim_{s(x)}(s(\mathcal{X})),$$

where $s(\mathcal{X})$ is the Noetherian scheme associated to $\mathcal{X}$ (§6.6), and $\dim_{s(x)}(s(\mathcal{X}))$ denotes the dimension at $s(x)$ of the Noetherian scheme $s(\mathcal{X})$ in the usual sense (cf. [54], IV, §5.1).

**Proof.** By 8.3.1 we have

$$\dim(\mathcal{O}_{\mathcal{X},x}) = \dim(\mathcal{O}_{\mathcal{X},x}) = \dim(\mathcal{O}_{\mathcal{X},s(x)}) = \dim_{s(x)}(s(\mathcal{X})),$$

which shows the assertion. \qed

**Proposition 10.1.9.** Let $\mathcal{X}$ be a rigid space of type (V) or of type (N). Then,

$$\dim(\mathcal{X}) = \sup_{x \in (\mathcal{X})^{\text{cl}}} \dim_x(\mathcal{X}).$$

**Proof.** The inequality $\sup_{x \in (\mathcal{X})^{\text{cl}}} \dim_x(\mathcal{X}) \leq \sup_{x \in (\mathcal{X})} \dim_x(\mathcal{X})$ is trivial. We need to show the opposite inequality. For any $x \in (\mathcal{X})$, there exists a Stein affinoid $\mathcal{U} = (\text{Spf } A)^{\text{rig}}$ neighborhood of $x$ such that

$$\dim_x(\mathcal{X}) \leq \dim(\mathcal{U}, \mathcal{O}_{\mathcal{U}}) = \sup_{z \in (\mathcal{U})^{\text{cl}}} \dim(\mathcal{O}_{s(\mathcal{U}),z}),$$

due to 10.1.5. Since, as in 8.2.10 (1), any closed point of $s(\mathcal{U})$ gives rise to a closed classical point of $\mathcal{U}$, and hence a classical point of $\mathcal{X}$, we have, by 10.1.8,

$$\sup_{z \in (\mathcal{U})^{\text{cl}}} \dim(\mathcal{O}_{s(\mathcal{U}),z}) \leq \sup_{y \in (\mathcal{X})^{\text{cl}}} \dim(\mathcal{O}_{\mathcal{X},y}),$$

from which we deduce the desired inequality. \qed

**Corollary 10.1.10.** Let $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$ be an affinoid of type (V), or an affinoid of type (N) having a distinguished Noetherian formal model $\mathcal{X}$, together with an ideal of definition $\mathcal{I}$, such that the scheme $X_0 = (X, \mathcal{O}_X / \mathcal{I})$ is Jacobson. Then we have

$$\dim(\mathcal{X}) = \dim(s(\mathcal{X})).$$

**Proof.** If $\mathcal{X}$ is a finite type affinoid over $(\text{Spf } V)^{\text{rig}}$, where $V$ is an $a$-adically complete valuation ring, we may replace $V$ by its height one localization $V_p$, where $p = \sqrt{(a)}$ is the associated height one prime, and thus may assume that $\mathcal{X}$ is of type $(V_p)$. Then, by 8.2.11, 10.1.9 and 10.1.8, we have

$$\dim(\mathcal{X}) = \sup_{x \in (\mathcal{X})^{\text{cl}}} \dim_x(\mathcal{X}) = \sup_{z \in (s(\mathcal{X}))^{\text{cl}}} \dim(\mathcal{O}_{s(\mathcal{X}),z}) = \dim(s(\mathcal{X})),$$

as desired. \qed
Corollary 10.1.11. Let $X$ be a coherent rigid space of type $(V_R)$, and $X$ a distinguished formal model of $X$, which is a finite type formal scheme over an $a$-adically complete valuation ring $V$ of height 1. Denote by $X_k = X \otimes_V k$ the closed fiber of $X$, which is a finite type scheme over the residue field $k = V/m_V$. Then we have

$$\dim(X) = \dim(X_k).$$

Proof. We may assume $X = (\text{Spf } A)^{\text{rig}}$, where $A$ is an $a$-torsion free topologically of finite type $V$-algebra. Take a finite injection $V \langle X_1, \ldots, X_d \rangle \hookrightarrow A$ with the $V$-flat kernel (0.9.2.10). By 10.1.10, we have $\dim(X) = \dim(\text{Spec } A[\frac{1}{a}]) = d$. On the other hand, since we have a finite injection $k[X_1, \ldots, X_d] \hookrightarrow A_k = A \otimes_V k$, we have $\dim(X_k) = d$, thereby the claim. □

10.1. (d) Calculation of the dimension

Proposition 10.1.12. Let $V$ be an $a$-adically complete valuation ring, and consider the unit disk $\mathbb{D}^n_S$ over $S = (\text{Spf } V)^{\text{rig}}$ (cf. §2.5. (c)). We have

$$\dim(\mathbb{D}^n_S) = n.$$

Once this proposition is established, one can apply the Noether normalization theorem (0.9.2.10) to compute dimensions of rigid spaces of type (V).

To show the proposition, we consider the ring $V \langle X_1, \ldots, X_n \rangle$ of restricted formal power series (0, §8.4); we have $\mathbb{D}^n_S = (\text{Spf } V \langle X_1, \ldots, X_n \rangle)^{\text{rig}}$. By 10.1.10,

$$\dim(\mathbb{D}^n_S) = \dim(\text{Spec } V \langle X_1, \ldots, X_n \rangle[\frac{1}{a}]).$$

Since $V \langle X_1, \ldots, X_n \rangle[\frac{1}{a}] = V_p \langle X_1, \ldots, X_n \rangle[\frac{1}{a}]$, where $p = \sqrt{a}$ (cf. 0.9.1.10), we may assume that $V$ is of height one. Now the ring $V \langle X_1, \ldots, X_n \rangle[\frac{1}{a}]$, the Tate algebra (cf. 0, §9.3. (a)), is Noetherian, and hence it suffices to show the following proposition.

Proposition 10.1.13. Let $V$ be an $a$-adically complete valuation ring of height one, and $K = V[\frac{1}{a}] (= \text{Frac}(V)$, cf. 0.6.7.2).

(1) The maximal ideals of $V \langle X_1, \ldots, X_n \rangle[\frac{1}{a}]$ are exactly the kernels of $K$-algebra homomorphisms of the form $V \langle X_1, \ldots, X_n \rangle[\frac{1}{a}] \rightarrow K'$, where $K'$ is a finite extension of $K$.

(2) For any closed point $z$ of $\text{Spec } V \langle X_1, \ldots, X_n \rangle[\frac{1}{a}]$ we have

$$\dim_z(\text{Spec } V \langle X_1, \ldots, X_n \rangle[\frac{1}{a}]) = n.$$
Corollary 10.1.14. Let $A$ be a topologically finitely generated algebra over an $a$-adically complete valuation ring $V$. Let $V'$ be an $a'$-adically complete valuation ring and $V \to V'$ an adic homomorphism. Set $A' = A \hat{\otimes}_V V'$. Then
\[
\dim((\text{Spf} A)^{\text{rig}}) = \dim((\text{Spf} A')^{\text{rig}}).
\]

Proof. We may assume that $A$ is $a$-torsion free (hence $V$-flat). Let $p = \sqrt{\langle a \rangle}$ be the associated height-one prime of $V$, and consider the height-one valuation ring $V_p$. Since $V_p \otimes_V V'$ is just the height-one localization of $V'$, we may assume that $V$ and $V'$ are of height one. By Noether normalization (0.9.2.10), there exists an injective finite map $V\langle X_1, \ldots, X_n \rangle \to A$. Then we have $\dim((\text{Spf} A)^{\text{rig}}) = \dim(\text{Spec} A[\frac{1}{a}]) = n$. On the other hand, by applying $\hat{\otimes}_V V'$, we have the injective finite homomorphism $V'\langle X_1, \ldots, X_n \rangle \to A'$ (1.4.2.4 (4)); note that the morphism $V \to V'$ is injective (since it is adic (0.6.7.6)) and hence is flat. Therefore, we have $\dim((\text{Spf} A')^{\text{rig}}) = n$, as desired.

The following corollary follows immediately from 10.1.14.

Corollary 10.1.15. Let $\mathcal{X}$ be a rigid space locally of finite type over $S = (\text{Spf} V)^{\text{rig}}$, where $V$ is an $a$-adically complete valuation ring. Let $V'$ be an $a'$-adically complete valuation ring, and $V \to V'$ an adic map. Set $S' = (\text{Spf} V')^{\text{rig}}$. Then
\[
\dim(\mathcal{X}) = \dim(\mathcal{X} \times_S S').
\]

Corollary 10.1.16. Let $V$ be an $a$-adically complete valuation ring of height one, and $A$ an $a$-torsion free topologically finitely generated $V$-algebra, which is further assumed to be an integral domain. Set $\mathcal{X} = (\text{Spf} A)^{\text{rig}}$, and $B = A[\frac{1}{a}] = \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. Then $\dim_x(\mathcal{X})$ is constant for $x \in (\mathcal{X})^{\text{cl}}$, and is equal to $\dim(\mathcal{X})$, which is further equal to the Krull dimension $\dim B$.

Proof. In view of 10.1.8, we need to show that $\dim_z(\text{Spec} B)$ is constant for $z \in (\text{Spec} B)^{\text{cl}}$; the other assertions follow from 10.1.9 and 10.1.10. The claim is clear in the case $A = V\langle X_1, \ldots, X_n \rangle$ by 10.1.13 (2). In general, by Noether normalization (0.9.3.6), we have a finite injective map $K\langle X_1, \ldots, X_n \rangle \to B$. Then, since $K\langle X_1, \ldots, X_n \rangle$ is integrally closed (0.9.3.12), the ring extension $B/K\langle X_1, \ldots, X_n \rangle$ satisfies going-up and going-down properties (cf. [81], Theorem 9.4), from which the desired constancy of the dimension follows.

Corollary 10.1.17. Let $V$ and $\mathcal{X}$ be as in 10.1.16, and $U \subseteq \mathcal{X}$ a non-empty open subspace of $\mathcal{X}$. Then we have
\[
\dim(\mathcal{X}) = \dim(U).
\]
10.1. (e) GAGA comparison of the dimensions

**Proposition 10.1.18.** Let $V$ be an $a$-adically complete valuation ring and let $A$ be a $V$-algebra of finite type. Consider the morphism of schemes

$$
\rho: \text{Spec } \hat{A}[\frac{1}{a}] \longrightarrow \text{Spec } A[\frac{1}{a}],
$$

where $\hat{A}$ is the $a$-adic completion of $A$.

1. The morphism $\rho$ maps closed points of $\text{Spec } \hat{A}[\frac{1}{a}]$ injectively to closed points of $\text{Spec } A[\frac{1}{a}]$.

2. For any closed point $x \in \text{Spec } \hat{A}[\frac{1}{a}]$, the induced map between the completed local rings is the isomorphism

$$
\hat{\mathcal{O}}_{\text{Spec } \hat{A}[\frac{1}{a}], x} \cong \hat{\mathcal{O}}_{\text{Spec } A[\frac{1}{a}], \rho(x)},
$$

where the completions are taken with respect to the adic topology defined by the maximal ideals. In particular,

$$
\dim_x (\text{Spec } \hat{A}[\frac{1}{a}]) = \dim_{\rho(x)} (\text{Spec } A[\frac{1}{a}]).
$$

**Proof.** Clearly, we may assume that $V$ is of height one and that $A$ is $a$-torsion free (that is, flat over $V$). Write $A = V[X_1, \ldots, X_n]/J$ for an $a$-saturated (hence finitely generated) ideal $J$. We have $\hat{A} = V\langle X_1, \ldots, X_n \rangle/JV\langle X_1, \ldots, X_n \rangle$. Hence the map in question is rewritten as

$$
\text{Spec } V\langle X_1, \ldots, X_n \rangle[\frac{1}{a}]/JV\langle X_1, \ldots, X_n \rangle[\frac{1}{a}] 
\longrightarrow 
\text{Spec } V[X_1, \ldots, X_n][\frac{1}{a}]/J V[X_1, \ldots, X_n][\frac{1}{a}].
$$

By 10.1.13 (1), the maximal ideals of $\hat{A}[\frac{1}{a}]$ are exactly the kernels of the map of the form $\hat{A}[\frac{1}{a}] \rightarrow K'$, where $K'$ is a finite extension of $K = V[\frac{1}{a}]$. Hence one gets a maximal ideal of $A[\frac{1}{a}]$, which is a $K$-algebra of finite type. Since the map above is uniquely determined by the images of the $X_i$’s, this correspondence between maximal ideals is injective. Thus we have shown (1). On the other hand, we already know that (2) is true when $A = V[X_1, \ldots, X_n]$, since, in this case, the completed local rings in question are isomorphic to the ring of formal power series. By taking the quotient modulo $J$, we get the desired assertion (since $J$ is finitely generated). \( \square \)
Now let us consider the following situation. Let $V$ be an $a$-adically complete valuation ring of height one, and set $K = \text{Frac}(V) = V\left[\frac{1}{a}\right]$. Let

$$f : X \longrightarrow \text{Spec } K$$

be a $K$-scheme of finite type. Then using the GAGA functor ($\S 9.1.(e)$) one can consider the rigid space

$$f^\text{an} : X^\text{an} \longrightarrow (\text{Spf } V)^\text{rig}$$

of finite type over $V$.

**Proposition 10.1.19.** For any classical point $x \in (X^\text{an})^\text{cl}$ of $X^\text{an}$ we have the canonical isomorphism

$$\hat{\Theta}_{X,\rho_X(x)} \cong \hat{\Theta}_{X^\text{an},x}$$

of complete local rings. (Here $\rho_X : (X^\text{an}) \to X$ is the comparison map ($\S 9.3.(a)$).)

In particular,

$$\dim_x(X^\text{an}) = \dim_{\rho_X(x)}(X).$$

**Proof.** Since the question is local, we may assume that $X$ is affine; replacing by a suitable (e.g., projective) compactification, we may furthermore assume that $X$ is proper. One can choose an affinoid neighborhood $\mathcal{U} = (\text{Spf } \hat{A})^{\text{rig}}$ of $x$, which comes from an affine open neighborhood $\text{Spec } A \subseteq X$ of $\rho(x)$. Then we need to show that

$$\hat{\Theta}_{\mathcal{U},x} \cong \hat{\Theta}_{\text{Spec } A\left[\frac{1}{a}\right],\rho(x)}.$$ 

But this follows from 8.3.1 and 10.1.18 (2), since $s(\mathcal{U}) = \text{Spec } \hat{A}\left[\frac{1}{a}\right].$ \qed

**Theorem 10.1.20** (GAGA comparison of dimensions). Let $V$ be an $a$-adically complete valuation ring, and $X$ a finite type scheme over $K = \text{Frac}(V) = V\left[\frac{1}{a}\right]$. Then we have

$$\dim(X^\text{an}) = \dim(X).$$

To show the theorem, we may assume that $V$ is of height one. Then in view of 10.1.19 the theorem follows from the following lemma, which in turn is an easy consequence of 9.2.2.

**Lemma 10.1.21.** In the above situation, the comparison map $\rho_X : (X^\text{an}) \to X$ maps the set of all classical points $(X^\text{an})^\text{cl}$ bijectively onto the set of all closed points $X^\text{cl}$ of $X$. 


10.1. (f) Dimension function

Definition 10.1.22. Let $X$ be a rigid space of type (V) or of type (N). We define the function

$$d = d_X: (X) \longrightarrow \mathbb{Z}_{\geq 0} \cup \{-\infty\},$$

which we call the dimension function on $X$, by

$$d(x) = \inf_{x \in (U)} \dim(U)$$

for $x \in (X)$, where $U$ runs through all open neighborhood $U$ of $x$.

Proposition 10.1.23. Let $V$ be an $a$-adically complete valuation ring of height one, and $A$ an $a$-torsion free topologically finitely generated $V$-algebra. Suppose that $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$ is reduced (that is, the nilpotent radical of $B = A[\frac{1}{a}]$ is zero; cf. 8.3.5). Let

$$\mathcal{X} = \bigcup_{i=1}^{r} \mathcal{X}_i$$

be the irreducible decomposition (as in §8.3.(b)). Then, for any $x \in (\mathcal{X})$, we have

$$d(x) = \max\{\dim(\mathcal{X}_i): i = 1, \ldots, r, x \in (\mathcal{X}_i)\}.$$  

Proof. Since closed rigid subspaces are overconvergent closed subsets (7.3.10), we have $\{i: x \in (\mathcal{X}_i)\} = \{i: G_x \cap (\mathcal{X}_i) \neq \emptyset\}$ (where $G_x$ denotes, as before, the set of all generizations of $x$), which we denote by $I_x$. Since $I_x$ is the set of indices $i = 1, \ldots, r$ such that $\langle U \rangle \cap (\mathcal{X}_i) \neq \emptyset$ for any open neighborhood $U$ of $x$, one can choose a sufficiently small affinoid open neighborhood $U$ of $x$ such that (i) $d(x) = \dim(U)$ and (ii) $I_x = \{i: \langle U \rangle \cap (\mathcal{X}_i) \neq \emptyset\}$ hold. By 10.1.10, we have

$$d(x) = \dim \Gamma(U, \mathcal{O}_U)$$

(Krull dimension), and hence

$$d(x) = \max_{i \in I_x} \dim \Gamma(\mathcal{X}_i \cap U, \mathcal{O}_{\mathcal{X}_i \cap U})$$

$$= \max_{i \in I_x} \dim \Gamma(\mathcal{X}_i, \mathcal{O}_{\mathcal{X}_i}) = \max_{i \in I_x} \dim(\mathcal{X}_i),$$

where the second equality is due to 10.1.17, and the third one again by 10.1.10. □

Corollary 10.1.24. Let $\mathcal{X}$ be a rigid space of type (V$_\mathbb{R}$). The dimension function $d = d_\mathcal{X}$ is upper-semi continuous on $(\mathcal{X})$; moreover, $\{x \in (\mathcal{X}): d(x) \geq c\}$ for any $c \in \mathbb{R}$ is the underlying topological space of a closed rigid subspace of $\mathcal{X}$. 

10.2 Codimension

Definition 10.2.1. Let $\mathcal{X}$ be a locally universally Noetherian rigid space.

(1) Let $x \in \langle \mathcal{X} \rangle$ be a point, and $\mathcal{Y}_x$ a germ of a rigid subspace $\mathcal{Y}$ at $x$. The codimension of $\mathcal{Y}_x$ in $\mathcal{X}$ at $x$, denoted by

$$\text{codim}_x(\mathcal{Y}, \mathcal{X}),$$

is the number defined as follows. If $\mathcal{Y}_x$ is a prime germ, then it is the supremum of the lengths of all possible chains of prime germs of rigid subspaces at $x$

$$Z_{0,x} \hookrightarrow Z_{1,x} \hookrightarrow \cdots \hookrightarrow Z_{n,x},$$

where $Z_{0,x} = \mathcal{Y}_x$; in general, it is the infimum of the codimensions in $\mathcal{X}$ of $Z_x$ at $x$, where $Z_x$ runs over the set of all prime germs that are contained in $\mathcal{Y}_x$.

(2) Let $\mathcal{Y}$ be a closed subspace of $\mathcal{X}$. Then the codimension of $\mathcal{Y}$ in $\mathcal{X}$, denoted by $\text{codim}(\mathcal{Y}, \mathcal{X})$, is defined by

$$\text{codim}(\mathcal{Y}, \mathcal{X}) = \inf_{x \in \langle \mathcal{X} \rangle} \text{codim}_x(\mathcal{Y}, \mathcal{X}).$$

10.3 Relative dimension

Proposition 10.3.1. Let $\varphi: \mathcal{X} \to \mathcal{Y}$ be a locally of finite type morphism between locally universally Noetherian rigid spaces, and $y \in \langle \mathcal{Y} \rangle$. Let

$$\alpha: S = (\text{Spf } V)^\text{rig} \to \mathcal{Y}$$

be a rigid point such that $\text{Spf } V \to \langle \mathcal{Y} \rangle$ maps the closed point to $y$. Then the number $\dim(\mathcal{X} \times_y S)$ depends only on $\varphi$ and $y$, and not on the choice of the rigid point $\alpha$.

Proof. Since any rigid point factors through the associated rigid point $\alpha_y$ (3.3.5), it suffices to show that the number in question is the same as the one with $\alpha$ replaced by $\alpha_y$. However, this follows immediately from 10.1.15. 

$\square$

Definition 10.3.2. Let $\varphi: \mathcal{X} \to \mathcal{Y}$ be a morphism locally of finite type between rigid spaces, and $y \in \langle \mathcal{Y} \rangle$. Then the relative dimension of $\varphi$ at $y$, denoted by

$$\dim_y(\varphi)$$

(or by $\dim_y(\mathcal{X})$), is the number $\dim(\mathcal{X} \times_y S)$, where $\alpha: S = (\text{Spf } V)^\text{rig} \to \mathcal{Y}$ is a rigid point such that $\text{Spf } V \to \langle \mathcal{Y} \rangle$ maps the closed point to $y$ (which is independent of the choice of $\alpha$ due to 10.3.1).
Theorem 10.3.3 (GAGA comparison of relative dimension). Let $S = \text{Spec } A$ where $A$ is an adic ring with a finitely generated ideal of definition $I \subseteq A$, $D = V(I)$, and $U = S \setminus D$. Let $X, Y$ be $U$-schemes of finite type, and $f : X \to Y$ a morphism over $U$. Then for any $y \in \langle Y^\text{an} \rangle$ we have

$$\dim_y(f^\text{an}) = \dim_{\rho_Y(y)}(f).$$

Proof. Let $\text{Spf } \widehat{V}_y \to \langle Y^\text{an} \rangle$ be the rigid point associated to the point $y$. Then $\rho_Y(y)$ is the image of $\text{Spec } \widehat{V}_y[\frac{1}{a}] \to Y$, and hence we need to prove is that the dimension of the rigid space $X^\text{an} \times_{Y^\text{an}} (\text{Spf } \widehat{V}_y)$ of type $(V)$ is equal to the dimension of the scheme $X \times_Y \text{Spec } \widehat{V}_y[\frac{1}{a}]$ of finite type over the field $\widehat{V}_y[\frac{1}{a}]$. Hence in view of 9.1.10 we may assume $B = V = \widehat{V}_y$ and $Y = \text{Spec } V[\frac{1}{a}]$. But the assertion in this case is nothing but 10.1.20. \hfill \square

11 Maximum modulus principle

In this section we discuss the maximal modulus principle for coherent rigid spaces of type of type $(V_R)$. The statement is given in 11.2.4 below. To show the theorem, we first give a classification of points on rigid spaces of type $(V)$ in the spirit of the classical classification of valuations as in 0, §6.6. This part of the discussion is interesting in its own right and can be read independently. Especially, in the case of the unit disk ($\S$11.1 (c)), one finds a strong analogy between our classification and Berkovich’s classification of points in [11], 1.4.4. The proof of the maximal modulus principle is based on the so-called spectral seminorm formula (11.2.1), which says that the spectral seminorm takes the maximum value of the norms at at most finitely many divisorial points.

11.1 Classification of points

11.1. (a) Basic inequality. Let $V$ be an $a$-adically complete valuation ring of height-one, $K = \text{Frac}(V)$ the fractional field, and $k = V/m_V$ the residue field. We denote by $\Gamma$ the value group of $V$, that is, $\Gamma = K^*/V^x$. Let $\mathcal{X}$ be a coherent rigid space of finite type over $\mathcal{S} = (\text{Spf } V)^{\text{rig}}$. For any point $x \in \langle \mathcal{X} \rangle$ we use the notation as in 3.2.13. We set

- $d(x)$: the dimension function (10.1.22);
- $t(x) = \text{tr.deg}_kk_x (k_x$ is the residue field of $\mathcal{O}_{\mathcal{X},x}^{\text{int}}$, or equivalently, of $V_x$);
- $\Gamma_x = K_x^*/V_x^x$, the value group of the valuation ring $V_x$;
- $\text{rat-rank}(V_x \mid V) = \dim_{\mathbb{Q}}(\Gamma_x / \Gamma) \otimes \mathbb{Q}$. 

The valuation ring $V_x$ at $x$ is $a$-adically separated (3.2.8) and has the associated height-one prime $p_x = \sqrt{aV_x}$. We set

- $\overline{V}_x = V_x/p_x$,

which is a valuation ring such that $\text{ht}(V_x) = \text{ht}(\overline{V}_x) + 1$. One easily sees that

$$\text{rat-rank}(\overline{V}_x) \leq \text{rat-rank}(V_x | V).$$

**Theorem 11.1.1.** (1) We have the inequality

$$\text{rat-rank}(V_x | V) + t(x) \leq d(x).$$

In particular, $\text{rat-rank}(\overline{V}_x)$ and $\text{ht}(V_x) = \text{ht}(\overline{V}_x) + 1$ are finite, and we have

$$\text{ht}(V_x) + t(x) \leq d(x) + 1.$$

(2) If $\text{rat-rank}(V_x | V) + t(x) = d(x)$, then $\Gamma_x/\Gamma$ is a finitely generated $\mathbb{Z}$-module, and $k_x$ is a finitely generated extension of $k$.

(3) If $\text{rat-rank}(\overline{V}_x) + t(x) = d(x)$, then the value group of $\overline{V}_x$ is isomorphic as a group to $\mathbb{Z}^d$ for some $d \geq 0$.

(4) If $\text{ht}(V_x) + t(x) = d(x) + 1$, then the value group of $\overline{V}_x$ is isomorphic as an ordered group to $\mathbb{Z}^d$ (with $d = \text{ht}(\overline{V}_x)$) equipped with the lexicographical order (cf. 0.6.1.3).

**Proof.** First let us show (1). We take an affinoid open neighborhood $U = (\text{Spf} A)^{\text{rig}}$ of $x$, where $A$ is an $a$-torsion free topologically finitely generated $V$-algebra, such that $d(x) = \text{dim}(U)$. Since we may work locally around $x$, we may set $X = U$ without loss of generality. By Noether normalization 0.9.2.10, we may assume that $A = V \langle X_1, \ldots, X_n \rangle$. We argue by induction with respect to $n$ as follows. Let $R = V \langle X_1, \ldots, X_{n-1} \rangle$ so that $A = R \langle X \rangle$ ($X = X_n$). Set $Y = (\text{Spf} R)^{\text{rig}}$, and let $y$ be the image of the point $x$ under the map $\langle X \rangle \to \langle Y \rangle$ induced by the canonical inclusion $R \hookrightarrow A = R \langle X \rangle$; note that the induced morphism $V_y \to V_x$ is $a$-adic, and hence is injective. By induction it suffices to show the inequality

$$\dim_{\mathbb{Q}}(\Gamma_x/\Gamma_y) \otimes \mathbb{Q} + \text{tr.deg}_{k_x} k_x \leq 1. \quad (\ast)$$

To show this, consider the valuation $v_x$ on $A$ associated to the valuation ring $V_x$, and its restriction $v'$ on $A' = R[X]$. Let $V'$ be the valuation ring of the valuation $v'$, and $\Gamma_{V'}$ and $k_{V'}$ the value group and the residue field of the valuation $v'$, respectively.
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Claim. We have $\Gamma_x = \Gamma_{V'}$ and $k_x = k_{V'}$.

Indeed, since $V'$ is $a$-adically dense in $V_x$ and $a \in m_{V_x}$, we have $k_x = k_{V'}$. Moreover, since $V_x$ is $a$-adically separated, for any $f \in V_x$ there exists $N \geq 1$ such that $v_x(f) < v_x(a^N)$. If $f = g + a^N h$ with $g \in V'$, then $v_x(f) = v_x(g) = v'(g)$. Hence we have the other equality.

Now we apply 0.6.5.3 to get the desired inequality (*). If equality holds in (*), then $k_x$ is a finitely generated field extension of $k_y$, and $\Gamma_x / \Gamma_y$ is finitely generated. Hence by induction we deduce (2).

Let us show (3). We may replace $X$ by an affinoid neighborhood $X = (\text{Spf } A)^{\text{rig}}$ for an $a$-torsion free $A$. Let $K$ be the kernel of the morphism $A \to \widehat{V}_x$, and set $A_0 = A/K$. Since $\sqrt{\alpha A} \subset K$, $A_0$ is a finite type domain over $k$ with the valuation $A_0 \leftrightarrow \widehat{V}_x$. Let $R$ be the localization of $A_0$ at the prime ideal $m_{\widehat{V}_x} \cap A_0$. Then, as can be shown easily with the aid of the Noether normalization 0.9.2.10, one has $\dim(R) \leq d(x)$. Hence we have the desired result by 0.6.5.2 (2). Similarly, (4) follows from 0.6.5.2 (3).

11.1. (b) Divisorial points. Let $V$ be an $a$-adically complete valuation ring of height one, and $X$ a coherent rigid space of finite type over $S = (\text{Spf } V)^{\text{rig}}$. We use the notation as in the previous paragraph.

Definition 11.1.2. (1) A point $x \in \langle X \rangle$ is said to be divisorial (over $V$) if $t(x) = d(x)$.

(2) Let $X$ be a distinguished formal model of $X$, and $x \in \langle X \rangle$ a divisorial point. We say that $x$ is residually algebraic over $X$ if $k_x$ is an algebraic extension of the residue field $k_{\text{sp}_X(x)}$ at $\text{sp}_X(x) \in X$.

By 11.1.1, we have the following proposition.

Proposition 11.1.3. If $x \in \langle X \rangle$ is divisorial, then $x$ is of height one (4.1.6 (1)) and $\Gamma_x / \Gamma$ is a finite group. If, moreover, $x$ is residually algebraic over a distinguished formal model $X$ of $X$, then $k_x$ is a finite extension of $k_{\text{sp}_X(x)}$.

11.1. (c) Example: Unit disk. Let $V$ be an $a$-adically complete valuation ring of height one, and $K = \text{Frac}(V)$. We consider the unit disk $\mathbb{D}^1_K = \mathbb{D}^1_S = (\text{Spf } V \langle T \rangle)^{\text{rig}}$ over $S = (\text{Spf } V)^{\text{rig}}$ (cf. §2.5. (c)).

Proposition 11.1.4. (1) For $x \in \langle \mathbb{D}^1_K \rangle$ one and only one of the following cases occurs.

(a) Divisorial case. $x$ is divisorial. In this case, $x$ is of height one, $t(x) = d(x) = 1$, and $\Gamma_x / \Gamma$ is finite.
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(b) Subject-to-divisorial case. \( x \) is of height two. In this case, \( t(x) = 0 \), and \( \Gamma_x/\Gamma \) is finitely generated, with \( \text{rat-rank}(V_x|V) = 1 \). Moreover, the maximal generalization \( \tilde{x} \) of \( x \) (4.1.6 (2)) is divisorial.

(c) Irrational case. \( x \) is of height one, \( t(x) = 0 \), and \( \text{rat-rank}(V_x|V) = 1 \). In this case, \( \Gamma_x/\Gamma \) is finitely generated.

(d) Limit case. \( x \) is of height one, \( t(x) = 0 \), and \( \text{rat-rank}(V_x|V) = 0 \).

(2) Classical case. If the kernel of \( V \langle T \rangle \to V_x \) is non-trivial, then \( x \) falls in case (d). In this case \( x \) is a classical point (8.2.8), and \( \Gamma_x/\Gamma \) is finite.

Remark 11.1.5. Note the similarity between the classification as in (1) and the classification of valuations in 0, §6.6 (c).

Proof of Proposition 11.1.4. First suppose that the kernel \( J \) of \( A = V \langle T \rangle \to V_x \) is non-trivial. Since \( J \) is clearly \( a \)-saturated, \( J \) is a finitely generated prime ideal of \( A \). Since \( \dim(\mathbb{D}^1_K) = \dim(\text{Spec } K \langle T \rangle) = 1 \) (10.1.12), \( J \) defines a closed point on the associated scheme \( s(\mathbb{D}^1_K) = \text{Spec } K \langle T \rangle \). It follows as in the proof of 8.2.10 that \( W = A/J \) is finite over \( V \) and that \( (\text{Spf } W)^\text{rig} \leftrightarrow \mathbb{D}^1_K \) defines a classical point, which is nothing but \( x \). By 11.1.1,

\[
\text{rat-rank}(V_x|V) + t(x) = \dim((\text{Spf } W)^\text{rig}) = 0,
\]

and hence \( x \) is of height one, \( t(x) = 0 \), and \( \Gamma_x/\Gamma \) is finite, which proves (2).

Let us show (1). Note first that \( \text{ht}(V_x) \leq 2 \) by 11.1.1 (1) and that we always have \( d(x) = 1 \). If \( \text{ht}(V_x) = 2 \), then we have the equalities in 11.1.1 with \( t(x) = 0 \).

Since \( \text{rat-rank}(V_x|V) \geq 1 \), we have \( \text{rat-rank}(V_x|V) = 1 \), and hence \( \Gamma_x/\Gamma \) is finitely generated by 11.1.1 (2).

Next, we assume \( \text{ht}(V_x) = 1 \). Then \( t(x) = 0 \) or 1. If \( t(x) = 1 \), then \( d(x) = t(x) \), and hence \( x \) is divisorial. We have the equality as in 11.1.1 (2) with \( \text{rat-rank}(V_x|V) = 0 \), and hence \( \Gamma_x/\Gamma \) is finite (case (a)). Suppose \( t(x) = 0 \). Then \( \text{rat-rank}(V_x|V) = 0 \) or 1. If \( \text{rat-rank}(V_x|V) = 1 \), then \( t(x) = 0 \), and hence \( \Gamma_x/\Gamma \) is finitely generated (11.1.1 (2)), which is case (c). If \( \text{rat-rank}(V_x|V) = 0 \), we are in case (d).

It remains to show that if \( \text{ht}(V_x) = 2 \), then the maximal generalization \( \tilde{x} \) of \( x \) is divisorial. We have \( \text{ht}(V_{\tilde{x}}) = 1 \). Since \( V_x \) is a composite of \( V_{\tilde{x}} \) and a height-one valuation ring for \( k_{\tilde{x}} \) over \( k \), \( k_{\tilde{x}} \) is not algebraic over \( k \). Hence we have \( t(\tilde{x}) = 1 \), which means that \( \tilde{x} \) is divisorial.

\[ \square \]

Remark 11.1.6. The above classification compares with the Berkovich’s classification of points on a unit disk ([11], 1.4.4) as follows. Berkovich deals only with height-one points, and so case (b) does not occur.

- Berkovich’s type (1) and type (4) points fall in the limit case (d); moreover, type (1) points are precisely the points in the classical case (2).
Type (2) points correspond to the divisorial case (a).
Type (3) points correspond to the irrational case (c).

The comparison is carried out as follows. We assume for simplicity that $K$ is algebraically closed; note that in this case, if $x \in \mathfrak{m}_V \setminus \{0\}$ and a real number $0 < c < 1$ and consider the valuation $\| \cdot \|: K \to \mathbb{R}_{\geq 0}$ associated to the valuation ring $V$ such that $|a| = c$ (cf. 0.6.3. (c)).

For any height-one point $x \in \langle \mathcal{D}^1_K \rangle$ lies in $\langle \mathcal{D}(u, |b|) \rangle$, the closure of $\langle \mathcal{D}(u, |b|) \rangle = \text{sp}_{X}^{-1}(U)$ in $\langle \mathcal{D}^1_K \rangle$, if and only if
\[ \|T - u\|_x \leq |b|, \]
where $\| \cdot \|_x = \| \cdot \|_{x,I,c}$ (with $\mathcal{I} = a \mathfrak{m}_K$) is the seminorm associated to the point $x$ (§3.3. (b)). This leads one to consider the overconvergent closed subsets of $\langle \mathcal{D}_K^1 \rangle$ of the form
\[ D(u, r) = \{ y \in \langle \mathcal{D}_K^1 \rangle : \|T - u\|_y \leq r \} \]
for $u \in V$ and $r \in \mathbb{R}_{\geq 0}$.

For any height-one point $x \in \langle \mathcal{D}_K^1 \rangle$, consider the family of subsets $\{ E(f) \}$ indexed by $f \in A = V \langle T \rangle$, where
\[ E(f) = \{ y \in \langle \mathcal{D}_K^1 \rangle : \|f\|_y \leq \|f\|_x \}. \]

By the Weierstrass preparation theorem (Exercise 0.A.3), this family is determined only by $E(f)$'s for polynomials $f \in V[T]$. Moreover, if $f = g_1 \cdots g_r$ in $V[T]$, then $E(f)$ contains the intersection $E(g_1) \cap \cdots \cap E(g_r)$. Hence the family $\{ E(f) \}$ is filtered, and the subfamily $\{E(T - u)\}_{u \in V}$ consisting only of the subsets determined by linear polynomials is cofinal. Since $\|T - u\|_x \leq \|T - v\|_x$ implies $E(T - u) \subseteq E(T - v)$, $\{E(T - u)\}_{u \in V}$ is a ‘family of embedded disks’ (according to Berkovich’s terminology [11], 1.4.4).
We fix once for all a real number \(0 < c < 1\) with \(f\) norm \(j\) then, since \(X\) ring of height one, and type schemes over \(k\) /TAB 11.2. (a) Spectral seminorm formula. Let \(X\) consider the set of radii \(\{\|T - u\|_X\}_{u \in V}\); we set \(r_x = \inf \{\|T - u\|_X : u \in V\}\).

Note that this value \(r_x\) lies in \(\|K\|_x\). If \(\{T-u_n\}_{n \geq 0}\) is such that \(\lim \|T-u_n\|_x = r_x\), then, since \(|u_n - u_m| = \|u_n - u_m\|_x = \|(T - u_m) - (T - u_n)\|_x\) for \(n, m \geq 0\), \(\{u_n\}_{n \geq 0}\) is a Cauchy sequence in \(V\), and we have \(r_x = \|T - u\|_x\) for \(u = \lim u_n\).

(i) Suppose that \(r_x \not\in \|K\|_x\). Then since the value group of \(V_x\) is strictly larger than \(\Gamma\), the point \(x\), which is of type (3) in Berkovich’s classification, falls into the irrational case (c). Note that, in this case, for any \(u \in V\) and for any \(b \in V\) such that \(\|T - u\|_x > |b|\), the rigid point \(\text{Spf } \overline{V}_x \to X_{u,b}\) hits the closed fiber \(X_{u,b} \otimes_V k\) at the double point (the intersection of the two irreducible components).

(ii) If \(r_x \in \|K\|_x\), then \(x\) is of Berkovich type (2). For \(u \in V\) and \(b \in V \setminus \{0\}\) such that \(r_x = |b| = \|T - u\|_x\), the rigid point \(\text{Spf } \overline{V}_x \to X_{u,b}\) hits the generic point of \(U' = \text{Spf } V \langle \frac{T - u}{b} \rangle\).

In particular, \(x\) is divisorial (that is, \(t(x) = 1\)).

(iii) Suppose \(r_x = 0\). In this case, for any \(b \in V \setminus \{0\}\) there exists \(u \in V\) such that \(\|T - u\|_x \leq |b|\); then the rigid point \(\text{Spf } \overline{V}_x \to X_{u,b}\) hits the closed fiber \(X_{u,b} \otimes_V k\) at a closed point on the affine part \(U' \otimes_V k\). The points in this case, of type (1) or (4) in Berkovich’s classification, fall into case (d) (limit case).

11.2 Maximum modulus principle

11.2. (a) Spectral seminorm formula. Let \(V\) be an \(a\)-adically complete valuation ring of height one, and \(X\) a coherent rigid space of finite type over \(S = (\text{Spf } V)_{\text{rig}}\). We fix once for all a real number \(0 < c < 1\), which gives rise to a non-archimedean norm \(\| \cdot \| : K = V[\frac{1}{a}] \to \mathbb{R}_{\geq 0}\) with \(|a| = c\), and the spectral seminorm \(\| \cdot \|_{\text{sp}} = \| \cdot \|_{\text{sp}, I, c}\) (§3.3.(c)), where \(I = a\mathcal{O}_{X, \text{rig}}^\text{int}\).

Let \(X\) be a flat and of finite type formal scheme giving a distinguished formal model of \(X\), \(X_k = X \otimes_V k\) its closed fiber, where \(k\) is the residue field \(k = V/\mathfrak{m}_V\) of \(V\), and \((X_k)_{\text{red}}\) the reduced model of \(X_k\). Notice that \(X_k\) and \((X_k)_{\text{red}}\) are finite type schemes over \(k\). We define
• \( I(X) = \) the set of all points of \((X_k)_{\text{red}}\) of which the closure is an irreducible component of \((X_k)_{\text{red}}\);
• \( D(X) = \) the set of all divisorial points in \( (X)\) that are residually algebraic over \(X\) (11.1.2).

Notice that, due to 11.1.3, one can equivalently define \( D(X) \) as the set of all divisorial points of \((X)\) that are residually finite over \(X\).

**Theorem 11.2.1.** (1) For any \( z \in I(X) \), the set \( \text{sp}_X^{-1}(z) \), where \( \text{sp}_X : (X) \to X \) is the specialization map, is non-empty, and is contained in \( D(X) \).

(2) We have
\[
D(X) = \bigcup_{z \in I(X)} \text{sp}_X^{-1}(z).
\]

(3) (Spectral seminorm formula) The set \( D(X) \) is finite. Moreover, for any \( f \in \Gamma(X, \mathcal{O}_X) \), we have
\[
\| f \|_{\text{sp}} = \max_{x \in D(X)} \| f \|_x,
\]
where \( \| \cdot \|_x = \| \cdot \|_{x, \mathcal{I}_x, \mathcal{E}} \) is the seminorm at the point \( x \) (§3.3.(b)).

To show the theorem, we need the following lemma.

**Lemma 11.2.2.** Let \( A \) be an \( a \)-torsion free topologically of finite type integral domain over \( V \), and set \( X = (\text{Spf } A)^{\text{rig}} \). Then, for any proper closed subspace \( Z \subsetneq X \), we have \( \dim(Z) < \dim(X) \).

**Proof.** Let \( d = \dim(X) \), and take a finite injection \( V \langle X_1, \ldots, X_d \rangle \hookrightarrow A \) with \( V \)-flat cokernel (see 0.9.2.10). Any proper closed subspace \( Z \subsetneq X \) corresponds to a proper closed subscheme \( Z \subset \text{Spec } A[\frac{1}{a}] \) (7.3.15) such that \( \dim(Z) = \dim(Z) \) (10.1.10), and hence to a \( V \)-flat proper closed subscheme \( \overline{Z} \subset \text{Spec } A \). Since a proper closed subscheme of \( \text{Spec } A \) does not contain the generic point of \( \text{Spec } A \), its image under the finite and dominant map \( X \to A = \text{Spf } V \langle X_1, \ldots, X_d \rangle \) does not contain the generic point, and hence is a proper closed subspace of \( A \). Hence, to show the lemma, one can reduce to the case \( X = A \), and the assertion in this case is clear.

**Proof of Theorem 11.2.1.** First notice that, for any open subset \( U \subset X \), we have \( I(U) = I(X) \cap (U_k)_{\text{red}} \) and \( D(U) = D(X) \cap (U^{\text{rig}}) \). In particular, if \( X = \bigcup_{\alpha \in L} U_\alpha \) is a finite open covering, we have \( I(X) = \bigcup_{\alpha \in L} I(U_\alpha) \) and \( D(X) = \bigcup_{\alpha \in L} D(U_\alpha) \). Hence the assertions (1), (2), and the finiteness (3) are reduced to those for each \( U_\alpha \). Moreover, since \( [X] = \bigcup_{\alpha \in L} [U_\alpha^{\text{rig}}] \), the spectral seminorm \( \| f \|_{\text{sp}} \) on \( X \) is equal to the maximum of the spectral seminorms of \( f \) on \( U_\alpha \)'s.
Hence the last assertion of (3) can also be reduced to that on each $U_a$. We can therefore assume that $X$ is affine, say $X = \text{Spf} A$, where $A$ is an $a$-torsion free topologically finitely generated $V$-algebra.

Let us first show the theorem under the additional assumption that $A$ is an integral domain. If $d$ is the Krull dimension of $\mathfrak{a} = A[\frac{1}{a}]$, then we have $\dim(\mathcal{X}) = d$ by 10.1.10. Take a finite injection $V \langle X_1, \ldots, X_d \rangle \hookrightarrow A$ with $V$-flat cokernel (see 0.9.2.10), and set $\mathbb{A} = \text{Spf} V \langle X_1, \ldots, X_d \rangle$. Notice that, passing to the closed fibers, we still have finite injection $k[X_1, \ldots, X_d] \hookrightarrow A_k = A \otimes_V k$. Notice that, by 10.1.23, the dimension function $d(x)$ is constant, that is, $d(x) = d$ for all $x \in \langle \mathcal{X} \rangle$.

Let us show (1). Since the specialization map $\text{Sp} : \langle \mathcal{X} \rangle \to X$ is surjective (see 3.1.5), $\text{Sp}^{-1}(z)$ for $z \in I(X)$ is non-empty. Let us show the inclusion $\text{Sp}^{-1}(s) \subseteq D(X)$. For any $z \in I(X)$, let $Z_z$ be the closure $\{z\}$ in $(X_k)_{\text{red}}$, which is an irreducible component of $(X_k)_{\text{red}}$. Set

$$C_z = \bigcup_{z' \in I(X) \setminus \{z\}} Z_{z'},$$

and let $U$ be the open complement of $C_z$ in $X$, which is an open formal subscheme of $X$. Then $(U_k)_{\text{red}} = Z_z \setminus C_z$ is irreducible and non-empty. Since $\dim(U)_{\text{rig}} = d$ holds by 10.1.17, we have $\dim((U_k)_{\text{red}}) = d$ by 10.1.11. Since $U$ satisfies $(U_k)_{\text{red}} \cap I(X) = \{z\}$, we have $t(x) \geq d$ for any $x \in \text{Sp}^{-1}(z)$. Such an $x$ is, since $d(x) = d$, always divisorial by the inequality in 11.1.1, hence belonging to $D(X)$.

Next we show (2). For $x \in D(X)$, we have, by constancy of the dimension function (10.1.23), $d(x) = d$, hence $t(x) = d$. This means that $x$ dominates the generic point of $\mathbb{A}_k$, the closed fiber of $\mathbb{A}$, hence belonging to $\text{Sp}^{-1}(z)$ for some $z \in I(X)$. Thus we have $D(X) \subseteq \bigcup_{z \in I(X)} \text{Sp}^{-1}(z)$. The other inclusion follows from (1).

Let us show (3). We show that the set $D(X)$ is finite. First, notice that $D(\mathbb{A})$ consists of a single point, which we denote by $w$. Since $\mathcal{X} \to \mathbb{A}^{\text{rig}}$ is finite, we readily see that $D(X)$ is the preimage of $w$ by the map $\langle \mathcal{X} \rangle \to \langle \mathbb{A}^{\text{rig}} \rangle$, which is finite due to 7.2.7.

Next, let us show the spectral seminorm formula. Since by definition $\|f\|_{\text{Sp}} \geq \max_{x \in D(X)} \|f\|_x$, we need to show $\|f\|_{\text{Sp}} \leq \max_{x \in D(X)} \|f\|_x$. Suppose that for any $x \in D(X)$ we have $\|f^n\|_x \leq |a|^m$ for $n, m \geq 1$. Since by 8.2.16

$$A^{\text{int}} = \{f \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) : f \text{ is integral at each } x \in D(X)\},$$

this implies $f^n/a^m \in A^{\text{int}}$. Therefore, $\|f^n\|_{\text{Sp}} \leq |a|^m$. We have shown that, for any $n, m \geq 1$, $\max_{x \in D(X)} \|f\|_x \leq |a|^{\frac{m}{n}}$ implies $\|f\|_{\text{Sp}} \leq |a|^{\frac{m}{n}}$. By taking the limit $|a|^{\frac{m}{n}} \to \max_{x \in D(X)} \|f\|_x$, the desired inequality follows.
Finally, let us discuss the general case $X = \text{Spf} A$, where $A$ is not necessarily an integral domain. We may assume that $A$ is reduced. Let

$$\text{Spec } A = \bigcup_{i \in I} \bar{Z}_i, \quad \bar{Z}_i = \text{Spec } A_i \quad (i \in I),$$

where $I$ is a finite set, be the irreducible decomposition; each $\bar{Z}_i$ is the closure of an irreducible component of the Noetherian scheme $\text{Spec } A[\frac{1}{a}]$. For $i \in I$, set

$$\bar{S}_i = \bigcup_{j \neq i} (\bar{Z}_j \cap \bar{Z}_i),$$

and set $Z_i = \bar{Z}_i$ and $S_i = \bar{S}_i$. Since $\bar{Z}_i$ is irreducible and $\bar{S}_i$ is a $V$-flat proper closed subscheme of $\bar{Z}_i$, we have, by 11.2.2, $\dim(S_i^{\text{rig}}) < \dim(Z_i^{\text{rig}})$. In particular, we have $\dim((S_i,k)_{\text{red}}) < \dim(Z_i^{\text{rig}})$ (10.1.11). By the above-discussed case of integral domains, any irreducible component of $(Z_i,k)_{\text{red}}$ is of dimension $d_i = \dim(Z_i^{\text{rig}})$, which implies that the complement $U_i$ of $(S_i,k)_{\text{red}}$ in $Z_i$ is dense in $Z_i$, and that $D(Z_i) = D(U_i)$.

We claim that the following equalities hold:

$$D(X) = \bigcup_{i \in I} D(Z_i), \quad I(X) = \bigcup_{i \in I} I(Z_i). \quad (*)$$

Indeed, by 10.1.23, we have $D(X) \subseteq \bigcup_{i \in I} D(Z_i)$, and the other inclusion follows by

$$\bigcup_{i \in I} D(Z_i) = \bigcup_{i \in I} D(U_i) = D\left(\bigcup_{i \in I} U_i\right) \subseteq D(X).$$

Since $U_i$ is dense in $Z_i$, we have $I(Z_i,\text{red}) = I(U_i,\text{red})$ for $i \in I$. Hence

$$\bigcup_{i \in I} I(Z_i) = \bigcup_{i \in I} I(U_i) = I\left(\bigcup_{i \in I} U_i\right) = I(X),$$

where the last equality follows from the fact that $\bigcup_{i \in I} U_i$ is dense in $X$.

By the equalities $(*)$ and the above argument on the case where $A$ is an integral domain, one deduces the assertions (1), (2), and the finiteness in (3). To show the spectral seminorm formula, notice first that we have an irreducible decomposition

$$\langle X \rangle = \bigcup_{i \in I} \langle Z_i^{\text{rig}} \rangle \quad (**)$$

of the rigid space $X = (\text{Spf } A)^{\text{rig}}$ (see §8.3. (b)). Then the decompositions $(*)$ and $(**)$ and the spectral seminorm formula in the integral domain case imply the spectral seminorm formula in general.
11. Maximum modulus principle

**Remark 11.2.3.** The theorem shows that there exists a canonical surjective map

\[ D(X) \longrightarrow I(X), \]

that is, any point in \( D(X) \) dominates an irreducible component of \((X_k)_{\text{red}}\), hence a point in \( I(X) \).

11.2. (b) Maximum modulus principle. We continue to use the notation fixed in the beginning of §11.2. (a).

**Theorem 11.2.4** (maximum modulus principle). Let \( \mathcal{X} \) be a coherent rigid space over \( S = (\text{Spf } V)_{\text{rig}} \).

1. There exists a positive integer \( e \geq 1 \) such that for any \( f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \) the value \( \| f \|_{\text{Sp}} \) belongs to \( |K^\times|^\frac{1}{e} \cup \{0\} \).

2. For any \( f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \) there exists a non-empty quasi-compact open subspace \( U \subset \mathcal{X} \) such that

\[ \langle U \rangle = \{ x \in (\mathcal{X}) : \| f \|_x = \| f \|_{\text{Sp}} \}. \]

3. For any \( f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \) the set \( (\mathcal{X})^{\text{cl}} \cap \{ x \in (\mathcal{X}) : \| f \|_x = \| f \|_{\text{Sp}} \} \) is non-empty, that is, the maximum value of \( \| f \|_x \) is attained at a classical point.

**Proof.** Since the group \( \Gamma_x / \Gamma \) is finite at a divisorial point, (1) follows immediately from 11.2.1. By 8.2.13, assertion (3) will follow from (2). Thus it suffices to show (2).

By a reduction process similar to that in the beginning of the proof of 11.2.1, we may assume that \( \mathcal{X} = X_{\text{rig}} \), with \( X = \text{Spf } A \). Let \( f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = A[\frac{1}{a}] \). Since the case \( \| f \|_{\text{Sp}} = 0 \) is trivial, we assume \( \| f \|_{\text{Sp}} \neq 0 \). By (1), there exist \( b \in K^\times \) and \( e \geq 1 \) such that \( \| f^e \|_{\text{Sp}} = |b| \). We may replace \( f \) by \( f^e / b \) without changing the right-hand set of the desired equality and thus may assume that \( \| f \|_{\text{Sp}} = 1 \). Then 8.2.16 implies that \( f \in A^{\text{int}} \). By 6.2.7, there exists an admissible blow-up \( X' \rightarrow X \) such that \( f \in \Gamma(X', \mathcal{O}_{X'}) \). We take the non-empty localization \( X'_f \) by \( f \), and consider the corresponding non-empty quasi-compact open subspace \( \mathcal{U} = (X'_f)_{\text{rig}} \) of \( \mathcal{X} \). It is then obvious that \( \langle \mathcal{U} \rangle = \{ x \in (\mathcal{X}) : \| f \|_x = 1 \} \). \( \square \)

11.2. (c) Reduction scheme. We use the same notation as in §11.2. (a). The following theorem says that birational geometry (in our sense; see Introduction) is easier around residually finite divisorial points.
Theorem 11.2.5. Let $\mathcal{X}$ be a coherent rigid space of finite type over $S = (\text{Spf } V)^{\text{rig}}$, $X$ a distinguished formal model of $\mathcal{X}$, and $x \in (\mathcal{X})$ a divisorial point that is residually finite over $X$. Then any admissible blow-up $\pi : X' \to X$ is finite near $\text{sp}_X(x)$.

Proof. Let $U = \text{Spf } A$ be an open neighborhood of $z = \text{sp}_X(x)$ that does not intersect any other components $\{z'\}$ with $z' \in I(X)$ and $z' \neq z$. Take a finite dominant morphism $U \to \mathbb{A} = \text{Spf } V\langle X_1, \ldots, X_d \rangle$ (by Noether normalization; see 0.9.2.10). Note that $D.A/ consists of a single point, which we denote by $w$. Set $W = \text{Spf } V_w$, and consider the canonical morphism $W \to A$. Then $X'_W = X' \times_A W$ is an admissible blow-up of $X_W = X \times_A W$. Since $X_W$ is finite flat over $W$, $X'_W$ is finite over $W$. In particular, $X'_W \to X_W$ is finite. Since $X' \to X$ is of finite type, there exists an open neighborhood $U'_k$ of $z$ in $U_k$ ($= \text{the closed fiber of } U$) such that $\pi_k : X'_k \to X_k$ is finite over $U'_k$. Hence $\pi$ is finite over the open subspace $U'$ of $U$ corresponding $U'_k$. □

Definition 11.2.6. Let $\mathcal{X}$ be a coherent rigid space of finite type over $S = (\text{Spf } V)^{\text{rig}}$.

(1) Two distinguished formal models $X_i$ ($i = 1, 2$) of $\mathcal{X}$ are said to be strictly equivalent if there exists a third distinguished formal model $X_3$ of $\mathcal{X}$ with finite morphisms $X_3 \to X_i$ ($i = 1, 2$).

(2) For a distinguished formal model $X$ of $\mathcal{X}$, let $\mathcal{C}_X$ be the category of distinguished formal models $X'$ of $\mathcal{X}$ finite over $X$, and morphisms over $X$. Set

$$R(\mathcal{X}, X) = \lim_{\longrightarrow} (X'_k)_{\text{red}}$$

(where $(X'_k)_{\text{red}}$ denotes the reduced model of the closed fiber $X'_k$ of $X'$), which is representable in the category of schemes. We call $R(\mathcal{X}, X)$ the reduction scheme of $\mathcal{X}$ relative to $X$.

Notice that the strict equivalence is an equivalence relation, and that the category $\mathcal{C}_X$ is directed; indeed, for two finite $X'_1 \to X$ and $X'_2 \to X$ in $\mathcal{C}_X$, there exists an admissible blow-up $W \to X$ dominating both $X'_1$ and $X'_2$; then take the Stein factorization $W \to Z \to X$ (see I.11.3.2), and observe that $Z$ is finite over $X$, and that $Z$ dominates both $X'_i$ ($i = 1, 2$).

Notice also that the scheme $R(\mathcal{X}, X)$ depends only on the strict equivalence class of $X$, and the construction is canonical; that is, for a finite type morphism $f : X \to Y$ between coherent formal schemes of finite type over $V$, we have the canonical morphism

$$R(f) : R(\mathcal{X}, X) \to R(\mathcal{Y}, Y).$$
where \( \mathcal{Y} = Y^{\text{rig}} \). Indeed, for any object \( Y' \rightarrow Y \) in \( \mathcal{C}_Y \), the objects in \( \mathcal{C}_X \) dominating \( X \times_Y Y' \) form a cofinal subcategory \( \mathcal{C}'_X \), and hence one obtains

\[
R(X, X) = \lim_{X' \in \mathcal{C}'_X} (X'_k)_{\text{red}} \quad \text{and} \quad R(Y, Y) = \lim_{Y' \in \mathcal{C}'_Y} (Y'_k)_{\text{red}}.
\]

**Remark 11.2.7.** If \( X = (\text{Spf } A)^{\text{rig}} \), then the reduction scheme \( R(X, X) \) coincides with the spectrum of the \( k \)-algebra \( \mathcal{G} \) (where \( \mathcal{G} \) is the associated classical affinoid algebra \( \mathcal{G} = A[\frac{1}{a}] \)) discussed in [18], §6.3. See A.4.10 in the appendix.

**Proposition 11.2.8.** Let \( \mathcal{X} \) be a coherent rigid space of finite type over \( S = (\text{Spf } V)^{\text{rig}} \), and \( X \) a distinguished formal model of \( \mathcal{X} \) over \( V \). Then there exists a distinguished formal model \( X' \) with the following properties.

(a) The formal model \( X' \) dominates \( X \) by a finite morphism \( X' \rightarrow X \).

(b) The canonical map \( D(X') \rightarrow I(X') \) induced by \( \text{sp}_{X'} \) is bijective.

(c) For any \( x \in D(X') \), the residue field \( k_x \) is isomorphic to the residue field \( k_{\text{sp}_{X'}(x)} \) at \( \text{sp}_{X'}(x) \in X' \).

**Proof.** Let us first try to find an admissible blow-up \( Z \rightarrow X \) that satisfies (b) and (c). Notice that, by the construction of the Zariski-Riemann space \( \langle X \rangle \), for two distinct points \( x, y \in \langle X \rangle \), there exists an admissible blow-up \( Z \rightarrow X \) such that \( \text{sp}_Z(x) \neq \text{sp}_Z(y) \). Since \( D(X) \) is a finite subset of \( \langle X \rangle \), one can take an admissible blow-up \( Z_0 \rightarrow X \) such that \( D(X) \rightarrow I(Z_0) \) is injective. Notice that

\[
\lim_{Z} k_{\text{sp}_Z(x)} = k_x
\]

holds, where \( Z \) runs through distinguished formal model dominating \( Z_0 \). Since, for any \( x \in D(X) \), \( k_x / k_{\text{sp}_{Z_0}(x)} \) is a finite extension, we can take an admissible blow-up \( Z \) dominating \( Z_0 \) such that \( k_{\text{sp}_Z(x)} = k_x \) for any \( x \in D(X) \). Thus we have the desired admissible blow-up \( Z \rightarrow X \) with the properties (b) and (c). Then, the formal model \( X' \) in the Stein factorization \( Z \rightarrow X' \rightarrow X \) (see I.11.3.2) satisfies all of the required properties, since \( Z \) and \( X' \) are isomorphic over an open subspace of \( X \) that contains \( I(X) = \text{sp}_X(D(X)) \); see 11.2.1.

**Remark 11.2.9.** The proposition and the proof of 11.2.1 show that, if \( X \) is affine, then the spectral seminorm formula (11.2.1 (3)) with \( D(X) \) replaced by a proper subset of it, does not hold.
**Theorem 11.2.10.** Let $\mathcal{X}$ be a coherent rigid space of finite type over $S = \text{Spf } V^{\text{rig}}$, and $X$ a distinguished formal model of $\mathcal{X}$ over $V$.

1. (finiteness) The reduction scheme $R(\mathcal{X}, X)$ is a scheme of finite type over $k$.

2. (stability) There exists a distinguished formal model $Z'$ finite over $X$ such that $(Z'_k)^{\text{red}} = R(\mathcal{X}, X)$. Moreover, for any distinguished formal model $Z$ finite over $Z'$, we have $(Z_k)^{\text{red}} = R(\mathcal{X}, X)$. (We call such a formal model $Z'$ a stabilized model.)

**Proof.** Let $X'$ be a distinguished formal model with the properties (a), (b), and (c) in 11.2.8. For any object $Z$ of $\mathcal{C}_X$ dominating $X'$, the finite map $(Z_k)^{\text{red}} \to (X'_k)^{\text{red}}$ is birational, and thus $(Z_k)^{\text{red}}$ is dominated by the normalization $W$ of $(X'_k)^{\text{red}}$. Hence $R(\mathcal{X}, X)$ is dominated by $W$. Since $W$ is finite over $(X'_k)^{\text{red}}$, $R(\mathcal{X}, X)$ is finite over $(X'_k)^{\text{red}}$, which shows (1). (2) is clear, since any increasing sequence of quasi-coherent algebras between $\mathcal{O}_{(X'_k)^{\text{red}}}$ and $\mathcal{O}_W$ is stationary.

**Theorem 11.2.11.** For any morphism $f: X \to Y$ of coherent formal schemes of finite type over $V$, the following conditions are equivalent.

(a) $f$ is proper (resp. separated, resp. affine, resp. finite).

(b) $R(f)$ is proper (resp. separated, resp. affine, resp. finite).

**Proof.** Take a formal model $f': X' \to Y'$ of $f^{\text{rig}}: X^{\text{rig}} \to Y^{\text{rig}}$, where $X'$ and $Y'$ are stabilized model of $X^{\text{rig}}$ and $Y^{\text{rig}}$, respectively, so that $R(f)$ is given by $(f'_k)^{\text{red}}: (X'_k)^{\text{red}} \to (Y'_k)^{\text{red}}$. Then the assertion follows from the fact that a morphism $g: S \to T$ of schemes is proper (resp. separated, resp. affine, resp. finite) if and only if so is $g^{\text{red}}: S^{\text{red}} \to T^{\text{red}}$. For separatedness, see [54], I, (5.5.1), and for properness, see [54], II, (5.4.6). As for affineness, one first uses the absolute Noetherian approximation ([98], C.9), [54], IV, (8.10.5), and Serre’s criterion [54], II, (5.2.1). Finally, the claim for finiteness follows from F.4.1 in the appendix.

**Remark 11.2.12.** One can define the so-called reduction map

$$\text{red}(\mathcal{X}, X): (\mathcal{X}) \longrightarrow R(\mathcal{X}, X)$$

by $\text{sp}_{Z}$, where $X$ is a stabilized model. It is straightforward to see that the map $\text{red}(\mathcal{X}, X)$ is continuous; it is, moreover, surjective, even restricted on the subset $[\mathcal{X}]$ consisting of maximal points; see 8.2.15.
Exercises

Exercise II.11.1. Let $V$ be an $a$-adically complete valuation ring of height one, $K = \text{Frac}(V) = V\left[ \frac{1}{a} \right]$ its fraction field, and $k = V/m_V$ its the residue field. Set $A = V\langle T_1, \ldots, T_n \rangle$ and $\mathfrak{A} = A\left[ \frac{1}{a} \right] = K\langle T_1, \ldots, T_n \rangle$ (the Tate algebra), and consider the corresponding coherent rigid space $\mathbb{D}^1_K = (\text{Spf} A)^\text{rig}$ (the unit disk). Let $| \cdot | : K \to \mathbb{R}_{\geq 0}$ be the non-Archimedean norm such that $|a| = c$ for a fixed real number $0 < c < 1$, and consider the spectral seminorm $\| \cdot \|_{\text{Sp}} = \| \cdot \|_{\text{Sp}, I, c}$, where $I = a\mathcal{O}_{\mathbb{D}^1_K}^{\text{int}}$, and the Gauss norm $\| \cdot \|_{\text{Gauss}} (0, \S 9.3. (a))$ on $\mathfrak{A} = \Gamma(\mathbb{D}^1_K, \mathcal{O}_{\mathbb{D}^1_K})$ (cf. 6.4.1).

1. Show that $\| \cdot \|_{\text{Sp}} = \| \cdot \|_{\text{Gauss}}$.

2. Let $x \in (\mathbb{D}^1_K)$ be such that the rigid point $\text{Spf} \mathcal{V}_x \to \text{Spf} A$ hits the closed fiber $\text{Spec} A \otimes_V k = \text{Spec} k[T_1, \ldots, T_n]$ at the generic point. Show that the seminorm $\| \cdot \|_x = \| \cdot \|_{x, I, c}$ coincides with the spectral seminorm $\| \cdot \|_{\text{Sp}}$.

Exercise II.11.2. Consider the situation as in Exercise II.11.1.

1. For $f \in \mathfrak{A}$, $\inf_{x \in \mathcal{X}} \| f \|_x$ is positive if and only if $f \in \mathfrak{A}^\times$.

2. Consider the map $\| f(\cdot) \| : \mathcal{X} \to \mathbb{R}_{\geq 0}$ given by $x \mapsto \| f \|_x$. Then $f$ is unit in $\mathfrak{A}$ if and only if the image of $\| f(\cdot) \|$ consists only of one non-zero value; otherwise, the image is the closed interval $[0, \| f \|_{\text{Gauss}}]$.

3. Suppose $|\overline{K}^\times| \neq \mathbb{R}_{>0}$, where $\overline{K}$ is the algebraic closure of $K$. Then for a non-zero non-unit $f \in \mathfrak{A}$ and $r \in (0, \| f \|_{\text{Gauss}}) \setminus |\overline{K}^\times|$, the open subspace $\mathcal{U}$ such that $\mathcal{U} = \{ x \in \mathcal{X} : \| f \|_x \neq r \}$ is a non-empty overconvergent open subspace that fails to enjoy the following property (the so-called $\{ \cdot \}^\text{cl}$-admissibility; cf. §B.1. (a) below): for any coherent rigid space $\mathcal{V}$ of finite type over $\mathcal{S} = (\text{Spf} V)^\text{rig}$ and any $\mathcal{S}$-morphism $\varphi : \mathcal{V} \to \mathcal{X}$ that maps $\varphi((\mathcal{V})^\text{cl}) \subseteq (\mathcal{U})^\text{cl}$, we have $\varphi(\mathcal{V}) \subseteq \mathcal{U}$.

Exercise II.11.3. Let $V$ be an $a$-adically complete valuation ring of height one, and $K = V\left[ \frac{1}{a} \right]$ the fraction field. Let $\mathcal{X}$ be a rigid space of finite type over $\mathcal{S} = (\text{Spf} V)^\text{rig}$. Consider an extension $V \subseteq V'$ of $a$-adically complete valuation rings, and let $\mathcal{X}'$ be the base change of $\mathcal{X}$ on $\mathcal{S}' = (\text{Spf} V')^\text{rig}$.

1. Show that if the extension $V \subseteq V'$ is finite, then an open subspace $\mathcal{U}$ of $\mathcal{X}$ is a tube open subset if and only if its inverse image $\mathcal{U}'$ by the morphism $\mathcal{X}' \to \mathcal{X}$ is a tube open subset of $\mathcal{X}'$. 
(2) Find an example of an overconvergent open subspace $U$ of $X$ that is not a tube open subset, but for which the inverse image $U'$ is a tube open subset of $X'$.

Exercise II.11.4. Use 11.2.4 (1) and 6.4.1 to deduce the following elementary version of the reduced fiber theorem (cf. [23]). Let $V$ be an $a$-adically complete valuation ring of height one, $X$ a coherent finite type flat formal scheme over $V$, and $\mathcal{X} = \mathcal{X}^{\text{rig}}$ the associated coherent rigid space over $K = \text{Frac}(V)$. Suppose that $\mathcal{X}$ is geometrically reduced over $K$, that is, for any finite extension $K'/K$ the base-change $\mathcal{X}_{K'}$ is reduced. Then there exist a finite separable extension $K'/K$ and an admissible blow-up $X' \to X \otimes_V V'$, where $V'$ denotes the integral closure of $V$ in $K'$, such that $X' \to \text{Spf } V'$ is flat and has the reduced geometric fiber.

A Appendix: Adic spaces

In this section we explain the relationship between our rigid spaces and adic spaces [59], [60], and [61]. One of our goals is to show that the formation of the Zariski–Riemann triple $((\mathcal{X}), \mathcal{O}^{\text{int}}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$ associated to a rigid space $\mathcal{X}$ gives a functor

$$\text{ZR}: \text{RigNoeRf} \longrightarrow \text{Adsp}$$

from the category of locally universally Noetherian rigid spaces (2.2.23) to the category of adic spaces.

A.1 Triples

As we saw in §3.2 (c), to any rigid space $\mathcal{X}$ is canonically associated the Zariski–Riemann triple $\text{ZR}(\mathcal{X}) = ((\mathcal{X}), \mathcal{O}^{\text{int}}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$ (3.2.11). Huber’s adic spaces are, on the other hand, defined as a similar kind of triples. Therefore, it will be convenient, for the comparison between rigid spaces and adic spaces to establish a general theory of triples.

Definition A.1.1. (1) A triple is a data $(X, \mathcal{O}^+_X, \mathcal{O}_X)$ consisting of a topological space $X$ and two sheaves $\mathcal{O}^+_X$ and $\mathcal{O}_X$ of topological rings on $X$ together with an injective morphism $\iota: \mathcal{O}^+_X \hookrightarrow \mathcal{O}_X$ of sheaves of topological rings such that

(a) the injection $\iota$ maps $\mathcal{O}^+_X$ isomorphically onto an open subsheaf of $\mathcal{O}_X$, and

(b) $X^+ = (X, \mathcal{O}^+_X)$ and $X = (X, \mathcal{O}_X)$ are topologically locally ringed spaces.

(2) A morphism of triples $(X, \mathcal{O}^+_X, \mathcal{O}_X) \to (Y, \mathcal{O}^+_Y, \mathcal{O}_Y)$ is a morphism of topologically locally ringed spaces

$$\varphi = (\varphi, h): (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$
(where $h: \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$) such that

(a) $h(\varphi^{-1}\mathcal{O}_Y^+) \subseteq \mathcal{O}_X^+$, and

(b) the induced morphism $\varphi^+ = (\varphi, h): (X, \mathcal{O}_X^+) \to (Y, \mathcal{O}_Y^+)$ is a morphism of topologically locally ringed spaces.

We denote by $\text{Tri}$ the category of triples. The following notion of triples will be essential in discussing adic spaces.

**Definition A.1.2.** (1) A valued triple is a triple $(X, \mathcal{O}_X^+, \mathcal{O}_X)$ together with a set $\{v_x\}_{x \in X}$ consisting of, for each $x \in X$, a continuous (additive) valuation $v_x$ of $\mathcal{O}_{X,x}$ such that $\mathcal{O}_{X,x}^+ = \{s \in \mathcal{O}_{X,x} : v_x(s) \geq 0\}$ and $m_{X,x}^+ (= \text{the maximal ideal of } \mathcal{O}_{X,x}^+) = \{s \in \mathcal{O}_{X,x} : v_x(s) > 0\}$. Here, a valuation $v$ on a topological ring $A$ is continuous if for any $\gamma \in \Gamma_v$, where $\Gamma_v$ is the value group, there exists an open neighborhood $U$ of $0$ in $A$ such that $v(x) > \gamma$ for every $x \in U$ (cf. [59], §3).

(2) A morphism $\zeta = (\varphi, h): ((X, \mathcal{O}_X^+, \mathcal{O}_X), \{v_x\}_{x \in X}) \to ((Y, \mathcal{O}_Y^+, \mathcal{O}_Y), \{v_y\}_{y \in Y})$ of valued triples is a morphism of triples such that

(*) for any $x \in X$, $v_{\varphi(x)}$ is equivalent\(^3\) to $v_x \circ h_x$.

We denote by $\text{VTri}$ the category of valued triples.

**Remark A.1.3.** Note that, if

$$((X, \mathcal{O}_X^+, \mathcal{O}_X), \{v_x\}_{x \in X}) \text{ and } ((Y, \mathcal{O}_Y^+, \mathcal{O}_Y), \{v_y\}_{y \in Y})$$

are valued triples, any morphism of topologically locally ringed spaces,

$$\varphi = (\varphi, h): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y),$$

with property (*) as above gives automatically a morphism of valued triples.

**Definition A.1.4.** A triple $(X, \mathcal{O}_X^+, \mathcal{O}_X)$ is said to be analytic if for any $x \in X$ there exist an open neighborhood $U \subseteq X$ of $x$ and an open ideal sheaf $I$ of $\mathcal{O}_X^+$|$_U$ such that

(a) for any $y \in U$, the ideal $I_y \subseteq \mathcal{O}_{X,x,y}^+$ is finitely generated, and the topology of $\mathcal{O}_{X,x,y}^+$ is $I_y$-adic, and

\(^3\)See, e.g., [59], §2, for the equivalence of valuations; in our case, the condition means that there exists an ordered isomorphism $\phi: \Gamma_{v_{\varphi(x)}} \cup \{\infty\} \to \Gamma_{v_x} \cup \{\infty\}$ (where $\Gamma_{v_x}$ etc. are taken to be the value groups (0, §6.2.(b)) of the valuations) such that $\phi \circ v_{\varphi(x)} = v_x \circ h_x$. 
(b) for any \( y \in U \), the ring \( \mathcal{O}_{X,y}^+ \) is \( I_y \)-valuative (0.8.7.1) and
\[
\mathcal{O}_{X,y}^+ = \lim_{\longrightarrow} \text{Hom}(I_y^n, \mathcal{O}_{X,y}^+).
\]

The ideal sheaf \( I \) as above will be called an ideal of definition over \( U \). We denote by \( \text{AnTri} \) the full subcategory of \( \text{Tri} \) consisting of analytic triples.

**Remark A.1.5.** If \( (X, \mathcal{O}_X^+, \mathcal{O}_X) \) is an analytic triple, then for any \( x \in X \) the topological ring \( \mathcal{O}_{X,x} \) is an extremal \( f \)-adic ring (0, §B.1.(c)) with \( \mathcal{O}_{X,x}^+ \) a ring of definition. In particular, by 0.B.1.6, for any morphism
\[
\varphi: (X, \mathcal{O}_X^+, \mathcal{O}_X) \to (Y, \mathcal{O}_Y^+, \mathcal{O}_Y)
\]
of triples between analytic triples, the map
\[
\mathcal{O}_{Y,\varphi(x)}^+ \to \mathcal{O}_{X,x}^+
\]
is adic for any \( x \in X \).

Let \( (X, \mathcal{O}_X^+, \mathcal{O}_X) \) be an analytic triple. Then, by 0.8.7.8, for any \( x \in X \) the ring \( B_x = \mathcal{O}_{X,x} \) has the canonical valuation
\[
v_x: \mathcal{O}_{X,x} \to \Gamma_x \cup \{\infty\}
\]
such that
\[
\mathcal{O}_{X,x}^+ = \{ f \in B_x : v_x(f) \geq 0 \}
\]
and
\[
m_{X,x}^+ = \{ f \in B_x : v_x(f) > 0 \}.
\]
Indeed, since \( A_x = \mathcal{O}_{X,x}^+ \) is \( a \)-valuative and \( B_x = A_x[\frac{1}{a}] \) (where \( I_x = (a) \)), the ring \( V_x = A_x/J_x \), where \( J_x = \bigcap_{n \geq 1} a^n A_x \), is an \( a \)-adically separated valuation ring of the fraction field \( K_x = B_x/J_x \), and so induces the valuation \( v_x \) as above. Moreover, it follows from A.1.5 and Exercise 0.8.8 that for any morphism \( (\varphi, h): (X, \mathcal{O}_X^+, \mathcal{O}_X) \to (Y, \mathcal{O}_Y^+, \mathcal{O}_Y) \) of analytic triples and for any \( x \in X \), the valuations \( v_{\varphi(x)} \) and \( v_x \circ h_x \) are equivalent. Thus we have a canonical functor \( \text{AnTri} \to \text{VTri} \), which is clearly fully faithful.

To sum up, we have the commutative diagram of categories

\[
\begin{array}{ccc}
\text{VTri} & \longrightarrow & \text{Tri} \\
\text{AnTri} \downarrow & & \downarrow \\
\end{array}
\]

where the arrows denoted by \( \iff \) are fully faithful.
Theorem A.1.6. (1) By $\mathcal{X} \mapsto ZR(\mathcal{X}) = (\langle \mathcal{X} \rangle, \mathcal{O}_X^{\text{int}}, \mathcal{O}_X)$, we have a functor, again defined by $ZR$, from the category $\text{Rf}$ of rigid spaces to the category $\text{AnTri}$ of analytic triples.

(2) The functor $ZR: \text{RigNoeRf} \to \text{AnTri}$, restricted to the category of locally universally Noetherian rigid spaces, is faithful.

Proof. (1) Clear.

(2) By the patching argument, it is enough to discuss morphisms between Stein affinoids of the form $\mathcal{X} = (\text{Spf} \ A)^\text{rig}$, where $A$ has an invertible ideal of definition $aA$. By 6.2.6 and 6.4.1, we have $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) = A[\frac{1}{a}]$ and $\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}^{\text{int}}) = A^{\text{int}}$, where $A^{\text{int}}$ is the integral closure of $A$ in $A[\frac{1}{a}]$. Suppose we are given two morphisms $f, g: B \to A$, respectively. If $ZR(f)$ and $ZR(g)$ give the same morphisms of triples, then we have $f[\frac{1}{a}] = g[\frac{1}{a}]$, as a homomorphism $B[\frac{1}{a}] \to A[\frac{1}{a}]$. Since $aA$ is an invertible ideal of $A$, $A \to A[\frac{1}{a}]$ is injective, and hence $f = g$. □

A.2 Rigid f-adic rings

A.2. (a) T.u. rigid-Noetherian f-adic rings. First recall that for a complete f-adic ring $A$ (0, §B.1. (d)) we have defined in 0, §B.1. (d) the restricted power series ring $A\langle X_1, \ldots, X_n \rangle$. If $A_0 \subseteq A$ is a ring of definition (0, §B.1. (b)), then $A_0$ is an adic ring of finite ideal type (I.1.1.6), and

$$A\langle X_1, \ldots, X_n \rangle = A_0\langle X_1, \ldots, X_n \rangle \otimes_{A_0} A.$$  

Recall also that for an ideal of definition $I_0 \subseteq A_0$ we have the equality

$$\text{Spec } A \langle X_1, \ldots, X_n \rangle \setminus V(I_0 A \langle X_1, \ldots, X_n \rangle) = \text{Spec } A_0 \langle X_1, \ldots, X_n \rangle \setminus V(I_0 A_0 \langle X_1, \ldots, X_n \rangle)$$

due to 0.B.1.1. Since the topology on $A$ coincides with the one induced by the filtration $\{I_0^k\}_{k \geq 0}$, the scheme $\text{Spec } A \langle X_1, \ldots, X_n \rangle \setminus V(I_0 A \langle X_1, \ldots, X_n \rangle)$ does not depend on the choice of the ideal of definition $I_0$.

Definition A.2.2. A complete f-adic ring $A$ is said to be t.u. rigid-Noetherian if $\text{Spec } A \langle X_1, \ldots, X_n \rangle \setminus V(I_0 A \langle X_1, \ldots, X_n \rangle)$ (defined as above) is a Noetherian scheme for any $n \geq 0$. 
Note that, if $A$ is a bounded complete f-adic ring (0, §B.1.(c)) or, equivalently, an adic ring of finite ideal type, then the above definition coincides with the one in I.2.1.1, since we have $A = A_0$ in the above notation. Note also that, if $A$ is a Tate ring, this notion coincides with that of ‘strongly Noetherian’ defined in [60], §2. The following proposition follows immediately from 0.B.1.1.

**Proposition A.2.2.** A complete f-adic ring $A$ is t.u. rigid-Noetherian if and only if any ring of definition $A_0 \subseteq A$ is t.u. rigid-Noetherian (in the sense as in (I.2.1.1)).

### A.2. (b) Finite type extensions.

In this paragraph we discuss finite type morphisms between complete f-adic rings in a special situation; for the general definition, see [60], §3.

Let $A$ be a t.u. rigid Noetherian f-adic ring, $A_0 \subseteq A$ a ring of definition, and $I_0 \subseteq A_0$ a finitely generated ideal of definition. Consider an ideal $J \subseteq A$, and set $J_0 = J \cap A_0$. As we saw in I.2.1.3, the ideal $J_0 \subseteq A_0$ is closed in $A_0$, and the quotient $B_0 = A_0/J_0$ is again t.u. rigid-Noetherian. The quotient $B = A/J$ is obviously an f-adic ring, and $B_0 \subseteq B$ is a ring of definition. Then, by 0.B.1.7, $B$ is again complete. Since $B_0$ is t.u. rigid-Noetherian, $B$ is a t.u. rigid Noetherian f-adic ring by A.2.2.

Let $\varphi : A \to B$ be a continuous homomorphism between t.u. rigid Noetherian extremal f-adic rings; note that, by 0.B.1.6, the map $\varphi$ is automatically adic. We say that $\varphi$ is of finite type if the ring $B$ is, as an $A$-algebra, isomorphic to an $A$-algebra of the form

$$A \llangle X_1, \ldots, X_n \rrangle / \mathfrak{a}$$

for an ideal $\mathfrak{a} \subseteq A \llangle X_1, \ldots, X_n \rrangle$ (cf. [60], 3.3 (iii)). By the above observation and 0.B.2.5, the topological ring $B$ is topologically isomorphic, hence isomorphic as an f-adic ring, to the above quotient of $A \llangle X_1, \ldots, X_n \rrangle$.

### A.2. (c) Rigidification of f-adic rings

**Definition A.2.3.** Let $A$ be a t.u. rigid-Noetherian f-adic ring, and $A_0, A_1 \subseteq A$ two rings of definition of $A$. We say that $A_0$ and $A_1$ are strictly (resp. finitely) equivalent if there exists a diagram $A_0 \leftrightarrow A_2 \leftrightarrow A_1$ consisting of strict (resp. finite) weak isomorphisms (6.2.3) between rings of definition of $A$.

It follows from 6.2.2 that, if $A_0$ and $A_1$ are t.u. adhesive (I.2.1.1), then strict equivalence is equivalent to finite equivalence. For a t.u. rigid-Noetherian f-adic ring $A$ and a ring of definition $B \subseteq A$, we denote by $C(B)$ the strict equivalence class that contains $B$. 

**Definition A.2.4.** (1) A strict equivalence class of rings of definition of a t.u. rigid-Noetherian $f$-adic ring $A$ is called a **rigidification** of $A$.

(2) A pair $(A, C)$ consisting of t.u. rigid-Noetherian $f$-adic ring $A$ and a rigidification of $A$ is called an **$f$-$r$-pair**.

**Example A.2.5.** Let $V$ be an $a$-adically complete valuation ring ($a \in \mathfrak{m}_V \setminus \{0\}$), and $K = \text{Frac}(V)$. Then $K$ with the topology defined by the filtration $\{a^nV\}_{n \geq 0}$ is a t.u. rigid-Noetherian $f$-adic ring having $V$ as a ring of definition. Let $p = \sqrt{(a)}$ be the associated height-one prime ($0.6.7.4$), and consider the height-one localization $\tilde{V} = V_p$. Then $\tilde{V}$ coincides with the set $K^o$ of all power-bounded elements and is also a ring of definition; if $\text{ht}(V) > 1$, then $V$ and $\tilde{V}$ are not strictly equivalent, since $(\text{Spf } V)^{\text{rig}} \not\cong (\text{Spf } \tilde{V})^{\text{rig}}$.

If $(A, C)$ is an $f$-$r$-pair, then the affinoid $X = (\text{Spf } A_0)^{\text{rig}}$ for $A_0 \in C$ does not depend, up to isomorphisms, on the choice of the ring of definition $A_0$. The affinoid $X$ thus obtained will be called the **affinoid associated to the $f$-$r$-pair** $(A, C)$.

**Definition A.2.6.** Let $(A, C)$ and $(A', C')$ be two $f$-$r$-pairs. A **continuous homomorphism** $\varphi: A \rightarrow A'$ is said to be **rigid** if there exist $A_0 \in C$ and $A'_0 \in C'$ such that $\varphi(A_0) \subseteq A'_0$ and the induced morphism $A_0 \rightarrow A'_0$ is adic.

Let $\varphi: A \rightarrow B$ be a finite type morphism between t.u. rigid-Noetherian extremal $f$-adic rings (§A.2. (b)). If $A$ is equipped with a rigidification (A.2.4) $C$, then $B$ has a canonically induced rigidification defined as follows. As we have seen in §A.2. (b), $B$ is topologically isomorphic to an $f$-adic ring of the form $A\langle X_1, \ldots, X_n \rangle/a$. Then for any $A_0 \subseteq C$ the ring $A_0\langle X_1, \ldots, X_n \rangle/a_0$, where $a_0 = a \cap A_0\langle X_1, \ldots, X_n \rangle$, is a ring of definition of $B$ and hence defines a rigidification of $B$. Note that the rigidification of $B$ thus obtained does not depend on the choice of $A_0 \in C$.

**A.3 Adic spaces**

**A.3. (a) Affinoid rings**

**Definition A.3.1.** An **affinoid ring** is a pair $\mathcal{G} = (\mathcal{G}^\pm, \mathcal{G}^+)$ consisting of an $f$-adic ring $\mathcal{G}^\pm$ and a subring $\mathcal{G}^+ \subset \mathcal{G}^\pm$ that is open, integrally closed in $\mathcal{G}^\pm$, and contained in $(\mathcal{G}^\pm)^o$ (the subset of $\mathcal{G}^\pm$ consisting of power-bounded elements (0, §B.1. (b))).$^4$ The ring $\mathcal{G}^+$ is called the **subring of integral elements** of $\mathcal{G}$.

Let $\mathcal{G} = (\mathcal{G}^\pm, \mathcal{G}^+)$ be an affinoid ring. By a **ring of definition** of $\mathcal{G}$ we mean a ring of definition of the $f$-adic ring $\mathcal{G}^\pm$ contained in $\mathcal{G}^+$. When $A_0$ is a ring of definition of $\mathcal{G}$, a finitely generated ideal of definition $I_0 \subseteq A_0$ is called an **ideal of definition** of $\mathcal{G}$.

$^4$In Huber’s original notation (such as in [59], §3), affinoid rings are written as $\mathcal{G} = (\mathcal{G}^\circ, \mathcal{G}^+)$. Here, in this book, we prefer to denote the ring in the first entry by $\mathcal{G}^\pm$ instead of $\mathcal{G}^\circ$. 
We say that an affinoid ring \( A = (A^\pm, A^+) \) is complete (resp. extremal) if the \( f \)-adic ring \( A^\pm \) is complete (resp. extremal \( f \)-adic). For an affinoid ring \( A = (A^\pm, A^+) \) its completion is given by the pair \( \widehat{A} = (\widehat{A}^\pm, \widehat{A}^+) \) consisting of the completions of the respective topological rings. It turns out that this is again an affinoid ring, hence a complete affinoid ring.

**Definition A.3.2.** Let \( A = (A^\pm, A^+) \), \( B = (B^\pm, B^+) \) be affinoid rings. A **homomorphism** \( \varphi: A \to B \) of affinoid rings is a continuous homomorphism \( \varphi: \widehat{A} \to \widehat{B} \) such that \( \varphi(A^+) \subset B^+ \). It is said to be **adic** if \( \varphi: A^\pm \to B^\pm \) is adic (0, §B.1. (b)).

A.3. (b) **Adic spectrum.** Let \( A = (A^\pm, A^+) \) be an affinoid ring. The associated **adic spectrum** \( \text{Spa } A \) is the topological space defined as follows.

- As a set, it consists of all (equivalence classes of) valuations\(^5\)
  \[ v: A^\pm \to \Gamma \cup \{\infty\} \]
  (cf. 0.6.2.4) that satisfy \( v(x) \geq 0 \) for \( x \in A^+ \) and are continuous (in the sense as in A.1.2 (1)).

- The topology is the one generated by the subsets of the form
  \[ \{v: v(x) \geq v(y) \neq \infty\} \]
  for any \( x, y \in A^\pm \).

It is well known that \( \text{Spa } A \) is a coherent sober topological space (cf. [59], Theorem 3.5 (i)).

**Definition A.3.3.** A subset \( U \) of \( \text{Spa } A \) is said to be **rational** if there exist \( f_0, \ldots, f_n \in A^\pm \) such that the ideal \( (f_0, \ldots, f_n) \) is open and
\[
U = \{v \in \text{Spa } A: v(f_i) \geq v(f_0) \neq \infty, i = 1, \ldots, n\}.
\]

The rational subset as above is often denoted by \( R(f_1, \ldots, f_n) \). Clearly, these subsets form a basis of the topology on \( \text{Spa } A \).

Let \( U = R(f_1, \ldots, f_n) \) be a rational subset of \( X = \text{Spa } A \). Define the topological ring \( A^\pm(f_1, \ldots, f_n) \) as follows.

- As a ring, \( A^\pm(f_1, \ldots, f_n) \) is \( \widehat{A}^\pm[f_1, \ldots, f_n] \).
- \( A^\pm(f_1, \ldots, f_n) \) has the ring of definition \( B[f_1, \ldots, f_n] \) with the ideal of definition \( IB[f_1, \ldots, f_n] \), where \( B \) is a ring of definition of \( A^\pm \) with the ideal of definition \( I \subseteq B \).

---

\(^5\)Although in [59] valuations are written multiplicatively, here we prefer to write them additively.
We denote the completion of $\mathfrak{A}^\pm\left(\frac{f_1,\ldots,f_n}{f_0}\right)$ by

$$\mathfrak{A}^\pm\left(\frac{f_1,\ldots,f_n}{f_0}\right).$$

It turns out that this ring is an $f$-adic ring determined only by $U$. Define the presheaf $\mathcal{O}_X$ by

$$\mathcal{O}_X(U) = \mathfrak{A}^\pm\left(\frac{f_1,\ldots,f_n}{f_0}\right)$$

for any rational subset $U \subseteq X = \text{Spf} \mathfrak{A}$ written as above, and $\mathcal{O}_X(V)$ for any open subset $V$ by the projective limit $\lim_{\leftarrow U \subseteq V} \mathcal{O}_X(U)$ taken over all rational subsets contained in $V$ endowed with the projective limit topology. We also define the presheaf $\mathcal{O}_X^+$ by

$$\mathcal{O}_X^+(U) = \{ f \in \mathcal{O}_X(U) : v(f) \geq 0 \text{ for any } v \in U \}$$

for any open subset $U$.

If $\mathfrak{A}$ is a complete affinoid ring, then

$$\Gamma(\text{Spa} \mathfrak{A}, \mathcal{O}_X) = \mathfrak{A}^\pm \quad \text{and} \quad \Gamma(\text{Spa} \mathfrak{A}, \mathcal{O}_X^+) = \mathfrak{A}^+$$

(cf. [60], Proposition 1.6 (iv)).

By definition, for any $x \in X = \text{Spa} \mathfrak{A}$, the stalk $\mathcal{O}_{X,x}$ is canonically equipped with the continuous valuation

$$v_x : \mathcal{O}_{X,x} \rightarrow \Gamma_x \cup \{\infty\},$$

and we have

$$\mathcal{O}_{X,x}^+ = \{ f \in \mathcal{O}_{X,x} : v_x(f) \geq 0 \}.$$

**Proposition A.3.4** ([60], Proposition 1.6 (i) and (ii)). (1) For any $x \in X = \text{Spa} \mathfrak{A}$, the stalk $\mathcal{O}_{X,x}$ is a local ring with the maximal ideal

$$m_{X,x} = \{ f \in \mathcal{O}_{X,x} : v_x(f) = \infty \}.$$

(2) For any $x \in X = \text{Spa} \mathfrak{A}$, the stalk $\mathcal{O}_{X,x}^+$ is a local ring with the maximal ideal

$$m_{X,x}^+ = \{ f \in \mathcal{O}_{X,x} : v_x(f) > 0 \}.$$

**A.3. (c) Adic spaces.** The presheaf $\mathcal{O}_X$ on $X = \text{Spa} \mathfrak{A}$ may not be a sheaf (cf. [60], §1), but if it is a sheaf, then so is $\mathcal{O}_X^+$. In this case, one obtains a valued triple (A.1.2)

$$((X = \text{Spa} \mathfrak{A}, \mathcal{O}_X^+, \mathcal{O}_X), \{v_x\}_{x \in X}). \quad (\ast)$$

**Definition A.3.5.** An affinoid adic space is a valued triple that is isomorphic in $V\text{Tri}$ to a triple of the form $(\ast)$ as above, where $\mathcal{O}_X$ is assumed to be a sheaf.
Note that if \( X = \text{Spa} \mathcal{O} \) is an affinoid adic space, then any rational subdomain \( U \subseteq X \), endowed with the structures induced from \( X \), is also an affinoid adic space.

**Remark A.3.6.** In [60], §2, and [61], §1.1, Huber defined affinoid adic spaces as objects of a category \( \mathcal{V} \), which is defined as follows. Objects of \( \mathcal{V} \) are triples of the form \((X, \mathcal{O}_X, \{v_x\}_{x \in X})\) consisting of a topological space \( X \), a sheaf of complete topological rings \( \mathcal{O}_X \), and a collection of valuations \( \{v_x\}_{x \in X} \) of the stalks \( \mathcal{O}_{X,x} \). A morphism \( \varphi = (\varphi, h): (X, \mathcal{O}_X, \{v_x\}_{x \in X}) \to (Y, \mathcal{O}_Y, \{v_y\}_{y \in Y}) \) in \( \mathcal{V} \) is a morphism of topologically ringed spaces satisfying the compatibility condition \((*)\) as in A.1.2 (2). However, by what we have remarked in A.1.3, our definition is equivalent to Huber’s. The same remark also applies to the definitions of morphisms of affinoid adic spaces, of adic spaces, and of morphisms of adic spaces, which will be given soon below.

Morphisms of affinoid adic spaces are defined to be morphisms in \( \mathcal{V}_{\text{Tri}} \), in view of the following proposition.

**Proposition A.3.7** ([60] Proposition 2.1 (i)). Let \( X = \text{Spa} \mathcal{O} \) and \( Y = \text{Spa} \mathcal{B} \) be affinoid adic spaces, where \( \mathcal{O} \) is complete.

1. Every homomorphism \( \mathcal{B} \to \mathcal{O} \) canonically induces a morphism \( X \to Y \) of valued triples.

2. The mapping 
\[
\{ \mathcal{B} \to \mathcal{O} \colon \text{homomorphism of affinoid rings} \} \to \text{Hom}_{\mathcal{V}_{\text{Tri}}}(X, Y)
\]

thus obtained is bijective.

**Definition A.3.8.** (1) An adic space is a valued triple \((X, \mathcal{O}_X^+, \mathcal{O}_X), \{v_x\}_{x \in X}\) that is locally an affinoid adic space.

(2) A morphism \((X, \mathcal{O}_X^+, \mathcal{O}_X), \{v_x\}_{x \in X}\) \to \((Y, \mathcal{O}_Y^+, \mathcal{O}_Y), \{v_y\}_{y \in Y}\) of adic spaces is a morphism of valued triples.

We denote by \( \text{Adsp} \) the category of adic spaces.

**Proposition A.3.9.** (1) The underlying topological space of an adic space is a valuative space (0.2.3.1).

(2) The underlying continuous map of a morphism of adic spaces is valuative (0.2.3.21) and locally quasi-compact (0.2.2.24).
Proof. (1) The underlying topological space of an adic space \( X \) is clearly a locally coherent sober space. It follows from [61], Lemma 1.1.10 (i), that, for any \( x \in X \), the set \( G_x \) of all generizations of \( x \) is totally ordered. Then the assertion follows from 0.2.3.2 (1).

(2) It follows from [61], Lemma 1.1.10 (iv), that the underlying continuous map of a morphism of adic spaces is valuative. It is clearly locally quasi-compact, for morphisms between affinoid adic spaces are quasi-compact. \( \square \)

A.3. (d) Analytic adic spaces. In the sequel, we will call that an affinoid ring \( \mathcal{A} = (\mathcal{A}^+, \mathcal{A}^+) \) extremal (resp. Tate) if the \( f \)-adic ring \( \mathcal{A}^\pm \) is extremal (resp. Tate) (0, §B.1.(c))

Definition A.3.10. Let \( X = ((X, \mathcal{O}_X^+, \mathcal{O}_X), \{v_x\}_{x \in X}) \) be an adic space. A point \( x \in X \) is said to be analytic if there exists an affinoid open neighborhood \( U = \text{Spa} \mathcal{A} \) of \( x \) given by an extremal affinoid ring \( \mathcal{A} \). If all points of \( X \) are analytic, we say that the adic space \( X \) is analytic.

Note that a point \( x \in X \) is analytic if and only if there exists an affinoid open neighborhood \( U = \text{Spa} \mathcal{A} \) of \( x \) given by a Tate affinoid ring \( \mathcal{A} \). Indeed, \( U = \text{Spa} \mathcal{A} \) is an affinoid open neighborhood of \( x \) by given an extremal affinoid ring \( \mathcal{A} \), and \( I = (f_0, \ldots, f_n) \) is an ideal of definition of a ring of definition of \( \mathcal{A}^\pm \), and then the rational subsets \( U\left(\frac{f_0, \ldots, f_n}{f_i} \right) (i = 0, \ldots, n) \) cover \( U \), and the corresponding \( f \)-adic rings \( \mathcal{A}^\pm \left(\frac{f_0, \ldots, f_n}{f_i} \right) \) are Tate rings.

Note also that any morphism

\[
((X, \mathcal{O}_X^+, \mathcal{O}_X), \{v_x\}_{x \in X}) \longrightarrow ((Y, \mathcal{O}_Y^+, \mathcal{O}_Y), \{v_y\}_{y \in Y})
\]

of analytic adic spaces is adic, that is, for any affinoid open \( U \subseteq X \) and affinoid open \( V \subseteq Y \) with \( U \subseteq V \), \( \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(U) \) is adic (due to 0.B.1.6; cf. [60], Proposition 3.2 (i)).

We denote by \( \text{AnAdsp} \) the full subcategory of \( \text{Adsp} \) consisting of analytic adic spaces.

Proposition A.3.11. Let \( \mathcal{A} = (\mathcal{A}^\pm, \mathcal{A}^+) \) be an extremal affinoid ring, and \( I \subseteq \mathcal{A}^\pm \) be an ideal of definition of a ring of definition of \( \mathcal{A}^\pm \). Consider the associated affinoid adic space \( ((X = \text{Spa} \mathcal{A}, \mathcal{O}_X^+, \mathcal{O}_X), \{v_x\}_{x \in X}) \), and \( x \in X \).

1. The ring \( \mathcal{O}_{X,x}^+ \) is \( I\mathcal{O}_{X,x}^+ \)-valuative (0.8.7.1). Moreover,

\[
\mathcal{O}_{X,x} = \lim_{\longrightarrow \atop n \geq 1} \text{Hom}(I^n \mathcal{O}_{X,x}^+, \mathcal{O}_{X,x}^+).
\]

2. The valuation on \( \mathcal{O}_{X,x} \) induced by the valuation ring \( V_x = \mathcal{O}_{X,x}^+ / J_x \) (where \( J_x = \bigcap_{n \geq 1} I^n \mathcal{O}_{X,x}^+ \)) is equivalent to \( v_x \).
Proof. Set $A_x = \mathcal{O}_{X,x}^+$ and $B_x = \mathcal{O}_{X,x}$. Then $IA_x = (a)$ for a non-zero-divisor $a \in A_x$. By 0.B.1.1, $B_x = \bigcup_{n \geq 0}[A_x : I^nA_x]$. By this and the fact that $IB_x = B_x$, we have $B_x = \lim_{\to n \geq 1} \text{Hom}(I^nA_x, A_x)$. Since the valuation $v_x$ is continuous, $J_x = \bigcap_{n \geq 1} I^nA_x = \{ f \in B_x : v_x(f) = \infty \}$, which is a prime ideal both in $A_x$ and in $B_x$. By [60], Proposition 1.6 (i), $J_x$ is a maximal ideal of $B_x$. Hence $K_x = B_x/J_x$ is a field, which contains $V_x = \mathcal{O}_{X,x}/J_x$. Since $V_x = \{ s \in K_x : v_x(s) \geq 0 \}$, $V_x$ is a valuation ring, which is clearly $\alpha$-adically separated; in particular, $K_x = V_x[\frac{1}{a}] = \text{Frac}(V_x)$. Then by 0.8.7.8 (2), $A_x$ is $IA_x$-valuative, which yields (1).

(2) is clear. □

It follows from the proposition that analytic adic spaces are analytic triples, see A.1.4. By A.3.11 and Exercise 0.8.8 (cf. A.1.5), we have the following corollary.

Corollary A.3.12. The forgetful functor

$$(((X, \mathcal{O}_X^+, \mathcal{O}_X), \{v_x\}_{x \in X}) \longmapsto (X, \mathcal{O}_X^+, \mathcal{O}_X))$$

from the category $\text{AnAdsp}$ of analytic adic spaces to the category $\text{AnTri}$ of triples is fully faithful.

For the reader’s convenience, we insert here the definition of locally of finite type morphisms between analytic adic spaces; see [60], §3, for the general definition. For a complete affinoid ring $\mathfrak{A} = (\mathfrak{A}^-, \mathfrak{A}^+)$, we have the affinoid ring

$$\mathfrak{A}\langle X_1, \ldots, X_n \rangle = (\mathfrak{A}^\langle X_1, \ldots, X_n \rangle, \mathfrak{A}^+\langle X_1, \ldots, X_n \rangle),$$

defined as follows: $\mathfrak{A}^\langle X_1, \ldots, X_n \rangle$ was already defined in 0.B.1.(d), and $\mathfrak{A}^+\langle X_1, \ldots, X_n \rangle$ is the subring consisting of power series with coefficients in $\mathfrak{A}^+$. A homomorphism $\mathfrak{A} \rightarrow \mathfrak{B}$ of affinoid rings is called a quotient mapping if $\mathfrak{A}^\pm \rightarrow \mathfrak{B}^\pm$ is surjective, continuous, and open, and the integral closure in $\mathfrak{B}^\pm$ of the image of $\mathfrak{A}^+$ coincides with $\mathfrak{B}^+$. A homomorphism $\mathfrak{A} \rightarrow \mathfrak{B}$ of complete extremal affinoid rings is said to be (topologically) of finite type if it factors through a quotient mapping $\mathfrak{A}\langle X_1, \ldots, X_n \rangle \rightarrow \mathfrak{B}$ (cf. [60], 3.5 (iii); see also §A.2.(b)); note that, in this case, $\mathfrak{B}^\pm$ is generated by finitely many elements over $\mathfrak{A}^\pm$ (cf. 0, §B.1.(d)).

Definition A.3.13. A morphism $f : X \rightarrow Y$ between analytic adic spaces is said to be locally of finite type if for any $x \in X$ there exist an open affinoid neighborhood $U \cong \text{Spa} \mathfrak{A}$ of $x$ and an affinoid open subset $V \cong \text{Spa} \mathfrak{B}$ of $Y$ such that $f(U) \subseteq V$ and the induced homomorphism $\mathfrak{B} \rightarrow \mathfrak{A}$ is topologically of finite type. If, moreover, (the underlying continuous mapping of) $f$ is quasi-compact (0.2.1.4 (2)), we say that $f$ is of finite type.
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A.4 Rigid geometry and affinoid rings

A.4. (a) Affinoid rings associated to f-r-pairs

Proposition A.4.1. Let \((A, C)\) be an f-r-pair (A.2.4), and set 

\[ \mathcal{A}^\pm = A, \quad \mathcal{A}^+ = \lim_{B \in C} B. \]

Then \(\mathcal{A} = (\mathcal{A}^\pm, \mathcal{A}^+)\) is a complete affinoid ring.

Proof. It is clear that \(\mathcal{A}^+\) is open in \(\mathcal{A}^\pm\). To show that \(\mathcal{A}^+\) is integrally closed in \(\mathcal{A}^\pm\), suppose \(x \in \mathcal{A}^+\) is integral over \(\mathcal{A}^\pm\). Then \(x\) is integral over some \(B \in C\). The homomorphism \(B \to B[x]\) is finite, and hence \(B[x]\) is open and bounded in \(\mathcal{A}^\pm\). For a finitely generated ideal of definition \(I \subset B\), we have by 0.B.1.1 the equality \(\text{Spec } B \setminus V(I) = \text{Spec } B[x] \setminus V(IB[x])\), and hence \(B \to B[x]\) is isomorphic outside \(I\) (6.2.1). Hence, by 6.2.4, we deduce \(B[x] \in C\) and thus \(x \in \mathcal{A}^+\). To verify that \(\mathcal{A}^+\) is contained in \(\mathcal{A}^\pm\), we only have to remark that any element of \(\mathcal{A}^+\) is contained in a ring of definition, and hence is power-bounded. \(\square\)

Definition A.4.2. (1) An a-r-pair is a pair \(\mathcal{R} = (\mathcal{A}_R, \mathcal{C}_R)\) consisting of a complete affinoid ring \(\mathcal{A}_R = (\mathcal{A}_R^\pm, \mathcal{A}_R^+)\) with \(\mathcal{A}_R^\pm\) t.u. rigid-Noetherian, and a rigidification \(\mathcal{C}_R\) of \(\mathcal{A}_R^\pm\) that satisfies the compatibility condition \(\mathcal{A}_R^+ = \lim_{\to} B \in \mathcal{C}_R\).

(2) For two a-r-pairs \(\mathcal{R} = (\mathcal{A}_R, \mathcal{C}_R)\) and \(\mathcal{R}' = (\mathcal{A}_{R'}, \mathcal{C}_{R'})\), a homomorphism of a-r-pairs is a homomorphism \(\varphi: \mathcal{A}_R \to \mathcal{A}_{R'}\) enjoying the following property: there exist \(B \in \mathcal{C}_R\) and \(B' \in \mathcal{C}_{R'}\) such that \(\varphi(B) \subset B'\) and the induced homomorphism \(B \to B'\) is adic (cf. A.2.6).

Note that any homomorphism of a-r-pairs is adic. The proposition A.4.1 shows that any f-r-pair \((A, C)\) canonically induces an a-r-pair \(\mathcal{R} = (\mathcal{A}_R, \mathcal{C}_R)\) with \(\mathcal{A}_R^\pm = A\) and \(\mathcal{C}_R = C\). We call this a-r-pair the a-r-pair associated to the f-r-pair \((A, C)\).

A.4. (b) Stein affinoids and analytic affinoid pairs. Let \(X\) be a Stein affinoid (6.5.2) \(X = (\text{Spf } A)^\text{rig}\) with \(\text{Spec } A \setminus V(I)\) affine, where \(I \subset A\) a finitely generated ideal of definition. We may assume that \(A\) is \(I\)-torsion free (6.1.5). As we saw in 0, §B.1. (b), the ring \(B\) with \(\text{Spec } B = \text{Spec } A \setminus V(I)\) is a complete extremal f-adic ring (0, §B.1. (c)) having \(A\) (resp. \(I\)) as a ring (resp. an ideal) of definition. By 6.4.1,

\[ \Gamma(X, \mathcal{O}_X) = B. \]

Moreover, by 6.2.2,

\[ \Gamma(X, \mathcal{O}_X^{\text{int}}) = \lim_{X' \to \text{Spf } A} \Gamma(X', \mathcal{O}_{X'}) = \lim_{A' \in C(A)} A'. \]
where \( X' \to \text{Spf} \ A \) runs through all admissible blow-ups of \( \text{Spf} \ A \) (see §A.2.3 for the definition of \( C(A) \)). By 6.2.6, the following results hold true.

**Proposition A.4.3.** In the situation as above, \( \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\text{int}}) \) coincides with the integral closure of \( A \) in \( \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \).

**Theorem A.4.4.** Let \( \mathcal{X} \) be a Stein affinoid.

1. The pair \( \mathcal{A}_{\mathcal{X}} = (\mathcal{A}_{\mathcal{X}}^0, \mathcal{A}_{\mathcal{X}}^{\pm}) \) defined by
   \[ \mathcal{A}_{\mathcal{X}}^0 = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}), \quad \mathcal{A}_{\mathcal{X}}^{\pm} = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\text{int}}), \]
   is a complete extremal affinoid ring such that \( \mathcal{A}_{\mathcal{X}}^{\pm} \) is t.u. rigid-Noetherian.

2. Any distinguished affine formal model \( \mathcal{X} = \text{Spf} \ A \) of \( \mathcal{X} \) induces a strict equivalence class \( C(A) \) of \( \mathcal{A}_{\mathcal{X}}^0 \) defined by \( A \), which depends only on \( \mathcal{X} \). Moreover, if we denote by \( \mathcal{C}_{\mathcal{X}} = C(A) \) the rigidification thus obtained, then
   \[ \mathcal{A}_{\mathcal{R}}(\mathcal{X}) = (\mathcal{A}_{\mathcal{X}}, \mathcal{C}_{\mathcal{X}}) \]
   is an a-r-pair.

**Definition A.4.5.** (1) An affinoid ring isomorphic to \( \mathcal{A}_{\mathcal{X}} \) as in A.4.4 (1) is called an analytic affinoid ring associated to \( \mathcal{X} \).

(2) An a-r-pair isomorphic to \( \mathcal{A}_{\mathcal{R}}(\mathcal{X}) \) as defined in A.4.4 (2) is called an analytic affinoid pair associated to \( \mathcal{X} \).

It is clear that for any morphism \( \varphi: \mathcal{X} \to \mathcal{Y} \) of Stein affinoids there is a canonical induced homomorphism \( \mathcal{A}_{\mathcal{R}}(\varphi): \mathcal{A}_{\mathcal{R}}(\mathcal{Y}) \to \mathcal{A}_{\mathcal{R}}(\mathcal{X}) \) of a-r-pairs. Thus, \( \mathcal{A}_{\mathcal{R}} \) gives rise to a functor from the opposite category of the category of Stein affinoids to the category of a-r-pairs.

**Theorem A.4.6.** The functor \( \mathcal{A}_{\mathcal{R}} \) thus obtained is fully faithful with the essential image being the totality of analytic affinoid pairs.

**Proof.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Stein affinoids, and consider the map
   \[ \text{Hom}_{\mathcal{CR}}(\mathcal{X}, \mathcal{Y}) \to \text{Hom}(\mathcal{A}_{\mathcal{R}}(\mathcal{Y}), \mathcal{A}_{\mathcal{R}}(\mathcal{X})), \]
   where the last set is the set of homomorphisms of a-r-pairs. It is not difficult to see that this map is injective (left to the reader). To show the surjectivity, take a homomorphism \( \psi: \mathcal{A}_{\mathcal{R}}(\mathcal{Y}) \to \mathcal{A}_{\mathcal{R}}(\mathcal{X}) \) of a-r-pairs. Let \( \text{Spf} \ A \) and \( \text{Spf} \ B \) be distinguished affine formal models of \( \mathcal{X} \) and \( \mathcal{Y} \), respectively. On the other hand, let \( A_0 \) and \( B_0 \) be rings of definition in \( \mathcal{C}_{\mathcal{X}} \) and \( \mathcal{C}_{\mathcal{Y}} \), respectively, such that \( \psi(B_0) \subseteq A_0 \) and the map \( B_0 \to A_0 \) is adic. One can replace \( B_0 \) and \( A_0 \) by strict weak equivalence and thus assume that \( B = B_0 \) and \( A = A_0 \) (cf. 6.2.5 and A.2.2). Hence we get the map \( \varphi: \mathcal{X} \to \mathcal{Y} \) induced by the adic map \( B \to A \). It is straightforward to check that \( \mathcal{A}_{\mathcal{R}}(\varphi) = \psi \). \( \square \)
A.4. (c) Visualization and adic spectrum

Theorem A.4.7. Let \( \mathcal{X} \) be a Stein affinoid, and consider the analytic affinoid pair \( \mathcal{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X}) \) associated to \( \mathcal{X} \). Then the adic spectrum \( X = \operatorname{Spa} \mathcal{X} \) is canonically homeomorphic to \( \langle \mathcal{X} \rangle \). Moreover, under this identification, the sheaf \( \mathcal{O}_X^{\text{int}} \) (resp. \( \mathcal{O}_\mathcal{X} \)) coincides with the presheaf \( \mathcal{O}_X^+ \) (resp. \( \mathcal{O}_\mathcal{X} \)); see §A.3. (b).

Proof. Set \( B = \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) \). To construct the map \( \langle \mathcal{X} \rangle \to X = \operatorname{Spa} \mathcal{X} \), take \( x \in \langle \mathcal{X} \rangle \). Then one has the associated rigid point \( \alpha_x : \operatorname{Spf} \hat{V}_x \to \langle \mathcal{X} \rangle \). Let \( \operatorname{Spf} A \) be a distinguished formal affine model of \( \mathcal{X} \), and \( I \) a finitely generated ideal of definition of \( A \). Then the rigid point induces an adic map \( A \to \hat{V}_x \). Since \( \hat{V}_x \) is \( a \)-adically separated, where \( I \hat{V}_x = (a) \), and since \( B = \lim_{\longleftarrow n \geq 1} \operatorname{Hom}(I^n, A) \), this adic map gives rise to a continuous valuation \( v_x : B \to \Gamma_x \cup \{ \infty \} \), where \( \Gamma_x \) is the value group of the valuation ring \( \hat{V}_x \), whence the desired map \( x \mapsto v_x \). It is easy to see that this map is continuous.

To construct the inverse map \( X = \operatorname{Spa} \mathcal{X} \to \langle \mathcal{X} \rangle \), take a continuous valuation \( v : B \to \Gamma \cup \{ \infty \} \), and consider the composition \( v : A \to \Gamma \cup \{ \infty \} \). The kernel \( v^{-1}(0) \) is a prime ideal, and thus the map factors through a field; taking the inverse image of the subset consisting of positive values, one gets a valuation ring \( V \) that factors through \( v \). Set \( IV = (a) \); we have \( a \neq 0 \) and \( a \notin V^\times \). Then \( V \) is \( a \)-adically separated due to the continuity of \( v \). By 0.9.1.1, the \( a \)-adic completion \( \hat{V} \) is an \( a \)-adically complete valuation ring having the same value group as \( V \), and thus we get the rigid point \( \alpha : \operatorname{Spf} \hat{V} \to \operatorname{Spf} A \), whence \( \alpha : \operatorname{Spf} \hat{V} \to \langle \mathcal{X} \rangle \). It is straightforward to see that the maps thus obtained are continuous and inverses to one another, and thus the first assertion of A.4.7 is shown.

Next we compare the structural presheaves. To this end, since the issue is local, we may replace \( A \) by an affine part of the admissible blow-up along the ideal of definition \( I \), and thus we may assume that \( I = (a) \) is invertible. Take \( f_0, \ldots, f_n \in B \) such that the ideal \( (f_0, \ldots, f_n) \) in \( B \) is the unit ideal (hence is open). In this situation we consider the rational subdomains

\[
U_k = \mathcal{X}(f_0 f_k^{-1}, \ldots, f_{k+1} f_k^{-1}, \ldots, f_n f_k^{-1})
\]

for \( k = 0, \ldots, n \) (6.1.7 (3)), which give a covering \( \langle \mathcal{X} \rangle = \bigcup_{k=0}^n \langle U_k \rangle \) by quasi-compact open subsets. Note that the rational subdomains \( \langle U_k \rangle \) correspond to the rational subsets \( U_k = \{ v \in \operatorname{Spa} \mathcal{X} : v(f_i) \geq v(f_k) \neq \infty, i \neq k \} \) of \( X = \operatorname{Spa} \mathcal{X} \) under the map constructed above. Since this covering comes from the admissible blow-up along the admissible ideal \( (a^l f_0, \ldots, a^l f_n) \) (with \( l \geq 0 \) sufficiently large), the open coverings of this form are cofinal, and hence it is enough to show that the presheaves take the same values on the subsets \( \langle U_k \rangle \).
For \( k = 0, \ldots, n \) we have
\[
\Gamma(U_k, \mathcal{O}_X) = B\left(\frac{f_0 \ldots f_{k-1}, f_{k+1} \ldots f_n}{f_k}\right)
\]
by definition (§A.3.(b)). On the other hand,
\[
\Gamma(\mathcal{U}_k, \mathcal{O}_X) = A\left(\frac{a^f f_0}{a^f f_k}, \ldots, \frac{a^f f_{k-1}}{a^f f_k}, \frac{a^f f_{k+1}}{a^f f_k}, \ldots, \frac{a^f f_n}{a^f f_k}\right) \otimes_A B
\]
by 6.5.7 (1), which is nothing but the right-hand side of (\`). Hence we have \( \mathcal{O}_X \). Since \( \mathcal{O}_X \) consists of elements in \( \mathcal{O}_X \) that have non-negative values at any \( v \in U_k \), it coincides with \( \mathcal{O}_X \), because in view of 0.8.7.8 the inequality \( v_x(f_x) \geq 0 \) for \( f \in \Gamma(\mathcal{U}_k, \mathcal{O}_X) \) holds if and only if \( f_x \in \mathcal{O}_X^\text{int}_{X,x} \).

**Corollary A.4.8.** Let \( \mathcal{O}_X \) be the analytic affinoid ring associated to a Stein affinoid \( X \), and consider \( X = \text{Spa} \mathcal{O}_X \). Then the presheaves \( \mathcal{O}_X \) and \( \mathcal{O}_X^\text{int} \) are sheaves, thereby defining the affinoid adic space (A.3.5) \( (X, \mathcal{O}_X^+, \mathcal{O}_X, \{v_x\}_{x \in X}) \), which is, as a valued triple, canonically isomorphic to the Zariski–Riemann triple \( \text{ZR}(X) \) associated to \( X \).

Notice that the Zariski–Riemann triple \( \text{ZR}(X) \) is an analytic triple (A.1.6), and hence is naturally regarded as a valued triple.

**Proof.** The first assertion follows immediately from A.4.7. One can check the second assertion by a calculation as in the proof of A.4.7. \( \square \)

**A.4.(d) Description of power-bounded elements.** Let \( X = (\text{Spf} A)^{\text{rig}} \) be a Stein affinoid, where \( A \) is an \( I \)-torsion free (where \( I \subseteq A \) is a finitely generated ideal of definition) t.u. rigid-Noetherian ring such that \( \text{Spec} A \setminus V(I) \) is affine. In this paragraph, we are interested in the subring \( \mathcal{O}_X^\text{int}_{X,x} \) of power-bounded elements (0. §B.1.(b)) in the \( f \)-adic ring \( \Gamma(X, \mathcal{O}_X) \). We first remark that the following inclusion holds:
\[
\Gamma(X, \mathcal{O}_X)^0 \subseteq \{ f \in \Gamma(X, \mathcal{O}_X) : f_x \in \mathcal{O}_X^\text{int}_{X,x} \text{ for any } x \in [X] \}.
\]
Indeed, if \( x \in [X] \), then the valuation at \( x \) (cf. 3.2.13) is of height one, and may take values in \( \mathbb{R} \). If \( f \in \Gamma(X, \mathcal{O}_X) \) is power-bounded, then the value of \( f_x \) is non-negative, which implies that \( f_x \in \mathcal{O}_X^\text{int}_{X,x} \).

**Proposition A.4.9.** Suppose that there exists a real valued spectral functor (8.1.9) defined on the category of quasi-compact open subspaces of \( X \). Then we have the equality
\[
\Gamma(X, \mathcal{O}_X)^0 = \Gamma(X, \mathcal{O}_X^\text{int}).
\]
Moreover, \( \Gamma(X, \mathcal{O}_X)^0 \) coincides with the integral closure of \( A \) in \( \Gamma(X, \mathcal{O}_X) \).
Proof. The first assertion follows from the above remark and 8.1.11. The second one follows from A.4.3. □

By 8.2.19, we have the following corollary.

**Corollary A.4.10.** Let \( X \) be a Stein affinoid of type \((\mathbb{V}_R)\) or of type \((N)\). Then

\[
\Gamma(X, \mathcal{O}_X)^0 = \Gamma(X, \mathcal{O}_X^{\text{int}}).
\]

Moreover, if \( X = \text{Spec} \, A \) is a distinguished affine formal model of \( X \), where \( A \) is a topologically of finite type algebra over an \( \alpha \)-adic complete height one valuation ring in type \((\mathbb{V}_R)\) case, or, in type \((N)\) case, an \( I \)-adically complete Noetherian ring (where \( I \subseteq A \) is a finitely generated ideal of definition) such that \( X \setminus V(I) \) is affine, then \( \Gamma(X, \mathcal{O}_X)^0 \) is the integral closure of \( A = \Gamma(X, \mathcal{O}_X) \) in \( \Gamma(X, \mathcal{O}_X) \).

**A.4. (e) Rigidification and finite type extensions**

**Theorem A.4.11.** Let \( A \) be a t.u. rigid-Noetherian extremal \( f \)-adic ring, and \( C \) a rigidification (A.2.4) of \( A \). Let \( \mathfrak{A} = (\mathfrak{A}^-, \mathfrak{A}^+) \) be the associated affinoid ring (A.4.1). Then the following categories are canonically equivalent.

(a) The category of \( A \)-algebras of finite type (cf. §A.2.(b)) with rigid (A.2.6) \( A \)-algebra homomorphisms (note that, as we have seen in §A.2.(c), any finite type extension of \( A \) has the canonical rigidification induced from \( C \)).

(b) The category of finite type affinoid rings (§A.3.(d)) over \( \mathfrak{A} \).

(c) The category of finite type affinoid rings over \( \mathfrak{A} \) with discrete \( A \)-algebra homomorphisms that preserve the positive parts.

We will use the following lemma for the proof of the theorem.

**Lemma A.4.12.** Let \( A \) be as in A.4.11, and \( B \) an \( A \)-algebra of finite type. Let \( A_0 \subseteq A \) and \( B_0 \subseteq B \) be respective rings of definition, and \( I_0 \subseteq A_0 \) a finitely generated ideal of definition. Consider elements \( f_1, \ldots, f_n \in B_0^{\text{int}} \) in the integral closure of \( B_0 \) in \( B \). Then the subring \( B_1 \) of \( B \) weakly generated over \( B_0 \) by \( f_1, \ldots, f_n \) (0, §B.1.(d)) is a ring of definition of \( B \). Moreover, \( B_1 \) is generated over \( B_0 \) by \( f_1, \ldots, f_n \), and defines the same rigidification as \( B_0 \).

Proof. Since \( f_1, \ldots, f_n \in B_0^{\text{int}} \), the ring \( B_0[f_1, \ldots, f_n] \) is finite over \( B_0 \) and hence is \( I_0 \)-adically complete. Consequently, \( B_1 = B_0[f_1, \ldots, f_n] \), which is a ring of definition. It is clear that \( B_1 \) is generated by \( f_1, \ldots, f_n \) and \( B_0 \). Since \( B_1 \) is finite over \( B_0 \), they define the same rigidification by 6.2.4. □
Chapter II. Rigid spaces

Proof of Theorem A.4.11. It is clear that there exist the canonical functors, one from category (a) to category (b), and another from category (b) to category (c). We are going to show that there exists the canonical functor from (c) to (a), which gives a quasi-inverse to the composite functor from category (a) to category (c). Let $\mathcal{B} = (\mathcal{B}^\pm, \mathcal{B}^\prime)$ and $\mathcal{B}' = (\mathcal{B}'^\pm, \mathcal{B}'^\prime)$ be affinoid rings of finite type over $\mathfrak{A}$, and $\varphi: \mathcal{B}^\pm \to \mathcal{B}'^\pm$ an $A$-algebra homomorphism such that $\varphi(\mathcal{B}^\pm) \subseteq \mathcal{B}'^\prime$. Replacing $A_0$ in the same rigidification class if necessary, one can take topologically finitely generated rings of definition $B_0 \subseteq \mathcal{B}^+$ and $B'_0 \subseteq \mathcal{B}'^+$ over $A_0$. Then replace $B'_0$ by the ring generated by $B'_0$ and $\varphi(B_0)$, which is, by A.4.12, again a ring of definition that gives rise to the same rigidification. Since $\varphi(B_0) \subseteq B'_0$, it defines a morphism in category (a).

Corollary A.4.13. Let $A$ be a t.u. rigid-Noetherian extremal f-adic ring, and $C$ a rigidification of $A$. Let $\mathfrak{A} = (\mathfrak{A}^\pm, \mathfrak{A}^\prime)$ be the associated affinoid ring. Suppose that there exists a real valued spectral functor (8.1.9) defined on the category of all rigid spaces of finite type over the affinoid associated to $(A, C)$ ($\S A.2. (c)$). Then the following categories are equivalent:

(a) the category of affinoid rings of finite type over $\mathfrak{A}$, and

(b) the category of $A$-algebras of finite type with continuous $A$-algebra homomorphisms.

Proof. By A.4.11, it suffices to check that morphisms in the category (b) preserve the positive part. This follows from A.4.9.

The significance of the corollary lies in that the latter category does not refer to rigidifications; in other words, the rigidifications in this situation are canonical. Note that, by 8.2.19, the hypothesis in A.4.13 is automatic in the type $(V_R)$ and type (N) cases.

A.4. (f) Analytic rings of type (N)

Definition A.4.14. An f-adic ring $A$ is called an analytic ring of type (N) if it is complete, extremal (0, §B.1. (c)), and admits a Noetherian ring of definition.

Proposition A.4.15. Let $A$ be an analytic ring of type (N), and $A_0$ a Noetherian ring of definition of $A$. For any ring of definition $R$ of $A$ contained in $A^0$, the subring generated by $A_0$ and $R$ in $A^0$ is finite over $A_0$.

Proof. Let $J$ be an ideal of definition of $R$. Replacing $J$ by a power of $J$ if necessary, we may assume that $J \subseteq A_0$ and that $JA_0$ is an ideal of definition of $A_0$. The subring $R$ is a subset of $M_0 = \{ x \in A_0^{\text{int}}, Jx \subseteq A_0 \}$, which is an $A_0$-submodule of $A_0^{\text{int}}$. Let $X$ be the blow-up of Spec $A$ centered at the ideal $JA$. 


We have $J\mathcal{O}_X = \mathcal{O}_X(1)$. Then $M_0$ is an $A_0$-submodule of $\Gamma(X, \mathcal{O}(-1))$, and the last module is a finitely generated $A_0$-module. Since $A_0$ is Noetherian, $M_0$ is a finitely generated $A_0$-module, and hence the ring generated by $A_0$ and $R$ is finite over $A_0$.

**Corollary A.4.16.** Let $A$ be an analytic ring of type (N). Then the strict equivalence class containing a Noetherian ring of definition is unique.

For an analytic ring $A$ of type (N) we denote by $C_0(A)$ the finite equivalence class (A.2.3) containing a Noetherian ring of definition.

**Proposition A.4.17.** Let $A$ be an analytic ring of type (N). Then every ring of definition of $A$ belonging to $C_0(A)$ is Noetherian.

**Proof.** The assertion follows immediately from the Eakin–Nagata theorem, which guarantees that for a finite ring extension $A \subseteq B$ of rings with $B$ Noetherian, $A$ is Noetherian.

**Corollary A.4.18.** Let $X$ be a universally adhesive affinoid. If $X$ has at least one Noetherian affine formal model, then any distinguished affine formal model $X$ of $X$ is Noetherian.

**Proof.** This follows from 6.2.6, 6.2.8, and A.4.17.

**Proposition A.4.19.** Let $A$ and $A'$ be analytic rings of type (N). Suppose we are given rigidifications induced from $C_0(A)$ and $C_0(A')$ of $A$ and $A'$, respectively. Then any homomorphism of affinoid rings $\varphi: (A, A^o) \to (A', A'^o)$ is a homomorphism of a-r-pairs (A.4.2).

**Proof.** By 0.B.1.4 (2) and 0.B.1.6, $\varphi$ is adic. Let $A_0$ and $A'_0$ be Noetherian rings of definition of $A$ and $A'$, respectively. Let $A''_0$ be the subring of $A'$ generated by $A'_0$ and $\varphi(A_0)$. We claim that $A''_0$ is finite over $A'_0$. Indeed, since $\varphi(A_0)$ is bounded, there exists $n \geq 0$ such that $I^n_0 \varphi(A_0) \subseteq A'_0$, where $I'_0 \subseteq A'_0$ is an ideal of definition. This means that $A''_0 \subseteq [A'_0 : I'^n_0]$. But since $[A'_0 : I'^n_0]$ is obviously finite over $A'_0$ (because $\text{Spec} A' = \text{Spec} A'_0 \setminus V(I'_0)$), we see that $A''_0$ is finite.

The discussion above yields the following result (cf. A.4.13 and 8.2.19).

**Theorem A.4.20.** The following categories are canonically equivalent.

(a) The category of a-r-pairs given by analytic rings of type (N) with homomorphisms of a-r-pairs.

(b) The category of affinoid rings given by analytic rings of type (N) (without rigidifications).

(c) The category of analytic rings of type (N) with continuous ring homomorphisms.
A.4. (g) Canonical rigidifications of classical affinoid algebras. Let $V$ be an $a$-adically complete valuation ring of height one ($a \in m_V \setminus \{0\}$), and $K = \text{Frac}(V)$. Recall that any classical affinoid algebra $\mathcal{O}$ over $K$ (0.9.3.1) is a complete Tate ring (0, §B.1.(c)). For any subring $A \subseteq \mathcal{O}$ topologically of finite type over $V$ such that $A[f] = \mathcal{O}$, the rigid space $\mathcal{X} = (\text{Spf} A)^{\text{rig}}$ is a Stein affinoid, and hence by A.4.4 (1) and A.4.10, the pair $(\mathcal{O}, \mathcal{O}^{\text{rig}})$ is an affinoid ring (A.3.1).

**Definition A.4.21** (canonical subring). For a classical affinoid algebra $\mathcal{O}$ over $K$ we define

$$\mathcal{O}^{\text{can}} = \{ f \in \mathcal{O} : (f \mod m) \text{ is integral over } V \text{ for any maximal ideal } m \subseteq \mathcal{O} \},$$

and call $\mathcal{O}^{\text{can}}$ the *canonical subring* of $\mathcal{O}$.

Note that the subring $\mathcal{O}^{\text{can}}$ depends only on the $K$-algebra structure, and not on the topological (f-adic) structure of $\mathcal{O}$.

**Proposition A.4.22** (Uniqueness of affinoid ring structure). Let $\mathcal{O}$ be a classical affinoid algebra over $K$, and choose a ring of definition $A \subseteq \mathcal{O}$ that is topologically finitely generated over $V$. Let $\mathcal{X} = (\text{Spf} A)^{\text{rig}}$ be the associated Stein affinoid. Then

$$\mathcal{O}^{\text{can}} = \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})^{\text{rig}} (= \mathcal{O}^{\text{rig}}) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\text{int}}).$$

The significance of the proposition lies in that the affinoid ring structure $(\mathcal{O}, \mathcal{O}^{\text{rig}})$ actually does not depend on the topology on $\mathcal{O}$; combined with the previously obtained results in §A.4.(e), this shows that such an $\mathcal{O}$ also possesses the canonical rigidification, that is to say, the assignment $\mathcal{O} \mapsto \mathcal{X}$ is canonical, depending only on the $K$-algebra structure of $\mathcal{O}$.

**Proof.** For any $x \in (\mathcal{X})^{\text{cl}}$, denote by $m_x \subseteq \mathcal{O}$ the corresponding maximal ideal; see 8.2.11. Let $f \in \mathcal{O} = \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. If $f \in \mathcal{O}^{\text{can}}$, then the image of $f$ in $\mathcal{O}_{\mathcal{X},x}/m_{\mathcal{X},x}$ is finite over $V_x = \mathcal{O}_{\mathcal{X},x}^{\text{int}}/m_{\mathcal{X},x}$, the valuation ring at $x$, and hence belongs to $V_x$ (here we partially used the notation as in 3.2.13). Hence the germ $f_x \in \mathcal{O}_{\mathcal{X},x}$ lies in $\mathcal{O}_{\mathcal{X},x}^{\text{int}}$. Now by 8.1.11 and 8.2.21 we have $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\text{int}})$. This shows the inclusion $\mathcal{O}^{\text{can}} \subseteq \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\text{int}})$, while the opposite inclusion is easy to see. The other equality follows from A.4.10. □

**Corollary A.4.23.** For any $K$-algebra homomorphism $\varphi : \mathcal{O} \rightarrow \mathcal{O}'$ between classical affinoid algebras over $K$, we have $\varphi(\mathcal{O}^{\text{rig}}) \subseteq \mathcal{O}'^{\text{rig}}$. In particular, $\varphi$ induces a morphism of affinoid rings $(\mathcal{O}, \mathcal{O}^{\text{rig}}) \rightarrow (\mathcal{O}', \mathcal{O}'^{\text{rig}})$.

Thus one deduces, in particular, that any $K$-algebra homomorphism between classical affinoid algebras is continuous. This assertion is considered to be one of the most fundamental results in classical rigid analytic geometry.
Proof. In view of 8.2.11 and 8.2.14, the inclusion \( \mathcal{O}(\mathcal{O}') \subseteq \mathcal{O}'^{\circ} \) follows immediately from A.4.22. For the second, one needs to check that the topologies on \( \mathcal{O} \) and \( \mathcal{O}' \) are uniquely determined and that a ring of definition of finite type over \( V \) is mapped into a ring of definition of finite type over \( V' \). This follows from A.4.11.

\[ \square \]

**Corollary A.4.24.** In the setting of A.4.23, the morphism

\[ \varphi: (\mathcal{O}, \mathcal{O}^{\circ}) \longrightarrow (\mathcal{O}', \mathcal{O}'^{\circ}) \]

is a map of \( \alpha \)-r-pairs with respect to the canonical rigidifications (§A.2.(b)).

Compiling all results so far obtained, we have the following theorem.

**Theorem A.4.25.** Let \( V \) be an \( \alpha \)-adically complete valuation ring of height one, and \( K = \text{Frac}(V) \) its fraction field. Set \( S = (\text{Spf} \, V)^{\text{rig}} \). Then the following categories are canonically equivalent:

(a) the category of affinoids of finite type over \( S \) and morphisms over \( S \);

(b) the opposite category of the category of classical affinoid algebras over \( K \) and \( K \)-algebra homomorphisms.

**A.5 Rigid geometry and adic spaces**

We already know that for a Stein affinoid \( \mathcal{X} \) the Zariski–Riemann triple \( \mathcal{ZR}(\mathcal{X}) \) is canonically identified with the affinoid adic space

\[ ((\mathcal{X} = \text{Spa} \, \mathcal{O}_X, \mathcal{O}^+_X, \mathcal{O}_X), \{v_x\}_{x \in \mathcal{X}}) \]

associated to the analytic affinoid pair \( \mathcal{O}(\mathcal{X}) = (\mathcal{O}_X, \mathcal{C}_X) \) (A.4.8). In particular, we know that in this situation the presheaves \( \mathcal{O}^+_X \) and \( \mathcal{O}_X \) are sheaves. Therefore, by the definition of adic spaces (A.3.8), we have the following theorem.

**Theorem A.5.1.** (1) Let \( \mathcal{X} \) be a locally universally Noetherian rigid space. Then the associated valued triple \( \mathcal{ZR}(\mathcal{X}) = (\mathcal{ZR}(\mathcal{X}), \{v_x\}_{x \in \mathcal{X}}) \) is an analytic adic space (A.3.10).

(2) Let \( \varphi: \mathcal{X} \rightarrow \mathcal{Y} \) be a morphism between locally universally Noetherian rigid spaces. Then the morphism \( \mathcal{ZR}(\varphi): \mathcal{ZR}(\mathcal{X}) \rightarrow \mathcal{ZR}(\mathcal{Y}) \) of valued triples is a morphism of adic spaces.

**Proof.** (1) follows immediately from A.4.8. Since morphisms of adic spaces are, by definition, morphisms of valued triples, (2) follows as well. \[ \square \]
Thus we have the functor

\[ \text{ZR: RigNoeRf} \longrightarrow \text{AnAdsp} \]

from the category of locally universally Noetherian rigid spaces to the category of analytic adic spaces, which is faithful due to \( A.1.6 (2) \).

**Theorem A.5.2.** Let \( S \) be a locally universally Noetherian rigid space. Then the visualization functor \( \text{ZR} \) establishes a categorical equivalence between the category of locally of finite type rigid spaces over \( S \) and the category of adic spaces locally of finite type \( \text{(A.3.13)} \) over \( \text{ZR}(S) \).

**Proof.** In view of \( A.4.13 \), it suffices to show that the functor in question is fully faithful, for then the essential surjectivity follows by a standard patching argument. It is obvious that for any locally of finite type \( X \) over \( S \), the adic space \( \text{ZR}(X) \) is locally of finite type over \( \text{ZR}(S) \). We need to show that, for any \( X \) and \( Y \) locally of finite type over \( S \), the map

\[ \text{Hom}_S(X, Y) \longrightarrow \text{Hom}_{\text{ZR}(S)}(\text{ZR}(X), \text{ZR}(Y)) \]

already known to be injective, is bijective. We may assume that \( S, X, \) and \( Y \) are Stein affinoids. Since the functor \( \text{AR} \) is fully faithful (\( A.4.6 \)), we only need to check that any morphism \( \varphi: \text{ZR}(X) \rightarrow \text{ZR}(Y) \) canonically gives a map \( \mathfrak{a} \mathfrak{R}(Y) \rightarrow \mathfrak{a} \mathfrak{R}(X) \) of a-r-pairs (which is automatically adic). Since \( \mathfrak{a} \mathfrak{R}(X) = (\Gamma(X, \mathcal{O}_X), \Gamma(X, \mathcal{O}_X^{\text{int}})) \) etc., the morphism \( \varphi \) induces a homomorphism \( \mathfrak{a}_Y \rightarrow \mathfrak{a}_X \) of affinoid rings. As we have seen in \( \S A.2 \) (c), the rigidifications \( C_X \) and \( C_Y \) are canonically determined by the rigidification \( C_S \) of \( \mathfrak{a} \mathfrak{R}(S) \), and it is clear that, in view of the definition of topologically of finite type affinoid rings (cf. [60], \S 3, p. 534ff), these rigidifications are preserved by \( \text{ZR}(S) \)-morphisms of adic spaces. \( \square \)

**Theorem A.5.3.** The category of analytic spaces of type (N) and the category of rigid spaces of type (N) \( (2.5.2) \) are equivalent.

Here by an analytic space of type (N) we mean an adic space that is locally isomorphic to the adic spectrum of a complete affinoid ring \( \mathfrak{a} = (\mathfrak{a}^+, \mathfrak{a}^\circ) \), where \( \mathfrak{a}^\pm \) is an analytic ring of type (N) \( (A.4.14) \) and \( \mathfrak{a}^+ = A^\circ \).

**Proof.** Due to \( A.4.19 \), one can show by an argument similar to that in the proof of \( A.5.2 \) that the functor \( \text{ZR} \) is fully faithful from the category of rigid spaces of type (N) to the category of analytic spaces of type (N). It is clear that the functor in question is essentially surjective. \( \square \)

**Remark A.5.4.** Let \( X \) be a coherent universally rigid-Noetherian formal scheme \( X \) such that \( \mathcal{O}_X \) is \( I \)-torsion free, where \( I \) is an ideal of definition. We do not know the answer to the question: if the associated rigid space \( X^{\text{rig}} \) is of type (N), is \( X \) Noetherian as a formal scheme? Notice that this is true if \( X \) is universally adhesive.
B Appendix: Tate’s rigid analytic geometry

B.1 Admissibility

B.1. (a) Admissibility with respect to a spectral functor. Throughout this paragraph we fix

- an O-stable subcategory \( \mathcal{C} \) of the category of rigid spaces (8.1.1 (1)) consisting of quasi-separated rigid spaces,
- a continuous spectral functor \( S \) (8.1.2, 8.1.4) defined on \( \mathcal{C} \).

We assume that the data \( (\mathcal{C}, S) \) satisfy the following weak left-exactness.

(a) For any arrow \( \psi: \mathcal{X} \to \mathcal{Y} \) in \( \mathcal{C} \) and any open subspace \( \mathcal{V} \subseteq \mathcal{Y} \), we have \( S(\psi^{-1}(\mathcal{V})) = S(\psi)^{-1}(S(\mathcal{V})) \).

(b) For any family of open subspaces \( \{\mathcal{U}_a\}_{a \in L} \) of \( \mathcal{X} \) belonging to \( \mathcal{C} \), we have \( S(\bigcup_{a \in L} \mathcal{U}_a) = \bigcup_{a \in L} S(\mathcal{U}_a) \).

Definition B.1.1. Let \( \mathcal{X} \in \text{obj}(\mathcal{C}) \). An open subspace \( \mathcal{U} \subseteq \mathcal{X} \) is said to be \((\mathcal{C}, S)\)-admissible if

- for any arrow \( \varphi: \mathcal{V} \to \mathcal{X} \) in \( \mathcal{C} \), where \( \mathcal{V} \) is coherent and \( S(\varphi)(S(\mathcal{V})) \subseteq S(\mathcal{U}) \), we have \( \varphi(\mathcal{V}) \subseteq \mathcal{U} \).

It is clear that if \( \psi: \mathcal{Y} \to \mathcal{X} \) is a morphism in \( \mathcal{C} \) and \( \mathcal{U} \subseteq \mathcal{X} \) is a \((\mathcal{C}, S)\)-admissible open subspace, then \( \psi^{-1}(\mathcal{U}) \) is a \((\mathcal{C}, S)\)-admissible open subspace of \( \mathcal{Y} \). Note that, for any \( \mathcal{X} \in \text{obj}(\mathcal{C}) \), any quasi-compact open subspace \( \mathcal{U} \subseteq \mathcal{X} \) is \((\mathcal{C}, S)\)-admissible; indeed, for \( \varphi: \mathcal{V} \to \mathcal{X} \) as above, since \( S(\varphi^{-1}((\mathcal{U})) = S(\varphi)^{-1}(S(\mathcal{U})) = S(\mathcal{V}) \), we have \( \varphi^{-1}(\mathcal{U}) = \mathcal{V} \) due to 8.1.6.

Proposition B.1.2. Let \( \mathcal{X} \) be a rigid space in \( \mathcal{C} \) and \( \mathcal{U} \) a \((\mathcal{C}, S)\)-admissible open subspace of \( \mathcal{X} \). Then \( \mathcal{U} \) is the maximal open subspace of \( \mathcal{X} \) among the open subspaces \( \mathcal{V} \subseteq \mathcal{X} \) such that \( S(\mathcal{V}) \subseteq S(\mathcal{U}) \).

Proof. For any quasi-compact open subspace \( \mathcal{V} \subseteq \mathcal{X} \) such that \( S(\mathcal{V}) \subseteq S(\mathcal{U}) \), we have \( \mathcal{V} \subseteq \mathcal{U} \). Thanks to the continuity of \( S \), \( \mathcal{U} \) is the union of all such quasi-compact subspaces. \( \square \)

Remark B.1.3. One can similarly show the following fact: for \( \mathcal{X} \in \text{obj}(\mathcal{C}) \), an open subspace \( \mathcal{U} \subseteq \mathcal{X} \) is \((\mathcal{C}, S)\)-admissible if and only if for any \( \varphi: \mathcal{V} \to \mathcal{X} \) in \( \mathcal{C} \), where \( \mathcal{V} \) is coherent, \( \varphi^{-1}(\mathcal{U}) \) is the maximal open subspace of \( \mathcal{V} \) among the open subspaces \( \mathcal{W} \subseteq \mathcal{V} \) such that \( S(\mathcal{W}) = S(\varphi^{-1}(\mathcal{U})) \).

For \( \mathcal{X} \in \text{obj}(\mathcal{C}) \) we denote by \( \mathfrak{A}(\mathcal{X}) \) the category of \((\mathcal{C}, S)\)-admissible open subspaces in \( \mathcal{X} \) and open immersions.
**Corollary B.1.4.** The spectral functor $S$ is conservative on $\mathfrak{A}(\mathcal{C})$, that is, for $\mathcal{U}, \mathcal{U}' \in \text{obj}(\mathfrak{A}(\mathcal{C}))$, $S(\mathcal{U}) = S(\mathcal{U}')$ implies $\mathcal{U} = \mathcal{U}'$.

**Definition B.1.5.** Let $\mathcal{X}$ be a rigid space in $\mathcal{C}$, and $\mathcal{U} \subseteq \mathcal{X}$ a $(\mathcal{C}, S)$-admissible open subspace. An open covering $\mathcal{U} = \bigcup_{\alpha \in L} \mathcal{U}_\alpha$ is said to be $(\mathcal{C}, S)$-admissible if every $\mathcal{U}_\alpha$ ($\alpha \in L$) is $(\mathcal{C}, S)$-admissible.

By B.1.4 and the weak left-exactness of $S$ we have the following result.

**Proposition B.1.6.** By attaching to each $\mathcal{X} \in \text{obj}(\mathcal{C})$ the category $\mathfrak{A}(\mathcal{X})$ together with the notion of $(\mathcal{C}, S)$-admissible coverings, we have a Grothendieck topology on the category $\mathcal{C}$. The associated topos is equivalent to the large admissible topos (2.2.25) restricted to $\mathcal{C}$.

As we remarked before, any quasi-compact open subspace is $(\mathcal{C}, S)$-admissible. Here is another example of $(\mathcal{C}, S)$-admissible subspaces.

**Proposition B.1.7.** Any tube open subset (4.2.4) of $\mathcal{X}$ is $(\mathcal{C}, S)$-admissible.

**Proof.** Let $T = \langle \mathcal{X} \rangle \setminus \langle \mathcal{U} \rangle$ be a tube open subset (4.2.4), where $\mathcal{U} \subseteq \mathcal{X}$ is a retrocompact open subspace. Since the inverse image of a tube open subset by a morphism of coherent rigid spaces is again a tube open subset (cf. Exercise II.4.2), it suffices to show, by B.1.3, that $T$ is the maximal among all open subsets $\mathcal{W} \subseteq \mathcal{X}$ such that $S(\mathcal{W}) = S(T)$. For any quasi-compact open subspace $\mathcal{V} \subseteq \mathcal{X}$ such that $S(\mathcal{V}) \subseteq S(T)$, we have $S(\mathcal{U} \cap \mathcal{V}) = S(\mathcal{U}) \cap S(\mathcal{V}) = \emptyset$. This shows that $\mathcal{U} \cap \mathcal{V} = \emptyset$ and hence that $\langle \mathcal{V} \rangle \subseteq T$. Then the assertion follows from continuity of $S$. $\square$

**B.1. (b) G-topology on a topological space.** Let us recall the definition of $G$-topologies on a topological space $X$ (cf. [18], (9.1.1)). Let $\text{Ouv}(X)$ be the category of open subsets of $X$; the objects of $\text{Ouv}(X)$ are the open subsets of $X$, and for two open subsets $\mathcal{U}, \mathcal{V} \subseteq X$

$$\text{Hom}_{\text{Ouv}(X)}(\mathcal{U}, \mathcal{V}) = \begin{cases} \{\ast\}, & \text{if } \mathcal{U} \subseteq \mathcal{V}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

A $G$-topology on $X$ is a Grothendieck topology $\tau$ on a full subcategory $\mathfrak{A}$ of $\text{Ouv}(X)$ such that $U, V \in \text{obj}(\mathfrak{A})$ implies $U \cap V \in \text{obj}(\mathfrak{A})$; open subsets in $\mathfrak{A}$ (resp. coverings of $\tau$) are called $\tau$-admissible open subsets (resp. $\tau$-admissible open coverings).

For example, the topology on $X$ itself gives rise to a $G$-topology in an obvious way. We call this $G$-topology the canonical $G$-topology.
If \( \tau \) is a G-topology on a topological space \( X \) and \( Y \subseteq X \) is a subset, then one can consider the *induced G-topology* on \( Y \), denoted by \( \tau \mid_Y \), as follows.

- \( \tau \mid_Y \)-admissible open subsets are the open subsets of \( Y \) of the form \( U \cap Y \) for a \( \tau \)-admissible open subset \( U \).
- \( \tau \mid_Y \)-admissible coverings of an \( \tau \mid_Y \)-admissible open subset \( U \cap Y \) are the open coverings of the form \( \{ Y \cap U_\alpha \}_{\alpha \in L} \) for a \( \tau \)-admissible covering \( \{ U_\alpha \}_{\alpha \in L} \) of \( U \).

Let \( X \) be a topological space and \( \tau \) a G-topology on \( X \) such that the associated topos \( \text{top}(\tau) \) is equivalent to the canonical topos \( \text{top}(X) \). Then it may be the case, for a subset \( Y \subseteq X \), that the topos \( \text{top}(\tau \mid_Y) \) and \( \text{top}(Y) \) are not equivalent. For example, consider \( X = \mathbb{R} \) with the topology \( \tau \) given by the open intervals with rational extremities; although \( \tau \) gives the topos equivalent to the one given by \( \mathbb{R} \) itself, the topos \( \text{top}(\tau \mid_{\mathbb{Q}}) \) and \( \text{top}(\mathbb{Q}) \) are different from each other, for the former is equivalent to \( \text{top}(\tau) \) (and hence the rational line \( \mathbb{Q} \) with this G-topology is connected (cf. [18], (9.1.1))), whereas the topological space \( \mathbb{Q} \) is, as is well known, totally disconnected.

In general, for a topological space \( X \), a subset \( Y \subseteq X \), and a G-topology \( \tau \) on \( X \) the associated topos of which is equivalent to the canonical one, the topos \( \text{top}(\tau \mid_Y) \) is equivalent to \( \text{top}(X) \) if \( \tau \) satisfies the following condition: if \( U, V \) are \( \tau \)-admissible open subsets, then \( Y \cap U = Y \cap V \) implies \( U = V \).

### B.1. (c) G-topology associated to a spectral functor.

Now we return to the situation considered in the beginning of §B.1. (a). We introduce the notion of \( (\mathcal{C}, S) \)-admissibility (and thus a G-topology) on each topological space \( S(\mathcal{X}) \) with \( \mathcal{X} \in \text{obj}(\mathcal{C}) \).

**Definition B.1.8.** For \( \mathcal{X} \in \text{obj}(\mathcal{C}) \), a (not necessarily open) subset \( U \subseteq S(\mathcal{X}) \) is said to be \( (\mathcal{C}, S) \)-admissible if the following conditions are satisfied.

1. There exists an open subspace \( U \subseteq \mathcal{X} \) such that \( S(U) = U \).
2. For any morphism \( \varphi : \mathcal{V} \rightarrow \mathcal{X} \) in \( \mathcal{C} \), where \( \mathcal{V} \) is coherent and such that \( S(\varphi)(S(\mathcal{V})) \subseteq U \), there exists a collection of quasi-compact open subspaces \( \{ U_\alpha \}_{\alpha \in L} \) such that \( U = \bigcup_{\alpha \in L} S(U_\alpha) \) and the covering \( \{ S(\varphi^{-1}(U_\alpha)) \}_{\alpha \in L} \) of \( S(\mathcal{V}) \) (which is a covering due to the weak left-exactness assumption in §B.1. (a)) is refined by a finite covering of the form \( \{ S(\mathcal{V}_j) \}_{j \in J} \) given by quasi-compact open subspaces \( \mathcal{V}_j \subseteq \mathcal{V} \).

Note here that a priori the collection \( \{ U_\alpha \}_{\alpha \in L} \) of quasi-compact open subspaces of \( \mathcal{X} \) may depend on the map \( \varphi \); we will show in B.1.10 that actually such a collection can be taken independently on \( \varphi \).
Proposition B.1.9. For \( X \in \text{obj}(\mathcal{C}) \), a subset \( U \subseteq S(X) \) is \((\mathcal{C}, S)\)-admissible if and only if it is of the form \( U = S(U) \) for a \((\mathcal{C}, S)\)-admissible open subspace \( U \subseteq X \).

Note that thanks to B.1.4 the \((\mathcal{C}, S)\)-admissible open subspace \( U \) is uniquely determined by \( U \). To prove the proposition, we need the following lemma.

Lemma B.1.10. Let \( X \) be a coherent rigid space in \( \mathcal{C} \) and \( \{U_\alpha\}_{\alpha \in L} \) a collection of quasi-compact open subspaces of \( X \). Then \( \{U_\alpha\}_{\alpha \in L} \) covers \( X \) if and only if

(a) \( S(X) = \bigcup_{\alpha \in L} S(U_\alpha) \) and

(b) \( \{S(U_\alpha)\}_{\alpha \in L} \) admits a finite refinement of the form \( \{S(V_j)\}_{j \in J} \), where \( V_j \) are quasi-compact open subspaces of \( X \).

Proof. The ‘only if’ part is clear. Let us show the ‘if’ part. Take for each \( j \in J \) an index \( \alpha(j) \in L \) such that \( S(V_j) \subseteq S(U_{\alpha(j)}) \). Since \( S \) is spectral and since \( V_j \) and \( U_{\alpha(j)} \) are quasi-compact, we have \( V_j \subseteq U_{\alpha(j)} \). From \( S(X) = \bigcup_{j \in J} S(V_j) = S(\bigcup_{j \in J} V_j) \), we deduce that \( X = \bigcup_{j \in J} V_j \), since \( S \) is spectral, and thus \( X = \bigcup_{\alpha \in L} U_\alpha \), as desired. \( \square \)

Proof of Proposition B.1.9. Suppose that there exists a \((\mathcal{C}, S)\)-admissible open subspace \( U \subseteq X \) such that \( U = S(U) \). Then any open covering \( \{U_\alpha\}_{\alpha \in L} \) of \( U \) consisting of quasi-compact open subspaces fulfills B.1.8 (b), and thus the ‘if’ part is clear. To show the ‘only if’ part, consider the set \( \{U : S(U) = U\} \) (non-empty by the definition B.1.8 (a)). Due to the continuity of \( S \), this set has the maximal element \( U \). We need to show that this \( U \) is \((\mathcal{C}, S)\)-admissible. Let \( V \) be a coherent rigid space in \( \mathcal{C} \), and \( \varphi : V \to X \) a morphism in \( \mathcal{C} \) such that \( S(\varphi)(S(V)) \subseteq U \). Let \( \{U_\alpha\}_{\alpha \in L} \) be as in B.1.8 (b), and set \( U' = \bigcup_{\alpha \in L} U_\alpha \). Since \( S(U) = S(U') = U \), we have \( U' \subseteq U \). Since the covering \( \{S(\varphi^{-1}(U_\alpha))\}_{\alpha \in L} \) of \( S(V) \) admits a finite refinement \( \{S(V_j)\}_{j \in J} \) as in B.1.8 (b), B.1.10 implies that \( V = \bigcup_{\alpha \in L} \varphi^{-1}(U_\alpha) = \varphi^{-1}(U) \) and thus \( \varphi(V) \subseteq U' \subseteq U \). \( \square \)

Definition B.1.11. Let \( X \) be a rigid space in \( \mathcal{C} \) and \( U \subseteq S(X) \) a \((\mathcal{C}, S)\)-admissible subset. A collection \( \{U_\alpha\}_{\alpha \in L} \) of subsets of \( S(X) \) is called a \((\mathcal{C}, S)\)-admissible covering of \( U \) if the following conditions are satisfied.

(a) Each \( U_\alpha \) is a \((\mathcal{C}, S)\)-admissible subset of \( S(X) \).

(b) For any morphism \( \varphi : V \to X \) in \( \mathcal{C} \) from a coherent rigid space such that \( S(\varphi)(S(V)) \subseteq U \), \( \{S(\varphi^{-1}(U_\alpha))\}_{\alpha \in L} \) admits a finite refinement of the form \( \{S(V_j)\}_{j \in J} \), where each \( V_j \subseteq V \) is a quasi-compact open subspace, that covers \( S(V) \).

One has the following proposition, which can be shown by an argument similar to that in B.1.10.
Proposition B.1.12. Let $X$ be a rigid space in $\mathcal{C}$ and $\{U_\alpha\}_{\alpha \in \mathcal{L}}$ a family of $({\mathcal{C}}, S)$-admissible open subspaces of $X$. Then $\{U_\alpha\}_{\alpha \in \mathcal{L}}$ is a $({\mathcal{C}}, S)$-admissible covering of $X$ if and only if $S(U_\alpha)_{\alpha \in \mathcal{L}}$ is a $({\mathcal{C}}, S)$-admissible covering of $S(X)$.

B.2 Rigid analytic geometry

B.2. (a) Classical affinoids. Let $V$ be an $a$-adically complete valuation ring of height one ($a \in m_V \setminus \{0\}$), and $K = \text{Frac}(V)$. As in 0, §6.3. (c), the field $K$ has a non-Archimedean valuation $\| \cdot \|: K \to \mathbb{R}_{\geq 0}$ and is complete with respect to the associated metric topology.

Let $\text{Aff}_K$ be the category of classical affinoid algebras over $K$ (0.9.3.1) and $K$-algebra homomorphisms. We consider the dual category $\text{Aff}^\text{opp}_K$, called the category of (classical) affinoids over $K$. For a classical affinoid algebra $\mathcal{O}$, we consider the set $\text{Spm} \mathcal{O}$ of all closed points in $\text{Spec} \mathcal{O}$; recall that any classical affinoid algebra is a Jacobson ring (0.9.3.10). Then for any morphism $\varphi: \mathcal{O} \to \mathfrak{B}$ in $\text{Aff}_K$, the corresponding morphism in $\text{Aff}^\text{opp}_K$ is interpreted as the map $\text{Spm} \mathfrak{B} \to \text{Spm} \mathcal{O}$ $(m \mapsto \varphi^{-1}(m))$, the existence of which is guaranteed by 0.9.3.7.

Subsets of an affinoid $\text{Spm} \mathcal{O}$ of the form

$$\{ x \in \text{Spm} \mathcal{O}: \| f_1(x) \| \leq 1, \ldots, \| f_r(x) \| \leq 1 \}$$

with $f_1, \ldots, f_r \in \mathcal{O}$ form a basis of a topology on the set $\text{Spm} \mathcal{O}$, called the canonical topology (cf. [94], §9, and [18], 7.2.1). Here, for an element $f \in \mathcal{O}$ and a point $x = m \in \text{Spm} \mathcal{O}$, we write $f(x) = (f \mod m)$, which lies the finite extension $K' = \mathcal{O}/m$ of $K$ (0.9.3.7), and denote, by a slight abuse of notation, the unique extension of the norm $\| \cdot \|$ on $K$ to $K'$ by the same symbol. Note that any morphism $\text{Spm} \mathfrak{B} \to \text{Spm} \mathcal{O}$ in $\text{Aff}^\text{opp}_K$ is continuous with respect to the canonical topologies.

B.2. (b) Affinoid subdomains

Definition B.2.1. A subset $U \subseteq \text{Spm} \mathcal{O}$ is said to be an affinoid subdomain if the functor $F_U: \text{Aff}_K \to \text{Sets}$ defined by

$$F_U(\mathfrak{B}) = \{ \varphi \in \text{Hom}_K(\mathcal{O}, \mathfrak{B}): \varphi^*(\text{Spm} \mathfrak{B}) \subseteq U \}$$

for $\mathfrak{B} \in \text{obj}(\text{Aff}_K)$ is representable by a classical affinoid algebra over $K$; in other words, these exists a $K$-algebra homomorphism $\mathcal{O} \to \mathfrak{B}_U$ of classical affinoid algebras over $K$ that satisfies the following universal mapping property: for any $K$-algebra homomorphism $\mathcal{O} \to \mathfrak{B}$ of classical affinoid algebras over $K$ such that the image of the induced map $\text{Spm} \mathfrak{B} \to \text{Spm} \mathcal{O}$ lies in $U$, there exists a unique
$K$-algebra homomorphism $\mathcal{O}_U \to \mathcal{B}$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{O} & \longrightarrow & \mathcal{O}_U \\
\downarrow & & \downarrow \\
\mathcal{B} & & \\
\end{array}
$$

commutes.

The classical affinoid algebra $\mathcal{O}_U$ as above is uniquely determined up to isomorphisms. It is, moreover, known that the map $\text{Spm } \mathcal{O}_U \to \text{Spm } \mathcal{O}$ gives a homeomorphism onto $U$ (cf. [18], 7.2.2 and 7.2.5).

Let $\mathcal{O}$ be a classical affinoid algebra over $K$, and $A$ a topologically finitely generated $V$-algebra such that $\mathcal{O} = A[\frac{1}{\alpha}]$ (cf. 0.9.3.1). Consider the affinoid $X = (\text{Spf } A)^\text{rig}$ in the sense of 6.1.1. Then, by 8.2.11, the set $\text{Spm } \mathcal{O}$ coincides with the set of all classical points $X_0$ of $X$. It is straightforward to see, using A.4.25, that for any affinoid subdomain (in the sense of 6.1.6) $U = (\text{Spf } B)^\text{rig} \subseteq X$, the subset $U = U_0 \subseteq X_0$ of classical points gives an affinoid subdomain of $\text{Spm } \mathcal{O}$, with the classical affinoid algebra $\mathcal{O}_U = B[\frac{1}{\alpha}]$. The famous Gerritzen–Grauert theorem [45] states the converse.

**Theorem B.2.2** (cf. Gerritzen and Grauert [45]). In the situation as above, a subset $U \subseteq \text{Spm } \mathcal{O}$ is an affinoid subdomain if and only if there exists an affinoid subdomain $U = (\text{Spf } B)^\text{rig} \subseteq X = (\text{Spf } A)^\text{rig}$ (in the sense of 6.1.6) such that $U$ coincides with the set of classical points $U_0$ of $U$; in this case, the related classical affinoid algebra is given by $\mathcal{O}_U = B[\frac{1}{\alpha}]$.

We will prove this theorem independently from [45] in the next volume (cf. Introduction). But, for the reader’s convenience, let us include here the argument to deduce this theorem from the classical Gerritzen–Grauert theorem (e.g., [18], 7.3.5).

Suppose $U = \text{Spm } \mathcal{B} \hookrightarrow X = \text{Spm } \mathcal{O}$ is an affinoid subdomain, and take $U = (\text{Spf } B)^\text{rig} \to X = (\text{Spf } A)^\text{rig}$ such that $U_0 \to X_0$ is the given $U \hookrightarrow X$. Take a finite covering by rational subdomains $X = \bigcup_{\alpha \in L} X_\alpha$, where $X_\alpha = \text{Spm } \mathcal{O}_\alpha$, such that $U \cap X_\alpha = \text{Spm } \mathcal{O}_\alpha \otimes_{\mathcal{O}_\alpha} \mathcal{B}$ is a Weierstrass domain of $X_\alpha = \text{Spm } \mathcal{O}_\alpha$ ([18], 7.3.5/2). Since $X_\alpha$ is a rational subdomain of $X$, there exists $A_\alpha$ as in 6.1.7 (3) such that $X_\alpha = (\text{Spf } A_\alpha)^\text{rig}$ gives a rational subdomain of $X$ and $X_\alpha = (X_\alpha)_0$ for $\alpha \in L$. Since $U \cap X_\alpha$ is a Weierstrass domain, there exists $B_\alpha$ as in 6.1.7 (1) such that $U_\alpha = (\text{Spf } B_\alpha)^\text{rig}$ gives a rational subdomain of $X_\alpha$ and $U \cap X_\alpha = (U_\alpha)_0$ for $\alpha \in L$. Since $U = \bigcup_{\alpha \in L} U_\alpha$, and $U_\alpha \hookrightarrow X_\alpha$ and $X_\alpha \hookrightarrow X$ are all open immersions for $\alpha \in L$, we deduce that $U \hookrightarrow X$ is an open immersion, as desired.
B.2. (e) Rigid analytic spaces. We next introduce the so-called weak G-topology and strong G-topology on affinoids.

- [18], 9.1.4. The weak G-topology on an affinoid $X = \text{Spm } \mathcal{O}$ is defined as follows:
  - the admissible open subsets are the affinoid subdomains;
  - the admissible coverings are the finite coverings by affinoid subdomains.

- [18], 9.1.4/2. The strong G-topology on an affinoid $X = \text{Spm } \mathcal{O}$ is defined as follows:
  - a subset $U \subseteq X$ is an admissible open subset if and only if it has a covering $U = \bigcup_{\alpha \in L} U_\alpha$ by affinoid subdomains of $X$ such that for any morphism $f: Y \to X$ of affinoids with $f(Y) \subseteq U$, the induced covering $\{f^{-1}(U_\alpha)\}_{\alpha \in L}$ is refined by an admissible covering of weak G-topology on $Y$;
  - a covering $U = \bigcup_{\alpha \in L} V_\alpha$ of an admissible open subset $V$ by admissible open subsets is an admissible covering if and only if for any morphism $f: Y \to X$ of affinoids with $f(Y) \subseteq U$, the induced covering $\{f^{-1}(V_\alpha)\}_{\alpha \in L}$ is refined by an admissible covering in the weak G-topology on $Y$.

The structure presheaf $\mathcal{O}_X$ on the affinoid $X = \text{Spm } \mathcal{O}$ is defined as follows. For any affinoid subdomain $U \subseteq X$, we set $\mathcal{O}_X(U) = \mathcal{O}_U$. It follows from the universal mapping property of affinoid subdomains that this indeed yields a presheaf $\mathcal{O}_X$.

**Theorem B.2.3** (Tate’s acyclicity theorem [94], §8). The presheaf $\mathcal{O}_X$ is a sheaf with respect to the weak G-topology, and hence uniquely extends to a sheaf with respect to the strong G-topology (cf. [18], 9.2.3).

**Definition B.2.4** (cf. [18], 9.3.1/4). (1) An affinoid space over $K$ is a locally G-ringed space $(X, \mathcal{O}_X)$ over $K$, that is, a pair consisting of a G-topological space $X$ and a sheaf of local rings over $K$ on $X$ with respect to the G-topology, that is $K$-isomorphic to the locally G-ringed space of the form $(\text{Spm } \mathcal{O}, \mathcal{O}_{\text{Spm } \mathcal{O}})$ considered with the strong G-topology.

(2) A rigid analytic space over $K$ is a locally G-ringed space $(X, \mathcal{O}_X)$ over $K$ such that the following conditions are satisfied.

(a) The G-topology of $X$ enjoys the following properties.

$(G_0)$ $\emptyset$ and $X$ are admissible open subsets of $X$.
(G₁) For an admissible open subset \( U \subseteq X \) and a subset \( V \subseteq U \), \( V \) is admissible open in \( X \) if there exists an admissible covering \( \{U_\alpha\}_{\alpha \in \mathcal{L}} \) of \( U \) such that each \( V \cap U_\alpha \) is admissible open in \( X \).

(G₂) If an open covering of an admissible open subset \( U \) consisting of admissible open subsets admits a refining admissible covering, it is an admissible covering of \( U \).

(b) There exists an admissible covering \( \{X_\alpha\}_{\alpha \in \mathcal{L}} \) of \( X \) such that, for each \( \alpha \in \mathcal{L} \), \((X_\alpha, \mathcal{O}_{X}|_{X_\alpha})\) is an affinoid space over \( K \).

(3) A morphism of rigid analytic spaces \((X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) over \( K \) is a \( K \)-morphism of locally \( G \)-ringed spaces over \( K \).

B.2. (d) Comparison with rigid spaces. Let \( V \) and \( K \) be as in the beginning of §B.2. (c). Set \( S = (\text{Spf } V)^{\text{rig}} \), and let \( \mathcal{C} \) be the category of rigid spaces locally of finite type over \( S \). By 8.2.21, the functor \( S = (\cdot)^{\text{cl}} \) is a continuous spectral functor defined on \( \mathcal{C} \) which satisfies properties (a) and (b) in the beginning of §B.1. (a).

For a rigid space \( \mathcal{X} \) in \( \mathcal{C} \), let

\[ \mathcal{X}_0 = (\mathcal{X}_0, \tau_0, \mathcal{O}_{\mathcal{X}_0}) \]

be the \( G \)-ringed space over \( K \) defined as follows.

- As a set, \( \mathcal{X}_0 \) is the set of all classical points of \( \mathcal{X} \):

\[ \mathcal{X}_0 = (\mathcal{X})^{\text{cl}}. \]

- The \( G \)-topology \( \tau_0 \) is the topology on \( \mathcal{X}_0 = S(\mathcal{X}) \) given by the admissible open subsets as in B.1.8 and the admissible coverings as in B.1.11. (It is clear that \((\mathcal{C}, S)\)-admissible subsets of \( \mathcal{X}_0 = S(\mathcal{X}) \) are open subsets with respect to the subspace topology on \( \mathcal{X}_0 \) induced by the topology on the Zariski–Riemann space \((\mathcal{X})\)).

- \( \mathcal{O}_{\mathcal{X}_0} \) is the sheaf of local rings defined as follows: for any quasi-compact open subspace \( \mathcal{U} \subseteq \mathcal{X} \),

\[ \mathcal{O}_{\mathcal{X}_0}((\mathcal{U})^{\text{cl}}) = \mathcal{O}_\mathcal{X}((\mathcal{U})) \]

(note that quasi-compact open subspaces are \((\mathcal{C}, S)\)-admissible).

It is clear by the construction that for an \( S \)-morphism \( \varphi: \mathcal{X} \to \mathcal{Y} \) of locally of finite type rigid spaces, one has the uniquely induced morphism \( \varphi_0: \mathcal{X}_0 \to \mathcal{Y}_0 \) of \( G \)-ringed spaces over \( K \). Moreover, if \( \varphi \) is an open immersion, then so is \( \varphi_0 \).
Theorem B.2.5. For a locally of finite type $S$-rigid space $\mathcal{X}$, the G-ringed space $X_0 = (X_0, \tau_0, \mathcal{O}_X)$ thus obtained is a rigid analytic space over $K$ in the sense of [18], (9.3.1/4). Thus we have a functor

$$X \mapsto X_0$$

from the category of locally of finite type $S$-rigid spaces to the category of rigid analytic spaces over $K$. Moreover, this functor gives a categorical equivalence from the category of quasi-separated locally of finite type $S$-rigid spaces to the category of quasi-separated rigid analytic spaces over $K$.

Here, a rigid analytic space is said to be quasi-separated (in the sense of Tate) if the intersection of two affinoid subdomains is a finite union of affinoid subdomains. Notice that the quasi-separatedness in the last assertion is necessary due to the following example. Let $X$ be any non-empty finite type affinoid over $V$, and $U = \mathcal{X} \setminus \{x\}$, where $x$ is a non-classical closed point of $X$. Then the gluing of two copies of $X$ along $U$ is a non-quasi-separated $S$-rigid space, of which the associated rigid analytic space is the same as that of $X$.

Proof of Theorem B.2.5. Let us first show step by step that $X_0 = (X_0, \tau_0, \mathcal{O}_X)$ is a rigid analytic space over $K$ and, at the same time, that the functor $X \mapsto X_0$ is fully faithful.

Step 1. Suppose $X$ is an affinoid, $X = (\text{Spf } A)^\text{rig}$, where $A$ is a topologically finitely generated algebra over $V$. In this case the ring $\mathcal{O} = A[\frac{1}{a}]$ is a classical affinoid algebra ([0], § 9.3. (a)). By 8.2.11, the set $X_0$ is canonically identified with the set $\text{Spm } \mathcal{O}$ of all closed points in $\text{Spec } \mathcal{O}$.

Claim. The G-topology $\tau_0$ coincides with the strong G-topology on the affinoid $X_0 = \text{Spm } \mathcal{O}$.

To show the claim, first note the following facts.

- For a quasi-compact open subspace $U \subseteq X$, $U_0 \subseteq X_0$ is an affinoid subdomain in the sense of B.2.1 if and only if $U$ is an affinoid subdomain of $X$ in the sense of 6.1.6 (Gerritzen–Grauert theorem [45]; cf. B.2.2).
- For a finite collection $\{U_a\}_{a \in L}$ of quasi-compact open subspaces of $X$, $X_0 = \bigcup_{a \in L} U_0$ if and only if $X = \bigcup_{a \in L} U_a$; this follows from the fact that $S = \{ \cdot \}^{\text{cl}}$ is a spectral functor.

Then the claim is evident thanks to the definitions of admissible open subsets (B.1.8) and admissible coverings (B.1.11).

Now it is immediate from 6.4.1 and the definition of the sheaf $\mathcal{O}_X$ as above that the resulting G-ringed space $(X_0, \tau_0, \mathcal{O}_X)$ is an affinoid space in the sense of B.2.4 (1).
Finally, we show that the functor $\mathcal{X} \mapsto \mathcal{X}_0$ restricted to the full subcategory consisting of finite type affinoids over $\mathcal{S}$ is fully faithful (with the essential image being the full subcategory consisting of affinoid spaces over $K$). Consider affinoids $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$ and $\mathcal{Y} = (\text{Spf } B)^{\text{rig}}$, where $A, B$ are topologically finitely generated $V$-algebras. It is well known in classical rigid analytic geometry that morphisms $\mathcal{X}_0 = \text{Spm } \mathcal{A} \to \mathcal{Y}_0 = \text{Spm } \mathcal{B}$ (where $\mathcal{A} = A[1/\alpha]$ and $\mathcal{B} = B[1/\beta]$) of affinoid spaces over $K$ are in canonical one-to-one correspondence with $K$-algebra homomorphisms $\mathcal{B} \to \mathcal{A}$. Based on this fact, the desired full faithfulness follows from A.4.25.

Step 2. Suppose $\mathcal{X}$ is a quasi-compact open subspace of an affinoid. In this case, since $\mathcal{X}$ is the union of a finite collection of affinoids $\mathcal{U}_{\alpha}$ with the property that any intersection $\mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta}$ is again an affinoid, it is easy to see that $(\mathcal{X}_0, \tau_0, \mathcal{O}_{\mathcal{X}_0})$ is actually a rigid analytic space over $K$. One can also show that the functor $\mathcal{X} \mapsto \mathcal{X}_0$ restricted to the full subcategory consisting of rigid spaces of this type is fully faithful; indeed, this follows from the affinoid case by a standard patching argument.

Step 3. Suppose $\mathcal{X}$ is coherent. In this case $\mathcal{X}$ is the union of a finite collection of affinoids $\mathcal{U}_{\alpha}$ with the property that any intersection $\mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta}$ is quasi-compact. Hence the assertion follows from Step 1 and Step 2.

Step 4. If $\mathcal{X}$ is quasi-separated, then by 3.5.3 it is a stretch of coherent rigid spaces (2.2.17 (1)), and thus the assertion follows easily. Then the assertion in the general case also follows in view of the definition of general rigid spaces as in 2.2.18.

Next we show the other assertion, that is, the essential surjectivity. By [18], (9.3.2) and (9.3.3), (by patching of analytic spaces and analytic mappings) we may restrict to the affinoid case. But then the assertion is clear.

The following result is a consequence of the proof.

**Corollary B.2.6.** Under the categorical equivalence in B.2.5, affinoids (resp. separated spaces, resp. coherent spaces) corresponds to affinoid spaces (resp. separated analytic spaces, resp. coherent analytic spaces).

B.2. (e) **Coherent sheaves.** Let $\mathcal{X}$ be a rigid space locally of finite type over $\mathcal{S} = (\text{Spf } V)^{\text{rig}}$, and $\mathcal{X}_0 = (\mathcal{X}_0, \tau_0, \mathcal{O}_{\mathcal{X}_0})$ the associated rigid analytic space. By 8.2.23, the subset $\mathcal{X}_0 = (\mathcal{X})^{\text{cl}}$ of $(\mathcal{X})$ is very dense. Hence, it follows from the definitions of the G-topology $\tau_0$ and the structure sheaf $\mathcal{O}_{\mathcal{X}_0}$ that we have the canonical equivalence between the locally ringed topoi

$$(\mathcal{X}_{\text{ad}}, \mathcal{O}_{\mathcal{X}}) \cong (\mathcal{X}_{0, \text{ad}}, \mathcal{O}_{\mathcal{X}_0})$$
C Appendix: Non-archimedean analytic spaces of Banach type

In this appendix, we will introduce and discuss metrized analytic spaces, and the relationship with Berkovich’s analytic geometry. Our construction is based on a spectral theory of filtered rings (§§C.1–C.4). It will turn out that the new notion of spectra, the so-called valuative spectra, is, under certain conditions, equivalent to that of reified adic spectra introduced by Kedlaya [67], and can be seen as a globalization of Temkin’s work [96]. The interpretation of Berkovich spaces is given in §C.6.

C.1 Seminorms and norms

Let us first recall some basic notions and terminology related to seminormed and normed rings (cf. [18], [11]).

Let $M$ be an abelian group, written additively. A seminorm on $M$ is a function

$$\| \cdot \| : M \rightarrow \mathbb{R}_{\geq 0}, \quad x \mapsto |x|,$$

satisfying the following conditions:

(a) $\| 0 \| = 0$;

(b) for $x, y \in M$, $\| x - y \| \leq \| x \| + \| y \|$. 

A seminorm $\| \cdot \|$ is called a norm if $\{ x \in M : \| x \| = 0 \}$ coincides with $\{0\}$. A seminorm or norm $\| \cdot \|$ is said to be non-archimedean if it satisfies the following condition:

(b)' $\| x - y \| \leq \max\{\| x \|, \| y \|\}$ for any $x, y \in M$.

Two seminorms $\| \cdot \|$ and $\| \cdot \|'$ on $M$ are said to be equivalent if there exist real numbers $C, C' > 0$ such that, for any $x \in A$,

$$\| x \|' \leq C \| x \| \leq C' \| x \|'.$$

The zero seminorm is the seminorm $\| \cdot \|$ such that $\| x \| = 0$ for all $x \in M$. The trivial norm is a norm $\| \cdot \|$ defined for $x \in M$ by

$$\| x \| = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise}. \end{cases}$$
A seminormed group (resp. normed group) is a pair \((M, \| \cdot \|)\) consisting of an abelian group \(M\) and a seminorm (resp. norm) \(\| \cdot \| \) on \(M\).

Let \(A\) be ring. A (ring) seminorm on \(A\) is a seminorm on the underlying additive group of \(A\) such that

(c) \(|1| \leq 1\);
(d) \(|xy| \leq |x| \cdot |y|\) for any \(x, y \in A\).

It follows that \(|1|\) is either 0 or 1, and it is 0 if and only if \(\| \cdot \|\) is the zero seminorm.

A seminorm \(\| \cdot \|\) on a ring \(A\) is said to be power-multiplicative if \(|1| = 1\) and \(|x^n| = |x|^n\) for any \(x \in A\) and \(n \geq 1\). It is multiplicative if \(|1| = 1\) and \(|xy| = |x| \cdot |y|\) for any \(x, y \in A\). Multiplicative norm are called a valuations. If \(\| \cdot \|\) and \(\| \cdot \|^\prime\) are seminorms on \(A\) with \(\| \cdot \|^\prime\) power-bounded, and if there exists a real number \(C > 0\) such that \(|x| \leq C|x|^\prime\) for all \(x \in A\), then \(|x| \leq |x|^\prime\) for all \(x \in A\) (that is, \(C\) can be chosen to be 1). Indeed, for any positive integer \(n\), we have \(|x|^n \leq C|x|^n\), hence \(|x| \leq \sqrt[n]{C}|x|^\prime\), thereby the claim. In particular, two power-bounded seminorms on a ring are equivalent if and only if they are equal.

A seminormed ring (resp. normed ring) is a pair \((A, \| \cdot \|)\) consisting of a ring \(A\) and a seminorm (resp. norm) \(\| \cdot \| \) on \(A\).

For a seminormed group \((M, \| \cdot \|)\) and a surjective homomorphism \(\pi: M \to N\) of abelian groups, the residue seminorm \(\| \cdot \|_{\text{res}}\) on \(N\) induced from \(\| \cdot \|\) is the seminorm on \(N\) defined for \(y \in N\) by

\[|y|_{\text{res}} = \inf\{|x|: \pi(x) = y\}.\]

When \(\pi\) is a ring homomorphism, the residue seminorm induced from a ring seminorm is again a ring seminorm.

Let \((A, \| \cdot \|_A)\) and \((B, \| \cdot \|_B)\) be seminormed groups (resp. seminormed rings), and \(f: A \to B\) a group (resp. ring) homomorphism.

- \(f\) is said to be bounded if there exists a real number \(C > 0\) such that

\[|f(x)|_B \leq C|x|_A\]

for all \(x \in A\). (By an argument similar to the one above, the number \(C\) can be 1, if \(\| \cdot \|_B\) is power-bounded.)

- \(f\) is said to be admissible if the residue seminorm on the image \(f(A)\) induced from \(\| \cdot \|_A\) is equivalent to the restriction of \(\| \cdot \|_B\). Admissible injective (resp. surjective) homomorphisms are often called admissible monomorphisms (resp. admissible epimorphisms).
A *seminormed module* over a seminormed ring \((A, | \cdot |_A)\) is a seminormed group \((M, | \cdot |_M)\) such that

(e) there exists \(C > 0\) such that \(|ax|_M \leq C|a|_A|x|_M\) for all \(a \in A\) and all \(x \in M\).

Let \((M, | \cdot |_M)\) and \((N, | \cdot |_N)\) be non-archimedean seminormed modules over a non-archimedean seminormed ring \((A, | \cdot |_A)\). Then, the tensor product \(M \otimes_A N\), can be endowed with following seminorm, called the *tensor-product seminorm*: for \(x \in M \otimes_A N\),

\[
|x| = \inf \left\{ \max_{1 \leq i \leq n} |m_i|_M |n_i|_N \mid x = \sum_{i=1}^{n} m_i \otimes n_i, \; m_i \in M, \; n_i \in N \right\}.
\]

The completion of \(M \otimes_A N\) with respect to the tensor-product seminorm will be denoted by

\[
M \hat{\otimes}_A N,
\]

and called the *complete tensor product*.

A *Banach ring* is a normed ring \((A, | \cdot |)\) such that \(A\) is complete with respect to the metric defined by the norm. Note that the zero ring \(\{0\}\) with the zero map as its norm is a Banach ring, which can be characterized as a Banach ring \((A, | \cdot |_A)\) such that \(|1|_A < 1\). Moreover, any ring \(A\) with the trivial norm \(| \cdot |_{\text{triv}}\) is a Banach ring.

### C.2 Graded valuations

**C.2. (a) Graded rings and modules.** Many notions on rings and modules admit ‘graded versions,’ which generalize those of the classical theory of rings and modules. Here we recall some of the basic material on graded rings and modules, especially on graded fields and graded valuation rings (cf. [86], [87], [96]).

Let \(\Delta\) be an abelian group, written multiplicatively, with the unit element \(1 \in \Delta\). A \(\Delta\)-*graded ring* is a commutative ring with \(1 = 1_G \in G\) of the form

\[
G = \bigoplus_{d \in \Delta} G_d,
\]

where \(G_d\) for \(d \in \Delta\) are additive subgroups of \(G\) such that \(G_d \cdot G_{d'} \subseteq G_{dd'}\) for any \(d, d' \in \Delta\). Similarly, a \(\Delta\)-*graded \(G\)-module* is a \(G\)-module of the form \(M = \bigoplus_{d \in \Delta} M_d\) satisfying \(G_d \cdot M_{d'} \subseteq M_{dd'}\). Note that, in this situation, the unit-element part \(G_1\) is a commutative ring, and all pieces \(G_d\) and \(M_d\) for \(d \in \Delta\) are \(G_1\)-modules; in fact, one can easily show that \(1_G \in G_1\) (cf. [87], 1.1.1). Note also that, for a graded ideal \(I = \bigoplus_{d \in \Delta} I_d\) of \(G, I = G\) if and only if \(I_1 = G_1\).

Notice that, if \(\Delta = \{1\}\) (the trivial group), then all that follows throughout this subsection reduces to the classical theory of rings and modules.
For $\Delta$-graded rings $G, G'$, a homomorphism $f: G \to G'$ of $\Delta$-graded rings is defined to be a ring homomorphism that preserves the grading. One can similarly define morphisms of $\Delta$-graded $G$-modules. One has thus categories of $\Delta$-graded rings and modules. If $\Delta'$ is a subgroup of $\Delta$, then by restriction of the grading one obtains a functor from the category of $\Delta$-graded rings to the category of $\Delta'$-graded rings. A left adjoint to this functor is given by ‘0-extension’ of the gradings, that is, to any $\Delta'$-graded ring $G'$, we associate $G = \bigoplus_{d \in \Delta} G_d$ with

$$G_d = \begin{cases} G_d & \text{if } d \in \Delta', \\ \{0\} & \text{otherwise} \end{cases}$$

(cf. [87], 1.1.2).

A $\Delta$-graded integral domain is a non-zero $\Delta$-graded ring $G = \bigoplus G_d$ such that the product $ab$ of any non-zero homogeneous elements $a, b \in G$ is non-zero. A $\Delta$-graded field is a non-zero $\Delta$-graded ring $G$ such that any non-zero homogeneous element is invertible. Note that a $\Delta$-graded integral domain (resp. $\Delta$-graded field) $G$ is not necessarily an integral domain (resp. a field), but the unit-element part $G_1$ is always an integral domain (resp. a field) in the usual sense.

A homogeneous ideal $p \subseteq G$ of a $\Delta$-graded ring $G$ is prime if $p \neq G$ and the set $S_p$ of all homogeneous elements of $G \setminus p$ is closed under multiplication, or equivalently, the quotient $\Delta$-graded ring $G/p = \bigoplus_{d \in \Delta} G_d/(p \cap G_d)$ is a $\Delta$-graded integral domain. A maximal element in the set of all homogeneous proper ideals in $G$ is called a maximal homogeneous ideal. It follows that a homogeneous ideal $m \subseteq G$ is maximal if and only if $G/m$ is a $\Delta$-graded field.

By a standard argument using Zorn’s lemma, one can show that any non-zero $\Delta$-graded ring has a maximal homogeneous ideal.

Let $G$ be a $\Delta$-graded ring, and $m \subseteq G$ a homogeneous ideal. Then $m$ is the unique maximal homogeneous ideal of $G$ if and only if $1 \not\in m$ and any non-invertible homogeneous element of $G$ belongs to $m$. In this situation, $G$ is said to be local. In other words, a local $\Delta$-graded ring is a $\Delta$-graded ring that has a unique maximal homogeneous ideal.

Let $G$ be a $\Delta$-graded ring, and $S \subseteq G$ a multiplicative subset consisting of homogeneous elements. Then one has the natural grading by $\Delta$ on the localization $G_S = S^{-1}G$ given by $(G_S)_d = \sum_{d' = d} (S \cap G_{d'}^{-1}) G_{d'}$ for $d \in \Delta$ (cf. [87], 8.1). For example, if $p \subseteq G$ is a prime homogeneous ideal, then one has the set $S_p$ as above, and one can form the localization $S_p^{-1}G$, which we denote by $G_p$. Notice that $G_p$ is a local $\Delta$-graded ring, in which $pG_p$ is the unique maximal homogeneous ideal. In particular, if $G$ is a $\Delta$-graded integral domain, then $G_{\langle 0 \rangle}$, the graded localization by the prime homogeneous ideal (0), is a $\Delta$-graded field, called the graded fraction field, and denoted by $\text{Frac}_\Delta(G)$.
C.2. (b) **Graded valuation rings.** Let $\Delta$ be an abelian group. We only consider gradings by $\Delta$, and drop `$\Delta$' from the notation.

**Definition C.2.1** ([96], §1). (1) Let $K$ be a graded field. A graded subring $V$ of $K$ is called a **graded valuation ring for $K$** if for any homogeneous non-zero $x \in K$, either $x$ or $x^{-1}$ is contained in $V$.

(2) A graded valuation ring is a graded integral domain $V$ that is a graded valuation ring for its graded fraction field $K = \text{Frac}_\Delta(V)$.

Notice that, in the situation as in (1), any graded subring $W$ of $K$ that contains $V$ is again a graded valuation ring. Notice also that, if $V$ is a graded valuation ring for $K$, then the unit-element part $V_1$ is a valuation ring for $K_1$ (in the usual sense; see 0, §6.2. (a)).

Let $(G_1, m_{G_1})$ and $(G_2, m_{G_2})$ be local graded subrings of a graded field $K$. We say that $G_1$ dominates $G_2$ if $G_2 \subseteq G_1$ and $m_{G_2} = m_{G_1} \cap G_2$.

**Proposition C.2.2.** Let $K$ be a graded field, and $V$ a graded subring of $K$. Then the following conditions are equivalent:

(a) $V$ is a graded valuation ring for $K$;

(b) $\text{Frac}_\Delta(V) = K$, and the set of all homogeneous principal ideals of $V$ is totally ordered with respect to inclusion;

(c) $\text{Frac}_\Delta(V) = K$, and the set of all homogeneous ideals of $V$ is totally ordered with respect to inclusion;

(d) $V$ is local, and $(V, m_V)$ is its maximal element with respect to the order by domination in the set of all local graded subrings of $K$.

We provide the proof, which mimics the proof of the classical case (cf. [27], Chapter VI, Theorem 1), for the reader’s convenience.

**Proof.** (a) $\implies$ (b). For two non-zero principal homogeneous ideals $aV, bV \subseteq V$, we have $aV \subseteq bV$ or $bV \subseteq aV$, according to whether $a/b \in V$ or $b/a \in V$.

(b) $\implies$ (c). Let $I, J \subseteq V$ be two homogeneous ideals such that $I \not\subseteq J$. Take a homogeneous $a \in I \setminus J$. For any non-zero $b \in J$, either $a/b$ or $b/a$ belongs to $V$. But $b/a \in V$ has to hold, since $a \not\in J$. Hence $b = a \cdot (b/a) \in I$, which shows that $J \subseteq I$.

(c) $\implies$ (a). For any non-zero homogeneous $a/b \in K$, where $a, b$ are homogeneous elements in $V$, we have $a/b \in V$ or $b/a \in V$, according to whether $aV \subseteq bV$ or $bV \subseteq aV$. 

(c) $\implies$ (d). Take a maximal homogeneous ideal $m$ of $V$, which is unique due to the assumption. Hence, $V$ is a local graded ring. Let $W$ be a local graded subring of $K$ that dominates $V$. For any non-zero homogeneous $x \in W$, either $x \in V$ or $x^{-1} \in V$. Since $m_V W \subseteq m_W \neq W$ and $x^{-1} \not\in m_W$, we have $x \in V$, which implies $W = V$.

(d) $\implies$ (a). Let $x \in K \setminus V$ be a homogeneous element. We want to show that $x^{-1} \in V$. We first show that $x$ is not integral over $V$. Indeed, if $x$ is integral over $V$, then the graded subring $W = V[x]$ of $K$ is finite over $V$. One can take a maximal ideal $m$ (in the usual sense) of $V$ containing $m_V$; since $m_W \neq W$, we have $m_V W \neq W$. (The last inequality also follows from the graded version of Nakayama’s lemma; see [86], 8.4.) Thus there exists a maximal homogeneous ideal $m'$ of $W$ that contains $m_V W$. Since $W_{m'}$ dominates $V$, we have $x \in W_{m'} = V$ by the assumption, a contradiction.

Thus $x$ is not integral over $V$, and hence by [27], Chapter VI, §1.2, Lemma 1, the ideal of $V[x^{-1}]$ generated by $m_V$ and $x^{-1}$ is not equal to $V[x^{-1}]$ itself, and hence there exists a maximal homogeneous ideal $m''$ of $V[x^{-1}]$ that dominates $V$. Therefore, $x^{-1} \in V[x^{-1}]$ implies $V$, as desired.

The last part of the above proof shows that the following statement holds.

**Proposition C.2.3.** Let $V$ be a graded valuation ring for $K = \text{Frac}_\Delta(V)$. Then $V$ is graded integrally closed, i.e., any homogeneous element $x \in K$ integral over $V$ lies in $V$.

Finally, let us state the existence of graded valuation rings that dominate given local graded rings.

**Proposition C.2.4.** Let $K$ be a graded field, and $A \subseteq K$ a local graded subring. Then there exists a graded valuation ring $V$ for $K$ that dominates $A$.

**Proof.** By a standard argument using Zorn’s lemma, one can show that there exists a maximal local graded subring $V \subseteq K$ that dominates $A$. By C.2.2, $V$ is a graded valuation ring. □

C.2. (c) **Graded valuations.** For a graded ring $G$, let us denote the set of all homogeneous elements by

$$h(G) = \bigcup_{d \in \Delta} G_d.$$
Definition C.2.5. Let $K$ be a graded field, and $\Gamma$ a totally ordered commutative group $(0, \S 6.1.(a))$. A graded valuation $v$ of $K$ with value target group $\Gamma$ is a mapping
\[ v: h(K) \longrightarrow \Gamma \cup \{ \infty \} \]
which maps non-zero elements to elements in $\Gamma$ and satisfies the following conditions:

(a) $v(xy) = v(x) + v(y)$ for all $x, y \in h(K)$;

(b) $v(x + y) \geq \inf\{v(x), v(y)\}$ for all $x, y \in h(K)$ of the same degree;

(c) $v(1) = 0$ and $v(0) = \infty$.

If $v$ is a graded valuation of $K$, then, for $d \in \Delta$,
\[ V_d = \{ x \in K_d : v(x) \geq 0 \} \]
is a subgroup of $K_d$ (due to (b) and (c)), and $V = \bigoplus_{d \in \Delta} V_d$ is a graded subring of $K$ (due to (a)). One deduces, using the conditions (a), (b), and (c), that, for any non-zero homogeneous $x \in K$, we have $v(x) \geq 0$ or $v(x^{-1}) \geq 0$. Hence, $V$ is a graded valuation ring for $K$. The maximal homogeneous ideal of $V$ is the homogeneous ideal generated by $\{ x \in h(K) : v(x) > 0 \}$.

Conversely, for a graded valuation ring $V$, one has a graded valuation $v$ of the graded fraction field $K = \text{Frac}_\Delta(V)$, which gives back, in the manner described above, the given $V$ up to isomorphism; the construction is as follows. Let $U(K)$ be the multiplicative group of the non-zero homogeneous elements of $K$, and $U(V)$ the multiplicative group of the invertible homogeneous elements of $V$. By C.2.2 (b), one can show that
\[ \Gamma_V = U(K)/U(V) \]
is a totally ordered commutative group with the property that the image of $h(V) \setminus \{ 0 \}$ is the set of all positive elements. Then the mapping
\[ v: h(K) \longrightarrow \Gamma_V \cup \{ \infty \}, \quad v(x) = \begin{cases} \left[ x \right] (= x \mod U(V)) & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{cases} \]
is a graded valuation of $K$. It is clear that $V$ is the ‘non-negative’ part; that is, $V_d = \{ x \in K_d : v(x) \geq 0 \}$ for all $d \in \Delta$.

As in the classical case, one can define the notion of equivalence of graded valuations in such a way that, by the above correspondence, the set of all equivalence classes of graded valuations of $K$ is mapped bijectively onto the set of all graded valuation rings for $K$. 
(d) **Generization and specialization of graded valuations.** Similarly to the classical case, we have the following result.

**Proposition C.2.6.** Let $V$ be a graded valuation ring for a graded field $K$. Then there exist canonical order-preserving bijections among the following sets:

1. the set of all homogeneous prime ideals of $V$ with the inclusion order;
2. the set of all graded subrings ($\neq V$) lying between $V$ and $K$ (which are automatically graded valuation rings) with the reversed inclusion order;
3. the set of all proper isolated subgroups of $\Gamma_V$ with the reversed inclusion order.

The bijections are described similarly to the ones in 0.6.2.9.

**Corollary C.2.7.** Let $V$ be a graded valuation ring for a graded field $K$. Then any graded subring $W$ of $K$ containing $V$ is a graded valuation ring for $K$, and the set of all such subrings is totally ordered with respect to inclusion.

Based on these results, we can introduce the following notions similarly to the classical case.

**Definition C.2.8.** Let $K$ be a graded field.

1. For two graded valuation rings $V, W$ for $K$, we say $V$ is a **specialization** of $W$, or $W$ is a **generization** of $V$, if $V \subseteq W$.

2. The **height** of a graded valuation ring $V$, denoted by $\text{ht}_\Delta(V)$, is the height of the value group $\Gamma_V$.

Note that $\text{ht}_\Delta(V) = 0$ if and only if $V = K$.

(e) **Unit-element part.** As mentioned before, the unit-element part $V_1$ of a graded valuation ring $V$ for the graded field $K$ is a valuation ring (in the usual sense) for the unit-part $K_1$ of $K$. The maximal ideal $m_{V_1}$ of $V_1$ is equal to $m_V \cap V_1$.

In general, for a $\Delta$-graded field $F$, we set

$$\Delta_F = \{d \in \Delta: F_d \neq \{0\}\}.$$  

Then $\Delta_F$ is a subgroup of $\Delta$ (since $F$ is a graded field). Note that, if $d \in \Delta_F$, we have $F_d \cdot F_{d-1} = F_1$, and $F_d$ is a one dimensional vector space over $F_1$. 
Proposition C.2.9. Let $V$ be a graded valuation ring for a graded field $K$, and $k = V/m_k$ the graded residue field of $V$.

(1) We have
\[ \Delta_k = \{ d \in \Delta : V_d \cdot V_{d-1} = V_1 \}. \]

In particular, $\Delta_k$ is a subgroup of $\Delta_K$ (which may not coincide with $\Delta_K$).

(2) For any $d \in \Delta_k$, $V_d$ is a free $V_1$-module of rank one.

(3) For any $d \in \Delta$,
\[ (m_V)_d = \begin{cases} m_{V_1}V_d & \text{if } d \in \Delta_k, \\ V_d & \text{if } d \notin \Delta_k. \end{cases} \]

Proof. (1) If $d \in \Delta_k$, we have $(m_V)_d = m_V \cap V_d \neq V_d$, and hence there exists $u \in V_d$ that is invertible in $V$. For any $x \in V_1$, we have $x = (xu) \cdot u^{-1} \in V_d \cdot V_{d-1}$, hence $V_1 = V_d \cdot V_{d-1}$. Conversely, if $V_d \cdot V_{d-1} = V_1$, there exist $u \in V_d$ invertible in $V$, and hence $k_d = V_d/(m_V \cap V_d) \neq 0$.

(2) Since $m_{V_1}V_d \subseteq m_V \cap V_d$, we have $m_{V_1}V_d \neq V_d$ for $d \in \Delta_k$. Take $x \in V_d \setminus m_{V_1}V_d$. Let us show that $x$ freely generates $V_d$. For $y \in V_d$, we have $y = ux$ or $x = uy$ for some $u \in V_1$ according to whether $y/x \in V$ or $x/y \in V$. If $y \in V_d \setminus m_{V_1}V_d$, then $u$ must be invertible in $V$, and thus $y \in xV_1$. If $y \in m_{V_1}V_d$, then $y = ux$ should hold, and hence $y \in xV_1$. Now, since $V$ is a graded integral domain, $V_d$ is $V_1$-torsion free. Hence $V_d$ is freely generated by $x$.

(3) It suffices to show that $m_V \cap V_d = m_{V_1}V_d$ for $d \in \Delta_k$. The inclusion $m_{V_1}V_d \subseteq m_V \cap V_d$ is clear. In the proof of (2), we have shown that $V_d/m_{V_1}V_d$ is, as a vector space over $k_1 = V_1/m_{V_1}$, spanned by the residue class of $x$; as it is non-zero (since $d \in \Delta_k$), it has to be of dimension one. Hence the surjection $V_d/m_{V_1}V_d \to V_d/(m_V \cap V_d)$ is an isomorphism, which shows that $m_V \cap V_d = m_{V_1}V_d$, as claimed.

Definition C.2.10. Let $V$ be a graded valuation ring for a graded field $K$, and $k = V/m_k$ the graded residue field of $V$. We say $V$ is non-degenerate if the equality $\Delta_k = \Delta_K$ holds.

As a corollary of C.2.9, we have the following result.

Corollary C.2.11. In the setting C.2.9, $V$ is non-degenerate if and only if $m_{V_1}V = m_V$. 
Proposition C.2.12. Let $V$ be a graded valuation ring for a graded field $K$. Suppose $V$ is non-degenerate.

(1) Any generization $W$ of $V$ is also non-degenerate.

(2) The canonical map $V_d \otimes_{V_1} V_{d'} \to V_{dd'}$ is an isomorphism, whenever $V_d$ and $V_{d'}$ are non-zero.

Proof. (1) Any generization $W$ of $V$ is of the form $W = V_p$ for a homogeneous prime ideal $p \subseteq V$. The graded residue field $k_W$ of $W$ is then the graded fraction field of $V = p$. For $d \in \Delta$, $(k_V)_d \neq 0$ implies $(V/p)_d \neq 0$, and hence $(k_W)_d \neq 0$.

(2) By C.2.9 (2), it suffices to show that $V_d \otimes_{V_1} V_{d'} = V_{dd'}$ is surjective. This follows from the equality $m_{V_1} V = m_V$ (C.2.11) and the fact that $V/m_V = k_V$ is a graded field.

Remark C.2.13. Note that a graded valuation ring $V$ for a graded field $K$ is non-degenerate if and only if it enjoys the following properties:

(a) each $V_d$ for $d \in \Delta_K$ is a free $V_1$-module;

(b) for any $d, d' \in \Delta_K$, $V_d \otimes_{V_1} V_{d'} \cong V_{dd'}$.

Indeed, the ‘only if’ part was already shown in C.2.9 (2) and C.2.12(2). Conversely, if (a) and (b) hold, then for any $d \in \Delta_K$, one can take the free generator $x$ of $V_d$ as a $V_1$-module. The condition (b) with $d = d^{-1}$ implies that $x$ is invertible in $V$. Then, for any $ax \in (m_V)_d$, since $v(ax) = v(a) > 0$, we have $a \in m_{V_1}$, showing that $(m_V)_d = m_{V_1} V_d$. Hence $V$ is non-degenerate by C.2.11.

Proposition C.2.14. Let $V$ be a graded valuation ring for a graded field $K$. Suppose $V$ is non-degenerate. Then the inclusion $K_1^\times \hookrightarrow U(K)$ induces an isomorphism

$$i : K_1^\times / V_1^\times \sim \Gamma_V$$

(see §C.2.(c) for the notation.)

Notice that, since $U(V) \cap K_1^\times = V_1^\times$, the map in question is always injective. But, as the following proof indicates, non-degeneracy of $V$ is essential for $i$ to be surjective.

Proof. To prove that $i$ is surjective, it suffices to show that the mod $U(V)$ class of any non-zero $x \in V_d$ with $d \in \Delta_K$ is contained in the image of $i$. Since $d \in \Delta_K = \Delta_d$, one can take a free generator $u$ of $V_d$ as a $V_1$-module (see C.2.9 (2)). Note that $u$ is invertible in $V$, that is, $u \in U(V)$. Then $x/u \in K_1^\times$, and its modulo $V_1^\times$ class does not depend on the choice of $u$. Thus, $x$ mod $U(V)$ is contained in the image of $i$. \qed
By 0.6.2.9, C.2.6, and C.2.14, we have the following result.

**Corollary C.2.15.** Let $V$ be a graded valuation ring for a graded field $K$. Suppose $V$ is non-degenerate.

1. The mapping $p \mapsto p_1 = p \cap V_1$ gives a bijection from the set of all homogeneous prime ideals of $V$ to the set of prime ideals of $V_1$.

2. The mapping $W_1 \mapsto W_1 \otimes_{V_1} V = \bigoplus_{d \in \Delta} (W_1 \otimes_{V_1} V_d)$ gives a bijection from the set of all valuation rings for $K_1$ containing $V_1$ to the set of all graded valuation rings for $K$ containing $V$.

C.2. (f) The space of graded valuations. We continue the discussion of graded objects with the grading by a fixed abelian group $\Gamma$. Let $A$ be a graded ring, and $K$ a graded $A$-algebra that is a graded field; notice that we do not assume that the homomorphism $A \to K$ is injective. Define

$$
ZR(K, A) = \left\{ V \subseteq K \mid V \text{ is a graded valuation ring for } K, \text{ and is a graded } A\text{-subalgebra of } K \right\}.
$$

For any graded $A$-subalgebra $B$ of $K$, set

$$
U(B) = ZR(K, B),
$$

which is a subset of $ZR(K, A)$. The subsets of the form $U(B)$ are closed under finite intersection. Indeed, if $B_1, B_2$ are graded $A$-subalgebras of $K$, then

$$
U(B_1) \cap U(B_2) = U(B),
$$

where $B$ is the graded $A$-subalgebra of $K$ generated by (homogeneous generators of) $B_1$ and $B_2$.

Note that, in case $\Delta = \{1\}$ and $A$ is a subring of $K$, then $ZR(K, A)$ is the classical Zariski-Riemann space associated to $X = \text{Spec } A$ (see E.2.3 below). In general, similar to the classical case, we consider the topology on $ZR(K, A)$ generated by subsets of the form $U(B)$ with $B$ of finite type (viz., generated by finitely many homogeneous elements) over $A$.

**Theorem C.2.16** (cf. [96], §2). The topological space $ZR(K, A)$ is a coherent valuative space (see 0.2.3.1 for the definition of valuative spaces).

**Proof.** Set $X = ZR(K, A)$, and consider the set $h(K)$ of homogeneous elements of $K$. For any non-zero $f \in h(K)$, the two open sets $U(A[f])$ and $U(A[f^{-1}])$ cover $X$. More precisely, one has

$$
X \setminus U(A[f]) = \{ V \in U(A[f^{-1}]) \mid f^{-1} \in m_V \}.
$$
For any \( f \in h(K) \), set \( C(f) = X \setminus U(A[f]) \), and for any family of elements \( \{ f_{\alpha} \}_{\alpha \in L} \) of \( h(K) \), set
\[
C(\{ f_{\alpha} \}_{\alpha \in L}) = \bigcap_{\alpha \in L} C(f_{\alpha}).
\]
If none of the \( f_{\alpha} \)'s is zero, let \( J \) be the ideal of \( B = A[f_{\alpha}: \alpha \in L] \) generated by \( f_{\alpha}^{-1} \) for all \( \alpha \in L \). It follows from the above description that
\[
C(\{ f_{\alpha} \}_{\alpha \in L}) = \begin{cases}
\{ V \in X : J \subseteq \mathfrak{m}_V \} & \text{if } f_{\alpha} \neq 0 \text{ for all } \alpha \in L, \\
\emptyset & \text{otherwise.}
\end{cases}
\]
In particular, in view of C.2.4, the family of open subsets \( \{ U(A[f_{\alpha}]) \}_{\alpha \in L} \) covers \( X \) if and only if \( J = B \).

Now let us show that \( X \) is quasi-compact. Take a family \( \{ C_{\alpha} \}_{\alpha \in L} \) of closed subsets of \( X \) with the finite intersection property, that is, \( \bigcap_{\alpha \in L'} C_{\alpha} \neq \emptyset \) for any finite subset \( L' \subseteq L \). We want to show that \( \bigcap_{\alpha \in L} C_{\alpha} \) is non-empty. Assume the contrary. As in [24], Chap. I, §9, Exer. 1, we may reduce to the case where \( C_{\alpha} = C(f_{\alpha}) \) while \( f_{\alpha} \in h(K) \) for \( \alpha \in L \). Then, by what we have seen above, we have \( f_{\alpha} \neq 0 \) for all \( \alpha \in L \), and we have the relation
\[
1 = \sum_{\alpha \in L'} c_{\alpha} f_{\alpha}^{-1}
\]
for a finite subset \( L' \subset L \) and \( c_{\alpha} \in A[f_{\alpha}^{-1}: \alpha \in L'] \). But this implies \( \bigcap_{\alpha \in L'} C_{\alpha} = \emptyset \), contradicting the assumed finite intersection property. It follows that \( X \) is quasi-compact. Moreover, since this implies that each \( U(B) \), where \( B \) is a finite-type graded \( A \)-subalgebra of \( K \), is quasi-compact, we have also shown that \( X \) has an open basis consisting of quasi-compact open subsets.

It is then easy to see that \( X \) is quasi-separated. Indeed, it suffices to show that the intersection of open subsets \( U(B_1) \cap U(B_2) \), where \( B_1, B_2 \) are finite type graded \( A \)-subalgebras of \( K \), is quasi-compact, which follows from what we have seen in (*) above.

So far, we have shown that \( X \) is a coherent topological space. Next, let us show that \( X \) is sober. It is easy to see that \( X \) is a \( T_0 \)-space (0, §2.1. (b)). Let \( Z \) be an irreducible closed subset of \( X \). Let \( F_Z \) be the set of all open subsets \( U \) of \( X \) such that \( U \cap Z \neq \emptyset \). Then, since \( Z \) is irreducible, \( F_Z \) is a prime filter (see 0, §2.2. (b)). Now, referring to the fact that \( F_Z \) has to be generated by open subsets of the form \( U(B) \) for a finite type graded \( A \)-subalgebra \( B \) of \( K \), we define \( V \) to be the union of all such \( B \)'s such that \( U(B) \in F_Z \). Since \( U(A[f]) \cap Z \neq \emptyset \) for any \( f \in B \) with \( U(B) \in F_Z \), it follows that \( V \) is a graded \( A \)-subalgebra of \( K \). We claim that \( V \) is a graded valuation ring for \( K \). Indeed, as we have seen above, any non-zero \( f \in h(K) \) gives an open covering \( \{ U(A[f]), U(A[f^{-1}]) \} \) of \( X \), and hence at least one of \( f \) and \( f^{-1} \) belongs to \( V \). Since the point in \( X \) corresponding to \( V \) belongs
to all open subsets $U$ such that $U \cap Z \neq \emptyset$, it is the unique generic point of $Z$. Thus, we have shown that $X$ is a sober space.

Finally, let us remark that, to show that $X$ is valuative, we only have to invoke C.2.7, for it asserts that the set of generizations of a point in $X$ is totally ordered; see 0.2.3.2 (1).

C.3 Filtered valuations

Throughout this subsection, we take $\Delta = \mathbb{R}_+$, the multiplicative group of positive real numbers, and all gradings and filtrations are indexed by $\mathbb{R}_+$, often without mentioning it.

C.3. (a) Filtered rings. In the sequel, we will consider filtered objects indexed by $\mathbb{R}_+$. For each filtered object $(M, F)$, where $F = \{F_r\}_{r > 0}$ is an increasing (viz., $F_r \subseteq F_{r'}$ for $r \leq r'$) filtration by subobjects of $M$ indexed by $\mathbb{R}_+$, the associated graded object $\text{Gr}_F M$ is defined by

$$\text{Gr}_F M = \bigoplus_{r \in \mathbb{R}_+} F_r / F_{<r},$$

where $F_{<r} = \bigcup_{r' < r} F_{r'}$.

For $f \in F_r$, we set

$$[f]_r = (f \mod F_{<r}),$$

which is an element of $\text{Gr}_F r M = F_r / F_{<r}$.

Notation C.3.1. For two filtration $F = \{F_r\}_{r > 0}$ and $F' = \{F'_r\}_{r > 0}$, we will often write

$$F \subseteq F',$$

or say ‘$F$ is contained in $F'$,’ to mean that $F_r \subseteq F'_r$ for any $r > 0$.

Definition C.3.2 (cf. [86], Chapter I, 1.1). (1) An $\mathbb{R}_+$-filtered ring, or filtered ring for short, is a pair $(A, F)$ consisting of a (commutative) ring $A$ with unit $1 = 1_A$ and an ascending filtration $F$ by additive subgroups satisfying the following conditions:

(a) $F$ is $\mathbb{R}_+$-multiplicative, that is, $1 \in F_1$, and $F_r \cdot F_{r'} \subseteq F_{rr'}$ for $r, r' > 0$;

(b) $F$ is exhaustive (cf. 0, §7.1. (a)), that is, $\bigcup_{r > 0} F_r = A$.

(2) A morphism $f : (A, F_A) \rightarrow (B, F_B)$ between filtered rings is a ring homomorphism $f : A \rightarrow B$ such that $f((F_A)_r) \subseteq (F_B)_r$ for any $r > 0$. In this situation, as usual, we often say that $(B, F_B)$ is an $(A, F_A)$-algebra.
Given a filtered ring $(A, F)$, one can endow $A$ with the linear topology defined by the filtration $F = \{F_r\}_{r \in \mathbb{R}_+}$ (see 0, §7.1. (b)). We call such a topological ring an $\mathbb{R}_+$-linearly topologized ring. For an $\mathbb{R}_+$-linearly topologized ring $A$, a filtration of definition is a multiplicative and exhaustive filtration $F$ on $A$ for which the induced linear topology coincides with the topology of $A$.

It can be shown that
\begin{itemize}
  \item if $F_1, F_2$ are filtrations of definition and $F_1 \subseteq F \subseteq F_2$, then $F$ is also a filtration of definition;
  \item if $F_1, F_2$ are filtrations of definition, then $F_1 \cap F_2 = \{(F_1)_r \cap (F_2)_r\}_{r \in \mathbb{R}_+}$ is also a filtration of definition.
\end{itemize}

We say that two filtration of definition $F$ and $F'$ are equivalent if there exist positive numbers $c, c' \geq 0$ such that $F_r \subseteq F'_r$, and $F'_r \subseteq F_r$ for all $r > 0$.

Let $(A, F)$ be a filtered ring. We set
$$F_0 = \bigcap_{r > 0} F_r.$$ 
We say that $(A, F)$ is separated if $F_0 = \{0\}$.

Let $(A, F)$ be a filtered ring, and $I \subseteq A$ an ideal. Then $A/I$ has the induced filtration $\hat{F}$ (cf. 0, §7.1. (a)), defined by $F_r = (F_r + I)/I$ for $r > 0$, which makes the pair $(A/I, \hat{F})$ an $(A, F)$-algebra.

A filtered multiplicative system of $(A, F)$ is a collection $S = \{S_r\}_{r > 0}$ of subsets of $A$ such that the following conditions are satisfied:
\begin{itemize}
  \item $S_r \subseteq F_r$ for all $r > 0$;
  \item $1 \in S_1$, and $S_r S_{r'} \subseteq S_{rr'}$ for all $r, r' > 0$.
\end{itemize}
In this situation, one has the induced filtration $S^{-1} F$ on $S^{-1} A$ given by
$$(S^{-1} F)_r = \sum_{r' > 0} S^{-1}_{r'} F_{rr'},$$
for $r > 0$, and thus we obtain an $(A, F)$-algebra $(S^{-1} A, S^{-1} F)$. Note that the canonical map
$$(\bar{S})^{-1} \text{Gr}_F A \to \text{Gr}_{S^{-1} F} S^{-1} A,$$
where $\bar{S}$ is the multiplicative subset of $\text{Gr}_F A$ induced from $S$, is an isomorphism of graded rings.

The completion of a filtered ring $(A, F)$ is the filtered ring $(\hat{A}, \hat{F})$ given by
$$\hat{A} = \lim_{r > 0} A/F_r, \quad \hat{F}_r = \lim_{r \to r'} F_r/F_{r'}, \quad \text{for } r > 0.$$ 
There exists a canonical homomorphism $(A, F) \to (\hat{A}, \hat{F})$ of filtered rings. If this morphism is an isomorphism, we say that $(A, F)$ is complete.
More generally, if $F'$ is another filtration on $A$ with $F \subseteq F'$, one has the induced filtration $F_F'$ on $\hat{A}$ given by $(F_F')_r = \lim_{r' \to r} F_{r'} / F_r$, for $r > 0$.

If $(B_1, F_1)$ and $(B_2, F_2)$ are $(A, F)$-algebras, then $B_1 \otimes_A B_2$ has the filtration $\tilde{F}$ defined by

$$\tilde{F}_r = \sum_{s,t = r} \text{image}((F_1)_s \otimes (F_2)_t \to B_1 \otimes_A B_2)$$

for any $r > 0$, which gives rise to a filtered ring $(B_1 \otimes_A B_2, \tilde{F})$ sitting in the canonical commutative square

$$(B_1, F_1) \to (B_1 \otimes_A B_2, \tilde{F})$$

$$(A, F) \to (B_2, F_2).$$

The filtration $\tilde{F}$ is called the tensor-product filtration. The completion of $(B_1 \otimes_A B_2, \tilde{F})$, denoted by

$$(B_1 \otimes_A B_2, F_1 \otimes_A F_2),$$

is called the complete tensor product of $(B_1, F_1)$ and $(B_2, F_2)$ over $(A, F)$.

**C.3. (b) Filtrations and seminorms.** Let $(A, F)$ be a filtered ring. Define a new filtration $F^+$ by

$$F^+_r = \bigcap_{r' > r} F_{r'}, \quad \text{for any } r > 0.$$  

The resulting object $(A, F^+)$ is a filtered ring, which is a filtered $(A, F)$-algebra. Moreover, we have

$$F^+_r = F_{<r}, \quad \text{for any } r > 0,$$

and hence the induced morphism $\text{Gr}_F A \to \text{Gr}_{F^+} A$ is always injective, viz., we can regard $\text{Gr}_F A$ as a subring of $\text{Gr}_{F^+} A$.

Let $(A, F)$ be a filtered ring. Since the filtration $F$ is exhaustive, one can define a function $\nu: A \to \mathbb{R}_{\geq 0}$ by

$$\nu(x) = \inf\{r: x \in F_r\}$$

for $x \in A$. Since $F$ is multiplicative, one can easily show that the mapping $\nu$ is a non-archimedean seminorm (§C.1) on the ring $A$. It is a norm if and only if $(A, F)$ is separated, and is the trivial norm if and only if $F$ is trivial; here, we say that the filtration $F$ is trivial if $F_r = \{0\}$ for $0 < r < 1$ and $F_r = A$ for $r \geq 1$. 


Conversely, if we are given a ring $A$ and a non-archimedean seminorm $\nu$ on $A$, one can define the filtration $F_\nu$ by

$$(F_\nu)_r = \nu^{-1}([0, r]) = \{ f \in A : \nu(f) \leq r \}$$

for $r > 0$, which makes the pair $(A, F_\nu)$ a filtered ring. It is easy to see that, starting from a filtered ring $(A, F)$, the resulting filtration $F_\nu$, where $\nu$ is the associated seminorm of $F$, coincides with $F^+$, and that this gives rise to the bijection between the following sets:

- the set of all filtrations $F$ on $A$ such that $F = F^+$;
- the set of all non-archimedean seminorms $\nu$ on $A$.

In this way, one sees that the notion of filtered rings gives a refinement of the notion of non-archimedean seminormed rings. Let us say that a filtration $F$ in a filtered ring $(A, F)$ is of seminorm type if $F = F^+$.

If $F$ is a filtration of definition of an $\mathbb{R}_+$-linearly topologized ring $A$, then so is $F^+$, and $F$ and $F^+$ are equivalent. Indeed, we have $F \subseteq F^+$ and $(F^+_r)_{r > 0} \subseteq F_{cr}$ for any $c > 1$ and $r > 0$. Thus, it follows that equivalence classes of filtrations of definition are in one-to-one correspondence with the equivalence classes of seminorms on $A$.

Note that the condition $f((F_A)_r) \subseteq (F_B)_r$ for filtered morphisms as in C.3.2 (2) implies $\nu_B(f(x)) \leq \nu_A(x)$ for any $x \in A$, where $\nu_A$ and $\nu_B$ are the associated seminorms of $F_A$ and $F_B$, respectively. In particular, filtered homomorphisms induce bounded homomorphisms between the corresponding seminormed rings.

Note also that the completion $(\hat{A}, \hat{F})$ of $(A, F)$ is isomorphic to the completion of $(A, F^+)$, and hence $\hat{A}$ is a Banach ring, isomorphic to the completion of $A$ with respect to the seminorm $\nu$ associated to $F$. (Caution: the induced filtration $\hat{F}$ may not be of seminorm type.)

For a filtered ring $(A, F_A)$ and a surjective ring homomorphism $A \to B$, consider the induced filtration $F_B$ on $B$ as in §C.3.(a). If $\nu_A$ is the seminorm corresponding to $F_A^+$, then the seminorm on $B$ corresponding to $F_B^+$ is the residue seminorm induced from $\nu_A$ (§C.1).

Let $(B_1, F_1)$ and $(B_2, F_2)$ be $(A, F)$-algebras, and consider the tensor-product filtration $\bar{F}$. Then the seminorm on $B_1 \otimes_A B_2$ corresponding to $\bar{F}^+$ is nothing but the tensor-product seminorm (§C.1) constructed from the seminorms $\nu_1$, $\nu_2$, and $\nu$ corresponding to $F_1^+$, $F_2^+$, and $F^+$, respectively. Hence, in the setting considered at the end of §C.3.(a), the complete tensor product $B_1 \hat{\otimes}_A B_2$ coincides with the complete tensor product of the corresponding seminormed rings (as in §C.1).
C.3. (c) **Filtered polynomial and power series algebras.** Let $X$ be a set, and fix a function $w: X \to \mathbb{R}_+$, which we call a weight function. Let $\mathbb{Z}[X]$ be the $\mathbb{Z}$-polynomial ring with the variables in $X$, where the variable corresponding to $x \in X$ is denoted by $X_x$. Let $M_X$ be the free (additive) commutative monoid with the basis $X$, which is identified with the set of all monomials by the correspondence

$$M_X \ni \alpha = \sum_{x \in X} m_x [x] \mapsto X^\alpha = \prod_{x \in X} (X_x)^{m_x} \in \mathbb{Z}[X].$$

We extend the function $w$ to $M_X$, hence to the set of all monomials in $\mathbb{Z}[X]$, in such a way that $w(X^\alpha + \beta) = w(X^\alpha w(X^\beta)$ for all $\alpha, \beta \in M_X$. For any $f = \sum \alpha a_\alpha X^\alpha \in \mathbb{Z}[X]$, we define the weight of $f$ by

$$w(f) = \begin{cases} \sup \{w(X^\alpha) : a_\alpha \neq 0\} & \text{if } f \neq 0, \\ 0 & \text{if } f = 0, \end{cases}$$

which induces a filtration $F_w$ on $\mathbb{Z}[X]$ defined by

$$(F_w)_r = \{ f \in \mathbb{Z}[X] : w(f) \leq r \}$$

for any $r > 0$.

For a filtered ring $(A, F)$, consider the tensor product $A[X] = A \otimes_{\mathbb{Z}} \mathbb{Z}[X]$ with the tensor product filtration (§C.3. (a)), also denoted by $F_w$. Explicitly, $f = \sum \alpha a_\alpha X^\alpha \in A[X]$ belong to $(F_w)_r$ if and only if $a_\alpha \in F_{r/w(X^\alpha)}$ for any $\alpha$. The associated seminorm, which corresponds to $F^+_w$ by the correspondence in §C.3. (b), is nothing but the Gauss seminorm $\| \cdot \|_{\text{Gauss}}$, given by

$$\| f \|_{\text{Gauss}} = \max \{ w(X^\alpha) \|a_\alpha\| : \alpha \in M_X \}$$

for $f = \sum \alpha a_\alpha X^\alpha \in A[X]$, where $\| \cdot \|$ denotes the seminorm on $A$ associated to the filtration $F$.

Note that, by construction, we have

$$(A[X], F_w) \cong \lim_{X' \subset X} (A[X'], F_{w\mid_{X'}}),$$

where $X'$ runs through all finite subsets of $X$. Note also that we have an isomorphism of graded rings

$$\operatorname{Gr}_{F_w} A[X] \sim (\operatorname{Gr}_F A)[X],$$

where the grading on $(\operatorname{Gr}_F A)[X]$ is defined by the grading of $\operatorname{Gr}_F A$ and the weight by $a X^\alpha (a \in F_r)$ mapped to $[a], X^\alpha$. 
If \((A, F)\) is complete, then the completion of \((A[X], F_w)\) is called the weighted power series algebra over \((A, F)\), and is denoted by 

\[ A \ll (X, w) \rr. \]

As an \(A\)-algebra, \(A \ll (X, w) \rr\) is isomorphic to the completion of \(A[X]\) with respect to the Gauss seminorm \(\| \cdot \|_{Gauss}\).

In the sequel, these notions and statements will be considered often in the case where \(X = \{1, 2, \ldots, n\}\) and \(w(i) = r_i > 0\) for \(i = 1, 2, \ldots, n\). In this case, as usual, the weighted power series algebra \(A \ll (X, w) \rr\) will be denoted by 

\[ A \ll r^{-1} X \rr = A \ll r_1^{-1} X_1, \ldots, r_n^{-1} X_n \rr \]

\[ = \left\{ \sum_{m \in \mathbb{N}^n} a_m X^m \in A[X] \mid r^m \|a_m\|_A \to 0 \text{ as } |m| \to \infty \right\}, \]

where \(r^m = r_1^{m_1} \cdots r_n^{m_n}, X^m = X_1^{m_1} \cdots X_n^{m_n}\), and \(|m| = m_1 + \cdots + m_n\) for \(m = (m_1, \ldots, m_n) \in \mathbb{N}^n\). This algebra will play an important role in §C.4.(b) below.

### C.3. (d) Filtered valuation fields

**Definition C.3.3.** A filtered field is a separated filtered ring \((K, F)\) with \(K\) a field (in the usual sense).

**Definition C.3.4.** A filtered valuation field is a filtered field \((K, F)\) that satisfies the following conditions:

(a) \(\text{Gr}_{F^+} K\) is a graded field (§C.2.(a));

(b) \(\text{Gr}_F K\) is a graded valuation ring for \(\text{Gr}_{F^+} K\) (C.2.1 (1)).

Note that, as we have seen in §C.3.(b), \(\text{Gr}_F K\) is, in general, regarded as a graded subring of \(\text{Gr}_{F^+} K\). Note also that, in (b) above, we allow the case \(\text{Gr}_F K = \text{Gr}_{F^+} K\), that is, \(F = F^+\). In this case, we say that the filtered valuation field \((K, F)\) is of maximal type. If \((K, F)\) is a filtered valuation field of maximal type, then \(F = F^+\) is of seminorm type, that is, \(F\) corresponds to a seminorm \(\nu: K \to \mathbb{R}_{\geq 0}\) on \(K\) (§C.3.(b)). Since the filtration \(F\) is separated, and \(K\) is a field, one sees that the seminorm \(\nu\) is a valuation (absolute value). Note that \(\nu\) may be a trivial one, that is, \(\nu(x) = 1\) for all non-zero \(x \in K\).

Conversely, as we saw in §C.3.(b), any non-archimedean valued field \((K, \nu)\) gives rise to a filtered valuation field \((K, F, \nu)\) of maximal type. Thus we see that filtered valuation fields of maximal type are virtually the same objects as non-archimedean valued fields. The notion of filtered valuation field in general gives, therefore, a refinement of the notion of non-archimedean valued field. In general,
for a filtered valuation field \((K, F)\), \((K, F^+)\) is a filtered valuation field of maximal type, called the associated maximal type of \((K, F)\). The valuation \(v\) corresponding to the maximal type will be called the associated absolute value of the filtered valuation field \((K, F)\).

**Proposition C.3.5.** If \((K, F)\) is a filtered valuation field, then the seminorm \(v\) corresponding to the filtration \(F^+\) is a non-archimedean absolute value on \(K\). Conversely, given a non-archimedean absolute value \(v\) on \(K\) and a graded valuation subring \(G\) for \(\text{Gr}_F K\), there exists a unique filtration \(F\) on \(K\) that makes the pair \((K, F)\) a filtered valuation field such that \(G = \text{Gr}_F K\) and that \(F^+ = F\).

**Proof.** The first assertion is easy to see, as we have already discussed above. Let \(v\) be a non-archimedean absolute value on \(K\), and \(G\) a graded valuation ring for \(\text{Gr}_F K\). Define the filtration \(F = \{F_r\}_{r \geq 0}\) so that, for each \(r > 0\), \(F_r\) is the preimage of \(G_r\) under the map \(\phi : F_r/(F_r)^{-1} \rightarrow \text{Gr}_F K\). Then, clearly, \((K, F)\) is a filtered valuation field with \(\text{Gr}_F K = G\). One easily sees that \(F^+ = F\). The uniqueness is clear.

**Corollary C.3.6.** Let \((K, v)\) be a non-archimedean valued field with the absolute value \(v : K \rightarrow \mathbb{R}_{\geq 0}\), and \(\hat{K}\) the completion of \(K\) with respect to \(v\). Then there exists a natural bijection between the set of all filtrations \(F\) on \(K\) that make the resulting pair \((K, F)\) a filtered valuation field with the associated absolute value \(v\), and the set of all filtrations \(\widehat{F}\) on \(\hat{K}\) that make the pair \((\hat{K}, \widehat{F})\) a filtered valuation field with the associated absolute value equal to the extension \(\hat{v}\) of \(v\) on \(\hat{K}\).

**Proof.** This follows from C.3.5 and the isomorphism \(\text{Gr}_{F_v} K \cong \text{Gr}_{F_v} \hat{K}\).

**Remark C.3.7.** Note that, if \((K, F)\) is a filtered valuation field, then \(F_1\) is a valuation ring for \(K\) in the usual sense. Indeed, \((F_1,F_{<1})\) is a valuation ring for the field \((F_v)_1/(F_v)_{<1}\), where \(v\) is the associated absolute value (see §C.2.(b)). On the other hand, \((F_v)_1 = \{x \in K : v(x) \leq 1\}\) is clearly a valuation ring for \(K\), with residue field \((F_v)_1/(F_v)_{<1}\). Hence, by the patching argument (0.6.4.2), \(F_1\) is a valuation ring.

Based on this, we call \(F_1\) the associated valuation ring of the filtered valuation field \((K, F)\). It is a valuation ring for the field \(K\) (in the usual sense).

Proposition C.3.5 suggests that a filtered valuation field \((K', F')\) should be said to dominate \((K, F)\) by a morphism \((K, F) \rightarrow (K', F')\) when the absolute value \(v_{F'}\) extends \(v_F\) and \(\text{Gr}_{F'} K\) dominates \(\text{Gr}_F K\) in the sense as in §C.2.(b). This last condition can be boiled down to the following one.
**Definition C.3.8.** Let \((K, F)\) and \((K', F')\) be filtered valuation fields. We say that \((K', F')\) *dominates* \((K, F)\) by a morphism \((K, F) \to (K', F')\) if \(F' \cap K = F_r\) for any \(r > 0\).

**Definition C.3.9.** Let \((K, F)\) is a filtered valuation field, and \(v = v_F\) the associated absolute value on \(K\). The *height* of \((K, F)\), denoted by \(ht(K, F)\), is the pair \((ht(v), ht(Gr_F K))\).

Here, \(ht(v)\) denotes the height of the corresponding valuation ring \((F_v)_1 = \{x \in K : v(x) \leq 1\}\), which is 0 or 1, according as \(v\) is trivial or non-trivial.

**Definition C.3.10.** Let \(K\) be a field, and suppose we have two filtrations \(F_1\) and \(F_2\) on \(K\) such that \((K, F_i)\) \((i = 1, 2)\) are filtered valuation fields with the same maximal type. We say \(F_1\) is a *specialization* of \(F_2\), or \(F_2\) is a *generalization* of \(F_1\), if \(F_1 \subseteq F_2\).

Notice that the ‘same maximal type’ condition is equivalent to that these filtrations induce the same absolute value \(v\) on \(K\). Thus \(F_1\) is a specialization of \(F_2\) if and only if the valuation ring \(Gr_{F_1} K\) is a specialization of the valuation ring \(Gr_{F_2} K\). In particular, the set of all generalizations of a fixed \((K, F)\) is a totally ordered with respect to the inclusion order (see C.2.7).

**C.3. (e) Filtered valuation via valuation.** Let us first remark that, when we consider absolute values \(v: K \to \mathbb{R}_{\geq 0}\) as a valuation (of height 0 or 1), the ordering of \(\mathbb{R}_+\) is the reverse of the standard ordering, hence is the reverse to the index-ordering for the filtrations \(F = \{F_r\}_{r > 0}\); in fact, \(\mathbb{R}_+\) as a value target group is the one isomorphic to the standard \(\mathbb{R}\) via

\[
(\mathbb{R}_+, \leq_{\text{rev}}) \sim (\mathbb{R}, \leq), \quad x \mapsto \log_c x
\]

for some \(0 < c < 1\).

**Definition C.3.11.** Let \(K\) be a field.

(1) An \(\mathbb{R}_+\)-valuation \(v\) of \(K\) is a valuation of the form

\[
v: K \to (\mathbb{R}_+ \times \Gamma) \cup \{(0, +\infty)\},
\]

where \(\Gamma\) is a totally ordered commutative group, and \(\mathbb{R}_+ \times \Gamma\) is endowed with the lexicographic order. The induced valuation \(v^\mathbb{R}_+: K \to \mathbb{R}_{\geq 0}\) (resp. \(v^{Gr}: K \to \Gamma \cup \{+\infty\}\)) given by the first (resp. second) projection is called the *absolute value* (resp. graded part) associated to \(v\).
(2) Let \((K, v)\) and \((K', v')\) be two fields with \(\mathbb{R}_+\)-valuations with value target groups \(\Gamma_v = \mathbb{R}_+ \times \Gamma\) and \(\Gamma_{v'} = \mathbb{R}_+ \times \Gamma'\), respectively. A homomorphism \((\varphi, \phi): (K, v) \to (K', v')\) of \(\mathbb{R}_+\)-valuation fields consists of a field homomorphism \(\varphi: K \to K'\) and a homomorphism of ordered groups \(\phi: \Gamma_v \to \Gamma_{v'}\) such that the following conditions are satisfied:

(a) \(\phi\) is of the form \(\phi = \text{id}_{\mathbb{R}_+} \times \psi\), where \(\psi: \Gamma \to \Gamma'\) is a homomorphism of ordered groups;

(b) \(v' \circ \varphi = \phi \circ v\).

We say that \(v'\) dominates \(v\) if, for \(f \in K\), \(v^\text{Gr}(f) > 0\) implies \(v'^\text{Gr}(\varphi(f)) > 0\).

(3) Two \(\mathbb{R}_+\)-valuations \(v_1\) and \(v_2\) on \(K\) are said to be equivalent if there exist an \(\mathbb{R}_+\)-valuation \(v\) on \(K\) and homomorphisms \((K, v) \to (K_i, v_i)\) by which \((K_i, v_i)\) dominates \((K, v)\) for \(i = 1, 2\).

Let \((K, F)\) be a filtered valuation field. One has the associated absolute value \(v = v_F: K \to \mathbb{R}_{\geq 0}\), and the graded valuation ring \(\text{Gr}_F K\) for \(\text{Gr}_{F+} K\), hence the value group \(\Gamma\) of the associated graded valuation \(v_F\) (see §C.2.(c)). Then one can define an \(\mathbb{R}_+\)-valuation on \(K\) with value target group \(\mathbb{R}_+ \times \Gamma\) by

\[
K \ni f \mapsto v(f) = (v(f), v_F([f]_{v(f)})) \in (\mathbb{R}_+ \times \Gamma) \cup \{(0, +\infty)\}.
\]

Notice that, from this \(\mathbb{R}_+\)-valuation \(v\), the filtration \(F\) is recovered by

\[
F_r = \{0\} \cup v^{-1}((0, r) \times \Gamma) \cup v^{-1}(\{r\} \times \mathbb{R}_{\geq 0})
\]

for \(r > 0\). By this, one can show the following result.

**Proposition C.3.12.** The construction above gives rise to a bijection from the set of all filtrations \(F\) on \(K\) that make the pair \((K, F)\) a filtered valuation field, to the set of all equivalence classes of \(\mathbb{R}_+\)-valuations on \(K\).

**Remark C.3.13.** \(\mathbb{R}_+\)-valuations can be regarded as so-called reified valuations by K. Kedlaya [67] with an extra structure. According to [67], 5.1, a reified valuation on a field \(K\) is a valuation \(v: K \to \Gamma \cup \{0\}\), where \(\Gamma\) is written multiplicatively, with a reification, viz., an order-preserving homomorphism \(r: \mathbb{R}_+ \to \Gamma\). In terms of this, an \(\mathbb{R}_+\)-valuation on \(K\) is interpreted as a reified valuation \((v, r)\) on \(K\) with an order-preserving homomorphism \(p: \Gamma \to \mathbb{R}_+\) such that \(p \circ r = \text{id}_{\mathbb{R}_+}\).
C.3. (f) Non-degenerate filtered valuations

**Proposition C.3.14.** Let \((K, F)\) be a filtered valuation field, \(v = v_F\) the associated absolute value on \(K\), and \(v: K \to (\mathbb{R}_+ \times \Gamma) \cup \{(0, +\infty)\}\) the associated \(\mathbb{R}_+\)-valuation. Then the following conditions are equivalent:

(a) the graded valuation ring \(\text{Gr}_F K\) is non-degenerate (see C.2.10);

(b) for any \(r \in v(K^\times)\), there exists \(f_r \in K\) such that \(v(f_r) = (r, 0)\);

(c) the value group \(v(K^\times)\) of \(v\) is isomorphic to the product \(v(K^\times) \times v^{\text{Gr}}(K^\times)\) (with the lexicographic order) as a totally ordered group.

**Proof.** (a) \(\implies\) (b). Let \(r \in v(K^\times)\). Then \(\text{Gr}_{F+r} K \neq \{0\}\). Since \(\text{Gr}_F K\) is non-degenerate, there exists an element in \(\text{Gr}_{F+r} K\), that is invertible in \(\text{Gr}_F K\) (see C.2.9 (1)). This means that there exists \(f_r \in K\) such that \(v(f_r) = (r, 0)\).

(b) \(\implies\) (c). By assumption, we have a splitting \(v(K^\times) \hookrightarrow v(K^\times)\) by \(r \mapsto (r, 0)\), from which the assertion follows by a standard argument.

(c) \(\implies\) (b). is clear.

(b) \(\implies\) (a). For any \(r \in v(K^\times)\), there exists \(f_r \in K\) such that \(v(f) = r\) and \([f]_r\) is invertible in \(\text{Gr}_F K\), from which the non-degeneracy of \(\text{Gr}_F K\) follows by C.2.9 (1).

**Definition C.3.15.** A filtered valuation field \((K, F)\) is said to be **non-degenerate** if it satisfies one, hence all, of the conditions in C.3.14.

Note that there exist filtered valuation fields that are not non-degenerate. Indeed, they can be obtained by an \(\mathbb{R}_+\)-valuation \(v: K \to (\mathbb{R}_+ \times \Gamma) \cup \{(0, +\infty)\}\) with the value group not being a product of totally ordered abelian groups; cf. 0.6.5.2.

**Proposition C.3.16.** Let \((K, F)\) be a non-degenerate filtered valuation field. Then any generization of \((K, F)\) is again non-degenerate. Moreover, there exists a natural bijection between the set of all generizations of \((K, F)\) and the set of all generizations (in the usual sense) of \(F_1\).

**Proof.** As we have remarked at the end of §C.3. (d), \(F'\) is a generization of \(F\) if and only if \(\text{Gr}_{F'} K\) is a generization of \(\text{Gr}_F K\). Hence the last assertion of the proposition follows from C.2.15 (2) and what we have seen in C.3.7. The first assertion follows from C.2.12 (1).
Proposition C.3.17. Let \((K, F)\) be a non-degenerate filtered valuation field, and \(v\) the associated absolute value.

(1) For any \(r \in \Gamma_v = v(K^\times)\), \(F_r\) is a free \(F_1\)-module of rank one.

(2) For any \(r, r' \in \Gamma_v = v(K^\times)\), the map by multiplication \(F_r \otimes_{F_1} F_{r'} \to F_{rr'}\) is an isomorphism.

Proof. Take \(f_r\) as in C.3.14 (b). For any \(x \in F_r\), we have \(f_r^{-1}x \in F_1\), and hence \(F_r = F_1 f_r\). Thus we have shown the first assertion, and also the second assertion follows.

Remark C.3.18. In summary, a non-degenerate filtered valuation field amounts to the following data (cf. C.2.13):

- a field \(K\) equipped with a non-archimedean absolute value \(v: K \to \mathbb{R}_{\geq 0}\);
- for any \(r \geq 0\), a submodule \(F_r\) of \((F_v)_r = \{x \in K : v(x) \leq r\}\), such that
  (a) for any \(r \geq 0\), \(F_r = \bigcup_{r' \in \Gamma_v \cap [0, r]} F_{r'}\);
  (b) \(F_1\) is a valuation ring for \(K\), and the valuation ring \((F_v)_1\) of \(v\) is the generization of \(F_1\);
  (c) for \(r \in \Gamma_v = v(K^\times)\), \(F_r\) is a free \(F_1\)-submodule of \((F_v)_r\) of rank one, and \(F_r \otimes_{F_1} (F_v)_1 \cong (F_v)_r\);
  (d) for \(r, r' \in \Gamma_v\), \(F_r \otimes_{F_1} F_{r'} \cong F_{rr'}\) compatibly with the multiplication map \((F_v)_r \otimes_{(F_v)_1} (F_v)_{r'} \cong (F_v)_{rr'}\).

Finally, let us show that, in a certain situation, the formation of filtered valuations reduces to that of usual valuations. Let \((K, F)\) be a non-degenerate filtered valuation field with the associated absolute value \(v_K\), and \(L/K\) a field extension with an absolute value \(v_L\) that extends \(v_K\). We assume \(v_K(K^\times) = v_L(L^\times)\). Take a valuation ring \(V\) for \(L\) such that

- \(V\) contains \(F_1\);
- \(v_L\) is a generization of \(V\).

Then the filtration \(F_L\) of \(L\) defined by

\[(F_L)_r = F_r \cdot V \cong F_r \otimes_{F_1} V\]

for \(r \in \mathbb{R}_+\) gives a filtered valuation field \((L, F_L)\) such that \((F_L)_1 = V\). Since each \((F_L)_r\) for \(r \in v_L(L^\times)\) is free of rank one over \(V\), and \((F_L)_{rr'} = (F_L)_r \cdot (F_L)_{r'}\), \((L, F_L)\) is non-degenerate.
Proposition C.3.19. Let \((K, F)\) be a non-degenerate filtered valuation field with the associated absolute value \(v_K\), and \(L/K\) a field extension with an absolute value \(v_L\) that extends \(v_K\). Suppose \(v_K(K^\times) = v_L(L^\times)\). Then any filtration \(\widetilde{F}\) of \(L\) that makes the pair \((L, \widetilde{F})\) a filtered valuation field over \((K, F)\) is obtained as above.

Proof. By assumption, for any \(r \in v_L(L^\times)\), \(\widetilde{F}_r\) contains \(F_r\), and \(F_r \cdot \widetilde{F}_1 \subseteq \widetilde{F}_r\). Hence \((L, \widetilde{F})\) is a generization of \((L, F_L)\) obtained as above from \(V = \widetilde{F}_1\). By C.3.16, both of them are non-degenerate. Moreover, they have the same \(F_1\)-part. Hence, again by C.3.16, they coincide with each other, which is what we wanted to show. \(\square\)

C.3. (g) Examples of filtered valuations. Let \((K, F)\) be a filtered valuation field, 

\[v: K \longrightarrow (\mathbb{R}_+ \times \Gamma) \cup \{(0, +\infty)\}\]

the associated \(\mathbb{R}_+\)-valuation, and \(v = v^{\mathbb{R}_+}\) the associated absolute value. Let \(X\) be a set with a weight function \(w: X \to \mathbb{R}_+\), and consider the polynomial algebra \((K[X], F_w)\) as in §C.3. (c). The seminorm corresponding to \(F_w^+\) is the Gauss norm \(|| \cdot \||_{\text{Gauss}}\), and we have an isomorphism

\[\text{Gr}_{F_w^+}(K[X]) \xrightarrow{\sim} (\text{Gr}_v(K)[X],\]

mapping \(aX^\alpha (v(a) = r)\) to \([a]_rX^\alpha\). For any non-zero \(f = \sum_{\alpha \in M_X} a_\alpha X^\alpha \in K[X]\), we have

\[[f] = \sum_{\alpha \in M_X} [a_\alpha]_{w(\alpha) - 1}r \cdot X^\alpha,\]

where \([f] = [f]_r\) with \(r = \|f\|_{\text{Gauss}}\). We have

(a) \([fg] = [f][g]\) for non-zero \(f, g \in K[X]\);
(b) if \(\|f\|_{\text{Gauss}} = \|g\|_{\text{Gauss}}\) and \([f] + [g] \neq 0\), then \([f + g] = [f] + [g]\) and the function

\[v_{X,w}(f) = \begin{cases} \(\|f\|_{\text{Gauss}}, v^\text{Gr}(\text{cont}^\text{Gr}([f]))\) & \text{if } f \neq 0, \\ (0, +\infty) & \text{if } f = 0, \end{cases}\]

defined on \(K[X]\) extends to an \(\mathbb{R}_+\)-valuation of the quotient field \(K(X)\) with value target group \(\mathbb{R}_+ \times \Gamma\).

Here \(\text{cont}^\text{Gr}(H)\) for homogeneous \(H \in (\text{Gr}_v(K)[X])\) is the graded \(\text{Gr}_F K\)-submodule of \(\text{Gr}_v K\) generated by the classes of coefficients of \(H\), and \(v^\text{Gr}(\text{cont}^\text{Gr}(H))\) is defined to be \(v^\text{Gr}(a)\), where \(a\) is a generator of \(\text{cont}^\text{Gr}(H)\). (One easily checks, as in the classical case, that \(\text{cont}^\text{Gr}(HH') = \text{cont}^\text{Gr}(H) \cdot \text{cont}^\text{Gr}(H')\) for homogeneous \(H, H'\).) The \(\mathbb{R}_+\)-valuation \(v_{X,w}\) will be called the Gauss valuation of \(K(X)\),
and often denoted by \( v_{\text{Gauss}} \). The absolute value part of \( v_{\text{Gauss}} \) is the Gauss norm \( \| \cdot \|_{\text{Gauss}} \).

Note that the equality (a), which simply follows from the fact that \( \text{Gr}_{F,v} K[X] \) is a graded integral domain, implies the Gauss lemma for the Gauss seminorm.

**Example C.3.20.** We apply the above construction to the extreme case \( X = \mathbb{R}_+ \) and \( w = \text{id}_{\mathbb{R}_+} \). In this situation, the Gauss norm \( \| \cdot \|_{\text{Gauss}} \) on \( K(X) \) has the value group equal to the whole \( \mathbb{R}_+ \). Moreover, if we denote by \( X_r \) the indeterminacy corresponding to \( r \in \mathbb{R}_+ \), then we have \( v_{\text{Gauss}}(X_r) = (r, 0) \), which shows that the filtered valuation field given by \( (K(X), v_{\text{Gauss}}) \) is non-degenerate (due to C.3.14). We denote the filtered valuation field over \( (K, F) \) thus obtained by

\[ K^{st} = (K(\mathbb{R}_+), v_{\text{Gauss}}), \]

and call it the **standard extension**. The standard extension is a non-degenerate filtered valuation field over any given \( (K, F) \), having \( \mathbb{R}_+ \) as its value group of the absolute value part. Moreover, the formation \( K \mapsto K^{st} \) is functorial.

Note that the existence of standard extension also shows that any filtered valuation field can be embedded into a non-degenerate isometric extension.

**C.4 Valuative spectrum of non-archimedean Banach rings**

In this subsection, we define and discuss valuative spectra of non-archimedean commutative Banach rings.

**C.4. (a) Gelfand–Berkovich spectrum.**

**Definition C.4.1** ([11], §1.2). Let \( (A, \| \cdot \|_A) \) be a commutative Banach ring. The **Gelfand–Berkovich spectrum** \( M(A) \) of \( A \) is

- the set of all bounded multiplicative seminorms on \( A \),
- with the weakest topology such that all the functions

\[ |f(\cdot)|: M(A) \longrightarrow \mathbb{R}_{\geq 0}, \quad z = \| \cdot \|_z \mapsto |f|_z \]

with \( f \in A \) are continuous.

Here, a seminorm \( | \cdot |: A \rightarrow \mathbb{R}_{\geq 0} \) is **bounded** if there exists \( C > 0 \) such that \( |f| \leq C \| f \|_A \) for any \( f \in A \).

As indicated above, for any point \( z \in M(A) \), the corresponding seminorm on \( A \) is often denoted by \( | \cdot |_z \). The kernel \( \ker(| \cdot |_z) = \{ f \in A: |f|_z = 0 \} \) is a closed prime ideal of \( A \), and the seminorm \( | \cdot |_z \) induces a valuation on the integral domain \( A/\ker(| \cdot |_z) \), and hence on its fraction field. The Hausdorff completion of this field
with respect to the valuation is denoted by \( \mathcal{H}(z) \), and called the \textit{complete residue field at} \( z \). As usual, the image of an element \( f \in A \) in \( \mathcal{H}(z) \) is denoted by \( f(z) \); thus, \( |f(z)| \) denotes the value \( |f|_z \).

The topological space \( \mathcal{M}(A) \) is compact, and is non-empty if \( A \neq \{0\} \) ([11], 1.2.1). Note that \( f \in A \) is invertible in \( A \) if and only if the function \( z \mapsto |f(z)| \) is everywhere non-zero on \( \mathcal{M}(A) \). If \( A \) is non-archimedean, then \( \mathcal{H}(z) \) for any \( z \in \mathcal{M}(A) \) is non-archimedean.

In the sequel, all seminorms and norms are assumed to be non-archimedean, unless otherwise clearly stated. Note that the norm \( |\cdot|_z \) on the complete residue field \( \mathcal{H}(z) \) may be a trivial one.

For \( f \in A \), we will denote by \( \|f\|_{\text{Sp}} \) the \textit{spectral seminorm}

\[
\|f\|_{\text{Sp}} = \sup_{z \in \mathcal{M}(A)} |f(z)|,
\]

which is equal to the \textit{spectral radius}

\[
\rho(f) = \inf_{n \geq 1} \|f^n\|_A^{\frac{1}{n}} = \lim_{n \to \infty} \|f^n\|_A^{\frac{1}{n}};
\]

see [11], §1.3. By this we can define an \( \mathbb{R}_+ \)-filtration \( F_{A}^{\text{Sp}} \) on \( A \), called the \textit{spectral filtration}, as

\[
(F_{A}^{\text{Sp}})_r = \{ f \in A : \|f\|_{\text{Sp}} \leq r \},
\]

which yields an \( \mathbb{R}_+ \)-filtered ring (C.3.2) \( (A, F_{A}^{\text{Sp}}) \). It follows that, for any \( z \in \mathcal{M}(A) \), the map \( f \mapsto f(z) \) maps \( (F_{A}^{\text{Sp}})_r \) to \( (F_{|z|}^{\text{Sp}})_r \) for any \( r > 0 \), that is, the residue map \( A \to \mathcal{H}(z) \) is a morphism of filtered rings (C.3.2). Note that the filtered field \( \mathcal{H}(z) \) with the spectral filtration coincides with \( (\mathcal{H}(z), F_{|z|}^{\text{Sp}}) \), hence is a filtered valuation field of maximal type (C.3.4).

In general, any bounded homomorphism \( f : A \to B \) induces the morphism

\[
f : (A, F_{A}^{\text{Sp}}) \longrightarrow (B, F_{B}^{\text{Sp}})
\]

of filtered rings, and hence the morphism

\[
\text{Gr}_{F_{A}^{\text{Sp}}} A \longrightarrow \text{Gr}_{F_{B}^{\text{Sp}}} B
\]

of graded rings.

For a commutative Banach ring \( A \), denote by \( A_{\text{Sp}} \) the completion of \( A \) with respect to the spectral filtration \( F_{A}^{\text{Sp}} \) (the so-called \textit{uniform completion}). The canonical morphism \( A \to A_{\text{Sp}} \) induces an isomorphism

\[
\mathcal{M}(A_{\text{Sp}}) \sim \mathcal{M}(A),
\]
and we have \((A^{\text{Sp}})^{\text{Sp}} = A^{\text{Sp}}\). A Banach ring \(A\) is said to be uniform, or a Banach function ring, if \(A \to A^{\text{Sp}}\) is a bounded isomorphism.

Let \((A, F)\) be a filtered ring, and consider filtered homomorphisms of the form \((A, F) \to (K, F_v)\), where \(K\) is a filtered valuation field of maximal type. For two such homomorphisms, \((A, F) \to (K, F_v)\) and \((A, F) \to (K', F_{v'})\), consider the relation by domination, that is, the relation given by a commutative diagram

\[
\begin{array}{ccc}
(A, F) & \to & (K, F_v) \\
\downarrow & & \downarrow \\
(K', F_{v'}) & \to & \\
\end{array}
\]

where by the down-arrow \((K', F_{v'})\) dominates \((K, F)\) (see C.3.8). Consider the equivalence relation generated by this relation, and set \(M(A, F) = \{\text{equivalence class of filtered homomorphisms of the form (}A, F) \to (K, F_v)\text{ to a filtered valuation field of maximal type}\}\).

Consider the completion \(\hat{A} = \hat{A}_{F}\) of \(A\) with respect to the filtration \(F\). Then \(\hat{A}\) is a Banach ring with respect to the norm induced from the seminorm \(v_F\) associated to \(F\). For any \((A, F) \to (K, F_v)\), the norm \(v\) on \(K\) induces a seminorm on \(\hat{A}\), which is clearly multiplicative and bounded. Hence, we have the natural map

\[
\mathcal{M}(A, F) \to \mathcal{M}(\hat{A}).
\]

**Theorem C.4.2.** The map \((\ast)\) is a bijection. In particular, if \(\text{Gr}_F A \neq \{0\}\), then \(\mathcal{M}(A, F)\) is non-empty.

**Proof.** Let \(z \in \mathcal{M}(\hat{A})\) be given by \(\alpha: \hat{A} \to \mathcal{K}(z)\). We want to show that the composition

\[
A \to \hat{A} \xrightarrow{\alpha} \mathcal{K}(z)
\]

extends to a filtered map \((A, F) \to (\mathcal{K}(z), F_{\mid z})\). Since \(A \to \hat{A}\) obviously respects the filtrations, we may assume \(A = \hat{A}\). Consider the uniform completion \(A \to A^{\text{Sp}}\), which factorizes \(A \to \mathcal{K}(z)\). The morphism \(A \to A^{\text{Sp}}\) respects filtration, because \(\|f\|_A \geq \|f\|_{A^{\text{Sp}}}\). The morphism \(A^{\text{Sp}} \to \mathcal{K}(z)\) also respects the filtration, since the norms on both sides are power-multiplicative, and hence \(\|f\|_{A^{\text{Sp}}} \geq |f(z)|\). Hence we have shown the first assertion. Since \(\text{Gr}_F A \neq \{0\}\) implies \(\hat{A} \neq \{0\}\), the second assertion follows from [11], 1.2.1. \(\square\)

Next let us give a few technical corollaries.

**Corollary C.4.3.** Let \((A, F)\) be a filtered ring. For a graded prime ideal \(p\) of \(\text{Gr}_F A\), there exists a filtered valuation field \((K, F_K)\) over \((A, F)\) such that the graded valuation ring \(\text{Gr}_{F_K} K\) dominates \((\text{Gr}_F A)_p\).
Proof. Let $\overline{S}_p$ be the graded multiplicative system corresponding to the ideal $p$ (cf. §C.2.(a)), and $S_p = \{(S_p)_r \}$ the corresponding filtered multiplicative system of $A$, where $(S_p)_r$ is the preimage of $(\overline{S}_p)_r$ under the map $F_r \to F_r / F_{<r} = \text{Gr}_{F,r} A$. The filtered localization $A' = S_p^{-1} A$ has $(\text{Gr}_F A)_p$ as its associated graded ring, which is non-zero. By C.4.2, there exists a filtered homomorphism $(A', F') \to K$, where $K = (K, F_v)$ is a filtered valuation field of maximal type. The image of $\text{Gr}_{F'} A'$ in $\text{Gr}_{F,v} K$ is a local graded subring. Now by C.2.4 and C.3.5, there exists a filtered valuation $(K, F_K)$ over $(A', F')$ such that $\text{Gr}_{F_K} K$ dominates $(\text{Gr}_F A)_p$. \qed

**Corollary C.4.4.** Let $(K, F_K)$ be a filtered valuation field, and $(M, F_M)$ and $(L, F_L)$ filtered valuation fields that dominate $(K, F_K)$. Then there exists a filtered valuation field $(N, F_N)$ sitting in the commutative diagram

$$
\begin{array}{ccc}
(M, F_M) & \longrightarrow & (N, F_N) \\
\uparrow & & \uparrow \\
(K, F_K) & \longrightarrow & (L, F_L)
\end{array}
$$

consisting of dominating filtered homomorphisms.

Proof. Let us denote by $v_K, v_L, v_M$ the associated absolute values of $K, L, M$, respectively, and set $V_K = (F_K)_1, V_L = (F_L)_1, V_M = (F_M)_1$. Replacing $K, L, M$ by the respective fields $K^{st}, L^{st}, M^{st}$ constructed as in C.3.20, we may assume $K, L, M$ are all non-degenerate, and having the whole $\mathbb{R}_+$ as their value group. By C.3.19, we have

$$(F_L)_r = (F_K)_r \otimes_{V_K} V_L, \quad (F_M)_r = (F_K)_r \otimes_{V_K} V_M$$

for all $r > 0$. Set $A = L \otimes_K M$, with the tensor product filtration $F_A$. Since $(F_A)_r = (F_K)_r \otimes_{V_K} (V_L \otimes_{V_K} V_M)$ for $r > 0$, the canonical surjection

$$\mu : \text{Gr}_{F,L} L \otimes_{\text{Gr}_{F,M} M} \text{Gr}_{F,M} M \longrightarrow \text{Gr}_{F,A} A$$

is an isomorphism.

Now, take a graded prime ideal $p$ of $\text{Gr}_{F,A} A$ lying above the graded maximal ideals of $\text{Gr}_{F,L} L$ and of $\text{Gr}_{F,M} M$. By C.4.3, one has a filtered homomorphism $(A, F_A) \to (N, F_N)$ to a filtered valuation field such that $\text{Gr}_{F_N} N$ dominates $(\text{Gr}_{F_A} A)_p$. This $(N, F_N)$ gives the commutative square as asserted. \qed
Let $A$ be a Banach ring. Recall that, for an $n$-tuple $r = (r_1, \ldots, r_n)$ of positive real numbers, the $R_+$-power series ring in the variables $T = (T_1, \ldots, T_n)$ with coefficients in $A$ is

$$A\langle r^{-1}T \rangle = A\langle r_1^{-1}T_1, \ldots, r_n^{-1}T_n \rangle = \left\{ \sum_{m \in \mathbb{N}^n} a_m T^m : m \in \mathbb{N}^n, a_m \in A[T], r^m \|a_m\|_A \to 0 \text{ as } |m| \to \infty \right\},$$

where $r^m = r_1^{m_1} \cdots r_n^{m_n}$, $T^m = T_1^{m_1} \cdots T_n^{m_n}$, and $|m| = m_1 + \cdots + m_n$ for $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$. It is a Banach ring with the Gauss norm

$$\|f\|_{A\langle r^{-1}T \rangle} = \sup_{m \in \mathbb{N}^n} r^m \|a_m\|_A$$

for $f = \sum_{m \in \mathbb{N}^n} a_m T^m$; see §C.3. (c). By (11.2.1 (3)), we have

$$\|f\|_{A\langle r^{-1}T \rangle, \mathbb{S}_p} = \sup_{m \in \mathbb{N}^n} r^m \|a_m\|_{A, \mathbb{S}_p}$$

for $f = \sum_{m \in \mathbb{N}^n} a_m T^m$ (Exercise II.C.3), and the filtered ring $(A\langle r^{-1}T \rangle, F^A_{A\langle r^{-1}T \rangle})$ is isomorphic to the completion of $(A[X], F^A_w)$, where $F^A_w$ is the tensor product filtration of $F^A_{A\langle r^{-1}T \rangle} \otimes A$ and $F^A_w$ on $A$ and $F^A_w$ on $\mathbb{Z}[X]$; see §C.3. (c). Passing to the associated graded algebras, we have an isomorphism

$$\text{Gr}_{F^A_{A\langle r^{-1}T \rangle}} A\langle r^{-1}T \rangle \sim \text{(Gr}_{F^A_{A\langle r^{-1}T \rangle}} A)[X], \quad [T_i]_{r_i} \mapsto X_i,$$

where each $X_i$ on the right-hand side is of degree $r_i$ ($i = 1, \ldots, n$).

**Definition C.4.5.** A Banach $A$-algebra $B$ is said to be of $R_+$-finite type if it is isomorphic to a Banach $A$-algebra of the form

$$A\langle r^{-1}T \rangle / \mathfrak{a}$$

by a bounded isomorphism, where $\mathfrak{a} \subseteq A\langle r^{-1}T \rangle$ is a closed ideal. If one can take such an isomorphism with $r = (1, \ldots, 1)$, then we say $B$ is of finite type.

An important class of $R_+$-finite type $A$-algebra is now introduced as follows. Let $f = (f_0, f_1, \ldots, f_n)$ be an $(n+1)$-tuple of elements of $A$ such that

$$(f_0, f_1, \ldots, f_n) = A,$$

and $r = (r_1, \ldots, r_n)$ an $n$-tuple of positive real numbers. The $R_+$-rational algebra over $A$ corresponding $(f, r)$ is

$$A\langle r^{-1}f_1 f_0^{-1}, \ldots, r_n^{-1}f_n f_0^{-1} \rangle = A\langle r^{-1}T \rangle / (f_0 T_1 f_1, \ldots, f_0 T_n - f_n),$$

which is a Banach $A$-algebra by the residue norm. An $R_+$-finite type $A$-algebra isomorphic to the one of this form will be called an $R_+$-rational localization of $A$. 

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**C.4. (b) $R_+$-finite type algebras.** Let $A$ be a Banach ring. Recall that, for an $n$-tuple $r = (r_1, \ldots, r_n)$ of positive real numbers, the $R_+$-power series ring in the variables $T = (T_1, \ldots, T_n)$ with coefficients in $A$ is
Note that
\[ U_0(f, r) = \mathcal{M}\left( A\left( r_1^{-1} \frac{f_1}{f_0}, \ldots, r_n^{-1} \frac{f_n}{f_0} \right) \right) \]
is identified with the closed subset
\[ \{ z \in \mathcal{M}(A) : |f_i(z)| \leq r_i \cdot |f_0(z)| \} \]
of \( \mathcal{M}(A) \), the so-called \( \mathbb{R}_+ \)-rational subdomain of \( \mathcal{M}(A) \).

**C.4. (c) Integrally closed filtrations**

**Definition C.4.6.** Let \( (A, F) \) be a filtered ring, and \( r \in \mathbb{R}_+ \). We say \( x \in A \) is \( r \)-integral over \( F \) if an equality of the form
\[ x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0 \tag{*} \]
holds for some \( n \geq 1 \) and \( a_i \in F_{r,i} \) for \( i = 1, \ldots, n \).

**Lemma C.4.7.** Let \( (A, F) \) be a filtered ring, and \( x \in A \). Then \( x \) is \( r \)-integral \( (r > 0) \) if and only if \( x \in F_{r,+}^+ \) and \( [x]_r \in \text{Gr}_F^+ A \) is integral over the graded subring \( \text{Gr}_F A \).

**Proof.** To show the ’only if’ part, suppose \( x \) is \( r \)-integral, and the equality \( (*) \) above holds for \( a_i \in F_{r,i} \) \( (i = 1, \ldots, n) \). Let \( \| \cdot \| \) be the seminorm corresponding to \( F^+ \) (see §C.3.(b)). Since \( \|a_i\| \leq r^i \) for \( i = 1, \ldots, n \), we have
\[ \|x^n\| \leq \sup_{i=1,\ldots,n} r^i \|x^{n-i}\|, \]
from which \( \|x\| \leq r \) follows. Hence \( x \in F_{r,+}^+ \). It is then clear that \( [x]_r \) is integral over \( \text{Gr}_F A \).

To show the converse, choose \( a_i \in F_{r,i} \) for \( i = 1, \ldots, n \) such that
\[ [x]_r^n + [a_1]_r [x]_r^{n-1} + \cdots + [a_{n-1}]_r [x]_r + [a_n]_r = 0. \]
Then \( \alpha = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \) belongs to \( F_{<r^n} \), and the equality
\[ x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n - \alpha = 0 \]
implies that \( x \) is \( r \)-integral. \( \square \)

**Definition C.4.8.** Let \( (A, F) \) be a filtered ring.

(1) For \( r > 0 \), define
\[ F_r^{\text{int}} = \{ x \in A : x \text{ is } r \text{-integral over } F \}. \]

Then \( F^{\text{int}} = \{ F_r^{\text{int}} \}_{r > 0} \) is, by C.4.7, an \( \mathbb{R}_+ \)-multiplicative filtration on \( A \) containing \( F \). The filtration \( F \) is said to be integrally closed if \( F = F^{\text{int}} \).
(2) Let $F'$ be an $\mathbb{R}_+$-multiplicative (see C.3.2 (a)) filtration of $A$ containing $F$. The integral closure of $F$ in $F'$, denoted by $F_\text{int}^{r,F'}$, is defined by
\[
F_\text{int}^{r,F'} = F_r' \cap F_r^{\text{int}}
\]
for any $r > 0$.

**Proposition C.4.9.** (1) If $(A, F)$ is a filtered ring with $F$ integrally closed, then for any filtered multiplicative system $S = \{S_r\}_{r>0}$ of $A$ (see §C.3. (a)), the filtration $S^{-1}F$ on $S^{-1}A$ is integrally closed.

(2) Let $(A, F_0)$ be a filtered ring, and $F$ an integrally closed $\mathbb{R}_+$-multiplicative filtration containing $F_0$. Then the completion $\hat{F}_{F_0}^\Lambda$ of $F$ with respect to $F_0$ (see §C.3. (a)) is integrally closed.

**Proof.** (1) can be shown similarly to the classical case. To show (2), let $x$ be an element of $(F_{F_0}^\Lambda)_r$ satisfying $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ for $a_i \in (F_{F_0}^\Lambda)_r$ ($i = 1, \ldots, n$). Take $z \in A$ such that $x - z \in (F_{F_0}^\Lambda)_r$, and $b_i \in F_r$ such that $a_i - b_i \in (F_{F_0}^\Lambda)_r$ ($i = 1, \ldots, n$). Then we have
\[
[z]^n_r + [b_1]_r[z]^{n-1}_r + \cdots + [b_{n-1}]_r[z]_r + [b_n]_r = 0
\]
in $\text{Gr}_{F_{F_0}^\Lambda} A_{F_0}^\Lambda = \text{Gr}_F A$. Since $F$ is integrally closed, $z \in F_r$ by C.4.7, whence $x \in (F_{F_0}^\Lambda)_r$, as desired. \qed

**Proposition C.4.10.** Let $(A, F_0)$ be a filtered ring, and $F$ an $\mathbb{R}_+$-multiplicative filtration containing $F_0$. Consider the set $M(A, F_0)$ as in C.4. (a), and the associated spectral filtration $F_{(A,F_0)}^{\text{Sp}}$.

(1) If $F$ is contained in $F_{(A,F_0)}^{\text{Sp}}$, then $F^{\text{int}} \subseteq F_{(A,F_0)}^{\text{Sp}}$.

(2) For any $r > 0$, any element of $(F_{(A,F_0)}^{\text{Sp}})_r$ is $r$-integral over $F$. In particular, $F^+ = F_{(A,F_0)}^{\text{Sp}}$ if $F$ is integrally closed and $F \subseteq F_{(A,F_0)}^{\text{Sp}}$.

(3) If $\tilde{F}$ is an $\mathbb{R}_+$-multiplicative filtration containing $F$ such that $\tilde{F}^+ = F_{(A,F_0)}^{\text{Sp}}$, then
\[
(F^{\text{int}},\tilde{F})_r = \{x \in \tilde{F}_r: [x]_r \text{ is integral over } \text{Gr}_{F,r} A\}
\]
for $r > 0$.

**Proof.** (1) Take a point $z: (A, F_0) \rightarrow K_z$ of $\mathcal{M}(A, F_0)$, where $K_z$ is a filtered valuation field of maximal type. By assumption, the image of $F$ is contained in $F_{K_z}$. Observe that the homomorphism $(A, F) \rightarrow (K_z, F_{K_z})$ factors through $(A, F^+)$. 


Since \( r \)-integral elements belong to \( F^+_r \) for any \( r > 0 \) (C.4.7), it follows that the image of \( F^{\text{int}} \) is contained in \( F_{K_0} \), from which the assertion follows.

(2) Let \( \nu \) be the seminorm on \( A \) corresponding to \( F^+_0 \). If \( x \in (F_{(A,F_0)})^+_r \), then \( \nu(x^N) < r^N \) for some \( N \geq 1 \) (see §C.4 (a)). Then \( \alpha = x^n \) is contained in \( (F_0)_r \), and \( x^n - \alpha = 0 \) shows that \( x \) is \( r \)-integral over \( F_0 \), hence over \( F \). If \( F \) is integrally closed and \( F \subseteq (F_{(A,F_0)})^+_r \), then we have \( F^<_r = (F_{(A,F_0)})^+_r \) for any \( r > 0 \), which implies \( F^+_r = (F_{(A,F_0)})^+_r \).

(3) Let \( x \) be an element in the right-hand set of (**). As in the proof of C.4.7, one has \( a_i \in F_{z_i} \) for \( i = 1, \ldots, n \) such that \( \alpha = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \) belongs to \( F^<_r \). Since \( F^<_r = (F_{(A,F_0)})^+_r \), there exists \( N \geq 1 \) such that \( \beta = \alpha^N \in F^<_r \), by an argument similar to that in the proof of (2). Then \((x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n)^N - \beta = 0 \) shows that \( x \in (F^{\text{int}}, \tilde{F})_r \). The other inclusion is easy to see.

**Theorem C.4.11.** Let \( (A,F_0) \) be a filtered ring, and \( F \) and \( \tilde{F} \) be \( \mathbb{R}_+ \)-multiplicative filtrations on \( A \) such that \( F_0 \subseteq F \subseteq \tilde{F} \) and \( \tilde{F}^+ = (F_{(A,F_0)})^+_r \). Then

\[
(F^{\text{int}}, \tilde{F})_r = \left\{ x \in \tilde{F}_r \mid \text{for any morphism } (G,F) \to (K,F_K) \text{ to a filtered valuation field, the image of } x \text{ is contained in } (F_K)_r \right\}
\]

for \( r > 0 \).

**Proof.** First note that the filtration \( F_K \) in a filtered valuation field \((K,F_K)\) is integrally closed, due to C.4.7 and C.2.3. By this, one sees that \((F^{\text{int}}, \tilde{F})_r \) is contained in the right-hand set. To show the opposite inclusion, take \( x \in \tilde{F}_r \) such that \([x]_r \) is not integral over \( F \) in \( \tilde{F} \). By [27], Chap. VI, §1.2, Lemma 1, \((\text{Gr}_F \mathcal{G})_[x]_r \) is non-zero, and \([x]^{-1}_r \) generates a proper graded ideal of \((\text{Gr}_F \mathcal{G})_[x]^{-1}_r \).

Let \( S = \bigcup_{r>0} S_r \) be the filtered multiplicative subset (§C.3 (a)) generated by \( x \), such that \( x \in S_r \) and \( A_x = S^{-1} A \) with the filtration \( F_x \) induced from \( F \). We have \( \text{Gr}_{F_x} A_x \cong (\text{Gr}_F A)_[x]_r \). Since the graded ideal of \( \text{Gr}_{F_x} A_x \) generated by \([x]^{-1}_r \) is a proper ideal, by C.4.3, there exists a filtered valuation field \((K,F_K)\) over \((A_x, F_x)\) such that the graded maximal ideal of \( \text{Gr}_{F_K} K \) contains the image of \([x]^{-1}_r \), hence the image of \( x \) is not contained in \((F_K)_r \), since, if it is, then the image of \( x \) in \( \text{Gr}_{F_K} K \) is invertible.

**Corollary C.4.12.** Let \( (A,F_0) \) be a filtered ring, \( F \) an \( \mathbb{R}_+ \)-multiplicative filtration on \( A \) such that \( F_0 \subseteq F \subseteq F^{\text{Sp}}_{(A,F_0)} \). Suppose \((A,F)\) is integrally closed. Then the filtration \( F_w \) in the weighted polynomial algebra \((A[X], F_w)\) as in §C.3 (c) is integrally closed. Similarly, the filtration \( \hat{F}_w \) in the weighted power series algebra \((A[[X,w]]), \hat{F}_w)\), the completion of \((A[X], F_w)\) with respect to the filtration \( F_{0,w} \) induced from \( F_0 \) is integrally closed.
Proof. Let us show the assertion under the assumption that \((A, F)\) is a filtered valuation field; the general case can be reduced to this case by C.4.11. Let \(v\) be the corresponding \(\mathbb{R}_+\)-valuation, and \(v_{\text{Gauss}}\) the Gauss valuation of \(A[X]\) (see §C.3. (g)). Take \(f \in A[X]\) such that \(f^n + a_1 f^{n-1} + \cdots + a_0 = 0\) for \(a_i \in (F_w)_r\). From this and the fact that \(v_{\text{Gauss}}(a_i) \geq (r^i, 0)\), we have \(v_{\text{Gauss}}(f) \geq (r, 0)\), hence \(f \in (F_w)_r\). This shows that \(F_w\) is integrally closed.

Finally, \(\widehat{F}_w\) in \(A\langle(X, w)\rangle\) is integrally closed due to C.4.9 (2).

C.4. (d) Power bounded filtration

Definition C.4.13. (1) Let \((A, \| \cdot \|)\) be a seminormed ring, and \(r > 0\) a positive real number. An element \(f \in A\) is said to be \(r\)-power bounded if

\[
\sup_{n \geq 1} r^{-n} \| f^n \| < +\infty.
\]

(2) Let \((A, F)\) be a filtered ring, and \(\| \cdot \|\) the seminorm corresponding to the filtration \(F^+\) (see §C.3. (b)). Then we say \(f \in A\) is \(r\)-power bounded if it is \(r\)-power bounded as an element of the seminormed ring \((A, \| \cdot \|)\).

Similarly to the classical notion of power-boundedness, the \(r\)-power-boundedness is closely related to the universality property of power series algebras.

Proposition C.4.14. Let \(A\) be a Banach ring. For \(X = (X_1, \ldots, X_n)\), \(r = (r_1, \ldots, r_n)\), and a Banach ring \(B\), there exists a natural bijection between the following sets:

(a) the set of all bounded homomorphisms of the form \(A\langle r^{-1} X \rangle \to B\);

(b) the set of all \(n\)-tuples \(f = (f_1, \ldots, f_n)\) of elements of \(B\), where \(f_i\) is \(r_i\)-power bounded for \(i = 1, \ldots, n\).

This follows from the following lemma.

Lemma C.4.15. For \(f \in A\) and \(r > 0\), the following conditions are equivalent:

(a) \(f\) is \(r\)-power bounded;

(b) the canonical homomorphism

\[
A \to A\langle r^{-1} T \rangle / (T - f)
\]

is a bounded isomorphism.
Proof. Suppose (b) holds, and set \( B = A \langle r^{-1}T \rangle / \langle T - f \rangle \). The residue norm on \( B \) induced from the Gauss norm on \( A \langle r^{-1}T \rangle \) is equivalent to the norm on \( A \). Since \( \| T^n \|_B \leq r^n \) for \( n \geq 1 \), \( f \) is \( r \)-power bounded. Conversely, suppose \( f \) is \( r \)-power bounded, and consider the homomorphism

\[
A \langle r^{-1}S \rangle \longrightarrow A \langle r^{-1}T \rangle, \quad S \mapsto T - f,
\]

and observe that it is a bounded isomorphism with the inverse given by \( T \mapsto S + f \). Passing to the quotient by \( \langle S \rangle \), we get the bounded isomorphism

\[
A \overset{\sim}{\longrightarrow} A \langle r^{-1}S \rangle / \langle S \rangle \overset{\sim}{\longrightarrow} A \langle r^{-1}T \rangle / \langle T - f \rangle,
\]

as desired. \( \square \)

The following is a corollary of the last proposition.

**Corollary C.4.16.** Let \( A \) be a Banach ring, \( f = (f_0, \ldots, f_n) \) an \( (n + 1) \)-tuple of elements of \( A \), that generate the unit ideal \( A \), and \( r = (r_1, \ldots, r_n) \) an \( n \)-tuple of positive real numbers. Then, for a Banach \( A \)-algebra \( B \), there exists a natural bijection between the following sets:

(a) the set of all bounded homomorphisms of the form

\[
A \left\{ r_{1}^{-1} \frac{f_1}{f_0}, \ldots, r_{n}^{-1} \frac{f_n}{f_0} \right\} \longrightarrow B;
\]

(b) the set of all \( n \)-tuples \( z = (z_1, \ldots, z_n) \) of elements of \( B \) such that, for \( i = 1, \ldots, n \), \( z_i \) is \( r_i \)-power bounded and \( f_0 z_i = f_i \).

In particular, any bounded \( A \)-algebra homomorphism between \( \mathbb{R}_+ \)-rational localization over \( A \) (see §C.4.(b)) is an epimorphism in the category of Banach rings.

**Definition C.4.17** (power bounded filtration). For a seminormed ring \( (A, \| \cdot \|) \), the **power bounded filtration** on \( A \), denoted by \( F_A^o \), is a filtration \( \{ F_A^o \} \) \( r \in \mathbb{R}_+ \) defined by

\[
(F_A^o)_r = \{ x \in A : x \text{ is } r \text{-power bounded} \}.
\]

For a filtered ring \( (A, F = F_A) \), the power bounded filtration on \( A \) is defined similarly, in reference to the \( r \)-power-boundedness as in C.4.13 (2), and is denoted by \( F^o = F_A^o \).

Note that the power bounded filtration \( F^o \) is an exhaustive and \( \mathbb{R}_+ \)-multiplicative filtration, hence defining a filtered ring \( (A, F^o) \) (see C.3.2). Moreover, for a filtered ring \( (A, F) \), we have

\[
F^o \subseteq F^{Sp}, \quad F^o_{< r} \subseteq F^o_r
\]
for $r > 0$, since the spectral seminorm $\| \cdot \|_{Sp}$ bounds from above the seminorm $\| \cdot \|$, and is power-multiplicative. It follows from the following proposition that $F^o$ is integrally closed (C.4.8), and, for any filtration of definition $F_0$, the inclusions

$$(F_0)^{int} \subseteq F^o \subseteq F^{Sp}$$

hold.

**Proposition C.4.18.** Let $(A, \| \cdot \|)$ be a seminormed ring, and $F_0$ a filtration of definition of $A$. Let $F$ be an $\mathbb{R}_{+}$-multiplicative filtration on $A$ such that $F_0 \subseteq F \subseteq F^o$. Then $F^{int} \subseteq F^o$. In particular, $F^o$ is integrally closed.

**Proof.** Let $x \in F_r^+$ satisfy an equality of the form

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0,$$

where $a_i \in F_r$ for $i = 1, \ldots, n$. For any $k \geq n$, we have an expression

$$x^k = \sum_{i=1}^{n} \left( \sum_{\lambda} C_{\lambda} a^{\lambda} \right) x^{n-i}$$

where $C_{\lambda} \in \mathbb{Z}$ and $\lambda$ runs over all partitions of $k - (n - i)$ such as $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$ and $\sum_{j=1}^{n} \lambda_j \cdot j = k - (n - i)$, and $a^{\lambda} = \prod_{j=1}^{n} a_j^{\lambda_j}$. By assumption, there exists $C \geq 1$ such that $\|a_i^\lambda\| \leq Cr^{\lambda_i}$. Then

$$\|x^k\| \leq Dr^k$$

with $D = (r^{-1}C)^n \sup_{i=1,\ldots,n} r^i \|x^{n-i}\|$, which shows that $x \in F_r^o$. □

**Corollary C.4.19.** Let $(A, F_0)$ be a filtered ring. Let $F$ be an $\mathbb{R}_{+}$-multiplicative filtration on $A$ such that $F_0 \subseteq F \subseteq F^o$. Then $F^{int} \subseteq F^o$. In particular, $F^o$ is integrally closed.

**Proposition C.4.20.** Let $(A, F)$ be a filtered ring.

1. For a filtered multiplicative subset $S = \bigcup_{r>0} S_r$ (see §C.3. (a)), the filtration $S^{-1}F^o$ on $S^{-1}A$ is contained in $(S^{-1}F)^o$.

2. Passing to the completion with respect to $F$, we have $(\hat{F})^o = (F^o)_{\hat{F}}$.

3. The induced filtration $(F^o)_w$ on the weighted polynomial algebra (resp. power series algebra) $A[X]$ (resp. $A \langle \langle X, w \rangle \rangle$) coincides with the power bounded filtration of $A[X]$ (resp. $A \langle \langle X, w \rangle \rangle$).

**Proof.** (1) and (2) are easy to see. For (3), it suffices, due to (2), to discuss the case of the polynomial algebra. For a monomial $cX^\alpha$ in $(F_w)_r$, we have $c \in F_s$, where $s = w(X^\alpha)^{-1} r$, and $X^\alpha$ is $w(X^\alpha)$-power bounded. By assumption, $c$ is $s$-power bounded, and hence the claim follows. □
Chapter II. Rigid spaces

C.4. (e) \( \mathbb{R}_+ \)-affinoid rings

**Definition C.4.21.** An \( \mathbb{R}_+ \)-affinoid ring, or an affinoid Banach ring, is a triple \( \Omega = (A, F_0, F) \) consisting of the following data:

- a Banach ring \( A \) with a filtration \( F_0 \) of definition;
- an integrally closed filtration \( F \) such that \( F^\text{int} \subseteq F \subseteq F^\text{Sp} \).

Note that, by C.4.11, \( F_r \) contains \( (F^\text{Sp})_{<r} \), and any \( \mathbb{R}_+ \)-affinoid ring structure of \( A \) is obtained from that of the uniform completion \( A^\text{Sp} \) by taking the preimage under the map \( A \to A^\text{Sp} \).

In the sequel, we will often drop the filtration of definition \( F_0 \) from the notation of \( \mathbb{R}_+ \)-affinoid rings, and write them as a pair \( \Omega = (A, F) \). The underlying Banach ring \( A \) of an \( \mathbb{R}_+ \)-affinoid ring \( \Omega \) will be denoted by \( \Omega^B \).

The graded ring \( \text{Gr}_{F_0} \Omega^B \) will be simply denoted by \( \text{Gr} \Omega \).

**Definition C.4.22.** Let \( \Omega = (A, F_\Omega) \) and \( \mathcal{B} = (B, F_\mathcal{B}) \) be two \( \mathbb{R}_+ \)-affinoid rings. A homomorphism \( \varphi: \Omega \to \mathcal{B} \) of \( \mathbb{R}_+ \)-affinoid rings is a bounded homomorphism \( \varphi^B: A \to B \) that respects the filtrations, that is, \( \varphi^B((F_\Omega)_r) \subseteq (F_\mathcal{B})_r \) for all \( r > 0 \).

In this situation, the Banach affinoid ring \( \mathcal{B} \) is also called an \( \Omega \)-affinoid algebra.

We denote by

\[
\mathbb{R}_+\text{-Aff}
\]

the category of \( \mathbb{R}_+ \)-affinoid rings. We have a forgetful functor

\[
\Omega \mapsto \Omega^B
\]

from \( \mathbb{R}_+\text{-Aff} \) to the category of Banach rings with bounded homomorphisms. Note that, if \( \varphi: \Omega \to \mathcal{B} \) is a homomorphism of \( \mathbb{R}_+ \)-affinoid rings, then, replacing the filtration of definition \( F_{A,0} \) of \( A \) by \( F_{A,0} \cap (\varphi^B)^{-1}(F_{B,0}) \), which is also a filtration of definition of \( A \), one can always assume that \( \varphi^B \) respects filtrations of definition.

Note also that any complete filtered valuation field \( (K = \hat{K}, F) \) can be uniquely regarded as an \( \mathbb{R}_+ \)-affinoid ring, since \( F_0 = F_0^\text{int} = F \subseteq F^\text{Sp} = F^+ \), where the first equality is due to C.2.3 and C.4.7.

The category of \( \mathbb{R}_+ \)-affinoid rings has tensor products. For two \( \mathbb{R}_+ \)-affinoid rings \( \Omega_i = (A_i, F_{\Omega_i}) \) (\( i = 1, 2 \)) over \( \Omega = (A, F_\Omega) \), we first take, as above, filtrations of definition of \( A, A_1, \) and \( A_2 \) such that \( (A_i, F_{A_i,0}) \) are filtered algebras over \( (A, F_{A_0}) \), and then take the complete tensor product \( \mathcal{B} = A_1 \hat{\otimes} A_2 \) with respect to these filtrations of definition; then the completion of \( F_{\Omega_1} \otimes F_{\Omega_2} \) \( F_{\Omega_2} \) with respect to
the filtration of definition of $B$ gives a filtration, the integral closure of which $F_{\mathfrak{B}}$ gives an $\mathbb{R}_+$-affinoid ring $\mathfrak{B}$. Then the pair $\mathfrak{B} = (B, F_{\mathfrak{B}})$ gives the desired tensor product of the $\mathbb{R}_+$-affinoid rings $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}$.

For an $\mathbb{R}_+$-affinoid ring $\mathfrak{A} = (A, F_{\mathfrak{A}})$, the power series algebra $A\langle\langle r^{-1}X \rangle\rangle$ as in §C.4.21) has the canonical $\mathfrak{A}$-affinoid ring structure by $F_{\mathfrak{A}, w}$; see §C.3.2 for the definition of the filtration $F_{\mathfrak{A}, w}$.

For any $\mathbb{R}_+$-finite type $A$-algebra with a presentation (that is, an admissible epimorphism (see §C.1) of $A$-algebras)

$$\phi: A\langle\langle r^{-1}X \rangle\rangle \rightarrow B$$

has an $\mathfrak{A}$-affinoid ring structure by the integral closure of the filtration induced from $F_{\mathfrak{A}, w}$; note that the $\mathfrak{A}$-affinoid algebra structure on $B$ may depend on the choice of the presentation $\phi$. In particular, any $\mathbb{R}_+$-rational localization $B = A\langle\langle r_1^{-1}f_1, \ldots, r_n^{-1}f_n \rangle\rangle$ is equipped with a structure of $\mathfrak{A}$-affinoid algebra, which we henceforth denote by

$$\mathfrak{A}\left(\frac{r_1^{-1}f_1, \ldots, r_n^{-1}f_n}{f_0}\right),$$

and call it the $\mathbb{R}_+$-rational localization of $\mathfrak{A}$.

**Proposition C.4.23.** Let $\mathfrak{A} = (A, F_{\mathfrak{A}})$ be an $\mathbb{R}_+$-affinoid ring (C.4.21) with $F_{\mathfrak{A}} \subseteq (F_{\mathfrak{A}})^0$. Then, for any $\mathbb{R}_+$-finite type $A$-algebra $B$ and any $\mathfrak{A}$-affinoid algebra structure $\mathfrak{B} = (B, F_{\mathfrak{B}})$ on $B$, we have $F_{\mathfrak{B}} \subseteq (F_{\mathfrak{B}})^0$.

**Proof.** Let $\mathfrak{A}\langle\langle r^{-1}X \rangle\rangle \rightarrow \mathfrak{B}$ be a presentation. By C.4.20 (3), we have $(F_{\mathfrak{A}})_{w} \subseteq (F_{\mathfrak{A}})^0_{w}$. Passing to the quotient, we have that the image of $(F_{\mathfrak{A}})_{w}$, and hence also the integral closure $F_{\mathfrak{B}}$, is contained in $(F_{\mathfrak{B}})^0$. $\square$

**C.4. (f) Valuative spectrum.** In the sequel, for an $\mathbb{R}_+$-affinoid ring $\mathfrak{A}$, we write $\mathcal{M}(\mathfrak{A}) = \mathcal{M}(\mathfrak{A}^B)$.

**Definition C.4.24.** Let $\mathfrak{A} = (A, F_{\mathfrak{A}})$ be an $\mathbb{R}_+$-affinoid ring.

1. A valuation $v$ of $\mathfrak{A}$ is a pair $v = (z, F)$, where $z \in \mathcal{M}(\mathfrak{A})$ and $F$ is a filtration on the complete residue field $\mathcal{H}(z)$ that makes the pair $(\mathcal{H}(z), F)$ a filtered valuation field over $(A, F_{\mathfrak{A}})$.

2. We set $\text{Spec}^\text{val} \mathfrak{A}$ the set of all valuation of $\mathfrak{A}$, and call it the valuative spectrum of the $\mathbb{R}_+$-affinoid ring $\mathfrak{A} = (A, F)$. 
For a filtered ring \((A, F_0)\), consider an integrally closed multiplicative filtration \(F\) on \(A\) such that \(F^\text{int} \subseteq F \subseteq F^\text{Sp}_0\). We define \(\text{Spec}^{\text{val}}(A, F) = \text{Spec}^{\text{val}}(\hat{A}, \hat{F})\).

We are going to define a topology on the set \(\text{Spec}^{\text{val}} A\). For an \((n + 1)\)-tuple \(f = (f_0, f_1, \ldots, f_n)\) of elements in \(A = \mathbb{Q}^B\) such that \((f_0, f_1, \ldots, f_n) = A\), and \(n\)-tuple \(r = (r_1, \ldots, r_n)\) of positive real numbers, define a subset

\[
U_0(f, r) = \left\{ x = (z, F) \in \text{Spec}^{\text{val}} A \mid \frac{f_j(z)}{f_0(z)} \in F_{r_j} \text{ for } j = 1, \ldots, n \right\}
\]

of \(\text{Spec}^{\text{val}} A\); note that \(f_0(z) \neq 0\) on \(U_0(f, r)\). We call a subset of this form a rational subdomain (or more precisely, \(\mathbb{R}_+\)-rational subdomain). For example, for \(f, g \in A\) and \(r, s > 0\),

\[
U_0((g, f, 1), (r, s)) = \left\{ (z, F) \mid g(z) \neq 0, \frac{f(z)}{g(z)} \in F_r, \frac{1}{g(z)} \in F_s \right\}
\]

is a rational subdomain.

Note that the intersection of finitely many rational subdomains is again a rational subdomain; e.g., for \(f = (f_0, f_1, \ldots, f_n)\), \(r = (r_1, \ldots, r_n)\) and \(f' = (f'_0, f'_1, \ldots, f'_m), r' = (r'_1, \ldots, r'_m)\), we have

\[
U_0(f, r) \cap U_0(f', r') = U_0(h, s),
\]

where

\[
h = (f_0 f'_0, f_i f'_i) : 0 \leq i \leq n, 0 \leq j \leq m, (i, j) \neq (0, 0)),
\]

\[
s = (r_i r'_i) : 0 \leq i \leq n, 0 \leq j \leq m, (i, j) \neq (0, 0)).
\]

**Definition C.4.25.** We endow a topology on \(\text{Spec}^{\text{val}} A\) in such a way that the set of all \(\mathbb{R}_+\)-rational subdomains forms an open basis.

Note also that the homomorphism \(\hat{A} \to \hat{A}(\frac{r_1^{-1} f_1 \ldots r_n^{-1} f_n}{f_0})\) of \(\hat{A}\)-affinoid algebras induces a bijection

\[
\text{Spec}^{\text{val}} \hat{A}(\frac{r_1^{-1} f_1 \ldots r_n^{-1} f_n}{f_0}) \sim U_0(f, r).
\]

One can show, similarly to the classical case as in [18], 7.2.4, that a rational subdomain of a rational subdomain is, via the bijection as above, again a rational subdomain. From this, one sees that the above bijection is in fact a homeomorphism.
If \( B \) and \( B' \) are \( \mathbb{R}_+ \)-rational localizations of \( A \), the complete tensor product \( B \hat{\otimes}_A B' \) provides the \( \mathbb{R}_+ \)-rational localization that gives the intersection \( U \cap U' \), where \( U = \text{Spec}^{\text{val}} B \) and \( U' = \text{Spec}^{\text{val}} B' \) are the corresponding \( \mathbb{R}_+ \)-rational subdomains.

Subsets of the following form are useful for our later argument; for \( f, g \in \mathcal{A} \) and \( r > 0 \), define

\[
B(g, f, r) = \left\{ (z, F) \in \text{Spec}^{\text{val}} \mathcal{A} \mid g(z) \neq 0, \frac{f(z)}{g(z)} \in F_r \right\},
\]

and call it a basic subset. A basic subset is an open subset, since

\[
B(g, f, r) = \bigcup_{s > 0} U_0((g, f, 1), (r, s)).
\]

Moreover, since

\[
U_0(f, r) = \bigcap_{i=1}^n B(f_0, f_i, r_i)
\]

holds for \( f = (f_0, f_1, \ldots, f_n) \) and \( r = (r_1, \ldots, r_n) \), the basic subsets form a subbase of the topology on \( \text{Spec}^{\text{val}} \mathcal{A} \).

There is a canonical map

\[
\text{sep}_\mathcal{A} : \text{Spec}^{\text{val}} \mathcal{A} \rightarrow \mathcal{M}(\mathcal{A}),
\]

which sends \( x = (z, F) \) into \( z \). This map is surjective; indeed, there exists a (non-continuous, in general) section of \( \text{sep}_\mathcal{A} \), which maps \( z \) to \( (z, F_z) \), where \( F_z \) is the filtration on \( \mathcal{H}(z) \) corresponding to the absolute value of \( \mathcal{H}(z) \), that is, the filtration such that \( (\mathcal{H}(z), F_z) \) is of maximal type. We can view \( \mathcal{M}(\mathcal{A}) \) as a subset of \( \text{Spec}^{\text{val}} \mathcal{A} \) in this manner. For a rational subdomain \( U = U_0(f, r) \) as above, the image \( \text{sep}_\mathcal{A}(U) \) coincides with \( \mathcal{M}(\mathcal{A}((\frac{1}{r}f_1, \ldots, \frac{1}{r}f_n))) \); in particular, \( \text{sep}_\mathcal{A}(U) \) is a closed subset in \( \mathcal{M}(\mathcal{A}) \).

**Proposition C.4.26.** The map \( \text{sep}_\mathcal{A} \) is continuous.

**Proof.** By the definition of the topology of \( \mathcal{M}(\mathcal{A}) \), subsets of the form

\[
X(f^\epsilon, r^\epsilon) = \{ z \in \mathcal{M}(\mathcal{A}) : |f(z)|^\epsilon < r^\epsilon \}
\]

for \( f \in \mathcal{A} \), \( \epsilon = \pm 1 \) (where, if \( \epsilon = -1 \), we tacitly suppose \( f(z) \neq 0 \) in the left-hand set), and \( r > 0 \), form an open basis of the topology of \( \mathcal{M}(\mathcal{A}) \). In accordance with this, define a subset \( X(f^\epsilon, r^\epsilon) \) of \( \text{Spec}^{\text{val}} \mathcal{A} \) by

\[
X(f^\epsilon, r^\epsilon) = \begin{cases} 
\bigcup_{s < r} U_0((1, f), s) & \text{if } \epsilon = 1, \\
\bigcup_{s-1 < r-1} U_0((f, 1), s^{-1}) & \text{if } \epsilon = -1.
\end{cases}
\]
Since for any \((z, F) \in \text{Spec}^{\text{val}} \mathcal{A}\) and \(r > 0\) one has \(F_{<r} = (F_z)_{<r}\), we have
\[
\text{sep}_{\mathcal{A}}^{-1}(X(f^\varepsilon, r^\varepsilon)) = X(f^\varepsilon, r^\varepsilon),
\]
from which the assertion follows.

\[\square\]

**C.4. (g) Basic properties of the valuative spectrum.** We first note the following topological property of valuative spectra.

**Lemma C.4.27.** Let \(\mathcal{A}\) be an \(\mathbb{R}_+\)-affinoid ring, and \(\mathcal{A}_1\) and \(\mathcal{A}_2\) be \(\mathcal{A}\)-affinoid algebras. Then the canonical mapping of topological spaces
\[
\text{Spec}^{\text{val}} \mathcal{A}_1 \hat{\otimes}_\mathcal{A} \mathcal{A}_2 \longrightarrow \text{Spec}^{\text{val}} \mathcal{A}_1 \times_{\text{Spec}^{\text{val}} \mathcal{A}} \text{Spec}^{\text{val}} \mathcal{A}_2
\]
is surjective.

**Proof.** This follows from C.4.4.

For \(z \in \mathcal{M}(\mathcal{A})\), we set
\[
C_z = \text{sep}_{\mathcal{A}}^{-1}(z) = \{x = (z, F) \in \text{Spec}^{\text{val}} \mathcal{A}\},
\]
with the subspace topology as a subspace of \(\text{Spec}^{\text{val}} \mathcal{A}\).

**Lemma C.4.28.** There exists a canonical homeomorphism
\[
\psi_z: C_z \sim \text{ZR}(\text{Gr}_{F_z} \mathcal{H}(z), \text{Gr} \mathcal{A}),
\]
where \(F_z\) is the filtration on \(\mathcal{H}(z)\) induced from the absolute value; see §C.2. (f).

In particular, \(C_z\) is a valuative space.

**Proof.** The last assertion is a consequence of C.2.16. We need to construct a homeomorphism as in the first assertion. Set \(\tilde{\mathcal{H}}(z) = \text{Gr}_{F_z} \mathcal{H}(z)\), which is a graded field, and \(\tilde{\mathcal{A}} = \text{Gr} \mathcal{A}\). Consider the map
\[
\psi_z: C_z \longrightarrow \text{ZR}(\tilde{\mathcal{H}}(z), \tilde{\mathcal{A}}), \quad (z, F) \longmapsto (\tilde{\mathcal{A}} \longrightarrow \text{Gr}_F \mathcal{H}(z)).
\]
We claim that \(\psi_z\) is bijective. First, notice that, since \(f_z: \mathcal{A} \to (\mathcal{H}(z), F_z)\) is a filtered homomorphism, it induces a graded homomorphism \(\tilde{f}_z: \tilde{\mathcal{A}} \to \tilde{\mathcal{H}}(z)\). For a filtered valuation \(F\) on \(\mathcal{H}(z)\), \(f_z((F_\mathcal{A})_r) \subseteq F_r\) if and only if \((\tilde{f}_z)_r(\tilde{\mathcal{A}}_r) \subseteq (\text{Gr}_{F_z}(\mathcal{H}(z)))_r\). Hence the claim follows from C.3.5.
Next, we show that $\psi_z$ is an open map. It suffices to show that $\psi_z(U)$ is open for $U = C_z \cap U_0(f, r)$, where $f = (f_0, f_1, \ldots, f_n)$ and $r = (r_1, \ldots, r_n)$. For $x = (z, F) \in C_z$, $x \in U$ if and only if

(a) $f_0(z) \neq 0$ and $|f_i(z)/f_0(z)| \leq r_i$ for $i = 1, \ldots, n$,

(b) $f_i(z)/f_0(z) \in F_{r_i}$.

Notice that, under (a), the condition (b) is equivalent to the following one:

(b)' the residue class $[f_i(z)/f_0(z)]_{r_i} \in (\tilde{\mathcal{H}}(z))_{r_i}$ belongs to $(\text{Gr}_F \mathcal{H}(z))_{r_i}$ for $i = 1, \ldots, n$.

Hence, if we introduce the graded $\mathcal{G}$-subalgebra $B$ of $\tilde{\mathcal{H}}(z)$ by

$$B = \begin{cases} \mathcal{G}[f_1(z)/f_0(z), \ldots, f_n(z)/f_0(z)] & \text{if (a) holds}, \\ \tilde{\mathcal{H}}(z) & \text{otherwise}, \end{cases}$$

we have $\psi_z(U) = U(B)$, which shows the claim.

Finally, to conclude the proof, we need to show that any open subset of $Z(\text{Gr}_{F_z} \mathcal{H}(z), \text{Gr} \mathcal{G})$ can be obtained in this way. Since subsets of the form $U(\mathcal{G}[\alpha])$ for $\alpha \in \tilde{\mathcal{H}}(z)$ give an open basis of the topology of $Z(\text{Gr}_{F_z} \mathcal{H}(z), \text{Gr} \mathcal{G})$, we need to find an open subset $U$ of $C_z$ such that $\psi_z(U) = U(\mathcal{G}[\alpha])$. Let $A$ be the image of $\mathcal{G}$ in $\mathcal{H}(z)$. Since the fraction field of $A$ is dense in $\mathcal{H}(z)$, there exists $f, g \in \mathcal{G}$ such that $g(z) \neq 0$ and $f(z)/g(z)$ represents $\alpha$. Then we have $\psi_z(B(g, f, r)) = U(\mathcal{G}[\alpha])$. \qed

In the next paragraph, we will show the following theorem.

**Theorem C.4.29.** Let $\mathcal{G}$ be an $\mathbb{R}_+$-affinoid ring. Then $\text{Spec}^{\text{val}} \mathcal{G}$ is a coherent valuative space.

From this theorem and C.4.28 we obtain the following corollary.

**Corollary C.4.30.** Let $\mathcal{G}$ be an $\mathbb{R}_+$-affinoid ring, and $x = (z, F) \in \text{Spec}^{\text{val}} \mathcal{G}$. Then the homomorphism $\mathcal{G} \to (\mathcal{H}(z), F)$ induces a homeomorphism from $\text{Spec}^{\text{val}}(\mathcal{H}(z), F)$ to the set $G_x$ of all generalizations of $x$ with the subspace topology. In particular, $\text{sep}_\mathcal{G} : \text{Spec}^{\text{val}} \mathcal{G} \to \mathcal{M}(\mathcal{G})$ gives the $T_1$-quotient (0, §2.3. (c)).

Finally, by 0.2.4.6, we have the following statement.

**Corollary C.4.31.** $(\text{Spec}^{\text{val}} \mathcal{G})^{\text{ref}}$ is a coherent valuative space.
Lemma C.4.32. Let $z \in M(\mathfrak{A})$. If a finite family $\{U_j\}_{j \in J}$ of rational subdomains of $\text{Spec}^\text{val} \mathfrak{A}$ covers $C_z$, then there exists an open neighborhood $V$ of $z$ in $M(\mathfrak{A})$ such that $\{U_j\}_{j \in J}$ covers $\text{sep}^{-1}_\mathfrak{A}(V)$.

Proof. Set $X = \text{Spec}^\text{val} \mathfrak{A}$. Write each $U_j$ as a finite intersection of basic subsets,

$$U_j = \bigcap_{i \in I_j} B_i$$

for $j \in J$. For $k = (i_j)_{j \in J} \in \prod_{j \in J} I_j$, we set $B_k = \bigcup_{j \in J} B_{i_j}$. We have

$$\bigcup_{j \in J} U_j = \bigcup_{j \in J} \bigcap_{i \in I_j} B_i = \bigcap_{k \in K} B_k.$$

Hence, to show the lemma, we only need to show the following claim.

**Claim.** Let $B$ be a finite union of basic subsets of $X$. If $B$ contains $\text{sep}^{-1}_\mathfrak{A}(z)$, then there exists an open neighborhood $V$ of $z$ in $M(\mathfrak{A})$ such that $\text{sep}^{-1}_\mathfrak{A}(V)$ is contained in $B$.

To prove this, write $B$ as a finite union of basic subsets

$$B = \bigcup_{i \in I} B_i,$$

where $B_i = (B, g_i, f_i, r_i)$ for $i \in I$. If there is an $i_0 \in I$ such that $|f_{i_0}(z)/g_{i_0}(z)| < r_{i_0}$, then

$$V = \{w \in M(\mathfrak{A}): |f_{i_0}(w)/g_{i_0}(w)| < r_{i_0}\}$$

enjoys the desired property.

We may thus assume that $|f_i(z)/g_i(z)| \geq r_i$ for any $i \in I$. Let $T' = (T'_i)_{i \in I}$ and $r' = (r_i)_{i \in I}$. For $z' \in \text{sep}(X)$ with $f_i(z') \neq 0$ and $|g_i(z')/f_i(z')| \leq r_i^{-1}$ for any $i \in I$, define

$$\varphi_{z'}: \mathfrak{A}(\langle r'T' \rangle) \longrightarrow \mathfrak{H}(z'), \quad T'_i \mapsto \frac{g_i(z')}{f_i(z')}.$$ 

and set

$$\mathfrak{A}(\langle r'T' \rangle) = \text{Gr}_{(F_{z'})_{r_i^{-1}}} \mathfrak{A}(\langle r'T' \rangle).$$

Let $G_{z'}$ be the image of $\mathfrak{A}(\langle r'T' \rangle)$ in $\tilde{\mathfrak{H}}(z') = \text{Gr}_{F_{z'}} \mathfrak{H}(z')$ under the homomorphism induced from $\varphi_{z'}$. Set $h_i(z') = \varphi_{z'}(T'_i)$ for $i \in I$. Then $h_i(z')$ belongs to $(F_{z'})_{r_i^{-1}}$. 


Now, the following two conditions are equivalent:

(a) \( \{ B_i \}_{i \in I} \) covers \( \mathcal{C}_{z'} \);

(b) the homogeneous ideal of \( G_{z'} \) generated by \( [h_i(z')]_{r_i^{-1}} \) for \( i \in I \) is \( G_{z'} \) itself.

Indeed, with the map \( \psi_{z'} \) as in C.4.28, we have \( \psi_{z'}(B_i) = B(\mathfrak{A}(\mathfrak{r}T'))(\alpha_i) \), where \( \alpha_i \) is the class of \( f_i(z')/g_i(z') \) in \( \mathcal{H}(z') \). If (a) holds, then for the subset \( I' = \{ i \in I : |f_i(z')/g_i(z')| = r_i \} \), \( \{ B_i \}_{i \in I'} \) already covers \( \mathcal{C}_{z'} \), since we assumed that \( |h_i(z')| = |g_i(z')/f_i(z')| \leq r_i^{-1} \). Since \( \alpha_i \neq 0 \) for \( i \in I' \), we obtain (b) as in the proof of C.2.16. Conversely, if (b) holds, then similarly, for \( I'' = \{ i \in I : [h_i(z')]_{r_i^{-1}} \neq 0 \}, \{ B_i \}_{i \in I''} \) covers \( \mathcal{C}_{z'} \).

Now, by the assumption that \( \{ B_i \}_{i \in I} \) covers \( \mathcal{C}_z \), there exists an expression

\[
1 = \sum_{i \in I} \tilde{a}_i [h_i(z)]_{r_i^{-1}}
\]

for some \( \tilde{a}_i \in \mathfrak{A}(\mathfrak{r}T') \) of degree \( r_i \) for \( i \in I \). One can take the preimage \( a_i \in (F_{\mathfrak{A}(\mathfrak{r}T')})_{r_i} \) of \( \tilde{a}_i \) by polynomials in \( T' \). Then the polynomial

\[
P(T') = \left( \sum_{i \in I} a_i T_i' \right) - 1
\]

in \( T' \) belongs to \( (F_{\mathfrak{A}(\mathfrak{r}T')})_{1} \) and satisfies

\[
|\varphi_z(P)(z)| < 1.
\]

Define

\[
V = \left\{ z' \in \mathcal{M}(\mathfrak{A}) \mid f_i(z') \neq 0 (i \in I), \left( P \left|_{T_i' = \frac{a_i}{T_i'}} \right. \right)(z') < 1 \right\}
\]

which is an open neighborhood of \( z \) in \( \mathcal{M}(\mathfrak{A}) \). To show that this \( V \) has the desired property, suppose the contrary, i.e., there exists \( x' = (z', F) \in \text{sep}_{\mathfrak{A}}^{-1}(V) \) not covered by \( \{ B_i \}_{i \in I} \). Since \( x' \in \cap_{i \in I}(X \setminus B_i) \), we have \( |g_i(z')/f_i(z')| \leq r_i^{-1} \) for any \( i \in I \), and hence we have \( \varphi_{z'}: \mathfrak{A}(\mathfrak{r}T') \rightarrow \mathcal{H}(z') \) as above. Moreover, \( |\varphi_{z'}(P)(z')| < 1 \) holds by the definition of \( V \). For \( a_i(z') = \varphi_{z'}(a_i) \in (F_{z'})_{r_i} \), the classes \( [a_i(z')]_{r_i} \) and \( [h_i(z')]_{r_i^{-1}} \) satisfy

\[
\sum_{i \in I} [a_i(z')]_{r_i} [h_i(z')]_{r_i^{-1}} - 1 = (\varphi_{z'}(P)(z')) \mod (F_{z'})_{<1} = 0.
\]

Thus we see that the homogeneous ideal in \( G_{z'} \) generated by \( [h_i(z')]_{r_i^{-1}} \) for \( i \in I \) is \( G_{z'} \) itself. But this is absurd in view of the equivalence of the conditions (a) and (b) above. We have therefore proved the claim, and hence the lemma.
We have seen above, a closed map, $z \rightarrow \pi$ due to C.4.28 and C.2.16. Hence, by [24], Chap. I, §10.2, Theorem 1, it suffices to show that $\pi$ is a closed map. Assume the contrary. Let $C \subseteq X$ be a closed subset such that $\pi(C)$ is not closed in $\mathcal{M}(\mathfrak{A})$, and take $z \in \pi(C) \setminus \pi(C)$, where $\pi(C)$ denotes the closure of $\pi(C)$. Let $\{W_j\}_{j \in J}$ be an open covering of $W = X \setminus C$ by rational subdomains. By our assumption, $\pi^{-1}(z) \subseteq C$, and there is a finite subset $J_z \subseteq J$ such that $\bigcup_{j \in J_z} W_j$ contains $\pi^{-1}(z)$. By C.4.32, there exists an open neighborhood $V_z$ of $z$ in $\mathcal{M}(\mathfrak{A})$ such that $\bigcup_{j \in J_z} W_j$ contains $\pi^{-1}(V_z)$. Hence $C \cap \pi^{-1}(V_z) = \emptyset$, that is, $z \in \pi(C)$, which is absurd. Thus, we have shown that $\pi$ is closed, and hence that any rational subdomain of $X$ is quasi-compact. Notice that rational subdomains are closed under finite intersection.

Next, we show that $X$ is $T_0$. Take $x = (z, F)$ and $z' = (z', F')$. If $z \neq z'$, then we can find open neighborhoods $V_z$ and $V_{z'}$ of $z$ and $z'$, respectively, such that $V_z \cap V_{z'} = \emptyset$. Then $\pi^{-1}(V_z)$ and $\pi^{-1}(V_{z'})$ separate the points $x$ and $x'$. If $z = z'$, then we only need to invoke the $T_0$-ness of $C_z \cong \text{Spec}_F \mathfrak{H}(z, \text{Gr}_F \mathfrak{A})$.

Hence, we have shown that $X$ is a coherent topological space. Next, we show that $X$ is sober. Let $Z$ be an irreducible closed subset of $X$. Since $\pi = \text{sep}_\mathfrak{A}$ is, as we have seen above, a closed map, $\pi(Z)$ is an irreducible closed subset of $\mathcal{M}(\mathfrak{A})$. Since $\mathcal{M}(\mathfrak{A})$ is Hausdorff, $\pi(Z) = \{z\}$ for some $z \in \mathcal{M}(\mathfrak{A})$. Since $C_z = \pi^{-1}(z)$ is sober (C.2.16), $Z$ has the unique generic point, as desired.

Finally, by C.4.28 and C.2.16, we deduce that $X$ is valutive.

\textbf{C.4. (i) Relation with adic spectrum.} Let $A$ be a complete $f$-adic ring. Then $A$ is a Banach ring with respect to the following norm. Let $A_0$ be a ring of definition of $A$ with an ideal of definition $I_0$ (see 0.B.1.2), and fix a real number $0 < c < 1$. Define a decreasing filtration $\{F^{m}_{(A_0, I_0)}\}_{m \in \mathbb{Z}}$ indexed by integers by

$$F^{m}_{(A_0, I_0)} = \begin{cases} I_0^m & \text{if } m \geq 0, \\ [A_0 : I_0^{-m}] & \text{if } m < 0. \end{cases}$$

Then the norm $\| \cdot \|_A$ on $A$ is defined for $f \in A$ by

$$\| f \|_A = \begin{cases} c^n, & \text{where } n = \inf\{m \in \mathbb{Z} : f \in F^{m}_{(A_0, I_0)}\} \text{ if } f \neq 0, \\ 0 & \text{if } f = 0. \end{cases}$$
Note that the norm \( \| \cdot \|_A \) actually depends on the choice of \((A_0, I_0)\). Note also that a homomorphism \( \varphi: A \to B \) inducing an adic morphism \((A_0, I_0) \to (B_0, I_0 B_0)\) between the rings of definition, respects the filtrations \( F_{(A_0, I_0)} \) and \( F_{(B_0, I_0 B_0)} \), and gives rise to a bounded homomorphism

\[
\varphi: (A, \| \cdot \|_A) \to (B, \| \cdot \|_B).
\]

For a Huber affinoid ring \((\mathcal{A}^\pm, \mathcal{A}^+)\) (in the sense as in A.3.1), an ideal \( I \) of \( \mathcal{A}^+ \) is called an ideal of definition, if there exists a pair \((A_0, I_0)\) consisting of a ring of definition and an ideal of definition, such that \( I = I_0 \mathcal{A}^+ \). Then the equivalence class of the norm defined as above from \((A_0, I_0)\) depends only on the data \(((\mathcal{A}^\pm, \mathcal{A}^+), I)\).

**Definition C.4.33.** (1) A Banach ring is said to be of f-adic type if it is isomorphic to an f-adic ring as a topological ring.

(2) An \( \mathbb{R}_+ \)-affinoid ring \( \mathcal{A} = (\mathcal{A}^B, F_\mathcal{A}) \) is said to be of adic type if there exist an affinoid ring \((\mathcal{A}^\pm, \mathcal{A}^+)\) in the sense as in A.3.1 and an topological isomorphism \( \mathcal{A}^B = \mathcal{A}^\pm \) that induces \( (F_\mathcal{A})_1 = \mathcal{A}^+ \) and \( \mathcal{M}(\mathcal{A}) = [\text{Spa}(\mathcal{A}^\pm, \mathcal{A}^+)] \).

For an affinoid ring \((\mathcal{A}^\pm, \mathcal{A}^+)\) and an ideal of definition \( I \subset \mathcal{A}^+ \), one can construct an \( \mathbb{R}_+ \)-affinoid ring \( \mathcal{A} = (\mathcal{A}^B, F_\mathcal{A}) \) of adic type such that \( \mathcal{A}^B = \mathcal{A}^\pm \), \( (F_\mathcal{A})_1 = \mathcal{A}^+ \), and \( \mathcal{M}(\mathcal{A}) = [\text{Spa}(\mathcal{A}^\pm, \mathcal{A}^+)] \) as follows.

Set \( A = \mathcal{A}^\pm \) and choose \((A_0, I_0)\) such that \( I = I_0 \mathcal{A}^+ \). Consider the norm \( \| \cdot \| \) on \( A = \mathcal{A}^\pm \) defined as above, and the filtration \( F_0 = F_0^+ \) on \( A \) corresponding to this norm. Then we define the multiplicative filtration \( F_\mathcal{A} \) on \( A \) as the integral closure of the one generated by \( \mathcal{A}^+ \) and \( F_0 \). This gives an \( \mathbb{R}_+ \)-affinoid ring \( \mathcal{A} = (A, F_\mathcal{A}) \) such that \( (F_\mathcal{A})_1 = \mathcal{A}^+ \). Note that the filtration \( F_\mathcal{A} \) does not depend on the choice of \((A_0, I_0)\), and contains \( F_\mathcal{A}^0 \).

We need to check \( \mathcal{M}(\mathcal{A}) = [\text{Spa}(\mathcal{A}^\pm, \mathcal{A}^+)] \). Observe first that, for any \( x \in \text{Spa}(\mathcal{A}^\pm, \mathcal{A}^+) \), the maximal generization \( \tilde{x} \) of \( x \) corresponds to a continuous valuation of height zero or one (since the topology is adic). Let \( \varphi_x: A \to K_x \) be the morphism to the corresponding valuation field. Since \( \varphi_x \) is continuous, then there exists a ring of definition \( A'_0 \) of \( A \) such that \( \varphi_x(A'_0) \subseteq V_x \), where \( V_x \) is the valuation ring of \( K_x \). Since \( \varphi_x \) maps \( A^0 \) to \( V_x \), and \( A_0 \subseteq A^0 \), it follows from \( \varphi_x \) induces a continuous homomorphism \( A_0 \to V_x \), which is, moreover, adic, since the height of \( V \) is one or zero.

Since \( A_0 \to V \) is adic, the valuation \( \| \cdot \|_x \) on \( K_x \) has the following description. Consider the norm \( \| \cdot \|_x \) on \( K_x \) determined as above by the ring of definition \( V_x \) and the ideal of definition \( I_0 V_x \). Then, for any \( f \in K_x \),

\[
\| f \|_x = \lim_{n \to \infty} \| f^n \|_x^{\frac{1}{2}} = \inf_{n \geq 1} \| f^n \|_x^{\frac{1}{2}},
\]

that is, \( \| \cdot \|_x \) is the power-multiplicative norm associated to \( \| \cdot \|_x \).
Hence \( \tilde{x} \) gives a multiplicative and bounded seminorm

\[ \| \cdot \|_{\tilde{x}} : A \to \mathbb{R}_{\geq 0}. \]

Thus we have the map

\[ \text{Spa}(\mathcal{A}^\pm, \mathcal{A}^+) \to \mathcal{M}(A), \quad x \mapsto \| \cdot \|_{\tilde{x}}. \]

It is clear by definition that this map factors through the separated quotient \([\text{Spa}(A, A^o)]\).

**Proposition C.4.34.** The resulting map

\[ [\text{Spa}(\mathcal{A}^\pm, \mathcal{A}^+)] \to \mathcal{M}(A) \quad (\ast) \]

is a homeomorphism.

**Proof.** The map \((\ast)\) is injective by the boundedness of \(\| \cdot \|_{\tilde{x}}\). For \(y \in \mathcal{M}(A)\), the corresponding bounded multiplicative seminorm \(y = \| \cdot \|_y : A \to \mathbb{R}_{\geq 0}\) defines a (maximal) point of \(\text{Spa}(\mathcal{A}^\pm, \mathcal{A}^+)\); note that, since boundedness implies \(\| \cdot \|_y \leq \| \cdot \|_A\), we have \(|f|_y \leq 1\) for any \(f \in A^o\). Thus the map \((\ast)\) is bijective. To compare the topologies of \([\text{Spa}(\mathcal{A}^\pm, \mathcal{A}^+)\] and \(\mathcal{M}(A)\), it suffices to show that the subsets of the form \(\{x \in [\text{Spa}(\mathcal{A}^\pm, \mathcal{A}^+)]: |f(x)| < 1\} \) with \(f \in A^o\) form an open basis of the topology of \([\text{Spa}(\mathcal{A}^\pm, \mathcal{A}^+)\], which is easy to see. \(\square\)

If \(\mathcal{A} = (\mathcal{A}^B, F_\mathcal{A})\) is an \(\mathbb{R}_+\)-affinoid ring of adic type, we write, by a slight abuse of notation,

\[ F_1(\mathcal{A}) = (\mathcal{A}^B, (F_\mathcal{A})_1), \]

which is an affinoid ring in the sense of Huber. The notation indicates that Huber’s affinoid structure is obtained by taking the \(F_1\)-part of \(\mathbb{R}_+\)-affinoid structures. We have a continuous mapping

\[ \text{Spec}^{\text{val}} \mathcal{A} \to \text{Spa} F_1(\mathcal{A}), \quad (\ast\ast) \]

called the 1-restriction map, which maps \(x = (z, F)\) to the induced valuation \((F_\mathcal{A})_1 \to F_1\) on \(\mathcal{A}^+ = (F_\mathcal{A})_1\).

**Proposition C.4.35.** The 1-restriction map \((\ast\ast)\) is surjective.

**Proof.** For \(x \in \text{Spa} F_1(\mathcal{A})\), let \(V\) be the corresponding valuation ring, and \(K = \text{Frac}(V)\). Let \(V_1\) be the maximal generization, which corresponds to the point \(\text{sep}(x)\) in \(\mathcal{M}(\mathcal{A}) \cong [\text{Spa} F_1(\mathcal{A})]\), and \(v: K \to \mathbb{R}_+\) be the corresponding absolute value. Consider the subring \(G_1 = V/(F_\mathcal{A})_{<0}\) of \(K_\mathcal{A} = \text{Gr}_{F_\mathcal{A}} K\) in degree 1, and let \(G\) be the graded subring over \(\text{Gr} \mathcal{A}\) generated by \(G_1\) in \(K_\mathcal{A}\). Note that \(G_1\) is a valuation ring, and is the degree-1 part of the graded ring \(G\). Take a graded valuation ring \(\tilde{V}\) of \(K_\mathcal{A}\) such that \(m_{G_1} G \subseteq m_{\tilde{V}}\). Then \(\tilde{V}\) is a graded valuation ring with \(\tilde{V}_1 = G_1\), and the associated filtered valuation \(F\) of \(K\) gives a lift of \(x\) to \(\text{Spec}^{\text{val}} \mathcal{A}\). \(\square\)
Finally, let us describe the relation between valuative spectra and the reified adic spectra of [67]. In [67], 6.1, Kedlaya introduced the notion of *affinoid seminormed ring*, which can be interpreted in our language as a filtered ring \((A, F)\) with a seminormed type integrally closed filtration \(F\) contained in \(F_A^o\). The Banach ring case, the so-called *affinoid Banach rings*, are pairs of the form

\[(A, A^{Gr})\]

consisting of a non-archimedean Banach ring \(A\) and a graded integrally closed subring \(A^{Gr}\) of the graded ring \(Gr_{F^o} A\). A morphism \((A, A^{Gr}) \rightarrow (B, B^{Gr})\) between affinoid Banach rings in this sense is a bounded homomorphism \(A \rightarrow B\) that induces \(A^{Gr} \rightarrow B^{Gr}\).

If \((A, A^{Gr})\) is an affinoid Banach ring in the sense of Kedlaya, then one can define a multiplicative filtration \(F\) on \(A\) by

\[F_r = \text{the preimage of } (A^{Gr})_r \text{ under } F_r^o \longrightarrow (Gr_{F^o} A)_r = F_r^o / F_{<r}.\]

By C.4.18, the resulting filtration \(F\) is integrally closed, and so it yields an \(\mathbb{R}_+\)-affinoid ring \((A, F)\). In this way, one can establish a categorical equivalence between the category of affinoid Banach rings in the sense of Kedlaya and the category of \(\mathbb{R}_+\)-affinoid rings \(\mathcal{A} = (A, F_\mathcal{A})\) such that \(F_\mathcal{A} \subseteq F_A^o\).

Given an affinoid Banach ring \((A, A^{Gr})\) in the sense of Kedlaya, the *reified adic spectrum* \(S_{pra}(A, A^{Gr})\) is the set of reified valuations, that is, pairs \((v, r)\) consisting of a valuation

\[v: A \longrightarrow \Gamma \cup \{0\}\]

with a value target group \(\Gamma\) (written multiplicatively), and a order-preserving homomorphism (called a *reification*)

\[r: \mathbb{R}_+ \longrightarrow \Gamma,\]

such that the following conditions are satisfied:

(a) for any \(r > 0\), and \(f \in F_r\) (where \(F\) is the filtration on \(A\) constructed from \(A^{Gr}\) as above), \(v(f) \leq r\);

(b) for any \(\gamma \in \Gamma_v (= \text{the ordered subgroup generated by } r(\mathbb{R}_+) \text{ and the image of } v)\), there exists \(r \in \mathbb{R}_+\) such that \(r \leq \gamma\).

By (b), one has for any \(\gamma \in \Gamma_v\) a positive real number

\[p(\gamma) = \inf\{r \in \mathbb{R}_+: \gamma \leq r\},\]

\footnote{It was Temkin [96] who first considered filtrations on Berkovich \(K\)-affinoid algebras of this kind and, in particular, the associated graded rings.}
which gives a splitting \( p: \Gamma_v \rightarrow \mathbb{R}_+ \) of the reification \( r \). Hence, as we have seen in C.3.13, the reified valuation \( v \) gives an \( \mathbb{R}_+ \)-valuation (C.3.11). Taking the condition (a) into account, one can thus establish a canonical bijection

\[
\text{Spr}_v(A, A^{Gr}) \cong \text{Spec}^{val}(A, F).
\]

Moreover, comparing the notions of ‘rational subdomains’ on both sides, one can show that this bijection gives a homeomorphism.

C.4. (j) \( \mathbb{R}_+ \)-affinoid algebras of \( \mathbb{R}_+ \)-finite type over \( K \). Let \( K = (K, v = | \cdot |) \) be a non-archimedean Banach field. We allow the case where the valuation \( | \cdot | \) is trivial.

We call a Banach \( K \)-algebra of the form

\[
K \langle \langle r^{-1} X \rangle \rangle = K \langle \langle r_1^{-1} X_1, \ldots, r_n^{-1} X_n \rangle \rangle
\]

(defined in §C.3.(c)) a Berkovich algebra, which is a Tate algebra (0, §9.3.(a)), if \( r_1 = \cdots = r_n = 1 \).

**Definition C.4.36** ([11], §2.1). (1) A Berkovich \( K \)-affinoid algebra (or simply, \( K \)-affinoid algebra, if there is no danger of confusion) is a commutative Banach \( K \)-algebra \( A \) that admits an admissible epimorphism (called presentation)

\[
K \langle \langle r_1^{-1} X_1, \ldots, r_n^{-1} X_n \rangle \rangle \longrightarrow A.
\]

(2) If a presentation (*) can be found with

\[
r_1 = \cdots = r_n = 1,
\]

or equivalently, with

\[
r_1, \ldots, r_n \in \sqrt{|K^\times|} = \{ r \in \mathbb{R}_+: r^n \in |K^\times| \text{ for some } n \geq 1 \}
\]

(cf. [18], 6.1.5/4), then \( A \) is called a strictly \( K \)-affinoid algebra.

We denote by

\[
\text{Aff}_K^B
\]

the category of Berkovich \( K \)-affinoid algebras with bounded \( K \)-algebra homomorphisms. The category \( \text{Aff}_K^B \) contains \( \text{Aff}_K \), the category of classical affinoid algebras over \( K \) (0.9.3.1) and \( K \)-algebra homomorphisms, as a full subcategory (cf. [18], 6.1.3/1).

Note that, if the norm \( | \cdot | \) of \( K \) is non-trivial, then strictly \( K \)-affinoid algebras are nothing but classical affinoid algebras as discussed already in 0, §9.3.(a).
The Berkovich algebra $K\llangle r^{-1}X \rrangle$ is equipped with Gauss norm $\| \cdot \|_{Gauss}$, and hence the filtration $F_w = F^{Sp}$ (see §C.3.(c)), which makes the pair

$$(K\llangle r^{-1}X \rrangle, F_K\llangle r^{-1}X \rrangle)$$

(where $F_K\llangle r^{-1}X \rrangle = F^{Sp}_{K\llangle r^{-1}X \rrangle}$) an $\mathbb{R}_+$-affinoid algebra over $K = (K, F_v)$. 

For a Berkovich $K$-affinoid algebra $A$ with a presentation as in (§C.3.10), one has a filtration on $A$ induced from the filtration $F_{K\llangle r^{-1}X \rrangle}$ (see §C.3.10), and thus $A$ can be regarded as an $\mathbb{R}_+$-affinoid algebra of $\mathbb{R}_+$-finite type over $K = (K, F_v)$. The next theorem shows that this $\mathbb{R}_+$-affinoid algebra structure does not depend on the choice of the presentation.

**Theorem C.4.37.** Let $\mathfrak{G} = (A, F^\mathfrak{G})$ be an $\mathbb{R}_+$-affinoid algebra of $\mathbb{R}_+$-finite type over $K = (K, F_v)$. Then $F^\mathfrak{G} = F^A_{Sp}$.

**Proof.** Choose a presentation $K\llangle r^{-1}X \rrangle \to A$ as in (§C.3.10). We first assume that $|K^\times| = \mathbb{R}_+$. Take elements $a_i \in K$ such that $|a_i| = r_i$ for $i = 1, \ldots, n$, which give rise to an isometry

$$K\llangle r^{-1}X \rrangle \overset{\sim}{\to} K\llangle Y \rrangle,$$

where $Y = (Y_1, \ldots, Y_n)$, mapping $X_i$ to $a_iY_i$ ($i = 1, \ldots, n$). In this way, we can assume that the affinoid structure comes from a presentation as in (§C.3.10) with $r_1 = \cdots = r_n = 1$. In this case, $(F^\mathfrak{G})_1$ is the integral closure of $A_0 = \text{the image of } V\llangle X \rrangle$ in $A$, where $V = (F_K)_1$ is the associated valuation ring of $K$. Moreover, it is easy to see that $F^{Sp}_A r = (F_K)_r \otimes (F_K)_1 (F^{Sp}_A)_1$ for any $r > 0$. By A.4.22, we have $(F^\mathfrak{G})_1 = (F^A_{Sp})_1$, and hence $F^\mathfrak{G} = F^A_{Sp}$.

In general, take an extension of Banach fields $L/K$ such that $|L^\times| = \mathbb{R}_+$; e.g., the completion of $K^\ast$ as in C.3.20. Suppose $F^\mathfrak{G} \neq F^A_{Sp}$. Then, by C.4.11, there exists a filtered valuation field $(M, F_M)$ over $\mathfrak{G}$ such that the image of $F^A_{Sp}$ is not contained in $F_M$. Consider the base change $\mathfrak{G}_L = (A \hat{\otimes}_KL, F_L)$ with the tensor product filtration (see §C.3.(a)). By C.4.4, there exists a filtered valuation field $(N, F_N)$ dominating $(M, F_M)$, which sits in the commutative diagram

$$(M, F_M) \overset{(N, F_N)}{\longrightarrow} (\mathfrak{G}_L, F_{\mathfrak{G}_L})$$

Since, as we have already seen, $F_L = F^A_{Sp}, \mathfrak{G} \to (N, F_N)$ factors through $(\mathfrak{G}, F^A_{Sp})$. This means that the image of $F^A_{Sp}$ in $N$ is contained in $F_M$, which is absurd. \qed
When considering Berkovich $K$-affinoid algebras, we henceforth regard them as $\mathbb{R}_+$-affinoid algebras of $\mathbb{R}_+$-finite type over $K$ in the canonical way as in the theorem. Note that one can also regard Berkovich $K$-affinoid algebras as $\mathbb{R}_+$-affinoid algebras of adic type (C.4.33 (2)), if the valuation of $K$ is non-trivial.

**Corollary C.4.38.** Let $\mathcal{A}$ be a Berkovich $K$-affinoid algebra, and $L/K$ an extension of Banach fields. Set $\mathcal{A}_L = \mathcal{A} \otimes_K L$. Then the canonical map

$$\text{Spec}^{\text{val}} \mathcal{A}_L \longrightarrow \text{Spec}^{\text{val}} \mathcal{A}$$

is surjective.

**Proof.** The $\mathbb{R}_+$-affinoid ring structure on $\mathcal{A}_L$ is the integral closure of the complete tensor product filtration of $F_{\mathcal{A}}$ and $F_L$. Hence the claim follows from C.4.27. □

**Proposition C.4.39.** Let $\mathcal{A}$ be a Berkovich $K$-affinoid algebra. Suppose $|K^\times| = \mathbb{R}_+$. Then the 1-restriction map (§C.4 (i))

$$\text{Spec}^{\text{val}} \mathcal{A} \longrightarrow \text{Spa} F_1(\mathcal{A})$$

is a homeomorphism.

**Proof.** By C.3.19, the map in question is bijective. For $f = (f_0, f_1, \ldots, f_n)$ and $r = (r_1, \ldots, r_n)$, one can take $a_i \in K$ such that $|a_i| = r_i$ for $i = 1, \ldots, n$, hence

$$\text{Spec}^{\text{val}} \mathcal{A} \left( \frac{r_1^{-1} f_1, \ldots, r_n^{-1} f_n}{f_0} \right) = \text{Spec}^{\text{val}} \mathcal{A} \left( \frac{1^{-1} g_1, \ldots, 1^{-1} g_n}{f_0} \right) \cong \text{Spa} F_1(\mathcal{A}) \left( \frac{g_1, \ldots, g_n}{f_0} \right),$$

where $g_i = a_i f_i$ for $i = 1, \ldots, n$. This means that the ational subsets on both sides are the same; see §A.3 (b). □

**Corollary C.4.40.** For a Berkovich $K$-affinoid algebra $\mathcal{A}$, the valuative space $\text{Spec}^{\text{val}} \mathcal{A}$ is reflexive (0.2.4.1).

**Proof.** Take an extension $L/K$ of Banach fields with $|L^\times| = \mathbb{R}_+$. By C.4.38, $\text{Spec}^{\text{val}} \mathcal{A}_L \to \text{Spec}^{\text{val}} \mathcal{A}$ is surjective, and $\text{Spec}^{\text{val}} \mathcal{A}_L$ is reflexive by C.4.39, A.5.2, and 8.2.19. □

**Corollary C.4.41.** Suppose $K$ has a non-trivial valuation. Then, for a Berkovich $K$-affinoid algebra $\mathcal{A}$, $\text{Spa} F_1(\mathcal{A})$ is reflexive.
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C.4. (k) Reflexivity of valuative spectrum.

Theorem C.4.42 (local affineness). Let $\mathcal{O}$ be an $\mathbb{R}_+$-affinoid ring, $X = \text{Spec}^{\text{val}} \mathcal{O}$, and $U \hookrightarrow V$ an open immersion of quasi-compact open subsets of $X$. Then, for any $x \in V$, there exists a quasi-compact open neighborhood $W_x$ of $x$ in $V$ such that $W_x$ and $U \cap W_x$ are $\mathbb{R}_+$-rational subdomains of $X$.

To show the theorem, we need the following lemma.

Lemma C.4.43. Let $X$ be coherent sober space, $U, V \subseteq X$ quasi-compact open subsets, and $x \in X$. Suppose $U \cap G_x = V \cap G_x$, where $G_x$ denotes the set of all generizations of $x$ in $X$. Then there exists a quasi-compact open neighborhood $W_x$ of $x$ such that $U \cap W_x = V \cap W_x$.

Proof. By considering $U \cap V$ instead of $U$, we may assume $U \subseteq V$. Set $C = X \setminus U$. Suppose the assertion is false. Then $C \cap V \cap W_x = \emptyset$ for any quasi-compact open neighborhood $W$ of $x$. Since $C$ is coherent and sober, this implies $G_x = \emptyset$ (0.2.2.10 (2)), which is absurd. □

Proof of Theorem C.4.42. We may assume $V = X$. Set $x = (z, F_x)$. The set $G_x$ of generizations of $x$ in $X$ is canonically identified with $\text{Spec}^{\text{val}}(K_z, F_x)$ regarded as a subset of $X$. Since $U \cap G_x$ is a quasi-compact open subset of $G_x$, there exist $g, f \in \mathcal{G}^B$ and $r > 0$ such that the basic subset (see §C.4. (f)) $B(g, f, r)$ contains $x$ and $U \cap G_x = B(f, g, r^{-1}) \cap G_x$. By C.4.43, there exists an $\mathbb{R}_+$-rational neighborhood $W_x$ of $x$ contained in $B(g, f, r)$ such that $U \cap W_x = B(f, g, r^{-1}) \cap W_x$. Notice that, since $W_x \subseteq B(g, f, r)$, $B(f, g, r^{-1}) \cap W_x$ is an $\mathbb{R}_+$-rational subdomain of $W_x$, and hence of $X$. □

Remark C.4.44. The proof indicates that, in the situation as in C.4.42, one can take $W_x = \text{Spec}^{\text{val}} \mathcal{B}$ such that $U \cap W_x$ is an $\mathbb{R}_+$-rational subdomain of the form $\text{Spec}^{\text{val}} \mathcal{B}(r^{-1})$ for some $f \in (F_{\mathcal{B}})_r$.

Proposition C.4.45. Let $\mathcal{O}$ be an $\mathbb{R}_+$-affinoid ring. Then the following conditions are equivalent:

(a) $X = \text{Spec}^{\text{val}} \mathcal{O}$ is reflexive;

(b) for any $\mathbb{R}_+$-rational localization $\mathcal{B} = (B, F_{\mathcal{B}})$ of $\mathcal{O}$,

$$\text{Spec}^{\text{val}}(B, F_B^{\text{Sp}}) \longrightarrow \text{Spec}^{\text{val}} \mathcal{B}$$

is bijective.

Proof. Suppose (a) holds, and let $\mathcal{B} = (B, F_{\mathcal{B}})$ be an $\mathbb{R}_+$-rational localization of $\mathcal{O}$. By assumption, the quasi-compact open subspace $Y = \text{Spec}^{\text{val}} \mathcal{B}$ of $X$ is also reflective. Since $[Y] = \mathcal{M}(B) = \text{Spec}^{\text{val}}(B, F_B^{\text{Sp}})$,

$$Y^{\text{ref}} = (\text{Spec}^{\text{val}}(B, F_B^{\text{Sp}}))^{\text{ref}} \hookrightarrow \text{Spec}^{\text{val}}(B, F_B^{\text{Sp}}).$$
This means that $Y^\text{ref} \hookrightarrow Y$ factors as

$$Y^\text{ref} \hookrightarrow \text{Spec}^\text{val}(B, F^\text{Sp}_B) \hookrightarrow Y,$$

and also $Y \cong \text{Spec}^\text{val}(B, F^\text{Sp}_B)$.

Conversely, suppose (b) holds, and take an open immersion $U \hookrightarrow V$ of quasi-compact open subsets of $X$ such that $[U] = [V]$. By C.4.42, one has a finite open covering $\{V_\alpha\}_{\alpha \in L}$ of $V$ such that $V_\alpha$ and $U \cap V_\alpha$ are $\mathbb{R}_+$-rational subdomains of $X$ for any $\alpha \in L$. We may assume that $U \cap V_\alpha \hookrightarrow V_\alpha$ is induced from a homomorphism $\mathfrak{B}_\alpha \to \mathfrak{B}'_\alpha$ of $\mathbb{R}_+$-rational localizations of $\mathfrak{A}$. Our assumption $[U] = [V]$ implies $[\text{Spec}^\text{val} \mathfrak{B}_\alpha] = [\text{Spec}^\text{val} \mathfrak{B}'_\alpha]$. It follows from the assumption (b) that $\mathfrak{B}_\alpha$ and $\mathfrak{B}'_\alpha$ are of spectral type, viz., $F^\text{Sp}_{\mathfrak{B}_\alpha} = F^\text{Sp}_{\mathfrak{B}'_\alpha}$ and similarly for $F^\text{Sp}_{\mathfrak{B}'_\alpha}$. From this and the fact that $[\text{Spec}^\text{val} \mathfrak{B}_\alpha] = [\text{Spec}^\text{val} \mathfrak{B}'_\alpha]$ it follows that $\text{Spec}^\text{val} \mathfrak{B}_\alpha = \text{Spec}^\text{val} \mathfrak{B}'_\alpha$ for any $\alpha \in L$ (cf. C.4.46 below), thereby $U = V$. \qed

Note that the proposition, together with C.4.37, gives another proof of C.4.40, the reflexiveness of $\text{Spec}^\text{val} \mathfrak{A}$ for a Berkovich $K$-affinoid algebra $\mathfrak{A}$.

**Remark C.4.46.** For an $\mathbb{R}_+$-rational localization $\mathfrak{A} = (A, F_\mathfrak{A}) \to \mathfrak{A}' = (A', F_{\mathfrak{A}'})$, the following conditions are equivalent:

(a) $A^\text{Sp} \cong A'^\text{Sp}$;

(b) $\mathcal{M}(A)$ is homeomorphic to $\mathcal{M}(A')$;

(c) $\text{Spec}^\text{val}(A', F^\text{Sp}_{A'}) \to \text{Spec}^\text{val}(A, F^\text{Sp}_A)$ is bijective.

**C.5 Non-archimedean analytic space of Banach type**

**C.5. (a) Admissible site of $\mathbb{R}_+$-affinoid rings.** Let $\mathfrak{A} = (A, F_\mathfrak{A})$ be an $\mathbb{R}_+$-affinoid ring. We denote by $R_\mathfrak{A}$ the category of $\mathbb{R}_+$-rational localizations of $\mathfrak{A}$ (see §C.4. (e)). Morphisms in $R_\mathfrak{A}$ (resp. $R_\mathfrak{A}^\text{opp}$) are all epimorphisms (resp. monomorphisms) (C.4.16). When we regard an object $\mathfrak{B}$ of $R_\mathfrak{A}$ as an object of $R_\mathfrak{A}^\text{opp}$, we denote it by $S(\mathfrak{B})$; the same convention is applied also to arrows in $R_\mathfrak{A}$.

We define a notion of coverings in $R_\mathfrak{A}^\text{opp}$ as follows: a collection of arrows $\{S(\mathfrak{B}_\alpha) \to S(\mathfrak{B})\}_{\alpha \in L}$ is a covering of $S(\mathfrak{B})$ if

$$\text{Spec}^\text{val} \mathfrak{B} = \bigcup_{\alpha \in L} \text{Spec}^\text{val} \mathfrak{B}_\alpha.$$

It is straightforward to check that this notion of covering defines a Grothendieck topology $J_\mathfrak{A}$ on $R_\mathfrak{A}^\text{opp}$. We set

$$D_\mathfrak{A} = (R_\mathfrak{A}^\text{opp}, J_\mathfrak{A}),$$

and call it the *admissible site* of $\mathfrak{A}$.
Proposition C.5.1. There exists an equivalence of the associated topoi

\[ D_\mathfrak{A} \cong \text{top}(\text{Spec}^{\text{val}} \mathfrak{A}) \]

(see 0, §2.7. (a) for the notation).

Proof. Let \( R \) be the set of all \( \mathbb{R}_+ \)-rational subdomains of \( \text{Spec}^{\text{val}} \mathfrak{A} \) (§C.4. (f)). The set \( R \) is partially ordered by inclusion, and is regarded as a category. We further regard the poset \( R \) as a site in a standard manner; a covering of \( a \in R \) is a finite collection \( \{b_1, \ldots, b_r\} \) of elements in \( R \) such that \( a = \sup\{b_1, \ldots, b_r\} \). (Note that \( R \) is closed under finite intersections.) Since \( R \) generates the topology of \( \text{Spec}^{\text{val}} \mathfrak{A} \) (C.4.25) and any \( U \in R \) is coherent, the morphism of sites \( \text{Spec}^{\text{val}} \mathfrak{A} \to R \) induces an equivalence of the associated topoi. Clearly, there exists a morphism of sites \( j : D_\mathfrak{A} \to R \) associated to the functor \( D_\mathfrak{A} \to R \) that maps \( S(\mathfrak{B}) \) to \( \text{Spec}^{\text{val}} \mathfrak{B} \). Suppose \( S(\mathfrak{B}) \to S(\mathfrak{B}') \) induces \( \text{Spec}^{\text{val}} \mathfrak{B} \cong \text{Spec}^{\text{val}} \mathfrak{B}' \). Due to this and the fact that \( S(\mathfrak{B}) \to S(\mathfrak{B}') \) is a monomorphism (and so \( S(\mathfrak{B}) \times_{S(\mathfrak{B}')} S(\mathfrak{B}) \cong S(\mathfrak{B}) \)), \( S(\mathfrak{B}) \to S(\mathfrak{B}') \) is a covering arrow, and hence, for any sheaf \( F \) on \( D_\mathfrak{A} \), we have \( F(S(\mathfrak{B})) \cong F(S(\mathfrak{B}')) \) by the definition of the Grothendieck topology \( J_\mathfrak{A} \). Thus \( j^{-1} \) induces an equivalence of the associated topoi. Then we obtain the desired equivalence of topoi by composition of the equivalences obtained above. \( \square \)

We define a presheaf \( \tilde{\mathfrak{O}}_\mathfrak{A} \) on the site \( D_\mathfrak{A} \) by

\[ \tilde{\mathfrak{O}}_\mathfrak{A}(S(\mathfrak{B})) = \mathfrak{B}^B. \]

This is a presheaf of Banach rings on \( D_\mathfrak{A} \), and we denote by \( \mathfrak{O}_\mathfrak{A} \) the sheaf of rings given by sheafification of \( \tilde{\mathfrak{O}}_\mathfrak{A} \) seen as a presheaf of rings. By C.5.1, one can regard \( \mathfrak{O}_\mathfrak{A} \) as a sheaf on \( \text{Spec}^{\text{val}} \mathfrak{A} \).

For \( x \in \text{Spec}^{\text{val}} \mathfrak{A} \), the stalk \( \mathfrak{N}_x = \mathfrak{O}_{\mathfrak{A},x} \) at \( x \) is a ring that allows the following description. Let \( S_x \) be the category of \( \mathbb{R}_+ \)-rational localization \( \mathfrak{B} \) is such that \( x \) lies in the image of \( \text{Spec}^{\text{val}} \mathfrak{B} \). Then, by what we have seen in §C.4. (f), \( S_x \) is directed, and

\[ \mathfrak{N}_x = \lim_{\mathfrak{B} \in \text{obj}(S_x)} \mathfrak{B}^B, \]

where the inductive limit is taken in the category of rings. Notice that the ring \( \mathfrak{N}_x \) comes with a canonical ring homomorphism

\[ \mathfrak{N}_x \to \mathcal{H}(z), \]

where \( x = (z, F) \). Let us denote by \( m_x \) the kernel of this map, and set

\[ k(x) = \mathfrak{N}_x/m_x. \]
Proposition C.5.2. Let $\mathcal{O}$ be an $\mathbb{R}^+$-affinoid ring, and $x = (z, F) \in \text{Spec}^{\text{val}} \mathcal{O}$.

1. An element $h \in \mathcal{O}_x$ is invertible if and only if its image in $\mathcal{H}(z)$ is non-zero. In particular, $(\mathcal{O}_x, \mathfrak{m}_x)$ is a local ring, and thus $(\text{Spec}^{\text{val}} \mathcal{O}, \mathcal{O}_\mathcal{O})$ is a locally ringed space.

2. The field $k(x)$, view as a subfield of $\mathcal{H}(z)$, is dense in $\mathcal{H}(z)$.

Proof. (1) Let $B$ be an object of $S_x$, and let $H \in B$ represent $h \in \mathcal{O}_x$. Replacing $B$ further by a rational localization around $x$ if necessary, we may assume that $|H(w)|$, viewed as a real-valued function of $w \in \mathcal{M}(B)$, is non-zero everywhere on $\mathcal{M}(B)$. Then $H$ is invertible in $B$, and hence $h$ is invertible in $\mathcal{O}_x$.

(2) Let $A'$ be the image of $\mathcal{O} \to \mathcal{H}(z)$. Then, by the definition of the complete residue field $\mathcal{H}(z)$, the fraction field $K'$ of $A'$ is dense in $\mathcal{H}(z)$. Clearly, $K'$ is a subfield of $k(x)$, whence the claim. $\square$

Let $x = (z, F) \in \text{Spec}^{\text{val}} \mathcal{O}$. By C.5.2 (2), there exists a unique filtration $F_x$ on $k(x)$, the completion of which with respect to the seminorm $\| \cdot \|_z$ coincides with $F$, viz., there exists a unique filtered valuation field of the form $(k(x), F_x)$, the completion of which is the given $(\mathcal{H}(z), F)$. Define a filtration $F_{\mathcal{O}_x}$ on $\mathcal{O}_x$ by

$$(F_{\mathcal{O}_x})_r = \text{the preimage of } (F_x)_r \text{ under } \mathcal{O}_x \to k(x).$$

for $r > 0$, which obviously gives rise to a filtered ring $(\mathcal{O}_x, F_{\mathcal{O}_x})$.

Proposition C.5.3. (1) The filtered ring $(\mathcal{O}_x, F_{\mathcal{O}_x})$ coincides with

$$\lim_{\longrightarrow} \mathfrak{B},$$

where the inductive limit is taken in the category of filtered rings.

(2) For $r > 0$,$$
(F_{\mathcal{O}})_r = \{ f \in \mathcal{O} : f \in (F_{\mathcal{O}_x})_r \text{ for any } x \in \text{Spec}^{\text{val}} \mathcal{O} \}.$$

Proof. First notice that the category of filtered rings has small inductive limits, which commute with the inductive limits in the category of rings by the forgetful functor. Let $(\mathcal{O}_x, F')$ be the filtered inductive limit as in (1). By definition, we have $F' \subseteq F_{\mathcal{O}_x}$. We first check that $F'_{<r} = (F_{\mathcal{O}_x})_{<r}$ for $r > 0$. Let $h \in (F_{\mathcal{O}_x})_{<r}$, and take $\mathfrak{B}$ from $S_x$ and $H \in \mathfrak{B}$ such that $h$ is represented by $H$. Since $|H(z)| < r$, there is an overconvergent open neighborhood $V$ of $z$ in $\text{Spec}^{\text{val}} \mathcal{O}$ such that $|H(w)| < r$ for all $w \in \text{sep}_\mathcal{O}(V) \subseteq \mathcal{M}(\mathcal{O})$. Replacing $\mathfrak{B}$ by another one in $S_x$ with smaller valuative spectrum, we may assume that $\text{Spec}^{\text{val}} \mathfrak{B}$ is contained in $V$, and hence that $H \in (F_{\mathfrak{B}})_{<r} = (F_{\mathfrak{B}}^\text{sp})_{<r}$. Thus we have $h \in F'_{<r}$, as desired.
To show $F'_r = (F/A_x)_r$ for $r > 0$, take $\alpha \in (F/A_x)_r$. Then, by what we have seen in the proof of C.5.2 (2), $[\alpha]_r \in \text{Gr}_{F/A_x} A_x$ is represented by $\tilde{a}/\tilde{b}$, where $a, b \in A$, $\tilde{a}$ and $\tilde{b}$ denote the image of $a$ and $b$ in Gr $A$, and $\tilde{b} \neq 0$. Let $s$ be a positive real number such that $s > 1/|b(z)|$. Then $a/b$ in

$\mathfrak{B} = \mathfrak{A}\left(\frac{r^{-1}a, s^{-1}1}{b}\right)$

represents $\tilde{a}/\tilde{b}$, and $\alpha - a/b \in F'_{<r}$. Hence we have $\alpha \in F'_r$, as desired, and thus (1) is proved. (2) follows immediately from C.4.11.

The following proposition determines $\mathbb{R}_+$-rational localizations $\mathfrak{A} \to \mathfrak{B}$ with the same valuative spectrum, and thus clarifies the dependence of $\mathbb{R}_+$-rational subdomains on their presentations as the valuative spectra of $\mathbb{R}_+$-rational localizations.

**Proposition C.5.4.** Let $\mathfrak{A} \to \mathfrak{B}$ be an $\mathbb{R}_+$-rational localization. Then $\text{Spec}^{\text{val}} \mathfrak{B} = \text{Spec}^{\text{val}} \mathfrak{A}$ if and only if $\mathfrak{B}$ is isomorphic to an $\mathbb{R}_+$-rational localization of the form

$\mathfrak{A}\left(r_1^{-1}g_1, \ldots, r_n^{-1}g_n\right)$

for $g_i \in (F/A)_r$ ($i = 1, \ldots, n$).

**Proof.** Only the ‘if’ part calls for a proof. Set

$\mathfrak{B} = \mathfrak{A}\left(\frac{r_1^{-1}f_1, \ldots, r_n^{-1}f_n}{f_0}\right)$

as an $\mathfrak{A}$-affinoid algebra. The element $f_0$ takes no zero value on $\mathcal{M}(\mathfrak{B})$, and hence on $\mathcal{M}(\mathfrak{A})$, due to the assumption. This implies that $f_0$ is invertible in $\mathfrak{A}$. Set $g_i = f_0^{-1}f_i$ for $i = 1, \ldots, n$. The image of $g_i$ in $\mathfrak{B}$ is in $(F/A)_r$ for $i = 1, \ldots, n$, and thus $\mathfrak{B}$ is isomorphic to the $\mathbb{R}_+$-rational localization as above. By the equality $\text{Spec}^{\text{val}} \mathfrak{B} = \text{Spec}^{\text{val}} \mathfrak{A}$ and C.5.3 (2), we have $g_i \in (F/A)_r$ for $i = 1, \ldots, n$.

**C.5. (b) Sheaf condition of Banach type.** Let $\mathfrak{A} = (\mathfrak{G}, F/\mathfrak{A})$ be an $\mathbb{R}_+$-affinoid ring, and consider the presheaf $\tilde{\mathcal{O}}_{\mathfrak{A}}$ on $D_{\mathfrak{A}}$ as in §C.5 (a), which is a presheaf of Banach rings with all restriction maps being bounded homomorphisms.

**Definition C.5.5.** We say that $\tilde{\mathcal{O}}_{\mathfrak{A}}$ satisfies the *sheaf condition of Banach type*, or that the $\mathbb{R}_+$-affinoid ring $\mathfrak{A}$ is $\mathbb{R}_+$-*sheafy*, if, for any finite covering $\{S(\mathfrak{B}_\alpha)\}_{\alpha \in L}$ of $S(\mathfrak{B})$,

$$\tilde{\mathcal{O}}_{\mathfrak{A}}(S(\mathfrak{B})) \longrightarrow \ker \left( \prod_{\alpha \in L} \tilde{\mathcal{O}}_{\mathfrak{A}}(S(\mathfrak{B}_\alpha)) \longrightarrow \prod_{\alpha, \beta \in L} \tilde{\mathcal{O}}_{\mathfrak{A}}(S(\mathfrak{B}_\alpha \otimes_\mathfrak{B} \mathfrak{B}_\beta)) \right)$$

is a bounded isomorphism of Banach rings. (Notice that the right-hand side is a closed set of the Banach ring $\prod_{\alpha \in L} \tilde{\mathcal{O}}_{\mathfrak{A}}(S(\mathfrak{B}_\alpha))$, and hence is a Banach ring.)
Remark C.5.6. The ‘$\mathbb{R}_+$-sheafy’ condition is, in our situation, equivalent to ‘sheafy’ in [67], 3.19. Note that, as we have seen in the end of §C.4. (i), affinoid Banach rings in the sense of Kedlaya are nothing but $\mathbb{R}_+$-affinoid rings of the form $\mathcal{A} = (A, F_\mathcal{A})$ such that $F_\mathcal{A} \subseteq F_\mathcal{A}^\circ$. As the next proposition shows, the last condition in our situation is rather a consequence of $\mathbb{R}_+$-sheafiness.

We denote by

$$\mathbb{R}_+\text{-Aff}^\text{Sh}$$

the full subcategory of $\mathbb{R}_+\text{-Aff}$, the category of $\mathbb{R}_+$-affinoid rings, consisting of $\mathbb{R}_+$-sheafy $\mathbb{R}_+$-affinoid rings.

Proposition C.5.7. Let $\mathcal{A} = (\mathcal{A}, F_\mathcal{A})$ be an $\mathbb{R}_+$-affinoid ring, and suppose $\mathcal{A}$ is $\mathbb{R}_+$-sheafy.

(1) The inclusion $F_\mathcal{A} \subseteq F_\mathcal{A}^\circ$ holds.

(2) The functor $\mathfrak{B} \mapsto \text{Spec}^{\text{val}} \mathfrak{B}$ gives an equivalence from the site of $\mathbb{R}_+$-rational subdomains on $\text{Spec}^{\text{val}} \mathcal{A}$ to the site $D_\mathcal{A}$.

Proof. (1) Let $f \in (F_\mathcal{A})_r$. Then $\text{Spec}^{\text{val}} \mathcal{A} = \text{Spec}^{\text{val}} \mathcal{A}\langle (r^{-1}f) \rangle$, since, by the sheaf condition of Banach type, $\mathcal{A} \rightarrow \mathcal{A}\langle (r^{-1}f) \rangle$ is a bounded homomorphism. Hence $f$ is $r$-power bounded.

(2) By (1) and C.4.23, we have $F_\mathfrak{B} \subseteq F_\mathfrak{B}^\circ$ for any $\mathbb{R}_+$-rational localization $\mathfrak{B}$. Thus for any $r > 0$ and $f \in (F_\mathfrak{B})_r$, the morphism $\mathfrak{B} \rightarrow \mathfrak{B}\langle (r^{-1}f) \rangle$ is a bounded homomorphism. Hence for any rational localization $\mathfrak{B}'$ of $\mathfrak{B}$, $\text{Spec}^{\text{val}} \mathfrak{B}' = \text{Spec}^{\text{val}} \mathfrak{B}$ implies the existence of a bounded isomorphism $\mathfrak{B} \cong \mathfrak{B}'$ by C.5.4. □

By C.5.7 (2), we see that an $\mathbb{R}_+$-rational subdomain $U$ of $\text{Spec}^{\text{val}} \mathcal{A}$ determines an $\mathbb{R}_+$-rational localization $\mathfrak{B}$ up to isomorphism. In particular, for an $\mathbb{R}_+$-sheafy $\mathbb{R}_+$-affinoid ring $\mathcal{A}$, one can define a presheaf $\mathcal{O}_\mathcal{A}$ on $\text{Spec}^{\text{val}} \mathcal{A}$ such that, for any $\mathbb{R}_+$-rational subdomain $U = \text{Spec}^{\text{val}} \mathfrak{B}$,

$$\mathcal{O}_\mathcal{A}(U) = \mathfrak{B}^\mathfrak{B},$$

and for any open subset $V$,

$$\mathcal{O}_\mathcal{A}(V) = \lim_U \mathcal{O}_\mathcal{A}(U),$$

where $U$ runs through all $\mathbb{R}_+$-rational subdomains contained in $V$. Note that this presheaf $\mathcal{O}_\mathcal{A}$ is a sheaf, and is nothing but the one corresponding to the sheaf $\bar{\mathcal{O}}_\mathcal{A}$ on the site $D_\mathcal{A}$.

An important example of $\mathbb{R}_+$-sheafy $\mathbb{R}_+$-affinoid rings is provided by the Berkovich $K$-affinoid algebras (uniquely regarded as $\mathbb{R}_+$-affinoid rings as in §C.4. (j)).
Proposition C.5.8. Any Berkovich \( K \)-affinoid algebra \( \mathcal{O} \) is \( \mathbb{R}_+ \)-sheafy (C.5.5).

Proof. Note first that \( X = \text{Spec} \text{val}\mathcal{O} \) is reflexive (C.4.40), and that, for any Berkovich \( K \)-affinoid algebra \( \mathcal{B} \), \( F^\mathcal{B} = F^\mathcal{B} \) due to C.4.37. Hence we have a well-defined presheaf \( \widehat{\mathcal{O}} \) on the site of rational subdomains and finite coverings by rational subdomains of \( \mathcal{M}(\mathcal{O}) \) defined by, for any \( \mathbb{R}_+ \)-rational subdomain \( U = \text{Spec} \text{val}\mathcal{B} \) of \( X \), \( \widehat{\mathcal{O}}([U]) = \widehat{\mathcal{O}}(U) = \mathcal{B} \). The sheaf condition for \( \widehat{\mathcal{O}} \) as in [11], 2.2.5 is nothing but the sheaf condition of Banach type for \( \widehat{\mathcal{O}} \).

For more information on the sheaf condition, see [67], §8.

C.5. (c) Metrized Banach ringed spaces.

Definition C.5.9. (1) A locally ringed space \( (X, \mathcal{O}_X) \) is a Banach ringed space if it enjoys the following properties:

(a) \( X \) is a sober locally coherent space;

(b) for a coherent open subset \( U \subseteq X \), \( \mathcal{O}_X(U) \) is a Banach ring;

(c) the restriction map \( \mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U) \) for \( U \subseteq V \), where \( U \) and \( V \) are coherent open subsets, is a bounded homomorphism of Banach rings;

(d) for any finite open covering \( U = \bigcup_{\alpha \in L} U_\alpha \) of a coherent open subset \( U \) by coherent open subsets, the isomorphism

\[
\mathcal{O}(U) \cong \ker \left( \prod_{\alpha \in L} \mathcal{O}(U_\alpha) \rightarrow \prod_{\beta \in L} \mathcal{O}(U_\alpha \cap U_\beta) \right)
\]

is a bounded isomorphism. (Note that the ring in the right-hand side is a closed subring of \( \prod_{\alpha \in L} \mathcal{O}(U_\alpha) \).)

(2) A morphism \( f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y) \) of Banach ringed spaces is a morphism of locally ringed spaces, with locally quasi-compact underlying continuous mappings, having the following property:

(e) for any coherent open subset \( U \) of \( X \) and any coherent open subset \( V \) of \( Y \) such that \( U \subseteq f^{-1}(V) \), the induced homomorphism \( \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(U) \) is a bounded homomorphism.

By definition, any open subset of a Banach ringed space is naturally a Banach ringed space, which we call an open subspace.
**Definition C.5.10.** Let \( X = (X, \mathcal{O}_X) \) be a Banach ringed space satisfying the following conditions:

(a) the underlying topological space \( X \) is a valuative space;

(b) if \( x \in X \) is a generization of \( y \in X \), then the natural ring homomorphism \( \mathcal{O}_{X,y} \to \mathcal{O}_{X,x} \) is local.

An \( \mathbb{R}_+ \)-valuation \( v \) of \( X = (X, \mathcal{O}_X) \) is a family of \( \mathbb{R}_+ \)-filtered valuation fields \( v = \{v_x = (k(x), F_x)\}_{x \in X} \), where \( k(x) \) is the residue field at \( x \) for \( x \in X \), enjoying the following properties:

(c) if \( x \) is a maximal point, then \( v_x \) is a filtered valuation field of maximal type;

(d) if \( x \) is a generization of \( y \in X \), then the local homomorphism \( \mathcal{O}_{X,y} \to \mathcal{O}_{X,x} \) induces a filtered homomorphism \( v_y = (k(y), F_y) \to v_x = (k(x), F_x) \).

A Banach ringed space satisfying (a) and (b) and equipped with an \( \mathbb{R}_+ \)-filtered valuation \( \mathcal{X} = ((X, \mathcal{O}_X), v) \) is called an \( \mathbb{R}_+ \)-metrized Banach ringed space.

In this setting, the structure sheaf \( \mathcal{O}_X \) has the multiplicative filtration \( F_\mathcal{X} \) defined as follows. For any open subset \( U \subseteq X \) and \( r > 0 \),

\[
F_\mathcal{X}(U)_r = \{ f \in \mathcal{O}_X(U) : (f_x \mod m_{X,x}) \in (F_x)_r \text{ for any } x \in U \}.
\]

By definition, any open subset of an \( \mathbb{R}_+ \)-metrized Banach ringed space is naturally an \( \mathbb{R}_+ \)-metrized Banach ringed space, which we call an open subspace.

**Definition C.5.11.** A morphism

\[
f : \mathcal{X} = ((X, \mathcal{O}_X), v) \longrightarrow \mathcal{X}' = ((X', \mathcal{O}_{X'}), v')
\]

of \( \mathbb{R}_+ \)-metrized Banach ringed spaces is a morphism of Banach ringed spaces \( f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) satisfying the following conditions:

(a) the underlying continuous mapping of \( f \) is valuative (0.2.3.21);

(b) for any \( x \in X \), the local homomorphism \( \mathcal{O}_{X', f(x)} \to \mathcal{O}_{X, x} \) induces a dominat-
ing filtered homomorphism \( (k(f(x)), F_{f(x)}) \to (k(x), F_x) \).

Note that a morphism \( f : \mathcal{X} \to \mathcal{X}' \) of \( \mathbb{R}_+ \)-metrized Banach ringed spaces respects the filtrations \( F_\mathcal{X} \) and \( F_{\mathcal{X}'} \).
**Theorem C.5.12.** Let $\mathfrak{A}$ be an $\mathbb{R}_+$-affinoid ring, and suppose $\mathfrak{A}$ is $\mathbb{R}_+$-sheafy. Let $X = (\text{Spec}^{\text{val}} \mathfrak{A}, \mathcal{O}_\mathfrak{A})$ be the resulting local by ringed space.

(1) The locally ringed space $X$ is a Banach ringed space. Moreover, there exists a natural $\mathbb{R}_+$-valuation $v_\mathfrak{A}$ of $X$, which make $((X = \text{Spec}^{\text{val}} \mathfrak{A}, \mathcal{O}_\mathfrak{A}), v_\mathfrak{A})$ an $\mathbb{R}_+$-metrized Banach ringed space.

(2) The functor

$$\mathfrak{A} \mapsto ((\text{Spec}^{\text{val}} \mathfrak{A}, \mathcal{O}_\mathfrak{A}), v_\mathfrak{A})$$

from the category of $\mathbb{R}_+$-sheafy $\mathbb{R}_+$-affinoid rings to the category of $\mathbb{R}_+$-metrized Banach ringed spaces, is fully faithful.

**Proof.** (1) First, note that $X$ is a locally ringed space due to C.5.2. To show (1), we first check that $X$ is a Banach ringed space. The condition (a) of C.5.9 is clear, since $X$ is coherent. For any coherent open subset $U \subseteq X$, choose a finite open covering $\{U_\alpha\}_{\alpha \in L}$ by $\mathbb{R}_+$-rational subdomains, and define

$$\mathcal{O}(U) = \ker\left( \prod_{\alpha \in L} \mathcal{O}_\mathfrak{A}(U_\alpha) \rightarrow \prod_{\alpha, \beta \in L} \mathcal{O}_\mathfrak{A}(U_\alpha \cap U_\beta) \right).$$

Then, as a closed subring of $\prod_{\alpha \in L} \mathcal{O}_\mathfrak{A}(U_\alpha)$, $\mathcal{O}(U)$ is a Banach ring and coincides with $\mathcal{O}_\mathfrak{A}(U)$ if $U$ is an $\mathbb{R}_+$-rational subdomain. It is straightforward to see that, in general, $\mathcal{O}(U)$ is independent of the choice of the finite covering $\{U_\alpha\}_{\alpha \in L}$, and that the restriction map $\mathcal{O}(V) \rightarrow \mathcal{O}(U)$ for $U \subseteq V$ is bounded, due to the sheaf condition of Banach type. Thus, so far, we have shown that the conditions (b), (c), (d) of C.5.9 are satisfied, and thus that $X$ is a Banach ringed space.

Let us show the rest of (1). The condition (a) of C.5.10 follows from C.4.29. The condition (b) is satisfied due to C.5.2 (1). Now, for each $x \in \text{Spec}^{\text{val}} \mathfrak{A}$, we attach the filtered valuation field $v_x = (k(x), F_x)$ given in §C.5.(a). Then $v_\mathfrak{A} = \{v_x\}_{x \in X}$ satisfies the conditions (c) and (d) in C.5.10. Hence $((X = \text{Spec}^{\text{val}} \mathfrak{A}, \mathcal{O}_\mathfrak{A}, v_\mathfrak{A})$ an $\mathbb{R}_+$-metrized Banach ringed space.

(2) To check that any morphism $\varphi: \mathfrak{B} \rightarrow \mathfrak{A}$ of $\mathbb{R}_+$-sheafy $\mathbb{R}_+$-affinoid rings induces a morphism of $\mathbb{R}_+$-metrized Banach ringed spaces

$$f: X = ((\text{Spec}^{\text{val}} \mathfrak{A}, \mathcal{O}_\mathfrak{A}, v_\mathfrak{A}) \rightarrow Y = ((\text{Spec}^{\text{val}} \mathfrak{B}, \mathcal{O}_\mathfrak{B}, v_\mathfrak{B}),$$

we only need to verify the following: if $f(U) \subseteq V$, where $U = \text{Spec}^{\text{val}} \mathfrak{A}'$ (resp. $V = \text{Spec}^{\text{val}} \mathfrak{B}'$) is an $\mathbb{R}_+$-rational subdomain of $\text{Spec}^{\text{val}} \mathfrak{A}$ (resp. $\text{Spec}^{\text{val}} \mathfrak{B}$), then there exists uniquely a morphism $\varphi': \mathfrak{B}' \rightarrow \mathfrak{A}'$ of $\mathbb{R}_+$-affinoid rings such that

$$\begin{array}{c}
\mathfrak{B} \xleftarrow{\varphi} \mathfrak{A} \\
\downarrow \text{res} \\
\mathfrak{B}' \xrightarrow{\varphi'} \mathfrak{A'}
\end{array}$$
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commutes. Set \( \mathcal{B}' = \mathcal{B}(\frac{g^{-1} \varphi_1 \cdots g^{-1} \varphi_n}{g_0}) \). Let us show that \( \varphi(g_i)/\varphi(g_0) \in (F_{\mathcal{A}'})_r \) for \( i = 1, \ldots, n \). Indeed, for any \( x \in V = \text{Spec}^{\text{val}} \mathcal{A}' \), \( k(x) \) contains the image of \( \varphi(g_i)/\varphi(g_0) \) by the assumption, and since \( \varphi \) respects filtrations, it lies in \( (F_x)_r \). Hence \( \varphi(g_i)/\varphi(g_0) \in (F_{\mathcal{A}'})_r \) for \( i = 1, \ldots, n \). Now, set \( \mathcal{A}'' = \mathcal{A}'(\frac{\varphi_1^{-1} \varphi(g_1) \cdots \varphi_n^{-1} \varphi(g_n)}{\varphi(g_0)}) \), which allows a morphism \( \mathcal{B}' \rightarrow \mathcal{A}'' \). Since \( \varphi(g_i)/\varphi(g_0) \in (F_{\mathcal{A}'})_r \) for \( i = 1, \ldots, n \), we have \( U = \text{Spec}^{\text{val}} \mathcal{A}' = \text{Spec}^{\text{val}} \mathcal{A}'' \) due to C.5.4.

Then, by C.5.7 (2), we have \( \mathcal{A}'' \cong \mathcal{A}' \), whence the claim.

Conversely, suppose a morphism \( f \) as above is given. By definition, \( f \) induces a bounded homomorphism of Banach rings \( \varphi: \mathcal{B} \rightarrow \mathcal{A} \). Let \( x = (z, F) \in \text{Spec}^{\text{val}} \mathcal{A} \). By the condition (b) of C.5.11, the composition

\[
\mathcal{B} \rightarrow \mathcal{A} \rightarrow \mathcal{A}(z)
\]

and \( F \) on \( \mathcal{A}(z) \) define the point \( f(x) \) of \( \text{Spec}^{\text{val}} \mathcal{B} \). Based on this, let us show that \( \varphi \) preserves the filtrations \( F_{\mathcal{B}} \) and \( F_{\mathcal{A}} \), hence giving a morphism of \( \mathbb{R}_+ \)-affinoid rings \( \mathcal{B} \rightarrow \mathcal{A} \). For \( h \in (F_{\mathcal{B}})_r \) and \( y = f(x) \in \text{Spec}^{\text{val}} \mathcal{B} \), let \( \tilde{h}_y \) be the image of \( h \) in the residue field \( k(y) \). We know that \( \tilde{h}_y \in (F_{\mathcal{A}})_r \). Then the image of \( \varphi(h) \) in \( k(x) \) coincides with the image of \( \tilde{h}_y \), which belongs to \( (F_x)_r \) by the condition (b) of C.5.11. Hence, \( \varphi(h) \in (F_{\mathcal{A}})_r \).

Thus, \( \varphi \) induces a morphism \( g: X \rightarrow Y \) of \( \mathbb{R}_+ \)-metrized Banach ringed spaces. We need to show that \( f = g \). For \( x = (z, F) \in X \), consider \( f(x) = (f(z), F') \) and \( g(x) = (g(z), F'') \). Observe that the diagrams

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\varphi} & \mathcal{A} \\
\downarrow & & \downarrow \\
(k(f(z)), F') & \rightarrow & (k(z), F),
\end{array} \quad \begin{array}{ccc}
\mathcal{B} & \xrightarrow{\varphi} & \mathcal{A} \\
\downarrow & & \downarrow \\
(k(g(z)), F'') & \rightarrow & (k(z), F)
\end{array}
\]

are commutative with dominating lower horizontal arrows (see C.5.11 (b)). Then, both \( f(x) \) and \( g(x) \) are the filtered valuations on \( \mathcal{B} \) induced from the composition

\[
\mathcal{B} \rightarrow \mathcal{A} \rightarrow (k(z), F) \rightarrow (\mathcal{A}(z), F),
\]

which implies that \( f(x) = g(x) \).

Finally, we need to show that the morphism of the structure sheaves is the same as the one induced by \( \varphi \). For two \( \mathbb{R}_+ \)-rational subdomains \( U = \text{Spec}^{\text{val}} \mathcal{A}' \) and \( V = \text{Spec}^{\text{val}} \mathcal{B} \) of \( X \) and \( Y \), respectively, such that \( f(U) \subseteq V \), the square of bounded homomorphisms of Banach rings

\[
\begin{array}{ccc}
\mathcal{B}' & \xrightarrow{\varphi} & \mathcal{A}' \\
\text{res} & & \text{res} \\
\downarrow & & \downarrow \\
\mathcal{B} & \xrightarrow{\varphi} & \mathcal{A}
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\psi} & \mathcal{A}' \\
\downarrow & & \downarrow \\
\mathcal{B}' & \xrightarrow{\varphi} & \mathcal{A}
\end{array}
\]

are commutative, where \( \text{res} \) is the restriction morphism.
is commutative, where \( \psi \) is the bounded homomorphism induced by \( f \). Since \( \mathcal{B} \to \mathcal{B}' \) is an epimorphism by (C.4.16), \( \psi \) is equal to the homomorphism induced from \( \varphi \). Since this is valid for all \( \mathbb{R}_+ \)-rational subdomains \( U, V \) with \( f(U) \subseteq V \), one sees, by considering filters of \( \mathbb{R}_+ \)-rational subdomains, that \( f = g \), as desired.

**Notation C.5.13.** In the sequel, for an \( \mathbb{R}_+ \)-affinoid ring \( \mathcal{A} = (\mathcal{A}, F_\mathcal{A}) \), we simply denote

\[
\text{Spec}^{\text{val}} \mathcal{A} = ((\text{Spec}^{\text{val}} \mathcal{A}, \mathcal{O}_\mathcal{A}), v_\mathcal{A}),
\]

unless there is a danger of confusion.

**Definition C.5.14.** An \( \mathbb{R}_+ \)-metrized Banach ringed space that is isomorphic to \( \text{Spec}^{\text{val}} \mathcal{A} \) for an \( \mathbb{R}_+ \)-sheafy \( \mathbb{R}_+ \)-affinoid ring \( \mathcal{A} \) is called an \( \mathbb{R}_+ \)-metrized affinoid space. Especially, \( \text{Spec}^{\text{val}} \mathcal{A} \) is called the \( \mathbb{R}_+ \)-metrized affinoid space attached to \( \mathcal{A} \).

Any \( \mathbb{R}_+ \)-rational subdomain \( \text{Spec}^{\text{val}} \mathcal{B} \) of \( \text{Spec}^{\text{val}} \mathcal{A} \) is an \( \mathbb{R}_+ \)-metrized affinoid space. Note that, in this situation, even if \( F_\mathcal{A} = F_\mathcal{B}^\circ, F_\mathcal{A}^\circ \) and \( F_\mathcal{B}^\circ \) may not be equal to each other.

**Definition C.5.15.** An \( \mathbb{R}_+ \)-metrized Banach ringed space \( X \) is an \( \mathbb{R}_+ \)-metrized analytic space if it is covered by open subspaces that are \( \mathbb{R}_+ \)-metrized affinoid spaces.

By the discussion above, any open subset of an \( \mathbb{R}_+ \)-metrized analytic space is again an \( \mathbb{R}_+ \)-metrized analytic space, called an open subspace.

We denote by

\[ \mathcal{M} \text{Ansp}_{\mathbb{R}_+} \]

the full subcategory of the category of \( \mathbb{R}_+ \)-metrized Banach ringed spaces consisting of \( \mathbb{R}_+ \)-metrized analytic spaces. By what we have seen above, there exists a fully faithful functor

\[ (\mathbb{R}_+\text{-Aff}^{\text{Sh}})^{\text{opp}} \to \mathcal{M} \text{Ansp}_{\mathbb{R}_+}, \mathcal{A} \mapsto \text{Spec}^{\text{val}} \mathcal{A}, \]

with the essential image consisting of the \( \mathbb{R}_+ \)-metrized affinoid spaces.

For an \( \mathbb{R}_+ \)-metrized analytic space \( S \), one can consider the comma category

\[ \mathcal{M} \text{Ansp}_{\mathbb{R}_+}^S \]

of \( \mathbb{R}_+ \)-metrized analytic spaces over \( S \) with morphisms over \( S \).

**Definition C.5.16.** A morphism \( f: X \to Y \) of \( \mathbb{R}_+ \)-metrized analytic space is said to be locally of \( \mathbb{R}_+ \)-finite type if there exists an open covering \( Y = \bigcup_{i \in I} V_i \) and, for each \( i \in I \), an open covering \( f^{-1}(V_i) = \bigcup_{j \in J_i} U_{ij} \) such that the following conditions are satisfied:
(a) for $i \in I$ and $j \in J_i$, $V_i$ and $U_{ij}$ are $\mathbb{R}_+$-metrized affinoid spaces; say $V_i \cong \text{Spec}^\text{val} \mathfrak{B}_i$ and $U_{ij} = \text{Spec}^\text{val} \mathfrak{G}_{ij}$;

(b) by the homomorphism $\mathfrak{B}_i \to \mathfrak{G}_{ij}$ corresponding to $U_{ij} \to V_i$, $\mathfrak{G}_{ij}$ is of $\mathbb{R}_+$-finite type over $\mathfrak{B}_i$.

A morphism of $\mathbb{R}_+$-finite type is a locally of finite type morphism such that the underlying continuous map is quasi-compact.

C.5. (d) Relation with adic spaces

Definition C.5.17. An $\mathbb{R}_+$-metrized analytic space $X$ is said to be of adic type if there exists an open covering $X = \bigcup_{\alpha \in L} U_\alpha$ by open $\mathbb{R}_+$-affinoid subspaces with the following properties:

(a) for each $\alpha \in L$, $U_\alpha \cong \text{Spec}^\text{val} \mathfrak{G}_\alpha$, where $\mathfrak{G}_\alpha$ is an $\mathbb{R}_+$-affinoid ring of adic type (see C.4.33 (2));

(b) for $\alpha, \beta \in L$, $U_\alpha \cap U_\beta$ has an open covering $U_\alpha \cap U_\beta = \bigcup_{\lambda} V_\lambda$ with $V_\lambda = \text{Spec}^\text{val} \mathfrak{B}_\lambda$, where $\mathfrak{B}_\lambda$ is a finite type $\mathbb{R}_+$-rational localization of both $\mathfrak{G}_\alpha$ and $\mathfrak{G}_\beta$, viz., $\mathfrak{B}_\lambda = \mathfrak{G}_\alpha(\frac{1}{f_1, \ldots, f_n})$, and similarly over $\mathfrak{G}_\beta$.

If $X$ is an $\mathbb{R}_+$-metrized analytic space of adic type with an open affinoid covering $X = \bigcup_{\alpha \in L} U_\alpha$ as above, each affine piece $U_\alpha = \text{Spec}^\text{val} \mathfrak{G}_\alpha$ corresponds, by 1-restriction (§C.4. (i)), to an adic space Spa$_{\mathfrak{G}_\alpha}$; notice that the affinoid ring $F_1(\mathfrak{G}_\alpha)$ is sheafy due to the $\mathbb{R}_+$-sheafiness of $\mathfrak{G}_\alpha$. This construction can be globalized by patching, and we get an adic space, denoted $F_1(X)$

by gluing of $\{\text{Spf} F_1(\mathfrak{G}_\alpha)\}_{\alpha \in L}$.

We denote by

$\mathcal{M}\text{Ansp}^{\mathbb{R}_+, \text{adic}}$

the full subcategory of $\mathcal{M}\text{Ansp}^{\mathbb{R}_+}$ consisting of $\mathbb{R}_+$-metrized analytic space of adic type. The above construction gives a functor

$(-)_1: \mathcal{M}\text{Ansp}^{\mathbb{R}_+, \text{adic}} \longrightarrow \text{Adsp}$. \quad X \mapsto F_1(X),$

called the 1-restriction functor to the category of adic spaces. Note that by C.4.35 there exists a continuous surjective map

$X \longrightarrow F_1(X)$.
As indicated in §C.4. (i), from any affinoid ring \((\mathcal{O}^\pm, \mathcal{A}^+)\) in the sense of Huber equipped with an ideal of definition \(I\) (in the sense therein), one can construct in a standard way an \(\mathbb{R}^+\)-affinoid ring of adic type that gives back the original affinoid ring by \(1\)-restriction. This can also be globalized in the following way. Consider an affinoid \(X\) with an ideal of definition \(I\); here, by an ideal of definition of \(X\), we mean an ideal sheaf of \(\mathcal{O}_X^+\) such that there exists an open affinoid covering \(X = \bigcup_{\alpha \in L} U_\alpha\) with the property that each \(I|_{U_\alpha}\) comes from an ideal of definition of \(\mathcal{O}_\alpha\) (where \(U_\alpha = \text{Spf} \mathcal{O}_\alpha\)). For example, if the adic space comes from a rigid space by the functor \(\text{ZR}\) (§A.5), it is an ideal of definition in the sense of 3.2.3. We also have to fix a real number \(0 < c < 1\). Then one can construct an \(\mathbb{R}^+\)-metrized analytic space \(\tilde{X}\) such that \(F_1(\tilde{X}) = X\) by globalizing the local construction as above, and thus we obtain a functor

\[
\left\{\text{adic spaces with ideal of definition and morphisms respecting ideals of definition}\right\} \longrightarrow \mathcal{MAnsp}_{\mathbb{R}^+, \text{adic}}^{\mathbb{R}^+}.
\]

In particular, adic spaces that admit ideals of definition lie in the essential image of the functor \((\cdot)_1\).

In this way, \(\mathbb{R}^+\)-metrized analytic spaces are regarded as ‘adic spaces (rigid spaces) with extra higher structure,’ where the extra higher structure is an analogue of a metric.

An important example of the last-mentioned functor is the following. Let \(K = (K, v = | \cdot |)\) be a non-archimedean Banach field with non-trivial valuation \(| \cdot |: K \to \mathbb{R}_{\geq 0}\), and \(a \in K\) with \(0 < |a| < 1\), and set \(c = |a|\) (the following construction actually does not depend on the choice of \(a\), since the norm \(| \cdot |\) on \(K\) is fixed). Let \(\mathcal{X} = \text{Spa} \mathcal{O}\) be an affinoid adic space of finite type (A.3.13) over \(K\). Then \(\mathcal{O}^\pm\) is a classical affinoid algebra over \(K\) (or, equivalently, a strictly \(K\)-affinoid algebra), and \(\mathcal{O}^+ = \mathcal{O}^\circ\) (A.4.22). Moreover, by C.4.37, \(\mathcal{O}\) can be viewed uniquely as an \(\mathbb{R}^+\)-affinoid algebra with the filtration \(F\mathcal{A} = F\mathcal{A}^\circ = F\mathcal{A}^{\text{Sp}}\). Hence we have the functor (called the metrization functor)

\[
\mathcal{X} = \text{Spa} \mathcal{O} \mapsto \mathcal{X}_{\text{met}}^{\text{Sp}} = \text{Spec}^{\text{val}}(\mathcal{O}, F\mathcal{A}^\circ)
\]

from the category of affinoid adic spaces of finite type over \(K\) to the category of \(\mathbb{R}^+\)-metrized affinoid spaces of finite type over \(K\). Note that the sheaf condition on the adic space side (equivalent to Tate’s acyclicity) implies the sheaf condition of Banach type, since continuity implies boundedness for \(K\)-linear maps (cf. [18], 2.1.8). By C.5.12 (2), gluing yields canonical fully faithful functor

\[
(\cdot)_{\text{met}}: \text{Adsp}_{\mathbb{R}^+}^{\text{lt}} \longrightarrow \mathcal{MAnsp}_{\mathbb{R}^+}^{\mathbb{R}^+}
\]

from the category of locally of finite type adic spaces over \(K\) to the category of \(\mathbb{R}^+\)-metrized analytic spaces over \(K\).
Corollary C.5.18. The functor $\mathcal{X} \mapsto \mathcal{X}^{\text{met}}$ thus obtained is a fully faithful functor from the category of locally of finite type adic spaces over $K$ to the category of $\mathbb{R}_+$-metrized analytic spaces of $\mathbb{R}_+$-finite type over $K$. The essential image consists of $\mathbb{R}_+$-finite type $\mathbb{R}_+$-metrized analytic spaces $X$ over $K$ having an open $\mathbb{R}_+$-affinoid covering

$$X = \bigcup_{\alpha \in L} U_\alpha, \quad U_\alpha = \text{Spec}^\text{val} \mathcal{G}_\alpha$$

such that

(a) for each $\alpha \in L$, $U_\alpha$ is an $\mathbb{R}_+$-metrized affinoid space of finite type over $K$, viz., $\mathcal{G}_\alpha$ is of finite type over $K$ (see C.4.5), and

(b) for each $\alpha, \beta \in L$, $U_\alpha \cap U_\beta$ is also covered by $\mathbb{R}_+$-affinoid subdomains of finite type over $K$.

Moreover, we have a canonical continuous surjection $\mathcal{X}^{\text{met}} \to \mathcal{X}$ (by the 1-restriction map (§C.4. i)).

C.6 Berkovich analytic geometry

C.6. (a) Gerritzen–Grauert theorem. Let $A$ be a Berkovich $K$-affinoid algebra (C.4.36), and set $X = \mathcal{M}(A)$.

Definition C.6.1 ([11], 2.2.1). A closed subset $U \subseteq \mathcal{M}(A)$ is said to be an affinoid subdomain if there exists a bounded $K$-algebra homomorphism $A \to A_U$ of Berkovich $K$-affinoid algebras such that the following universal mapping property holds: for any Berkovich $L$-affinoid algebra $B$, where $L = (L, | \cdot |)$ is a complete isometric extension of $K$, and any bounded $K$-algebra homomorphism $A \to B$ such that the image of the induced map $\mathcal{M}(B) \to \mathcal{M}(A)$ lies in $U$, there exists a unique bounded $K$-algebra homomorphism $A_U \to B$ such that the diagram

$$\begin{array}{ccc}
A & \longrightarrow & A_U \\
\downarrow & & \downarrow \\
B & \longleftarrow & 
\end{array}$$

commutes.

One sees, as in the case of classical rigid geometry, that the Berkovich $K$-affinoid algebra $A_U$ is uniquely determined by $U$ up to unique isomorphisms, and that the map $\mathcal{M}(A_U) \to \mathcal{M}(A)$ gives a homeomorphism onto $U$ ([11], 2.2.4).
In the rest of this paragraph, we give an argument for the reducing Gerritzen–Grauert type theorem for Berkovich $K$-affinoid algebras to the theorem for Tate’s affinoid algebras, based on generalities of locally coherent spaces. For the preceding studies on affinoid subdomains and Gerritzen-Grauert type theorems in Berkovich geometry, see [35] and [97].

**Lemma C.6.2.** Let $X$ be a locally coherent space, and $W \subseteq X$ a subset of $X$ satisfying the following conditions:

(a) $X \setminus W$ with the subspace topology is sober and retrocompact (0.2.1.7) in $X$;

(b) $W$ is stable under generizations.

Then $W$ is open in $X$.

**Proof.** We may assume $X$ is coherent. Set $C = X \setminus W$. For any quasi-compact open subset $V \subseteq X$, $C \cap V$ is quasi-compact by (a). Since open sets of $C$ of this type form a basis of the topology which is stable under finite intersection, $C$ is coherent and sober. In particular, $C \cap V$ is coherent, for any quasi-compact open subset $V \subseteq X$.

Let $x \in W$. Since $C \cap G_x = \emptyset$ by (b) (where $G_x$ denotes, as before, the set of all generizations of $x$ in $X$), there exists a quasi-compact open neighborhood $V_x$ of $x$ such that $C \cap V_x = \emptyset$; indeed, otherwise $C \cap G_x = \lim_{\leftarrow V} C \cap V$, the inverse limit of non-empty coherent subsets with quasi-compact transition maps, has to be non-empty due to 0.2.2.10. Thus we have, for any $x \in W$, the existence of an open neighborhood $V_x$ contained in $W$, which shows the assertion. 

**Corollary C.6.3.** Let $f : X \rightarrow Y$ be a quasi-compact surjective map between locally coherent sober spaces. Assume $W \subseteq Y$ is stable under generizations, and $f^{-1}(W)$ is open in $X$. Then $W$ is open in $Y$.

**Proof.** We may assume that $Y$ is coherent. Set $C = Y \setminus W$. Let us check the conditions (a) and (b) in C.6.2. It is easy to see that $C \hookrightarrow Y$ is quasi-compact, since $f^{-1}(C) = X \setminus f^{-1}(W)$ is coherent and $f$ is surjective. The subspace $C$ is a $T_0$-space, as $Y$ is $T_0$. Let $Z \subseteq C$ be an irreducible closed subset. Consider the set $E$ of all quasi-compact open subsets of $C$, and set $E' = \{U \in E : U \cap Z \neq \emptyset\}$. Then $E'$ is a filter, and $\bigcap_{U \in E'} (U \cap Z) \neq \emptyset$; indeed, $f^{-1}(\bigcap_{U \in E'} (U \cap Z)) = \bigcap_{U \in E'} f^{-1}(U \cap Z) \neq \emptyset$ (0.2.2.10), since $f$ is surjective and quasi-compact. It is then easy to see that $Z = \{z\}$ for $z \in \bigcap_{U \in E'} (U \cap Z)$. Hence $C$ is sober, whence the claim. 


Theorem C.6.4. Let $K$ be a non-archimedean Banach field, $\mathfrak{G}$ a Berkovich $K$-affinoid algebra, and $\mathfrak{B}$ another Berkovich $K$-affinoid algebra that gives an affinoid subdomain in the sense of C.6.1. Then the induced mapping

$$\text{Spec}^\text{val} \mathfrak{B} \longrightarrow \text{Spec}^\text{val} \mathfrak{G}$$

is an open immersion of topological spaces, viz., a homeomorphism onto an open subset of $\text{Spec}^\text{val} \mathfrak{G}$.

Proof. Take an extension $L/K$ of Banach fields such that $|L^\times| = \mathbb{R}_+$. We first check that

(a) $f : \text{Spec}^\text{val} \mathfrak{B} \to \text{Spec}^\text{val} \mathfrak{G}$ is injective,

(b) $f_L : \text{Spec}^\text{val} \mathfrak{B}_L \to \text{Spec}^\text{val} \mathfrak{G}_L$ is an open immersion,

(c) $f_L(\text{Spec}^\text{val} \mathfrak{B}_L)$ is the inverse image of $f(\text{Spec}^\text{val} \mathfrak{B})$ under the canonical map $\text{Spec}^\text{val} \mathfrak{G}_L \to \text{Spec}^\text{val} \mathfrak{G}$.

For (a), observe first that $\mathcal{H}(f(z)) \cong \mathcal{H}(z)$ for $z \in \mathcal{M}(\mathfrak{B})$. Then the filtered valuation field structures on $\mathcal{H}(z)$ are the same as those of $\mathcal{H}(f(z))$, whence the injectivity in question.

For (b), first note that $\mathfrak{B}_L$ gives an affinoid subdomain of $\mathcal{M}(\mathfrak{G}_L)$ in the sense of Berkovich, and both $\mathfrak{G}_L$ and $\mathfrak{B}_L$ are Tate’s affinoid algebras over $L$. Thus, (b) is a consequence of the Gerritzen–Grauert theorem B.2.2 for Tate’s affinoid spaces, which we assume in this book, due to C.4.39 and A.5.2.

For (c), take $x = (z, F) \in \text{Spec}^\text{val} \mathfrak{G}_L$, which maps to a point in $f(\text{Spec}^\text{val} \mathfrak{B})$. By the argument used to prove (a) above, the filtered homomorphism $(\mathfrak{G}, F_{\mathfrak{G}}) \to (\mathcal{H}(z), F)$ factors through $\mathfrak{B}$, and $x$ lifts to a unique point of $\text{Spec}^\text{val} \mathfrak{B}_L$. Then (c) follows by C.4.37.

To conclude, we need to check that the subset $W = f(\text{Spec}^\text{val} \mathfrak{B})$ is stable under generizations. This also follows from the argument used to prove (a) above. Hence $W = f(\text{Spec}^\text{val} \mathfrak{B})$ is an open subset of $\text{Spec}^\text{val} \mathfrak{G}$ by C.4.38 and C.6.3. Replacing $\mathfrak{B}$ by Berkovich $K$-affinoid algebras giving affinoid subdomains of $\mathcal{M}(\mathfrak{G})$, one deduces that the map $f$ is an open map. □

C.6.(b) Berkovich analytic spaces

Definition C.6.5. (1) Let $X$ be a locally Hausdorff topological space, and $\tau$ a net (0.2.6.1 (2)) on $X$ consisting of compact subsets. A $K$-affinoid atlas $\mathcal{A}$ on $X$ with the net $\tau$ is given by the following data:

(a) a Berkovich $K$-affinoid algebra $A_U$ and a homeomorphism $U \to \mathcal{M}(A_U)$ for each $U \in \tau$;
(b) for each pair \( U, U' \in \tau \) with \( U \subseteq U' \), a bounded \( K \)-algebra homomorphism \( \rho_U^U: A_U' \to A_U \) that identifies \( U \) with an affinoid subdomain in \( \overline{U'} \cong \mathcal{M}(A_U') \) with the corresponding \( K \)-affinoid algebra \( A_U \);

these data are assumed to satisfy the following cocycle condition:

(c) for any triple \( U, U', U'' \in \tau \) with \( U \subseteq U' \subseteq U'' \), we have \( \rho_U^{U''} = \rho_U^{U'} \circ \rho_U^{U''} \).

(2) A (Berkovich) \( K \)-analytic space is a triple \( X = (X, \tau, A) \) consisting of a locally Hausdorff space \( X \), a net \( \tau \) on \( X \), and a \( K \)-affinoid atlas \( A \) on \( X \) with the net \( \tau \). For example, for any Berkovich \( K \)-affinoid algebra \( A \), the set \( X = \mathcal{M}(A) \) with the \( K \)-affinoid atlas given by affinoid subdomains gives a Berkovich \( K \)-analytic space, called a (Berkovich) \( K \)-affinoid space.

(3) A strong morphism of Berkovich \( K \)-analytic spaces \( (X, \tau, A) \to (X', \tau', A') \) consists of

(a) a continuous map \( \varphi: X \to X' \) with the property that for each \( U \in \tau \) there exists \( U' \in \tau' \) such that \( \varphi(U) \subseteq U' \) and

(b) for each pair \( (U, U') \) with \( U \in \tau, U' \in \tau' \), and \( \varphi(U) \subseteq U' \), a bounded \( K \)-algebra homomorphism \( \phi_U: A_U' \to A_U \) such that the induced map \( \mathcal{M}(A_U) \to \mathcal{M}(A_U') \) is identified with \( \varphi|_U: U \to U' \) via \( \mathcal{M}(A_U) \cong U \) and \( \mathcal{M}(A_U') \cong U' \).

(4) A strong morphism \( \varphi: X = (X, \tau, A) \to (X', \tau', A') \) is said to be a quasi-isomorphism if \( \varphi: X \to X' \) is a homeomorphism and, for any pair \( (U, U') \) as in (3) (b), the map \( \mathcal{M}(A_U) \to \mathcal{M}(A_U') \) identifies \( U \cong \mathcal{M}(A_U) \) with an affinoid subdomain in \( U' \).

(5) The category of (Berkovich) \( K \)-analytic spaces, denoted by \( \mathbf{Bsp}_K \), is the quotient category of the category of \( K \)-analytic spaces and strong morphisms by quasi-isomorphisms.

(6) The category of (Berkovich) strictly \( K \)-analytic spaces, denoted by \( \mathbf{Bsp}^*_K \), is defined in the similar way, where all \( K \)-affinoid algebras in the \( K \)-affinoid atlas are strictly \( K \)-affinoid.

Let \( X = (X, \tau, \mathfrak{G}) \) be a \( K \)-analytic space. A subset \( W \subseteq X \) is said to be \( \tau \)-special if it is compact and there exists a finite covering \( W = \bigcup_{i=1}^n W_i \) such that \( W_i \cap W_j \) \((i, j = 1, \ldots, n)\) belongs to \( \tau \) and that \( A_{W_i} \otimes_A A_{W_j} \to A_{W_i \cap W_j} \) is an admissible epimorphism. In this case, one has a commutative Banach \( K \)-algebra \( A_W \) defined by

\[
A_W = \ker \left( \prod_{i=1}^n A_{W_i} \longrightarrow \prod_{i,j} A_{W_i \cap W_j} \right).
\]
By the generalized Tate acyclicity theorem ([11], 2.2.5), the Banach $K$-algebra $A_W$ does not depend, up to isomorphism, on the choice of the covering $\{W_i\}_{i=1}^n$. We define

- the collection $\bar{\tau}$ to be the set of all subsets $W \subseteq X$ satisfying the following condition: there exists $U \in \tau$ with $W \subseteq U$ such that $W$ is identified with an affinoid subdomain of $\mathcal{M}(A_U)$;
- the collection $\hat{\tau}$ to be the set of all $\bar{\tau}$-special subsets $W$ such that the algebra $A_W$ is $K$-affinoid and, for any finite covering $W = \bigcup_{i=1}^n W_i$ as above, each $W_i$ is an affinoid subdomain of $\mathcal{M}(A_W)$ with the corresponding $K$-affinoid algebra $A_{W_i}$.

Then these collections are nets on $X$, and the $K$-affinoid atlas $\mathcal{A}$ extends uniquely to $K$-affinoid atlases $\mathcal{A}$ and $\hat{\mathcal{A}}$ with respect to the nets $\bar{\tau}$ and $\hat{\tau}$, respectively ([12], 1.2.6, 1.2.13).

- The elements of $\hat{\tau}$ are called $K$-affinoid domains in $X$.
- The $\hat{\tau}$-special subsets are called $K$-special domains in $X$.

Similarly, one defines the notions of strictly $K$-affinoid domains and strictly $K$-special domains.

C.6. (c) G-topology on Berkovich analytic spaces

**Definition C.6.6.** A subset $Y \subseteq X$ of a $K$-analytic space $X = (X, \tau, \mathcal{A})$ is called a $K$-analytic domain if $\hat{\tau}|_Y = \{V \in \hat{\tau}: V \subseteq Y\}$ is a net on $Y$.

In this case, $Y$ endowed with the subspace topology carries the induced structure of a $K$-analytic space $Y = (Y, \hat{\tau}|_Y, \mathcal{A}|_Y)$, where $\mathcal{A}|_Y$ is the induced atlas with the net $\hat{\tau}|_Y$, together with the canonical morphism $Y \to X$ of $K$-analytic spaces. Moreover,

- any open subset $U \subseteq X$ is a $K$-analytic domain (but not conversely in general);
- the intersection of finitely many $K$-analytic domains is again a $K$-analytic domain.

We now introduce the Grothendieck topology on $X$ generated by the pretopology defined as follows:

- admissible open subsets are $K$-analytic domains;
- an admissible covering of a $K$-analytic domain $Y \subseteq X$ is a family $\{Y_a\}_{a \in L}$ of analytic domains in $Y$ that is a quasi-net on $Y$.

We call this Grothendieck topology the $G$-topology on $X$; note that the admissible open sets in it are not necessarily open subsets of the topological space $X$. One has
then a natural sheaf $\mathcal{O}_{X_G}$, called the structure sheaf, on the resulting site $X_G$ in such a way that, for any $K$-affinoid domain $W \subseteq X$, we have

$$\mathcal{O}_{X_G}(W) = A_W.$$ 

**Definition C.6.7** ([12], p. 22). A $K$-analytic space $X = (X, \tau, \mathcal{A})$ is said to be **good** if every point $x \in X$ has a neighborhood $W$ from $\mathcal{O}_{X_G}$.

By definition, $K$-affinoid spaces are good. It is known that any $K$-analytic space coming from a $K$-scheme of locally of finite type through the GAGA functor (cf. [11], §3.4) is good. But there are many interesting non-good $K$-analytic spaces.

If $X$ is good, then one can restrict the $G$-topology as above to open subsets and open coverings to recover the usual topology of $X$. The resulting sheaf, obtained from $\mathcal{O}_{X_G}$ by the restriction, is denoted by $\mathcal{O}_X$. It is known that the category of coherent $\mathcal{O}_{X_G}$-modules and that of coherent $\mathcal{O}_X$-modules are equivalent to each other ([12], 1.3.4).

**C.6. (d) Berkovich analytic spaces and $\mathbb{R}_+$-metrized analytic spaces.** Let $\mathcal{G}$ be a Berkovich $K$-affinoid algebra. We regard $\mathcal{G}$ as an $\mathbb{R}_+$-affinoid ring of $\mathbb{R}_+$-finite type over $K$ in the unique manner, as indicated in §C.4. (j).

Let $X = (X, \tau, \mathcal{A})$ be a Berkovich $K$-analytic space (C.6.5 (2)). For any $U \in \tau$, one has the $K$-affinoid algebra $\mathcal{G}_U$, and the corresponding $\mathbb{R}_+$-metrized affinoid space $\text{Spec}^{\text{val}} \mathcal{G}_U$. Let $D_U$ be the distributive lattice of quasi-compact open subsets of $\text{Spec}^{\text{val}} \mathcal{G}_U$. Then, $D_U$ gives a valuation (0.2.6.3) of $U \cong \mathcal{M}(\mathcal{G}_U)$. Now $\text{Spec}^{\text{val}} \mathcal{G}_U$ is homeomorphic to $\text{Spec} D_U$, since the underlying topological space of $\text{Spec}^{\text{val}} \mathcal{G}_U$ is a reflexive valuative space (C.4.40). These data constitute a prevaluation $v_X = (\tau, \{D_U\}_{U \in \tau})$ (0.2.6.9 (1)) of the underlying topological space $X$. By 0.2.6.16, one has the reflexive valuative space $\text{Spec} v_X$ given by

$$\text{Spec} v_X = \lim_{\longrightarrow} \text{Spec} D_U$$

so that $[\text{Spec} v_X] \cong X$. By 0.2.6.18, $\text{Spec} v_X$ is locally strongly compact (0.2.5.1). By C.6.4 and C.5.8, the structures of $\mathbb{R}_+$-metrized analytic spaces on $\text{Spec}^{\text{val}} \mathcal{G}_U$ glue by open immersions, and hence we obtain an $\mathbb{R}_+$-metrized analytic space locally of $\mathbb{R}_+$-finite type over $K$ supported on $\text{Spec} v_X$, which we denote by $X^\text{met}$.

It is obvious that any strong morphism (C.6.5 (3)) $\varphi: X = (X, \tau, \mathcal{A}) \rightarrow X' = (X', \tau', \mathcal{A}')$ induces a morphism

$$\varphi^\text{met}: X^\text{met} \longrightarrow X'^\text{met}$$
of $\mathbb{R}_+$-metrized analytic spaces over $K$.\footnote{Notice that the underlying continuous map $\text{Spec}_k \mathcal{O}_X \to \text{Spec}_k \mathcal{O}_{X'}$ of $\varphi^\met$ is the one induced, via the functor considered in \textbf{0.2.6.19}, from the morphism $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of valued locally Hausdorff spaces (\textbf{0.2.6.11}), which can be constructed from the strong morphism $\varphi$.} Moreover, if $\varphi$ is a quasi-isomorphism (C.6.5 (4)), then $\varphi^\met$ is an isomorphism, since the underlying topological spaces are reflexive. Thus we obtain the functor

$$(\cdot)^\met: X \mapsto X^\met$$

acting from the category $\text{Bsp}_K$ of Berkovich $K$-analytic spaces to the category $\mathcal{M}\text{Ansp}_{K}^{\mathbb{R}_+-\text{lf}}$ of $\mathbb{R}_+$-metrized analytic spaces over $K$.

**Theorem C.6.8.** (1) The functor $(\cdot)^\met$ gives a fully faithful functor from the category $\text{Bsp}_K$ to the category $\mathcal{M}\text{Ansp}_{K}^{\mathbb{R}_+-\text{lf}}$. Moreover, the essential image of $(\cdot)^\met$ consists of $\mathbb{R}_+$-metrized analytic spaces locally of $\mathbb{R}_+$-finite type with locally strongly compact underlying topological spaces.

(2) For any Berkovich $K$-analytic space $X$, the admissible topos $X^\sim_G$ is equivalent to the admissible topos of $X^\met$ (cf. \S C.5. (a)) as ringed topos. In particular, it is spacial, and locally ringed.

(3) For any Berkovich $K$-analytic space $X$, the underlying topological space of $X^\met$ is a reflexive valuative space, and $[X^\met]$ is canonically homeomorphic to $X$. Moreover, the underlying topological space of $X^\met$ is quasi-separated (resp. paracompact, resp. coherent) if and only if $X$ is Hausdorff (resp. paracompact, resp. compact).

\textbf{Proof.} (1) First consider the functor $(\cdot)^\met$ restricted on Berkovich $K$-affinoid spaces $\mathcal{M}(\mathcal{O})$. In this case, the functor is given by $\mathcal{M}(\mathcal{O}) \mapsto \text{Spec}_k \mathcal{O}$, and the fully-faithfulness in follows immediately from C.5.12 (2).

In general, we need to check the following conditions:

(a) if $\varphi, \psi: (X, \tau, \mathcal{A}) \to (X', \tau', \mathcal{A}')$ are two strong morphisms of Berkovich $K$-analytic spaces such that the resulting morphisms $f = \varphi^\met$ and $g = \psi^\met$ coincide, then $\varphi = \psi$;

(b) for any valuative and locally quasi-compact morphism $f: X^\met \to X'^\met$ of $\mathbb{R}_+$-metrized analytic spaces locally of $\mathbb{R}_+$-finite type over $K$, there exists a diagram

$$X \leftarrow X'' \xrightarrow{\varphi} X'$$

of strong morphisms of Berkovich $K$-analytic spaces such that the first morphism is a quasi-isomorphism supported on the identity map $\text{id}_X$ and that $\varphi^\met = f$. 

\hfill \Box
(a) Since \( [X^\text{met}] \cong X \) etc. as a topological space, \( \varphi \) and \( \psi \) coincide as continuous mappings of the underlying topological spaces. For any \( U \in \tau \) and \( U' \in \tau' \) such that \( \varphi(U) \subseteq U' \) (and hence \( \psi(U) \subseteq U' \)), the maps \( f \) and \( g \) give the same morphism \( \text{Spec}^\text{val} \mathcal{A}_U \to \text{Spec}^\text{val} \mathcal{A}_{U'} \) as a morphism of \( \mathbb{R}_+ \)-metrized analytic spaces. Hence, by what we have already shown, \( \varphi = \psi \) as a strong morphism.

(b) Since \( f \) is valuative, it induces a continuous mapping \( \left[ f \right] : X \to X' \). Since \( f \) is locally quasi-compact, one can find a refinement \( \tau'' \) of \( \tau \) and the corresponding \( K \)-affinoid atlas \( \mathcal{A}'' \) with the property that, for any \( U \in \tau'' \), there exists \( U' \in \tau' \) such that \( f(U) \subseteq U' \). We then have \( \text{Spec}^\text{val} \mathcal{A}_U \to \text{Spec}^\text{val} \mathcal{A}_{U'} \), which induces \( \mathcal{A}'_{U'} \to \mathcal{A}_U \) (C.5.12 (2)). Hence we have a strong morphism \( \varphi : X'' = (X, \tau'', \mathcal{A}'') \to X' \). Since \( [\varphi^\text{met}] = [f] \) as a continuous mapping \( X \to X' \), by 0.2.4.12, we have \( \varphi^\text{met} = f \) as a continuous mapping \( X^\text{met} \to X'^\text{met} \). It is then obvious, by the affinoid case discussed above, that \( \varphi^\text{met} = f \) as a morphism of \( \mathbb{R}_+ \)-metrized analytic spaces.

\[
X'' = (X, \tau'', \mathcal{A}'') \longrightarrow X = (X, \tau, \mathcal{A})
\]
is clearly a quasi-isomorphism supported on the identity map \( \text{id}_X \), we have verified the claim.

Finally, by 0.2.6.18 the \( \mathbb{R}_+ \)-metrized analytic space \( X^\text{met} \) is locally strongly compact. Conversely, if an \( \mathbb{R}_+ \)-metrized analytic space \( W \) locally of \( \mathbb{R}_+ \)-finite type over \( K \) is locally strongly compact, by 0.2.6.19, \( W \) is homeomorphic to \( \text{Spec} \mathfrak{v} \) for a valued locally Hausdorff space \( (X = [W], \mathfrak{v} = (\tau(\mathfrak{v}), \{v_S\})) \), where \( \tau(\mathfrak{v}) = \{[U] : U \subseteq \text{QC} \mathfrak{O} \text{uv}(W)\} \) and \( v[U] = \{[V] : V \subseteq \text{QC} \mathfrak{O} \text{uv}(U)\} \). Hence, for any \( \mathbb{R}_+ \)-affinoid subdomain \( U \subseteq W \), one can set \( \mathfrak{g}[U] \) to be the corresponding \( K \)-affinoid algebra, and thus we get a Berkovich \( K \)-analytic space \( (X, \tau, \mathcal{A}) \) such that \( X^\text{met} \cong W \).

(2) By C.6.4, the site \( X_G \) is generated by rational subdomains, and hence the associated topos is equivalent to the admissible topos of \( X^\text{met} \).

(3) That \( X^\text{met} \) is reflexive and that \( [X^\text{met}] \cong X \) are consequences of the construction of \( X^\text{met} \). The other assertions follow from 0.2.6.19 and 0.2.5.17. \[ \square \]

Thus, in case the valuation on \( K \) is non-trivial, we have, combined with the functor \( \mathcal{X} \mapsto \mathcal{X}^\text{met} \) considered in C.5.18, the 2-cartesian diagram of categories

\[
\begin{array}{ccc}
\text{Bsp}_K \big& \longrightarrow & \text{MAnsp}_K^{\mathbb{R}_+} \\
\uparrow & & \uparrow \\
\text{Adsp}_K^{\text{lsc,lft}} \big& \longrightarrow & \text{Adsp}_K^{\text{lf}}
\end{array}
\]

consisting of fully faithful functors, where \( \text{Adsp}_K^{\text{lf}} \) is the category of locally of finite type adic spaces over \( K \), and \( \text{Adsp}_K^{\text{lsc,lft}} \) the full subcategory of \( \text{Adsp}_K^{\text{lf}} \) consisting
of adic spaces with locally strongly compact underlying topological spaces. The following corollary is easy to see.

**Corollary C.6.9.** The category $\text{Adsp}^{\text{lsc,lt}}_K$ of locally strongly compact and locally of finite type adic spaces over $K$ is naturally categorically equivalent to the category $\text{Bsp}^*_K$ of Berkovich strictly $K$-analytic spaces.

Note that the corollary is consistent with [61], 8.3.1, for the combination of ‘locally strongly compact’ and ‘quasi-separated’ is equivalent to ‘taut’ (cf. 0.2.5.6).

Note also that, since the category $\text{Adsp}^{\text{lt}}_K$ of locally of finite type adic spaces over $K$ is equivalent to the category of locally of finite type rigid spaces over $\text{Spf} V/\text{rig}$ (where $V$ is the valuation ring of $K$) by the functor $\text{ZR}$ (A.5.2), the above comparison also applies to the situation where $\text{Adsp}^{\text{lt}}_K$ (resp. $\text{Adsp}^{\text{lsc,lt}}_K$) is replaced by the category of locally of finite type (resp. and locally quasi-compact (4.4.1)) rigid spaces over $(\text{Spf} V)^{\text{rig}}$; see C.6.12 below.

**Remark C.6.10** (cf. [57], 4.4). Note that the fully faithful functor

$$
\mathcal{B}: \text{Bsp}_K \hookrightarrow \mathcal{M}\text{Ansp}^{\mathbb{R}_+^{\text{lt}}}_K
$$

is not a categorical equivalence. Indeed, there exists an $\mathbb{R}_+$-metrized analytic space locally of $\mathbb{R}_+$-finite type (e.g., a locally of finite type adic space) over $K$ whose underlying topological space is not locally strongly compact. For example, let $X$ be a locally strongly compact and locally of finite type adic space over $K$, and consider $Y = X \setminus \{x\}$, where $x \in X$ is a closed point of height larger than one. Then $Y$ is still a locally of finite type adic space, but is not locally strongly compact. Indeed, $X$ and $Y$ give rise to the same valuation on the locally Hausdorff space $[X] = [Y]$, and hence, if $Y$ is locally strongly compact, then by 0.2.6.19, $X$ and $Y$ have to be homeomorphic by a valuative map, which is absurd.

**C.6. (e) Comparison with rigid spaces.** Let $V$ be an $\alpha$-adically complete valuation ring of height one ($\alpha \in \mathfrak{m}_V \setminus \{0\}$), and $K = \text{Frac}(V)$. Let $X$ be a locally quasi-compact (4.4.1) locally of finite type rigid space over $S = (\text{Spf} V)^{\text{rig}}$; recall that the locally-quasi-compactness assumption is satisfied if, for example, $X$ is coherent (0.2.5.2). By 4.4.2, we know that the separated quotient $X = [X]$ is locally compact (and hence is locally Hausdorff). If, moreover, $X$ is quasi-separated, then $[X]$ is Hausdorff.

Let $\mathcal{U} = \{\mathcal{U}_\alpha = (\text{Spf} A_\alpha)^{\text{rig}}\}_\alpha \in L$ be an open covering of $X$ consisting of affinoids such that $\tau_\mathcal{U} = \{[\mathcal{U}_\alpha]\}_\alpha \in L$ is a net on $[X]$ (cf. 0.2.6.2). Then one has the strictly $K$-analytic space

$$
X_B = ([X], \tau_\mathcal{U}, \mathcal{G}_\mathcal{U})
$$

with the strictly $K$-affinoid atlas $\mathcal{G}_\mathcal{U}$ given by the affinoid algebra $\mathcal{G}_\alpha = A_\alpha[\frac{1}{\alpha}]$ and a homeomorphism $[\mathcal{U}_\alpha] \sim \mathcal{M}(\mathcal{G}_\alpha)$ (C.4.34 and A.4.22) for each $\alpha \in L$. 
It is clear by the construction that, if $V = \{V_\lambda\}_{\lambda \in \Lambda}$ is another affinoid open covering of $X$ that gives a refinement of $\{U_\alpha\}_{\alpha \in \mathcal{L}}$, then the similarly defined strictly $K$-analytic space $X'_B = ([X], \tau_V, \mathcal{O}_V)$ admits a canonical quasi-isomorphism $X'_B \to X_B$. Hence, the strictly $K$-analytic space $X_B$ is uniquely determined, up to quasi-isomorphism, by the rigid space $X$. It is then clear that, for an $S$-morphism $\phi: X \to Y$ of locally quasi-compact locally of finite type rigid spaces, we have the induced morphism $\varphi_B: X_B \to Y_B$ in the category of strictly $K$-analytic spaces.

**Remark C.6.11.** Let $X$ and $\tau = \tau_U$ be as above. We define collections of subsets $\bar{\tau}$ and $\tilde{\tau}$ of $X = [X]$ as follows:

- $\bar{\tau} = \{[V]: V$ is an affinoid subdomain of some $U_\alpha \in \tau\}$;
- $\tilde{\tau} = \{[V]: V$ is a quasi-compact separated open subspace of $X\}$.

Then these are nets on $X = [X]$; cf. [12], §1.2.

Note here that, if $j: U \hookrightarrow X$ is an open immersion of locally quasi-compact locally of finite type rigid spaces over $S$, then $j_B: U_B \hookrightarrow X_B$ is an open immersion in the sense as in [12], §1.3. Indeed, the net $\bar{\tau}$ on $X = [X]$ as above induces by restriction a net $\sigma$ of $U = [U]$; more precisely, $\sigma = \{[V] \in \bar{\tau} \mid V \subseteq U\}$.

**Theorem C.6.12** (cf. [12], 1.6.1). The functor $X \mapsto X_B$ establishes a categorical equivalence from the category of all locally quasi-compact locally of finite type rigid spaces over $S = (\text{Spf} V)^{\text{rig}}$ to the category $\mathbf{Bsp}_K^*$ of all strictly $K$-analytic spaces. Moreover, $X_B$ is Hausdorff (resp. paracompact Hausdorff, resp. compact) if and only if $X$ is quasi-separated (resp. paracompact and quasi-separated, resp. coherent).

Note that by 4.4.6 paracompact quasi-separated rigid spaces are locally quasi-compact, and hence one has the functor $X \mapsto X_B$ for paracompact rigid spaces as in the theorem.

**Proof.** Note that the functor $X \mapsto X_B$ followed by $(\cdot)^{\text{met}}: \mathbf{Bsp}_K \to \mathbf{MANsp}^{\mathbb{R},+}$ in §C.6. (d) coincides (up to natural equivalence) to the composition of $\mathbf{ZR}$ (cf. A.5.2) followed by $X \mapsto X^{\text{met}}$ (cf. C.5.18). Hence, the first assertion follows immediately by C.6.9. The other assertions follow from 0.2.6.19 and 0.2.5.17. □

**Proposition C.6.13.** Let $X$ be a quasi-separated locally quasi-compact locally of finite type rigid space over $S$. Then the following conditions are equivalent:

(a) the associated strictly $K$-analytic space $X_B$ is good (C.6.7);

(b) for any point $x \in (X)$, there exists a pair $(U, V)$ of affinoid open neighborhoods of $x$ such that $(\overline{U}) \subseteq (V)$, where $\overline{\phantom{x}}$ denotes the closure in $(X)$.
Proof. Let us first show (a) \( \Rightarrow \) (b). We may assume that \( x \) is a height-one point. There exists an affinoid open neighborhood \( \mathcal{V} \) of \( x \) such that \( x \), considered as a point of \( [\mathcal{X}] \), lies in the interior of \( [\mathcal{V}] \). This implies that \( x \in \text{int}_X(\mathcal{V}) \). Take an affinoid open neighborhood \( \mathcal{U} \) of \( x \) contained in the open subset \( \text{int}_X(\mathcal{V}) \). Then by 4.2.1 and 4.3.13 we have \( \langle \mathcal{U} \rangle \subseteq \langle \mathcal{V} \rangle \), as desired. To show the converse, take any height-one point \( x \in [\mathcal{X}] \) and a pair \( (\mathcal{U}, \mathcal{V}) \) of affinoid open neighborhoods as above. Since \( \langle \mathcal{U} \rangle \subseteq \langle \mathcal{V} \rangle \), \( x \) lies in the overconvergent interior \( \text{int}_X(\mathcal{V}) \) (4.3.13). Hence \( [\mathcal{V}] \) gives a strictly affinoid neighborhood of \( x \).

Exercises

Exercise II.C.1. Show that, for a filtered ring \( (A, F) \), the following conditions are equivalent:

(a) the associated graded ring \( \text{Gr}_F A \) is zero;
(b) \( F_0 = A \);
(c) \( 1 \in F_r \) for some \( r < 1 \).

(In this situation, we say that \( (A, F) \) is \textit{graded trivial}.)

Exercise II.C.2. Let \( K \) be a field, and \( V \) a valuation ring for \( K \). Suppose \( K \) has a non-archimedean absolute value \( \nu: K \to \mathbb{R}_{\geq 0} \) that is a localization of the valuation associated to \( V \) (of height 0 or 1). Show that there exists an \( \mathbb{R}_+ \)-valuation \( \nu \) on \( K \) with \( V \) equal to the associated valuation ring.

Exercise II.C.3. Let \( \mathcal{O} \) be a commutative Banach ring. Prove the spectral semi-norm formula

\[
\| f \|_{\mathcal{O}((r^{-1}T)), \text{Sp}} = \sup_{n \in \mathbb{N}} r^n \| a_m \|_{\mathcal{O}, \text{Sp}}
\]

for \( f = \sum_{m \in \mathbb{N}} a_m T^m \).

Exercise II.C.4. (1) Let \( \mathcal{O} \) be an \( \mathbb{R}_+ \)-affinoid ring. For \( f = (f_0, f_1, \ldots, f_n) \) that generates \( \mathcal{O} \), and for an \( n \)-tuple \( r = (r_1, \ldots, r_n) \) of positive real numbers, show that there exist \( m_1, \ldots, m_n > 0 \) such that

\[
U_0(f, r) = \bigcap_{i=1}^n B(f_0, f_i, r_i) = \bigcap_{i=1}^n U_0((f_0, f_i, 1), (r_i, s_i))
\]

for \( s_i > m_i \) (\( i = 1, \ldots, n \)).

(2) Show that a finite intersection of basic subsets is a rational subdomain if and only if its image under the separation map \( \text{sep}_{\mathcal{O}} \) is closed in \( M(\mathcal{O}) \).

Exercise II.C.5. For a \( \Delta \)-graded ring \( G \), let \( \text{Spec}_\Delta G \) be the graded spectrum, that is, the set of all homogeneous prime ideals of \( G \).
(1) For a homogeneous element \( f \in h(G) \), define
\[
D(f) = \{ p \in \text{Spec}_\Delta G : f \not\in p \}.
\]
The Zariski topology of \( \text{Spec}_\Delta G \) is the topology generated by the sets \( D(f) \) with \( f \in h(G) \). Show that \( \text{Spec}_\Delta G \) is coherent and sober.

(2) (reduction map) Let \( \mathcal{A} \) be an \( \mathbb{R}_+ \)-affinoid ring. For \( x = (z, F) \in \text{Spec}^{\text{val}} \mathcal{A} \), let \( P_x \) the kernel of the induced map \( \text{Gr}\mathcal{A} \to \text{Gr} F \mathcal{H}(z) \to k_x \), where \( k_x \) is the graded residue field of the graded valuation ring \( \text{Gr} F \mathcal{H}(z) \). Consider the map
\[
\text{sp}_\mathcal{A} : \text{Spec}^{\text{val}} \mathcal{A} \to \text{Spec}_{\mathbb{R}_+} \text{Gr} \mathcal{A}, \quad x \mapsto P_x,
\]
called the (graded) reduction map. Show that \( \text{sp}_\mathcal{A} \) is continuous, quasi-compact, and surjective.

(3) If \( \mathcal{A} \) is a Berkovich \( K \)-affinoid algebra, then
\[
\text{sp}_\mathcal{A} |_{\mathcal{M}(\mathcal{A})} : \mathcal{M}(\mathcal{A}) \to \text{Spec}_{\mathbb{R}_+} \text{Gr} \mathcal{A}
\]
is surjective.

D. Appendix: Rigid Zariskian spaces

D.1 Admissible blow-ups

In this section we will only deal with Zariskian schemes (I, §B) of finite ideal type; a Zariskian scheme \( X \) is said to be of finite ideal type if for any affine open set \( U = \text{Spz} A \) of \( X \) the ring \( A \) has a finitely generated ideal of definition. In the sequel, with a slight abuse of notation, we denote by \( \text{CZs}^* \) the category of coherent Zariskian schemes of finite ideal type and adic morphisms. Similarly to the formal schemes case (I.3.7.12) one can show the following result.

**Proposition D.1.1.** Let \( X \) be an object of \( \text{CZs}^* \). Then \( X \) has an ideal of definition \( I \) of finite type.

For an affine Zariskian scheme \( X = \text{Spz} A \) associated to a pair \( (A, I) \) and an admissible ideal \( J \), one has the so-called admissible blow-up \( \pi : X' \to X \); the Zariskian scheme \( X' \) is the one given by \( X' = Y^{(\Delta)}_{\text{V}(I)} \) (cf. I, §B.1. (b)), where \( Y' \to Y = \text{Spec} A \) is the blow-up along the ideal \( J \). This construction gives, by gluing, the general concept of admissible blow-ups \( \pi : X' \to X \) of arbitrary Zariskian schemes and admissible ideals; here, an ideal \( \mathfrak{j} \) of \( \mathcal{O}_X \) is said to be admissible if it is quasi-coherent of finite type and open or, equivalently, if it is locally representable as \( \mathfrak{j}^\Diamond \) (in the notation as in I, §B.1. (a)), where \( J \) is a finitely generated \( I \)-adically open ideal of \( A \).

Let \( X \) be an object of \( \text{CZs}^* \). We define \( \text{BL}_X \) to be the category of all admissible blow-ups of \( X \) as follows.
Objects of $\text{BL}_X$ are admissible blow-ups $\pi : X' \to X$.

A morphism $\pi' \to \pi$ between two objects $\pi : X' \to X$ and $\pi' : X'' \to X$ means a morphism that makes the resulting triangle commute.

An argument similar to that in 1.3.1 yields the following result.

**Proposition D.1.2.** (1) The category $\text{BL}_X$ is cofiltered (cf. 0, §1.3. (c)), and $\text{id}_X$ gives the final object.

(2) Let us define an order on the set $\text{AId}_X$ of all admissible ideals of $\mathcal{O}_X$ as follows: $\mathcal{J} \leq \mathcal{J}'$ if there exists an admissible ideal $\mathcal{J}''$ such that $\mathcal{J} = \mathcal{J}' \mathcal{J}''$. Then $\text{AId}_X^{\text{opp}}$ is a directed set, and the functor

$$\text{AId}_X \longrightarrow \text{BL}_X$$

that maps $\mathcal{J}$ to the admissible blow-up along $\mathcal{J}$ is cofinal.

### D.2 Coherent rigid Zariskian spaces

**D.2. (a) The category of coherent rigid Zariskian spaces**

**Definition D.2.1.** (1) We define the category $\text{CRz}$ as follows.

- Objects of $\text{CRz}$ are the same as those of $\text{CZs}^*$; that is,

$$\text{obj}(\text{CRz}) = \text{obj}(\text{CZs}^*).$$

For an object $X$ of $\text{CZs}^*$ we denote by $X^\text{rig}$ the same object regarded as an object of $\text{CRz}$.

- For $X, X' \in \text{obj}(\text{CZs}^*)$ we set

$$\text{Hom}_{\text{CRz}}(X^\text{rig}, X'^\text{rig}) = \lim_{\longrightarrow} \text{Hom}_{\text{CZs}^*}(-, X'),$$

where $\text{Hom}_{\text{CZs}^*}(-, X')$ is the functor

$$\text{Hom}_{\text{CZs}^*}(-, X') : \text{BL}_X^{\text{opp}} \longrightarrow \text{Sets}$$

that maps $\pi : X'' \to X$ into the set $\text{Hom}_{\text{CZs}^*}(X'', X')$. 
By D.1.2, the inductive limit in the above definition can be replaced by a filtered inductive limit along the directed set $\text{Ald}_X$. The composition law for morphisms in the category $\text{CRz}$

$$\text{Hom}_{\text{CRz}}(X^{\text{rig}}, X'^{\text{rig}}) \times \text{Hom}_{\text{CRz}}(X'^{\text{rig}}, X''^{\text{rig}}) \rightarrow \text{Hom}_{\text{CRz}}(X^{\text{rig}}, X''^{\text{rig}})$$

is described similarly to the case of rigid (formal) spaces (§2.1.1(a)).

**Definition D.2.2.** (1) An object of the category $\text{CRz}$ is called a **coherent rigid Zariskian space**.

(2) For an object $X$ of $\text{CZs}^*$ the coherent rigid Zariskian space $X^{\text{rig}}$ is called the associated (coherent) rigid space. Similarly, for a morphism $f : X \rightarrow X'$ in $\text{CZs}^*$ the associated morphism of rigid spaces is denoted by $f^{\text{rig}} : X^{\text{rig}} \rightarrow X'^{\text{rig}}$.

We often denote by

$$Q : \text{CZs}^* \rightarrow \text{CRz}, \quad X \mapsto Q(X) = X^{\text{rig}}$$

the canonical quotient functor.

Similarly to the case of rigid (formal) spaces, one also defines consistently the comma category $\text{CRz}_S$ of coherent rigid Zariskian spaces over a fixed coherent rigid Zariskian space $\mathcal{S}$.

The following assertions are shown similarly to 2.1.4, 2.1.5, and 2.1.6.

**Proposition D.2.3.** Let $X$ and $X'$ be objects of $\text{CZs}^*$, and consider the corresponding rigid spaces $X^{\text{rig}}$ and $Y^{\text{rig}}$. Then there exists an isomorphism $X^{\text{rig}} \rightarrow Y^{\text{rig}}$ in $\text{CRz}$ if and only if it is represented by a diagram in $\text{CZs}^*$

$$X'' \rightarrow X \rightarrow X',$$

where both arrows are admissible blow-ups.

**Corollary D.2.4.** Let $f : X \rightarrow X'$ be a morphism in $\text{CZs}^*$. Then

$$f^{\text{rig}} : X^{\text{rig}} \rightarrow X'^{\text{rig}}$$

is an isomorphism if and only there exists a commutative diagram

$$X'' \rightarrow X \rightarrow X'$$

where both $X'' \rightarrow X$ and $X'' \rightarrow X'$ are admissible blow-ups.
**Corollary D.2.5.** Consider the diagram

\[
\begin{array}{ccc}
X & \xleftarrow{f} & X' \\
\downarrow & & \downarrow \\
X'' & \xleftarrow{h} & X'
\end{array}
\]

If \( f \text{ rig} \) is an isomorphism in \( \text{CRz} \), then so is \( h \text{ rig} \).

**Definition D.2.6.**

1. Let \( X \) be a coherent rigid Zariskian space. A model of \( X \) is a couple \((X, \phi)\) consisting of \( X \in \text{obj}(\text{CZs}^\text{rig})\) and an isomorphism \( \phi: X \text{ rig} \xrightarrow{\sim} X \). A model \((X, \phi)\) is said to be distinguished if \( X \) is \( I \)-torsion free, where \( I \) denotes an ideal of definition of \( X \).

2. Let \( \varphi: X \rightarrow X' \) be a morphism of coherent rigid Zariskian spaces. A model of \( \varphi \) is a triple \((f, \phi, \psi)\) consisting of a morphism \( f: X \rightarrow X' \) of \( \text{CZs}^\text{rig} \) and isomorphisms \( \phi: X \text{ rig} \xrightarrow{\sim} X \) and \( \psi: X' \text{ rig} \xrightarrow{\sim} X' \) such that the resulting square

\[
\begin{array}{ccc}
X \text{ rig} & \xrightarrow{f \text{ rig}} & X' \text{ rig} \\
\downarrow{\phi} & & \downarrow{\psi} \\
X & \xrightarrow{\varphi} & X'
\end{array}
\]

commutes. A model \((f, \phi, \psi)\) is said to be distinguished if \( X \) and \( X' \) are distinguished models of \( X \) and \( X' \), respectively.

Similarly to the case of coherent rigid (formal) spaces, one can define the category \( M_X \) of models of \( X \) and the category \( M_\varphi \) of models of \( \varphi \); the details are left to the reader. It is easy to see that these categories are cofiltered. We denote by \( M_X^\text{dist} \) (resp. \( M_\varphi^\text{dist} \)) the full subcategory of \( M_X \) (resp. \( M_\varphi \)) consisting of distinguished models. The following propositions are the counterparts of 2.1.9 and 2.1.10, and are shown by a similar argument.

**Proposition D.2.7.**

1. If \((X, \phi)\) is a distinguished model of \( X \) and \( \pi: X' \rightarrow X \) is an admissible blow-up, then \((X', \phi \circ \pi \text{ rig})\) is a distinguished model of \( X \).

2. If, moreover, \( I \) (an ideal of definition of \( X \)) is an invertible ideal, then \( I \Theta_X \) is invertible.

**Proposition D.2.8.** The categories \( M_X^\text{dist} \) and \( M_\varphi^\text{dist} \) are cofiltered, and the inclusions \( M_X^\text{dist} \hookrightarrow M_X \) and \( M_\varphi^\text{dist} \hookrightarrow M_\varphi \) are cofinal.
D.2. (b) Visualization. Let \( \mathcal{X} \) be a coherent rigid Zariskian space. One can define the associated Zariski–Riemann space \( \langle \mathcal{X} \rangle \) in an entirely analogous way as in §3.1. Let

\[ \text{sp}_X : \langle \mathcal{X} \rangle \rightarrow X \]

be the specialization map, where \( X \) is a model of \( \mathcal{X} \). Similarly to 3.1.2 and 3.1.3, we have the following theorem.

**Theorem D.2.9.** Let \( \mathcal{X} \) be a coherent rigid Zariskian space, and \( \langle \mathcal{X} \rangle \) the associated Zariski–Riemann space.

1. The topological space \( \langle \mathcal{X} \rangle \) is coherent and sober (0.2.2.1).
2. The specialization map \( \text{sp}_X \) is quasi-compact (0.2.1.4 (2)) and closed for any model \( X \) of \( \mathcal{X} \).

**Proposition D.2.10.** (1) For any quasi-compact open subset \( U \) of \( \langle \mathcal{X} \rangle \) there exist a model \( X \) and a quasi-compact open subset \( U \) of \( X \) such that \( U = \text{sp}_X^{-1}(U) \).

(2) Let \( X \) be an object of \( \text{CZs}^* \), and \( U \) a quasi-compact open subset of \( X \). Set \( \mathcal{X} = X^\text{rig} \) and \( \mathcal{U} = U^\text{rig} \). Then the induced map \( \langle \mathcal{U} \rangle \rightarrow \langle \mathcal{X} \rangle \) maps \( \langle \mathcal{U} \rangle \) homeomorphically onto the quasi-compact open subset \( \text{sp}_X^{-1}(U) \).

One can define general rigid Zariskian spaces and their associated Zariski–Riemann triples in a similar way, as in the case of rigid (formal) spaces. The details are left to the reader.

E Appendix: Classical Zariski–Riemann spaces

E.1 Birational geometry

E.1. (a) Basic terminology. Throughout this section we fix once and for all a coherent (= quasi-compact and quasi-separated) scheme \( S \) and work entirely in the category \( \text{CAs}_S \) of coherent \( S \)-algebraic spaces with \( S \)-morphisms. Note that all arrows in \( \text{CAs}_S \) are automatically coherent.

Our main objects in this subsection are the pairs \( (X, U) \) consisting of an object \( X \in \text{obj}(\text{CAs}_S) \) and a quasi-compact open subspace \( U \) of \( X \); note that \( U \) is again coherent, hence belongs to \( \text{CAs}_S \). With the objects of this kind, we treat the theory of the so-called \( U \)-admissible birational geometry. Here, in order to include the usual birational geometry (without reference to any \( U \)), we admit the case where \( U = \emptyset \).

**Definition E.1.1.** Let \( X \) be a coherent \( S \)-algebraic space, and \( U \subseteq X \) a quasi-compact open subspace. A morphism \( f : Y \rightarrow X \) in \( \text{CAs}_S \) is said to be \( U \)-admissible if the induced morphism \( f^{-1}(U) \rightarrow U \) is an isomorphism.
Definition E.1.2 (cf. [54], I, §7). Let $X$ and $Y$ be coherent $S$-algebraic spaces, and $U \subseteq X$ a quasi-compact open subspace.

(1) A $(U$-admissible) rational map $f : Y \to X$ is the equivalence class of a pair $(V, f_V)$ consisting of a quasi-compact dense open subspace $V$ of $Y$ and a $U$-admissible $S$-morphism $f_V : V \to X$, where the equivalence relation is defined as follows: $(V, f_V)$ and $(V', f_{V'})$ are equivalent if the morphisms $f_V$ and $f_{V'}$ coincide on a quasi-compact dense open subspace $W$ of $Y$ such that $W \subseteq V \cap V'$; since the intersection of two quasi-compact dense open subspaces of $Y$ is again a quasi-compact open dense subspace, this indeed defines an equivalence relation.

(2) A $U$-admissible rational map $f : Y \to X$ is said to be birational if it is the equivalence class of a pair $(V, f_V)$ such that $f_V : V \to X$ is a $U$-admissible open immersion onto a quasi-compact dense open subspace of $X$.

(3) A $(U$-admissible) birational morphism $f : Y \to X$ is a $U$-admissible $S$-morphism that is birational, that is, there exists a quasi-compact open dense subspace $V$ of $Y$ such that $f|_V : V \to X$ is an open immersion onto a quasi-compact dense open subspace of $X$.

(4) A $U$-admissible $S$-morphism $f : Y \to X$ is called a $(U$-admissible) $S$-modification (or simply modification) if it is proper and birational.

The composition $g \circ f$ of a $U$-admissible rational map $f : Y \to X$ and an $f^{-1}(U)$-admissible rational map $g : Z \to Y$ is defined in an obvious way and is a $U$-admissible rational map. A $U$-admissible rational map $f : Y \to X$ is birational if and only if there exists an $f^{-1}(U)$-admissible rational map $g : X \to Y$ such that $g \circ f$ (resp. $f \circ g$) is represented by the identity map on a quasi-compact open dense subspace of $Y$ (resp. $X$). Note that when $X$ and $Y$ are schemes having finitely many irreducible components our definition of birational morphisms coincides with that in [53], (2.3.4).

E.1. (b) $U$-admissible blow-ups. We continue to work in the above setting; let $X$ be a coherent $S$-algebraic space, and $U$ a quasi-compact open subspace of $X$.

Definition E.1.3. A $U$-admissible ideal is a quasi-coherent ideal $\mathcal{J} \subset \mathcal{O}_X$ of finite type such that the closed subspace $V(\mathcal{J})$ of $X$ corresponding to $\mathcal{J}$ is disjoint from $U$.

We denote by $\text{AId}_{(X,U)}$ the set of all $U$-admissible ideals. One can introduce the ordering $\preceq$ to the set $\text{AId}_X$ in a similar manner as in 1.3.1. Moreover, similarly to I.3.7.6, for an $S$-morphism $Y \to X$ of $S$-algebraic spaces one has the induced map

$$\text{AId}_{(X,U)} \to \text{AId}_{(Y, f^{-1}(U))}, \quad \mathcal{J} \mapsto \mathcal{J} \mathcal{O}_Y.$$
Definition E.1.4. Let $X$ be a coherent $S$-algebraic space, and $U$ a quasi-compact open subspace of $X$. A $U$-admissible blow-up of $X$ is a blow-up of $X$ along a $U$-admissible ideal $\mathcal{J}$, that is, the morphism

$$X_{\mathcal{J}} = \text{Proj} \bigoplus_{n \geq 0} \mathcal{J}^n \longrightarrow X.$$

Clearly, the $U$-admissible blow-up is $U$-admissible as a morphism between $S$-algebraic spaces. Note that the $U$-admissible blow-up is proper, since $\mathcal{J}$ is of finite type. It is, in particular, birational if the open complement of the closed subscheme $V(\mathcal{J})$ in $X$ is dense.

Definition E.1.5. Let $X$ be a coherent $S$-algebraic space, $U \subseteq X$ a quasi-compact open subspace, and $\mathcal{J}$ a $U$-admissible ideal. Let $\pi: X' \to X$ the $U$-admissible blow-up of $X$ along $\mathcal{J}$.

1. For an $\mathcal{O}_X$-module $\mathcal{F}$ the strict transform of $\mathcal{F}$ by $\pi$ is the $\mathcal{O}_{X'}$-module

$$\pi'\mathcal{F} = \pi^* \mathcal{F} / (\pi^* \mathcal{F})_{\mathcal{J}\text{-tor}}.$$

Notice that $\pi'\mathcal{F}$ is quasi-coherent if $\mathcal{F}$ is quasi-coherent.

2. For a morphism $f: Y \to X$ of coherent $S$-algebraic spaces the strict transform of $f$ by $\pi$ is the composite morphism $Y' \hookrightarrow Y \times_X X' \to X'$, where the first arrow is the closed immersion given by the ideal $(\mathcal{O}_Y \times_X X')_{\mathcal{J}\text{-tor}}$.

Note that in the situation as in (2) the morphism $Y' \to Y$ is the $f^{-1}(U)$-admissible blow-up along the ideal $\mathcal{J}\mathcal{O}_Y$.

The following fact will be frequently used (often tacitly) in the sequel.

Proposition E.1.6. Let $X$ be a coherent algebraic space and $U$ a quasi-compact open subspace of $X$. Let $Y \hookrightarrow X$ be a quasi-compact open immersion, and set $V = Y \cap U$.

1. For any $U$-admissible blow-up $X' \to X$ the strict transform $Y' \to Y$ coincides with the base change $X' \times_X Y \to Y$ and is the $V$-admissible blow-up of $Y$ along the restriction of the blow-up center of $X' \to X$.

2. (extension of $U$-admissible blow-up) Conversely, for any $V$-admissible blow-up $Y' \to Y$ there exists a $U$-admissible blow-up $X' \to X$ that admits the Cartesian diagram

$$
\begin{array}{c}
Y' \searrow X' \\
Y \swarrow \\
X
\end{array}
$$
Proof. (1) is clear. To show (2), let $\mathcal{J}$ be the blow-up center of $Y' \to Y$. The direct image $j_*\mathcal{J}$ by the open immersion $j: Y \to X$ is a quasi-coherent ideal of $\mathcal{O}_X$. By 0.5.5.6, we have a quasi-coherent ideal $\mathcal{J}$ of $\mathcal{O}_X$ of finite type such that the support of the corresponding closed subspace $Z = V(\mathcal{J})$ is $X \setminus U$ and that $\mathcal{J}|_V \subseteq \mathcal{J}$. The closed subspace $Z$ is a coherent algebraic space, and $Z \cap Y$ is a quasi-compact open subspace of $Z$. Consider the quasi-coherent ideal $\mathcal{J}/(\mathcal{J}|_V)$ of finite type on $Z \cap Y$, and extend it to a quasi-coherent ideal of finite type on $Z$ (0.5.5.6 and 0.5.5.7). Pulling it back to $\mathcal{O}_X$, we get a $U$-admissible ideal $\widetilde{\mathcal{J}}$ that extends $\mathcal{J}$. Then the $U$-admissible blow-up $X' \to X$ along $\widetilde{\mathcal{J}}$ satisfies the properties as in (2).

Here we include some facts on $U$-admissible blow-ups quoted from [89], Première partie, §5, which are fundamental in the $U$-admissible birational geometry.

**Proposition E.1.7 ([89], Première partie, (5.1.4)).** Let $X$ be a coherent scheme and $U \subseteq X$ a quasi-compact open subset. Let $\pi: X' \to X$ be a $U$-admissible blow-up along a quasi-coherent ideal $\mathcal{J} \subseteq \mathcal{O}_X$ of finite type, and $X'' \to X'$ a $\pi^{-1}(U)$-admissible blow-up along a quasi-coherent ideal $\mathcal{J}' \subseteq \mathcal{O}_{X'}$ of finite type. Then there exists a quasi-coherent ideal $\mathcal{J}''$ of $\mathcal{O}_X$ of finite type such that $\mathcal{J}''\mathcal{O}_{X'}$ coincides with $\mathcal{J}'^m \cdot \mathcal{J}''^n \mathcal{O}_{X'}$, for some positive integers $m$ and $n$ and that the composition $X'' \to X$ coincides up to canonical isomorphisms with the $U$-admissible blow-up along the ideal $\mathcal{J} \cdot \mathcal{J}''$.

The proposition says, in particular, that the composition of two $U$-admissible blow-ups is again a $U$-admissible blow-up.

**Proposition E.1.8.** Let $X$ be a coherent algebraic space, $U$ a quasi-compact open subspace of $X$, and $f: Y \to X$ a separated morphism of algebraic spaces of finite type. Suppose that the induced morphism $f^{-1}(U) \to U$ is an open immersion. Then there exists an $U$-admissible blow-up $\pi: X' \to X$ such that, if $Y'$ denotes the strict transform of $Y$, the resulting morphism $Y' \to X'$ is an open immersion:

\[
\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
\pi' \downarrow & & \downarrow \pi \\
Y & \xrightarrow{f} & X.
\end{array}
\]

Moreover, if $U$ is dense in $X$, then the map $\pi': Y' \to Y$ is an $f^{-1}(U)$-admissible modification.

The first assertion is a special case of [89], Première partie, (5.7.11). The second is clear, since $Y' \to Y$ is proper.
Proposition E.1.9. Let $X$ be a coherent algebraic space, $U$ a quasi-compact open subspace of $X$, and $f: Y \to X$ a $U$-admissible proper morphism. Then there exists a $U$-admissible blow-up $\pi: X' \to X$ such that, if $Y'$ denotes the strict transform of $Y$, the resulting morphism $Y' \to X'$ is an isomorphism:

$$
\begin{array}{ccc}
Y & \xrightarrow{\pi} & X' \\
\downarrow f & & \downarrow \pi' \\
X
\end{array}
$$

In other words, there exists an $f^{-1}(U)$-admissible blow-up $\pi: X' \to Y$ such that the composition $X' \to Y \to X$ is a $U$-admissible blow-up.

This is a special case of [89], Première partie, (5.7.12).

E.1. (c) The correspondence diagram. Let $S$ be a coherent scheme, and consider the diagram

$$
\begin{array}{ccc}
& X \\
j & \searrow & \\
U & \downarrow & \\
& Y
\end{array}
$$

of coherent $S$-algebraic spaces, where $j$ is a quasi-compact open immersion, and $Y$ is separated of finite type over $S$. Consider the induced map

$$
i: U \longrightarrow X \times_S Y.
$$

Since the composition $U \xrightarrow{i} X \times_S Y \xrightarrow{p_1} X$, where $p_1$ is the first projection, is the open immersion $j$, the morphism $i$ is a quasi-compact immersion (cf. [53], (4.3.6) (iv)).

Definition E.1.10. The scheme-theoretic closure (cf. [72], II.4.6) of the image of $U$ in $X \times_S Y$ is called the join of $X$ and $Y$ along $U$ and is denoted by $X \ast^U Y$.

The join $X \ast^U Y$ comes with the morphisms

$$
p: X \ast^U Y \longrightarrow X \quad \text{and} \quad q: X \ast^U Y \longrightarrow Y
$$

induced, respectively, by the first and the second projections. The following proposition is clear, and the proof is left to the reader.
**Proposition E.1.11.** Diagram (*) extends to the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{p} & X_{*U} Y \\
\downarrow & & \downarrow \\
U & \xrightarrow{q} & X_{*U} Y \\
\end{array}
\]

of \(S\)-algebraic spaces, where the horizontal arrow is an open immersion. Moreover, we have the following properties.

1. The morphism \(p\) is \(U\)-admissible. If \(U \rightarrow Y\) is also an open immersion, then \(q\) is \(U\)-admissible.

2. If \(X\) (resp. \(Y\)) is proper over \(S\), then \(q\) (resp. \(p\)) is proper.

Note that, if \(X\) and \(Y\) are schemes, then all what we have done (and what we will do in this paragraph) can be done within the category of schemes.

**Proposition E.1.12.** Consider the diagram (*) of coherent \(S\)-algebraic spaces.

1. There exists a \(U\)-admissible blow-up \(X' \rightarrow X\) sitting in a commutative diagram of the form

\[
\begin{array}{ccc}
X' & \xrightarrow{p'} & X_{*U} Y \\
\downarrow & & \downarrow \\
U & \xleftarrow{q'} & Z, \\
\end{array}
\]

where \(p'\) is the strict transform of \(p: X_{*U} Y \rightarrow X\), which is an open immersion. Moreover

(a) if \(X\) is proper over \(S\), then \(q'\) is proper;

(b) if \(Y\) is proper over \(S\), then one can take \(X'\) as above such that the map \(p'\) is an isomorphism.

2. Suppose that the map \(U \rightarrow Y\) is an open immersion. Then there exist a \(U\)-admissible blow-up \(X'\) of \(X\) and a \(U\)-admissible blow-up \(Y'\) of \(Y\) sitting in a
commutative diagram of the form

\[
\begin{array}{ccc}
X' & \xrightarrow{p'} & U \\
\downarrow & & \downarrow \\
Y' & \xleftarrow{q'} & Z,
\end{array}
\]

consisting of open immersions, where \( Z \) is a \( U \)-admissible blow-up of the join \( X \ast_U Y \). Moreover, if \( X \) (resp. \( Y \)) is proper, then one can take \( X' \) and \( Y' \) as above such that \( q' \) (resp. \( p' \)) is an isomorphism.

Proof. (1) By E.1.8, there exists a \( U \)-admissible blow-up \( X' \) of \( X \) such that the strict transform \( Z \to X' \) of the map \( X \ast_U Y \to X \) is an open immersion. Assertion (a) is clear. If \( Y \) is proper, then the map \( X \ast_U Y \to X \) is proper and \( U \)-admissible. Then by E.1.9, one can take a \( U \)-admissible blow-up \( X' \) of \( X \) such that the strict transform \( p' \colon Z \to X' \) of \( X \ast_U Y \to X \) is an isomorphism, whence (b).

(2) First, as in (1), we take \( X' \) and \( Z \) such that the resulting map \( Z \to X \) is an open immersion. Since \( Z \to Y \) is also \( U \)-admissible, one can do the same for this morphism to obtain the strict transform \( Z' \to Y' \), which is an open immersion. Then by E.1.6 there exists a \( U \)-admissible blow-up \( X'' \to X' \) that induces the \( U \)-admissible blow-up \( Z' \to Z \). Hence we obtain the diagram of the form as in (2). The other assertion is shown similarly to (b) of (1); here we use E.1.6 to keep the resulting diagram consisting only of open immersions. \qed

E.1. (d) Birational category. Let \( S \) be a coherent scheme with finitely many irreducible components. Consider the category \( \mathcal{C} \) defined as follows.

- Objects are separated \( S \)-algebraic spaces \( X \) of finite type.
- Arrows \( Y \to X \) between objects \( Y \) and \( X \) are given by \((\emptyset\text{-admissible})\) rational maps \( f: Y \dasharrow X \).

We define the category \( \text{Bir}_S \) as the localized category of \( \mathcal{C} \) by the set of all \((\emptyset\text{-admissible})\) birational maps.

For any object \( X \) of the category \( \mathcal{C} \), that is, a separated \( S \)-algebraic space \( X \) of finite type, we denote by

\[ k(X) \]

the same object considered as an object in \( \text{Bir}_S \), following the customary notation in classical birational geometry.
**Definition E.1.13.** Let $\mathcal{X}$ be an object of $\text{Bir}_S$. A *model* of $\mathcal{X}$ is a pair $(X, \phi)$ consisting of a separated $S$-algebraic space $X$ of finite type and an isomorphism

$$\phi: k(X) \sim \mathcal{X}$$

in the category $\text{Bir}_S$.

Note that if $X_1$ and $X_2$ are models of $\mathcal{X}$, then there exist a separated $S$-algebraic space $U$ of finite type and birational open immersions

$$X_1 \xleftarrow{j_1} U \xrightarrow{j_2} X_2,$$

that is, the images in $j_1(U)$ and $j_2(U)$ are dense in $X_1$ and $X_2$, respectively. Hence one can consider the join $X_1 \ast X_2 = X_1 \ast^U X_2$, which is again a model of $\mathcal{X}$. Note that, as we discussed in §E.1(c), the join $X_1 \ast X_2$ depends only on the closures of $U$ (that is, $X_1$ and $X_2$), and hence is independent on the choice of $U$.

**E.2 Classical Zariski–Riemann spaces**

**E.2. (a) The cofiltered category of modifications.** Let $S$ be a coherent scheme $X$ a coherent $S$-algebraic space, and $U \subseteq X$ a quasi-compact open subspace. We define the category $\text{Mdf}_{(X,U)}$ as follows: objects are $U$-admissible modifications (E.1.2 (4)) $X' \to X$ and arrows between two such morphisms $X' \to X$ and $X'' \to X$ are $X$-morphisms $X' \to X''$.

**Proposition E.2.1.** (1) The category $\text{Mdf}_{(X,U)}$ is cofiltered, and $\text{id}_X$ gives the final object.

(2) The opposite ordered set $\text{Ald}^{\text{opp}}_{(X,U)}$ of the set of all $U$-admissible ideals with the above-mentioned ordering is a directed set, and the functor

$$\text{Ald}_{(X,U)} \longrightarrow \text{Mdf}_{(X,U)}$$

that maps $\mathcal{I}$ to the blow-up $X_\mathcal{I} \to X$ along $\mathcal{I}$ is cofinal.

**Proof.** (1) We need to show that the following conditions are satisfied.

(a) For two $U$-admissible modifications $X' \to X$ and $X'' \to X$ there exist a $U$-admissible modification $X''' \to X$ and $X$-morphisms $X''' \to X'$ and $X''' \to X''$.

(b) For two $U$-admissible modifications $X' \to X$ and $X'' \to X$ and two $X$-morphisms $f_0, f_1: X'' \to X'$ there exist a $U$-admissible modification $X'''' \to X$ and an $X$-morphism $g: X'''' \to X''$ such that $f_0 \circ g = f_1 \circ g$. 
Let us prove (a). By E.1.9, there exists a $U \times_X X'$-admissible blow-up $Y' \to X'$ (resp. a $U \times_X X''$-admissible blow-up $Y'' \to X''$) such that $Y' \to X' \to X$ (resp. $Y'' \to X'' \to X$) is an $U$-admissible blow-up. Let $\mathfrak{f}'$ (resp. $\mathfrak{f}''$) be the blow-up center of $Y' \to X$ (resp. $Y'' \to X$), and set $\mathfrak{f}''' = \mathfrak{f}' \mathfrak{f}''$. Let $X''' \to X$ be the $U$-admissible blow-up centered at $\mathfrak{f}'''$. Then there are $X$-morphisms $X''' \to Y'$ and $X''' \to Y''$ by the universality of blow-ups (cf. [89], Première partie, (5.1.2) (i)). The composites $X''' \to Y' \to X'$ and $X''' \to Y'' \to X''$ are the desired morphisms.

Next let us show (b). Take $Y', Y'', \mathfrak{f}'$, and $\mathfrak{f}''$ as above. The blow-up $Y' \to X'$ (resp. $Y'' \to X''$) is centered at $\mathfrak{f}' \mathfrak{o} X'$ (resp. $\mathfrak{f}'' \mathfrak{o} X''$). Hence, there exist two maps $h_0, h_1: Y'' \to Y'$, induced respectively by $f_0$ and $f_1$. Take the $U$-admissible blow-up $X''' \to X$ along $\mathfrak{f}''' = \mathfrak{f}' \mathfrak{f}''$. Then by the uniqueness in the universality of blow-ups, the two compositions $X''' \to Y'' \to Y'$, where the second maps are given by $h_0$ and $h_1$, coincide. Hence if one defines $g$ to be the composition $X''' \to Y'' \to X''$, we have $f_0 \circ g = f_1 \circ g$, as desired.

(2) is clear by the argument above.

**E.2. (b) The classical Zariski–Riemann spaces**

**Definition E.2.2.** Let $X$ be a coherent $S$-scheme and $U$ a quasi-compact open subset of $X$. Consider the functor $S(X,U): \text{Mdf}(X,U) \to \text{LRsp}$ that maps $X' \to X$ to the underlying locally ringed space of $X'$. The (classical) Zariski–Riemann space associated to the pair $(X,U)$, denoted by $\langle X \rangle_U$, is the underlying topological space of the limit

\[
\lim_{\rightarrow \mathfrak{A}(X,U)} S(X,U).
\]

This limit is canonically identified with the limit taken along a directed set

\[
\lim_{\mathfrak{A}(X,U)} X_{\mathfrak{f}},
\]

which guarantees the existence of the limit (cf. 0.4.1.10). We have for any modification $X' \to X$ of $X$ the canonical projection (specialization map) $\langle X \rangle_U \to X'$, which we denote by $\text{sp}_{X'}$. The inductive limit sheaf

\[
\mathfrak{O}(X)_U = \lim_{\mathfrak{A}(X,U)} \text{sp}^{-1}_{X_{\mathfrak{f}}} \mathfrak{O}_{X_{\mathfrak{f}}}
\]

is a sheaf of local rings, called the structure sheaf of $\langle X \rangle_U$.

Any inclusion $U_1 \hookrightarrow U_2$ of quasi-compact dense open subsets of $X$ induces a map

\[
\langle X \rangle_{U_1} \longrightarrow \langle X \rangle_{U_2},
\]

which is quasi-compact due to 0.2.2.13 (1) and (3).
Notice that, in the above construction, one can actually replace $X$ by the closure $\overline{U}$ of $U$ in $X$ without changing the formation of the locally ringed space $\langle X \rangle_U$; indeed, any irreducible component that does not touch $U$ can be effaced by a $U$-admissible blow-up. In particular, if $U = \emptyset$, then the resulting space $\langle X \rangle_U$ is empty, while if $U \neq \emptyset$, we have $\langle X \rangle_U \neq \emptyset$, since it contains $U$. Because of this, we should have a special treatment for what we really want to mean by $\langle X \rangle_{\emptyset}$, for which we offer the following definition.

**Definition E.2.3.** If $X$ is irreducible, we define

$$\langle k(X) \rangle = \lim_{\overline{U}} \langle X \rangle_U,$$

where $U$ runs over all non-empty quasi-compact open subsets, and call it the (classical) Zariski–Riemann space associated to $X$.

**Remark E.2.4.** Our classical Zariski–Riemann spaces defined as above generalizes the so-called abstract Riemann manifolds introduced by O. Zariski (cf. [107]) in the case where $S$ is the spectrum of an algebraically closed field $k$ and $X$ is an algebraic variety over $k$. Note that the first embryonic idea of abstract Riemann manifolds emerged from the notion of abstract Riemann surface in the works of Dedekind and Weber in the 19th century.

**Theorem E.2.5.** Let $X$ be a non-empty coherent $S$-scheme and $U$ a quasi-compact open subset of $X$. (We allow $U = \emptyset$, if $X$ is irreducible, with the Zariski–Riemann space $\langle X \rangle_{\emptyset}$ replaced by $\langle k(X) \rangle$ defined as in E.2.3.)

1. (Zariski [107]) The topological space $\langle X \rangle_U$ is coherent and sober (0.2.2.1).

2. For any object $X' \to X$ of $\text{Mdf}_{(X,U)}$ the specialization map $\text{sp}_{X'}$ is quasi-compact and closed.

**Proof.** The proof is given by an argument similar to that of 3.1.2 when $U \neq \emptyset$. The other case follows from 0.2.2.10 (1), 0.2.2.13 (1) and (3), and the fact that the map $\langle X \rangle_U_1 \to \langle X \rangle_U_2$ induced by an inclusion $U_1 \hookrightarrow U_2$ of quasi-compact open subsets of $X$ is quasi-compact and closed. 

An argument similar to that in 3.1.3 yields the following result.

**Proposition E.2.6.** A subset $\mathcal{V}$ of $\langle X \rangle_U$ is a quasi-compact open subset if and only if there exist an object $X'$ of $\text{Mdf}_{(X,U)}$ and a quasi-compact open subset $V'$ of $X'$ such that $\mathcal{V} = \text{sp}_{X'}^{-1}(V')$. 

Proposition E.2.7. Let $X$ be a coherent $S$-scheme, and $U \subseteq X$ a quasi-compact open subset.

1. Let $Y \subset X$ be a quasi-compact open subset. Then we have $\langle Y \rangle_{U \cap Y} = \operatorname{sp}_X^{-1}(Y)$, where $\operatorname{sp}_X : \langle X \rangle_U \to X$ is the specialization map, and a canonical open immersion

$$\langle (Y)_{U \cap Y}, \mathcal{O}_{(Y)_{U \cap Y}} \rangle \hookrightarrow \langle (X)_{U}, \mathcal{O}_{(X)_{U}} \rangle$$

of locally ringed spaces.

2. If $X' \to X$ is a proper $U$-admissible morphism, then the induced morphism

$$\langle (X')_{U}, \mathcal{O}_{(X')_{U}} \rangle \to \langle (X)_{U}, \mathcal{O}_{(X)_{U}} \rangle$$

is an isomorphism.

Proof. (1) follows from E.1.6, and (2) from E.1.9.

E.2. (c) Comparison maps. Let $X_1$ and $X_2$ be coherent $S$-schemes, and consider the diagram

$$X_1 \hookleftarrow U \hookrightarrow X_2$$

of quasi-compact $S$-open immersions onto dense open subsets. By E.1.12 (2) and E.1.6, we have the following result.

Proposition E.2.8. Suppose that $X_1$ is separated of finite type over $S$, and that $X_2$ is proper over $S$. Then there exists a canonical open immersion

$$\langle (X_1)_{U}, \mathcal{O}_{(X_1)_{U}} \rangle \hookrightarrow \langle (X_2)_{U}, \mathcal{O}_{(X_2)_{U}} \rangle$$

of locally ringed spaces.

We call the open immersion thus obtained the comparison map of the models $X_1$ and $X_2$.

Corollary E.2.9. In the above situation, suppose that both $X_1$ and $X_2$ are proper over $S$. Then the associated Zariski–Riemann space $\langle X_1 \rangle_U$ and $\langle X_2 \rangle_U$ are canonically isomorphic.

E.2. (d) Relation with rigid Zariskian spaces. Let $X$ be a coherent $S$-scheme and $U \subseteq X$ a quasi-compact dense open subset. Set $Z = X \setminus U$, which we regard as a closed subscheme of $X$ defined by a quasi-coherent ideal $I \subset \mathcal{O}_X$ of finite type. Let $Y = X^\text{Zar} \mid_Z$ be the Zariskian completion ($I$, §B.1. (b)) with the ideal of definition $I \mathcal{O}_Y$ ($I$.B.3.1), and $\mathcal{X} = Y^\text{rig}$ the associated rigid Zariskian
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space (§D.2.(a)). The open immersion \( j : U \hookrightarrow X \) lifts to an open immersion \( \tilde{j} : U \hookrightarrow (X)_U \) of locally ringed spaces such that the diagram

\[
\begin{array}{ccc}
& (X)_U & \\
\tilde{j} & \downarrow \sp_X & \\
U & \downarrow j & \rightarrow \\
& X & \\
\end{array}
\]

commutes. Consider the closed subset \( (X)_U \setminus U \) and the inclusion map

\( \tilde{i} : (X)_U \setminus U \hookrightarrow (X)_U \).

**Proposition E.2.10.** There exists a canonical isomorphism

\[
((X)_U \setminus U, \tilde{i}^{-1} \mathcal{O}_{(X)_U}) \sim ((X), \mathcal{O}^\text{int}_X)
\]

of locally ringed spaces.

**Proof.** Let \( \pi : X' \to X \) be a \( U \)-admissible blow-up, and \( \mathcal{I} \) the blow-up center. The morphism \( \pi \) induces a morphism \( \pi^\text{Zar} : Y' = X^\text{Zar}|_{\pi^{-1}(Z)} \to Y = X^\text{Zar}|_Z \) of Zariskian schemes, which is in fact an admissible blow-up along the admissible ideal \( \mathcal{I} \mathcal{O}_Y \). The topological space \( (X)_U \setminus U \) is nothing but the projective limit of the underlying topological spaces of the Zariskian schemes of the form \( Y' = X^\text{Zar}|_{\pi^{-1}(Z)} \), where \( \pi \) runs through the set of all \( U \)-admissible blow-ups of \( X \). Hence in order to show that there exists a canonical homeomorphism between the topological spaces \( (X)_U \setminus U \) and \( (X) \), it suffices to show that any admissible blow-up of the Zariskian scheme \( Y \) extends naturally to a \( U \)-admissible blow-up of \( X \).

Let \( Y' \to Y \) be an admissible blow-up of the Zariskian scheme \( Y \) along the admissible ideal \( \mathcal{I} \). Take \( k \geq 0 \) such that \( \mathcal{I} \supseteq \mathcal{I}^{k+1} \mathcal{O}_Y \). Consider the scheme \( Y_k = (Y, \mathcal{O}_Y/\mathcal{I}^{k+1} \mathcal{O}_Y) \), which is canonically a closed subscheme of \( X \). Let \( i_k : Y_k \hookrightarrow X \) be the corresponding closed immersion. Consider the surjective map \( \mathcal{O}_X \to i_k* \mathcal{O}_{Y_k} \) of sheaves on \( X \), and take the pull-back of the ideal \( \mathcal{I}_k = \mathcal{I}/\mathcal{I}^{k+1} \mathcal{O}_Y \) by this map. Denote the pull-back ideal by \( \tilde{\mathcal{I}}_k \). Then clearly \( \tilde{i}^{-1} \tilde{\mathcal{I}} = \mathcal{I} \), where \( i : Z \hookrightarrow X \) is the closed immersion; moreover, \( \tilde{\mathcal{I}} \) is a \( U \)-admissible ideal. Hence, if we denote by \( X' \to X \) the \( U \)-admissible blow-up along \( \tilde{\mathcal{I}} \), then it induces the admissible blow-up \( Y' \to Y \) that we have begun with, and the claim is proved.

Since the sheaf pull-back \( \tilde{i}^{-1} \) commutes with filtered inductive limits, we have \( \tilde{i}^{-1} \mathcal{O}_{(X)_U} \cong \mathcal{O}^\text{int}_X \), and thus the proposition is shown. \( \square \)
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E.2. (e) Points of the Zariski–Riemann space. Let $X$ be a coherent $S$-scheme, and $U$ a quasi-compact dense open subset of $X$. Let $I \subset \mathcal{O}_X$ be a quasi-coherent ideal of finite type such that $X \setminus U = V(I)$. Consider the set

$$\{ \text{$X$-isomorphism classes of morphisms of the form } \alpha: \text{Spec } V \to X, \text{ where } V \text{ is an } I\mathcal{V}\text{-adically separated valuation ring, which map the generic point to a point of } U \}.$$ 

There exists a map from this set to the Zariski–Riemann space $\langle X \rangle_U$, constructed as follows. Let $\alpha: \text{Spec } V \to X$ be given, and set $K = \text{Frac}(V)$. For any $U$-admissible modification $X' \to X$, we have a morphism $\text{Spec } K \to X'$ such that the diagram

$$\begin{array}{ccc}
\text{Spec } K & \longrightarrow & X' \\
\downarrow & & \downarrow \\
\text{Spec } V & \longrightarrow & X
\end{array}$$

commutes, since $X' \to X$ is isomorphic on $U$. Since $X' \to X$ is proper, we have a unique lift $\alpha_{X'}: \text{Spec } V \to X'$. Thus we have a projective system of morphisms $\{\alpha_{X'}: \text{Spec } V \to X'\}_{X'}$ and the map of locally ringed spaces $\text{Spec } V \to \langle X \rangle_U$. The sought-for point of $\langle X \rangle_U$ is the image of the closed point of $\text{Spec } V$.

We introduce an equivalence relation $\approx$ on the above set, which is generated by the relation $\sim$ defined as follows: given $\alpha: \text{Spec } V \to X$ and $\beta: \text{Spec } W \to X$, $\alpha \sim \beta$ if there exists a local injection $f: V \hookrightarrow W$ such that $\alpha \circ \text{Spec } f = \beta$ (cf. 0.6.4.5 (2)). There exists a map

$$\begin{array}{c}
\{ \text{$X$-isomorphism classes of morphisms of the form } \alpha: \text{Spec } V \to X, \text{ where } V \text{ is an } I\mathcal{V}\text{-adically separated valuation ring, which map the generic point to a point of } U \} \\
\approx \longrightarrow \langle X \rangle_U
\end{array}$$

induced by the mapping defined above. We equip the left-hand set with the weakest topology such that the map $(\star)$ is continuous. By E.2.5 (3) and E.2.6, this topology coincides with the one generated by subsets of the form

$$\approx\text{-equivalence class of } \alpha: \text{Spec } V \to X \text{ that extends to } \text{Spec } V \to X' \text{ in such a way that the image is contained in } Y'$$

where $X' \to X$ is a $U$-admissible modification and $Y'$ is an affine open subset of $X'$.

**Theorem E.2.11.** The map $(\star)$ is a homeomorphism.

The theorem can be verified by an argument similar to that in the proof of 3.3.6, with the aid of the following lemma, which can be shown similarly to 3.2.6.
Lemma E.2.12. For any point \( \xi \in \langle X \rangle_U \), the ring \( \mathcal{O}_{\langle X \rangle_U, \xi} \) is \( \mathcal{I}_{\langle X \rangle_U, \xi} \)-valuative, see 0.8.7.1.

Finally, let us include here, as a corollary, the classical version (where \( U = \emptyset \)) of the theorem, which is well known in classical birational geometry.

Corollary E.2.13. Let \( X \) be a coherent and integral \( S \)-scheme of finite type and \( k(X) \) its function field. Then the topological space \( \langle X \rangle \) is identified with the following space: as a set, it is the set of all valuation rings for \( k(X) \) dominating the local ring \( \mathcal{O}_{X, x} \) of a point \( x \) of \( X \). The topology is generated by subsets of the form

\[
\left\{ \text{valuation ring for } k(X) \text{ that contains } A \right\},
\]

where \( A \) varies among the subrings of \( k(X) \) arising from a dominant and finite type morphism \( \text{Spec } A \rightarrow X \) over \( S \).

Proof. Let \( U \) be a non-empty quasi-compact open subset of \( X \). The Zariski–Riemann space \( \langle X \rangle \) is the projective limit of all Zariski–Riemann spaces of the form \( \langle X \rangle_U \). Hence it is the projective limit of the spaces in the left-hand side of (\#) as above. Suppose \( \alpha: \text{Spec } V \rightarrow X \) belongs to this set. Then the image of the generic point of \( V \) lies in any quasi-compact open subset of \( X \) and hence is the generic point of \( X \). It follows that the fractional field \( K = \text{Frac}(V) \) of \( V \) contains the function field \( k(X) \). Set \( W = V \cap k(X) \). Then \( W \) is a valuation ring for \( k(X) \), and the map \( W \leftarrow V \) is local. Moreover, we have the morphism \( \beta: \text{Spec } W \rightarrow X \) that is \( \approx \)-equivalent to \( \alpha \). Conversely, if we are given a valuation ring \( W \) for \( k(X) \) dominating a point of \( X \), then the induced morphism \( \beta: \text{Spec } W \rightarrow X \) lies in \( \langle X \rangle_U \) for all \( U \).

\[ \square \]

F Appendix: Nagata’s embedding theorem

F.1 Statement of the theorem

In this section we give a complete proof of the following result, which generalizes a famous theorem due to M. Nagata (cf. [85]).

Theorem F.1.1 (Nagata’s embedding theorem). Let \( Y \) be a coherent scheme and \( f: X \rightarrow Y \) a separated morphism of finite type between schemes. Then there exists a proper \( Y \)-scheme \( \tilde{f}: \tilde{X} \rightarrow Y \) that admits a dense open immersion \( X \hookrightarrow \tilde{X} \) over \( Y \).
F. Appendix: Nagata’s embedding theorem

F.2 Preparation for the proof

F.2. (a) Canonical compactification. Let \( f : X \to Y \) a separated morphism of finite type between algebraic spaces, where \( Y \) is a coherent scheme. We say that \( f \) (or \( X \)) is compactifiable if there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{j} & \tilde{X} \\
\downarrow f & & \downarrow \tilde{f} \\
Y & & \\
\end{array}
\]

such that \( \tilde{f} \) is proper and \( j \) is an open immersion. If this is the case, since the scheme-theoretic closure of \( X \) in \( \tilde{X} \) is proper over \( Y \), one can take \( \tilde{f} \) such that \( j \) is a dense open immersion. In this situation, \( \tilde{X} \) is called a compactification of \( X \) over \( Y \). It is clear that, if \( f \) is affine, then \( f \) is compactifiable by a proper \( Y \)-scheme. If \( X \) is a scheme, \( X \) is covered by open subschemes affine over \( Y \), and hence is locally compactifiable.

Suppose that \( f : X \to Y \), where \( X \) is a scheme, is compactifiable by a proper \( Y \)-scheme \( f' : \tilde{X} \to Y \); here we assume, without loss of generality, that \( \tilde{X} \) contains \( X \) as a quasi-compact dense open subset. Then consider the Zariski–Riemann space \( \langle \tilde{X} \rangle_X \), which is the projective limit of all \( X \)-admissible blow-ups of the scheme \( \tilde{X} \). Let \( f'' : \tilde{X}' \to Y \) be another compactification of \( f \) by a proper \( Y \)-scheme. Then by E.2.9, the topological spaces \( \langle \tilde{X} \rangle_X \) and \( \langle \tilde{X}' \rangle_X \) are canonically isomorphic, that is, the space \( \langle \tilde{X} \rangle_X \) does not depend, up to canonical isomorphisms, on the choice of the compactification.

**Definition F.2.1** (canonical compactification). Let \( f : X \to Y \) be compactifiable by a proper \( Y \)-scheme \( f' : \tilde{X} \to Y \). We write

\[
\langle X \rangle_{\text{cpt}} = \langle \tilde{X} \rangle_X,
\]

and call it the **canonical compactification** of \( X \) over \( Y \).

Note that the topological space \( \langle X \rangle_{\text{cpt}} \) is coherent and sober (E.2.5 (1)) and has \( X \) as a quasi-compact open subspace. By E.2.10,

\[
\langle X \rangle_{\text{cpt}} \setminus X \xrightarrow{\sim} \langle \langle \tilde{X} |_{\text{Zar}} \setminus Z \rangle_{\text{rig}} \rangle,
\]

where \( Z \) is a finitely presented closed subscheme of \( \tilde{X} \) with the underlying topological space \( \tilde{X} \setminus X \). The topological space \( \langle X \rangle_{\text{cpt}} \) has the structure sheaf

\[
\mathcal{O}_{\langle X \rangle_{\text{cpt}}} = \mathcal{O}_{\langle \tilde{X} \rangle_X}.
\]

Note also that, if \( Z \hookrightarrow X \) is a closed immersion of separated and finite type \( Y \)-schemes, and if \( X \) is compactifiable by a proper \( Y \)-scheme, then \( Z \) is compactifiable by a proper \( Y \)-scheme, and \( \langle Z \rangle_{\text{cpt}} \) coincides with the closure of \( Z \) in \( \langle X \rangle_{\text{cpt}} \) with respect to the canonical inclusion \( Z \hookrightarrow \langle X \rangle_{\text{cpt}} \).
Proposition F.2.2. (1) Let $X \hookrightarrow X'$ be a $Y$-open immersion of separated $Y$-schemes of finite type. Suppose that $X'$ is compactifiable by a proper $Y$-scheme. Then $X$ is compactifiable by a proper $Y$-scheme, and we have the canonical closed map
\[(\langle X \rangle_{\text{cpt}}, \mathcal{O}_{\langle X \rangle_{\text{cpt}}}) \rightarrow (\langle X' \rangle_{\text{cpt}}, \mathcal{O}_{\langle X' \rangle_{\text{cpt}}})\]
of locally ringed spaces.

(2) Let $f : U \rightarrow Y$ be a separated $Y$-scheme of finite type that is compactifiable by a proper $Y$-scheme, and $j : U \hookrightarrow X$ an open immersion into a separated $Y$-scheme of finite type. Then we have the canonical open immersion
\[(\langle X \rangle_U, \mathcal{O}_{\langle X \rangle_U}) \hookrightarrow (\langle U \rangle_{\text{cpt}}, \mathcal{O}_{\langle U \rangle_{\text{cpt}}})\]
of locally ringed spaces.

Proof. Let $\bar{X}'$ be a compactification of $X'$ over $Y$. Since the closure $\bar{X}$ of $X$ in $\bar{X}'$ gives a compactification of $X$ over $Y$, $X$ is compactifiable by a proper $Y$-scheme. Since any $X'$-admissible blow-up $\bar{X}' \rightarrow \bar{X}'$ induces an $X$-admissible blow-up $\bar{X} \rightarrow \bar{X}$ by the strict transform, we have the desired map by passage to the projective limits. By 0.2.2.13 (3), this map is closed, whence (1).

(2) follows from E.2.8. \hfill \square

Note that the quasi-compact open subset $\langle X \rangle_U$ in $\langle U \rangle_{\text{cpt}}$ depends only on the closure of $U$ in $X$.

F.2. (b) General construction. Let $f : X \rightarrow Y$ be a separated morphism of finite type between coherent schemes. We are to construct the space $\langle X \rangle_{\text{cpt}}$, again called the canonical compactification, without assuming that $X$ is compactifiable.

Let $U$ be a quasi-compact open subset of $X$ that is compactifiable by a proper $Y$-scheme (e.g., affine over $Y$). By F.2.2 (2) the space $\langle X \rangle_U$ is regarded as a quasi-compact open subset of $\langle U \rangle_{\text{cpt}}$.

Definition F.2.3 (partial compactification). Set
\[\langle U \rangle^X_{\text{pc}} = \langle U \rangle_{\text{cpt}} \setminus \langle X \rangle_U \setminus U,\]
where the closure in the right-hand side is taken in $\langle U \rangle_{\text{cpt}}$, and call it the partial compactification of $U$ relative to $X$.

Proposition F.2.4. Let $X$ be a separated of finite type $Y$-scheme, and $U \subseteq X$ a quasi-compact open subset, which is assumed to be compactifiable by a proper $Y$-scheme.

(1) For any proper $U$-admissible morphism $X' \rightarrow X$, we have $\langle U \rangle^{X}_{\text{pc}} \subseteq \langle U \rangle^{X'}_{\text{pc}}$.

(2) There exists a proper $U$-admissible morphism $X' \rightarrow X$ such that $X'$ is compactifiable by a proper $Y$-scheme.
The proposition says that, whenever discussing the partial compactification \( \langle U \rangle_{pc}^X \), one may assume \( X \) is compactifiable without loss of generality.

**Proof.** (1) The assertion is clear if \( X' \to X \) is a \( U \)-admissible blow-up, since, then, the images of \( \langle X \rangle_U \) and \( \langle X' \rangle_U \) in \( \langle U \rangle_{cpt} \) coincide with each other. The general case reduces to this situation due to E.1.9.

(2) Take a proper \( Y \)-scheme \( \widetilde{U} \) that contains \( U \) as an open subset. Consider the join \( X *U \widetilde{U} \), and take a \( U \)-admissible blow-up \( \tilde{U}' \to \widetilde{U} \) such that the strict transform, denoted by \( X' \to \tilde{U}' \), of \( X *U \widetilde{U} \to \widetilde{U} \) is an open immersion, as in E.1.12 (1). Then \( X' \) is compactifiable by a proper \( Y \)-scheme (F.2.2 (1)), and \( X' \to X \) is proper \( U \)-admissible (E.1.12 (1) (a)). \( \square \)

**Lemma F.2.5.** Let \( X \) be a separated of finite type \( Y \)-scheme, and \( U \subseteq X \) a quasi-compact open subset. We assume that \( X \) is compactifiable by a proper \( Y \)-scheme.

(1) The partial compactification \( \langle U \rangle_{pc}^X \) coincides with the maximal open subset of \( \langle U \rangle_{cpt} \) on which the canonical map \( \langle U \rangle_{cpt} \to \langle X \rangle_{cpt} \) is an isomorphism.

(2) Set \( Z = X \setminus U \). Then we have

\[
\langle U \rangle_{pc}^X = \langle X \rangle_{cpt} \setminus \bar{Z},
\]

where \( \bar{Z} \) denotes the closure of \( Z \) in \( \langle X \rangle_{cpt} \) with respect to the canonical inclusion \( Z \hookrightarrow \langle X \rangle_{cpt} \).

**Proof.** (1) Take a compactification \( \bar{X} \) of \( X \) and the closure \( \bar{U} \) of \( U \) in \( \bar{X} \). We have \( \langle X \rangle_{cpt} = \langle \bar{X} \rangle_\bar{X} \) and \( \langle U \rangle_{cpt} = \langle \bar{U} \rangle_U = \langle \bar{X} \rangle_U \). For any \( U \)-admissible blow-up \( \bar{X}_1 \to \bar{X} \), we have \( \text{sp}_{\bar{X}_1}(\langle X \rangle_U \setminus U) = X_1 \setminus U \), where \( X_1 = X \times_{\bar{X}} \bar{X}_1 \), and

\[
\langle X \rangle_U \setminus U = \bigcap_{\bar{X}_1 \to \bar{X}} \text{sp}_{\bar{X}_1}^{-1}(X_1 \setminus U),
\]

(*)

where \( \bar{X}_1 \) of the left-hand side runs through all \( U \)-admissible blow-ups of \( \bar{X} \); see 0.2.2.19 (2).

Let \( W \) be a quasi-compact open subset of \( \langle U \rangle_{cpt} \), and take a \( U \)-admissible blow-up \( \bar{X}_1 \to \bar{X} \) and a quasi-compact open subset \( W_1 \subseteq \bar{X}_1 \) such that \( W = \text{sp}_{\bar{X}_1}^{-1}(W_1) \); see E.2.6. In view of E.2.7 (1), the map \( \langle U \rangle_{cpt} \to \langle X \rangle_{cpt} \) is an isomorphism on \( W \) if and only if

\[
\langle W_1 \rangle_{U \cap W_1} = \langle W_1 \rangle_{X_1 \cap W_1},
\]

which is, furthermore, equivalent to that any \( X_1 \cap W_1 \)-admissible blow-up of \( W_1 \) is dominated by a \( U \cap W_1 \)-admissible blow-up. This is possible if and only if \( U \cap W_1 = X_1 \cap W_1 \), or equivalently, \( W_1 \cap (X_1 \setminus U) = \emptyset \).
To sum up, we have shown the following: $\langle U \rangle_{\text{cpt}} \to \langle X \rangle_{\text{cpt}}$ is an isomorphism on $W$ if and only if $W \cap (X_1 \setminus U) = \emptyset$ for any $U$-admissible blow-up $\widetilde{X}_1 \to \widetilde{X}$ and any quasi-compact open subset $W_1 \subseteq \widetilde{X}_1$ such that $W = \text{sp}_{\widetilde{X}_1}^{-1}(W_1)$. By the equality $(*)$, the last condition is equivalent to that $W \subseteq \langle U \rangle_{\text{pc}}$.

(2) Restricting to $X$-admissible blow-ups, one can similarly show the following: a quasi-compact open $W \subseteq \langle X \rangle_{\text{cpt}} = \langle \widetilde{X} \rangle_X$ lies in the image of $\langle U \rangle_{\text{pc}}$ if and only if, for any $X$-admissible blow-up $\widetilde{X}_1 \to \widetilde{X}$ and a quasi-compact open subset $W_1 \subseteq \widetilde{X}_1$ such that $W = \text{sp}_{\widetilde{X}_1}^{-1}(W_1)$, we have $W_1 \cap (X \setminus U) = \emptyset$, which is equivalent to that $W \cap Z = \emptyset$.

Let $X$ be a separated of finite type $Y$-scheme, which we assume to be compactifiable by a proper $Y$-scheme $\bar{X}$. Since the open immersion $X \hookrightarrow \bar{X}$ is quasi-compact, one can take a quasi-coherent ideal $\mathfrak{J} \subseteq \mathcal{O}_{\bar{X}}$ of finite type that gives a closed subscheme with the underlying topological space $\bar{X} \setminus X$. Then, for any $x \in \langle X \rangle_{\text{cpt}} = \langle \bar{X} \rangle_X$, the local ring $\mathcal{O}_{\langle X \rangle_{\text{cpt}}, x}$ is $\mathfrak{J}_x \mathcal{O}_{\langle X \rangle_{\text{cpt}}, x}$-valuative; see E.2.12. Set
\[ J_x = \bigcap_{n \geq 1} \mathfrak{J}_x^n, \]
which is a prime ideal of $\mathcal{O}_{\langle X \rangle_{\text{cpt}}, x}$, giving the associated valuation ring $V_x = \mathcal{O}_{\langle X \rangle_{\text{cpt}}, x}/J_x$; see 0, §8.7. By the morphism as in §E.2. (e), the generic point of Spec $V_x$ defines a generization $\bar{x} \in \langle X \rangle_{\text{cpt}}$ of $x$.

**Lemma F.2.6.** In the above situation, let $U \subseteq X$ be a quasi-compact open subset. Then we have
\[ \langle U \rangle_{\text{pc}}^X = \{ x \in \langle X \rangle_{\text{cpt}} : \bar{x} \in U \}. \]

**Proof.** Let $x \in \langle X \rangle_{\text{cpt}}$. For any $X$-admissible blow-up $\widetilde{X}_1 \to \widetilde{X}$, we have $\text{sp}_{\widetilde{X}_1}(\bar{x}) \in X$. Due to F.2.5 (2), $x \in \langle U \rangle_{\text{pc}}^X$ if and only if $\text{sp}_{\widetilde{X}_1}(\bar{x}) \notin Z = X \setminus U$, that is, $\text{sp}_{\widetilde{X}_1}(\bar{x}) \in U$, whence the assertion. □

**Proposition F.2.7.** Let $X$ be a separated of finite type $Y$-scheme, and $U_1, U_2$ quasi-compact open subsets of $X$.

(1) Suppose that $U_2$ is compactifiable by a proper $Y$-scheme, and that $U_1 \subseteq U_2$. Then there exists a canonical open immersion $\langle U_1 \rangle_{\text{pc}}^X \hookrightarrow \langle U_2 \rangle_{\text{pc}}^X$ that extends the inclusion map $U_1 \hookrightarrow U_2$.

(2) Suppose $X$ is compactifiable by a proper $Y$-scheme. Then we have the following equalities in $\langle X \rangle_{\text{cpt}}$:
\[ \langle U_1 \rangle_{\text{pc}}^X \cap \langle U_2 \rangle_{\text{pc}}^X = \langle U_1 \cap U_2 \rangle_{\text{pc}}^X, \quad \langle U_1 \rangle_{\text{pc}}^X \cup \langle U_2 \rangle_{\text{pc}}^X = \langle U_1 \cup U_2 \rangle_{\text{pc}}^X. \]
Proof. (1) By F.2.4, we may assume without loss of generality that $X$ is compactifiable by a proper $Y$-scheme. Then, by F.2.6, we have the inclusion $\langle U_1 \rangle^X_{\text{pc}} \hookrightarrow \langle U_2 \rangle^X_{\text{pc}}$ in $\langle X \rangle_{\text{cpt}}$.

(2) follows immediately from F.2.6. \hfill \Box

Let $f: X \to Y$ be a separated and of finite type $Y$-scheme, and $\{ U_\alpha \}_{\alpha \in L}$ a finite open covering of $X$ such that each $U_\alpha$ is compactifiable by a proper $Y$-scheme. Define the topological space $\langle X \rangle_{\text{cpt}}$ by the following cokernel diagram in the category Top of topological spaces

$$
\langle X \rangle_{\text{cpt}} \leftarrow \coprod_{\alpha \in L} \langle U_\alpha \rangle^X_{\text{pc}} \leftarrow \coprod_{\alpha, \beta \in L} \langle U_\alpha \cap U_\beta \rangle^X_{\text{pc}},
$$

that is, by patching the partial compactifications $\langle U_\alpha \rangle^X_{\text{pc}}$ along the open subspaces $\langle U_\alpha \cap U_\beta \rangle^X_{\text{pc}}$. The space $\langle X \rangle_{\text{cpt}}$ has the structure sheaf $\mathcal{O}_{\langle X \rangle_{\text{cpt}}}$ by patching the structure sheaves $\mathcal{O}_{\langle U_\alpha \rangle_{\text{pc}}}$. $\langle U_\alpha \rangle^X_{\text{pc}}$.

Proposition F.2.8. (1) For any quasi-compact open subset $U$ of $X$ that is compactifiable by a proper $Y$-scheme, $\langle U \rangle^X_{\text{pc}}$ is canonically an open subspace of $\langle X \rangle_{\text{cpt}}$. In particular, the formation of $\langle X \rangle_{\text{cpt}}$ does not depend on the choice of the finite open covering $\{ U_\alpha \}_{\alpha \in L}$.

(2) If $X$ is compactifiable by a proper $Y$-scheme, then the above-defined $\langle X \rangle_{\text{cpt}}$ coincides with the one defined in F.2.1.

Proof. First notice that (2) follows immediately from F.2.7. As for (1), since $U$ and all $U_\alpha$ are compactifiable, we have $\langle U \rangle^X_{\text{pc}} = \langle \bigcup_{\alpha \in L} U \cap U_\alpha \rangle^X_{\text{pc}} = \bigcup_{\alpha \in L} \langle U \cap U_\alpha \rangle^X_{\text{pc}}$ by F.2.7 (2), and since each $\langle U \cap U_\alpha \rangle^X_{\text{pc}}$ is canonically an open subspace of $\langle U_\alpha \rangle^X_{\text{pc}}$ by F.2.7 (1), $\langle U \rangle^X_{\text{pc}}$ is an open subspace of $\langle X \rangle_{\text{cpt}}$. To show that the formation of $\langle X \rangle_{\text{cpt}}$ does not depend on the choice of the finite open covering $\{ U_\alpha \}_{\alpha \in L}$, one employs the inductive argument based on the following statement, which follows from what we have proven just now: for any quasi-compact open subset $U \subset X$, the space $\langle X \rangle_{\text{cpt}}$ constructed from the covering $\{ U_\alpha \}_{\alpha \in L}$ is canonically isomorphic to the one from the covering $\{ U_\alpha \}_{\alpha \in L} \cup \{ U \}$. \hfill \Box

Definition F.2.9. The topological space

$$
\langle X \rangle_{\text{cpt}}
$$

thus defined is called the canonical compactification of $X$ over $Y$.

Note that, by the construction, $\langle X \rangle_{\text{cpt}}$ contains $X$ as an open subset and that there exists the canonical morphism $\langle X \rangle_{\text{cpt}} \to Y$ of locally ringed spaces such that
the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{j} & (X)_{\text{cpt}} \\
\downarrow f & & \downarrow \\
Y & & 
\end{array}
\]

commutes.

Let \( U \) be a quasi-compact open subset of \( X \). For a finite open covering \( \{ U_\alpha \}_{\alpha \in L} \) of \( X \) as above, the collection of spaces \( \{ (U \cap U_\alpha)_{\text{pc}} \}_{\alpha \in L} \) defines an open subset of \((X)_{\text{cpt}}\), which we consistently denote by \((U)_{\text{pc}}\).

Note that the space \((U)_{\text{pc}}\) is, at the same time, an open subspace of \((U)_{\text{cpt}}\).

If \( X \hookrightarrow X' \) is an open immersion of separated \( Y \)-schemes of finite type, then an obvious patching argument shows that \((X')_X\) can be regarded as an open subset of \((X)_{\text{cpt}}\).

**F.2. (c) Properties of canonical compactification**

**Proposition F.2.10.** Any \( Y \)-open immersion \( X \hookrightarrow X' \) of separated \( Y \)-schemes of finite type canonically induces a closed map

\[
((X)_{\text{cpt}}, \mathcal{O}_{(X)_{\text{cpt}}}) \longrightarrow ((X')_{\text{cpt}}, \mathcal{O}_{(X')_{\text{cpt}}})
\]

of locally ringed spaces. Moreover, the open subspace \((X')_\text{pc}\) is the maximal one among the open subspaces of \((X)_{\text{cpt}}\) to which the restrictions of \((X)_{\text{cpt}} \to (X')_{\text{cpt}}\) are open immersions.

**Proof.** The first assertion follows from **F.2.2** (1) by patching. The last assertion follows from the proof of **F.2.7** (2). \(\square\)

**Corollary F.2.11.** The topological space \((X)_{\text{cpt}}\) is quasi-compact.

**Proof.** Take a finite open covering \( \{ U_\alpha \}_{\alpha \in L} \) of \( X \) such that each member \( U_i \) is compactifiable by a proper \( Y \)-scheme. Since \((X)_{\text{cpt}}\) is covered by the open subsets \((U_\alpha)_{\text{pc}}\), the map

\[
\bigsqcup_{\alpha \in L} (U)_{\text{pc}} \longrightarrow (X)_{\text{pc}}
\]

is surjective by **F.2.10**. Since each \((U_\alpha)_{\text{pc}}\) is quasi-compact, the result follows. \(\square\)
Proposition F.2.12 (valuative criterion). Let $V$ be a valuation ring, and set $K = \text{Frac}(V)$. Suppose we are given a commutative diagram

$$
\begin{array}{c}
\text{Spec } K \\
\downarrow \\
\text{Spec } V
\end{array}
\begin{array}{c}
\xrightarrow{\beta} \\
\downarrow \\
\xrightarrow{\alpha} \\
\xrightarrow{f} \\
Y
\end{array}
$$

Then there exists a morphism \(\text{Spec } V \to \langle X \rangle_{\text{cpt}}\) of locally ringed spaces such that the resulting diagram

$$
\begin{array}{c}
\langle X \rangle_{\text{cpt}} \\
\downarrow \\
\text{Spec } V
\end{array}
\begin{array}{c}
\xrightarrow{\alpha} \\
\xrightarrow{\beta}
\end{array}
\begin{array}{c}
\xrightarrow{f} \\
\xrightarrow{\gamma} \\
Y
\end{array}
$$

commutes. Moreover, this morphism is unique.

Proof. The assertion is clear if $X$ is compactifiable. In general, take a quasi-compact open subset $U$ of $X$ that contains the image of $\beta$. We may assume that $U$ is compactifiable by a proper $Y$-scheme. Then we have $\gamma: \text{Spec } V \to \langle U \rangle_{\text{cpt}}$ that lifts $\alpha$. On the other hand, since the open subset $\langle X \rangle_U$ of $\langle U \rangle_{\text{cpt}}$ is quasi-compact, $\gamma^{-1}(\langle X \rangle_U)$ has the minimal point $p$ (0.6.2.3). We may then replace $U$ by a compactifiable quasi-compact open subset of $X$ that contains the image $x$ of $\gamma(p)$ under the specialization map $sp_X: \langle X \rangle_U \to X$. Then $\overline{\{p\}} = \text{Spec } V/\overline{p} \subseteq \text{Spec } V$ is mapped to $\langle U \rangle_{\text{cpt}}$ by the map $\gamma$. It is easy to see by the construction that $\overline{\{p\}}$ is actually mapped into $\langle U \rangle_{\text{pc}}$. Then by patching the valuation rings $\text{Spec } V_p = \gamma^{-1}(\langle X \rangle_U) \to X$ and $\text{Spec } V/\overline{p} = \overline{\{p\}} \to \langle U \rangle_{\text{pc}}$ (cf. 0, §6.4), one obtains the desired lifting. \qed

This result and the usual valuative criterion of properness admit the following corollary.

Corollary F.2.13. Let $X \hookrightarrow X'$ be a dense $Y$-open immersion of separated $Y$-schemes of finite type. If $\langle X' \rangle_X = \langle X \rangle_{\text{cpt}}$, then $X'$ is proper over $Y$.

F.3 Proof of Theorem F.1.1

F.3. (a) Lemmas

Lemma F.3.1 (intersection lemma). Let $X \hookrightarrow X_1$ and $X \hookrightarrow X_2$ be dense $Y$-open immersions between separated $Y$-schemes of finite type, and consider the join $X_1 \ast_X X_2$ (cf. E.1.10). Then

$$
\langle X_1 \ast_X X_2 \rangle_X = \langle X_1 \rangle_X \cap \langle X_2 \rangle_X
$$
in $\langle X \rangle_{\text{cpt}}$. 

Proof. It follows from E.1.12 (2) and E.1.6 that there exists projective systems 
\( \{ X_{i,\lambda} \}_{\lambda \in \Lambda} \) \((i = 1, 2)\) and \( \{ Z_{\lambda} \}_{\lambda \in \Lambda} \) indexed by a directed set \( \Lambda \) and dense open immersions
\[
X_{1,\lambda} \leftarrow Z_{\lambda} \leftarrow X_{2,\lambda}
\]
that are compatible with the projection maps, such that \( (X_i)_X = \lim_{\lambda \in \Lambda} X_{i,\lambda} \) for \( i = 1, 2 \) and \( (X_1 \times X_2)_X = \lim_{\lambda \in \Lambda} Z_{\lambda} \). Then the claimed equality follows from the left-exactness of projective limits.

Lemma F.3.2 (patching lemma). Let \( X \leftarrow X_1 \) and \( X \leftarrow X_2 \) be dense \( \mathcal{Y} \)-open immersions between separated \( \mathcal{Y} \)-schemes of finite type. Then there exists another dense \( \mathcal{Y} \)-open immersion \( X \leftarrow Z \) of separated \( \mathcal{Y} \)-schemes of finite type such that
\[
\langle Z \rangle_X = \langle X_1 \rangle_X \cup \langle X_2 \rangle_X
\]
in \( \langle X \rangle_{\text{cpt}} \).

Proof. By E.1.12 (2), we have a diagram
\[
\X_1 \leftarrow \tilde{W} \leftarrow \X_2
\]
consisting of \( \mathcal{X} \)-admissible quasi-compact \( \mathcal{Y} \)-open immersions, where \( \X_1 \) and \( \X_2 \) are admissible blow-ups of \( X_1 \) and \( X_2 \), respectively. The sought-for \( Z \) is obtained by gluing \( \X_1 \) and \( \X_2 \) along \( \tilde{W} \) in the usual sense. Note that the scheme \( \tilde{W} \) is proper over the join \( W = X_1 \times X_2 \).

To conclude, we need to show that the model \( Z \) thus obtained is a separated \( S \)-scheme; the key point is that the intersection \( \langle X_1 \rangle_X \cap \langle X_2 \rangle_X \) is represented by the join \( X_1 \times X_2 \), as we have seen in F.3.1. To show the claim, we use the valuative criterion. Let \( V \) be a valuation ring and let \( K \) be its field of fractions. Suppose we have two morphisms \( \alpha_1, \alpha_2 : \text{Spec} \, V \to Z \) dominating the same maximal point such that \( \alpha_1 \otimes_V K = \alpha_2 \otimes_V K \). We need to show that \( \alpha_1 = \alpha_2 \). When the closed point of \( \text{Spec} \, V \) is mapped by both \( \alpha_1 \) and \( \alpha_2 \) into either one of \( \X_1 \) and \( \X_2 \), the claim immediately follows, since \( X_1 \) and \( X_2 \) are assumed to be separated. If, on the contrary, say, \( \alpha_1(\text{Spec} \, V) \subset \X_1 \) and \( \alpha_2(\text{Spec} \, V) \subset \X_2 \), since \( \alpha_1 \) and \( \alpha_2 \) coincide on \( \text{Spec} \, K \), we have a morphism
\[
(\alpha_1, \alpha_2) : \text{Spec} \, V \to X_1 \times X_2,
\]
and hence \( \beta : \text{Spec} \, V \to \tilde{W} \), since \( \tilde{W} \) is proper over \( W = X_1 \times X_2 \). Now by the construction the compositions
\[
\text{Spec} \, V \xrightarrow{\beta} \tilde{W} \leftarrow \X_i \leftarrow \X
\]
\((i = 1, 2)\) are nothing but \( \alpha_i \), thereby the claim.
Lemma F.3.3 (local extension lemma). For any point \( y \in \langle X \rangle_{\text{cpt}} \) there exists a dense \( Y \) -open immersion \( X \hookrightarrow X_y \) of separated \( Y \)-schemes of finite type such that \( \langle X_y \rangle_X \) contains the point \( y \).

Proof. Take a quasi-compact open subset \( U \) of \( X \) that is compactifiable by a proper \( Y \)-scheme \( \bar{U} \) such that \( y \) lies in \( \langle U \rangle_{\text{pc}} \). By the construction of \( \langle U \rangle_{\text{pc}} \) we have \( y \notin \langle X \rangle_{U \setminus \bar{U}} \). It follows that there exists a quasi-compact open neighborhood \( \mathcal{V} \) of \( y \) in \( \langle U \rangle_{\text{cpt}} \) that is disjoint from \( \langle X \rangle_{U \setminus \bar{U}} \); note that \( \mathcal{V} \subseteq \langle X \rangle_{\text{cpt}} \). Replacing the compactification \( \bar{U} \) of \( U \) by a \( U \)-admissible blow-up if necessary, we may assume that there exists a quasi-compact open subset \( V \) of \( \bar{U} \) such that \( \text{sp}^{-1}(V) = \mathcal{V} \) (0.2.2.9). Replacing \( V \) by \( V \cup U \) if necessary, we may assume that \( V \) contains \( U \). By the construction, \( \langle X \rangle_{U \cap V} = U \). Patching \( X \) and \( V \) along \( U \) birationally, we get a separated \( Y \)-scheme \( X \) of finite type such that \( \langle X \rangle_{X} = \langle X \rangle_{U \cap V} \). Since \( \langle X \rangle_{X} \cap \langle U \rangle_{\text{pc}} = \langle X \rangle_{U} \), \( \langle X \rangle_{X} \) contains \( y \), as claimed.

F.3. (b) Proof of the theorem. Now we proceed to the proof of F.1.1. Let \( f : X \to Y \) be a separated \( Y \)-scheme of finite type, where \( Y \) is a coherent scheme. First, for any \( y \in \langle X \rangle_{\text{cpt}} \) we take \( X_y \) as in F.3.3 containing \( X \) such that \( y \in \langle X_y \rangle_X \). Now the quasi-compactness of \( \langle X \rangle_{\text{cpt}} \) (F.2.11) implies that there exists a finite set \( I \) of \( \langle X \rangle_{\text{cpt}} \) such that \( \{ \langle X_y \rangle_X \}_{y \in I} \) gives a covering of \( \langle X \rangle_{\text{cpt}} \). Then, applying F.3.2 successively, we get a separated \( Y \)-scheme \( \bar{X} \) of finite type containing \( X \) such that

\[
\langle \bar{X} \rangle_X = \bigcup_{y \in I} \langle X_y \rangle_X = \langle X \rangle_{\text{cpt}}.
\]

By F.2.13, we conclude that \( \bar{X} \) is proper over \( Y \), which therefore gives the desired compactification.

F.4 Application: Removing the Noetherian hypothesis

Proposition F.4.1. Let \( Y \) be a quasi-compact scheme, and \( f : X \to Y \) a morphism. If \( f \) is proper and affine, then it is finite.

To show the proposition, we need several preparatory results.

Lemma F.4.2. Let \( X \) be a coherent scheme, and \( U \) a quasi-compact open subset of \( X \). Let \( \{ Z_\lambda \}_{\lambda \in \Lambda} \) be a filtered projective system consisting of closed subscheme of \( X \) indexed by a directed set \( \Lambda \), such that for each \( \lambda \geq \mu \) the transition maps \( i_{\lambda \mu} : Z_\lambda \to Z_\mu \) are closed immersions over \( X \). Then,

\[
\bigcap_{\lambda \in \Lambda} U \cap Z_\lambda = \bigcap_{\lambda \in \Lambda} (U \cap Z_\lambda) = \bar{U} \cap \left( \bigcap_{\lambda \in \Lambda} Z_\lambda \right).
\]
Proof. The inclusion \( \bigcap_{\lambda \in \Lambda} U \cap Z_{\lambda} \supseteq \bigcap_{\lambda \in \Lambda} (U \cap Z_{\lambda}) \) is obvious. Take any \( x \in \bigcap_{\lambda \in \Lambda} U \cap Z_{\lambda} \) and any quasi-compact open neighborhood \( V \) of \( x \). Then \( V \cap (U \cap Z_{\lambda}) \neq \emptyset \) for any \( \lambda \in \Lambda \). By \((6.9.15)\), and \((6.4.3)\). Since this holds for any quasi-compact open neighborhood \( V \) of \( x \), we have \( x \in \bigcap_{\lambda \in \Lambda} (U \cap Z_{\lambda}) \), whence the first equality. The fact that \( \bigcap_{\lambda \in \Lambda} U \cap Z_{\lambda} = \bigcap_{\lambda \in \Lambda} Z_{\lambda} \) is shown in a similar manner. \( \square \)

Proposition F.4.3. Let \( Y \) be a coherent scheme and \( f : X \to Y \) a separated morphism of finite type. Let \( \{Z_{\lambda}\}_{\lambda \in \Lambda} \) be a filtered projective system consisting of closed subschemes of \( X \) indexed by a directed set \( \Lambda \), such that for each \( \lambda \geq \mu \) the transition maps \( i_{\lambda,\mu} : Z_{\lambda} \to Z_{\mu} \) are closed immersions over \( X \). If the projective limit \( \lim_{\lambda \in \Lambda} Z_{\lambda} \) is proper over \( Y \), then there exists \( \lambda_0 \in \Lambda \) such that \( Z_{\lambda_0} \) is proper over \( Y \) for any \( \lambda \geq \lambda_0 \).

Proof. Take a proper \( Y \)-scheme \( \bar{f} : \bar{X} \to Y \) together with a dense \( Y \)-open immersion \( X \hookrightarrow \bar{X} \) (F.1.1). Let \( \bar{Z}_{\lambda} \) be the scheme-theoretic closure of \( Z_{\lambda} \) in \( \bar{X} \) for \( \lambda \in \Lambda \). Suppose that \( \bar{Z}_{\lambda} \cap (\bar{X} \setminus X) \neq \emptyset \) for any \( \lambda \in \Lambda \). Then by the quasi-compactness of \( X \) we have \( \bigcap_{\lambda \in \Lambda} \bar{Z}_{\lambda} \cap (\bar{X} \setminus X) \neq \emptyset \). On the other hand, we have by \((6.4.2)\) the equality \( \bigcap_{\lambda \in \Lambda} \bar{Z}_{\lambda} = \bigcap_{\lambda \in \Lambda} Z_{\lambda} \). Therefore, we have \( \bigcap_{\lambda \in \Lambda} \bar{Z}_{\lambda} \cap (\bar{X} \setminus X) = (\bigcap_{\lambda \in \Lambda} Z_{\lambda}) \cap (\bar{X} \setminus X) \neq \emptyset \). But since \( \bigcap_{\lambda \in \Lambda} Z_{\lambda} \) is proper over \( Y \), it is closed in \( \bar{X} \), and hence \( (\bigcap_{\lambda \in \Lambda} Z_{\lambda}) \cap (\bar{X} \setminus X) = \emptyset \), which is absurd. Therefore, there exists \( \lambda_0 \) such that \( \bar{Z}_{\lambda_0} \subseteq X \) for \( \lambda \geq \lambda_0 \), that is, \( Z_{\lambda} \) is proper. \( \square \)

Corollary F.4.4. Let \( Y \) be a coherent scheme, and \( f : X \to Y \) a separated morphism of finite type. Let \( Z \) be a closed subscheme of \( X \) that is proper over \( Y \). Then there exists a closed subscheme \( Z' \) of \( X \) of finite presentation that is proper over \( Y \) and contains \( Z \) scheme-theoretically.

Proof. This follows immediately from \((53), (6.9.15), \text{and F.4.3})\.

Proof of Proposition F.4.1. We may assume that \( Y \) is affine. If \( f \) is of finite presentation, then there exist a proper and affine morphism \( f' : X' \to Y' \) between Noetherian schemes ((\(54), \text{IV}, (8.10.5)) and a morphism \( Y \to Y' \) such that \( f = f' \). Since the assertion is well known in the Noetherian case, we have the desired result for the case when \( f \) is finitely presented.

In general, write \( X \) as a closed subscheme of \( \mathbb{A}_Y^n \) for some \( n \geq 0 \). By \((6.4.4)\), there exists a closed subscheme \( X' \), proper over \( Y \), of \( \mathbb{A}_Y^n \) of finite presentation that contains \( X \) scheme-theoretically. Since \( X' \) is finite over \( Y \), so is \( X \). \( \square \)

F.5 Nagata embedding for algebraic spaces

Finally, we include here the statement of the compactification theorem for algebraic spaces ([\(40 \text{ and } 30])\).
Theorem F.5.1 (Nagata’s embedding theorem for algebraic spaces). Let $Y$ be a coherent algebraic space, and $f: X \to Y$ a separated $Y$-algebraic space of finite type. Then there exists a proper $Y$-algebraic space

$$
\bar{f}: \bar{X} \to Y
$$

that admits a dense open immersion $X \subseteq \bar{X}$ over $Y$. Moreover,

(a) there exists a quasi-coherent ideal $I$ of $O_{\bar{X}}$ of finite type such that the underlying topological subspace of the corresponding closed subspace coincides with the boundary $\partial X = \bar{X} \setminus X$ and that $(\partial X, O_{\bar{X}}/I)$ is a scheme

(b) if $X$ is a scheme, then there exists a compactification $\bar{f}: \bar{X} \to Y$ as above such that $\bar{X}$ is a scheme.

Note that, in view of 0.5.5.9, the contents of (a) can be rephrased without ambiguity as the assertion that the boundary $\partial X = \bar{X} \setminus X$ ‘is’ a scheme.

Exercises

Exercise II.F.1. Let $Y$ be a coherent scheme, $f: X \to Y$ a separated $Y$-scheme of finite type, and $U \subseteq X$ a quasi-compact open subset. Then the open immersion $j: U \to X$ induces a closed map $(j)_{\text{cpt}}: \langle U \rangle_{\text{cpt}} \to \langle X \rangle_{\text{cpt}}$. Moreover, $(j)_{\text{cpt}}$ maps the open subset $\langle U \rangle_{\text{pc}}$ isomorphically onto an open subset of $\langle X \rangle_{\text{cpt}}$.

Exercise II.F.2. Let $Y$ be a coherent scheme, and $f: X \to Y$ a separated $Y$-algebraic space of finite type. Let $U$ be a quasi-compact open subspace of $X$ that is a scheme. Then there exists a $U$-admissible blow-up $X'$ of $X$ that is a scheme.
Solutions and hints for exercises

Chapter 0

Exercise 0.1.2. We may assume that $I$ is countable. To skip the trivial cases, we may further assume that $I$ does not have maximal elements. For any fixed $\alpha_0 \in I$ we may replace $I$ by the final subset $\{ \alpha \in I: \alpha \geq \alpha_0 \}$ and thus may assume that $I$ has the minimum element $\alpha_0$. Write $I = \{ \alpha_0, \alpha_1, \alpha_2, \ldots \}$. Set $L(0) = \alpha_0$ and $L(1) = \alpha_1$ and define $L(k+1) = \alpha_{l(k+1)}$ for $k \geq 1$ inductively as follows: $l(k+1)$ is the smallest number such that $\alpha_i < \alpha_{l(k+1)}$ for $l(k-1) + 1 \leq i \leq l(k)$. Then the resulting map $L: \mathbb{N} \to I$ is final.

Exercise 0.2.3. The projective limit $X$ is the underlying topological space of a scheme that is affine over a coherent scheme; see [54], II, §8.2.

Exercise 0.2.7. (1) Consider for each $i \in I$ the set $G_{x_i}$ of all generizations of $x_i$, which is coherent and sober due to 0.2.2.16. For $i \leq j$ there exists the canonical inclusion map $G_{x_j} \hookrightarrow G_{x_i}$, which is quasi-compact due to 0.2.1.6. Then the intersection $\bigcap_{i \in I} G_{x_i}$ is non-empty due to 0.2.2.10.

(2) Use (1) and apply Zorn’s lemma.

Exercise 0.2.8. Set $U = \bigcup_{k=1}^n U_k$, and apply 0.2.12 to the case when $V = X_i$.

Exercise 0.2.9. Suppose that $X$ is not connected, and take non-empty open subsets $U_0, U_1 \subseteq X$ such that $X = U_0 \cup U_1$ and $U_0 \cap U_1 = \emptyset$. Since $U_0, U_1$ are closed and $X$ is quasi-compact (0.2.10 (1)), $U_0, U_1$ are quasi-compact. By 0.2.2.9 there exist quasi-compact open subsets $U_{0i}, U_{1i} \subseteq X_i$ for some $i \in I$ such that $U_0 = p_i^{-1}(U_{0i})$ and $U_1 = p_i^{-1}(U_{1i})$, where $p_i: X \to X_i$ is the projection map. Since $p_i$ is surjective (0.2.14), one deduces easily that $X_i = U_{0i} \cup U_{1i}$ and $U_{0i} \cap U_{1i} = \emptyset$.

Exercise 0.2.10. Let $x, y \in [X]$ with $x \neq y$. Since $\text{sep}_X^{-1}(x) = \overline{\{x\}}$ is an overconvergent closed subset of $X$, by 0.2.3.13 there exists a unique open subset $U \subseteq [X]$ such that $\text{sep}_X^{-1}(U) = X \setminus \{x\}$. In particular, $y \in U$ and $x \notin U$.

Exercise 0.2.11. (1) follows from 0.2.5.7. To show (2), we may assume, in view of 0.2.5.7, that $X$ is quasi-separated, since the question is local on $[X]$. Using 0.2.5.4, take for any $x \in U \cap V$, a pair $(U_x, V_x)$ of coherent open neighborhoods in $V$ of the closure of $\{x\}$ in $V$ such that $V_x$ contains the closure $\overline{U_x}$ of $U_x$ in $V$. 
Then \( (U \cap U_x, U \cap V_x) \) gives a pair of open neighborhoods of \( x \) in \( U \cap V \). Since \( U \cap U_x \leftrightarrow U_x \) and \( U \cap V_x \leftrightarrow V_x \) are quasi-compact, \( U \cap U_x \) and \( U \cap V_x \) are coherent. As we have \( U \cap U_x \subseteq U \cap V_x \), \((U \cap U_x, U \cap V_x)\) satisfies the condition as in 0.2.5.4.

**Exercise 0.2.12.** (1) Since the question is local on \( Y \), we may assume that \( Y \) is Hausdorff. Replacing \( Y \) by the image of \( f \) endowed with the subspace topology from \( Y \), we may also assume that \( f \) is bijective. For any \( x \in X \), take a relatively compact open neighborhood \( U_x \) of \( x \) in \( X \). Then, since \( U_x \) is compact, \( f|_{U_x} \) is a homeomorphism onto its image. Hence \( f \) is a local homeomorphism. Since \( f \) is bijective, \( f \) is a homeomorphism.

(2) It suffices to show that a locally compact subspace of \( Y \) is open in its closure. One can reduce by localization to the case where \( Y \) is Hausdorff, and in this situation the claim is well known.

(3) By 0.2.5.7, \([U]\) and \([X]\) are locally compact locally Hausdorff spaces.

**Exercise 0.2.13.** (1) To show the ’if’ part, we may assume in view of 0.2.5.7 that \( X \) is quasi-separated, since the question is local on \([X]\). Let \( V \subseteq X \) be a coherent open subset. We need to show that \( U \cap V \) is quasi-compact. Since \([V]\) is compact and \([U]\) is closed, \([Z] \cap [U] = [Z \cap U] \) is compact. On the other hand, since \( U \cap V \) is locally strongly compact (Exercise 0.2.11), the separation map \( \text{sep}_{U \cap V} \) is proper (0.2.5.7), and hence \( U \cap V \) is quasi-compact, as desired.

To show the ’only if’ part, consider an open covering \( X = \bigcup_{\alpha \in L} U_\alpha \) by coherent open subsets. Since \( U \cap U_\alpha \) is quasi-compact, \([U] \cap [U_\alpha] = [U \cap U_\alpha] \) is compact, and hence is closed in the Hausdorff space \([U_\alpha]\) (0.2.3.18) for each \( \alpha \in L \). Then it follows from 0.2.3.26 that \([U]\) is closed in \([X]\).

(2) Take an open subset \( Z \subseteq [X] \) of \([X]\) that contains \([U]\) as a closed subset. Then \( Z = \text{sep}_X^{-1}(Z) \) satisfies the desired property.

**Exercise 0.2.14.** Let \( U, V \) be locally strongly compact open subsets of a locally strongly compact valuative space \( X \). We want to show that \( U \cap V \) is locally strongly compact. (Here, recall that \( X \) has an open basis consisting of coherent open subsets.) Since the problem is local on \([X]\), we may assume in view of Exercise 0.2.12 (3) that \([U]\) is closed in \([X]\). By Exercise 0.2.13 (1), the inclusion map \( U \leftrightarrow X \) is quasi-compact. Now apply Exercise 0.2.11 (2).

**Exercise 0.2.15.** We want to show that, for any coherent open subset \( U \subseteq X \), the inclusion map \( U \leftrightarrow X \) is quasi-compact. Since \( U \) is locally strongly compact (0.2.5.2), \([U]\) is identified with a subspace of \([X]\) (Exercise 0.2.12 (3)). Since \([U]\) is compact and \([X]\) is Hausdorff, \([U]\) is closed in \([X]\). Hence \( U \leftrightarrow X \) is quasi-compact by Exercise 0.2.13 (1).
Exercise 0.3.1. See [68], §2.

Exercise 0.3.3. Consider the homomorphism of the form $A^\otimes p \to A$. We want to show that its kernel is finitely generated. There exist $i \in I$ and a map $A_i^\otimes p \to A_i$ that induces the above map by the tensor product with $A$. Since $A_i$ is coherent, its kernel $K$ is finitely generated. Since $A$ is flat over $A_i$, the kernel of $A_i^\otimes p \to A$ is given by $K \otimes_{A_i} A$.

Exercise 0.3.4. The ring in question is coherent due to Exercise 0.3.3. To see that it is not Noetherian, consider a prime number $p$ and the sequence of ideals $(p) \subseteq (\sqrt{p}) \subseteq (\sqrt[2]{p}) \subseteq \cdots \subseteq (\sqrt[n]{p}) \subseteq \cdots$, which one can show is strictly increasing using the $p$-adic valuation.

Exercise 0.4.1. We may assume that $X$ is affine, $X = \text{Spf} A$, where $A$ is a Noetherian adic ring with an ideal of definition $I \subseteq A$. Let $p$ (resp. $q$) be the open prime ideal corresponding to $x$ (resp. $y$); we have $p \supseteq q$. Then $\mathcal{O}_{X,x} = A_{\{S\}}$ and $\mathcal{O}_{X,y} = A_{\{T\}}$ (in the notation as in [54], 0.1, §7.6; notice that these are Noetherian rings ([54], 0.1, (7.6.18))), where $S = A \setminus p$ and $T = A \setminus q$. We have a commutative square of rings

$$
\begin{array}{ccc}
(A_{\{S\}})_{q} & \longrightarrow & A_{\{T\}} \\
\downarrow & & \downarrow \\
\hat{A}_{p} & \longrightarrow & \hat{A}_{q}
\end{array}
$$

where $\hat{A}_{p}$ (resp. $\hat{A}_{q}$) denotes the $I$-adic completion of the localization $A_p$ (resp. $A_q$). One needs to show that the lower horizontal arrow is faithfully flat. By [54], 0.1, (7.6.2) and (7.6.18), the vertical arrows are faithfully flat. Moreover, since the $I$-adic completion of $(\hat{A}_{p})_{q} \hat{A}_{p}$ is nothing but $\hat{A}_{q}$, the upper horizontal arrow is faithfully flat ([27], Chapter III, §3.5, Proposition 9). Hence $(A_{\{p\}})_{q}A_{\{p\}} \to A_{\{q\}}$ is faithfully flat.

Exercise 0.4.2. Using 0.4.1.10, reduce the questions to questions for inductive limits of rings. Then use the results in 0, §3.1. (a).

Exercise 0.4.3. Let $p_i : X \to X_i$ be the projection for each $i \in I$. We need to show that, for any quasi-compact open subset $U \subseteq X$, the kernel of a morphism of $\mathcal{O}_U$-modules of the form

$$
\varphi : \mathcal{O}_U^\otimes p \to \mathcal{O}_U
$$

is of finite type. To show this, in view of 0.2.2.9 and 0.4.1.11, we may assume $U = X$. By 0.4.2.1, there exists $i \in I$ and a morphism $\varphi_i : \mathcal{F}_i \to \mathcal{G}_i$ of finitely presented $\mathcal{O}_{X_i}$-modules such that $p_i^* \varphi_i \cong \varphi$. Since $X_i$ is cohesive, these sheaves
are coherent $\mathcal{O}_X$-modules, and hence $\mathcal{K}_i = \ker \varphi_i$ is again coherent. Now by Exercise 0.4.2 (1) it follows that the induced sequence

$$0 \rightarrow p_i^* \mathcal{K}_i \rightarrow \mathcal{O}_X^{\oplus n} \rightarrow \mathcal{O}_X$$

is exact, where $p_i^* \mathcal{K}_i$ is clearly of finite type.

**Exercise 0.5.1.** The ‘if’ part is clear. Suppose $U$ is quasi-compact, and consider a closed subscheme $Y$ with the underlying topological space $X \setminus U$. Let $I$ be the defining ideal of $Y$ in $X$. By [54], I, (9.4.9), and IV, (1.7.7), one can write $I$ as a filtered inductive limit of quasi-coherent subideals $I_i$ of $\mathcal{O}_X$ of finite type. Set $U_i = X \setminus Y(I_i)$. Then $U = \bigcup U_i$. Since $U$ is quasi-compact, we have $U = U_i$ for some $i$.

**Exercise 0.5.2.** By [53], (2.3.5), we know that $f$ is an affine morphism. Thus we can assume that $X$ and $Y$ are affine, $X = \text{Spec } A$ and $Y = \text{Spec } B$. Let $I \subseteq A$ be the nilpotent finitely generated ideal such that $I^2 = 0$. The assumption is that the induced map $A/I \rightarrow B/IB$ is surjective. By [81], Theorem 8.4, (see 0.7.2.4), $B$ is finitely generated as an $A$-module. To show that $A \rightarrow B$ is surjective, it suffices to show that the map $I \rightarrow IB$ is surjective (by the snake lemma). Let $C$ be the cokernel of $I \rightarrow IB$. Then $C$ is a finitely generated $A$-module such that $IC = 0$. By Nakayama’s lemma, $C = 0$.

**Exercise 0.5.3.** We mimic the proof of [15], Exposé II, Corollary 2.2.2.1, which refers to [15], Exposé II, Proposition 2.2.2, in which the Noetherian hypothesis is used only in the end of the proof, where one refers to [54], I, (9.4.9). It uses the fact that quasi-coherent sheaves of finite type on locally Noetherian schemes are coherent. We replace this part of the proof by the following argument: for any quasi-coherent sheaf $\mathcal{F}$ of finite type, there exist a coherent sheaf $\mathcal{G}$ and a surjective morphism $\mathcal{G} \rightarrow \mathcal{F}$. This follows from [54], II, (2.7.9).

**Exercise 0.6.2.** (3) Embed $\text{Spec } A$ into $(\mathbb{P}_V^1)^n$, and show that the closure is finite using the projections $(\mathbb{P}_V^1)^n \rightarrow \mathbb{P}_V^1$.

**Exercise 0.6.3.** By Exercise 0.6.2 (1), it suffices to show that $M$ is torsion free if and only if $M$ is $a$-torsion free. This follows from the following observation: for any $x \in V \setminus \{0\}$, since $\bigcap_{n \geq 0} (a^n) = 0$, there exists $n \geq 0$ such that $x \notin (a^n)$; then we have $a^n = xy$ for some $y \in V$.

**Exercise 0.6.4.** (1) By 0.6.1.4, the totally order commutative group $\Gamma_V$ is order isomorphic to an ordered subgroup of $\mathbb{R}^d$ (where $d = \text{ht}(V)$) with the lexicographical order (cf. 0.6.1.3). Take $a \in V$ such that $v(a) = (i_1, \ldots, i_d)$ with $i_1 > 0$. Then for any element $b \in V \setminus \{0\}$ there exists $n \geq 0$ such that $v(b) < v(a^n)$, that is, $b \notin a^nV$. Hence, $\bigcap_{n \geq 0} a^nV = \{0\}$. 


(2) Consider the quotient ring \( V/p \), which is again a valuation ring such that \( 0 < \text{ht}(V/p) < +\infty \) (0.6.4.1 (1) and 0.6.4.3). Now apply (1).

**Exercise 0.7.1.** An element \( x \in M \) lies in \( \hat{N} \) if and only if \((x + F^\lambda) \cap N \neq \emptyset\) for any \( \lambda \in \Lambda \), where the last condition is equivalent to \( x \in N + F^\lambda \) for any \( \lambda \in \Lambda \).

**Exercise 0.7.2.** Consider the closure \( J = \bigcap_{\lambda \in \Lambda} (gA + I^{(\lambda)}) \) of the ideal \( gA \). Let \( x \in J \). For any \( \lambda \in \Lambda \), we have \( a_\lambda \in A \) and \( b_\lambda \in I^{(\lambda)} \) such that \( x = ga_\lambda + b_\lambda \). For \( \lambda \leq \mu \), we have \( g(a_\lambda - a_\mu) = b_\mu - b_\lambda \in I^{(\lambda)} \). Since \( (g \bmod I^{(\lambda)}) \) is a non-zero-divisor in \( A/I^{(\lambda)} \), we have \( a_\lambda - a_\mu \in I^{(\lambda)} \) for any \( \mu \geq \lambda \). Hence \( \{a_\lambda\}_{\lambda \in \Lambda} \) is a Cauchy sequence, converging to an element \( a \in A \). Since \( x - ga \in \bigcap_{\lambda \in \Lambda} I^{(\lambda)} = \{0\} \), we have \( x \in gA \).

**Exercise 0.7.5.** (1) By 0.3.2.8, the exact sequences

\[
0 \longrightarrow N/F = M/F \longrightarrow M/N \longrightarrow 0
\]

induce, by passage to the projective limits, the exact sequence

\[
0 \longrightarrow N_{\hat{F}^\bullet} \longrightarrow M_{\hat{F}^\bullet} \longrightarrow M/N \longrightarrow 0
\]

(cf. 0, §7.1. (d)).

(2) If \( F^\bullet \) is separated, then the canonical maps \( M \rightarrow M_{\hat{F}^\bullet} \) and \( N \rightarrow N_{\hat{F}^\bullet} \) are injective. Since \( M/N \cong M_{\hat{F}^\bullet}/N_{\hat{F}^\bullet} \), we have \( N_{\hat{F}^\bullet} \cap M = N \).

**Exercise 0.7.6.** See [54], 01, §7.7.

**Exercise 0.7.7.** The \( a \)-adic completion \( \hat{V} \) coincides with the \( a \)-adic completion of \( V/p \), where \( p = \bigcap_{n \geq 0} (a^n) \).

**Exercise 0.7.8.** We show that the condition implies (AP) (resp. (APf)). Let \( M \) be a finitely generated \( A \)-module and \( N \subseteq M \) an \( A \)-module (resp. a finitely generated \( A \)-submodule). For any fixed \( n \geq 0 \), consider \( \bar{N} = N/I^nN \), which is an \( A \)-submodule of \( \bar{M} = M/I^nN \). As \( \bar{N} \cap I^m\bar{M} = 0 \) for some \( m \geq 0 \), we have \( N \cap I^mM \subseteq I^nN \).

**Exercise 0.7.9.** Consider an exact sequence of the form

\[
0 \longrightarrow K \longrightarrow A^{\oplus n} \longrightarrow M \longrightarrow 0
\]

and the induced commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & K_{\hat{F}^\bullet} & \longrightarrow & A_{\hat{F}^\bullet}^{\oplus n} & \longrightarrow & M_{\hat{F}^\bullet} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & K \otimes_A \hat{A} & \longrightarrow & A_{\hat{F}^\bullet}^{\oplus n} & \longrightarrow & M \otimes_A \hat{A} & \longrightarrow & 0 \\
& & \text{(*)} & & \end{array}
\]
where $K_{n=0}^{\Lambda} A_{n} \cap K$ denotes the Hausdorff completion of $K$ with respect to the induced filtration $I^{\bullet} A_{n}^{\oplus} \cap K$. By 0.7.4.18, the image of $(\ast)$ is closed in $A_{n}^{\oplus}$ with respect to the $I$-adic topology. This implies that the left-hand vertical map is surjective. The right-hand vertical map is an isomorphism by the snake lemma.

Exercise 0.8.1. We need to show that the map $\prod_{i=1}^{r} \text{Spec} (A_{i})^{Zar} \to \text{Spec} A$ is surjective. It suffices to verify that for any prime ideal $p \subset A$ there exists $i$ $(1 \leq i \leq r)$ such that $f_{i}^{m} + ax \not\in p$ for any $n \geq 0, a \in I$, and $x \in A$. Suppose, by contradiction, that for each $1 \leq i \leq r$ we have $f_{i}^{m} + a_{i}x_{i} \in p$ for some $n_{i}, a_{i},$ and $x_{i}$. One can assume $n_{1} = \cdots = n_{r} = n$. Since $(f_{1}^{m}, \ldots, f_{r}^{m}) = A$, there exists $b_{1}, \ldots, b_{n} \in A$ such that $\sum_{i=1}^{r} b_{i}f_{i}^{m} = 1$. Then $1 + \sum_{i=1}^{r} a_{i}b_{i}x_{i} \in p \cap (1 + I)$, which is absurd, because $A$ is $I$-adically Zariskian.

Exercise 0.8.2. Set $L = \text{image}(f)$. By 0.8.2.14, the topology on $L$ defined by the filtration $\{L \cap a^{n}N\}_{n \geq 0}$ coincides with the $a$-adic topology. Since $N$ is $a$-adically separated, $L$ is $a$-adically separated, and hence the kernel $\ker(f)$ is closed in $M$ with respect to the $a$-adic topology. By 0.7.4.6, $L$ is $a$-adically complete and hence is Hausdorff complete with respect to the filtration $\{L \cap a^{n}N\}_{n \geq 0}$. This implies that $L$ is closed in $N$ (cf. Exercise 0.7.1).

Exercise 0.8.3. We may assume $I \subseteq (f)$. Suppose $f^{m}x \in \mathfrak{a}$ for $x \in A$ and $m \geq 1$. By Exercise 0.7.1, for any $n \geq 1$ we have $f^{m}x = a_{n} + y_{n}$, where $a_{n} \in \mathfrak{a}$ and $y_{n} \in I^{n}$. Since $I \subseteq (f)$, one can write $y_{n} = f^{m}z_{n}$ for any $n \geq m$, where $z_{n} \in I^{n-m}$. Then for $n > m$ we have $f^{m}(x-z_{n}) = y_{n} \in \mathfrak{a}$. Since $a$ is $f$-saturated, $x-z_{n} \in \mathfrak{a}$ for any $n \geq m$, whence $x \in \mathfrak{a}$.

Exercise 0.8.4. Due to Exercise 0.6.4 (2) there exists $a \in m_{V}$ such that $\bigcap_{n \geq 0} a^{n}V$ is a height-one prime ideal. Show that $V[f_{a}] = W$ and apply 0.8.5.15.

Exercise 0.8.6. Consider the exact sequence $0 \to N \cap P \to N \oplus P \to N + P \to 0$. Since $N + P \subseteq M$ is $I$-torsion free, it is finitely presented.

Exercise 0.8.8. (1) Since $h$ is adic, $h(J) \subseteq J'$, hence $h$ induces a local homomorphisms $V \to V'$. Moreover, $h$ induces $g = h[\frac{1}{a}]: B \to B'$, which is local, for $h(J) \subseteq J'$. Since $K = B/J \to B'/J' = K'$, being a homomorphism between fields, is injective, $V \to V'$ is also injective. Moreover, since $V \to V'$ is local between valuation rings, one can easily show that $V = K \cap V'$ in $K'$.

(2) Since $A = \{f \in B: (f \mod J) \in V\}$ and $V = K \cap V'$, we can deduce that $g^{-1}(A') = A$. 

Exercise 0.9.1. We first show the claim in the case where both $V$ and $V'$ are of height one. In this case, the real valued valuation on $V$ restricts to that of $V'$, and hence Cauchy sequences in $V$ are Cauchy sequences in $V'$. Hence we have $\hat{V} \subseteq \hat{V}'$ in this case. In general, take the associated height one primes $p \subseteq V$ and $p' \subseteq V'$, respectively. By 0.9.1.10, we have $\hat{V} \subseteq \hat{V}'$ and similarly for $V'$. By this and the above-discussed height one case, we have the desired result.

Exercise 0.9.2. (1) Let $V$ be the filtered inductive limit of subrings $V = \lim_{\lambda \in \Lambda} A_\lambda$, where each $A_\lambda$ is finitely generated over $\mathbb{Z}$ (and hence Noetherian). Localizing at the prime ideal $m_V \cap A_\lambda$, we may assume that $A_\lambda$ are local rings and that the maps $A_\lambda \to A_\mu$ and $A_\lambda \to V$ are local. Let $K_\lambda = \text{Frac}(A_\lambda)$ for $\lambda \in \Lambda$, and consider the composite valuation $K_\lambda \hookrightarrow K \to \Gamma_V \cup \{-\infty\}$, which defines, as the subset of all elements in $K_\lambda$ with non-negative values, a valuation ring $V_\lambda$ for $K_\lambda$ that dominates the local ring $A_\lambda$. Since $A_\lambda$ is Noetherian local, we know by 0.6.5.2 and 0.6.2.8 that each $V_\lambda$ is of finite height. Clearly, we have $V = \lim_{\lambda \in \Lambda} V_\lambda$, since $A_\lambda \subseteq V_\lambda \subseteq V$ for each $\lambda \in \Lambda$.

(2) Suppose $V$ is $a$-adically separated, and write $V = \lim_{\lambda \in \Lambda} V_\lambda$ as in (1). We may assume that $a \in V_\lambda$ for any $\lambda \in \Lambda$. Then each $V_\lambda$ is $a$-adically separated, since $\bigcap_{n \geq 0} a^n V_\lambda \subseteq \bigcap_{n \geq 0} a^n V = \{0\}$. Suppose $V$ is $a$-adically complete. Then by Exercise 0.9.1, we have $\hat{V}_\lambda \subseteq V$, each of which is of finite height (see 0.9.1.1 (5)), and $V = \lim_{\lambda \in \Lambda} \hat{V}_\lambda$.

Exercise 0.9.3. First note that for any non-zero element $b \in V$ there exists $m \geq 0$ such that $a^m$ divides $b$, but $a^{m+1}$ does not; in fact, since $\bigcap_{m \geq 0} a^m V = 0$, we have $b \in a^m V \setminus a^{m+1} V$ for some $m \geq 0$. Let $f = \sum_{v_1, \ldots, v_n} b_{v_1, \ldots, v_n} X_1^{v_1} \cdots X_n^{v_n}$. The the numbers $m$ such that $b_{v_1, \ldots, v_n} \in a^m V \setminus a^{m+1} V$ increase as $|v_1 + \cdots + v_n| \to \infty$. Then the ideal $\text{cont}(f)$ is generated by $b_{v_1, \ldots, v_n}$’s that have the minimal $m$.

Exercise 0.9.4. (1) The case $k = 0$ follows from [18], (7.1.1/2); then the equality $m_0^{k+1} \mathcal{A} = m_0^{k+1}$ for any $k \geq 0$ follows immediately. For $k > 0$ we look at the following commutative diagram with exact rows:

$$
0 \to m^k/m^{k+1} \to \mathcal{A}/m^{k+1} \to \mathcal{A}/m^k \to 0
$$

$$
0 \to m_0^k/m_0^{k+1} \to A_0/m_0^{k+1} \to A_0/m_0^k \to 0.
$$

The rightmost vertical arrow is an isomorphism by induction with respect to $k$. It suffices to show that the map $m_0^k/m_0^{k+1} \to m^k/m^{k+1}$ between finite-dimensional vector spaces over the residue field $\mathcal{A}/m = A_0/m_0$ is an isomorphism. Since $m_0^{k+1} \mathcal{A} = m_0^{k+1}$, it is surjective. Since one sees easily that $m_0^{k+1} = m^{k+1} \cap A_0$, it is injective, too.
(2) For any maximal ideal $m \subseteq \mathcal{O}$, the $m$-adic completion $\widehat{\mathcal{O}}_m$ of $\mathcal{O}_m$ is isomorphic to the $m_0$-adic completion of the regular local ring $(A_0)_{m_0}$, where $m_0 = m \cap A_0$, and hence $\mathcal{O}_m$ is regular. Now apply Serre’s theorem ([81], Theorem 19.3).

**Exercise 0.A.2.** Let us set $A' = A \otimes \kappa V'$ and $p = \sqrt{aV}$, the associated height one prime of $V$. Note that $V' = V_p$.

(1) Let us first assume that $N$ is a prime of $V$ and $A$ is $N$-adically complete, since so are $N^l$. Moreover, one has the exact sequence

$$\begin{array}{c}
\text{Tor}_1(B, V/a) \longrightarrow J \longrightarrow A/aA \longrightarrow B/aB \longrightarrow 0,
\end{array}$$

with $\text{Tor}_1(B, V/a) = 0$. We know $B$ is of finite type over $V$, and take a generating set \{x_1, \ldots, x_n\}. Take $y_i \in A$ ($i = 1, \ldots, n$) that is mapped to $x_i$ in $\hat{B}$. Let $z_1, \ldots, z_m$ be the topological generator of $A$ over $V$, and consider the subring $C = V[z_1, \ldots, z_m, y_1, \ldots, y_n] \subseteq A$. In view of $(*)$, we deduce that $J \subseteq C$ and $C/J \cong B$.

(3) By height-one localization and patching argument as above, one can reduce to the situation where $V$ is of height one. Since $B = A/J$ is $V$-flat and finite outside $aV$, it is quasi-finite over $V$. According to [54], IV, (18.12.3), one has a decomposition $B = B' \times B''$, where $B'$ is finite over $V$, and $B'' \otimes_V k = 0$ ($k = V/m_V$ is the residue field). We have $\hat{B}' = \hat{B}$ and $\hat{B}'' = \ker(B \to \hat{B})$. Now, set $A'' = \ker(A \to \hat{A})$; $A''$ is an ideal of $A$ consisting of $a$-divisible elements, that is, for any $f \in A''$ and any $n \geq 1$, there exists $g \in A''$ such that $f = a^n g$. Since the $a$-torsion part $J$ of $A$ is bounded, we have $A'' \cap J = \{0\}$. By this and $J = \ker(\hat{A} \to \hat{B})$, one has $A'' \cong B''$, which gives a section to the surjective morphism $A \to B''$ by the composition $A \to B = B' \times B''$. Hence we have a decomposition $A \cong A' \times A''$ with $A' \hookrightarrow \hat{A}$. Since $A''$ is $a$-divisible, we have $A'' \otimes_V (V/aV) = 0$. Moreover, one has the exact sequence

$$\begin{array}{c}
0 \longrightarrow J \longrightarrow A' \longrightarrow B' \longrightarrow 0.
\end{array}$$

In particular, $A'$ is $a$-adically complete, since so are $J$ and $B'$.

**Exercise 0.A.3.** Apply 0.A.2.1 to get $X^{v_1(f)} = uf + r$, where $r$ has no exponent in $(v_1(f), 0, \ldots, 0) + \mathfrak{m}^n$, that is, $r$ is a polynomial in $X_1$ of degree $< v_1(f)$. Set $g = X_1^{v_1(f)} - r$. We have $uf = g$. Dividing out by $m_r$, we have the equality of polynomials $\tilde{u} \tilde{f} = \tilde{g}$ with coefficients in $k = V/m_V$; since the leading degrees in $X_1$ of $\tilde{f}$ and $\tilde{g}$ coincides and their leading coefficients are units, we deduce that
\( \bar{u} \in k^\times \) and hence that \( u \) is a unit. For the uniqueness, observe that for a given \( f \) the equality \( X^{v_1(f)} = uf + r \) with \( r \) being a polynomial in \( X_1 \) of degree \( < v_1(f) \) determines \( u \) and \( r \).

**Chapter I**

**Exercise I.1.3.** We may assume that \( X = \text{Spf} \ A \) for an admissible ring \( A \) and that \( I = I^\Delta \) and \( I' = I'^\Delta \) for ideals of definition \( I, I' \subseteq A \). It is then easy to see that \( I \cap I' \) is an ideal of definition. If \( F^* = \{ F^{\lambda} \}_{\lambda \in \Lambda} \) is a descending filtration of ideals that defines the topology on \( A \), then

\[
(I \cap I')^\Delta = \lim_{\leftarrow} I \cap I'/F^\lambda,
\]

\[
= \lim_{\leftarrow} I/F^\lambda \cap I'/F^\lambda,
\]

\[
= (\lim_{\leftarrow} I/F^\lambda) \cap (\lim_{\leftarrow} I'/F^\lambda),
\]

\[
= I^\Delta \cap I'^\Delta,
\]

which shows that \( I \cap I' \) is an ideal of definition.

**Exercise I.1.4.** We may assume that \( X = \text{Spf} \ A \) and \( I = I^\Delta \), where \( I \) is an ideal of definition of the adic ring \( A \). Since \( A \) is \( I \)-adically complete, \( I \) is finitely generated if \( I/I^2 \) is so (0.7.2.4). Then apply I.1.1.21.

**Exercise I.1.5.** See [53], (3.2.4).

**Exercise I.1.6.** Take \( g \in A \) such that \( f = j(g) \) belongs to \( I.A. \). Then for an open prime ideal \( p \) of \( A \) the element \( f \) does not belong to \( p \) if and only if \( j(g) \) does not belong to \( p \), since \( p \) contains the ideal of definition \( I.A. \).

**Exercise I.1.8.** By I.1.4.3, the space \( X \) is an adic formal schemes of finite ideal type. To show that \( X \to Y \) is adic, we may work in the affine situation, and the claim follows from 0.7.2.12.

**Exercise I.3.1.** We may assume that \( X \) is affine, \( X = \text{Spf} \ A \), where \( A \) is an adic ring with the finitely generated ideal of definition \( I \) such that \( I = I^\Delta \). Then by I.3.2.7 we have \( B = B^\Delta \) for an \( I \)-adically complete \( A \)-algebra \( B \). It follows from I.3.2.8 (2) that \( B \) is an adically quasi-coherent sheaf of finite type if and only if \( B \) is finitely generated as an \( A \)-module, which is further equivalent to \( B/I.B \) being finitely generated (0.7.2.4). Since \( B/I.B = B/I.B \) (I.3.2.2), the claim follows from [53], (1.4.3).
Exercise I.3.2. We may assume that \( X \) is affine, \( X = \text{Spf} \, A \), where \( A \) is a t.u. rigid-Noetherian ring, and that \( I = I^\Delta \), where \( I \subseteq A \) is a finitely generated ideal of definition. By I.3.5.6, the morphism \( \varphi \) comes from a morphism \( f: M \to N \) between finitely generated \( A \)-modules, and we have \( \ker(f) = \ker(f)^\Delta \), which is an a.q.c. sheaf of finite type; in particular, its \( I \)-torsion part is bounded, since \( \ker(f)_{I\text{-tor}} \) is bounded \( I \)-torsion. Let \( K = \ker(f) \). By I.3.5.3, we deduce that \( K^\Delta = \ker(\varphi) \). Since \( M_{I\text{-tor}} \) is bounded \( I \)-torsion, \( K_{I\text{-tor}} \) is also bounded \( I \)-torsion.

Exercise I.3.3. Set \( X_0 = (X, \mathcal{O}_X/I) \), and define \( U_0 \) similarly. Then \( \mathcal{F}_0 = \mathcal{F}/I\mathcal{F} \) is a quasi-coherent sheaf on \( X_0 \), and \( \mathcal{G}_0 = \mathcal{G}/I\mathcal{F}|_U \) is a quasi-coherent subsheaf of \( \mathcal{F}_0 \) of finite type. Since \( X_0 \) is a coherent scheme (I.1.6.9) and \( U_0 \) is a quasi-compact open subset of \( X_0 \), we may apply [54], I, (9.4.7), and IV, (1.7.7), to deduce that there exists a quasi-coherent subsheaf \( \mathcal{G}_0' \) of \( \mathcal{F}_0 \) of finite type that extends \( \mathcal{G}_0 \). Let \( \mathcal{G}' \) be the inverse image sheaf of \( \mathcal{G}_0' \) by the canonical map \( \mathcal{F} \to \mathcal{F}_0 \). By I.3.7.2 it is an a.q.c. subsheaf of \( \mathcal{F} \) of finite type.

Exercise I.3.4. For any open subset \( U \subseteq X \), consider the set \( S_U \) of all finite type a.q.c. subsheaves \( \mathcal{H} \) of \( \mathcal{G}|_U \) such that \( I^s|_U (\mathcal{G}/\mathcal{H}) = 0 \) for some \( s > 0 \). If \( U \) is affine, one has \( S_U \neq \emptyset \). To show that \( S_X \neq \emptyset \) by induction with respect to the number of quasi-compact open subsets in a finite covering of \( X \), we may assume that \( X \) is covered by two quasi-compact open subsets \( U_1 \) and \( U_2 \) such that \( S_{U_1} \) and \( S_{U_2} \) are non-empty. Take \( \mathcal{H}_i \in S_X \) for \( i = 1, 2 \). Let \( \mathcal{H}_{12} \) be the a.q.c. subsheaf of finite type of \( \mathcal{G}|_{U_1 \cap U_2} \) on \( U_1 \cap U_2 \) generated by \( \mathcal{H}_1|_{U_1 \cap U_2} \) and \( \mathcal{H}_2|_{U_1 \cap U_2} \). Since \( \mathcal{H}_{12}/\mathcal{H}_1|_{U_1 \cap U_2} \) is bounded \( I \)-torsion, by Exercise I.3.3 one may extend the quasi-coherent sheaf \( \mathcal{H}_{12} \) (\( i = 1, 2 \)) onto \( U_i \) to get an a.q.c. subsheaf of finite type \( \mathcal{H}_{1} \) of \( \mathcal{G}|_{U_i} \) that contains \( \mathcal{H}_i \). The sheaves \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) patch together to an element of \( S_X \). For \( \mathcal{H} \in S_X \) the sheaf \( \mathcal{G}/\mathcal{H} \) is annihilated by \( I^n \) for some \( n > 0 \). Hence it defines a quasi-coherent sheaf on the scheme \( (X, \mathcal{O}_X/I^n) \). As it is an inductive limit of quasi-coherent subsheaves of finite type, we get the desired result by pull-back (cf. I.3.7.2).

Exercise I.3.5. Let \( M \subseteq L \), where \( L \) is a finitely generated \( A \)-module. Consider the commutative diagram with exact rows (due to I.3.5.3)

\[
\begin{array}{c}
0 & \to & (M/N)^\Delta & \to & (L/N)^\Delta & \to & (L/M)^\Delta & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M^\Delta & \to & L^\Delta & \to & (L/M)^\Delta & \to & 0.
\end{array}
\]

The desired exact sequence follows by the snake lemma.
**Exercise I.3.6.** We may assume that $X$ is affine, $X = \text{Spf } A$, where $A$ is a t.u. rigid-Noetherian ring with a finitely generated ideal of definition $I \subseteq A$.

(1) Consider for each $k \geq 0$ the induced exact sequence

$$0 \longrightarrow \mathcal{F}/\mathcal{F} \cap \mathcal{I}^{k+1} \longrightarrow \mathcal{G}/\mathcal{I}^{k+1} \longrightarrow \mathcal{H}/\mathcal{I}^{k+1} \longrightarrow 0,$$

where $\mathcal{I} = I^\Delta$. Since each $\mathcal{F}/\mathcal{F} \cap \mathcal{I}^{k+1} \mathcal{G}$ is quasi-coherent, from I.1.1.23 we deduce that the sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H}) \longrightarrow 0$$

is exact. By I.3.5.3, we have $\mathcal{F} = \Gamma(X, \mathcal{F})^\Delta$, which is a.q.c..

(2) Let $N = \Gamma(X, \mathcal{F}) \hookrightarrow M = \Gamma(X, \mathcal{G})$; $M$ is finitely generated. We have $N^\Delta = \mathcal{F}$ and $M^\Delta = \mathcal{G}$. Hence the claim follows from I.3.5.3.

**Exercise I.3.7.** (1) See [53], (10.10.2.7).

(2) Use [53], (10.10.2.9), and I.3.5.6.

(3) It suffices to show that $\mathcal{J}$ is of finite type. We may assume that $X$ is affine, $X = \text{Spf } A$, where $A$ is a Noetherian adic ring. Set $J = \Gamma(X, \mathcal{J})$, which is an ideal of $A$, and hence is finitely generated. We have the canonical morphism $J^\Delta \rightarrow \mathcal{J}$. Now show that this is an isomorphism.

**Exercise I.3.8.** We may assume that $X$ is affine, $X = \text{Spf } A$, where $A$ is a t.u. adhesive ring with a finitely generated ideal of definition $I \subseteq A$ and $A$ is $I$-torsion free. Consider the exact sequence

$$0 \longrightarrow \mathcal{J} \cap \mathcal{J}' \longrightarrow \mathcal{J} \oplus \mathcal{J}' \longrightarrow \mathcal{J} + \mathcal{J}' \longrightarrow 0,$$

and compare it with the exact sequence

$$0 \longrightarrow J \cap J' \longrightarrow J \oplus J' \longrightarrow J + J' \longrightarrow 0,$$

where $J = \Gamma(X, \mathcal{J})$ and $J' = \Gamma(X, \mathcal{J}')$, which are $I$-admissible ideals of $A$. By Exercise 0.8.6, the intersection $J \cap J'$ is finitely generated. Hence, applying the exact functor $(\ast)$ in I.3.5.6, we get the first exact sequence from the second one. This implies that $\mathcal{J} \cap \mathcal{J}' = (J \cap J')^\Delta$, which is an admissible ideal.

**Exercise I.4.1.** We may assume that $X$ and $Y$ are affine, $X = \text{Spf } A$ and $Y = \text{Spf } B$, and that the morphism $f: X \rightarrow Y$ comes from an adic map $B \rightarrow A$ between adic rings of finite ideal type. Let $I \subseteq B$ be a finitely generated ideal of definition of $B$ such that $I^\Delta = I$, and $M$ be the $A$-module such that $\mathcal{F} = M^\Delta$. Then we have $f_*M^\Delta = M^\Delta_B$, where $M_B$ is the module $M$ regarded as a $B$-module (I.4.1.4). By I.3.2.4, $f_*I^{k+1}M^\Delta = (I^{k+1}M)^\Delta_B = I^{k+1}M^\Delta_B = I^{k+1}f_*M^\Delta$, which shows the assertion for the module sheaf $\mathcal{F}$. The other case is similar.
Exercise I.4.2. All the assertion follow from I.3.5.3.

Exercise I.4.3. We consider the conormal cones
\[ \text{gr}_I^*(A) = \bigoplus_{n \geq 0} I^n/I^{n+1} \quad \text{and} \quad \text{gr}_B^*(B) = \bigoplus_{n \geq 0} I^n B/I^{n+1} B \]
(cf. 0, §7.5). Since \( \text{gr}_B^*(B) = \text{gr}_I^*(A) \otimes A B = \text{gr}_I^*(A) \otimes_{A_0} B_0 \), where \( A_0 = A/I \) and \( B_0 = B/IB \), and since \( \text{Spf} B \to \text{Spf} A \) is adically faithfully flat, we deduce that the map \( \text{gr}_I^*(A) \to \text{gr}_B^*(B) \) is faithfully flat. Now consider
\[ \text{gr}_F^*(A) = \bigoplus_{n \geq 0} F^n / F^{n+1} \quad \text{and} \quad \text{gr}_F^*(B) = \bigoplus_{n \geq 0} F^n B / F^{n+1} B, \]
which are a graded \( \text{gr}_I^*(A) \)-module and a graded \( \text{gr}_B^*(B) \)-module, respectively. Since \( \text{gr}_{F,B}^*(B) = \text{gr}_F^*(A) \otimes_{\text{gr}_{I,B}^*(B)} \text{gr}_B^*(B) \), \( \text{gr}_F^*(A) \) is finitely generated as a \( \text{gr}_I^*(A) \)-module if and only if \( \text{gr}_{F,B}^*(B) \) is finitely generated as a \( \text{gr}_{I,B}^*(B) \)-module. Hence the assertion follows from 0.7.5.2.

Exercise I.5.4. Set \( X_k = (X, \mathcal{O}_X / I^{k+1}\mathcal{O}_X) \) and \( Y_k = (Y, \mathcal{O}_Y / I^{k+1}\mathcal{O}_Y) \) for \( k \geq 0 \) (where \( I \subseteq A \) is a finitely generated ideal of definition), and consider the induced map \( f_k : X_k \to Y_k \). Then \( f \) is proper if and only if \( f_0 \) is proper. Then the claim follows from [54], II, (5.6.3) (where we first reduce to the case where \( Y \) is affine, and use [54], IV, (8.10.5.1), instead of [54], II, (5.6.1)).

Exercise I.6.2. Use the same technique as in the proof of I.6.3.5; cf. [72], II.1.7.

Exercise I.6.3. We may assume that \( Y \) has an ideal of definition of finite type \( I \). By I.3.7.10, we have \( \hat{f}^* I^n = I^n \mathcal{O}_X \) for \( n \geq 1 \). Since \( \hat{f}^* \mathcal{J} \) is an ideal of definition of \( X \), there exists \( n \geq 1 \) such that \( \hat{f}^* I^n \subseteq \hat{f}^* \mathcal{J} \subseteq \mathcal{O}_X \). By this, one has an a.q.c. ideal sheaf \( \mathcal{H} \) containing \( \mathcal{J}^n \) such that \( \hat{f}^* \mathcal{J} = \hat{f}^* \mathcal{H} \), whence \( \mathcal{H} = \mathcal{J} \). Notice that \( \mathcal{J} \) is an open ideal of \( \mathcal{O}_Y \). In particular, \( \mathcal{J}^m \) for any \( m \geq 1 \) is again an open a.q.c. ideal. For a sufficiently large \( m \geq 1 \) we have \( (\hat{f}^* \mathcal{J})^m \subseteq \hat{f}^* I \). Combined with the canonical morphism \( \hat{f}^* \mathcal{J}^m \to (\hat{f}^* \mathcal{J})^m \), we conclude again by I.6.1.11 that \( \mathcal{J}^m \subseteq I \).

Exercise I.6.4. See [72], II.3.13.

Exercise I.6.6. Reduce to the situation where \( X \) has an ideal of definition, and apply [72], II.6.7, and I.6.3.21; cf. [89], Première partie, (5.7.7).

Exercise I.8.1. For any coherent sheaf \( \mathcal{F} \) on \( X \) there exists, by I.8.1.2, an integer \( n \) such that \( (f^* f_* \mathcal{F}(n))(-n) \to \mathcal{F} \) is surjective. Let \( M \) be the \( B \)-module such that \( \hat{M} = f_* \mathcal{F}(n) \). Find a finitely generated \( B \)-submodule \( N \) of \( M \) such that the map \( (f^* \hat{N})(-n) \to \mathcal{F} \) is surjective. Then replace \( N \) by a free \( B \)-module of finite rank. Prove in this way that any coherent sheaf on \( X \) admits a resolution by coherent sheaves of the form \( \mathcal{O}(n) \oplus m \), which are projective objects in the category \( \text{QCoh}_X \). Then use 0.C.5.3 (2).
**Exercise I.8.2.** Take a quasi-compact representable étale covering $Y \to X$, and set $R = Y \times_X Y$. The projections $p_1, p_2: R \to Y$ give an étale equivalence relation (defining $X$) such that $(p_1, p_2): R \to Y \times_A Y$ is an immersion (resp. closed immersion). Since $Y$ and $R$ are finitely presented over $A$, there exist a filtered family $\{A_\lambda\}_{\lambda \in \Lambda}$ of subrings of $A$ of finite type over $\mathbb{Z}$ and a projective system $\{p_{1, \lambda}, p_{2, \lambda}: R_\lambda \to Y_\lambda\}_{\lambda \in \Lambda}$ of diagrams that converges to $p_1, p_2: R \to Y$, such that for each $\lambda \in \Lambda$, $R_\lambda$ and $Y_\lambda$ are finite type over $A_\lambda$. Then by Exercise 0.1.3 $p_{1, \lambda}, p_{2, \lambda}: R_\lambda \to Y_\lambda$ defines an étale equivalence relation for sufficiently large $\lambda \in \Lambda$.

**Exercise I.10.1.** The ‘if’ part is clear. To show the converse, by a reduction argument similar to that in [54], III, (4.6.8), we may assume that $Y = D = Z$ and that it is affine, $Y = Z = \text{Spec } A$. Let $I \subseteq A$ be the finitely generated ideal that defines $W \subseteq Z$. Since $f^{-1}(y)$ is a singleton set for any $y \in W$, there exists an open neighborhood $U$ of $W$ such that $f^{-1}(U) \to U$ is quasi-finite (this follows from [54], III, (4.4.11), applied with the standard limit argument (cf. [54], IV, §8.10)). Hence, we may assume $f$ is quasi-finite; since $f$ is proper of finite presentation, one deduces by Zariski’s Main Theorem ([54], IV, (8.12.6)) that $f$ is finite. Hence $X = \text{Spec } B$, where $B$ is a finite $A$-algebra. Now since $A$ is $I$-adically universally adhesive (and so is $B$), the map $A^{\text{Zar}} \to \hat{A}$ is faithfully flat (and so is $B^{\text{Zar}} \to \hat{B}$), where $A^{\text{Zar}}$ is the associated Zariskian ring (0.7.3.8 (2) and 0.8.2.18 (2)). Moreover, by 0.8.2.18 (1) we know that $\hat{B} = B^{\text{Zar}} \otimes_{A^{\text{Zar}}} \hat{A}$. Hence the map $\hat{A} \to \hat{B}$ is an isomorphism (resp. surjective) if and only if $A^{\text{Zar}} \to B^{\text{Zar}}$ is an isomorphism (resp. surjective). Since $A^{\text{Zar}}$ is the inductive limit of the rings of the form $A_{(1+a)}$, where $a \in I$, we have the desired result.

**Exercise I.C.1.** It suffices to show the assertion in the following two cases:

- (a) $f$ is injective;
- (b) $f$ is surjective.

Case (a). Since $M/N$ is bounded $I$-torsion, we have $I^n M \subseteq N$ for a sufficiently large $n \gg 0$. This implies that the subspace topology on $N$ induced from the $I$-adic topology on $M$ is the $I$-adic topology, since $I^{n+k} N \subseteq I^{n+k} M = I^{n+k} M \cap N \subseteq I^k N$ for any $k \geq 0$. Hence by 0.3.2.8 and 0.7.2.3 the sequence

$$0 \longrightarrow \hat{N} \longrightarrow \hat{M} \longrightarrow M/N \longrightarrow 0$$

is exact, where $\hat{\cdot}$ denotes the $I$-adic completion. The assertion follows from this.

Case (b). In this case, $K = \ker(f)$ is bounded $I$-torsion. Since the subspace topology on $K$ induced from the $I$-adic topology on $N$ is the $I$-adic topology arguing in much the same way as above we have the exact sequence

$$0 \longrightarrow K \longrightarrow \hat{N} \longrightarrow \hat{M} \longrightarrow 0.$$
Exercise I.C.2. Take any FP-approximations $\alpha: \mathcal{F}' \to \mathcal{F}$ and $\beta: \mathcal{G}'' \to \mathcal{G}$, and consider the fiber product $\mathcal{K}$ of the maps $\varphi \circ \alpha$ and $\beta'$, which is an a.q.c. subsheaf of the direct sum $\mathcal{F}' \oplus \mathcal{G}''$. Then mimic the argument as in the proof of II.C.2.3 (use Exercise I.3.4 in the formal scheme case).

Chapter II

Exercise II.1.2. Let $\mathcal{I}$ be an admissible ideal that gives the admissible blow-up $\pi: X' \to X$.

Consider the admissible ideals $\mathcal{I}\mathcal{O}_Y$ on $Y$ and $\mathcal{I}\mathcal{O}_{Y'}$ on $Y'$, and let $Z \to Y$ and $Z' \to Y'$ the respective admissible blow-ups. We want to show that $Z$ and $Z'$ are isomorphic. By II.1.1.4 (3), there exists $Z' \to Z$ that makes the resulting square commutative. Similarly, by II.1.1.4 (3), we have $Z \to X'$, which induces $Z \to Y'$. Then one sees, again by II.1.1.4 (3), that there exists an arrow $Z \to Z'$. It is then easy to verify that these arrows are inverse to each other.

Exercise II.1.3. Consider $U = \text{Spf} A_{(f)} \subseteq X$ for any $f \in A$. The module $\Gamma(U, \mathcal{F}_{\mathcal{I}\text{-tor}})$ is the $\mathcal{I}$-torsion part of $\Gamma(U, \mathcal{F}) = M \otimes_A A_{(f)} = M \otimes_A A_{(f)}$; see (0.8.2.18 (1)). Since $A_{(f)}$ is flat over $A$ (0.8.2.18 (2)), we have $\Gamma(U, \mathcal{F}_{\mathcal{I}\text{-tor}}) = M_{\mathcal{I}\text{-tor}} \otimes_A A_{(f)}$. On the other hand, since $J$ is $I$-admissible, we have $M_{\mathcal{I}\text{-tor}} \subseteq M_{I\text{-tor}}$, and $M_{I\text{-tor}} \otimes_A A_{(f)}$ is bounded $I$-torsion. Then, by 0.8.1.4 and I.3.2.1, we have $\Gamma(U, (M_{I\text{-tor}})^\Delta) = M_{I\text{-tor}} \otimes_A A_{(f)}$. Therefore, $\mathcal{F}_{\mathcal{I}\text{-tor}} = (M_{I\text{-tor}})^\Delta$, as desired.

Exercise II.1.4. We may work in the affine situation $X = \text{Spf} A$ where $A$ is a t.u. rigid-Noetherian ring. Then the claim follows from Exercise II.1.3 and I.3.5.3.

Exercise II.1.5. By I.3.6.2, we know that $\pi^* \mathcal{F}$ is an a.q.c. sheaf of finite type on $X'$. By I.3.7.6, the sheaf $\mathcal{I}\mathcal{O}_{X'}$ is an admissible ideal of $\mathcal{O}_{X'}$. Hence the assertion follows from Exercise II.1.4.

Exercise II.1.6. Since $B$ is $J$-torsion free, we have $K = (J \otimes_A B)_{J\text{-tor}}$. In particular, we can deduce, similarly as in the hint for Exercise II.1.3, that $K$ is a bounded $I$-torsion module and hence is $I$-adically complete. From this it follows, by an argument similar to that in the hint for Exercise II.1.3, that $\mathcal{K} = K^\Delta$, whence (1). Then by Exercise II.1.3 one has (2).

Exercise II.2.2. For a quasi-separated $X$, cover $X$ by coherent open subsets, and then construct $X^{\text{rig}}$ as a stretch of coherent rigid spaces. In general, cover $X$ by coherent open subsets (e.g., affine), and mimic the construction as in II.2.2.18.
Exercise II.3.1. Mimic the proof of [53], (6.9.17), with the following modifications:

- consider the modules $R$ and $S$ as in [53], p. 323; it follows that $R/S$ is $\mathcal{J}$-torsion; since $R/S$ is a submodule in the finitely generated $A$-module $A^{\otimes mn}/S$, the $\mathcal{J}$-torsion is bounded;

- use $(\text{AP})$ instead of Artin–Rees lemma in [53], p. 324.

Exercise II.3.2. We may assume that $\mathcal{X}$ is a coherent rigid space. Then there exists a formal model $X$ of $\mathcal{X}$ on which there exist ideals of definition $\mathcal{I}_X^m$ and $\mathcal{I}_X^m$ such that $\mathcal{I}_X^m \subseteq \mathcal{I}_X^m \subseteq \mathcal{I}_X$. Then the assertion follows from the fact that there exist $n, m > 0$ such that $\mathcal{I}_X^m \subseteq \mathcal{I}_X^m \subseteq \mathcal{I}_X$.

Exercise II.3.4. If $\mathcal{X}$ has a Noetherian formal model $X$, then any admissible blow-up $X'$ of $X$ is again Noetherian, since $X'$ is of finite type over $X$ (cf. [54], I, (10.13.2)), whence (a) $\implies$ (b). Next, suppose (b) holds, and let $\mathcal{U} \subseteq \mathcal{X}$ be a quasi-compact open subspace. Then there exists a Noetherian formal model $X$ of $\mathcal{X}$ having a quasi-compact open subset $U$ that corresponds to $\mathcal{U}$ (II.3.1.3), whence (b) $\implies$ (c). Implication (c) $\implies$ (d) is clear. To show (d) $\implies$ (a), take a finite open covering $\{\mathcal{U}_\alpha\}_{\alpha \in \Lambda}$ as in (d). By II.3.1.3, there exists a formal model $X$ and an open covering $\{\mathcal{U}_\alpha\}_{\alpha \in \Lambda}$ of $X$ such that each $\mathcal{U}_\alpha$ gives a formal model of $\mathcal{U}_\alpha$ (cf. II, §3.4). Applying (a) $\implies$ (b) to each $\mathcal{U}_\alpha$ and replacing $X$ by an admissible blow-up if necessary, we may assume that each $\mathcal{U}_\alpha$ is Noetherian. Then $X$ is Noetherian, as desired.

Exercise II.4.2. Cf. 0.2.3.24.

Exercise II.5.1. Replacing $X$ by admissible blow-ups and the sheaves by their strict transforms, one can assume that $I^n \mathcal{F}_X|_U \subseteq I^m \mathcal{G}_X|_U \subseteq \mathcal{F}_X|_U$ holds for positive integers $n, m$. Then since $\mathcal{F}_X + \mathcal{G}_X$ is obviously a.q.c. of finite type and also $I_X$-torsion free, it gives a lattice model. By the exact sequence

$$0 \longrightarrow \mathcal{F}_X \cap \mathcal{G}_X \longrightarrow \mathcal{F}_X \oplus \mathcal{G}_X \longrightarrow \mathcal{F}_X + \mathcal{G}_X \longrightarrow 0,$$

$\mathcal{F}_X \cap \mathcal{G}_X$ is of finite type, since $\mathcal{F}_X + \mathcal{G}_X$ if finitely presented. Hence the assertion for $\mathcal{F}_X \cap \mathcal{G}_X$ follows.

Exercise II.5.2. We can start from the following situation (similarly to that in Step 3 of the proof of II.5.2.4): $X$ is a distinguished formal model of $\mathcal{X}$ having an invertible ideal of definition $I_X$, $\langle \mathcal{X} \rangle = \bigcup_{\alpha=1}^{n} \mathcal{U}_\alpha$ is a finite open covering, and for each $\alpha$ we have

- an admissible blow-up $X_\alpha \rightarrow X$ and a quasi-compact open subset $U_\alpha$ of $X_\alpha$ such that $\mathcal{U}_\alpha = \text{sp}^{-1}(U_\alpha)$;

- a positive integer $s$ (independent on $\alpha$);
• a formal model $\mathcal{F}_{X_\alpha}$ of $\mathcal{F}$ on $X_\alpha$ and a weak isomorphism

$$\tilde{\varphi}_\alpha: I_X^a \mathcal{O}_X^{\oplus r} |_{U_\alpha} \rightarrow \mathcal{F}_{X_\alpha} |_{U_\alpha}$$

such that $\tilde{\varphi}_\alpha^{\text{reg}} = \varphi |_{U_\alpha}$.

Note that, since $I_X$ is invertible, each $I_X^a \mathcal{O}_X^{\oplus r} |_{U_\alpha}$ is locally free. By an argument similar to that in Step 3 of the proof of II.5.2.4, one finds a further admissible blow-up on which the strict transforms of $I_X^a \mathcal{O}_X^{\oplus r} |_{U_\alpha}$ and $\mathcal{F}_{X_\alpha} |_{U_\alpha}$ glue together and gives rise to a weak isomorphism between the resulting lattice models. Taking the admissible blow-up to be a distinguished formal model of $X$, one sees that the resulting gluing of the strict transforms of $I_X^a \mathcal{O}_X^{\oplus r} |_{U_\alpha}$ is locally free.

Exercise II.6.1. See [18], (7.2.3/7).

Exercise II.6.2. Let $A' = \Gamma(X', \mathcal{O}_{X'})$, and consider the morphism

$$q: X'' = \text{Spf } A' \rightarrow \text{Spf } A$$

induced by the strict weak isomorphism $A \rightarrow A'$ (II.6.2.2). Then we have $\pi_* \mathcal{O}_{X'} = q_* \mathcal{O}_{X''}$. Since $q$ is affine, $q_* \mathcal{O}_{X''}$ is a.q.c. by I.4.1.5.

Exercise II.6.3. Set $\mathcal{X} = (\text{Spf } A)^{\text{rig}}$. For any closed point $x \in s(\mathcal{X}) = \text{Spec } A \setminus V(I)$, take a map $\text{Spec } V \rightarrow \text{Spec } A$ from the spectrum of a valuation ring such that the generic point is mapped to $x$ and the closed points are mapped in $V(I)$. Then we have a rigid point $\text{Spf } \hat{V} \rightarrow (\mathcal{X})$.

Exercise II.7.2. Easy by II.7.5.16.

Exercise II.9.1. Take an object $(X \hookrightarrow \tilde{X})$ of $\textbf{Emb}_{X|S}$, and let $Z$ and $\bar{Z}$ be as in II.9.1.2. Then, by II.9.1.9, we have $X^{\text{an}} = (\tilde{X}_U)^{\text{an}} \setminus Z^{\text{an}}$. Let $\mathcal{I}$ be the ideal defining $\bar{Z}$ in $\tilde{X}$, which we may assume to be invertible; we may moreover assume $I \mathcal{O}_{\tilde{X}}$ is invertible. Let $\tilde{X}_n$ for each $n \geq 1$ be the blow-up of $\tilde{X}$ along the ideal $\mathcal{I}^n + I \mathcal{O}_{\tilde{X}}$, and $\bar{X}_n$ the maximal open set of $\tilde{X}_n$ where $\mathcal{I}^n \mathcal{O}_{\bar{X}_n}$ generates the ideal $\mathcal{I}^n \mathcal{O}_{\bar{X}_n} + I \mathcal{O}_{\bar{X}_n}$. Then $\bar{X}_n \rightarrow \bar{X}$ is affine for any $n \geq 1$ and we have $\lim_{\rightarrow} (\tilde{X}_n)^{\text{rig}} = X^{\text{an}}$.

Exercise II.C.1. Only (a) $\implies$ (c) calls for a hint. Suppose $\text{Gr}_F A = \{0\}$. Then one can show $F = F^\perp$. Consider the seminorm $\nu$ associated to $F$. Since $1 \in F^{-1} = F^\perp$, $\nu(1) < 1$, which implies (c).
Exercise II.C.2. Let $F_v$ be the filtration induced from $v$. Then $(K, F_v)$ is a filtered valuation field of maximal type, and we have $V \subseteq (F_v)_1$. Consider $\overline{V} = V/(F_v)_{<1}$, which is a subring of $\text{Gr}_{F_v, 1} K$. We regard $\overline{V}$ as a graded local subring of $\text{Gr}_{F_v} K$ by 0-extension (cf. II, §C.2. (a)). By II.C.2.4, there exists a graded valuation subring of $\text{Gr}_{F_v} K$, whose unit-element part coincides with $\overline{V}$. Hence by II.C.3.5, we have a filtration $F$ on $K$ such that $(K, F)$ is a filtered valuation field with $F_1 = V$.

Exercise II.C.3. Reduce to the case where $\mathcal{O}$ is a Banach field by the proper continuous mapping $\mathcal{M}(\mathcal{O} \langle \langle r^{-1} T \rangle \rangle) \to \mathcal{M}(\mathcal{O})$.

Exercise II.C.5. (3) Consider $L = K^{\text{st}}$ as in C.3.20. We first show that

$$\text{Spec}_{\mathbb{R}^+} \text{Gr} \mathcal{O}_L \longrightarrow \text{Spec}_{\mathbb{R}^+} \text{Gr} \mathcal{O}$$

is surjective. This follows from the surjectivity of $\text{sp}_\mathcal{O}$ and of

$$\text{Spec}^{\text{val}} \mathcal{O}_L \longrightarrow \text{Spec}^{\text{val}} \mathcal{O}.$$  

Then the claim reduces to the strict case 11.2.12.

Exercise II.E.2. Take a proper $Y$-scheme $\overline{U}$ that contains $U$ as an open subset. Then, by II.E.1.12 (2), replacing $\overline{U}$ by a $U$-admissible blow-up if necessary, we can find a $U$-admissible blow-up $X'$ of $X$ that admits an open immersion $X' \hookrightarrow \overline{U}$. Since $\overline{U}$ is a scheme, $X'$ is a scheme.
Bibliography


— B —


— C —


— G —


— H —


— I —


— J —


— K —


— N —


— R —


— S —

[90] W. Schöbe, *Beiträge zur Funktionentheorie in nichtarchimedisch bewerteten Körpern*. Helios Verlag, Münster, 1930. JFM 56.0141.03

http://journals.cambridge.org/action/displayAbstract?fromPage=online&aid=8920245


— T —


— U —

— V —


— W —


— Z —


# List of Notations

## Categories

— Sets and Spaces

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<td>Locally ringed spaces</td>
<td>Local morphisms</td>
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| Tri | Triples | Morphism of triples | II.A.1.1 |
| VTri | Valued triples | Morphism of valued triples | II.A.1.2 |
| AnTri | Analytic triples | Morphism of triples | II.A.1.4 |

| Sch | Schemes | Morphisms of schemes | 0, §5.1. (a) |
| SchS | S-schemes | S-morphisms | 0, §5.1. (a) |
| As | Algebraic spaces | Morphisms of algebraic spaces | 0, §5.2. (a) |
| CAs | Coherent | Morphisms of algebraic spaces | II, §E.1. (a) |
| AsS | S-algebraic spaces | S-morphisms | 0, §5.2. (a) |

| Zs | Zariskian schemes | Morphisms of Zariskian schemes | I, §B.1. (b) |
| AZs | Affine | Morphisms of Zariskian schemes | I, §B.1. (b) |
| CZs | Coherent | Morphisms of Zariskian schemes | I, §B.1. (b) |

| Fs | Formal schemes | Morphisms of formal schemes | I, §1.5. (a) |
| AcFs | Adic — of finite ideal type | Morphisms of formal schemes | I, §1.5. (a) |
| AcFs* | Adic — of finite ideal type | Adic morphisms | I, §1.5. (a) |


**List of Notations**

- **AcFs** — adic over $S$ ........................................ Adic morphisms .............................. I, §1.5.(a)
- **RigNoeFs** — Morphisms of formal schemes ............. I, §2.1.(c)
- **RigNoeFs** — Adic morphisms ............................ I, §2.1.(c)
- **RigNoeFs** — adic over $S$ ................................ Adic morphisms .............................. I, §2.1.(c)
- **AdhFs** — Universally adhesive — ....................... Morphisms of formal schemes ............. I, §2.1.(c)
- **AdhFs** — Adic morphisms ................................. I, §2.1.(c)
- **AdhFs** — adic over $S$ ................................ Adic morphisms .............................. I, §2.1.(c)
- **CFs** — Coherent formal schemes ....................... Morphisms of formal schemes ............. I, §1.6.(c)
- **AcCFs** — Adic — of finite ideal type ................ Morphisms of formal schemes ............. I, §1.6.(c)
- **AcCFs** — Adic morphisms ................................. I, §1.6.(c)
- **AcCFs** — adic over $S$ ................................ Adic morphisms .............................. I, §1.6.(c)
- **RigNoeCFs** — Morphisms of formal schemes ............. I, §2.1.(c)
- **AdhCFs** — Universally adhesive — ....................... Morphisms of formal schemes ............. I, §2.1.(c)
- **Affs** — Affine formal schemes ......................... Morphisms of formal schemes ............. I, §1.5.(c)
- **AcFAs** — Coherent — ................................ Morphisms of sheaves .......................... I, §6.3.(a)
- **AcFAs** — $S$ ................................................ Morphisms of sheaves .......................... I, §6.3.(b)
- **Rf** — Rigid spaces .................................... Morphisms of rigid spaces .................... II, §2.2.(c)
- **ARf** — Affinoids — .................................. Morphisms of rigid spaces .................... II, §6.1.(a)
- **CRf** — Coherent rigid spaces — ....................... Morphisms of rigid spaces .................... II, §2.2.1

**— Sheaves**

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<td>AQCohX</td>
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#### Operator symbols

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<td>$\text{AId}_X$</td>
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<td>$\text{C}^*(\mathcal{A})$</td>
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<td>$\text{card}(x)$</td>
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<td>Class of objects in the category $\mathcal{C}$</td>
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<td>Differential module</td>
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<td>$\text{ouv}(E)$</td>
<td>Set of all isom. classes of subobj’s of a final object</td>
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<td>Ouv((X))</td>
<td>Set of all open subsets of (X)</td>
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<td>(\mathbb{P}^n_S)</td>
<td>Rigid analytic projective (n)-space over (S)</td>
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<td>Proj (S)</td>
<td>Homogeneous prime spectrum of (S)</td>
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<td>Set of all isomorphism classes of points of (E)</td>
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Foundations of Rigid Geometry I

Rigid geometry is one of the modern branches of algebraic and arithmetic geometry. It has its historical origin in J. Tate's rigid analytic geometry, which aimed at developing an analytic geometry over non-archimedean valued fields. Nowadays, rigid geometry is a discipline in its own right and has acquired vast and rich structures, based on discoveries of its relationship with birational and formal geometries.

In this research monograph, foundational aspects of rigid geometry are discussed, putting emphasis on birational and topological features of rigid spaces. Besides the rigid geometry itself, topics include the general theory of formal schemes and formal algebraic spaces, based on a theory of complete rings which are not necessarily Noetherian. Also included is a discussion on the relationship with Tate's original rigid analytic geometry, V.G. Berkovich's analytic geometry and R. Huber's adic spaces. As a model example of applications, a proof of Nagata's compactification theorem for schemes is given in the appendix. The book is encyclopedic and almost self-contained.