Approximate Calculation of Integrals

V. I. Krylov

Translated by Arthur H. Stroud
DOVER BOOKS ON MATHEMATICS

HANDBOOK OF MATHEMATICAL FUNCTIONS, Milton Abramowitz and Irene A. Stegun. (0-486-61272-4)
TENSOR ANALYSIS ON MANIFOLDS, Richard L. Bishop and Samuel I. Goldberg. (0-486-64039-6)
VECTOR AND TENSOR ANALYSIS WITH APPLICATIONS, A. I. Borisenko and I. E. Tarapov. (0-486-63833-2)
THE HISTORY OF THE CALCULUS AND ITS CONCEPTUAL DEVELOPMENT, Carl B. Boyer. (0-486-60509-4)
THE QUALITATIVE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS: AN INTRODUCTION, Fred Brauer and John A. Nohel. (0-486-65846-5)
ALGORITHMS FOR MINIMIZATION WITHOUT DERIVATIVES, Richard P. Brent. (0-486-41998-3)
PRINCIPLES OF STATISTICS, M. G. Bulmer. (0-486-63760-3)
THE THEORY OF SPINORS, Élie Cartan. (0-486-64070-1)
ADVANCED NUMBER THEORY, Harvey Cohn. (0-486-64023-X)
STATISTICS MANUAL, Edwin L. Crow, Francis Davis, and Margaret Maxfield. (0-486-60599-X)
FOURIER SERIES AND ORTHOGONAL FUNCTIONS, Harry F. Davis. (0-486-65973-9)
COMPUTABILITY AND UNSOLVABILITY, Martin Davis. (0-486-61471-9)
ASYMPTOTIC METHODS IN ANALYSIS, N. G. de Bruijn. (0-486-64221-6)
PROBLEMS IN GROUP THEORY, John D. Dixon. (0-486-61574-X)
THE MATHEMATICS OF GAMES OF STRATEGY, Melvin Dresher. (0-486-64216-X)
APPLIED PARTIAL DIFFERENTIAL EQUATIONS, Paul DuChateau and David Zachmann. (0-486-41976-2)
ASYMPTOTIC EXPANSIONS, A. Erdélyi. (0-486-60318-0)
COMPLEX VARIABLES: HARMONIC AND ANALYTIC FUNCTIONS, Francis J. Flanigan. (0-486-61388-7)
ON FORMALLY UNDECIDABLE PROPOSITIONS OF PRINCIPIA MATHEMATICA AND RELATED SYSTEMS, Kurt Gödel. (0-486-66980-7)
A HISTORY OF GREEK MATHEMATICS, Sir Thomas Heath. (0-486-24073-8, 0-486-24074-6) Two-volume set
PROBABILITY: ELEMENTS OF THE MATHEMATICAL THEORY, C. R. Heathcote. (0-486-41149-4)
INTRODUCTION TO NUMERICAL ANALYSIS, Francis B. Hildebrand. (0-486-65363-3)
METHODS OF APPLIED MATHEMATICS, Francis B. Hildebrand. (0-486-67002-3)
TOPOLOGY, John G. Hocking and Gail S. Young. (0-486-65676-4)
MATHEMATICS AND LOGIC, Mark Kac and Stanislaw M. Ulam. (0-486-67085-6)
MATHEMATICAL FOUNDATIONS OF INFORMATION THEORY, A. I. Khinchin. (0-486-60434-9)
ARITHMETIC REFRESHER, A. Albert Klaaf. (0-486-21241-6)
CALCULUS REFRESHER, A. Albert Klaaf. (0-486-20370-0)

(continued on back flap)
APPROXIMATE CALCULATION OF INTEGRALS

Vladimir Ivanovich Krylov

Translated by
Arthur H. Stroud

DOVER PUBLICATIONS, INC.
Mineola, New York
The author attempts in this book to introduce the reader to the principal ideas and results of the contemporary theory of approximate integration and to provide a useful reference for practical computations.

In this book we consider only the problem of approximate integration of functions of a single variable. We almost completely ignore the more difficult problem of approximate integration of functions of more than one variable, a problem about which much less is known. Only in one place do we mention double and triple integrals in connection with their reduction to single integrals.

But even for single integrals the author has omitted many interesting considerations. Problems not touched upon are, for example, methods of integration of rapidly oscillating functions, the calculation of contour integrals of analytic functions, the application of random methods, and others. The book is devoted for the most part to methods of mechanical quadrature where the integral is approximated by a linear combination of a finite number of values of the integrand.

The contents of the book are divided into three parts. The first part presents concepts and theorems that are met with in the theory of quadrature, but are at least partially outside of the programs of higher academic institutions.

The second part is devoted to the problem of calculation of definite integrals. Here we consider, in essence, three basic topics: the theory of the construction of mechanical quadrature formulas for sufficiently smooth integrand functions, the problem of increasing the precision of quadratures, and the convergence of the quadrature process.

In the third part of the book we study methods for the calculation of indefinite integrals. Here we confine ourselves for the most part to a study of methods for constructing computational formulas. In addition we indicate stability criterions and the convergence of the computational process.

My colleagues in this work, M. K. Gavurin and I. P. Mysovskich, examined a large part of the manuscript and I am very thankful for their remarks and advice.

Academy of Sciences of the
Byelorussian Socialist Soviet Republic

V. I. KRYLOV
TRANSLATOR’S PREFACE

This book provides a systematic introduction to the subject of approximate integration, an important branch of numerical analysis. Such an introduction was not available previously. The manner in which the book is written makes it ideally suited as a text for a graduate seminar course on this subject.

A more exact title for this book would be Approximate Integration of Functions of One Variable. As in many aspects of the theory of functions the theory developed here for functions of one variable is very difficult to extend to functions of more than one variable, and the corresponding results are mostly unknown. Several years from now, after methods for integration of functions of more than one variable have been investigated more thoroughly, a book entirely devoted to this subject will be needed.

As a source of reference for other topics concerning approximate integration see "A Bibliography on Approximate Integration," Mathematics of Computation (vol. 15, 1961, pp. 52–80), which was compiled by the translator. This is a reasonably complete bibliography, particularly for papers published during the past several decades.

The only significant change in this translation from the original is the inclusion in the appendices of slightly more extensive tables of Gaussian quadrature formulas. The formulas in Appendix A for constant weight function are taken from a memorandum by H. J. Gawlik and are published with the permission of the Controller of Her Britannic Majesty’s Stationery Office, and the British Crown copyright is reserved.

I wish to thank Dr. V. I. Krylov for the assistance he provided in furnishing a list of corrections to the original edition. I am also indebted to Professor G. E. Forsythe for the interest he expressed on behalf of the Association for Computing Machinery in having this book published in the present monograph series. Finally I am indebted to James T. Day for his interest in this book and for his assistance in reading parts of the manuscript.

University of Wisconsin
Madison, Wisconsin

A. H. STROUD

vi
CONTENTS

Preface v
Translator's Preface vi

PART ONE. PRELIMINARY INFORMATION

Chapter 1. Bernoulli Numbers and Bernoulli Polynomials 3
1.1. Bernoulli numbers 3
1.2. Bernoulli polynomials 6
1.3. Periodic functions related to Bernoulli polynomials 13
1.4. Expansion of an arbitrary function in Bernoulli polynomials 15

Chapter 2. Orthogonal Polynomials
2.1. General theorems about orthogonal polynomials 18
2.2. Jacobi and Legendre polynomials 23
2.3. Chebyshev polynomials 26
2.4. Chebyshev-Hermite polynomials 33
2.5. Chebyshev-Laguerre polynomials 34

Chapter 3. Interpolation of Functions 37
3.1. Finite differences and divided differences 37
3.2. The interpolating polynomial and its remainder 42
3.3. Interpolation with multiple nodes 45

Chapter 4. Linear Normed Spaces. Linear Operators 50
4.1. Linear normed spaces 50
4.2. Linear operators 54
4.3. Convergence of a sequence of linear operators 59
PART TWO. APPROXIMATE CALCULATION OF DEFINITE INTEGRALS

Chapter 5. Quadrature Sums and Problems Related to Them. The Remainder in Approximate Quadrature

5.1. Quadrature sums
5.2. Remarks on the approximate integration of periodic functions
5.3. The remainder in approximate quadrature and its representation

Chapter 6. Interpolatory Quadratures

6.1. Interpolatory quadrature formulas and their remainder terms
6.2. Newton-Cotes formulas
6.3. Certain of the simplest Newton-Cotes formulas

Chapter 7. Quadratures of the Highest Algebraic Degree of Precision

7.1. General theorems
7.2. Constant weight function
7.3. Integrals of the form \( \int_a^b (b - x)^\alpha (x - a)^\beta f(x) \, dx \) and their application to the calculation of multiple integrals
7.4. The integral \( \int_{-\infty}^{\infty} e^{-x^2} f(x) \, dx \)
7.5. Integrals of the form \( \int_0^{\infty} x^\alpha e^{-x} f(x) \, dx \)

Chapter 8. Quadrature Formulas with Least Estimate of the Remainder

8.1. Minimization of the remainder of quadrature formulas
8.2. Minimization of the remainder in the class \( L_q^{(r)} \)
8.3. Minimization of the remainder in the class \( C_r \)
8.4. The problem of minimizing the estimate of the remainder for quadrature with fixed nodes

Chapter 9. Quadrature Formulas Containing Preassigned Nodes

9.1. General theorems
9.2. Formulas of special form
9.3. Remarks on integrals with weight functions that change sign
16.3. The number of interpolating polynomials of the highest degree of precision 326
16.4. The remainder of the interpolation and minimization of its estimate 327
16.5. Conditions for which the coefficients $\alpha_j$ are positive 329
16.6. Connection with the existence of a polynomial solution to a certain differential equation 331
16.7. Some particular formulas 333

Appendix A. Gaussian Quadrature Formulas for Constant Weight Function 337
Appendix B. Gaussian-Hermite Quadrature Formulas 343
Appendix C. Gaussian-Laguerre Quadrature Formulas 347

Index. 353
Part One of this book presents certain selected results from the following special mathematical topics: Bernoulli numbers and Bernoulli polynomials, orthogonal polynomials, interpolation, linear operators and convergence of sequences of such operators. These topics are needed to construct the theory of approximate integration and are presented only to the extent required to understand the other chapters. The results developed here can be found in special literature, but we think it is useful to present them in this book to free the reader from the inconvenience of looking up literature references.
1.1. BERNOULLI NUMBERS

Bernoulli polynomials and Bernoulli numbers are needed in later chapters (Sections 6.3 and 11.3) to construct Euler-Maclaurin formulas and other similar formulas which serve to increase the accuracy of approximate quadrature.

The Bernoulli numbers can be defined by means of the following generating function. Let \( t \) be a complex parameter. Consider the function

\[
g(t) = \frac{t}{e^t - 1}.
\]

(1.1.1)

For \( k \), an integer, the points \( t = 2k\pi i \) are zeros of the denominator. All of these are simple zeros because the derivative of the denominator is \( e^t \) and is different from zero for all finite \( t \). The point \( t = 0 \) is not a singular point of \( g(t) \) because \( \lim_{t \to 0} \frac{t}{e^t - 1} = 1 \).

The function \( g(t) \) is holomorphic in the circle \( |t| < 2\pi \) and thus can be expanded there in a power series in \( t \). We write the expansion in the form

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n, \quad |t| < 2\pi.
\]

(1.1.2)

The numbers \( B_n \) defined by this equation are called Bernoulli numbers.

If both sides of (1.1.2) are multiplied by \( e^t - 1 = \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} \), then we
obtain the equation
\[
\left( \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right) \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = t,
\]
valid for all \( t \) in the circle \(|t| < 2\pi\). After multiplying out the power series on the left side of this equation there must remain only the first power of \( t \) with coefficient of unity. Thus the powers of \( t \) higher than the first must all become zero: \( B_0 = 1 \) and for \( n = 2, 3, \ldots \) we must have
\[
\frac{B_0}{n!} + \frac{B_1}{(n-1)!} + \frac{B_2}{(n-2)!} + \cdots + \frac{B_{n-1}}{1!(n-1)!} = 0.
\]

This last equation permits us to sequentially calculate all of the Bernoulli numbers. We can obtain other forms which are more convenient for some purposes. Multiplying the last equation by \( n! \) and adding \( B_n \) to both sides we obtain
\[
\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} B_k = B_n.
\]

Comparing this equation with the binomial expansion we see that it can be written in the form
\[
(B + 1)^n = B_n \tag{1.1.3}
\]
if we interpret this equation to mean that after raising the binomial \( B + 1 \) to the \( n \)th power the \( k \)th power of \( B \) is the Bernoulli number \( B_k \) (\( k = 0, 1, \ldots, n \)).

We can easily verify that all Bernoulli numbers with odd indices, greater than unity, are equal to zero:
\[
B_{2k+1} = 0, \quad k > 0. \tag{1.1.4}
\]

In order to show this replace \( t \) by \(-t\) in (1.1.2):
\[
\frac{-t}{e^{-t} - 1} = \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} t^n.
\]

On the other hand
\[
\frac{-t}{e^{-t} - 1} = \frac{te^t}{e^t - 1} = t + \frac{t}{e^t - 1} = t + \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n,
\]
and therefore we must have
Comparing the coefficients of $t^n$, for $n > 1$, gives

$$B_n = (-1)^n B_n.$$ 

When $n$ is an odd integer $2k + 1$ ($k > 0$) we have

$$B_{2k + 1} = -B_{2k + 1},$$

which is equivalent to (1.1.4).

The values of the nonzero Bernoulli numbers for $n \leq 30$ are:

<table>
<thead>
<tr>
<th>$B_0$</th>
<th>$B_{10}$</th>
<th>$B_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{5}{66}$</td>
<td>$\frac{854513}{138}$</td>
</tr>
<tr>
<td>$B_1$</td>
<td>$-\frac{1}{2}$</td>
<td>$B_{12}$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$\frac{1}{6}$</td>
<td>$B_{14}$</td>
</tr>
<tr>
<td>$B_4$</td>
<td>$-\frac{1}{30}$</td>
<td>$B_{16}$</td>
</tr>
<tr>
<td>$B_6$</td>
<td>$\frac{1}{42}$</td>
<td>$B_{18}$</td>
</tr>
<tr>
<td>$B_8$</td>
<td>$-\frac{1}{30}$</td>
<td>$B_{20}$</td>
</tr>
</tbody>
</table>

The Bernoulli numbers with even indices are related to sums of even negative powers of the natural numbers by the following remarkable identity:

$$B_{2k} = \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}} \left( 1 + 2^{-2k} + 3^{-2k} + 4^{-2k} + \cdots \right).$$

From this it is seen that for increasing $k$ the Bernoulli numbers $B_{2k}$ will increase in size and for large $k$ will asymptotically approach

$$B_{2k} \approx 2(-1)^{k-1}(2k)!((2\pi)^{-2k}).$$

Equation (1.1.5) follows at once from the expansion (1.3.1) which we will obtain for Bernoulli polynomials in a trigonometric series on the segment $[0, 1]$. 

$$t + \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} t^n.$$
1.2. BERNOULLI POLYNOMIALS

Bernoulli polynomials can be defined by various methods, but for our purpose it is convenient to define them by means of a generating function. We introduce the function

\[ g(x, t) = e^{xt} \frac{t}{e^t - 1} \]  

This differs from (1.1.1) by the factor \( e^{xt} \) which does not vanish, so \( g(x, t) \) has the same singular points as \( g(t) \). In particular it is holomorphic in the circle \( |t| < 2\pi \) and can be expanded there in a power series in \( t \):

\[ g(x, t) = e^{xt} \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n. \]  

In the next paragraph we will see that the functions \( B_n(x) \) are polynomials of degree \( n \). They are called the Bernoulli polynomials.

If in \( g(x, t) \) the factor \( e^{xt} \) is replaced by the series \( \sum_{\nu=0}^{\infty} \frac{x^{\nu}t^\nu}{\nu!} \) and \( \frac{t}{e^t - 1} \) is replaced by the expansion (1.1.2), then we obtain the identity

\[ \sum_{\nu=0}^{\infty} \frac{x^{\nu}t^\nu}{\nu!} \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \quad |t| < 2\pi. \]

Comparing the coefficients of \( t^n \) leads to the equation

\[ \frac{B_n(x)}{n!} = \frac{x^n B_0}{n!} + \frac{x^{n-1} B_1}{(n-1)!1!} + \cdots + \frac{B_n}{n!}. \]

After multiplying by \( n! \) we obtain the following expression for \( B_n(x) \)

\[ B_n(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} B_{n-k} x^k, \]  

which shows that \( B_n(x) \) is indeed a polynomial of degree \( n \). The expression (1.2.3) can be written in a simpler form

\[ B_n(x) = (x + B)^n \]  

if we agree to consider that after raising the binomial \( x + B \) to the \( n^{th} \) power that the \( k^{th} \) power of \( B \) is taken to be the \( k^{th} \) Bernoulli number \( B_k \).
1.2. Bernoulli Polynomials

We will need to be familiar with certain properties of the Bernoulli polynomials; these will now be developed.

1. The value of a Bernoulli polynomial.

For $x = 0$ the value of a Bernoulli polynomial is the corresponding Bernoulli number:

$$B_n(0) = B_n$$  \hspace{1cm}(1.2.5)$$

which we see from (1.2.3).

2. Differentiation and integration of $B_n(x)$.

Differentiating (1.2.2) with respect to $x$ gives

$$te^{xt} \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B'_n(x)}{n!} t^n.$$  \hspace{1cm}(1.2.5.1)$$

The left-hand side of this equation is different from $g(x, t)$ only by the factor $t$ and therefore must be

$$te^{xt} \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^{n+1}.$$  \hspace{1cm}(1.2.5.2)$$

The power expansions of the two previous equations must be identically equal and thus

$$\frac{B'_n(x)}{n!} = \frac{B_{n-1}(x)}{(n - 1)!}$$

or

$$B'_n(x) = nB_{n-1}(x).$$  \hspace{1cm}(1.2.6)$$

From this and from (1.2.5) we immediately have the following relationship for the integration of Bernoulli polynomials

$$B_n(x) = B_n + n \int_0^x B_{n-1}(t) \, dt.$$  \hspace{1cm}(1.2.7)$$

3. Multiplication of the argument by a constant.

Let $m$ be any positive integer

$$e^{mxt} \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(mx)}{n!} t^n.$$
By a very simple transformation we can obtain another expansion
\[ e^{mzt} \frac{t}{e^t - 1} = \frac{1}{m} e^{mzt} \left[ \frac{m t (1 + e^t + \cdots + e^{(m-1)t})}{e^{mt} - 1} \right] = \]
\[ = \frac{1}{m} \sum_{s=0}^{m-1} \frac{e^{(x+s/m)mt}}{e^{mt} - 1} = \frac{1}{m} \sum_{s=0}^{m-1} \sum_{n=0}^{\infty} \frac{m^n B_n \left( x + \frac{s}{m} \right)}{n!} t^n. \]

From these two expansions we deduce the relationship for multiplication of the argument by a constant factor:
\[ B_n(mx) = m^{n-1} \sum_{s=0}^{m-1} B_n \left( x + \frac{s}{m} \right). \quad (1.2.8) \]

4. Representations for the polynomials \( B_n(x) \).

In order to study the behavior of \( B_n(x) \) it is convenient to replace the variable \( x \) by a new variable \( z = x(1-x) \). We will show the validity of the following assertions concerning representations for Bernoulli polynomials in the variable \( z \).

Each polynomial \( B_n(x) \) of even order \( n = 2k \) can be expanded in powers of \( z \):
\[ (-1)^k [B_{2k}(x) - B_{2k}] = \sum_{\nu=0}^{k-2} F_{k, \nu} z^{k-\nu-\nu} \quad (1.2.9) \]
where \( F_{k,0} = 1 \) and \( F_{k, \nu} > 0 \) \((\nu = 1, 2, \ldots, k-2)\). Each Bernoulli polynomial of odd order \( n = 2k - 1 \) can be represented in the form:
\[ (-1)^k B_{2k-1}(x) = (1 - 2x) \sum_{\nu=0}^{k-2} H_{k, \nu} z^{k-\nu-1} \quad (1.2.10) \]
where all the coefficients \( H_{k, \nu} \) \((\nu = 0, 1, \ldots, k-2)\) are positive.

Let us verify the first of these assertions concerning the polynomials \( B_{2k}(x) \) of even order. To simplify the discussion we introduce the auxiliary variable \( \xi \), setting \( x = 1/2 + \xi \). The variables \( \xi \) and \( z \) are related by
\[ z = x(1-x) = \frac{1}{4} - \xi^2. \]

In order to see that \( B_{2k}(x) \) is a polynomial in the variable \( z \) it is sufficient to establish that the expansion of \( B_{2k}(x) \) in powers of \( \xi \) will contain only even powers of \( \xi \).
1.2. Bernoulli Polynomials

Differentiating the function (1.2.1) with respect to the variable $\xi$ gives the following expression

$$g(x, t) = e^{\left(\frac{1}{2} + \xi\right)t} \frac{t}{e^t - 1} = e^{\xi t} \frac{te^{\frac{1}{2}t}}{e^t - 1} =$$

$$= e^{\xi t} \frac{t}{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}} = e^{\xi t} \frac{t/2}{\sinh t/2}.$$

$B_{2k}(x)$ is the coefficient of $t^{2k}$ in the expansion of $g(x, t)$ in powers of $t$. The factor $\frac{t/2}{\sinh t/2}$ is an even function of $t$ and its power series in $t$ will contain only even powers of $t$. After multiplication of this series by

$$e^{\xi t} = \sum_{\nu=0}^{\infty} \frac{\xi^\nu t^\nu}{\nu!},$$

in order to obtain the term in $t^{2k}$ we must take from the series for $e^{\xi t}$ only terms with even powers of $t$. But all of these also contain only even powers of $\xi$, and thus $B_{2k}(x)$ will contain only even powers of $\xi$.

For $x = 0$ we also have $z = 0$, and hence the difference $B_{2k}(x) - B_{2k}$ will be a polynomial in $z$ without a constant term and must have the form

$$(-1)^k[B_{2k}(x) - B_{2k}] = \sum_{\nu=0}^{k-1} F_{k, \nu} z^{k-\nu}.$$

There remains only to verify the assertion about $F_{k, \nu}$. The coefficient in $B_{2k}(x)$ of the highest degree (that is the coefficient of $x^{2k}$) is equal to unity, and therefore we must have $F_{k, 0} = 1$. In addition the coefficient of $x$ in $B_{2k}(x)$ is $2kB_{2k-1} = 0$ and because the first power of $x$ on the righthand side can be only contained in the term corresponding to $\nu = k - 1$, then $F_{k, k-1} = 0$. We may construct a recursion relation to find the remaining $F_{k, \nu}$. Let us calculate the second derivative with respect to $x$ of both sides of (1.2.9). Because

$$B_{2k}''(x) = 2k(2k - 1)B_{2k-2}(x)$$

and because the operators of differentiation with respect to $x$ and $z$ are related by
\[
\frac{d}{dx} = \frac{dz}{dx} \frac{d}{dz} = (1 - 2x) \frac{d}{dz},
\]
\[
\frac{d^2}{dx^2} = (1 - 2x)^2 \frac{d^2}{dz^2} - 2 \frac{d}{dx} = (1 - 4x) \frac{d^2}{dz^2} - 2 \frac{d}{dz}
\]

then we obtain:
\[
(-1)^k 2k(2k - 1)B_{2k-2}(x) = \sum_{\nu=1}^{k-1} F_{k,\nu-1}(k - \nu + 1)(k - \nu)z^{k-\nu-1} - \\
- \sum_{\nu=0}^{k-2} F_{k,\nu}(2k - 2\nu)(2k - 2\nu - 1)z^{k-\nu-1}.
\]

Comparing this with (1.2.9) for \(B_{2k-2}(x)\), namely with
\[
(-1)^k [-1]B_{2k-2}(x) - B_{2k-2}] = \sum_{\nu=0}^{k-2} F_{k-1,\nu} z^{k-\nu-1},
\]
we obtain the desired recursion relation for \(F_{k,\nu}\)
\[
(2k - 2\nu)(2k - 2\nu - 1)F_{k,\nu} = \\
= 2k(2k - 1)F_{k-1,\nu} + (k - \nu + 1)(k - \nu)F_{k,\nu-1}.
\]
Hence knowing \(F_{k,0} = 1\) and \(F_{k,k-1} = 0\) \((k = 1, 2, \ldots)\) we can sequentially find \(F_{k,\nu}\) \((k = 3, 4, \ldots; \nu = 1, 2, \ldots, k-2)\), and all of them turn out to be positive.

To establish the representation for \(B_{2k-1}(x)\) it suffices to differentiate both sides of (1.2.9) with respect to \(x\):
\[
(-1)^k 2kB_{2k-1}(x) = \sum_{\nu=0}^{k-2} F_{k,\nu}(1 - 2x)(k - \nu)z^{k-\nu-1}.
\]
Hence we see that (1.2.10) is valid with
\[
H_{k,\nu} = \frac{(k - \nu)F_{k,\nu}}{2k} > 0.
\]

5. Symmetry of \(B_n(x)\).

Consider the point \(x = 1/2\) on the \(x\) axis. The points \(x\) and \(1 - x\) are symmetrically situated with respect to this point. The parameter \(z = x(1 - x)\) does not change in value if we replace \(x\) by \(1 - x\). Thus from
We have obtained
\[ B_{2k}(1 - x) = B_{2k}(x); \quad (1.2.11) \]
the graph of \( B_{2k}(x) \) is symmetric with respect to the line \( x = 1/2 \).

The factor \( \sum_{\nu=0}^{k-2} H_k, \nu z^{k-\nu-1} \) in (1.2.10) has the same value at the points \( x \) and \( 1 - x \). The factor \( (1 - 2x) \) has the same absolute value but opposite sign at these points. Therefore
\[ B_{2k-1}(1 - x) = -B_{2k-1}(x). \quad (1.2.12) \]
Thus the graph of \( B_{2k-1}(x) \) is centrally symmetric with respect to the point \( x = 1/2 \).

From (1.2.5) and (1.2.11) we obtain
\[ B_{2k}(1) = B_{2k}, \]
and from (1.2.12) for \( k \geq 2 \) we obtain
\[ B_{2k-1}(1) = -B_{2k-1}. \]
Thus each Bernoulli polynomial, except \( B_1(x) \), has equal values at the ends of the segment \([0, 1]\).
\[ B_n(1) = B_n(0) = B_n. \quad (1.2.13) \]

6. The behavior of the Bernoulli polynomials on the segment \([0, 1]\).

We will need to know the value \( B_n(1/2) \) which can be easily calculated from (1.2.8). If in (1.2.8) we substitute \( m = 2 \) and \( x = 1/2 \) we obtain
\[ B_n(1) = 2^{n-1} \left[ B_n \left( \frac{1}{2} \right) + B_n(1) \right]. \]
But since
\[ B_n(1) = B_n(n > 1), \]
then for every \( n \)
\[ B_n \left( \frac{1}{2} \right) = -(1 - 2^{-n+1}) B_n. \quad (1.2.14) \]
We will also need some properties of the polynomials
\[ \gamma_n(x) = B_n(x) - B_n \]
which are essentially the same as \( B_n(x) \), but which are more convenient for some purposes. Consider, at first, the polynomial of even order \( n = 2k \), which by (1.2.9) is
The points $x = 0$ and $x = 1$ are zeros of $y_{2k}(x)$:

$$y_{2k}(0) = B_{2k}(0) - B_{2k} = B_{2k} - B_{2k} = 0$$

$$y_{2k}(1) = B_{2k}(1) - B_{2k} = B_{2k} - B_{2k} = 0.$$  

It is easily seen that for $k \geq 2$ both of these points are zeros of multiplicity two; for example for $x = 0$

$$y'_{2k}(0) = 2kB_{2k-1}(0) = 0$$

$$y''_{2k}(0) = 2k(2k - 1)B_{2k-2}(0) = 2k(2k - 1)B_{2k-2} \neq 0.$$  

By (1.2.11) the same holds true for $x = 1$. For $0 < x < 1$ the parameter $z$ will lie within the limits $0 < z \leq 1/4$, and since $F_{k,v} > 0$  

$$(-1)^{k}y_{2k}(x) > 0 \quad \text{for } 0 < x < 1.$$ 

In the open segment $0 < x < 1$ the polynomial $y_{2k}(x)$ has no zeros and has the same sign as $(-1)^k$.

When $x$ varies from zero up to $1/2$, the function $z = x(1 - x)$ will increase from zero up to $1/4$, and as $x$ varies from $1/2$ up to $1$, the function $z$ will decrease from $1/4$ to zero.

As can be seen from (1.2.15) as $x$ varies from zero up to $1/2$ the polynomial $(-1)^k y_{2k}(x)$ will increase from zero up to $(-1)^k y_{2k}(1/2) = |B_{2k}(1/2) - B_{2k}| = (2 - 2^{-2k+1})|B_{2k}|$. When $x$ varies from $1/2$ up to $1$, the polynomial $(-1)^k y_{2k}(x)$ will decrease again to zero. Each value $\alpha$ in the range $0 < \alpha < (2 - 2^{-2k+1})|B_{2k}|$ will be taken on twice by $y_{2k}(x)$, on the segment $(0, 1)$, at two points which are symmetrically located with respect to $x = 1/2$.

Let us consider now a polynomial $y_n(x)$ of odd order $n = 2k - 1$. If we take $k \geq 2$ then

$$y_{2k-1}(x) = B_{2k-1}(x)$$

and

$$(-1)^k y_{2k-1}(x) = (1 - 2x) \sum_{\nu=0}^{k-2} H_{k,\nu} z^{k-\nu-1}. \tag{1.2.16}$$

The points $x = 0$ and $x = 1$ will be zeros of $y_{2k-1}(x)$, and we can see that they both will be zeros of multiplicity one. In fact

$$y'_{2k-1}(0) = y'_{2k-1}(1) = (2k - 1)B_{2k-2}(0) \neq 0.$$  

In addition, from (1.2.16) and from $H_{k,\nu} > 0$, we see that $x = 1/2$ is a simple zero of $y_{2k-1}(x)$ and these are the only zeros of this polynomial on the closed segment $0 \leq x \leq 1$. The sign of $y_{2k-1}(x)$ is given by
1.3. Periodic Functions Related to Bernoulli Polynomials

\[ (-1)^k y_{2k-1}(x) > 0 \quad \text{for } 0 < x < \frac{1}{2}, \]

\[ (-1)^k y_{2k-1}(x) < 0 \quad \text{for } \frac{1}{2} < x < 1. \]

Here we give a table of the first ten Bernoulli polynomials.

\[
\begin{align*}
B_0(x) &= 1 \\
B_1(x) &= x - 1/2 \\
B_2(x) &= x^2 - x + 1/6 \\
B_3(x) &= x^3 - 3/2 x^2 + 1/2 x \\
B_4(x) &= x^4 - 2 x^3 + x^2 - 1/30 \\
B_5(x) &= x^5 - 5/2 x^4 + 5/3 x^3 - 1/6 x \\
B_6(x) &= x^6 - 3 x^5 + 5/2 x^4 - 1/2 x^2 + 1/42 \\
B_7(x) &= x^7 - 7/2 x^6 + 7/2 x^5 - 7/6 x^3 + 1/6 x \\
B_8(x) &= x^8 - 4 x^7 + 14/3 x^6 - 7/3 x^4 + 2/3 x^2 - 1/30 \\
B_9(x) &= x^9 - 9/2 x^8 + 6 x^7 - 21/5 x^5 + 2 x^3 - 3/10 x \\
B_{10}(x) &= x^{10} - 5 x^9 + 15/2 x^8 - 7 x^6 + 5 x^4 - 3/2 x^2 + 5/66.
\end{align*}
\]

Figure 1 illustrates the behavior of the Bernoulli polynomials \( B_n(x) \) on the segment \([0, 1]\).
Let us construct the trigonometric Fourier series for $B^*_n(x)$. For this purpose we construct the Fourier series for the generating function

$$
g(x, t) = e^{xt} \frac{t}{e^t - 1}
$$

for $0 \leq x < 1$. To do this we expand $g(x, t)$ in an exponential series

$$
g(x, t) = \sum_{m=-\infty}^{+\infty} C_m e^{i2\pi mx};$$

where

$$
C_m = \int_0^1 g(x, t) e^{-i2\pi mx} \, dx = \frac{t}{e^t - 1} \int_0^1 e^{xt} e^{-i2\pi mx} \, dx = \frac{t}{e^t - 1} \left[ e^{xt-i2\pi m} \right]_0^1 = \frac{t}{t - i2\pi m}.
$$

By singling out the summand $C_0 = 1$ and combining the terms in the series corresponding to the indices $m$ and $-m$ we obtain

$$
g(x, t) = 1 + \sum_{m=1}^{\infty} \left[ \frac{t}{t - i2\pi m} e^{i2\pi mx} + \frac{t}{t + i2\pi m} e^{-i2\pi mx} \right].
$$

It can be shown that for any value of $x$ on the segment $0 \leq x \leq 1$ the series on the right hand side of this equation will converge for all $t$ distinct from $i2k\pi$ ($k = 0, \pm1, \pm2, \ldots$). To prove this we take any bounded part $\sigma$ of the plane $t$ and exclude from the series the terms at the beginning which have poles in this part of the plane; the terms which remain will converge uniformly relative to $t$ in $\sigma$. From these remarks it is easy to justify the change of order of the summation which we will make below in the construction of a power series for $g(x, t)$.

If we consider $|t| < 2\pi$ and expand the right side of the last equation in powers of $t$ then the coefficient of $t^n$ will be a trigonometric series for $B_n(x)/n!$. It will also give a representation for $B^*_n(x)/n!$ for all $x$. 
1.4. Expansion of an Arbitrary Function in Bernoulli Polynomials

\[
\frac{t}{t + i2\pi m} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{t}{i2\pi m}\right)^n
\]

\[
g(x, t) = 1 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ \frac{(-1)^{n-1}}{(i2\pi m)^n} t^n e^{-i2\pi mx} - \frac{1}{(i2\pi m)^n} t^n e^{i2\pi mx} \right] =
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{t^n}{(i2\pi m)^n} \sum_{m=1}^{\infty} \left[ \frac{(-1)^{n-1}}{m^n} e^{-i2\pi mx} - \frac{1}{m^n} e^{i2\pi mx} \right] \]

thus, for \( n > 1, \)

\[
B_n(x) = \frac{n!}{(2\pi i)^n} \sum_{m=1}^{\infty} \left[ \frac{(-1)^{n-1}}{m^n} e^{-i2\pi mx} - \frac{1}{m^n} e^{i2\pi mx} \right].
\]

For even and odd orders the calculations give the following results

\[
B_{2k}^*(x) = \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}} \sum_{m=1}^{\infty} \cos \frac{2\pi mx}{m^{2k}}, \quad (1.3.1)
\]

\[
B_{2k+1}^*(x) = \frac{(-1)^{k-1}(2k+1)!}{2^{2k+1}\pi^{2k+1}} \sum_{m=1}^{\infty} \sin \frac{2\pi mx}{m^{2k+1}}. \quad (1.3.2)
\]

From this we obtain, for \( x = 0, \) the series (1.1.5) for the Bernoulli numbers.

1.4. EXPANSION OF AN ARBITRARY FUNCTION IN BERNOULLI POLYNOMIALS

Theorem 1. If \( f \) has a continuous derivative of order \( \nu \) on \([0, 1]\) then for any \( x \in [0, 1] \) we have the relation

\[
f(x) = \int_0^1 f(t) \, dt + \sum_{k=1}^{\nu-1} \frac{B_k(x)}{k!} \left[ f^{(k-1)}(1) - f^{(k-1)}(0) \right] - \frac{1}{\nu!} \int_0^1 f^{(\nu)}(t) [B_{\nu}^*(x - t) - B_{\nu}^*(x)] \, dt. \quad (1.4.1)
\]

Proof. Consider the integral

\[\rho_{\nu}(x) = \rho_{\nu} = \frac{1}{\nu!} \int_0^1 f^{(\nu)}(t) B_{\nu}^*(x - t) \, dt.\]

Considering that \( \nu > 1, \) we integrate by parts. Because
\[
\frac{d}{dt} B^*_\nu(x - t) = -\nu B^*_{\nu-1}(x - t)
\]

then

\[
B^*_\nu(x - 1) = B^*_\nu(x) = B(x)
\]

\[
\rho_\nu = \frac{B^*_\nu(x)}{\nu!} \left[ f^{(\nu-1)}(1) - f^{(\nu-1)}(0) \right] + \\
+ \frac{1}{(\nu - 1)!} \int_0^1 f^{(\nu-1)}(t) B^*_{\nu-1}(x - t) dt = \\
= \frac{B_\nu(x)}{\nu!} \left[ f^{(\nu-1)}(1) - f^{(\nu-1)}(0) \right] + \rho_{\nu-1}.
\]

We carry out this operation \(\nu - 1\) times:

\[
\rho_\nu = \sum_{k=2}^{\nu} \frac{B_k(x)}{k!} \left[ f^{(k-1)}(1) - f^{(k-1)}(0) \right] + \rho_1.
\]

The function \(B^*_1(x)\) has a jump of \(-1\) at the integers; at all other points it has a derivative equal to \(+1\),

\[
B^*_1(+0) - B^*_1(-0) = -1, \quad \frac{d}{dt} B^*_1(x - t) = -1.
\]

In order to calculate \(\rho_1\) we suppose at first \(0 < x < 1\)

\[
\rho_1(x) = \int_0^x f'(t) B^*_1(x - t) dt + \int_x^1 f'(t) B^*_1(x - t) dt = \\
= B^*_1(+0) f(x) - B^*_1(x) f(0) + \int_0^x f(t) dt + \\
+ B^*_1(x - 1) f(1) - B^*_1(-0) f(x) + \int_x^1 f(t) dt = \\
= [B^*_1(+0) - B^*_1(-0)] f(x) + B_1(x) [f(1) - f(0)] + \int_0^1 f(t) dt
\]

For \(\rho_\nu\ (\nu = 1, 2, \ldots)\) we finally obtain

\[
\rho_\nu = \frac{1}{\nu!} \int_0^1 f^{(\nu)}(t) B^*_\nu(x - t) dt
\]
1.4. Expansion of an Arbitrary Function in Bernoulli Polynomials

\[ \rho_v = \sum_{k=1}^{\nu} \frac{B_k(x)}{k!} \left[ f^{(k-1)}(1) - f^{(k-1)}(0) \right] - f(x) + \int_0^1 f(t) \, dt. \]

This result differs only in form from (1.4.1). The proof was carried out for the open segment 0 < x < 1, but by continuity equation (1.4.1) is valid also for the closed segment 0 ≤ x ≤ 1. This proves Theorem 1.

If \( f \) is defined on an arbitrary finite segment \([a, b]\) and has \( \nu \) continuous derivatives there, then its expansion on \([a, b]\) in Bernoulli polynomials is obtained from (1.4.1) by means of a linear transformation of the variable

\[
f(x) = \frac{1}{h} \int_a^b f(t) \, dt + \sum_{k=1}^{\nu-1} \frac{h^{k-1} B_k \left( \frac{x-a}{h} \right)}{k!} \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right] - \frac{h^{\nu-1}}{\nu!} \int_a^b f^{(\nu)}(t) \left[ B^*_\nu \left( \frac{x-t}{h} \right) - B^*_\nu \left( \frac{x-a}{h} \right) \right] \, dt, \tag{1.4.2} \]

where \( h = b - a \).

REFERENCES


CHAPTER 2

Orthogonal Polynomials

2.1. GENERAL THEOREMS ABOUT ORTHOGONAL POLYNOMIALS

Much of this book is devoted to a study of integrals of the form

\[ \int_{a}^{b} p(x) f(x) dx \]  

(2.1.1)

where \( p(x) \) is a given fixed function and \( f(x) \) is an arbitrary function of some wide class. The theory of approximate evaluation of this type of integral is closely related to the theory of orthogonal polynomials.

The function \( p(x) \) is called a weight function. We will usually restrict ourselves to nonnegative weight functions except in a few cases which will be specifically mentioned.

The theory of orthogonal polynomials for nonnegative weight functions has been developed to a high degree. We will discuss only the small portion of this theory which is necessary to construct certain special approximate integration formulas.

Let \([a, b]\) be any finite or infinite segment. For the present it suffices to assume that the weight function \( p(x) \) satisfies the two conditions:\n
1. \( p(x) \) is nonnegative, measurable, and not identically zero on the segment \([a, b]\),
2. the products \( p(x)x^m \), for any nonnegative integer \( m \), are summable on \([a, b]\).

The functions \( f(x) \) and \( g(x) \) are said to be orthogonal on the segment

\[ \int_{a}^{b} p(x)(x)|x|^m dx \] is finite for \( m = 0, 1, 2, \ldots \).
[a, b] with respect to the weight function \( p(x) \) if the product \( p(x)f(x) \times g(x) \) is summable and

\[
\int_a^b p(x)f(x)g(x)dx = 0. \tag{2.1.2}
\]

The function \( f(x) \) is said to be normalized on \([a, b]\) with respect to \( p(x) \) if \( p(x)f^2(x) \) is summable and

\[
\int_a^b p(x)f^2(x)dx = 1. \tag{2.1.3}
\]

Hereafter, if it is clear which function is taken as the weight function, the phrase "with respect to the weight function \( p(x) \)" will be omitted.

We introduce the notation

\[
c_m = \int_a^b p(x)x^m dx \quad (m = 0, 1, 2, \ldots),
\]

and let us consider the determinant

\[
\Delta_n = \begin{vmatrix}
c_0 & c_1 & \cdots & c_n \\
c_1 & c_2 & \cdots & c_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
c_n & c_{n+1} & \cdots & c_{2n}
\end{vmatrix}.
\]

It is not difficult to see that \( \Delta_n \) is different from zero. For this purpose we construct the homogeneous system of \( n + 1 \) equations in the \( n + 1 \) unknowns \( a_0, a_1, \ldots, a_n \)

\[
\begin{align*}
a_0c_0 + a_1c_1 + \cdots + a_nc_n &= 0 \\
a_0c_1 + a_1c_2 + \cdots + a_nc_{n+1} &= 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_0c_n + a_1c_{n+1} + \cdots + a_nc_{2n} &= 0 \tag{2.1.4}
\end{align*}
\]

If it were true that \( \Delta_n = 0 \), then this system would have nontrivial solutions, which we can show is impossible. Indeed, if we substitute in (2.1.4), the integrals which the \( c_m \) represent then the system becomes

\[
\begin{align*}
\int_a^b p(x)[a_0 + a_1x + \cdots + a_nx^n]dx &= 0 \\
\int_a^b p(x)x[a_0 + a_1x + \cdots + a_nx^n]dx &= 0
\end{align*}
\]
\[
\int_{a}^{b} p(x)x^n[a_0 + a_1x + \cdots + a_nx^n]dx = 0
\]

Multiplying these equations respectively by \(a_0, a_1, \ldots, a_n\) and adding we obtain

\[
\int_{a}^{b} p(x)[a_0 + a_1x + \cdots + a_nx^n]^2dx = 0
\]

which is possible only if the polynomial \(a_0 + a_1x + \cdots + a_nx^n\) is identically zero and consequently only if all of its coefficients \(a_0, a_1, \ldots, a_n\) are zero. Therefore the system (2.1.4) can have only the trivial solution, and \(\Delta_n \neq 0\).

Let \(n\) be any positive integer. In order to solve one of the problems in the theory of approximate integration it will be necessary to construct a polynomial of degree \(n\):

\[
P_n(x) = a_0 + a_1x + \cdots + a_nx^n, \quad a_n \neq 0, \tag{2.1.5}
\]

which will be orthogonal on \([a, b]\) to all polynomials of degree < \(n\). This is the same as requiring \(P_n(x)\) to satisfy the conditions

\[
\int_{a}^{b} p(x)P_n(x)x^mdx = 0 \quad (m = 0, 1, \ldots, n - 1). \tag{2.1.6}
\]

The coefficients \(a_k\) are determined by the linear system of \(n\) equations in \(n + 1\) unknowns:

\[
a_0c_0 + a_1c_1 + \cdots + a_{n-1}c_{n-1} + a_nc_n = 0
\]

\[
a_0c_1 + a_1c_2 + \cdots + a_{n-1}c_n + a_nc_{n+1} = 0
\]

\[
\vdots
\]

\[
a_0c_{n-1} + a_1c_n + \cdots + a_{n-1}c_{2n-2} + a_nc_{2n-1} = 0. \tag{2.1.7}
\]

This is a homogeneous system, and since the number of equations is less than the number of unknowns it will have a nontrivial solution. This is true even if \(p(x)\) changes sign on the interval of integration. However, without some additional assumption about \(p(x)\) it is impossible to make any definite statement about the number of linearly independent solutions of the system or about the degree of the polynomial (2.1.5).

The determinant of the coefficients of \(a_0, a_1, \ldots, a_{n-1}\) is \(\Delta_{n-1}\). If \(p(x)\) is nonnegative then \(\Delta_{n-1} \neq 0\). If \(a_n\) is fixed then the system will have a unique solution \(a_0, a_1, \ldots, a_{n-1}\). The orthogonality conditions (2.1.6) determine \(P_n(x)\) to within a constant factor; we will choose this factor so that
2.1. General Theorems about Orthogonal Polynomials

\[ a_n > 0 \quad \text{and} \quad \int_a^b p(x) P_n^2(x) \, dx = 1. \]

We can prove the following theorem about the roots of \( P_n(x) \).

Theorem 1. *If the polynomial \( P_n(x) \) is orthogonal on the segment \([a, b]\) to all polynomials of degree less than \( n \), with respect to the nonnegative weight function \( p(x) \), then all the roots of \( P_n(x) \) are real and distinct and lie inside \([a, b]\).*

**Proof.** Let us consider the roots of \( P_n(x) \) which lie inside \([a, b]\) and which have odd multiplicities to be

\[ \xi_1, \xi_2, \ldots, \xi_m. \]

To establish the theorem it suffices to prove that the number of such roots \( m \) is not less than \( n \).

Let us assume the contrary: \( m < n \). We can show that this is inconsistent with the orthogonality assumption. We construct the polynomial of degree \( m \)

\[ Q_m(x) = (x - \xi_1)(x - \xi_2)\cdots(x - \xi_m). \]

\( Q_m(x) \) changes sign at the same points inside \([a, b]\) as does \( P_n(x) \). The product \( P_n(x)Q_m(x) \) does not change sign inside \([a, b]\) and therefore the integral \( \int_a^b p(x) P_n(x)Q_m(x) \, dx \) is different from zero. Because \( Q_m(x) \) has degree \( < n \) this contradicts the assumption that \( P_n(x) \) is orthogonal to each polynomial of degree less than \( n \). This proves the theorem.

The system of polynomials

\[ P_0(x), P_1(x), \ldots, P_n(x), \ldots \quad (2.1.8) \]

is called an orthogonal and normalized system, or, for short, an orthonormal system, if it satisfies the requirements:

1. \( P_n(x) \) is a polynomial of degree \( n \).

2. \[ \int_a^b p(x) P_n(x) P_m(x) \, dx = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n. \end{cases} \]

We will write the \( n^{th} \) degree polynomial of an orthonormal system in the form

\[ P_n(x) = a_n x^n + b_n x^{n-1} + \cdots \quad (2.1.9) \]

We now prove that three consecutive polynomials of an orthonormal system satisfy a recursion relation
\[ xP_n(x) = \frac{a_n}{a_{n+1}} P_{n+1}(x) + \left(\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}\right) P_n(x) + \frac{a_{n-1}}{a_n} P_{n-1}(x). \quad (2.1.10) \]

In fact, \( xP_n(x) \) is a polynomial of degree \( n + 1 \) and can be represented in the form

\[ xP_n(x) = \sum_{k=0}^{n+1} c_{n,k} P_k(x). \]

The coefficients \( c_{n,k} \) are the Fourier coefficients:

\[ c_{n,k} = \int_a^b \rho(x) xP_n(x) P_k(x) \, dx. \]

If \( k < n - 1 \) then \( xP_k(x) \) is a polynomial of degree \( k + 1 < n \) and \( c_{n,k} = 0 \) because \( P_n(x) \) is orthogonal to each polynomial of degree less than \( n \),

\[ xP_n(x) = c_{n,n+1} P_{n+1}(x) + c_{n,n} P_n(x) + c_{n,n-1} P_{n-1}(x). \]

Let us substitute for \( P_s(x) \) \((s = n - 1, n, n + 1)\) its representation \((2.1.9)\).

Comparing the coefficients of the highest degree gives \( c_{n,n+1} = \frac{a_n}{a_{n+1}} \).

Since for any \( n \) and \( k \) we have the relation \( c_{n,k} = c_{k,n} \) then we also have \( c_{n,n-1} = \frac{a_{n-1}}{a_n} \). To obtain \( c_{n,n} \) we can compare the coefficients of \( x^n \); this gives

\[ c_{n,n} = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}. \]

This establishes \((2.1.10)\) for \( n = 1, 2, \ldots \), but that equation is also valid for \( n = 0 \) if we assume \( a_{-1} = 0 \) and \( P_{-1}(x) \equiv 0 \).

To calculate the coefficients in certain approximate integration formulas the Christoffel-Darboux relationship will be useful. To establish this relationship let us, at first, multiply the recursion relation \((2.1.10)\) by \( P_n(t) \).

\[ xP_n(x) P_n(t) = \frac{a_n}{a_{n+1}} P_{n+1}(x) P_n(t) + \]

\[ + \left(\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}\right) P_n(x) P_n(t) + \frac{a_{n-1}}{a_n} P_{n-1}(x) P_n(t) \]

Let us form an equation similar to this by interchanging \( x \) and \( t \) in the above and then subtracting the resulting equation from the above. The middle terms will cancel and we will have
2.2. Jacobi and Legendre Polynomials

\[(x - t) P_n(x) P_n(t) = \frac{a_n}{a_{n+1}} \left[ P_{n+1}(x) P_n(t) - P_n(x) P_{n+1}(t) \right] - \frac{a_{n-1}}{a_n} \left[ P_n(x) P_{n-1}(t) - P_{n-1}(x) P_n(t) \right]. \]

Let us write equations similar to the previous by replacing \( n \) in turn by \( n - 1, n - 2, \ldots, 0 \). If we add all of the resulting equations we obtain the Christoffel-Darboux identity

\[(x - t) \sum_{k=0}^{n} P_k(x) P_k(t) = \frac{a_n}{a_{n+1}} \left[ P_{n+1}(x) P_n(t) - P_n(x) P_{n+1}(t) \right]. \]

2.2. JACOBI AND LEGENDRE POLYNOMIALS

Jacobi polynomials are polynomials which form an orthogonal system on the segment \([-1, +1]\) with respect to the weight function \( p(x) = (1-x)^{\alpha}(1+x)^{\beta} \). They depend on two parameters \( \alpha \) and \( \beta \), and for any values of these parameters we can determine the function

\[ P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}]. \]  

(2.2.1)

This equation is called the Rodriguez formula for the Jacobi polynomial. Usually we take those branches of this many-valued function for which \( \arg(1-x) = \arg(1+x) = 0 \) for \(-1 < x < +1\).

Then (2.2.1) is a polynomial of degree not greater than \( n \):

\[ P_n^{(\alpha, \beta)}(x) = A_n x^n + B_n x^{n-1} + \cdots. \]

This can be seen by differentiating \( \frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}] \) by the rule of Leibnitz and substituting the result in (2.2.1),

\[ P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x \]

\[ \times (\alpha + n) \cdots (\alpha + n - k + 1)(-1)^k (1-x)^n-k \times \]

\[ \times (\beta + n) \cdots (\beta + k + 1)(1+x)^k. \]

The coefficient \( A_n \), of the highest order term \( x^n \), can be found if we take the highest order terms from the factors \((1-x)^{n-k}\) and \((1+x)^k\); these terms are respectively \((-1)^{n-k}x^{n-k}\) and \(x^k\):
A_n x^n = \frac{1}{2^n n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \times \\
\times (a+n) \cdots (a+n-k+1)x^{a-k}(\beta+n) \cdots (\beta+k+1)x^k.

The same result is obtained if we apply the rule of Leibnitz to calculate the derivative of order \( n \) in the function

\[
\frac{1}{2^n n!} x^{-a-\beta} \frac{d^n}{dx^n} (x^{a+n}x^{\beta+n}) = \frac{1}{2^n n!} x^{-a-\beta} \frac{d^n}{dx^n} (x^{a+\beta+2n}).
\]

Therefore

\[
A_n = \frac{1}{2^n n!} (a+\beta+2n)(a+\beta+2n-1) \cdots (a+\beta+n+1) = \\
= \frac{\Gamma(a+\beta+2n+1)}{2^n n! \Gamma(a+\beta+n+1)}. \tag{2.2.2}
\]

We will consider the parameters \( \alpha, \beta \) to be real and \( \alpha, \beta > -1 \) and show that the Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \) \( (n = 0, 1, 2, \ldots) \) form an orthogonal system on the segment \([-1, +1]\) with respect to the weight function \( p(x) = (1-x)^{\alpha}(1+x)^{\beta} \):

\[
I_{n,m} = \int_{-1}^{+1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) dx = 0 \tag{2.2.3}
\]

For convenience we write

\[
y_n = \frac{(-1)^n}{2^n n!} (1-x)^{\alpha+n}(1+x)^{\beta+n}.
\]

Then

\[
P_n^{(\alpha, \beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta} y_n^{(n)}. 
\]

Let us assume \( m \leq n \) and substitute in \( I_{n,m} \) the expression for \( P_n^{(\alpha, \beta)}(x) \) in terms of \( y_n \):

\[
I_{n,m} = \int_{-1}^{+1} y_n^{(n)} P_m^{(\alpha, \beta)}(x) dx.
\]

Integrating by parts gives

\[\text{To construct quadrature formulas for the integration of analytic functions of a complex variable it is necessary to take } \text{Re} \alpha, \text{Re} \beta > -1.\]
2.2. Jacobi and Legendre Polynomials

\[ I_{n,m} = \gamma_n^{(n-1)} P_m^{(\alpha, \beta)}(x) \left|_{-1}^{+1} \int_{-1}^{+1} y^{(n-1)}[P_m^{(\alpha, \beta)}(x)]' dx = \right. \]

\[ = - \int_{-1}^{+1} y^{(n-1)}[P_m^{(\alpha, \beta)}(x)]' dx. \]

The term which does not involve the integral vanishes because \( \alpha, \beta > -1. \)

Integrating by parts \( n \) times gives

\[ I_{n,m} = (-1)^n \int_{-1}^{+1} y_n[P_m^{(\alpha, \beta)}(x)]^{(n)} dx. \quad (2.2.4) \]

For \( m < n \) we have \( [P_m^{(\alpha, \beta)}(x)]^{(n)} = 0, \) and consequently \( I_{n,m} = 0, \) which proves orthogonality for two Jacobi polynomials of different degrees.

For \( m = n \) equation (2.2.3) gives

\[ I_{n,n} = \int_{-1}^{+1} (1 - x)^{\alpha}(1 + x)^{\beta}[P_n^{(\alpha, \beta)}(x)]^2 dx = \]

\[ = (-1)^n \int_{-1}^{+1} y_n n! A_n dx = \frac{n!}{2^n n!} A_n \int_{-1}^{+1} (1 - x)^{a+n}(1 + x)^{\beta+n} dx. \]

The last integral reduces to the Euler integral of the first kind. Let us substitute \( x = 2t - 1: \)

\[ \int_{-1}^{+1} (1 - x)^{a+n}(1 + x)^{\beta+n} dx = 2^{a+\beta+2n+1} \int_{0}^{1} t^{\beta+n}(1 - t)^{a+n} dt = \]

\[ = 2^{a+\beta+2n+1} B(\alpha + n + 1, \beta + n + 1). \]

Since

\[ B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)} \]

then

\[ I_{n,n} = \frac{2^{a+\beta+1} \Gamma(\alpha + n + 1)\Gamma(\beta + n + 1)}{\Gamma(\alpha + \beta + 2n + 1) n! \Gamma(\alpha + \beta + n + 1)}. \quad (2.2.5) \]

If \( n = 0 \) and \( \alpha + \beta + 1 = 0, \) then

\[ I_{0,0} = \Gamma(\alpha + 1)\Gamma(\beta + 1). \]

From (2.2.4) and (2.2.5) we see that an orthonormal system of polynomials on \([-1, +1]\) with respect to the weight function \((1 - x)^{\alpha}(1 + x)^{\beta}\) is given by
The leading coefficients of these are

\[ a_n = \frac{1}{\sqrt{T_{n,n}}} A_n \] (2.2.7)

Legendre polynomials are a special case of the Jacobi polynomials. They are the Jacobi polynomials for \( \alpha = 0, \beta = 0 \). The Rodriguez formula for Legendre polynomials is

\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \] (2.2.8)

From this equation it is easy to find the expansion for \( P_n(x) \) in powers of \( x \)

\[ P_n(x) = \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n - 2)!}{2^n (n - 1)! (n - 2)!} x^{n-2} + \cdots. \]

The Legendre polynomials are orthogonal on \([-1, +1]\) with respect to the constant weight function \( p(x) = 1 \). Equations (2.2.4) and (2.2.5) have the form:

\[ \int_{-1}^{+1} P_n(x) P_m(x) dx = \begin{cases} 0 & \text{for } m \neq n \\ \frac{2}{2n + 1} & \text{for } m = n. \end{cases} \] (2.2.9)

An orthonormal system on \([-1, +1]\) with constant weight function is given by the polynomials

\[ p_n(x) = \sqrt{\frac{2n + 1}{2}} P_n(x). \] (2.2.10)

The leading coefficient in \( p_n(x) \) is

\[ a_n = \sqrt{\frac{2n + 1}{2}} \frac{(2n)!}{2^n (n!)^2}. \] (2.2.11)

2.3. CHEBYSHEV POLYNOMIALS

The Chebyshev polynomials of the first kind can be defined by

\[ T_n(x) = \cos(n \arccos x) \quad (n = 0, 1, 2, \ldots). \] (2.3.1)

These polynomials are an orthogonal system on the segment \([-1, +1]\) with
respect to the weight function \( p(x) = \frac{1}{\sqrt{1 - x^2}} \). First of all, let us show that \( T_n(x) \) is indeed a polynomial of degree \( n \) in \( x \) and that the coefficient of the highest degree term is \( 2^{n-1} \):

\[
T_n(x) = 2^{n-1} x^n + \cdots. \tag{2.3.2}
\]

We use the elementary trigonometric identity

\[
\cos(n + 1)\theta + \cos(n - 1)\theta = 2\cos \theta \cos n\theta.
\]

If we put \( \theta = \arccos x \), we obtain the following recursion relation for \( T_n(x) \):

\[
T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).
\]

It is evident that equation (2.3.2) is valid for \( T_0(x) = 1 \) and \( T_1(x) = x \). By the recursion relation we can see that it is true for all \( n \).

We will now establish the orthogonality property for the polynomials \( T_n(x) \):

\[
\int_{-1}^{1} T_n(x)T_m(x) dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi/2 & \text{for } m = n. \end{cases} \tag{2.3.3}
\]

This is equivalent to the statement that the polynomials \( T_n(x) \) \( (n = 0, 1, 2, \ldots) \) form an orthogonal system on the segment \([-1, +1]\) with respect to the weight function \( p(x) = \frac{1}{\sqrt{1 - x^2}} \).

Let us change the variable of integration in \( I_{n,m} \) by substituting \( x = \cos \theta, \ \theta = \arccos x \). As \( x \) varies from \(-1\) to \(+1\) we can take \( \theta \) to vary from \( \pi \) to \( 0 \). Since \( T_n(x) = \cos n\theta \), \( T_m(x) = \cos m\theta \) and \( dx = -\sin \theta d\theta \), then

\[
I_{n,m} = \int_{0}^{\pi} \cos n\theta \cos m\theta d\theta = \begin{cases} 0 & \text{for } m \neq n \\ \pi/2 & \text{for } m = n, \end{cases}
\]

which establishes (2.3.3).

The weight function \( p(x) = (1 - x^2)^{-\frac{1}{2}} \) \((-1 \leq x \leq +1)\) is a special case of the Jacobi weight function \((1 - x)^{\alpha} (1 + x)^{\beta}\) for \( \alpha = \beta = -1/2 \). For a given weight function the polynomials of the corresponding orthogonal system are defined to within a constant factor. Therefore the Jacobi polynomials \( P_n^{(-1/2, -1/2)}(x) \) can differ from \( T_n(x) \) by only a constant factor

\[
P_n^{(-1/2, -1/2)}(x) = c_n T_n(x). \tag{2.3.4}
\]
In order to find \( c_n \) it is sufficient to compare the leading coefficients

\[
\frac{\Gamma(2n)}{2^n n! \Gamma(n)} = c_n 2^{n-1}
\]

\[
c_n = \frac{\Gamma(2n)}{2^{2n-1} \Gamma(n) \Gamma(n + 1)}.
\]

The polynomials \( T_n(x) \) were introduced by P. L. Chebyshev in connection with the solution of the following problem:

Among all the polynomials of degree \( n \) which have leading coefficient equal to unity

\[
P(x) = x^n + c_{n-1} x^{n-1} + \cdots
\]

determine those which deviate least from zero in absolute value on the segment \([-1, +1]\). That is, determine the polynomials for which

\[
\max_{-1 \leq x \leq 1} |P(x)|
\]

has the least possible value.

We will show that the polynomials

\[
T_n^*(x) = 2^{-n+1} T_n(x) = 2^{-n+1} \cos(n \arccos x)
\]

have this property. Indeed, \( \max_{[-1, +1]} |T_n^*(x)| = 2^{-n+1} \) and we also have

\[
T_n^* \left( \cos \frac{m\pi}{n} \right) = 2^{-n+1} (-1)^m \quad (m = 0, 1, \ldots, n).
\]

If there would be a polynomial \( P(x) \) which would satisfy the condition \( |P(x)| < 2^{-n+1} (-1 \leq x \leq +1) \), then the difference \( R(x) = T_n^*(x) - P(x) \) would be a polynomial of degree less than \( n \), for which

\[
(-1)^m R \left( \cos \frac{m\pi}{n} \right) > 0 \quad (m = 0, 1, \ldots, n).
\]

Then \( R(x) \) would have at least \( n \) roots in the interval \([-1, +1]\) which is impossible, because its degree is less than \( n \).

A similar argument establishes the uniqueness of the polynomials of least deviation. Let \( P(x) \) be an arbitrary polynomial of the indicated form for which

\[
\max_{[-1, +1]} |P(x)| = \max_{[-1, +1]} |T_n^*(x)| = 2^{-n+1}.
\]

Hence \( S(x) = P(x) - T_n^*(x) \) will have degree less than \( n \). At the points

\[
x_m = \cos \frac{m\pi}{n}
\]

\[
S(x_m) = (-1)^m 2^{-m+1} - P(x_m)
\]
and since \( |P(x_m)| \leq 2^{-n+1}, \)
\[ (-1)^m S(x_m) \geq 0 \quad (m = 0, 1, \ldots, n). \]
Hence it follows that \( S(x) \) has no fewer than \( n \) zeros, either distinct or coincident. But because the degree of \( S(x) \) is less than \( n \) then \( S(x) \) is identically zero and \( P(x) = T_n^*(x) \).

The Chebyshev polynomials of the second kind are defined as the polynomials
\[
U_n(x) = \frac{\sin \left[ (n + 1) \arccos x \right]}{\sqrt{1 - x^2}} \quad (n = 0, 1, 2, \ldots). \tag{2.3.5}
\]
It is possible to show that the functions \( U_n(x) \) are indeed polynomials of degree \( n \), having leading coefficient \( 2^n \). To do this we use the trigonometric identity
\[
\sin((n + 2)\theta) + \sin(n\theta) = 2\cos \theta \sin((n + 1)\theta).
\]
If we put \( \cos \theta = x, \theta = \arccos x \) and divide both sides by \( \sqrt{1 - x^2} \), then we obtain the recursion formula for \( U_n(x) \)
\[
U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x). \tag{2.3.6}
\]
We note that \( U_0(x) = 1 \) and \( U_1(x) = 2x \) have the indicated form. By means of induction it is easy to show from (2.3.6) that \( U_n(x) \) is indeed a polynomial of the form \( U_n(x) = 2^n x^n + \cdots \).

The polynomials \( U_n(x) \) satisfy the relationship
\[
I_{n,m} = \int_{-1}^{1} U_n(x)U_m(x)\sqrt{1 - x^2} \, dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi/2 & \text{for } m = n. \end{cases} \tag{2.3.7}
\]
In other words, the \( U_n(x) \) \((n = 0, 1, 2, \ldots)\) form an orthogonal system on the segment \([-1, +1]\) with respect to the weight function \( p(x) = \sqrt{1 - x^2} \). To prove this we change the variable of integration in the integral
\[
I_{n,m} = \int_{-1}^{1} \frac{\sin[(n + 1)\arccos x] \sin[(m + 1)\arccos x]}{\sqrt{1 - x^2}} \, dx
\]
by substituting \( x = \cos \theta \); then it changes to the form
\[
I_{n,m} = \int_{0}^{\pi} \sin(n + 1)\theta \sin(m + 1)\theta \, d\theta
\]
and equation (2.3.7) is verified without difficulty.

The weight function \( p(x) = \sqrt{1 - x^2} \) is also a Jacobi weight function for \( \alpha = \beta = 1/2 \). Therefore the polynomials \( U_n(x) \) can only differ
by a constant factor from the Jacobi polynomials $P_n^{(\frac{1}{2}, \frac{1}{2})}(x)$

$$P_n^{(\frac{1}{2}, \frac{1}{2})}(x) = e_n U_n(x).$$

Comparison of the leading coefficients gives

$$e_n = \frac{(2n + 1)!}{2^{2n} n!(n + 1)!}.$$

The polynomials $U_n(x)$ possess the following minimal property:

Among all polynomials $P(x)$ of degree $n$ with leading coefficient equal to unity, $2^{-n} U_n(x)$ minimizes the value of the integral

$$\int_{-1}^{1} |P(x)| dx.$$ \hspace{1cm} (2.3.8)

In order to prove this it will be necessary to establish certain auxiliary results.

1. We will need a trigonometric series for the function $\sin x \text{sign} \sin px$, where $p$ is an integer. In the theory of Fourier series the following expansion is known

$$\text{sign} \sin x = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin (2k + 1)x}{2k + 1}.$$ 

Hence we see that

$$\text{sign} \sin px = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin (2k + 1)px}{2k + 1}. \hspace{1cm} (2.3.9)$$

If this equation is multiplied by $\sin x$, then using the relation

$$2 \sin x \sin (2k + 1)px = \cos [(2k + 1)p - 1]x - \cos [(2k + 1)p + 1]x$$

we immediately obtain the desired trigonometric series

$$\sin x \text{sign} \sin px =$$

$$= \frac{2}{\pi} \sum_{k=0}^{\infty} (2k + 1)^{-1} \{\cos[(2k + 1)p - 1]x - \cos[(2k + 1)p + 1]x\}.$$

---

3The function $\text{sign} x$ is defined by

$$\text{sign} x = \begin{cases} 
-1 & \text{for } x < 0 \\
0 & \text{for } x = 0 \\
+1 & \text{for } x > 0 
\end{cases}$$
2. If \( n \) is a positive integer then for \( r = 0, 1, \ldots, n - 1 \) the following equation is satisfied:

\[
\int_{-1}^{1} x^r \text{sign } U_n(x) \, dx = 0. \tag{2.3.10}
\]

If we substitute \( x = \cos \theta \) in (2.3.10) we obtain

\[
\int_0^{\pi} \cos^r \theta \sin \theta \text{sign } \sin (n + 1) \theta \, d\theta = 0.
\]

The powers \( \cos^r \theta \) \( (r = 0, 1, \ldots, n - 1) \) can be linearly expressed in terms of \( \cos m \theta \) \( (m = 0, 1, \ldots, n - 1) \) and conversely. Therefore the last equation is equivalent to

\[
\int_0^{\pi} \cos m \theta \sin \theta \text{sign } \sin (n + 1) \theta \, d\theta = 0 \quad (m = 0, 1, \ldots, n - 1). \tag{2.3.11}
\]

Because the function under the integral sign is even, this is equivalent to

\[
\int_{-\pi}^{\pi} \cos m \theta \sin \theta \text{sign } \sin (n + 1) \theta \, d\theta = 0.
\]

The trigonometric series for \( \sin \theta \text{sign } \sin (n + 1) \theta \) is given by (2.3.9) for \( p = n + 1 \).

The smallest frequency in the terms of the series (2.3.9) is in the term corresponding to \( k = 0 \); this frequency is \( (n + 1) - 1 = n \). Therefore, for \( m = 0, 1, \ldots, n - 1 \), equation (2.3.11) is known to be satisfied.

Using (2.3.10) it is easy to prove the above stated minimal property for \( U_n(x) \). For simplicity we denote \( 2^{-n}U_n(x) = P(x) \) and let us take any polynomial \( P^*(x) \) of degree \( n \) which has leading coefficient equal to unity:

\[
\int_{-1}^{1} |P(x)| \, dx = \int_{-1}^{1} P(x) \text{sign } U_n(x) \, dx =
\]

\[
= \int_{-1}^{1} P^*(x) \text{sign } U_n(x) \, dx +
\]

\[
+ \int_{-1}^{1} [P(x) - P^*(x)] \text{sign } U_n(x) \, dx.
\]

The last of these integrals is equal to zero by (2.3.10) and by the fact that the difference \( P(x) - P^*(x) \) is a polynomial of degree less than \( n \).
Also
\[
\int_{-1}^{1} P^*(x) \text{ sign } U_n(x) \, dx \leq \int_{-1}^{1} P^*(x) \text{ sign } P^*(x) \, dx = \int_{-1}^{1} |P^*(x)| \, dx.
\]
Consequently
\[
\int_{-1}^{1} |P(x)| \, dx \leq \int_{-1}^{1} |P^*(x)| \, dx. \tag{2.3.12}
\]
This proves the assertion. We make two more remarks. From the above argument we see that equality is possible in (2.3.12) only when
\[
\text{sign } P^*(x) = \text{sign } U_n(x) \quad \text{for } -1 < x < 1.
\]
The polynomials
\[
U_n(x) = \sin \left[ \frac{(n + 1) \arccos x}{\sqrt{1 - x^2}} \right] = \frac{\sin (n + 1) \theta}{\sin \theta}
\]
have \( n \) roots \( x_k = \cos \frac{k\pi}{n + 1} \) \((k = 1, 2, \ldots, n)\) in the interval \(-1 < x < 1\).

If \( \text{sign } P^*(x) = \text{sign } U_n(x) \), \(-1 < x < +1\), then the points \( x_k \) must also be roots of \( P^*(x) \). The polynomial \( P^*(x) \) has degree \( n \) and therefore the \( x_k \) are roots of multiplicity one and \( P^*(x) \) has no other roots. Since \( P^*(x) \) and \( P(x) = 2^{-n}U_n(x) \) have identical leading coefficients we must have
\[
P^*(x) = P(x).
\]
Equality in (2.3.12) is possible only when \( P^*(x) = P(x) = 2^{-n}U_n(x) \).

Let us now calculate the minimal value of the integral (2.3.8):
\[
2^{-n} \int_{-1}^{1} |U_n(x)| \, dx = 2^{-n} \int_{0}^{\pi} |\sin (n + 1) \theta| \, d\theta =
\]
\[
= 2^{-n}(n + 1) \int_{0}^{\pi} \sin (n + 1) \theta \, d\theta =
\]
\[
= -2^{-n}(n + 1) \left[ \frac{\cos (n + 1) \theta}{n + 1} \right]_{0}^{\pi} = 2^{-n+1}.
\]
Thus we have proven the theorem:

**Theorem 2.** For any polynomial of degree \( n \) which has leading coefficient equal to unity
2.4. Chebyshev-Hermite Polynomials

The Chebyshev-Hermite polynomials are orthogonal on the entire line \(-\infty < x < \infty\) with respect to the weight function \(p(x) = e^{-x^2}\). These polynomials can be defined by the formula

\[
H_n(x) = (-1)^n e^{-x^2} \frac{d^n}{dx^n} e^{-x^2}.
\]  

(2.4.1)

Let us write \(\phi = e^{-x^2}\). Then \(\phi^{(n)} = (-1)^n e^{-x^2} H_n(x)\). Differentiating gives

\[
\phi^{(n+1)} = (-1)^n [-2x H_n(x) + H'_n(x)] e^{-x^2},
\]

and since \(\phi^{(n+1)} = (-1)^n e^{-x^2} H_{n+1}(x)\), then

\[
H_{n+1}(x) = 2x H_n(x) - H'_n(x).
\]  

(2.4.2)

Hence, from \(H_0(x) = 1\), it is easy to obtain, by induction, that \(H_n(x)\) is a polynomial of degree \(n\) of the form

\[
H_n(x) = 2^n x^n + \ldots.
\]

The polynomials \(H_n(x)\) satisfy the following relationship:

\[
\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \, dx = \begin{cases} 
0 & \text{for } m \neq n \\
2^n \sqrt{\pi} n! & \text{for } m = n.
\end{cases}
\]

In other words the \(H_n(x)\) \((n = 0, 1, 2, \ldots)\) form an orthogonal system on the entire line \((-\infty, +\infty)\) with respect to the weight function \(e^{-x^2}\). To prove this let us suppose \(m \leq n\):

\footnote{Sometimes other Chebyshev-Hermite polynomials are used:

\[
H_n^*(x) = (-1)^n \frac{x^{\frac{1}{2}}}{2^n} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.
\]

These are related to the polynomials (2.4.1) by \(H_n^*(x) = 2^{-\frac{1}{2}} H_n(x)\).}
\[ I = \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \, dx = (-1)^n \int_{-\infty}^{\infty} \phi^{(n)} H_m(x) \, dx = \]
\[ = (-1)^n \phi^{(n-1)} H_m(x) \bigg|_{-\infty}^{\infty} + (-1)^{n-1} \int_{-\infty}^{\infty} \phi^{(n-1)} H'_m(x) \, dx = \]
\[ = (-1)^{n-1} \int_{-\infty}^{\infty} \phi^{(n-1)} H'_m(x) \, dx = \cdots = \int_{-\infty}^{\infty} \phi H_m^{(n)}(x) \, dx. \]

For \( m < n, \) \( H_m^{(n)}(x) = 0 \) and thus \( I = 0. \) If \( m = n \) then
\[ I = 2^n n! \int_{-\infty}^{\infty} \phi \, dx = 2^n n! \int_{-\infty}^{\infty} e^{-x^2} \, dx = 2^n n! \sqrt{\pi}. \]

An orthonormal system is formed by the polynomials
\[ h_n(x) = \frac{H_n(x)}{ \frac{1}{2^n} \frac{1}{(n!)^2 \pi^{1/4}}}. \]  
(2.4.3)

The leading coefficients of these are
\[ a_n = 2^{\frac{1}{2}} (n!) \frac{-1}{\pi} \frac{-1}{4}. \]  
(2.4.4)

2.5. CHEBYSHEV-LAGUERRE POLYNOMIALS

The Chebyshev-Laguerre polynomials are orthogonal on the half-line \( 0 \leq x < \infty \) with respect to the weight function \( p(x) = x^a e^{-x}. \) Let \( a \) be any number. We choose the branch of the many-valued function \( x^a \) defined by the condition \( \arg x = 0, \) for \( x > 0. \) We can define the Chebyshev-Laguerre polynomials by the formula
\[ L_n^{(a)}(x) = (-1)^n x^{-a} e^x \frac{d^n}{dx^n} (x^{a+n} e^{-x}). \]  
(2.5.1)

Differentiating by the rule of Leibnitz we find the expansion of \( L_n^{(a)}(x) \) in powers of \( x \) to be
\[ L_n^{(a)}(x) = x^n - \frac{n}{1!} (n + a) x^{n-1} + \]
\[ + \frac{n(n-1)}{2!} (n + a) (n + a - 1) x^{n-2} - \cdots. \]  
(2.5.2)

We will consider \( a \) to be a real number \( a > -1. \) We can show that \( L_n^{(a)}(x) \) possesses the following property:
2.5. Chebyshev-Laguerre Polynomials

\[ l = \int_0^\infty x^a e^{-x} L_n^{(a)}(x) L_m^{(a)}(x) \, dx = \begin{cases} 0 & \text{for } m \neq n \\ n! \Gamma(n + \alpha + 1) & \text{for } m = n. \end{cases} \tag{2.5.3} \]

Let us denote, for simplicity, \( x^a + e^{-x} = \phi_n \). Then

\[ L_n^{(a)}(x) = (-1)^n x^{-a} e^x \phi_n^{(n)}. \]

Consider \( m \leq n \) and substitute, in \( l \), for the polynomial \( L_n^{(a)}(x) \) its expression in terms of \( \phi_n^{(n)} \):

\[ l = (-1)^n \int_0^\infty \phi_n^{(n)} L_m^{(a)}(x) \, dx = \]

\[ = (-1)^n \phi_n^{(n-1)} L_m^{(a)}(x) \bigg|_0^\infty + (-1)^{n-1} \int_0^\infty \phi_n^{(n-1)} [L_m^{(a)}(x)]' \, dx = \]

\[ = (-1)^{n-1} \int_0^\infty \phi_n^{(n-1)} [L_m^{(a)}(x)]' \, dx. \]

The term which does not involve the integral vanishes because \( \alpha > -1 \).

Carrying out the integration by parts \( n \) times we obtain

\[ l = \int_0^\infty \phi_n [L_m^{(a)}(x)]^{(n)} \, dx. \]

For \( m < n \), we have \([L_m^{(a)}(x)]^{(n)} = 0 \) and therefore \( l = 0 \). When \( m = n \),

\[ l = n! \int_0^\infty \phi_n \, dx = n! \int_0^\infty x^a + e^{-x} \, dx = n! \Gamma(\alpha + n + 1). \]

The orthonormal Chebyshev-Laguerre polynomials are

\[ l_n^{(a)}(x) = \frac{L_n^{(a)}(x)}{[n! \Gamma(\alpha + n + 1)]^{\frac{1}{2}}}. \tag{2.5.4} \]

The coefficients of \( x^n \) in these are

\[ a_n = [n! \Gamma(\alpha + n + 1)]^{-\frac{1}{2}}. \tag{2.5.5} \]

REFERENCES


3.1. FINITE DIFFERENCES AND DIVIDED DIFFERENCES

The theory of approximate integration uses in many ways results from the theory of interpolation which in turn makes wide use of finite differences. Here we develop only the simplest results from the theory of differences.

Suppose that we know the values of \( f(x) \) at the following equally spaced points of interval \( h \):

\[
x_k = x_0 + kh \quad (k = 0, 1, 2, \ldots)
\]

\[
f_0 = f(x_0), \quad f_1 = f(x_0 + h), \ldots, \quad f_k = f(x_0 + kh), \ldots
\]

We call the quantities

\[
\Delta f_0 = f_1 - f_0, \quad \Delta f_1 = f_2 - f_1, \ldots, \quad \Delta f_n = f_{n+1} - f_n, \ldots
\]

finite differences of the first order, and the quantities

\[
\Delta^2 f_0 = \Delta f_1 - \Delta f_0, \quad \Delta^2 f_1 = \Delta f_2 - \Delta f_1, \ldots, \quad \Delta^2 f_n = \Delta f_{n+1} - \Delta f_n, \ldots
\]

are called differences of the second order, and so forth.

Differences of order \( n \) are defined from differences of the preceding order by

\[
\Delta^n f_0 = \Delta^{n-1} f_1 - \Delta^{n-1} f_0, \quad \Delta^n f_1 = \Delta^{n-1} f_2 - \Delta^{n-1} f_1, \ldots
\]

This provides a recursive definition of finite differences of all orders. We can find an expression for differences of any order in terms of the values \( f_k \) of the function

\[
\Delta^n f = f_n - \frac{n}{1!} f_{n-1} + \frac{n(n - 1)}{2!} f_{n-2} - \frac{n(n - 1)(n - 2)}{3!} f_{n-3} + \cdots + (-1)^n f_0.
\]

(3.1.1)
This equation is obviously true for \( n = 1 \), and it can easily be proved for any \( n \) by induction. If we introduce the operator which increases the argument by step \( h \)

\[
Ef(x) = f(x + h) \quad \text{or} \quad Ef_k = f_{k+1}
\]

then (3.1.1) can be written in the symbolic form

\[
\Delta^n f_0 = (E - 1)^n f_0. \quad \text{(3.1.2)}
\]

It is also useful to note that any value of the function \( f_n \) can be expressed in terms of \( f_0 \) and the differences \( \Delta f_0, \Delta^2 f_0, \ldots \) by the relationship

\[
f_n = f_0 + \frac{n}{1!} \Delta f_0 + \frac{n(n-1)}{2!} \Delta^2 f_0 + \cdots + \Delta^n f_0. \quad \text{(3.1.3)}
\]

This equation is true for \( n = 1 \) since \( f_1 - f_0 = \Delta f_0 \), and it can be established for any \( n \) by induction. In symbolic form (3.1.3) is

\[
f_n = (1 + \Delta)^n f_0. \quad \text{(3.1.4)}
\]

In interpolation problems it is not always possible to use equally spaced values of the function. For example, one cannot always obtain astronomical observations at equally spaced intervals of time.

For unequally spaced values of the argument finite differences are replaced by quantities which are usually called divided differences or difference ratios.

Let \( x_0, x_1, x_2, \ldots, x_n, \ldots \) be arbitrary values of the argument. Divided differences of the first order are defined

\[
f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \ldots
\]

The quantities

\[
f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}
\]

\[
f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}
\]

are divided differences of the second order; and

\[
f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}
\]

is a divided difference of third order, and so forth.
3.1. Finite Differences and Divided Differences

The function \( f(x_0, x_1, \ldots, x_n) \) is a linear function of \( f(x_0), \ldots, f(x_n) \) and it can be shown that

\[
f(x_0, \ldots, x_n) = \sum_{\nu=0}^{n} \frac{f(x_{\nu})}{(x_{\nu} - x_0) \cdots (x_{\nu} - x_{\nu-1}) (x_{\nu} - x_{\nu+1}) \cdots (x_{\nu} - x_n)}.
\]

This equation is true for \( n = 1 \) since

\[
f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0)}{x_1 - x_0} + \frac{f(x_1)}{x_1 - x_0}.
\]

Assuming that (3.1.5) is true for divided differences of order \( n \) we can verify it for order \( n + 1 \):

\[
f(x_0, x_1, \ldots, x_{n+1}) = (x_{n+1} - x_0)^{-1} [f(x_1, x_2, \ldots, x_{n+1}) - f(x_0, x_1, \ldots, x_n)] =
\]

\[
= (x_{n+1} - x_0)^{-1} \left[ \sum_{\nu=1}^{n+1} \frac{f(x_{\nu})}{(x_{\nu} - x_1) \cdots (x_{\nu} - x_{n+1})} - \sum_{\nu=0}^{n} \frac{f(x_{\nu})}{(x_{\nu} - x_0) \cdots (x_{\nu} - x_n)} \right] =
\]

\[
= \sum_{\nu=0}^{n+1} \frac{f(x_{\nu})}{(x_{\nu} - x_0) \cdots (x_{\nu} - x_{n+1})}.
\]

Equation (3.1.5) can be written in a shorter form by introducing the polynomial

\[
\omega(x) = (x - x_0) (x - x_1) \cdots (x - x_n).
\]

Then

\[
f(x_0, x_1, \ldots, x_n) = \sum_{\nu=0}^{n} \frac{f(x_{\nu})}{\omega'(x_{\nu})}.
\]

A permutation of \( x_0, x_1, \ldots, x_n \) in the right side of (3.1.5) only changes the order of the summands, and therefore \( f(x_0, x_1, \ldots, x_n) \) is a symmetric function of its arguments \( x_0, x_1, \ldots, x_n \).

By means of induction we can also verify the following formula which expresses any value of the function \( f(x_n) \) in terms of \( f(x_0) \) and the divided differences \( f(x_0, x_1, \ldots, x_k) \) \( (k = 1, 2, \ldots, n) \):
\[ f(x_n) = f(x_0) + (x_n - x_0) f(x_0, x_1) + \\
\quad + (x_n - x_0) (x_n - x_1) f(x_0, x_1, x_2) + \cdots \\
\quad \cdots + (x_n - x_0) (x_n - x_1) \cdots (x_n - x_{n-1}) f(x_0, x_1, \ldots, x_n). \]  
(3.1.7)

When the values of the argument are equally spaced the divided differences can be simply expressed in terms of finite differences:

\[ f(x_0, x_0 + h) = \frac{f(x_0 + h) - f(x_0)}{h} = \Delta f_0 \]

\[ f(x_0, x_0 + h, x_0 + 2h) = \frac{f(x_0 + h, x_0 + 2h) - f(x_0, x_0 + h)}{2h} = \Delta^2 f_0 \]

\[ f(x_0, x_0 + h, \ldots, x_0 + nh) = \frac{\Delta^n f_0}{n! h^n}. \]

It is often useful in applications to be able to relate finite differences and divided differences to derivatives. We assume that the points \( x_0, x_1, \ldots, x_n \) lie in the segment \([a, b]\).

**Theorem 1.** If \( f(x) \) has a continuous derivative of order \( n \) on \([a, b]\) then the following equation is valid:

\[ f(x_0, \ldots, x_n) = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} f^{(n)} \left( x_0 + \sum_{\nu=1}^n t_\nu (x_\nu - x_{\nu-1}) \right) \times \\
\times dt_n \cdots dt_2 dt_1 \]  
(3.1.9)

**Proof:** This equation is easily verified for \( n = 1 \):

\[ \int_0^1 f'' (x_0 + t_1 (x_1 - x_0)) dt_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f(x_0, x_1). \]

Assuming that (3.1.9) is true for divided differences of order \( n - 1 \) we can show that it is true for differences of order \( n \). Denoting the integral on the right side of (3.1.9) by \( I(x_0, x_1, \ldots, x_n) \), and carrying out the integration with respect to \( t_n \) gives:

\[ I(x_0, x_1, \ldots, x_n) = \\
= \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-2}} (x_n - x_{n-1})^{-1} \{ f^{(n-1)} (x_0 + t_1 (x_1 - x_0) + \\
\quad + \cdots + t_{n-2} (x_n - x_{n-2}) - f^{(n-1)} (x_0 + t_1 (x_1 - x_0) + \\
\quad + \cdots + t_{n-2} (x_n - x_{n-2}) ) \} dt_{n-1} \cdots dt_2 dt_1 = \\
\]
### 3.1. Finite Differences and Divided Differences

\[
(x_n - x_{n-1})^{-1} \left[ f(x_0, \ldots, x_{n-2}, x_n) - f(x_0, \ldots, x_{n-2}, x_{n-1}) \right] = \\
f(x_{n-1}, x_0, x_1, \ldots, x_{n-2}, x_n) = \\
f(x_0, x_1, \ldots, x_{n-2}, x_{n-1}, x_n).
\]

This proves the theorem.

As a corollary to (3.1.9) we can obtain a simpler relationship between \( f(x_0, x_1, \ldots, x_n) \) and \( f^{(n)}(x) \). The region of integration in (3.1.9) is a simplex in the \( n \)-dimensional space \((t_1, t_2, \ldots, t_n)\). This is the simplex defined by the inequalities

\[
0 < t_n < t_{n-1} < \cdots < t_1 < 1. \tag{3.1.10}
\]

The volume of this simplex is

\[
\int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n \cdots dt_2 dt_1 = \frac{1}{n!}.
\]

Consider the quantity

\[
\xi = x_0 + \sum_{\nu=1}^{n} t_\nu (x_\nu - x_{\nu-1}) = \\
= (1 - t_1) x_0 + (t_1 - t_2) x_1 + \cdots + (t_{n-1} - t_n) x_{n-1} + t_n x_n.
\]

From (3.1.10) we see that the multipliers of all the \( x_k \) are nonnegative. Since the sum of these multipliers is unity, \( \xi \) is a weighted average of the abscissas \( x_k \) \((k = 0, 1, \ldots, n)\) and therefore \( \xi \) will certainly lie in the segment \([a, b]\). A point in the interior of the simplex (3.1.10) thus corresponds to an interior point \( \xi \) of \([a, b]\).

Applying the mean value theorem to the integral (3.1.9) gives:

**Theorem 2.** If \( f(x) \) has a continuous derivative of order \( n \) on \([a, b]\) then there exists an interior point \( \xi \) of \([a, b]\) for which

\[
f(x_0, x_1, \ldots, x_n) = \frac{f^{(n)}(\xi)}{n!}.
\]

The relationship between finite differences and derivatives then follows from (3.1.8), (3.1.9), and (3.1.11):

\[
\Delta^n f_0 = n! h^n \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} f^{(n)}(x + h \sum_{\nu=1}^{n} t_\nu) \times \\
\times dt_n \cdots dt_2 dt_1 = h^n f^{(n)}(\xi) \quad x_0 < \xi < x_0 + nh.
\]

Thus if we divide the interval size \( h \) by \( \lambda \), then the finite difference \( \Delta^n f_0 \) will be divided by about \( \lambda^n \).
3.2. THE INTERPOLATING POLYNOMIAL AND ITS REMAINDER

Suppose that for \( n + 1 \) arbitrary points \( x_0, x_1, \ldots, x_n \), which we will call the nodes (or points) of interpolation, we are given the values of the function \( f(x_k) \). We wish to construct a polynomial of degree \( \leq n \)

\[
P_n(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n = \sum_{\nu=1}^{n} a_{\nu} x^{n-\nu}
\]  

(3.2.1)

which has the same value as \( f(x) \) at the nodes \( x_k \):

\[
P_n(x_k) = f(x_k) \quad (k = 0, 1, \ldots, n).
\]  

(3.2.2)

To find the coefficients \( a_{\nu} \) of this polynomial we must solve the system of \( n + 1 \) linear equations

\[
\sum_{\nu=0}^{n} a_{\nu} x_k^{n-\nu} = f(x_k) \quad (k = 0, 1, \ldots, n).
\]

The determinant of this system is the Vandermonde determinant

\[
\begin{vmatrix}
  x_0^n & x_0^{n-1} & \cdots & 1 \\
  x_1^n & x_1^{n-1} & \cdots & 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  x_n^n & x_n^{n-1} & \cdots & 1 \\
\end{vmatrix}
\]

which is different from zero since no two of the nodes \( x_k \) coincide. Therefore for any values \( f(x_k) \) we can construct one and only one polynomial \( P_n(x) \) which satisfies (3.2.2).

The polynomial \( P_n(x) \) can be represented in different forms; the most convenient form depends on how it is to be used. Below we derive two of the most useful representations for \( P_n(x) \).

From the nodes \( x_k \) we construct the auxiliary polynomial \( \omega_k(x) \) defined by

\[
\omega_k(x_i) = \begin{cases} 
0 & \text{for } i \neq k \\
1 & \text{for } i = k.
\end{cases}
\]  

(3.2.3)

It is easy to see that this polynomial can be written in the form

\[
\omega_k(x) = \frac{(x - x_0) \cdots (x - x_{k-1}) (x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_{k-1}) (x_k - x_{k+1}) \cdots (x_k - x_n)}.
\]

or in terms of the polynomial \( \omega(x) = (x - x_0) (x - x_1) \cdots (x - x_n) \)
3.2. The Interpolating Polynomial and its Remainder

\[ \omega_k(x) = \frac{\omega(x)}{(x-x_k)\omega'(x_k)}. \]  

(3.2.4)

The polynomial \( \omega_k(x) \) is called the Lagrangian coefficient corresponding to the node \( x_k \).

The interpolating polynomial \( P_n(x) \) can now be written in the form

\[ P_n(x) = \sum_{k=0}^{n} \frac{\omega(x)}{(x-x_k)\omega'(x_k)} f(x_k) \]  

(3.2.5)

which is due to Lagrange.

Since the \( \omega_k(x) \) are polynomials of degree \( n \), then (3.2.5) is a polynomial of degree not greater than \( n \). From (3.2.3) it is easy to see that (3.2.5) satisfies conditions (3.2.2).

In some cases the Lagrangian representation for \( P_n(x) \) is inconvenient. It is often impossible to say beforehand how many nodes \( x_k \) will be necessary to achieve the desired precision in the interpolation. Suppose that for the number of nodes first chosen the required precision is not achieved. Then we must use one or more additional nodes. Introducing one more node completely changes all the terms in (3.2.5). It is desirable then to have a representation for \( P_n(x) \) for which the previous calculations do not have to be repeated with the addition of one more node but for which it is only necessary to add one new term.

Newton's representation of the interpolating polynomial has this property:

\[ P_n(x) = f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) + \cdots \]
\[ \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f(x_0, x_1, \ldots, x_n). \]  

(3.2.6)

The right side of (3.2.6) is a polynomial of degree not greater than \( n \). From (3.1.7) we can see that it indeed satisfies conditions (3.2.2) since for \( x = x_0 \) the right side of (3.2.6) reduces to \( f(x_0) \) and for \( x = x_1 \) it reduces to \( f(x_0) + (x_1 - x_0)f(x_0, x_1) = f(x_1) \), and so forth.

We call the difference between \( f(x) \) and the interpolating polynomial \( P_n(x) \) the remainder of the interpolation:

\[ R_n(x) = f(x) - P_n(x) = f(x) - \sum_{k=0}^{n} \frac{\omega(x)f(x_k)}{(x-x_k)\omega'(x_k)}. \]  

(3.2.7)

The remainder \( R_n(x) \) depends on the properties of the function \( f(x) \) and on the location of \( x \) and the nodes \( x_k \). It can be expected that \( R_n(x) \) will be smaller for functions \( f(x) \) which are smoother, that is for functions with higher order derivatives. When \( f(x) \) is analytic it can be expected that the further the singular points of \( f(x) \) lie from \( x \) and the \( x_k \) the smaller \( R_n(x) \) will be.
From the representation (3.2.7) for $R_n(x)$ it is difficult to see how the properties of $f(x)$ influence the remainder and it will be useful to have other representations from which we can more easily estimate $R_n(x)$ for different classes of functions. Many representations for $R_n(x)$ have been constructed. Here we derive only two of the simplest.

1. Let the point $x$ and the nodes $x_k \ (k = 0, 1, \ldots, n)$ belong to the segment $[a, b]$.

**Theorem 3.** If $f(x)$ has a continuous derivative of order $n + 1$ on $[a, b]$ then the remainder $R_n(x)$ of the interpolation can be written:

$$R_n(x) = \omega(x) \int_0^1 \int_0^1 \cdots \int_0^1 \times \ f^{(n+1)} \left( x + \sum_{\nu=0}^{n} t_{\nu+1} (x_\nu - x_{\nu-1}) \right) dt_{n+1} \cdots dt_2 dt_1$$

where $x_{-1} = x$.

**Proof.** This theorem can be obtained as a corollary to Theorem 1. Consider the values $f(x_0), f(x_1), \ldots, f(x_n), f(x)$ of the function. From (3.1.7) we see that

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + \cdots + (x - x_0) (x - x_1) \cdots (x - x_{n-1}) f(x_0, x_1, \ldots, x_n) +$$

$$+ (x - x_0) (x - x_1) \cdots (x - x_n) f(x_0, x_1, \ldots, x_n, x).$$

The terms on the right side of this equation with the last term omitted is Newton's form for $P_n(x)$. Therefore the last term must be the remainder of the interpolation:

$$R_n(x) = \omega(x) f(x_0, x_1, \ldots, x_n, x) = \omega(x) f(x, x_0, x_1, \ldots, x_n). \ (3.2.9)$$

This result combined with Theorem 1 gives (3.2.8).

If we apply the mean value theorem to the integral in (3.2.8) we obtain a simpler representation for $R_n(x)$.

**Theorem 4.** If $f(x)$ has a continuous derivative of order $n + 1$ on $[a, b]$ then there exists an interior point $\xi$ of $[a, b]$ for which

$$R_n(x) = \frac{\omega(x)}{(n + 1)!} f^{(n+1)}(\xi). \ (3.2.10)$$

This is often called the Lagrange form for $R_n(x)$.

---

1See the references at the end of this chapter.
Theorem 5. The remainder of the interpolation for \( f(z) \) at the point \( x \) can be represented as the contour integral

\[
R_n(x) = \frac{\omega(x)}{2\pi i} \int_{\gamma} \frac{f(z)}{\omega(z)(z-x)} \, dz
\]

(3.2.11)

where \( \gamma \) is any simple closed curve inside \( B \) which encloses \( x \) and \( x_k \) \((k = 0, 1, \ldots, n)\).

This theorem is easily proved by verifying (3.2.7). The function \( \frac{f(z)}{\omega(z)(z-x)} \) has simple poles at the points \( z = x, z = x_k \) \((k = 0, 1, \ldots, n)\). Thus calculating the integral in (3.2.11) by residues we at once obtain (3.2.7).

3.3. INTERPOLATION WITH MULTIPLE NODES

We assume that we are given \( m \) distinct nodes \( x_1, x_2, \ldots, x_m \) and that at the first node \( x_1 \) we are given the value of the function \( f(x) \) and its derivatives up to order \( a_1 - 1 \)

\[
f(x_1), f'(x_1), \ldots, f^{(a_1-1)}(x_1).
\]

At the second node \( x_2 \) we assume that we are given the value of \( f(x) \) and its derivatives up to order \( a_2 - 1 \)

\[
f(x_2), f'(x_2), \ldots, f^{(a_2-1)}(x_2),
\]

and so forth. The numbers \( a_1, a_2, \ldots, a_m \) are called the multiplicities of the nodes \( x_1, x_2, \ldots, x_m \).

Let \( n + 1 \) denote the number of conditions given about \( f(x) \):

\[
a_1 + a_2 + \cdots + a_m = n + 1.
\]

We wish to construct a polynomial \( P_n(f; x) \) of degree not greater than \( n \) which will satisfy the conditions \(^2\)

\[
P^{(i)}_n(f; x_k) = f^{(i)}(x_k), \quad i = 0, 1, \ldots, a_k - 1; \quad k = 1, 2, \ldots, m. \quad (3.3.1)
\]

That \( P_n(f; x) \) will be unique can be proved from well-known theorems of algebra. Suppose that there exists two polynomials \( P_n(f; x) \) which satisfy

\(^2\)We use the convention \( f^{(0)}(x) = f(x) \).
Preliminary Information

conditions (3.3.1) and let \( Q(x) \) be their difference. Then \( Q(x) \) is a polynomial of degree not greater than \( n \) which satisfies the conditions

\[
Q^{(i)}(x_k) = 0, \quad i = 0, 1, \ldots, a_k - 1; \ k = 1, 2, \ldots, m.
\]

Thus each node \( x_k \) is a zero of \( Q(x) \) of multiplicity not less than \( a_k \). The sum of the multiplicities of these zeros will be not less than \( a_1 + a_2 + \ldots + a_m = n + 1 \). But it is known that the sum of the multiplicities of the zeros can exceed the degree of the polynomial only when \( Q(x) \) is identically zero. This proves that \( P_n(f; x) \) will be unique.

It is clear that the interpolating polynomial \( P_n(f; x) \) can be written in the form

\[
P_n(f; x) = \sum_{k=1}^{m} \sum_{i=0}^{a_k-1} L_{k,i}(x)f^{(i)}(x_k)
\]

(3.3.2)

where the \( L_{k,i}(x) \) are polynomials of degree \( \leq n \). To construct these polynomials we assume at first that \( f(x) \) is an analytic function.

Assume that \( f(z) \) is a function of the complex variable \( z \) which is holomorphic in a certain domain \( B \) which contains in its interior the points \( x \) and \( x_k \) \((k = 1, \ldots, m)\). As above, we again assume that \( B \) is simply connected. We take any simple closed curve \( l \) contained in \( B \) which encloses \( x \) and the \( x_k \). Everywhere inside \( l \) the function \( f(z) \) can be represented as a Cauchy integral

\[
f(x) = \frac{1}{2\pi i} \int_{l} \frac{f(z)}{z-x} \, dz.
\]

(3.3.3)

This equation permits us to investigate \( f(z) \) by a study of the elementary function \( \frac{1}{z-x} \) which is often called the Cauchy kernel.

Instead of studying the interpolating polynomial for \( \frac{1}{z-x} \) it will be more convenient to study the remainder of the interpolation:

\[
R_n\left(\frac{1}{z-x}; x\right) = \frac{1}{z-x} - P_n\left(\frac{1}{z-x}; x\right) = \frac{1}{z-x} - \sum_{k=1}^{m} \sum_{i=0}^{a_k-1} L_{k,i}(x) \frac{i!}{(z-x_k)^{i+1}}.
\]

(3.3.4)

We consider (3.3.4) to be a function of the parameter \( z \). This is a proper rational fraction for which (3.3.4) is the expansion in sums of simple fractions. We note that the point \( z = x \) is a simple pole of \( R_n\left(\frac{1}{z-x}; x\right) \) with residue equal to unity.
We now reduce the fraction on the right of (3.3.4) to a common denominator.

Setting

\[ A(z) = \prod_{k=1}^{m} (z - x_k)^{a_k} \]

we obtain a fractional representation for \( R_n \) of the form

\[ R_n \left( \frac{1}{z - x}; x \right) = \frac{B(z, x)}{A(z)(z - x)}. \]  

(3.3.5)

Since the fraction (3.3.5) is proper the numerator \( B(z, x) \) is a polynomial in \( z \) of degree not greater than \( n + 1 \). We can show that \( B(z, x) \) is independent of \( z \) and equals \( A(x) \). To do this we will find an expansion of (3.3.5) for values of \( z \) with large modulus. If \( |z| \) is large then

\[ \frac{1}{z - x} = \sum_{\nu=0}^{\infty} \frac{z}{z^{\nu+1}}. \]

Because the remainder operator \( R_n \) is linear we can see that

\[ R_n \left( \frac{1}{z - x}; x \right) = \sum_{\nu=0}^{\infty} \frac{1}{z^{\nu+1}} R_n(x^\nu; x). \]

Here \( R_n(x^\nu; x) \) is the remainder of the interpolation for \( x^\nu \). But for a polynomial of degree not greater than \( n \) the interpolation is exact and therefore

\[ R_n(x^\nu; x) = 0, \quad \nu = 0, 1, \ldots, n \]

\[ R_n \left( \frac{1}{z - x}; x \right) = \sum_{\nu=n+1}^{\infty} \frac{1}{z^{\nu+1}} R_n(x^\nu; x). \]

The highest degree of \( 1/z \) in the expansion (3.3.5) for large \( |z| \) must be \( \frac{1}{z^{n+2}} \). This means that the degree of the numerator \( B(z, x) \) with respect to \( z \) must be \( n + 2 \) lower than the degree of the denominator and therefore must not depend on \( z \): \( B(z, x) = B(x) \). At the pole \( z = x \) the residue of (3.3.5) is unity and therefore \( B(x) = A(x) \) and

\[ R_n \left( \frac{1}{z - x}; x \right) = \frac{A(x)}{A(z)(z - x)}. \]  

(3.3.6)

Now we multiply (3.3.4) by \( \frac{f(z)}{2\pi i} \) and integrate around \( l \). Using (3.3.3)
and (3.3.6) we obtain an expression for the remainder of the interpolation for \( f(z) \) at the point \( x \) of the form

\[
R_n (f; x) = f(x) - \sum_{k=1}^{m} \sum_{i=1}^{a_{k-1}} L_{k, i}(x) f^{(i)}(x_k) = \\
= \frac{A(x)}{2\pi i} \int \frac{f(z)}{A(z) (z - x)} dz. \quad (3.3.7)
\]

Evaluating the integral in (3.3.7) by residues we can find the interpolating polynomial (3.3.4):

\[
P_n (f; x) = f(x) - R_n (f; x).
\]

At the pole \( z = x \) the residue of \( \frac{A(x)f(z)}{A(z) (z - x)} \) is \( f(x) \). Let us now find the residue of this function at the pole \( z = x_k \). For \( z \) close to \( x_k \) we have the following expansions in powers of \( z - x_k \).

\[
f(z) = \sum_{s=0}^{\infty} \frac{f^{(s)}(x_k)}{s!} (z - x_k)^s
\]

\[
\frac{1}{z - x} = \frac{1}{(z - x_k) - (x - x_k)} = - \sum_{s=0}^{\infty} \frac{(x - x_k)^s}{(x - x_k)^{s+1}}
\]

\[
\frac{(z - x_k)^{a_k}}{A(z)} = \sum_{s=0}^{\infty} c_s^{(k)} (z - x_k)^s.
\]

The residue of the function

\[
\frac{f(z)}{A(z) (z - x)} = \frac{1}{(z - x_k)^{a_k}} \frac{1}{A(z)} \frac{1}{z - x} f(z)
\]

is obtained by multiplying the above three series together and determining the coefficient of \( (x - x_k)^{a_k-1} \). A simple calculation shows that this coefficient is

\[
- \sum_{i=0}^{a_{k-1}} f^{(i)}(x_k) \frac{1}{i!} \sum_{s=0}^{a_{k-1}-i} c_s^{(k)} (x - x_k)^{-a_k+i+s}.
\]

The residue of \( \frac{A(x)f(z)}{A(z) (z - x)} \) is this expression multiplied by \( A(x) \).

Thus we have obtained the following expression for \( P_n (f; x) \) which is due to Hermite:
3.3. Interpolation with Multiple Nodes

\[ P_n(f; x) = \sum_{k=1}^{m} \sum_{i=0}^{a_{k-1}} f^{(i)}(x_k) \frac{A(x)}{i!(x-x_k)^{a_k}} \times \]

\[ \times \sum_{s=0}^{a_{k-1}-i} c_s^{(k)}(x-x_k)^{i+s}. \quad (3.3.8) \]

If \( f(x) \) is defined on the real line and is not an analytic function then the representation (3.3.8) for its interpolating polynomial remains valid, but the representation of the remainder for \( P_n(f; x) \) as a contour integral is no longer valid.

For a nonanalytic function \( f(x) \) we give another representation for \( R_n \) for functions with sufficiently high order derivatives.

Let the points \( x \) and \( x_k \) (\( k = 1, 2, \ldots, m \)) belong to a certain segment \([a, b]\).

**Theorem 6.** If \( f(x) \) has a continuous derivative of order \( n + 1 \) on \([a, b]\) then there exists an interior point \( \xi \) of \([a, b]\) for which

\[ R_n(f; x) = \frac{A(x)}{(n + 1)!} f^{(n+1)}(\xi). \quad (3.3.9) \]

The proof of this theorem follows from an application of the following variation of Rolle's theorem to the function

\[ F(z) = f(z) - P_n(f; z) - \frac{A(x)}{A(x)} [f(x) - P_n(f; x)]. \]

Let \( a_1 < a_2 < \cdots < a_m \) and let \( f(x) \) satisfy the conditions

\[ f^{(i)}(a_k) = 0, \quad (i = 0, 1, \ldots, a_k - 1; k = 1, 2, \ldots, m). \]

Then if \( f(x) \) has a continuous derivative of order \( r = a_1 + \cdots + a_m \) then between \( a_1 \) and \( a_m \) there exists a point \( \xi \) for which \( f^{(r)}(\xi) = 0 \).

**REFERENCES**


CHAPTER 4

Linear Normed Spaces.
Linear Operators

4.1. LINEAR NORMED SPACES

Functional analysis provides a useful method for studying certain questions related to quadrature formulas. Using concepts from this branch of mathematics we can study many different questions related to quadrature formulas from a single point of view. In this chapter we develop only a few of the simple concepts and results from functional analysis which will be needed in the remainder of this book.

Let $X = \{x\}$ be a set of certain "elements" $x$. The nature of these elements is arbitrary: they may be points, lines, functions or any other quantities.

The set $X$ is called linear if the following two operations are defined on the elements of $X$: addition $x + y$, and multiplication $\lambda x$ by a (real or complex) number $\lambda$, in such a way that the result of these operations produces a new element of the set. To be more specific these operations are required to satisfy:

1. associativity of addition $(x + y) + z = x + (y + z)$;
2. commutivity of addition $x + y = y + x$;
3. the existence of a zero element $0$, which for every $x \in X$ satisfies $x + 0 = x$;
4. for each $x$ of $X$ there exists an inverse element $-x$, for which $x + (-x) = 0$;
5. associativity of multiplication $\lambda (\mu x) = (\lambda \mu)x$;
4.1. Linear Normed Spaces

6. the distributive laws

\[(\lambda + \mu)x = \lambda x + \mu x, \quad \lambda(x + y) = \lambda x + \lambda y;\]

7. \(1 \cdot x = x;\)

8. \(0 \cdot x = \theta;\)

9. if \(\lambda x = \theta\) and \(x \neq \theta\), then \(\lambda = 0.\)

A linear set \(X\) is called a linear normed or vector space, if for each element \(x \in X\) there is defined a norm \(\|x\|\), that is a real number possessing the properties of the length of a vector:

1. \(\|x\| \geq 0,\) and \(\|x\| = 0\) if and only if \(x = \theta,\)

2. \(\|x + y\| \leq \|x\| + \|y\|,\)

3. \(\|\lambda x\| = |\lambda| \cdot \|x\|.\)

By means of the norm we can define convergence of a sequence of elements: we say that \(x_n \rightarrow x,\) or \(\lim_{n \to \infty} x_n = x,\) if \(\|x_n - x\| \to 0\) as \(n \to \infty.\)

Closely related to the concept of convergence is the concept of completeness of the space. If the sequence \(x_n (n = 1, 2, \ldots)\) converges to a certain element \(x,\) then such a sequence satisfies the Bolzano-Cauchy criterion: for each \(\epsilon > 0\) there exists an integer \(N(\epsilon)\) such that for \(n > N(\epsilon)\) and any \(m > 0\)

\[\|x_{n+m} - x_n\| < \epsilon.\]

The converse may be false: if a sequence \(x_n (n = 1, 2, \ldots)\) satisfies the Bolzano-Cauchy criterion then it is still possible that there does not exist an element \(x\) in \(X\) to which the sequence \(x_n\) converges as \(n \to \infty.\)

The space \(X\) is called complete if for every sequence \(x_n,\) which satisfies the Bolzano-Cauchy criterion, there exists an element \(x\) in \(X\) to which the sequence \(x_n\) converges as \(n \to \infty.\) A complete, normed, linear space is called a Banach space.

Let us give some examples of Banach spaces.

1. The space \(C.\)

Let \([a, b]\) be any finite segment. The elements of \(C\) are all continuous functions on \([a, b]\). Addition of the elements and multiplication of them by a number is the usual addition of functions and multiplication of functions by a number. For the norm of the function \(x = x(t)\) we take

\[\|x\| = \max_{t \in [a, b]} |x(t)|.\]  

(4.1.1)

Convergence of elements of \(C\) corresponds to uniform convergence of sequences of functions.
The space $C$ is complete. From
\[
\|x_{n+m} - x_n\| = \max_{t \in [a, b]} |x_{n+m}(t) - x_n(t)| < \varepsilon
\]
there follows the convergence of the sequence of functions $x_n(t)$ ($n = 1, 2, \ldots$) for every $t$: \( \lim_{n \to \infty} x_n(t) = x(t) \) and because the limit of a uniformly convergent sequence of continuous functions is also a continuous function then $x(t) \in C$.

2. The space $L_p$ ($p \geq 1$).

This is the space of measurable functions on $[a, b]$ which are $p^{th}$ power summable. Addition and multiplication by a number is also the usual addition of functions and multiplication of them by a number. The norm is defined by
\[
\|x\| = \left( \int_a^b |x(t)|^p \, dt \right)^{\frac{1}{p}}. \tag{4.1.2}
\]
Functions which differ only on a set of points of measure zero are considered equivalent.

The conditions which must be satisfied by the norm (4.1.2) are easily verified. Conditions 1 and 3 are obviously fulfilled. Condition 2 is the well known Minkowski inequality for integrals:\footnote{\textit{See, for example, L. A. Lyusternik and V. I. Sobolev, Elements of Functional Analysis, Gostekhizdat, Moscow, 1951 (Russian; or the German translation of this book, Elemente der Funktionalanalysis, Berlin, 1955, pp. 244–6).}}

\[
\left( \int_a^b |x(t) + y(t)|^p \, dt \right)^{\frac{1}{p}} \leq \left( \int_a^b |x(t)|^p \, dt \right)^{\frac{1}{p}} + \left( \int_a^b |y(t)|^p \, dt \right)^{\frac{1}{p}}.
\]

The space $L_p$ is complete.\footnote{\textit{See, for example, L. A. Lyusternik and V. I. Sobolev, ibid., pp. 35–7 (or in the German translation, pp. 18–19).}}

3. The space $L_2$.

These are the functions that are square summable and the special case of $L_p$ for $p = 2$. The norm in $L_2$ is
\[
\|x\| = \left( \int_a^b x^2(t) \, dt \right)^{\frac{1}{2}}. \tag{4.1.3}
\]
4.1. Linear Normed Spaces

Convergence of elements here means convergence of functions in the sense of mean square deviation.

4. The space \( L \) of summable functions on \([a, b]\).

It is also a particular case of \( L_p \) for \( p = 1 \). The norm in \( L \) is defined as:

\[
\|x\| = \int_a^b |x(t)| \, dt \tag{4.1.4}
\]

and has the geometric meaning of the area between the \( t \) axis and the graph of the function \( x(t) \) over the interval \( a \) to \( b \).

5. The space \( V \).

For functions of bounded variation on \([a, b]\), for which \( x(a) = 0 \), to obtain the norm in \( V \) we take the total variation of \( x(t) \) on \([a, b]\)

\[
\|x\| = \text{Var } x(t). \tag{4.1.5}
\]

It is clear that conditions 1 and 3 for the norm are fulfilled. The fulfillment of the second condition follows from the inequality

\[
\text{Var } [x(t) + y(t)] \leq \text{Var } x(t) + \text{Var } y(t).
\]

The space \( V \) is complete. Indeed, let the Bolzano-Cauchy criterion be satisfied for the sequence of elements \( x_n \) \((n = 1, 2, \ldots)\):

\[
\lim_{n \to \infty} x_n(t) = x(t).
\]

From the Bolzano-Cauchy criterion it must follow that the norms \( \|x_n\| \) \((n = 1, 2, \ldots)\) are bounded from above by a certain number\(^3\)

\(^3\)Take \( \epsilon > 0 \) and choose \( N \) so that for \( n, m > N \) we will have \( \|x_m - x_n\| < \epsilon \). Fix any value of \( m > N \). Thus \( \|x_n\| \leq \|x_m\| + \|x_m - x_n\| < \|x_m\| + \epsilon \). Let \( M \) be the greatest of the numbers \( \|x_1\|, \ldots, \|x_N\|, \|x_m\| + \epsilon \). Then for every \( n \) we will have \( \|x_n\| \leq M \).
\[ \|x_n\| \leq M \quad (n = 1, 2, \ldots). \]

Divide \([a, b]\) into parts by the points \(a = t_0 < t_1 < \cdots < t_k = b\). The following inequality holds for \(x_n(t)\):

\[
\sum_{i=0}^{k-1} |x_n(t_{i+1}) - x_n(t_i)| \leq \text{Var} x_n(t) = \|x_n\| \leq M.
\]

If we now pass to the limit as \(n \to \infty\) we obtain

\[
\sum_{i=0}^{k-1} |x(t_{i+1}) - x(t_i)| \leq M
\]

which is equivalent to \(\text{Var} x(t) \leq M\), and consequently \(x(t)\) is an element of the space \(V\).

Let \(\varepsilon > 0\) and choose the number \(N\) so that for \(n, m > N\) we will have \(\|x_m - x_n\| \leq \varepsilon\). If in the inequality

\[
\sum_{i=0}^{k-1} |[x_m(t_{i+1}) - x_n(t_{i+1})] - [x_m(t_i) - x_n(t_i)]| \leq \text{Var} [x_m(t) - x_n(t)] = \|x_m - x_n\| \leq \varepsilon
\]

we pass to the limit as \(m \to \infty\), then we easily obtain

\[
\sum_{i=0}^{k-1} |[x(t_{i+1}) - x_n(t_{i+1})] - [x(t_i) - x_n(t_i)]| \leq \varepsilon.
\]

Because this is valid for any choice of points \(t_i\) \((i = 0, 1, \ldots, k)\) then there follows

\[
\text{Var} [x(t) - x_n(t)] = \|x - x_n\|_{[a, b]}
\]

and consequently the functions \(x_n(t)\) converge to \(x(t)\) with respect to the norm (4.1.5).

4.2. LINEAR OPERATORS

Let \(X = \{x\}\) and \(Y = \{y\}\) be two arbitrary sets of elements \(x\) and \(y\). If for each element \(x\) there corresponds by some rule a certain element \(y\): \(y = H(x)\), then we will say that we are given an operator \(H\). The set \(X\) is the domain on which \(H\) is defined and the domain of values of \(H\) is the set \(Y\).
4.2. Linear Operators

In the particular case when $Y$ is the set of real or complex numbers so that to each element $x$ there corresponds a certain number, the operator $H$ is called a functional.

The concept of an operator is a direct and far-reaching generalization of the concept of a function.

If in the sets $X$ and $Y$ there is a rule for passing to the limit then we can define a continuous operator. The operator $H$ is called continuous if from $x_n \to x$ (in the set $X$) it follows that $H(x_n) \to H(x)$ (in the set $Y$).

Below we will always assume that $X$ and $Y$ are linear normed spaces. The operator $H$ is called additive if for any two elements $x_1$ and $x_2$ of $X$ we have:

$$H(x_1 + x_2) = Hx_1 + Hx_2.$$ 

The operator is called linear if it is additive and continuous.

If there exists a number $M$ so that for every $x$ there is satisfied the inequality

$$\|Hx\| \leq M \|x\|$$

then $H$ is called a bounded operator. Let us prove the assertion:

*In order that an additive operator be continuous it is necessary and sufficient that it be bounded.*

**Proof of necessity.** Let us suppose that the linear operator $H$ is unbounded and show that this leads to a contradiction. Thus we could find a sequence of elements $x_n$ for which

$$\|Hx_n\| \geq n \|x_n\|.$$ 

Consider the elements

$$x'_n = \frac{x_n}{n \|x_n\|}.$$ 

It is evident that $x'_n \to \theta$ as $n \to \infty$.

On the other hand

$$Hx'_n = \frac{1}{n \|x_n\|} Hx_n$$

and

$$\|Hx'_n\| = \frac{1}{n \|x_n\|} \|Hx_n\| \geq 1;$$

*We will often omit the parentheses around the argument of an operator.*
\[ \| H x_n \| \not\to 0 \text{ as } n \to \infty \text{ and the operator } H \text{ is not continuous at the zero element } \theta. \]

**Proof of sufficiency.** We will assume the operator \( H \) to be additive and bounded and take any element \( x \). If \( x_n \to x \), that is \( \| x_n - x \| \to 0 \), then
\[
\| H x_n - H x \| = \| H(x_n - x) \| \leq M \| x_n - x \| \to 0,
\]
as \( n \to \infty \). \( H x_n \to H x \) and the operator \( H \) is continuous.

If \( H \) is a linear operator then the smallest of the constants \( M \) which satisfy the inequality
\[
\| H x \| \leq M \| x \|,
\]
is called the norm of the operator \( H \) and is designated by \( \| H \| \):
\[
\| H \| = \min M.
\]

In certain cases the following easily proved relation can be useful to find the norm
\[
\| H \| = \sup_{\| x \| \leq 1} \| H x \|. \tag{4.2.1}
\]
Indeed, for \( \| x \| \leq 1 \), \( \| H x \| \leq \| H \| \| x \| \leq \| H \| \). Therefore
\[
\sup_{\| x \| \leq 1} \| H x \| \leq \| H \|. \tag{4.2.2}
\]

By the definition of the norm, for each \( \epsilon > 0 \) there exists an element \( x' \) for which
\[
\| H x' \| > (\| H \| - \epsilon) \| x' \|.
\]
Let us put
\[
x = \frac{x'}{\| x' \|},
\]
\[
\| H x \| = \frac{1}{\| x' \|} \| H x' \| > \frac{1}{\| x' \|} (\| H \| - \epsilon) \| x' \| = \| H \| - \epsilon.
\]
Because \( \| x \| = 1 \), then
\[
\sup_{\| x \| \leq 1} \| H x \| > \| H \| - \epsilon
\]
By the arbitrariness of \( \epsilon \) and by (4.2.2) we obtain (4.2.1).

Let us find the norms of certain linear functionals which we will encounter later.
4.2. Linear Operators

1. \( X \) is the space \( C[a, b] \).

Consider the functional

\[
F x = \int_{a}^{b} f(t) x(t) \, dt,
\]

where \( f(t) \) is a measurable and summable function on \([a, b]\). We have

\[
\|F\| = \sup_{\|x\| \leq 1} \left| \int_{a}^{b} f(t) x(t) \, dt \right| = \sup_{|x(t)| \leq 1} \int_{a}^{b} |f(t)| |x(t)| \, dt \leq \int_{a}^{b} |f(t)| \, dt.
\]

Consider the function \( \text{sign} \, f(t) \). It is measurable and \( |\text{sign} \, f(t)| \leq 1 \). For it

\[
\int_{a}^{b} f(t) \, \text{sign} \, f(t) \, dt = \int_{a}^{b} |f(t)| \, dt.
\]

Because \( \text{sign} \, f(t) \) is a measurable function there certainly exists a continuous function \( x^*(t) \) for which \( |x^*(t)| \leq 1 \) and which differs from \( \text{sign} \, f(t) \) only on a set of arbitrarily small measure. Such a function can always be found so that \( \int_{a}^{b} fx^* \, dt \) differs from \( \int_{a}^{b} f \, \text{sign} \, f \, dt \) by as little as we please. Therefore

\[
\sup_{|x(t)| \leq 1} \int_{a}^{b} f(t) x(t) \, dt \geq \int_{a}^{b} |f(t)| \, dt.
\]

and consequently

\[
\|F\| = \int_{a}^{b} |f(t)| \, dt. \tag{4.2.4}
\]

2. \( X \) is \( L[a, b] \).

\[
F x = \int_{a}^{b} f(t) x(t) \, dt. \tag{4.2.5}
\]

Here \( f \) is a continuous function on \([a, b]\). The norm in \( L \) is defined by equation (4.1.4). We have
\[ |F_x| = \left| \int_a^b f(x) dt \right| \leq \max_t |f(t)| \int_a^b |x(t)| dt = \max_t |f(t)| \cdot \|x\| \]

Hence we see that \( \|F\| \leq \max_t |f(t)| \).

We can convince ourselves that in this estimate for \( \|F\| \) the righthand side can not be decreased. Let \( \epsilon \) be any small positive number.

Let us put \( M = \max_t |f(t)| \) and let us suppose that this maximum is achieved at the point \( \xi \). In order to be definite let us consider that \( f(\xi) \) is a positive number: \( f(\xi) = M > 0 \). By the continuity of \( f(t) \), close to \( \xi \) there exists a segment \( \alpha < t < \beta \) in which

\[ f(t) > M - \epsilon. \]

We define the function \( x(t) \) by the relationships

\[ x(t) = \begin{cases} \frac{1}{\beta - \alpha} & \text{for } t \in (\alpha, \beta) \\ 0 & \text{for } t \in (\alpha, \beta). \end{cases} \]

Clearly

\[ \|x\| = \int_a^b |x(t)| dt = 1, \]

\[ |F_x| = \left| \int_a^b f(t) x(t) dt \right| = \frac{1}{\beta - \alpha} \int_a^\beta f(t) dt > \frac{1}{\beta - \alpha} (M - \epsilon) (\beta - \alpha) = M - \epsilon. \]

Therefore

\[ \|F\| = \sup_{\|x\| \leq 1} |F_x| > M - \epsilon, \]

and because \( \epsilon \) is an arbitrary number, from this and from the previously obtained upper bound for \( \|F\| \) we have

\[ \|F\| = M = \max_t |f(t)|. \quad (4.2.6) \]

3. Let \( X \) be the space \( V \).

Consider the functional

\[ F_x = \int_a^b f(t) dx(t) \quad (4.2.7) \]
4.3. Convergence of a Sequence of Linear Operators

where \( f(t) \) is a continuous function on \([a, b]\); we show that

\[
\| F \| = \max_t |f(t)|, \quad (4.2.8)
\]

\[
|Fx| = \left| \int_a^b f(t) d\xi \right| \leq \max_t |f(t)| \cdot \text{Var} \alpha(t) = \max_{t, [a, b]} |f(t)| \cdot \|x\|. \quad (4.2.9)
\]

Let us suppose that \( \max |f(t)| \) is achieved at the point \( t_0 \) and suppose that \( t_0 \) lies inside \([a, b]\). Taking

\[
x(t) = \begin{cases} 
0 & \text{for } t < t_0 \\
\frac{1}{2} & \text{for } t = t_0 \\
1 & \text{for } t > t_0
\end{cases}
\]

we can see that the upper estimate which we have obtained for \(|Fx|\) is achieved. This proves (4.2.8).

4.3. CONVERGENCE OF A SEQUENCE OF LINEAR OPERATORS

Let \( X \) and \( Y \) be Banach spaces. Consider the sequence of linear operators \( H_n \) \((n = 1, 2, \ldots)\) defined on \( X \) and taking values in \( Y \). The sequence \( H_n \) will be called convergent if for each \( x \in X \) there is a convergent sequence of elements \( y_n = H_n x \) (in the space \( Y \)). Let us denote \( \lim H_n x = y = H x \). The operator \( H \) is additive. In fact if in the equation

\[
H_n (x_1 + x_2) = H_n x_1 + H_n x_2
\]

we pass to the limit as \( n \to \infty \) then we obtain

\[
H (x_1 + x_2) = H x_1 + H x_2.
\]

We can also show that the operator \( H \) is linear. We prove a preliminary lemma.

**Lemma.** If the sequence of operators \( H_n \) \((n = 1, 2, \ldots)\) converges then their norms \( \|H_n\| \) \((n = 1, 2, \ldots)\) have a common bound:

\[
\|H_n\| \leq M. \quad (4.3.1)
\]

**Proof.** Let us suppose the contrary. The set of elements \( x \) which satisfy the condition \( \|x - x_0\| \leq \varepsilon \) will be called a closed sphere of radius \( \varepsilon \) with center \( x_0 \) and denoted by \( S(x_0, \varepsilon) \). We show that \( \|H_n x\| \) can not have a common bound in that closed sphere. In fact let

\[
\|H_n x\| \leq K \quad (4.3.2)
\]

for \( n = 1, 2, \ldots \), and for every \( x \) in the sphere \( S(x_0, \varepsilon) \). For any \( x \) in \( X \) the element

\[
x' = \frac{\varepsilon}{\|x\|} x + x_0
\]
belongs to $S(x_0, \varepsilon)$. Therefore

$$\|H_n x\| = \left\| \frac{\varepsilon}{\|x\|} H_n x + H_n x_0 \right\| \leq K$$

and

$$\frac{\varepsilon}{\|x\|} \left\| H_n x \right\| - \left\| H_n x_0 \right\| \leq K.$$

Hence

$$\left\| H_n x \right\| \leq \frac{K + \left\| H_n x_0 \right\|}{\varepsilon} \left\| x \right\|.$$

The sequence of elements $H_n x_0$ converges and their norms $\|H_n x_0\|$ have a common bound. There must, then, exist a number $K_1$ independent of $n$ and $x$ for which

$$\left\| H_n x \right\| \leq K_1 \|x\|.$$

Therefore

$$\|H_n\| = \sup_{\|x\| \leq 1} \left\| H_n x \right\| \leq K_1$$

and this contradicts the assumption and inequality (4.3.2) can not be valid.

Let us take an arbitrary closed sphere $S_0(x_0, \varepsilon)$. In this sphere the sequence $\|H_n x\|$ is unbounded. We can find, then, an operator $H_{n_1}$ and an element $x_1 \in S_0$ for which

$$\|H_{n_1} x_1\| > 1.$$

By the continuity of the operator $H_{n_1}$ this inequality will be satisfied in a certain closed sphere $S_1(x_1, \varepsilon_1)$ contained in $S_0$. By an analogous argument we can find an operator $H_{n_2}$ and an element $x_2 \in S_1$ for which

$$\|H_{n_2} x_2\| > 2$$

and so forth. We can assume that $\varepsilon_n \to 0$ as $n \to \infty$. The constructed sequence of elements $x_1, x_2, \ldots$ will satisfy the Bolzano-Cauchy criterion. The space $X$ is complete and the sequence will converge to a certain element $x^* \in X$,

$$x_n \to x^*, \quad \text{as} \quad n \to \infty;$$

$x^*$ belongs to all the spheres $S_k \quad (k = 1, 2, \ldots)$. Thus for the element $x^*$

$$\|H_{n_k} x^*\| > k.$$
This then contradicts the assumption that the sequence $H_nx$ converges for arbitrary $x \in X$.

The linearity of the limit operator $H$ is now easily proved by using the lemma. Passing to the limit as $n \to \infty$ in the inequality

$$\|H_nx\| \leq M\|x\|$$

we obtain

$$\|Hx\| \leq M\|x\|.$$ 

The operator $H$ is bounded and, in view of its additivity, is continuous and linear.

The conditions which must be satisfied by the sequence of operators $H_n$ $(n = 1, 2, \ldots)$ in order that they be convergent is expressed in the following theorem of Banach.

**Theorem 1.** In order that the sequence of linear operators $H_n$ $(n = 1, 2, \ldots)$ be convergent it is necessary and sufficient that they satisfy the two conditions:

1. The norms of the operators $\|H_n\|$ have a common bound.
2. $H_nx$ is convergent for each $x$ in a set $E$ which is everywhere dense in $X$.

**Proof.** The necessity of the second condition is obvious. The necessity of the first follows from the lemma.

The sufficiency of the conditions can be verified in the following way. Let $\|H_n\| \leq M$. Let us take an arbitrary $x \in X$ and select an element $x' \in E$ for which $\|x - x'\| < \frac{\varepsilon}{3M}$. The sequence $H_nx'$ converges by condition 2 and for large $n$ we will have

$$\|H_{n+m}x' - H_nx'\| < \frac{\varepsilon}{3}.$$ 

Then

$$\|H_{n+m}x - H_nx\| \leq \|H_{n+m}x - H_{n+m}x'\| + \|H_{n+m}x' - H_nx'\| +$$

$$+ \|H_nx' - H_nx\| \leq 2M\|x - x'\| + \frac{\varepsilon}{3} < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$ 

Therefore the sequence $H_nx$ satisfies the Bolzano-Cauchy condition and in view of the completeness of $Y$ there exists for each $x \in X$ a limit element $y = Hx = \lim_{n \to \infty} H_nx$.

As shown above the limit operator $H$ is linear.

---

5 A set $E$ is called everywhere dense in $X$ if every element $x \in X$ is arbitrarily close, with respect to the norm, to an element of $E$. 
L. A. Lyusternik and V. I. Sobolev, Elements of Functional Analysis, Gostekhizdat, Moscow, 1951 (Russian). (There is a German translation of this book, Elemente der Funktionalanalysis, Berlin, 1955.)
Part Two

APPROXIMATE CALCULATION OF DEFINITE INTEGRALS
5.1. QUADRATURE SUMS

The problem of finding the numerical value of the integral of a function of one variable, because of its geometrical meaning, is often for simplicity called quadrature. In this book we study methods of quadrature which are used to approximately evaluate the integral by means of a finite number of values of the integrand and derivatives of the integrand. These methods are universal and they can be applied where other methods for calculating integrals fail. In many cases these methods also require less work than other methods.

Let us consider an integral of the form

\[ \int_a^b p(x) f(x) \, dx \]

where \([a, b]\) is any finite or infinite segment of the real line, and \(f(x)\) is an arbitrary function of a certain class. To simplify the discussion we assume in the beginning of this chapter that all functions \(f(x)\) are continuous. We assume that \(p(x)\) is a certain fixed function, which is measurable on \([a, b]\) and is not the identically zero function, and that the product \(p(x)f(x)\) is summable on \([a, b]\). At first we will not make any additional assumptions about \(p(x)\).
Approximate Calculation of Definite Integrals

The most widely applied quadrature formulas are those which approximate the integral by a linear combination of values of the function

$$\int_a^b p(x)f(x)\,dx = \sum_{k=1}^n A_k f(x_k). \quad (5.1.1)$$

The sum $\sum_{k=1}^n A_k f(x_k)$ we will call a quadrature sum. Equations of the form (5.1.1) have received the name of mechanical quadrature formulas\(^1\). They contain the following $2n + 1$ parameters which can be selected in the construction: $n$ abscissa or "nodes" $x_k$, $n$ coefficients $A_k$ and the number $n$. It is necessary to choose all of these parameters so that formula (5.1.1) will give a "sufficiently small error" for all functions $f$ of a certain wide class. For the following discussion of ideas related to the construction of quadrature sums we will not precisely define the words "small error" and how wide the class of functions must be. The precise meaning of these words will be made clear later.

It is immediately clear, by counting the choices of the $x_k$ and $A_k$, that the larger the value of $n$ the more precise can (5.1.1) be made. Therefore for the construction of approximate quadrature formulas $n$ is considered an arbitrary but fixed natural number.

In applying (5.1.1) the greatest amount of difficulty is usually in finding the values $f(x_k)$ ($k = 1, 2, \ldots, n$). After the $f(x_k)$ have been found the construction of the quadrature sum $\sum_{k=1}^n A_k f(x_k)$, if $n$ is not very large, is carried out comparatively easily. Therefore it is natural to try to achieve the necessary precision in the calculation with as small a number of nodes $x_k$ as possible. For the construction of quadrature sums

\(^1\)It is easy to attach a mechanical meaning to (5.1.1). Let us introduce the quantity $P = \int_a^b p\,dx$ and write (5.1.1) in the form $P^{-1} \int_a^b p\,dx \approx \sum_{k=1}^n B_k f(x_k)$. Here the coefficients $B_k$ will be abstract numbers. Let us agree to interpret them as "weights" belonging to the corresponding values $f(x_k)$. If we require that the equation be exact whenever $f$ is a constant function then the $B_k$ must satisfy $\sum_{k=1}^n B_k = 1$. The sum $\sum_{k=1}^n B_k f(x_k)$ then will have the meaning of an average of the values $f(x_k)$. The problem of construction of the equation reduces to finding weights $B_k$ so that the average weighted value of $f(x_k)$ will approximately equal the mean integral value of $f$ on the segment $[a, b]$: $P^{-1} \int_a^b p\,dx$. 
5.1. Quadrature Sums

this is equivalent then to choosing the \( x_k \) and \( A_k \) to increase the precision of formula (5.1.1) for a given \( n \). We now discuss the principle ways that have been investigated to achieve this.

1. Let us suppose we have been given a certain class \( F \) of functions \( f \). In relation to this class we consider the system of functions

\[
\omega_m(x) \quad (m = 1, 2, \ldots)
\]

(5.1.2)

for which the products \( p(x) \omega_m(x) \) are summable on \([a, b]\). Let us form a linear combination

\[
s_n(x) = \sum_{k=1}^{n} a_k \omega_k(x).
\]

For the evaluation of the integral \( \int_a^b p(x)f(x)\,dx \) we take as the "distance" between \( f \) and \( s_n \) the value

\[
\rho(f, s_n) = \int_a^b |p(f - s_n)|\,dx.
\]

(5.1.3)

We will consider the system (5.1.2) to be complete in the class \( F \), that is for each function \( f \in F \) and any \( \epsilon > 0 \) there exists a linear combination \( s_n \) for which \( \rho(f, s_n) < \epsilon \).

In view of the inequality

\[
|\int_a^b pf\,dx - \int_a^b ps_n\,dx| \leq \int_a^b |p(f - s_n)|\,dx = \rho(f, s_n)
\]

it follows that the integral \( \int_a^b pf\,dx \) can be calculated to as high a degree of accuracy as desired, if the integrand \( f \) is replaced by an appropriate linear combination \( s_n \).

Thus it is evident that we can achieve a high degree of precision in the calculation by taking a large number of functions \( \omega_k \) in the formation of \( s_n \).

We can expect that if we choose the nodes \( x_k \) and coefficients \( A_k \) in the formula (5.1.1) to give good precision in integrating the functions \( \omega_m \), then the formula must also give good precision in the calculation of the integral for each function \( f \in F \). These simple considerations serve, of course, only for motivation and the error of the constructed formulas must be subjected to precise analysis and estimation. But it is useful to indicate a simple principle for the selection of the \( x_k \) and \( A_k \): we will
attempt to choose the $x_k$ and $A_k$ so that formula (5.1.1) gives an exact result for as many of the functions $\omega_m(x)$ as possible.

We say that equation (5.1.1) has degree of precision $m$ with respect to the functions (5.1.2) if it is exact for $\omega_1, \omega_2, \ldots, \omega_m$:

$$\int_a^b p \omega_i dx = \sum_{k=1}^n A_k \omega_i(x_k) \quad (i = 1, 2, \ldots, m)$$

and it is not exact for $\omega_{m+1}$. This way for choosing the $x_k$ and $A_k$ is a way to increase the degree of precision of equation (5.1.1). Of special interest are formulas of approximate quadrature which possess the highest possible degree of precision. Some formulas of this type will be discussed in Chapter 7.

If the class $F$ is given, then for the construction of equation (5.1.1) for the integration of the function $f$, the choice of the system of functions $\omega_n \ (n = 1, 2, \ldots)$ is still arbitrary. The requirement of completeness, which must be satisfied by the system, does not fully define it and there are still many ways to select the $\omega_n$.

Approximate quadrature formulas which we will now consider take into account the properties of the functions $\omega_n$. If we want the formulas to give good precision then the $\omega_n$ must necessarily be chosen so that the properties of $\omega_n$ will agree with the properties of $f$ and we can expect that the error in (5.1.1) will be smaller the more closely the linear combination $s_n$ approximates the function $f$ for a fixed $n$.

We mention now some examples of the choice of $\omega_n$. Let $[a, b]$ be any finite segment. It is known that for any continuous function $f$ on $[a, b]$ and for any $\epsilon > 0$ there exists a polynomial $P(x)$ which differs from $f(x)$, for any $x \in [a, b]$, by less than $\epsilon$:

$$|f(x) - P(x)| < \epsilon.$$ 

This is the property of completeness of algebraic polynomials in the space of continuous functions $C$. From this then there follows, at once, the completeness of the system of polynomials in the sense of the metric (5.1.3).

We take the system of powers of $x$: 1, $x$, $x^2$, ... as the functions $\omega_n$ and we will say that equation (5.1.1) has algebraic degree of precision $m$, if it is exact for all possible polynomials of degree $m$ and not exact for all polynomials of degree $m + 1$. This is equivalent to the equation

$$\int_a^b px^i dx = \sum_{k=1}^n A_k x_k^i$$

being fulfilled for $i = 0, 1, \ldots, m$ and not fulfilled for $i = m + 1$. 
5.1. Quadrature Sums

We can expect that (5.1.1) will have a smaller error for more continuous functions on \([a, b]\) for the higher the algebraic degree of precision.

The system of powers \(x^n (n = 0, 1, \ldots)\) are a sufficiently convenient basis for the construction of quadrature formulas of the highest degree of precision for any finite segment \([a, b]\).

Let us suppose now that the segment of integration is infinite, for example, let it be the segment \(0 \leq x < \infty\). We will take some subset \(F\) of continuous functions \(f\) on \([0, \infty)\). On each finite segment \(0 \leq x \leq b < \infty\) we can construct a polynomial \(P(x)\) which approximates \(f\) uniformly with any preassigned degree of precision. But \(P(x)\) can not give a uniform approximation to \(f\) on the entire half-line and the difference \(f - P\), for large \(x\), can have a large value. In spite of this, providing the weight function \(p(x)\) decreases sufficiently fast as \(x \to \infty\), it can happen that for any \(f \in F\) the integral \(\int_{0}^{\infty} |p(f - P)| \, dx\) can be made as small as we desire and the system of powers \(x^n\) will then be complete in the class \(F\) with respect to the metric (5.1.3). In this case the quadrature formulas of the highest algebraic degree of precision also can be applied for the approximate calculation of integrals of the form \(\int_{0}^{\infty} p(f) \, dx\). These formulas will be discussed in Chapter 7.

In this connection we wish to give an example to clarify how to choose the functions \(\omega_n\) so that their properties closely agree with the class \(F\) of functions to be integrated. Let us suppose \(f\) is continuous and has an asymptotic representation, on the segment \(0 < x < \infty\), of the form

\[
f(x) \sim c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots.
\]

Each polynomial \(P(x)\) of degree greater than zero grows without bound as \(x \to \infty\), and the order of growth is higher for the polynomials of higher degree. The behavior of the polynomials on the half-line \([0, \infty)\) naturally differs from the behavior of bounded functions and polynomials can not be successful for the approximation of bounded functions on \([0, \infty)\). For certain weight functions \(p(x)\) it can happen that approximate quadratures of the highest degree of accuracy, with basis functions the system of powers \(x^n\), will give slow convergence, as \(n \to \infty\), to the value of the integral and to obtain the necessary precision a large number of nodes will be required.

For the approximation of functions of the type mentioned above it is more suitable to use rational functions of some special form, for example, the functions \((x + 1)^{-k} (k = 0, 1, 2, \ldots)\). If we take these for the basis functions and construct the corresponding quadrature formulas of the
highest degree of precision\(^2\) then they might be expected to give better precision for the same number of nodes than formulas based on \(\omega_k(x) = x^k\).

We mention now another example for the choice of \(\omega_n(x)\). Let \(f\) be a periodic function of period \(2\pi\) and suppose that we want to evaluate the integral

\[
\int_0^{2\pi} f(x) \, dx.
\]

For the functions \(\omega_n\) it is then natural to choose the trigonometric functions \(\cos kx, \sin kx\) \((k = 0, 1, 2, \ldots)\). As it turns out in this case the formulas of the highest degree of accuracy are elementary and their construction is quite simple. Because of their simplicity we will not postpone their construction to a later chapter. However, in order that we do not interrupt the discussion related to the choice of the nodes and coefficients we delay the study of these formulas to the following section.

2. Let us suppose that we are given a class \(F\) of functions \(f\). We endeavor to construct a quadrature formula (5.1.1) which will be in some sense, which we will clarify below, "best" for a given class. For each function \(f\) the error in the formula (5.1.1) has the value

\[
R(f) = \int_a^b p(x) f(x) \, dx - \sum_{k=1}^n A_k f(x_k).
\]

\(^2\)Trans. note. A quadrature formula for the segment \(0 < x < \infty\) which is exact for \((1 + x)^{-k} (k = 2, 3, \ldots, m + 2)\):

\[
\int_0^\infty (1 + x)^{-k} \, dx = \sum_{i=1}^n A_i (1 + x_i)^{-k} \quad (k = 2, 3, \ldots, m + 2)
\]

may be obtained by a transformation of a formula for the segment \(0 \leq y \leq 1\) which is exact for \(y^k (k = 0, 1, \ldots, m)\):

\[
\int_0^1 y^k \, dy = \sum_{i=1}^n B_i y_i^k \quad (k = 0, 1, \ldots, m).
\]

Using the transformation \(y = \frac{1}{1 + x}\), \(dy = \frac{-dx}{(1 + x)^2}\) we see that

\[
\int_0^1 y^k \, dy = \int_0^\infty (1 + x)^{-k-2} \, dx \quad (k = 0, 1, 2, \ldots)
\]

and it is then not difficult to see that the nodes \(x_i\) and coefficients \(A_i\) given by

\[
x_i = \frac{1 - y_i}{y_i} \quad A_i = \frac{B_i}{y_i^2} \quad (i = 1, 2, \ldots, n)
\]

give the desired formula for \([0, \infty)\).
5.1. Quadrature Sums

As a quantity which characterizes the precision of the quadrature formula for all functions \( f \), we can take the upper bound of \( |R(f)| \).

\[
R = \sup_f |R(f)|.
\]

Here \( R \) depends on the \( x_k \) and \( A_k \). Desiring to achieve possibly better accuracy for all functions \( f \in F \) we can choose the \( x_k \) and \( A_k \) so that \( R \) will have the least possible value. Such formulas we will call formulas with least estimate of the remainder in the class \( F \).

3. We have now indicated two possibilities with regard to the choice of the nodes and coefficients. There are still other methods for constructing quadrature formulas by subjecting the nodes and coefficients to meet other demands. We indicate one problem of this type. First of all we note that to make formula (5.1.1) exact for functions having constant value on \([a, b]\) we have only the choice of the coefficients \( A_k \) at our disposal. If it is required that (5.1.1) be exact for \( f = 1 \), then we obtain the following condition:

\[
\sum_{k=1}^{n} A_k = \int_a^b p(x) \, dx. \tag{5.1.4}
\]

Let us assume that the values \( f(x_k) \), of the function \( f \), entering into the quadrature sum are to be found from measurements and contain accidental errors. Let us suppose in addition that all of the \( f(x_k) \) have been obtained as the result of measurements of equivalent precision.

The values of the quadrature sums will also contain accidental errors. We can state the problem thus: in what manner shall we choose the coefficients \( A_k \), which fulfill condition (5.1.4), so that the quadrature sum

\[
\sum_{k=1}^{n} A_k f(x_k)
\]

will have the least square error. It is known that if the arguments \( z_1, \ldots, z_n \) of a linear function \( y = a_1 z_1 + \cdots + a_n z_n \) are random quantities subjected to the law of normal distribution with one and the same standard deviation and if the coefficients of the linear function are subjected to the condition \( \sum_{k=1}^{n} a_k = 1 \), then the average squared error of the sum will be the least when all the coefficients are equal\(^3\).

\(^3\)If the random variables \( z_1, \ldots, z_n \) are normally distributed with standard deviations \( \sigma_1, \ldots, \sigma_n \) and if \( y \) is a linear function of them: \( y = a_1 z_1 + \cdots + a_n z_n \), then \( y \) is also normally distributed with standard deviation \( S = (a_1^2 \sigma_1^2 + \cdots + a_n^2 \sigma_n^2)^{\frac{1}{2}} \) (see, for example, S. N. Bernstein, *Theory of Probability* (in Russian), Gostekhizdat, Moscow, 1946, pp. 269-72; or H. D. Brunk, *An Introduction to Mathematical Statistics*, Ginn, 1960, pp. 88-9). For \( \sigma_1 = \sigma_2 = \cdots = \sigma_n = \sigma \) we will have \( S = \sigma (a_1^2 + \cdots + a_n^2)^{\frac{1}{2}} \) and for the condition \( a_1 + \cdots + a_n = 1 \), \( S \) will have a minimum in the case when all of the \( a_k \) are equal.
Therefore a quadrature formula with equal coefficients

\[ \int_a^b p(x) f(x) \, dx = C [f(x_1) + \cdots + f(x_n)] \]  

(5.1.5)

will have the least square error. At the same time such formulas are especially convenient for graphical calculation because the sum of the ordinates can be removed from the drawing with the help of the simplest graphical equipment.

We mention now one more property of quadrature sums which will have great value in the remainder of the book. For calculations, almost always, it is necessary to know the approximate values \( f(x_k) \) exact to a certain number of decimal places.

Let all values \( f(x_k) \) be known with error not exceeding in absolute value the number \( \epsilon \). Calculating, from the approximate values \( f(x_k) \), the quadrature sum \( \sum_{k=1}^n A_k f(x_k) \) we obtain its value with error which must be estimated by the quantity

\[ \epsilon \sum_{k=1}^n |A_k|. \]

Such an estimate is exact and can not be decreased. If the sum \( \sum_{k=1}^n |A_k| \) is very large, then even a small error in the values \( f(x_k) \) can lead to a large error in the approximate value of the integral. For the construction of quadrature formulas therefore we should always strive so that the sum of the absolute values of the coefficients will have the smallest possible value.

In one important special case it is easy to indicate the condition for which \( \sum_{k=1}^n |A_k| \) will have the smallest possible value. We will consider the weight function \( p(x) \) to be nonnegative

\[ p(x) \geq 0 \text{ for } x \in [a, b]. \]

In addition we suppose that the quadrature formula is exact for \( f(x) = 1 \), which is equivalent to equation (5.1.4) for the coefficients \( A_k \). Then, evidently, \( \sum_{k=1}^n |A_k| \) will have the least value when all the coefficients \( A_k \) are positive: \( A_k > 0 \). This fact is one of the reasons why quadrature formulas with positive coefficients are especially important for applications.
5.2. REMARKS ON THE APPROXIMATE INTEGRATION OF PERIODIC FUNCTIONS

Let the segment of integration [a, b] be finite. It is always possible by a linear transformation to transform this segment to the segment [0, 2\pi]. Let us consider integrals of the form

$$\int_{0}^{2\pi} f(x) \, dx$$

(5.2.1)

where \( f(x) \) is a function with period 2\pi. As above we will study approximate quadrature formulas of the form

$$\int_{0}^{2\pi} f(x) \, dx \approx \sum_{k=1}^{n} A_k f(x_k).$$

(5.2.2)

Here \( x_k \) belongs to the segment \( 0 \leq x \leq 2\pi \).

For the obvious reason, for the approximation of a periodic function we take not algebraic, but trigonometric polynomials. We recall that trigonometric polynomials of degree \( m \) are functions of the form

$$T_m(x) = a_0 + \sum_{k=1}^{m} (a_k \cos kx + b_k \sin kx)$$

(5.2.3)

where \( a_0, a_k, b_k \) \( (k = 1, \ldots, m) \) are certain constants.

We will say that formula (5.2.2) has trigonometric degree of precision \( m \) if it is exact for all possible trigonometric polynomials up to degree \( m \) inclusive and there exists a polynomial of degree \( m + 1 \) for which it is not exact.

It is easy to verify that no matter how we choose the nodes \( x_k \) and coefficients \( A_k \) formula (5.2.2) can not be exact for all trigonometric polynomials of degree \( n \).

Let us construct the function

$$T(x) = \prod_{k=1}^{n} \sin^2 \frac{x - x_k}{2}.$$

Because \( \sin^2 \frac{x - x_k}{2} = \frac{1}{2} [1 - \cos (x - x_k)] \) it is clear that \( T(x) \) is a polynomial of degree \( n \). Then the quadrature formula (5.2.2) can not be exact for it because \( \int_{0}^{2\pi} T(x) \, dx > 0 \), but \( \sum_{k=1}^{n} A_k T(x_k) = 0 \) because all of the nodes \( x_k \) are roots of the polynomial \( T(x) \).
The trigonometric degree of precision of (5.2.2) is therefore always less than \( n \) and the \( A_k \) and \( x_k \) can be taken to make the degree, at the most, equal to \( n - 1 \).

It turns out that the highest degree of precision \( n - 1 \) is achieved by the simplest quadrature formula with equal coefficients:

\[
A_k = \frac{2\pi}{n} \quad (k = 1, 2, \ldots, n)
\]

and equally spaced nodes.

Let us consider any set of equally-spaced points on the real axis with interval \( h = \frac{2\pi}{n} \). Let \( a \) be the point of the set nearest to the origin from the right or coinciding with the origin. The points \( a + kh \) \((k = 0, 1, \ldots, n - 1)\) lie in the segment \( 0 < x < 2\pi \). Let us take these as the nodes \( x_k \) and construct a quadrature formula

\[
\int_0^{2\pi} f(x) \, dx = \frac{2\pi}{n} \sum_{k=1}^{n} f\left(a + (k - 1) \frac{2\pi}{n}\right).
\]

(5.2.4)

We can show that it is exact for all trigonometric polynomials up to degree \( n - 1 \) inclusive. To do this it is sufficient to show that equation (5.2.4) will be exact for the functions \( e^{imx} \) \((m = 0, 1, \ldots, n - 1)\). For \( m = 0 \) the assertion is evidently true. For \( 1 \leq m \leq n - 1 \)

\[
\int_0^{2\pi} e^{imx} \, dx = \frac{1}{im} (e^{im2\pi} - 1) = 0
\]

and

\[
\sum_{k=1}^{n} e^{im[a + (k - 1)h]} = e^{ima} \sum_{k=1}^{n} e^{i(k - 1)m} = e^{ima} \frac{e^{imn} - 1}{e^{im} - 1} =
\]

\[
e^{ima} \frac{e^{im2\pi} - 1}{e^{im} - 1} = 0
\]

which then proves the assertion.

5.3. THE REMAINDER IN APPROXIMATE QUADRATURE AND ITS REPRESENTATION

The value of the remainder

\[
R(f) = \int_a^b p(x) f(x) \, dx - \sum_{k=1}^{n} A_k f(x_k)
\]

(5.3.1)

of the quadrature depends on the choice of the quadrature formula, that
is on the choice of the $x_k$ and $A_k$, and also on the properties of the integrand function $f$. Formula (5.3.1) is one of the possible representations of the remainder, but to determine from it the influence of the structural properties of $f(x)$ on $R(f)$ is very difficult. The expression (5.3.1) is defined for a very wide class of functions. It is valid for any function $f$ for which the integral $\int_a^b pf dx$ exists and which has finite value at each node $x_k$. Because of its generality it does not take into account other properties of $f$.

In order to simplify the study of $R(f)$ we will construct another representation for it by which we can easily study the influence on $R(f)$ of such properties of the function $f$ as its order of differentiability, the value of $\max_x |f(x)|$ and so forth. The representation which we will derive will be especially useful to determine how the structure of the class influences $R(f)$.

We will consider that we are given a set $F$ of integrand functions $f$. The remainder $R(f)$ is a functional defined on the set $F$. In functional analysis there are theorems concerning the general forms of linear functionals defined on certain concrete linear spaces. These theorems can be used, in many cases, to construct a representation of the remainder term $R(f)$ for the set $F$.

For this problem of finding the desired representation of the remainder we will make use of some simple methods of classical analysis.

If we consider a class $F$ of functions which possess some structural property then it is often possible to give a formula which will represent each function of the class $F$ and only functions of this class. Such a formula is called the characteristic representation of the class $F$ or its structural formula.

If the structural formula of the class $F$ is known then from it we can in principle obtain all of the necessary information about the class $F$ and in particular we can construct the representation of the remainder in the quadrature for functions of the class $F$. Such a representation will be constructed each time that it is required in the presentation.

We will now give one example to illustrate the above remarks.

We say that the function $f$ belongs to the class $C_r[a, b]$ if it has $r$ continuous derivatives on $[a, b]$. The characteristic representation of a function of this class is furnished by its Taylor series. If $f \in C_r[a, b]$ and if $a$ is any point of the segment $[a, b]$ then

---

4By "structural properties" of the function we mean such properties as bounded variation, absolute continuity, satisfaction of a Lipschitz condition, belonging to a certain class of differentiability and so forth.

5See the references at the end of this chapter.
Approximate Calculation of Definite Integrals

\[ f(x) = \sum_{i=0}^{r-1} \frac{f^{(i)}(a)}{i!} (x-a)^i + \int_a^b f^{(r)}(t) (x-t)^{r-1} \frac{dt}{(r-1)!}. \quad (5.3.2) \]

It will be convenient to replace the integral having a variable limit by a definite integral over the segment \([a, b]\). This can be done by introducing the "jump" function which annihilates the superfluous section of integration. We define \( E(x) \) by

\[ E(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0. \end{cases} \]

It is easy to verify that equation (5.3.2) can be written in the form

\[ f(x) = \sum_{i=0}^{r-1} \frac{f^{(i)}(a)}{i!} (x-a)^i + \int_a^b f^{(r)}(t) \left[ E(x-t) - E(a-t) \right] (x-t)^{r-1} \frac{dt}{(r-1)!}. \quad (5.3.3) \]

On the righthand side of (5.3.3) there are \( r \) numerical parameters \( f^{(i)}(a) \) \((i = 0, 1, \ldots, r-1)\) and the functional parameter \( f^{(r)}(t) \) which is a continuous function on \([a, b]\).

Each function \( f \) of \( C_r[a, b] \) has a representation of the form (5.3.3). Conversely, for any numerical parameters \( f^{(i)}(a) \) \((i = 0, 1, \ldots, r-1)\) and any function \( f^{(r)}(t) \) continuous on \([a, b]\), the function \( f(x) \) defined by equation (5.3.3) belongs to \( C_r[a, b] \).

If the interval of integration is not the entire real axis then, in order not to introduce an additional parameter, \( a \) is often taken as one of the end points of \([a, b]\). For example, if we take \( a \) as the left end point \( a \), then formula (5.3.3) has the simplified form:

\[ f(x) = \sum_{i=0}^{r-1} \frac{f^{(i)}(a)}{i!} (x-a)^i + \int_a^b f^{(r)}(t) E(x-t) (x-t)^{r-1} \frac{dt}{(r-1)!}. \quad (5.3.4) \]

where \( r \geq 1 \).

Where there is no ambiguity in the designation of the class of functions \( C_r[a, b] \) the symbol \([a, b]\) will be omitted.

Let the integrand \( f \) belong to the class \( C_r \). We will attempt to determine how the \( r \)-fold differentiability of \( f \) affects the remainder and the convergence of the quadrature process. To do this we obtain a representation for \( R(f) \) which is characteristic of the class \( C_r \). This can be found if we replace in (5.3.1) the expression (5.3.3) for \( f \):
5.3. Remainder in Approximate Quadrature

\[ R(f) = \sum_{i=0}^{r-1} \frac{f^{(i)}(a)}{i!} R[(x-a)^i] + \]

\[ + R \int_a^b f^{(r)}(t) [E(x-t) - E(a-t)] \frac{(x-t)^{r-1}}{(r-1)!} \, dt. \]  

(5.3.5)

In the double integral

\[ \int_a^b p(x) \int_a^b f^{(r)}(t) [E(x-t) - E(a-t)] \frac{(x-t)^{r-1}}{(r-1)!} \, dt \, dx \]

which occurs in the last term on the righthand side of (5.3.5), we assume that we can change the order of integration. By the assumptions that we have made about the weight function \( p(x) \) this is certainly possible if \([a, b]\) is a finite segment. Then (5.3.5) can be written as

\[ R(f) = \sum_{i=0}^{r-1} \frac{f^{(i)}(a)}{i!} R[(x-a)^i] + \int_a^b f^{(r)}(t) K(t) \, dt \]  

(5.3.6)

where the kernel \( K(t) \) has the form

\[ K(t) = \int_a^b p(x) \frac{E(x-t) - E(a-t)}{(r-1)!} \, dx - \sum_{k=1}^{n} A_k \frac{E(x_k-t) - E(a-t)}{(r-1)!}. \]  

(5.3.7)

If \( t \neq a \) and \( t \neq x_k \) \( (k = 1, \ldots, n) \) then we easily obtain the following equations for \( K(t) \):

\[ t < a, \quad K(t) = - \int_a^t p(x) \frac{(x-t)^{r-1}}{(r-1)!} \, dx + \sum_{x_k < t} A_k \frac{(x_k-t)^{r-1}}{(r-1)!}; \]  

(5.3.8)

\[ t > a, \quad K(t) = \int_t^b p(x) \frac{(x-t)^{r-1}}{(r-1)!} \, dx - \sum_{x_k > t} A_k \frac{(x_k-t)^{r-1}}{(r-1)!}. \]

(5.3.9)

Analogously, we can construct representations for the remainder for other classes of functions when we know a characteristic representation for them, for example, for analytic functions.

In Chapters 8 and 12 we will see that the specialized representation of the remainder which we discussed above permits a sufficiently simple solution of the problems of finding a precise estimate for \( R(f) \) and of convergence of the quadrature process for certain classes of functions.
REFERENCES


Interpolatory Quadratures

6.1. INTERPOLATORY QUADRATURE FORMULAS AND THEIR REMAINDER TERMS

Quadrature formulas are often constructed from interpolating polynomials. In this way we can, in many cases, obtain quadrature formulas which are convenient to use and which will give sufficiently accurate results.

Let us choose $n$ arbitrary points $x_1, x_2, \ldots, x_n$ in the segment $[a, b]$ and, using these points, construct the interpolating polynomial for $f(x)$:

$$f(x) = P(x) + r(x)$$

(6.1.1)

$$P(x) = \sum_{k=1}^{n} \frac{\omega(x)}{(x - x_k)\omega'(x_k)} f(x_k),$$

$$\omega(x) = (x - x_1) \cdots (x - x_k).$$

(6.1.2)

Here $r(x)$ is the remainder of the interpolation.

The exact value of the integral $\int_{a}^{b} p(x)f(x)dx$ is

$$\int_{a}^{b} p(x)f(x)dx = \int_{a}^{b} p(x)P(x)dx + \int_{a}^{b} p(x)r(x)dx.$$

1If we assume that $f(x)$ is defined only on the segment $[a, b]$ then we must choose the $x_k$ to belong to $[a, b]$. If $f(x)$ is also defined outside the segment of integration then it is not necessary that all the $x_k$ belong to $[a, b]$. Quadrature formulas which contain nodes lying outside $[a, b]$ can be used for the integration of analytic functions. It is usually desirable, however, to have the points belong to the segment of integration.
Approximate Calculation of Definite Integrals

If the interpolation (6.1.1) is sufficiently precise so that the remainder \( r(x) \) is small throughout the segment \([a, b]\) then the second term in this last equation can be neglected. Thus we obtain the approximate equation

\[
\int_a^b p(x)f(x)dx \approx \sum_{k=1}^n A_k f(x_k)
\]

(6.1.3)

where

\[
A_k = \int_a^b p(x) \frac{\omega(x)}{(x - x_k)\omega'(x_k)} \, dx.
\]

(6.1.4)

Quadrature formulas of the form (6.1.3), in which the coefficients have the form (6.1.4), are called \textit{interpolatory} quadrature formulas. Interpolatory quadrature formulas can be characterized by the following theorem:

**Theorem 1.** In order that the quadrature formula (6.1.3) be interpolatory it is necessary and sufficient that it be exact for all possible polynomials of degree less than or equal to \( n - 1 \).

**Proof.** Each polynomial \( P(x) \) of degree \( \leq n - 1 \) can be represented in the form

\[
P(x) = \sum_{k=1}^n \frac{\omega(x)}{(x - x_k)\omega'(x_k)} P(x_k).
\]

If we take the coefficients \( A_k \) to have the values (6.1.4) then the quadrature formula (6.1.3) will be exact for \( P(x) \).

In the previous paragraph the values \( P(x_k) \), in the representation for \( P(x) \), may be any real, finite numbers. The requirement that (6.1.3) be exact for all polynomials of degree \( \leq n - 1 \) is equivalent to requiring that the equation

\[
\int_a^b p(x) \sum_{k=1}^n \frac{\omega(x)}{(x - x_k)\omega'(x_k)} P(x_k) \, dx = \sum_{k=1}^n A_k P(x_k)
\]

be valid for every set of \( P(x_k) \). But then the coefficients \( A_k \) must have the values (6.1.4) and formula (6.1.3) will be interpolatory. This completes the proof.

This theorem shows that specifying the \( n \) nodes \( x_k \) will completely define the quadrature formula—that is, the coefficients \( A_k \) will also be completely determined—if we require that the formula be exact for each polynomial of degree \( \leq n - 1 \). The nodes, however, may still be chosen in any manner we desire in order to make the quadrature formula meet some special demand.

Everything that was said in Section 5.3 holds true for the remainder of an interpolatory quadrature formula (6.1.3). In addition we can obtain, for this type of quadrature formula, a few deeper results.
The remainder of the quadrature (6.1.3) is the integral of the remainder \( r(x) \) of the interpolation,

\[
R(f) = \int_a^b p(x)r(x)\,dx = \int_a^b p(x)\omega(x)f(x, x_1, \ldots, x_n)\,dx. \tag{6.1.5}
\]

To study \( R(f) \) we can now use theorems concerning the remainder \( r(x) \). For example, if \( f(x) \) has \( n \) continuous derivatives on \([a, b]\) then \( r(x) \) can be represented in the form (3.2.8). Using the notation of Chapter 3 we obtain the following expression for the remainder \( R(f) \):

\[
R(f) = \int_a^b \int_0^1 \int_{s_1}^{s_1} \cdots \int_{s_{n-1}}^{s_{n-1}} p(x)\omega(x) \times

\times f^{(n)} \left( x + \sum_{\nu=1}^{n} t_{\nu}(x_{\nu} - x_{\nu-1}) \right) dt_n \cdots dt_2 dt_1 \,dx \tag{6.1.6}
\]

where \( x = x_0 \). It is often preferable to use the simpler expression for \( R(f) \) which is obtained from the Lagrangian form of \( r(x) \):

\[
r(x) = \frac{1}{n!} \omega(x)f^{(n)}(\xi), \quad a < \xi < b
\]

\[
R(f) = \frac{1}{n!} \int_a^b p(x)\omega(x)f^{(n)}(\xi)\,dx. \tag{6.1.7}
\]

It is difficult to find an exact estimate for \( R(f) \) from (6.1.7) because we cannot determine how \( \xi \) depends on \( x \).

If the \( n \)th derivative of \( f(x) \) is bounded in absolute value on \([a, b]\) by the number \( M_n \):

\[
|f^{(n)}(x)| \leq M_n, \quad x \in [a, b] \tag{6.1.8}
\]

then from (6.1.7) we obtain the estimate

\[
|R(f)| \leq \frac{M_n}{n!} \int_a^b |p(x)\omega(x)|\,dx. \tag{6.1.9}
\]

If \( p(x)\omega(x) \) does not change sign on \([a, b]\) then the estimate (6.1.9) cannot be improved. For an arbitrary \( p(x) \) and an arbitrary set of \( n \) nodes \( x_k \) we can obtain a precise estimate for \( R(f) \) for any function satisfying (6.1.8). To do this we use (5.3.6). If in that equation we put \( r = n \) and use the fact that the remainder in the quadrature formula is zero for every polynomial of degree \( <n \), then we obtain
\[ R(f) = \int_{a}^{b} f^{(n)}(t) K(t) dt \]  
(6.1.10)

where \( K(t) \) has the form (5.3.7). For a function which satisfies (6.1.8) we obtain from (6.1.10) the precise estimate²

\[ |R(f)| \leq M_n \int_{a}^{b} |K(t)| dt. \]  
(6.1.11)

6.2. NEWTON-COTES FORMULAS

The earliest known quadrature formulas are those which are now known as the Newton-Cotes formulas. Some of these are still widely used because of their simplicity. They are formulas for a constant weight function and a finite interval of integration.

Let us consider the integral

\[ \int_{a}^{b} f(x) dx. \]  
(6.2.1)

Let us divide the segment \([a, b]\) into \(n\) equal subsegments, of length

\[ h = \frac{b - a}{n}, \]

with endpoints \(a, a + h, a + 2h, \ldots, a + nh = b\). We will construct an interpolatory quadrature formula using these points as the nodes. To find the values of the coefficients \(A_k\) in a form which is independent of the segment \([a, b]\) let us write (6.1.3) in the form

\[ \int_{a}^{b} f(x) dx = (b - a) \sum_{k=0}^{n} B^n_k f(a + kh). \]  
(6.2.2)

The coefficients \(B^n_k = (b - a)^{-1} A_k\) are given by:

\[ B^n_k = (b - a)^{-1} \int_{a}^{b} \frac{\omega(x)}{(x - a - kh)\omega(a + kh)} dx, \]

where \(\omega(x) = (x - a)(x - a - h) \cdots (x - a - nh)\). If we introduce a new variable \(t\), by substituting \(x = a + th\), then we will have

\[ x - a - kh = h(t - k), \]

\[ \omega(x) = h^{n+1} t(t - 1)(t - 2) \cdots (t - n), \]

²Trans. note: For a better discussion of this result, and some examples, see the book by S. M. Nikol'skii listed in the references at the end of this chapter.
6.2. Newton-Cotes Formulas

\[ \omega'(a + kh) = (-1)^{n-k} h^n k!(n - k)! . \]

This gives

\[ B_k^n = \frac{(-1)^{n-k}}{nk!(n-k)!} \int_0^n t(t-1) \cdots (t-k+1) \times \]
\[ \times (t-k-1) \cdots (t-n) \, dt. \]  \hspace{1cm} (6.2.3)

Here we give\(^3\) the values of the coefficients \(B_k^n\) for \(n = 1\) to 10. Since \(B_k^n = B_{n-k}^n\) we have tabulated only those coefficients for \(k \leq \frac{1}{2} n\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(B_0^n)</th>
<th>(B_1^n)</th>
<th>(B_2^n)</th>
<th>(B_3^n)</th>
<th>(B_4^n)</th>
<th>(B_5^n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{1}{2})</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(\frac{1}{6})</td>
<td>(\frac{4}{6})</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(\frac{1}{8})</td>
<td>(\frac{3}{8})</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(\frac{7}{90})</td>
<td>(\frac{32}{90})</td>
<td>(\frac{12}{90})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(\frac{19}{288})</td>
<td>(\frac{75}{288})</td>
<td>(\frac{50}{288})</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(\frac{41}{840})</td>
<td>(\frac{216}{840})</td>
<td>(\frac{27}{840})</td>
<td>(\frac{272}{840})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(\frac{751}{17280})</td>
<td>(\frac{3577}{17280})</td>
<td>(\frac{1323}{17280})</td>
<td>(\frac{2989}{17280})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>(\frac{989}{28350})</td>
<td>(\frac{5888}{28350})</td>
<td>(\frac{-928}{28350})</td>
<td>(\frac{10496}{28350})</td>
<td>(\frac{-4540}{28350})</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>(\frac{2857}{89600})</td>
<td>(\frac{15741}{89600})</td>
<td>(\frac{1080}{89600})</td>
<td>(\frac{19344}{89600})</td>
<td>(\frac{5778}{89600})</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>(\frac{16067}{598752})</td>
<td>(\frac{106300}{598752})</td>
<td>(\frac{-48525}{598752})</td>
<td>(\frac{272400}{598752})</td>
<td>(\frac{-260550}{598752})</td>
<td>(\frac{427368}{598752})</td>
</tr>
</tbody>
</table>

\(^3\)Trans. note: The values of \(B_k^n\) for \(n = 1\) to 20 are given in the paper by W. W. Johnson and in the book by Z. Kopal.
Even this short table of the $B_k^n$ shows the irregularity of these coefficients. In order to appraise the Newton-Cotes formulas for a large number of nodes we will derive asymptotic representations for the $B_k^n$ for large $n$. To do this let us transform the integral

\[ l = \int_0^n \frac{x(x-1) \cdots (x-n)}{x-k} \, dx \]

occurring in (6.2.3). Using the relationships

\[ x(x-1) \cdots (x-n) = \frac{\Gamma(x+1)}{\Gamma(x-n)} \]

\[ \frac{1}{\Gamma(z)} = \frac{\Gamma(1-z) \sin \pi z}{\pi} \]

we obtain

\[ x(x-1) \cdots (x-n) = \frac{(-1)^n}{\pi} \frac{\Gamma(x+1)\Gamma(n+1-x) \sin \pi x}{\pi(x-k)} \]

\[ l = (-1)^n \int_0^n \frac{\Gamma(x+1)\Gamma(n+1-x) \sin \pi x}{\pi(x-k)} \, dx. \]

Let us divide this integral into 3 parts:

\[ \int_0^n = \int_0^3 + \int_3^{n-3} + \int_{n-3}^n = \alpha + \beta + \gamma. \]

We will first obtain an estimate for the integral $\beta$. From the theory of the function $\Gamma(z)$ it is known that $\frac{\Gamma'(z)}{\Gamma(z)}$ is a monotonically increasing function for $z > 0$. Thus $\frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{\Gamma'(n+1-x)}{\Gamma(n+1-x)}$, for $-1 < x < \frac{n}{2}$, will be negative and, for $\frac{n}{2} < x < n + 1$, will be positive. From this it

---

4See the paper by R. O. Kuz'min.
5This can be seen from the expansion

\[ \frac{\Gamma''(z)}{\Gamma(z)} = -\frac{1}{z} - C + \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k + z} \right). \]

See, for example, V. I. Smirnov, Course of Higher Mathematics, Gostekhizdat, Moscow, 1949, Vol. 3, sec. 73 (Russian); or E. D. Rainville, Special Functions, Macmillan, New York, 1960, p. 10.
follows that \( \ln \Gamma(n + 1) \) and also \( \Gamma(x + 1) \Gamma(n + 1 - x) \) will, for \( 3 \leq x \leq n - 3 \), have its largest value on the end of this segment:

\[
0 < \Gamma(x + 1) \Gamma(n + 1 - x) \leq \Gamma(4) \Gamma(n - 2) = 6 \Gamma(n - 2).
\]

For every \( x \) we have

\[
\left| \frac{\sin \pi x}{\pi(x - k)} \right| \leq 1 \quad \text{and thus}
\]

\[
|\beta| \leq 6n \Gamma(n - 2) = \frac{6 \Gamma(n + 1)}{(n - 2)(n - 1)} = O\left(\frac{\Gamma(n + 1)}{n^2}\right).
\]

We will, at first, study the integrals \( \alpha \) and \( \gamma \) for \( 1 \leq k \leq n - 1 \). It will be sufficient to consider the integral \( \alpha \). Using Taylor's formula and the fact that the derivative of the function \( \frac{\Gamma'(z)}{\Gamma(z)} \) is, for large \( z \), of the order \( \frac{1}{z} \) we obtain:

\[\ln \Gamma(n + 1 - x) = \ln \Gamma(n + 1) - \frac{x \Gamma'(n + 1)}{\Gamma(n + 1)} + O\left(\frac{1}{n}\right).\]

Thus using the fact that for large \( z \) the approximation \( \frac{\Gamma'(z)}{\Gamma(z)} = \ln z + O\left(\frac{1}{z}\right) \) is valid,\(^6\) we obtain

\[
\Gamma(n + 1 - x) = \Gamma(n + 1) e^{-x \ln n} \left[ 1 + O\left(\frac{1}{n}\right) \right].
\]

For \( 0 \leq x \leq 3 \) it is then evident that we have

\[
\Gamma(x + 1) \frac{\sin \pi x}{\pi(x - k)} = -\frac{x}{k} + O\left(\frac{x^2}{k}\right),
\]

\[
\alpha = \int_{0}^{3} \Gamma(n + 1) e^{-x \ln n} \left[ 1 + O\left(\frac{1}{n}\right) \right] \left[ -\frac{x}{k} + O\left(\frac{x^2}{k}\right) \right] dx.
\]

Because

\[
\int_{0}^{3} e^{-x \ln n} x \, dx = \frac{1}{ln^2 n} - \frac{1}{n^3} \left[ \frac{3}{ln n} + \frac{1}{ln^2 n} \right]
\]

and
\[ \int_0^3 e^{-x} \ln n \ x^2 \ dx = \frac{2}{\ln^3 n} - \frac{1}{n^3} \left[ \frac{9}{\ln n} + \frac{6}{\ln^2 n} + \frac{2}{\ln^3 n} \right] \]

then
\[ a = -\frac{\Gamma(n + 1)}{k \ln^2 n} \left[ 1 + O\left( \frac{1}{\ln n} \right) \right] \]

for \( 1 \leq k \leq n - 1 \). In a similar way we can obtain the following estimate for the integral \( \gamma \):
\[ \gamma = (-1)^{n-1} \frac{\Gamma(n + 1)}{(n - k) \ln^2 n} \left[ 1 + O\left( \frac{1}{\ln n} \right) \right] \]

for \( 1 \leq k \leq n - 1 \).

If we use the estimate which we obtained above for the integral \( \beta \), then we obtain the following estimate for the integral \( l \):
\[ l = \frac{(-1)^{n-1} \Gamma(n + 1)}{\ln^2 n} \left[ \frac{1}{k} + \frac{(-1)^n}{n - k} \right] \left[ 1 + O\left( \frac{1}{\ln n} \right) \right]. \]

This leads to the asymptotic representation of the Newton-Cotes coefficients for \( 1 \leq k \leq n - 1 \):
\[ B_k^n = \frac{(-1)^{k-1} n!}{k! (n - k)! \ln^2 n} \left[ \frac{1}{k} + \frac{(-1)^n}{n - k} \right] \left[ 1 + O\left( \frac{1}{\ln n} \right) \right]. \quad (6.2.4) \]

A similar calculation for \( B_0^n \) and \( B_n^n \) gives
\[ B_0^n = B_n^n = \frac{1}{n \ln n} \left[ 1 + O\left( \frac{1}{\ln n} \right) \right]. \quad (6.2.5) \]

From these expressions for \( B_k^n \) we see that for large \( n \) the Newton-Cotes formulas will have both positive and negative coefficients which exceed in absolute value any arbitrary large number. Thus, for large \( n \), a small discrepancy in the values of the function \( f(a + kh) \) can lead to a large error in the quadrature sum. Therefore the Newton-Cotes formulas with large numbers of nodes are of little use for practical calculations.

The expression (6.1.5) for the remainder \( R(f) \) of the Newton-Cotes formulas is:
\[ R(f) = \int_a^b \omega(x) f(x, a, a + h, \ldots, a + nh) \ dx. \quad (6.2.6) \]

This equation can be reduced to a very simple form which is much more convenient for application.
6.2. Newton-Cotes Formulas

Let us consider, at first, the case when \( n \) is an even number; in this case the Newton-Cotes formulas have an odd number of nodes. The polynomial \( \omega(x) = (x - a)(x - a - h) \cdots (x - a - nh) \) possesses the property \( \omega(a + z) = -\omega(a + nh - z) \) and the graph of it will be symmetric with respect to the midpoint \( \frac{a + b}{2} \) of the segment \([a, b]\). An example of the form of the graph is illustrated by Figure 2.

![Figure 2](image)

Let us consider the function \( \Omega(x) = \int_a^x \omega(t) dt \). We note, first of all, that \( \Omega(a) = 0 \) and \( \Omega(a + nh) = \Omega(b) = 0 \). This last equation follows from the symmetry properties of the function \( \omega(x) \). We show now that \( \Omega(x) \) is not zero anywhere inside \([a, b]\). To do this consider the integrals \( I_\nu = \int_{a+\nu h}^{a+(\nu+1)h} \omega(x) dx \). The assertion will be proved if we establish that the sequence of numbers \( I_0, I_1, \ldots, I_{n/2-1} \) decrease in absolute value.

If in the integral \( I_\nu = \int_{a+\nu h}^{a+(\nu+1)h} (x - a)(x - a - h) \cdots (x - a - nh) dx \) we set \( x = y + h \), then the integral is transformed to the form

\[
I_\nu = \int_{a+(\nu-1)h}^{a+\nu h} (y - a + h) \cdots (y - a) \cdots (y - a - (n - 1)h) dy =
\]

\[
= \int_{a+(\nu-1)h}^{a+\nu h} \frac{y - a + h}{y - a - nh} \omega(y) dy = \frac{\eta - a + h}{\eta - a - nh} I_{\nu-1},
\]

where \( a + (\nu - 1)h < \eta < a + \nu h \). In order that \( |I_\nu| < |I_{\nu-1}| \) the inequality \( \eta - a + h < nh - \eta + a \) or \( \eta - a < \frac{1}{2}(n - 1)h \) must be satisfied. But this last inequality is indeed true because

\[
\eta < a + \nu h, \quad \eta - a < \nu h \leq \left( \frac{n}{2} - 1 \right) h.
\]

Let us integrate (6.2.6) by parts and apply the mean value theorem:
Approximate Calculation of Definite Integrals

\[ R(f) = \Omega(x) f(x, a, \ldots, a + nh) \left| b - \int_a^b f'(x, a, \ldots, a + nh) \Omega(x) \, dx \right| = \]
\[ = - f'(\eta, a, \ldots, a + nh) \int_a^b \Omega(x) \, dx, \quad a < \eta < b. \]

From

\[ f(x, a, \ldots, a + nh) = \]
\[ = \int_0^1 \cdots \int_0^{t_n} f^{(n+1)} \left( x + t_1(a - x) + \cdots + h \sum_{\nu=2}^{n+1} t_\nu \right) \, dt_{n+1} \cdots dt_1 \]

it follows that

\[ f'(x, a, \ldots, a + nh) = \int_0^1 \cdots \int_0^{t_n} (1 - t_1) \times \]
\[ \times f^{(n+2)} \left( x + t_1(a - x) + \cdots + h \sum_{\nu=2}^{n+1} t_\nu \right) \, dt_{n+1} \cdots dt_1 \]

and applying the mean value theorem to this last integral gives

\[ f'(\eta, a, \ldots, a + nh) = \frac{f^{(n+2)}(\xi)}{(n + 2)!}, \quad a < \xi < b. \]

Finally

\[ \int_a^b \Omega(x) \, dx = x \Omega(x) \left| b - \int_a^b x \Omega'(x) \, dx \right| = - \int_a^b x \omega(x) \, dx. \]

This proves that the remainder term of the Newton-Cotes quadrature formula, for an odd number of nodes, can be expressed as

\[ R(f) = \frac{f^{(n+2)}(\xi)}{(n + 2)!} \int_a^b x \omega(x) \, dx. \quad (6.2.7) \]

We will now find the sign of the coefficient of \( f^{(n+2)}(\xi) \). The function

\[ \Omega(x) = \int_a^x \omega(t) \, dt \]

does not change sign on the segment \([a, b]\) and therefore it is sufficient to calculate its sign at one point; let us use the point \( x = a + h \):

\[ \Omega(a + h) = \int_a^{a+h} \omega(t) \, dt. \]
For $a < t < a + h$ the first factor in the product $\omega(t) = (t - a)(t - a - h) \cdots (t - a - nh)$ is positive and all the remaining factors are negative. Thus the sign of $\Omega(t)$ is $(-1)^n$ for $t \in (a, b)$. Because

$$\int_a^b x \omega(x) \, dx = - \int_a^b \Omega(x) \, dx$$

it follows that the sign of $\int_a^b x \omega(x) \, dx$ is $(-1)^{n+1} = -1$ since $n$ is even.

Thus we have established the theorem:

**Theorem 2.** If the number of nodes, which is $n + 1$, in the Newton-Cotes formula (6.2.2) is odd and if the function $f(x)$ has a continuous derivative of order $n + 2$ on $[a, b]$, then the expression for $R(f)$ is given by (6.2.7) where $\xi$ is a point inside $[a, b]$. The coefficient of $f^{(n+2)}(\xi)$ is negative.

We indicate two consequences of this theorem.

1. If the number of nodes in formula (6.2.2) is odd then the algebraic degree of precision of this formula is $n + 1$.

From the representation (6.2.7) for the error, formula (6.2.2) will be exact whenever $f(x)$ is a polynomial of degree $\leq n + 1$. If $f(x)$ is a polynomial of degree $n + 2$ then $f^{(n+2)}(x)$ will be different from zero and $R(f) \neq 0$.

2. Let us assume that $f^{(n+2)}(x)$ exists and is continuous on $[a, b]$. We will construct the representation (5.3.6) for the error. We have $r = n + 2$ and for simplicity we take $a = a$. Because the degree of precision of (6.2.2) is $n + 1$, the terms under the summation sign in (5.3.6) will be zero and we will have the following expression for the error:

$$R(f) = \int_a^b f^{(n+2)}(t)K(t) \, dt.$$

(6.2.8)

Using the fact that $p(x) = 1$, the kernel $K(t)$ is calculated to be

$$K(t) = \frac{(b - t)^{n+2}}{(n + 2)!} - \sum_{k=1}^n A_k E(a + kh - t) \frac{(a + kh - t)^{n+1}}{(n + 1)!}.$$

We can show that the function $K(t)$ is nonpositive on $[a, b]$.

From (6.2.7) we see that if $f^{(n+2)}(x)$ does not become zero at any point of $[a, b]$ then $R(f) \neq 0$. If the function $K(t)$ would change sign on $[a, b]$ then there would exist a function $f^{(n+2)}(x)$, which is different from zero throughout $[a, b]$, for which $\int_a^b f^{(n+2)}(t)K(t) \, dt = 0$. From the derivative
Approximate Calculation of Definite Integrals

\( f^{(n+2)}(x) \) we could reconstruct the function \( f(x) \) in the usual way. For such a function \( R(f) = 0 \) which contradicts our assumption. Because the coefficient of \( f^{(n+2)}(\xi) \) in (6.2.7) is negative the kernel \( K(t) \) must be a nonpositive function on \([a, b]\):

\[
K(t) \leq 0.
\]

Let us now consider the case when \( n \) is an odd number; in this case there are an even number of nodes in formula (6.2.2). The polynomial \( \omega(x) \) takes on equal values at the points \( a + t \) and \( b - t \), \( t < b - a \). This means that the graph of \( \omega(x) \) is symmetric with respect to the line \( x = \frac{a + b}{2} \).

In order to simplify the expression (6.2.6) for the remainder let us split the segment \([a, b]\) into two parts \([a, a + (n - 1)h]\) and \([a + (n - 1)h, b]\). The polynomial \( \omega(x) \) does not change sign on the second part of the segment and we can apply the mean value theorem to the integral over this segment:

\[
R(f) = \int_a^{a + (n-1)h} \omega(x)f(x, a, \ldots, a + nh)\,dx + \frac{f^{(n+1)}(\xi_1)}{(n + 1)!} \int_{a + (n-1)h}^b \omega(x)\,dx = I + II.
\]

Let us now look at the integral over the first part of the segment. From \( \omega(x) \) we separate the factor \( x - a - nh \) and write \( \omega(x) = (x - a - nh)\omega_1(x) \). By the definition of the divided difference

\[
f(a + nh, \ldots, a, x) = \frac{f[a + (n - 1)h, \ldots, a, x] - f[a + nh, \ldots, a]}{x - a - nh}
\]

and thus

\[
I = \int_a^{a + (n-1)h} \omega_1(x)f(x, a, \ldots, a + (n - 1)h)\,dx - f(a, \ldots, a + nh)\int_a^{a + (n-1)h} \omega_1(x)\,dx.
\]

Because \( n \) is an even number \( \int_a^{a + (n-1)h} \omega_1(x)\,dx = 0 \), and the second term in the above expression for \( I \) vanishes. The first term is an integral of the form (6.2.6) for an odd number of nodes and it can be expressed as:
6.2. Newton-Cotes Formulas

\[
I = \frac{f^{(n+1)}(\xi_2)}{(n+1)!} \int_a^{a+(n-1)h} x \omega_1(x) \, dx,
\]

where we recall that the coefficient of \( f^{(n+1)}(\xi_2) \) is a negative number.

Since \( \int_a^{a+(n-1)h} \omega_1(x) \, dx = 0 \), we can replace \( x \omega_1(x) \) by \((x - a - nh) \times \omega_1(x) = \omega(x)\) in the integral \( I \). Thus we obtain the expression

\[
R(f) = \frac{f^{(n+1)}(\xi_2)}{(n+1)!} \int_a^{a+(n-1)h} \omega(x) \, dx + \frac{f^{(n+1)}(\xi_1)}{(n+1)!} \int_{a+(n-1)h}^{b} \omega(x) \, dx.
\]

For \( a + (n-1)h < x < b \) the last factor in

\[
\omega(x) = (x - a) (x - a - h) \cdots (x - a - nh)
\]

is negative and the other factors are positive. This means that

\[
\int_{a+(n-1)h}^{b} \omega(x) \, dx < 0.
\]

Since the coefficients of both \( f^{(n+1)}(\xi_2) \) and \( f^{(n+1)}(\xi_1) \), in the last expression for \( R(f) \), are different from zero and of the same sign and since \( f^{(n+1)}(x) \) is a continuous function, then between \( \xi_1 \) and \( \xi_2 \) there exists a point \( \xi \) for which

\[
R(f) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \int_a^b \omega(x) \, dx. \tag{6.2.9}
\]

This proves:

**Theorem 3.** If the number of nodes in the Newton-Cotes formula (6.2.2) is even and if \( f(x) \) has a continuous derivative of order \( n + 1 \) on \([a, b]\) then the remainder \( R(f) \) is given by (6.2.9) where \( \xi \) is a point inside the segment. The coefficient of \( f^{(n+1)}(\xi) \) in this expression is negative.

As in the case for an odd number of nodes there are two immediate consequences of this theorem.

1. If formula (6.2.2) has an even number of nodes \( n + 1 \), then its algebraic degree of precision is \( n + 1 \).

2. If (6.2.2) has an even number of nodes and if \( f(x) \) has a continuous derivative of order \( n + 1 \) on \([a, b]\) then the remainder in this formula can be represented in the form

\[
R(f) = \int_a^b f^{(n+1)}(t) K(t) \, dt \tag{6.2.10}
\]
where the kernel $K(t)$ is nonpositive on $[a, b]$ and is given by

$$K(t) = \frac{(b-t)^{n+1}}{(n+1)!} - \sum_{k=1}^{n} A_k E(a + kh - t) \frac{(a + kh - t)^n}{n!}. \quad (6.2.11)$$

### 6.3. CERTAIN OF THE SIMPLEST NEWTON-COTES FORMULAS

Newton-Cotes formulas with a large number of nodes are seldom applied in practical calculations for the reasons pointed out in the previous section. It is preferable to use a formula with a small number of nodes and to increase its accuracy split up the segment $[a, b]$ into many subintervals and apply the formula to each of these smaller intervals.

Let us consider, at first, the case $n = 1$. Here we interpolate $f(x)$ using its values at the endpoints $a, b$ of the segment of integration.

Equation (6.2.2) then becomes the well known formula:

$$\int_{a}^{b} f(x) \, dx = (b - a) \left[ \frac{1}{2} f(a) + \frac{1}{2} f(b) \right] \quad (6.3.1)$$

which is called the trapezoidal formula. In this case we have $\omega(x) = (x - a) (x - b)$ and (6.2.9) gives

$$R(f) = -\frac{(b-a)^3}{12} f''(\xi), \quad a < \xi < b. \quad (6.3.2)$$

To study the error in the formula (6.3.1) when the interval of integration is split up into subsegments we will obtain a representation for the remainder which is different from (6.2.10).

Let us assume that $f(x)$ has a continuous second derivative. We will expand it in Bernoulli polynomials using equation (1.4.2) with $\nu = 2$:

$$f(x) = (b - a)^{-1} \int_{a}^{b} f(t) \, dt + B_1 \left( \frac{x-a}{b-a} \right) \left[ f(b) - f(a) \right] - \frac{(b-a)}{2} \int_{a}^{b} f''(t) \left[ B^*_2 \left( \frac{x-t}{b-a} \right) - B^*_2 \left( \frac{x-a}{b-a} \right) \right] \, dt.$$ 

Because (6.3.1) is exact for linear functions, $R(f)$ reduces to the remainder of the quadrature for the last term on the right hand side of this equation. The remainder of this term is the same if we replace $B^*_2(x)$ by $\gamma^*_2(x) = B^*_2(x) - B_2$:

$$R(f) = -\frac{(b-a)}{2} \int_{a}^{b} f''(t) \, R_x \left[ \gamma^*_2 \left( \frac{x-t}{b-a} \right) - \gamma^*_2 \left( \frac{x-a}{b-a} \right) \right] \, dt.$$
6.3. The Simplest Newton-Cotes Formulas

The symbol \( R_x \) denotes the remainder of the quadrature with respect to the variable \( x \).

In the following calculations we use the rule for integration of Bernoulli polynomials; the fact that \( y_2^* \) and \( B_2^* \) have period one; and the relations \( y_2(1) = y_2(0) = 0 \)

\[ R_x \left[ y_2^* \left( \frac{x-t}{b-a} \right) - y_2^* \left( \frac{x-a}{b-a} \right) \right] = \int_a^b \left[ y_2^* \left( \frac{x-t}{b-a} \right) - y_2^* \left( \frac{x-a}{b-a} \right) \right] dt - \]

\[ - \frac{(b-a)^2}{2} \left[ y_2^* \left( \frac{a-t}{b-a} \right) - y_2^* (0) \right] + \frac{(b-a)}{2} \left[ y_2^* \left( \frac{b-t}{b-a} \right) - y_2^* (1) \right] = \]

\[ = - (b-a) y_2^* \left( \frac{b-t}{b-a} \right), \]

\[ R(f) = \frac{(b-a)^2}{2!} \int_a^b f''(t) y_2^* \left( \frac{b-t}{b-a} \right) dt. \] (6.3.3)

Now we split up the segment \([a, b]\) into \( n \) equal subsegments of length \( h = \frac{b-a}{n} \). Consider the segment \([a + kh, a + (k + 1)h]\) and let us apply formula (6.3.1)

\[ \int_{a+kh}^{a+(k+1)h} f(x) dx = \frac{h}{2} [f(a + kh) + f(a + (k + 1)h) + \]

\[ + \frac{h^2}{2!} \int_{a+kh}^{a+(k+1)h} f''(t) y_2^* \left( \frac{a + kh - t}{h} \right) dt. \]

Since \( y_2^*(x) \) has period one we have \( y_2^* \left( \frac{a + kh - t}{h} \right) = y_2^* \left( \frac{a - t}{h} \right) \). We carry out this calculation for each subsegment and sum the results to obtain the repeated trapezoidal formula with remainder in the form of a definite integral

\[ \int_a^b f(x) dx = h \left[ \frac{1}{2} f_0 + f_1 + \cdots + f_{n-1} + \frac{1}{2} f_n \right] + \]

\[ + \frac{h^2}{2!} \int_a^b f''(t) y_2^* \left( \frac{a - t}{h} \right) dt. \] (6.3.4)

where we have written \( f_k = f(a + kh) \). The kernel \( y_2^* \left( \frac{a - t}{h} \right) \) of the re-
remainder does not change sign and the mean value theorem can be applied to the integral in (6.3.4) to give

\[ R(f) = -\frac{(b-a)^3}{12n^2} f'''(\xi). \]

We go now to the case \( n = 2 \). Here we interpolate \( f(x) \) using its values at the three points \( a, \frac{a+b}{2}, b \).

Quadrature formula (6.2.2) then becomes

\[
\int_a^b f(x) \, dx = (b-a) \left[ \frac{1}{6} f(a) + \frac{4}{6} f \left( \frac{a+b}{2} \right) + \frac{1}{6} f(b) \right]
\]

which is known as Simpson's formula. The remainder is found by (6.2.7) to be

\[
R(f) = \frac{f^{(4)}(\xi)}{4!} \int_a^b x(x-a) \left( x - \frac{a+b}{2} \right) (x-b) \, dx = -\frac{1}{90} \left( \frac{b-a}{2} \right)^5 f^{(4)}(\xi).
\]

Assuming that \( f(x) \) has four continuous derivatives on \([a, b]\) we expand it in Bernoulli polynomials as follows:

\[
f(x) = (b-a)^{-1} \int_a^b f(t) \, dt + \\
+ \sum_{k=1}^{3} \frac{(b-a)^{k-1}}{k!} B_k^* \left( \frac{x-a}{b-a} \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] - \\
- \frac{(b-a)^3}{4!} \int_a^b f^{(4)}(t) \left[ y_4^* \left( \frac{x-t}{b-a} \right) - y_4^* \left( \frac{x-a}{b-a} \right) \right] \, dt.
\]

Equation (6.3.5) is exact for all polynomials of third degree. Therefore \( R(f) \) will be the remainder when the quadrature is applied to the last term on the right hand side of this equation:

\[
R(f) = -\frac{(b-a)^3}{4!} \int_a^b f^{(4)}(t) R_x \left[ y_4^* \left( \frac{x-t}{b-a} \right) - y_4^* \left( \frac{x-a}{b-a} \right) \right] \, dt.
\]

\[
R_x \left[ y_4^* \left( \frac{x-t}{b-a} \right) - y_4^* \left( \frac{x-a}{b-a} \right) \right] = \int_a^b y_4^* \left( \frac{x-t}{b-a} \right) - y_4^* \left( \frac{x-a}{b-a} \right) \, dx -
\]
6.3. The Simplest Newton-Cotes Formulas

\[-(b - a) \left\{ \frac{1}{6} \left[ \gamma_4^* \left( \frac{a - t}{b - a} \right) - \gamma_4^* (0) \right] + \right.\]
\[+ \frac{4}{6} \left[ \gamma_4^* \left( \frac{\frac{1}{2} (a + b) - t}{b - a} \right) - \gamma_4^* \left( \frac{1}{2} \right) \right] + \left.\right.\]
\[+ \frac{1}{6} \left[ \gamma_4^* \left( \frac{b - t}{b - a} \right) - \gamma_4^* (1) \right]\}

\[-(b - a) \left\{ \frac{1}{3} \gamma_4^* \left( \frac{a - t}{b - a} \right) + \frac{2}{3} \gamma_4^* \left( \frac{\frac{1}{2} (a + b) - t}{b - a} \right) - \frac{1}{24} \right\} .\]

In these calculations we have made use of the values \( B_n \left( \frac{1}{2} \right) \) given in (1.2.14):

\[\gamma_4^* \left( \frac{1}{2} \right) = B_4 \left( \frac{1}{2} \right) - B_4 = -(2 - 2^{-3}) B_4 = \frac{1}{16} ,\]

\[R(f) = \frac{(b - a)^4}{4!} \int_a^b f^{(4)}(t) \left\{ \frac{1}{3} \gamma_4^* \left( \frac{a - t}{b - a} \right) + \right.\]
\[+ \frac{2}{3} \gamma_4^* \left( \frac{\frac{1}{2} (a + b) - t}{b - a} \right) - \frac{1}{24} \right\} dt. \tag{6.3.9}\]

Let us divide \([a, b]\) into \( n \) equal subsegments of length \( h = \frac{b - a}{n} \) where \( n \) is an even integer. Let us apply formula (6.3.5) with remainder (6.3.9) to the segment \([a + (k - 1) h, a + (k + 1) h]\) which consists of an adjacent pair of subsegments:

\[\int_{a + (k - 1) h}^{a + (k + 1) h} f(x) dx = 2h \left[ \frac{1}{6} f_{k-1} + \frac{4}{6} f_k + \frac{1}{6} f_{k+1} \right] + \]
\[+ \frac{2}{9} h^4 \int_{a + (k - 1) h}^{a + (k + 1) h} f^{(4)}(t) \left\{ \gamma_4^* \left( \frac{a + (k - 1) h - t}{2h} \right) + \right.\]
\[+ 2 \gamma_4^* \left( \frac{a + kh - t}{2h} \right) - \frac{1}{8} \right\} dt. \]
Carrying out this last calculation for the segments
\[ [a, a + 2h], \ [a + 2h, a + 4h], \ldots, \ [a + (n - 2)h, a + nh] \]
and summing the results we obtain the repeated Simpson’s rule:
\[
\int_a^b f(x) \, dx = \frac{h}{3} [f_0 + f_n + 2(f_2 + f_4 + \cdots + f_{n-2}) + \\
+ 4(f_1 + f_3 + \cdots + f_{n-1})] + \frac{2}{9} h^4 \int_a^b f^{(4)}(t) \, dt
\]
(6.3.10)
The remainder term in (6.3.10) differs only in notation from (6.2.10) and the kernel of the remainder is therefore a function which does not change sign on the interval \([a, b]\). Applying the mean value theorem to this integral permits us to write the remainder term of (6.3.10) in the form
\[
R(f) = -\frac{(b - a)^5}{180 n^4} f^{(4)}(\xi).
\]
(6.3.11)
For \(n = 3\) we obtain a formula which is sometimes called the “three-eighths rule,”
\[
\int_a^b f(x) \, dx = H \left[ \frac{1}{8} f(a) + \frac{3}{8} f \left( a + \frac{1}{3} H \right) + \\
+ \frac{3}{8} f \left( a + \frac{2}{3} H \right) + \frac{1}{8} f(a + H) \right],
\]
(6.3.12)
\[
\omega(x) = (x - a) \left( x - a - \frac{1}{3} H \right) \left( x - a - \frac{2}{3} H \right) (x - a - H)
\]
\[
R(f) = \frac{f^{(4)}(\xi)}{4!} \int_a^b \omega(x) \, dx = -\frac{(b - a)^5}{6480} f^{(4)}(\xi)
\]
(6.3.13)
\[
H = b - a.
\]
In order to obtain the integral representation for the remainder \(R(f)\) in the repeated three-eighths rule we expand \(f(x)\) in Bernoulli polynomials in the form (6.3.7). Equation (6.3.12) is exact for all polynomials of degree \(\leq 3\) and \(R(f)\) has the form (6.3.8), but with other values of the inte-
grand. In the present case\footnote{Here we make use of the following relationships:
   a. $\gamma_4$ has period one, that is $\gamma_4(z + 1) = \gamma_4(z)$
   b. If we put $n = 4$, $x = 1/3$, $m = 3$ in (1.2.8) we find}
\[
R_x \left[ \gamma_4^* \left( \frac{x - t}{H} \right) - \gamma_4^* \left( \frac{x - a}{H} \right) \right] = \int_a^b \left[ \gamma_4^* \left( \frac{x - t}{H} \right) - \gamma_4^* \left( \frac{x - a}{H} \right) \right] dx - \\
- H \left\{ \frac{1}{8} \left[ \gamma_4^* \left( \frac{a - t}{H} \right) - \gamma_4^*(0) \right] + \frac{3}{8} \left[ \gamma_4^* \left( \frac{a - t}{H} + \frac{1}{3} \right) - \gamma_4^*(1) \right] + \\
+ \frac{3}{8} \left[ \gamma_4^* \left( \frac{a - t}{H} + \frac{2}{3} \right) - \gamma_4^*(2) \right] + \frac{1}{8} \left[ \gamma_4^* \left( \frac{a - t}{H} + 1 \right) - \gamma_4^*(1) \right] \right\} = \\
- \frac{H}{8} \left\{ 2\gamma_4^* \left( \frac{a - t}{H} \right) + 3\gamma_4^* \left( \frac{a - t}{H} + \frac{1}{3} \right) + 3\gamma_4^* \left( \frac{a - t}{H} + \frac{2}{3} \right) - \frac{8}{27} \right\}.
\]

Thus we obtain
\[
R(f) = \frac{H^4}{4 \cdot 18} \int_a^b f^{(4)}(t) \left\{ 2\gamma_4^* \left( \frac{a - t}{H} \right) + 3\gamma_4^* \left( \frac{a - t}{H} + \frac{1}{3} \right) + \\
+ 3\gamma_4^* \left( \frac{a - t}{H} + \frac{2}{3} \right) - \frac{8}{27} \right\} dt. \tag{6.3.14}
\]

Let $n$ be a multiple of three. We divide $[a, b]$ into $n$ equal parts of length $h = \frac{b - a}{n}$. Let us take the segment $[a + kh, a + (k + 3)h]$ and apply to it the three-eighths rule with remainder in the form (6.3.14)
\[
\int_a^{a+(k+3)h} f(x) dx = \frac{3h}{8} \left\{ f[a + kh] + 3f[a + (k + 1)h] + \\
+ 3f[a + (k + 2)h] + f[a + (k + 3)h] \right\} + \\
+ \frac{27h^4}{64} \int_a^{a+(k+3)h} f^{(4)}(t) \left\{ 2\gamma_4^* \left( \frac{a + kh - t}{3h} \right) \right\} + \\
\]
Approximate Calculation of Definite Integrals

\[
+ 3y_4^* \left( \frac{a + kh - t}{3h} + \frac{1}{3} \right) + \\
+ 3y_4^* \left( \frac{a + kh - t}{3h} + \frac{2}{3} \right) - \frac{8}{27} dt.
\]

Writing equations like this for the segments

\[
[a, a + 3h], [a + 3h, a + 6h], \ldots, [a + (n - 3)h, a + nh]
\]

and summing the results we obtain the repeated three-eighths rule:

\[
\int_a^b f(x)dx = \frac{3h}{8} \left\{ f_0 + f_n + 2(f_3 + f_6 + \cdots + f_{n-3}) + \\
+ 3(f_1 + f_2 + f_4 + f_5 + \cdots + f_{n-2} + f_{n-1}) \right\} + \\
+ \frac{27h^4}{64} \int_a^b f^{(4)}(t) \left\{ 2y_4^* \left( \frac{a - t}{3h} \right) + 3y_4^* \left( \frac{a - t}{3h} + \frac{1}{3} \right) + \\
+ 3y_4^* \left( \frac{a - t}{3h} + \frac{2}{3} \right) - \frac{8}{27} \right\} dt.
\]

(6.3.15)

We can also apply the mean value theorem to the integral representation for the remainder in the last expression. The remainder of the three-eighths rule can thus be written

\[
R(f) = -\frac{(b - a)^5}{80n^4} f^{(4)}(\xi), \quad a < \xi < b.
\]

(6.3.16)

When the number of segments \( n \) is a multiple of both 2 and 3 we can approximate the integral by both Simpson's rule and the three-eighths rule. Both of these formulas have the same algebraic degree of precision and are almost equally simple to use. The choice between these formulas must be based on the error of the final results. Comparison of the remainder terms (6.3.11) and (6.3.16) shows that use of the three-eighths rule may lead to an error which is more than twice as great as the error obtained by use of Simpson's rule. Thus we are forced to prefer Simpson's rule over the three-eighths rule.

REFERENCES


S. M. Nikol'skii, Quadrature Formulas, Fizmatgiz, Moscow, 1958 (Russian).
J. F. Steffensen, Interpolation, Williams and Wilkins, Baltimore, 1927.
7.1. GENERAL THEOREMS

In the beginning of this section we make the same assumptions about the weight function \( p(x) \) as we made in Chapter 5.

The quadrature formula

\[
\int_a^b p(x)f(x)dx \approx \sum_{k=1}^{n} A_k f(x_k),
\]

for a fixed \( n \), contains the \( 2n \) parameters \( A_k \) and \( x_k \) (\( k = 1, 2, \ldots, n \)). The problem is to select these parameters so that formula (7.1.1) will be exact for all polynomials of the highest possible degree (that is, for all polynomials of degree \( \leq k \), where \( k \) is as large as possible).

In Section 6.1 we showed, by counting the choices of the coefficients \( A_k \), that for any arrangement of \( x_k \) we can find an equation (7.1.1) which is exact for all polynomials of degree \( \leq n - 1 \). This requirement completely defines the coefficients \( A_k \): formula (7.1.1) must be interpolatory and its coefficients must be given by (6.1.4).

In order to increase the precision of (7.1.1) the choice of the nodes \( x_k \) is still at our disposal. We might hope that for some choice of these nodes the degree of precision can be increased by \( n \) and that the formula can be made exact for all polynomials of degree \( \leq 2n - 1 \). Under what circumstances this can be achieved will be seen below.

We will now establish the conditions which must be satisfied by the
7.1. General Theorems

\( A_k \) and \( x_k \) in order that the degree of precision of formula (7.1.1) will be not less than \( 2n - 1 \).

We prefer to consider the polynomial \( \omega(x) = (x - x_1)(x - x_2) \cdots (x - x_n) \) instead of the nodes \( x_k \) themselves. If we know the \( x_k \), then we can easily construct the polynomial \( \omega(x) \). Conversely, if we know the polynomial \( \omega(x) = x^n + a_1x^{n-1} + \ldots \), then determining the roots of \( \omega(x) \) will give us the \( x_k \).

We must remember that if we determine \( \omega(x) \) instead of the \( x_k \) directly then we must be careful that the roots of \( \omega(x) \) will be real, distinct and located in the segment \([a, b]\).

**Theorem 1.** If formula (7.1.1) is to be exact for all polynomials of degree \( \leq 2n - 1 \), then it is necessary and sufficient that (7.1.1) be interpolatory and that the polynomial \( \omega(x) \) be orthogonal with respect to \( p(x) \) to all polynomials \( Q(x) \) of degree \( < n \):

\[
\int_{a}^{b} p(x) \omega(x) Q(x) \, dx = 0.
\]

**Proof.** First we establish the necessity. If (7.1.1) is to be exact for all polynomials of degree \( \leq 2n - 1 \), then it is also exact for all polynomials of degree \( \leq n - 1 \) and therefore, by Theorem 1 of Chapter 6, it must be interpolatory.

Let \( Q(x) \) be any polynomial of degree \( \leq n - 1 \). The product \( f(x) = \omega(x) Q(x) \) is a polynomial of degree \( \leq 2n - 1 \) and equation (7.1.1) must be exact for it. But \( f(x_k) = 0 \) \( (k = 1, 2, \ldots, n) \) and hence

\[
\int_{a}^{b} p(x) \omega(x) Q(x) \, dx = 0.
\]

This shows the necessity of the orthogonality condition.

We now prove the sufficiency of the conditions. Let \( f(x) \) be an arbitrary polynomial of degree \( \leq 2n - 1 \). We can divide \( f(x) \) by \( \omega(x) \) and represent \( f(x) \) in the form

\[
f(x) = Q(x) \omega(x) + \rho(x)
\]

where \( Q(x) \) and \( \rho(x) \) are polynomials of degree \( \leq n - 1 \). Since \( \omega(x_k) = 0 \) we have

\[
f(x_k) = \rho(x_k), \quad (k = 1, 2, \ldots, n)
\]

\[
\int_{a}^{b} p(x)f(x) \, dx = \int_{a}^{b} p(x)\omega(x) Q(x) \, dx + \int_{a}^{b} p(x) \rho(x) \, dx.
\]
The first of the integrals on the right hand side is zero by the assumed orthogonality. Because the degree of \( p(x) \) is not greater than \( n - 1 \), and because formula (7.1.1) is assumed to be interpolatory, then the equation

\[
\int_a^b p(x) \rho(x) \, dx = \sum_{k=1}^n A_k \rho(x_k)
\]

must be exact. Since \( f(x_k) = \rho(x_k) \) we must also have

\[
\int_a^b p(x)f(x) \, dx = \sum_{k=1}^n A_k f(x_k)
\]

and formula (7.1.1) will indeed be exact for an arbitrary polynomial of degree \( \leq 2n - 1 \). This completes the proof.

The possibility of constructing formulas with degree of precision \( 2n - 1 \) is related to the existence of polynomials \( \omega(x) \) of degree \( n \) which possess the above stated orthogonality property. If the weight function \( p(x) \) changes sign on \([a, b]\) then such a polynomial \( \omega(x) \) may not exist. If such a polynomial does exist its roots might not satisfy the above requirements.

In the remainder of this section we will assume that the weight function \( p(x) \) is nonnegative on \([a, b]\):

\[
p(x) \geq 0, \quad \text{for } x \in [a, b].
\]

In this case, as was shown in Section 2.1, the polynomial \( \omega(x) \) of \( n^{th} \) degree, which is orthogonal on \([a, b]\) with respect to \( p(x) \) to all polynomials of lower degree, does exist for all \( n \). The roots of \( \omega(x) \) are real, distinct and lie inside the segment \([a, b]\).

These remarks are summarized by the following statement:

If \( p(x) \geq 0 \) for \( x \in [a, b] \), then a quadrature formula (7.1.1), which is exact for all polynomials of degree \( \leq 2n - 1 \), exists for all \( n \).

Up until now it has not been established that \( 2n - 1 \) is the highest degree for which formula (7.1.1) is exact. If \( p(x) \) changes sign on \([a, b]\) then this may not be true. But, if \( p(x) \) does not change sign then it is easy to prove the following:

**Theorem 2.** If \( p(x) \geq 0 \) then no matter how we choose the \( x_k \) and \( A_k \) equation (7.1.1) can not be exact for all polynomials of degree \( 2n \).

**Proof.** For the polynomial \( P(x) = \omega^2(x) \), which has degree \( 2n \), the integral \( \int_a^b p(x)\omega^2(x) \, dx > 0 \) because the function \( p(x) \) is nonnegative.
and not identically zero. The quadrature sum $\sum A_k P(x_k)$ is zero because $P(x_k) = 0$. Hence equation (7.1.1) cannot be exact for $P(x) = \omega^2(x)$.

Now we will discuss the construction of quadrature formulas which have the highest degree of precision. Let us consider the system of polynomials $P_n(x)$ ($n = 1, 2, \ldots$) which are orthogonal on $[a, b]$ with respect to the weight function $p(x)$. In order to be definite let us assume that this system is normalized. The $n^{th}$ degree polynomial of this system $P_n(x)$ can differ from $\omega(x)$ by only a constant multiple. The roots of $P_n(x)$ will thus be the nodes $x_k$ ($k = 1, 2, \ldots, n$) which are to be used in the quadrature formula.

The coefficients $A_k$ are determined by equation (6.1.4) or equivalently by

$$A_k = \int_a^b p(x) \frac{P_n(x)}{(x - x_k) P_n'(x_k)} dx. \quad (7.1.2)$$

In order to calculate $A_k$ by (7.1.2) we make use of the Christoffel-Darboux identity (2.1.11) by substituting in that equation $t = x_k$. After dividing by $x - x_k$ we obtain

$$\sum_{s=0}^{n-1} P_s(x) P_s(x_k) = -\frac{a_n}{a_{n+1}} \frac{P_n(x) P_{n+1}(x_k)}{x - x_k}$$

where $a_n$ is the coefficient of $x^n$ in $P_n(x)$.

Let us multiply this last equation by $p(x)$ and integrate over $[a, b]$. The integral $\int_a^b p(x) P_s(x) dx$ is zero for $s \geq 1$, by the orthogonality of $P_s(x)$, and is 1 for $s = 0$, by the normality of $P_0(x)$. After carrying out the integration we have

$$1 = -\frac{a_n}{a_{n+1}} P_{n+1}(x_k) \int_a^b p(x) \frac{P_n(x)}{x - x_k} dx.$$ 

Hence we obtain

$$A_k = -\frac{a_{n+1}}{a_n} \frac{1}{P_n'(x_k) P_{n+1}(x_k)}. \quad (7.1.3)$$

This expression for $A_k$ can be changed slightly by making use of the recursion relation (2.1.10) for orthonormal polynomials. Let us substitute the root $x_k$ of $P_n(x)$ in place of $x$ in (2.1.10). This gives
Approximate Calculation of Definite Integrals

\[
\frac{a_n}{a_{n+1}} P_{n+1}(x_k) + \frac{a_{n-1}}{a_n} P_{n-1}(x_k) = 0.
\]

From this relationship we can write (7.1.3) in the form

\[
A_k = \frac{a_n}{a_{n-1}} \frac{1}{P_n'(x_k) P_{n-1}(x_k)}. 
\]

(7.1.4)

An important fact is that a quadrature formula of the highest degree of precision has all positive coefficients:

**Theorem 3.** If the quadrature formula (7.1.1) is exact for all possible polynomials of degree \(\leq 2n - 2\) then all of the coefficients \(A_k\) are positive.

**Proof.** Consider the function \(f(x) = \left[\frac{\alpha(x)}{x - x_i}\right]^2\). This is a polynomial of degree \(2n - 2\) and hence equation (7.1.1) must be exact for it. But

\[
f(x_k) = \begin{cases} 
0 & \text{for } k \neq i \\
\alpha'(2)(x_i) & \text{for } k = i 
\end{cases}
\]

which means that

\[
\int_a^b p(x) \left[\frac{\alpha(x)}{x - x_i}\right]^2 dx = A_i \alpha'2(x_i)
\]

or

\[
A_i = \int_a^b p(x) \left[\frac{\alpha(x)}{(x - x_i) \alpha'(x_i)}\right]^2 dx > 0,
\]

which then proves the theorem.

We will now study the remainder of the quadrature. The segment of integration \([a, b]\) can be any finite or infinite segment. Let us assume that the product \(p(x)f(x)\) is summable on \([a, b]\).

**Theorem 4.** If \(f(x)\) has a continuous derivative of order \(2n\) on \([a, b]\) then there exists a point \(\eta\) in \([a, b]\) for which the remainder of the quadrature formula of the highest degree of precision is

\[
R(f) = \frac{f^{(2n)}(\eta)}{(2n)!} \int_a^b p(x) \alpha^2(x) dx. 
\]

(7.1.5)

**Proof.** Let us construct the interpolating polynomial \(H(x)\) of degree \(\leq 2n - 1\) which satisfies the conditions
7.1. General Theorems

\[ H(x_k) = f(x_k), \quad H'(x_k) = f'(x_k). \]

By Theorem 6 of Chapter 3, the remainder \( r(x) = f(x) - H(x) \) of the interpolation can be expressed as

\[ r(x) = \frac{f^{(2n)}(\xi)}{(2n)!} \omega^2(x) \]

where \( \xi \) belongs to the segment which contains \( x \) and the nodes \( x_k \).

Thus

\[
\int_a^b p(x)f(x)\,dx = \int_a^b p(x)H(x)\,dx + \frac{1}{(2n)!} \int_a^b p(x)f^{(2n)}(\xi)\omega^2(x)\,dx.
\]

Because the quadrature formula is exact for all polynomials of degree \( \leq 2n - 1 \) it is exact for \( H(x) \):

\[
\int_a^b p(x)H(x)\,dx = \sum_{k=1}^n A_k H(x_k) = \sum_{k=1}^n A_k f(x_k)
\]

and hence we obtain as the remainder of the quadrature

\[
R(f) = \frac{1}{(2n)!} \int_a^b f^{(2n)}(\xi) p(x) \omega^2(x)\,dx.
\]

By the usual reasoning\(^1\) it can be shown that there exists a point \( \eta \in [a, b] \) for which (7.1.5) is valid. This completes the proof.

We mention one other integral representation for the remainder. Everything we discussed in Section 5.3 holds true for the remainder of an arbitrary quadrature formula. Let us assume that \( f(x) \) has a continuous derivative of order \( 2n \) on \([a, b]\). Then, with \( r = 2n \), equation (5.3.6) gives an integral representation for \( R(f) \):

\[
R(f) = \int_a^b f^{(2n)}(t) K(t)\,dt.
\]

\(^1\)If \( n = \inf_{[a, b]} f^{(2n)} \) and \( M = \sup_{[a, b]} f^{(2n)} \) then

\[
m \int_a^b p(x) \omega^2(x)\,dx \leq \int_a^b p(x)f^{(2n)}(\xi) \omega^2(x)\,dx \leq M \int_a^b p(x) \omega^2(x)\,dx.
\]

Therefore

\[
\int_a^b f^{(2n)}(\xi) p(x) \omega^2(x)\,dx = T \int_a^b p(x) \omega^2(x)\,dx
\]

where \( m \leq T \leq M \). Thus it is easy to establish the existence of the point \( \eta \).
Approximate Calculation of Definite Integrals

If the segment \([a, b]\) is finite then such a representation is certainly possible.

From (7.1.5) we see that if \(f^{(2n)}(x)\) is different from zero throughout \([a, b]\) then \(R(f)\) is not zero and has the same sign as \(f^{(2n)}(x)\). Because this is true for an arbitrary function \(f^{(2n)}(x)\), which possesses the properties we have assumed, then the kernel \(K(t)\) of (7.1.6) must be nonnegative throughout \([a, b]\).

We will now establish a theorem on the convergence of the quadrature formula. This result could also be obtained as a corollary to a more general result of Chapter 12. We prove this theorem now, however, because we are able to use a much simpler argument than that in Chapter 12.

Let \(p(x)\) be a weight function which is nonnegative on \([a, b]\) and let \(\omega_n(x)\) \((n = 0, 1, \ldots)\) be the corresponding orthogonal system of polynomials. Also, let \(x_k^{(n)}\) \((k = 1, 2, \ldots, n)\) be the roots of the polynomial \(\omega_n(x)\) and let \(A_k^{(n)}\) \((k = 1, 2, \ldots, n)\) be the coefficients of the quadrature formula of the highest degree of precision.

**Theorem 5.** If the segment \([a, b]\) is finite and if \(f(x)\) is continuous on \([a, b]\) then

\[
\lim_{n \to \infty} \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) = \int_{a}^{b} p(x) f(x) \, dx.
\]

**Proof.** Since \(f(x)\) is continuous on \([a, b]\) for any \(\epsilon > 0\) we can find a polynomial \(P(x)\) with the property that for any \(x \in [a, b]\) we have

\[
|f(x) - P(x)| < \epsilon.
\]

Then

\[
\left| \int_{a}^{b} p(x) f(x) \, dx - \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) \right| \\
\leq \left| \int_{a}^{b} p(x) f(x) \, dx - \int_{a}^{b} p(x) P(x) \, dx \right| + \\
+ \left| \int_{a}^{b} p(x) P(x) \, dx - \sum_{k=1}^{n} A_k^{(n)} P(x_k^{(n)}) \right| + \\
+ \left| \sum_{k=1}^{n} A_k^{(n)} P(x_k^{(n)}) - \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) \right|.
\]

But, by (7.1.8),
7.2. Constant Weight Function

\[
\left| \int_a^b p(x)f(x) \, dx - \int_a^b p(x)P(x) \, dx \right| < \varepsilon \int_a^b p(x) \, dx
\]

and

\[
\left| \sum_{k=1}^n A_k^{(n)}P(x_k^{(n)}) - \sum_{k=1}^n A_k^{(n)}f(x_k^{(n)}) \right| \leq \\
\leq \varepsilon \sum_{k=1}^n A_k^{(n)} = \varepsilon \int_a^b p(x) \, dx.
\]

Now if \( m \) is the degree of the polynomial \( P(x) \), then for \( 2n - 1 \geq m \) we will have

\[
\int_a^b p(x)P(x) \, dx = \sum_{k=1}^n A_k^{(n)}P(x_k^{(n)}),
\]

and for such an \( n \)

\[
\left| \int_a^b p(x)f(x) \, dx - \sum_{k=1}^n A_k^{(n)}f(x_k^{(n)}) \right| < 2\varepsilon \int_a^b p(x) \, dx,
\]

which proves (7.1.7).

7.2. CONSTANT WEIGHT FUNCTION

The formulas of Gauss are historically the first formulas of the highest algebraic degree of precision. These formulas are used to approximate the integral

\[
\int_a^b f(x) \, dx
\]

where \([a, b]\) is a finite segment; here \( p(x) = 1 \).

By a linear transformation we can transform an arbitrary segment \([a, b]\) into any standard segment we choose. In order to make use of the symmetry of the nodes \( x_k \) and coefficients \( A_k \) we will take the standard segment to be \([-1, +1]\). Thus, we will assume that (7.2.1) has been transformed into the form

\[
\int_{-1}^{+1} f(x) \, dx
\]

The system of polynomials which are orthogonal on \([-1, +1]\) with respect to the constant weight function are the Legendre polynomials
The quadrature formula of the highest degree of precision $2n - 1$
\[ \int_{-1}^{+1} f(x) \, dx = \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) \]  
(7.2.3)

has for its $n$ nodes the roots of the Legendre polynomial of degree $n$:
\[ P_n(x_k^{(n)}) = 0. \]

The coefficients $A_k^{(n)}$ of this formula can be found from either equation (7.1.3) or (7.1.4); we must remember, however, that in those equations we used orthonormal polynomials. The orthonormal Legendre polynomials are the polynomials $p_n(x) = \sqrt{\frac{2n + 1}{2}} P_n(x)$. The leading coefficients of these are $a_n = \sqrt{\frac{2n + 1}{2}} \frac{(2n)!}{2^{2n(n!)^2}}$ (see equations (2.2.10) and (2.2.11)). A simple calculation then gives
\[ A_k^{(n)} = -\frac{2}{(n + 1) P_n'(x_k^{(n)}) P_{n+1}(x_k^{(n)})} = \frac{2}{n P_n'(x_k^{(n)}) P_{n-1}(x_k^{(n)})}. \]  
(7.2.4)

This can be simplified by use of the following relationship which is known from the theory of Legendre polynomials:
\[ (1 - x^2) P_n'(x) = (n + 1)[x P_n(x) - P_{n+1}(x)] = n[P_{n-1}(x) - x P_n'(x)]. \]

If we substitute $x_k^{(n)}$ for $x$ in this equation we obtain
\[ [1 - (x_k^{(n)})^2] P_n'(x_k^{(n)}) = -(n + 1) P_{n+1}(x_k^{(n)}) = n P_{n-1}(x_k^{(n)}). \]

This permits us to eliminate either $P_n'$, $P_{n+1}$, or $P_{n-1}$ from (7.2.4). We can obtain, for example,
\[ A_k^{(n)} = \frac{2}{[1 - (x_k^{(n)})^2] [P_n'(x_k^{(n)})]^2}. \]  
(7.2.5)

In Appendix A we give values of the nodes and coefficients for (7.2.3) for $n = 2(1)16(4)40(8)48$.

---


Writing $n = 2(1)16 \ldots$ means that $n$ takes values from 2 to 16 in steps of 1 and so forth. The original Russian edition of this book gave only the 15 decimal place values of the $x_k^{(n)}$ and $A_k^{(n)}$ for $n = 1(1)16$ tabulated by: A. N. Lowan, N. Davids and A. Levenson, "Table of the zeros of the Legendre polynomials of order 1 to 16 and the weight coefficients for the Gauss' mechanical quadrature formula," *Bull. Amer. Math. Soc.*, Vol. 48, 1942, pp. 739–43.
7.2. Constant Weight Function

If the integrand $f(x)$ has a continuous derivative of order $2n$ on $[-1, +1]$ then we can use equation (7.1.5) to find the remainder of the Gauss quadrature formula. In (7.1.5) we must now use $p(x) = 1$ and take as $\omega(x)$ the polynomial of degree $n$, with leading coefficient unity, which is orthogonal with respect to $p(x) = 1$ on $[-1, +1]$, to all polynomials of degree $\leq n - 1$. The polynomial $\omega(x)$ differs from the Legendre polynomial $P_n(x)$ by a constant multiple:

$$\omega(x) = \frac{2^n(n!)^2}{(2n)!} P_n(x).$$

Now, since

$$\int_{-1}^{+1} P_n^2(x) \, dx = \frac{2}{2n + 1},$$

we have the following representation for the remainder of the Gauss formula (7.2.3)

$$R(f) = \frac{2^{2n+1}}{(2n + 1) (2n)!} \left[ \frac{(n!)^2}{(2n)!} \right]^2 f^{(2n)}(\eta),$$

where $\eta$ is a point in the segment $[-1, +1]$.

**Example 1.** Suppose we wish to calculate the integral

$$J = \int_{0}^{1} \frac{dt}{1 + t} = \ln 2 \approx 0.69314718.$$

Let us use the 5-point Gauss formula. In order to use the nodes and coefficients which are tabulated in Appendix A we must transform the segment of integration $[0, 1]$ to the segment $[-1, +1]$. This is accomplished by the transformation

$$t = \frac{1}{2} (1 + x).$$

We then obtain

$$J = \int_{-1}^{+1} \frac{dx}{3 + x}.$$

The approximate value of $J$, using the 5-point Gauss formula, is then

$$J = A_1(5) (3 + x_1(5))^{-1} + A_2(5) (3 + x_2(5))^{-1} + \ldots + A_5(5) (3 + x_5(5))^{-1}.$$

After substituting the values from the table we obtain, to eight significant figures,

$$J \approx 0.69314717.$$
Approximate Calculation of Definite Integrals

We could approximately evaluate \( I \) in its original form as an integral over the segment \([0, 1]\) by transforming the Gauss formula for the segment \([-1, +1]\) into the corresponding formula for the segment \([0, 1]\). This would be done as follows:

\[
\begin{align*}
    u_k^{(5)} &= \frac{1}{2} [1 + x_k^{(5)}], \\
    B_k^{(5)} &= \frac{1}{2} A_k^{(5)} \\
    u_1^{(5)} &= 0.04691 \ldots \\
    u_2^{(5)} &= 0.23076 \ldots \\
    u_3^{(5)} &= 0.50000 \ldots \\
    u_4^{(5)} &= 0.76923 \ldots \\
    u_5^{(5)} &= 0.95308 \ldots \\
    B_1^{(5)} &= 0.11846 \ldots \\
    B_2^{(5)} &= 0.23931 \ldots \\
    B_3^{(5)} &= 0.28444 \ldots \\
    B_4^{(5)} &= 0.23931 \ldots \\
    B_5^{(5)} &= 0.11846 \ldots 
\end{align*}
\]

Now we can use these nodes and coefficients to calculate \( I \) in the original form

\[
I = B_1^{(5)} (1 + u_1^{(5)})^{-1} + B_2^{(5)} (1 + u_2^{(5)})^{-1} + \ldots + B_5^{(5)} (1 + u_5^{(5)})^{-1}. 
\]

**Example 2** The integral equation

\[
y(x) = f(x) + \int_a^b K(x, s)y(s)\, ds
\]

is often solved approximately by replacing it with a linear system\(^4\). Such a system can be constructed, for example, if we replace the integral by a quadrature sum:

\[
y(x) = f(x) + \sum_{j=1}^n A_j K(x, x_j) y(x_j) + R(x). 
\]

If we substitute, in turn, \( x = x_1, x_2, \ldots, x_n \) into this equation we obtain the linear system of equations

\[
y(x_i) = f(x_i) + \sum_{j=1}^n A_j K(x_i, x_j) y(x_j) + R(x_i), \quad (i = 1, 2, \ldots, n). 
\]

If we ignore the remainder terms \( R(x_i) \) then this is a system of \( n \) equations which have as unknowns the \( n \) approximate values \( \tilde{y}(x_i) \) of the unknown function \( y(x) \):

\[
\tilde{y}(x_i) = f(x_i) + \sum_{j=1}^n A_j K(x_i, x_j) \tilde{y}(x_j), \quad (i = 1, 2, \ldots, n). \tag{7.2.7}
\]

The magnitude of the remainders $R(x_i)$ depend on the precision of the quadrature formula and we can expect that the more precise the formula the more accurate will be the solution of the integral equation.

The solution of the linear system (7.2.7) becomes increasingly difficult as the number of equations increases. Therefore, if we wish to find the approximate solution of an integral equation by replacing it by a linear system it is desirable to use a quadrature formula of the highest degree of precision.

Let us consider the integral equation

$$y(x) - \frac{1}{2} \int_0^1 e^{xt} y(t) \, dt = \frac{1}{2x} (e^x - 1)$$

and let us use the Gauss 2-point formula to find its approximate solution. The nodes and coefficients of this formula for the segment $[0, 1]$ are:

$$A_1(2) = A_2(2) = \frac{1}{2}, \quad x_1(2) = 0.2113, \quad x_2(2) = 0.7887.$$ 

The system (7.2.7) has the form

$$(1 - \frac{1}{2} K_{1,1}) \tilde{y}_1 - \frac{1}{2} K_{1,2} \tilde{y}_2 = f_1$$

$$\frac{1}{2} K_{2,1} \tilde{y}_1 + (1 - \frac{1}{2} K_{2,2}) \tilde{y}_2 = f_2$$

where

$$\tilde{y}_i = \tilde{y}(x_i(2)), \quad K_{i,j} = K(x_i(2), x_j(2)), \quad K(t, x) = \frac{1}{2} e^{xt}, \quad f_i = f(x_i(2)).$$

After computing the coefficients this system becomes

$$0.7386 \tilde{y}_1 - 0.2954 \tilde{y}_2 = 0.4434$$

$$- 0.2954 \tilde{y}_1 + 0.5343 \tilde{y}_2 = 0.2384.$$ 

Solving these equations we find

$$\tilde{y}_1 = \tilde{y}(0.2113) = 0.9997, \quad \tilde{y}_2 = \tilde{y}(0.7887) = 0.9990.$$ 

The exact solution of the equation, as can be easily verified by substitution, is $y(x) = 1$.

7.3. Integrals $\int_a^b (b - x)^{\alpha}(x - a)^{\beta} f(x) \, dx$

AND THEIR APPLICATION TO THE CALCULATION OF MULTIPLE INTEGRALS

Let $[a, b]$ be an arbitrary finite segment and let us be given the corresponding weight function $p(x) = (b - x)^{\alpha}(x - a)^{\beta}$, $\alpha, \beta > -1$. In
order to study the integral \( \int_{a}^{b} (b - x)^{\alpha}(x - a)^{\beta} f(x) \, dx \) and for the construction of quadrature formulas for its approximation, one usually transforms the segment \([a, b]\) into the segment \([-1, +1]\) by the linear transformation

\[
x = \frac{1}{2} [a + b + t(b - a)], \quad -1 \leq t \leq +1.
\]

We will assume that such a transformation has been carried out and will restrict our attention to the integral

\[
\int_{-1}^{+1} (1 - x)^{\alpha}(1 + x)^{\beta} f(x) \, dx. \tag{7.3.1}
\]

The orthogonal system of polynomials which correspond to the segment \([-1, +1]\) and the weight function \((1 - x)^{\alpha}(1 + x)^{\beta}\) is the system of Jacobi polynomials \(P_{n}(\alpha, \beta)(x)\) \((n = 0, 1, 2, \ldots)\). A quadrature formula with \(n\) nodes

\[
\int_{-1}^{+1} (1 - x)^{\alpha}(1 + x)^{\beta} f(x) \, dx = \sum_{k=1}^{n} A_{k} f(x_{k}), \tag{7.3.2}
\]

which has the highest degree of precision \(2n - 1\), must have for its nodes \(x_{k}\) the roots of the Jacobi polynomial of degree \(n\)

\[
P_{n}(\alpha, \beta)(x_{k}) = 0.
\]

The coefficients \(A_{k}\) can be found from either equation \((7.1.3)\) or \((7.1.4)\).

The normalized Jacobi polynomials are \([\text{by } (2.2.2), (2.2.5) \text{ and } (2.2.7)]\):

\[
p_{n}(\alpha, \beta)(x) = \delta_{n}^{-\frac{1}{2}} P_{n}(\alpha, \beta)(x)
\]

where

\[
\delta_{n} = \frac{2^{\alpha+\beta+1} \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{(\alpha + \beta + 2n + 1) n! \Gamma(\alpha + \beta + n + 1)}.
\]

The leading coefficients of the normalized Jacobi polynomials are

\[
a_{n} = \delta_{n}^{-\frac{1}{2}} \frac{\Gamma(\alpha + \beta + 2n + 1)}{2^{n} n! \Gamma(\alpha + \beta + n + 1)}.
\]

\(^{8}\text{Trans. note: We omit the superscript } (n) \text{ from the symbols } x_{k}^{(n)} \text{ and } A_{k}^{(n)} \text{ whenever it is clear to which values of } n \text{ they correspond.}\)
7.3. Integrals \( \int_{a}^{b} (b - x)^{\alpha} (x - a)^{\beta} f(x) \, dx \)

We then find

\[
A_k = \frac{2^{\alpha+\beta+1} \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{n! \Gamma(\alpha + \beta + n + 1)} \frac{\Gamma(\alpha + \beta + n + 1)}{\Gamma(\alpha + \beta + 2n + 1)} [P_n^{(\alpha, \beta)}(x_k)]^2 \tag{7.3.3}
\]

This expression for the coefficients can be simplified somewhat if we make use of the relationship

\[
\frac{\Gamma(a + \beta + 2n)(1 - x^2)}{\Gamma(a + \beta + n + 1)} P_n^{(a, \beta)}(x) = -n[(a + \beta + 2n)x + \beta - a] P_n^{(a, \beta)}(x) + 2(a + n)(\beta + n) P_n^{(a, \beta)}(x). \tag{7.3.3}
\]

Substituting \( x = x_k \) we obtain

\[
(a + \beta + 2n)(1 - x_k^2) P_n^{(a, \beta)}(x_k) = 2(a + n)(\beta + n) P_n^{(a, \beta)}(x_k)
\]

which permits us to write \( A_k \) in the form

\[
A_k = \frac{2^{\alpha+\beta+1} \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{n! \Gamma(\alpha + \beta + n + 1)} \frac{\Gamma(\alpha + \beta + n + 1)}{\Gamma(\alpha + \beta + 2n + 1)} [P_n^{(\alpha, \beta)}(x_k)]^2 \tag{7.3.4}
\]

The leading coefficient of the polynomial \( P_n^{(\alpha, \beta)}(x) \) has the value (2.2.2). Therefore the polynomial

\[
\omega(x) = \frac{2^n n! \Gamma(\alpha + \beta + n + 1)}{\Gamma(\alpha + \beta + 2n + 1)} P_n^{(\alpha, \beta)}(x)
\]

has unity for its leading coefficient. If \( f(x) \) has a continuous derivative of order \( 2n \) on the segment \([-1, +1]\) then the remainder term of formula (7.3.2) is

\[
R(f) = \frac{f^{(2n)}(\eta)}{(2n)!} \left[ \frac{2^n n! \Gamma(\alpha + \beta + n + 1)}{\Gamma(\alpha + \beta + 2n + 1)} \right]^2 \times \int_{-1}^{1} (1 - x)^{\alpha}(1 + x)^{\beta}[P_n^{(\alpha, \beta)}(x)]^2 \, dx = \frac{f^{(2n)}(\eta)}{(2n)!} \times \frac{2^{\alpha+\beta+2n+1} n! \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1) \Gamma(\alpha + \beta + n + 1)}{(\alpha + \beta + 2n + 1)(\Gamma(\alpha + \beta + 2n + 1))^2} \tag{7.3.5}
\]

where \(-1 < \eta < +1\).

Let us now consider some special cases of quadrature formulas for use with Jacobi weight functions.

---

1. Quadrature formulas on \([-1, +1]\).

A. For \(\alpha = \beta = -1/2\) the weight function is \((1 - x^2)^{-\frac{1}{2}}\) and the corresponding Jacobi polynomials are a multiple of the Chebyshev polynomials of the first kind (see (2.3.4)):

\[
P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x) = C_n T_n(x) = C_n \cos(n \arccos x).
\]

The roots of \(T_n\) are the nodes to be used in the quadrature formula; these are

\[
x_k = \cos \frac{2k - 1}{2n} \pi \quad (k = 1, 2, \ldots, n).
\]

The coefficients \(A_k\) are easily computed. Since

\[
T'_n(x_k) = \frac{n \sin(n \arccos x_k)}{\sqrt{1 - x_k^2}} = (-1)^{k-1} \frac{n}{\sqrt{1 - x_k^2}}
\]

then

\[
(1 - x_k^2)[P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x_k)]^2 = C_n^2(1 - x_k^2)[T'_n(x_k)]^2 = C_n^2 n^2
\]

and

\[
A_k = \frac{2^n \left[\Gamma \left(n + \frac{1}{2}\right)\right]^2}{n! \Gamma(n) C_n^2 n^2}.
\]

The righthand side of this expression is independent of \(k\) and hence, for a fixed \(n\), \(A_1 = \ldots = A_n\). Let \(A\) denote the common value of the \(A_k\). The easiest way to find the value of \(A\) is to use the fact that the quadrature formula is exact for \(f(x) = 1\) and hence

\[
\sum_{k=1}^{n} A_k = nA = \int_{-1}^{+1} \frac{dx}{\sqrt{1 - x^2}} = \pi.
\]

Hence

\[
A = \frac{\pi}{n}.
\]

The quadrature formulas of the highest degree of precision for the weight function \((1 - x^2)^{-\frac{1}{2}}\) have the form\(^7\)

\(^7\)This formula was found by F. G. Mehler in 1864. See the reference at the end of this chapter.
7.3. Integrals \[ \int_a^b (b - x)^\alpha (x - a)^\beta f(x) \, dx \]

\[
\int_{-1}^{+1} \frac{f(x)}{\sqrt{1 - x^2}} \, dx = \frac{\pi}{n} \sum_{k=1}^{n} f\left(\cos \frac{2k - 1}{2n} \pi\right) + R(f). \quad (7.3.6)
\]

Using (7.3.5) we obtain the following expression for the remainder

\[ R(f) = \frac{\pi}{2^{2n-1}} \frac{f^{(2n)}(\eta)}{(2n)!}, \quad -1 < \eta < +1. \]

B. Let \( \alpha = \beta = \frac{1}{2} \) and \( p(x) = \sqrt{1 - x^2} \). The Jacobi polynomials \( P_{\frac{n}{2}, \frac{1}{2}}(x) \) are a multiple of the Chebyshev polynomials of the second kind [see (2.3.7)]:

\[
P_{\frac{n}{2}, \frac{1}{2}}(x) = \frac{(2n + 1)!}{2^{2n}n!(n + 1)!} U_n(x)
\]

\[ U_n(x) = \frac{\sin \left[ (n + 1) \arccos x \right]}{\sqrt{1 - x^2}}. \]

The roots of \( P_{\frac{n}{2}, \frac{1}{2}}(x) \) are \( x_k = \cos \frac{k}{n + 1} \pi \) \((k = 1, 2, \ldots, n)\).

The coefficients \( A_k \) can be computed from (7.3.4):

\[
A_k = \frac{\pi}{n + 1} \sin^2 \frac{k \pi}{n + 1}.
\]

The quadrature formulas have the form

\[
\int_{-1}^{+1} \frac{f(x)}{\sqrt{1 - x^2}} \, dx = \frac{\pi}{n + 1} \sum_{k=1}^{n} \sin^2 \frac{k \pi}{n + 1} f\left(\cos \frac{k \pi}{n + 1}\right) + R(f). \quad (7.3.7)
\]

The remainder \( R(f) \) can be found from (7.3.5)

\[ R(f) = \frac{\pi}{2^{2n}} \frac{f^{(2n)}(\eta)}{(2n)!}, \quad -1 < \eta < +1. \]

C. Let \( \alpha = \frac{1}{2} \), \( \beta = -\frac{1}{2} \) so that \( p(x) = \sqrt{\frac{1 - x}{1 + x}} \). As in the two
previous cases the Jacobi polynomials $P_n^{(1/2, -1/2)}(x)$ can be simply expressed in terms of trigonometric functions. If $Q(x)$ is an arbitrary polynomial of degree less than $n$ then the following orthogonality condition must be satisfied:

$$\int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}} P_n^{(1/2, -1/2)}(x) Q(x) \, dx =$$

$$= \int_{-1}^{+1} (1-x) P_n^{(1/2, -1/2)}(x) Q(x) \frac{dx}{\sqrt{1-x^2}} = 0.$$

Let us consider the polynomial $S(x) = (1-x) P_n^{(1/2, -1/2)}(x)$. This is a polynomial of degree $n + 1$ and it is orthogonal on the segment $[-1, +1]$ with respect to the weight function $(1-x^2)^{-1/2}$ to each polynomial $Q(x)$ of degree less than $n$. If $S(x)$ is expanded in terms of Chebyshev polynomials of the first kind $T_k(x)$ ($k = 0, 1, \ldots, n + 1$) then all the coefficients of the polynomials $T_k(x)$, for $k \leq n - 1$, in this expansion must be zero by the orthogonality properties of $S(x)$. Hence this expansion must have the form $S(x) = C_n T_n(x) + C_{n+1} T_{n+1}(x)$. Since $S(x)$ is divisible by $1-x$ we must have

$$S(1) = C_n T_n(1) + C_{n+1} T_{n+1}(1) = C_n + C_{n+1} = 0.$$

Therefore $C_{n+1} = -C_n$ and

$$P_n^{(1/2, -1/2)}(x) = C_n \frac{T_n(x) - T_{n+1}(x)}{1-x}.$$

If we equate the leading coefficients from (2.2.2) and (2.3.2) we find

$$C_n = \frac{(2n)!}{2^{2n} (n!)^2}.$$

Setting $x = \cos \theta$ and using the fact that $T_k(x) = \cos (k \text{ arc} \cos x) = \cos k \theta$ we obtain

$$P_n^{(1/2, -1/2)}(x) = \frac{(2n)!}{2^{2n} (n!)^2} \frac{\sin (2n + 1)\theta/2}{\sin \theta/2}.$$

The roots of this polynomial are

$$x_k = \cos \frac{2k}{2n + 1} \pi \quad (k = 1, 2, \ldots, n).$$
The coefficients of the quadrature formula can be computed from (7.3.4):

\[ A_k = \frac{4\pi}{2n+1} \sin^2 \frac{k\pi}{2n+1} . \]

Thus the quadrature formulas for use with the weight function \( \sqrt{\frac{1-x}{1+x}} \) have the form

\[
\int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}} f(x) \, dx =
\frac{4\pi}{2n+1} \sum_{k=1}^{n} \sin^2 \frac{k\pi}{2n+1} f\left(\cos \frac{2k\pi}{2n+1}\right) + R(f). \tag{7.3.8}
\]

The remainder term in this formula is

\[ R(f) = \frac{\pi}{2^{2n}} \frac{f^{(2n)}(\eta)}{(2n)!}, \quad -1 < \eta < +1. \]

2. Quadrature formulas on \([0, 1]\).

A. The first case we consider is \( \alpha = 0, \beta = \frac{1}{2} \); this corresponds to the integral \( \int_{0}^{1} \sqrt{x} f(x) \, dx \). The polynomials \( Q_n(x) \) which are orthogonal on the segment \([0, 1]\) with respect to the weight function \( \sqrt{x} \) are closely related to the Legendre polynomials \( P_n(x) \). Let us put \( m = 2n + 1 \) and consider the Legendre polynomials of odd order \( P_{2n+1}(y) \) \( (n = 0, 1, 2, \ldots) \). These are odd functions of \( y \) and the ratio \( P_{2n+1}(y)/y \) depends only on \( y^2 \). Let us replace \( y^2 \) by \( x \). We will show that \( Q_n(x) \) can be taken as the polynomial

\[ Q_n(x) = \frac{P_{2n+1}(\sqrt{x})}{\sqrt{x}}. \]

Using the substitution \( x = y^2 \) we obtain

\[
\int_{0}^{1} \sqrt{x} Q_n(x) Q_m(x) \, dx = \int_{0}^{1} P_{2n+1}(\sqrt{x}) P_{2m+1}(\sqrt{x}) \frac{dx}{\sqrt{x}} =
= 2 \int_{0}^{1} P_{2n+1}(y) P_{2m+1}(y) \, dy = \int_{-1}^{+1} P_{2n+1}(y) P_{2m+1}(y) \, dy = 0.
\]
This proves the orthogonality of $Q_n(x)$.

In the quadrature formula of the highest degree of precision

$$\int_0^1 \sqrt{x} f(x) \, dx = \sum_{k=1}^{n} A_k f(x_k) + R(f) \quad (7.3.9)$$

the nodes $x_k$ are the squares of the positive roots $y_k$ of the Legendre polynomial $P_{2n+1}(y)$:

$$x_k = y_k^2.$$

We can also show that the coefficients $A_k$

$$A_k = \int_0^1 \sqrt{x} \frac{Q_n(x)}{(x-x_k)Q'(x_k)} \, dx$$

are simply related to the coefficients of the Gauss formula (7.2.3) with $2n + 1$ nodes. Using the relationships

$$x_k = y_k^2, \quad Q'_n(x_k) = \frac{P'_{2n+1}(y_k)}{2y_k^2}$$

we obtain

$$A_k = 2y_k^2 \int_0^1 \frac{P_{2n+1}(y)}{P'_{2n+1}(y_k) \frac{2y}{y^2 - y_k^2}} \, dy.$$

This integral can be written as the sum of two integrals since

$$\frac{2y}{y^2 - y_k^2} = \frac{1}{y - y_k} + \frac{1}{y + y_k}.$$

If, in the second of these two integrals, we replace $y$ by $-y$ we obtain

$$A_k = 2y_k^2 \int_{-1}^{+1} \frac{P_{2n+1}(y)}{(y - y_k)P'_{2n+1}(y_k)} \, dy. \quad (7.3.10)$$

Let us write the coefficients of the Gauss formula (7.2.3) with $2n + 1$ nodes as $A^{(2n+1)}_k (k = -n, -n + 1, \ldots, -1, 0, 1, \ldots, n)$. Then the integral in (7.3.10) is equal to $A^{(2n+1)}_k$. Therefore

$$A_k = 2y_k^2 A^{(2n+1)}_k \quad (k = 1, 2, \ldots, n).$$

The remainder $R(f)$ of formula (7.3.9) can be found from the general expression (7.1.5) if we use the fact that the leading coefficient of $Q_n(x)$ is the same as the leading coefficient of $P_{2n+1}(y)$:
7.3. Integrals \( \int_a^b (b-x)^\alpha (x-a)^\beta f(x) \, dx \)

\[
\omega (x) = \frac{2^{2n+1}[(2n + 1)!]^2}{(4n + 2)!} \quad Q_n(x) = \frac{2^{2n+1}[(2n + 1)!]^2}{(4n + 2)!} \quad P_{2n+1}(\sqrt{x})
\]

Thus we obtain

\[
R(f) = f^{(2n)}(\eta) \left( \frac{2}{(2n)!} \frac{2^{2n+1}[(2n + 1)!]^2}{(4n + 3)!} \right)^{2} , \quad 0 < \eta < 1.
\]

Here we give values of the \( x_k \) and \( A_k \) in formula (7.3.9) for \( n = 1(1)8 \):

Quadrature Formulas for the Integral \( \int_0^1 \sqrt{x} f(x) \, dx \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_k^{(n)} )</th>
<th>( A_k^{(n)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.60000</td>
<td>0.666666 66667</td>
</tr>
<tr>
<td>2</td>
<td>0.28994 91979</td>
<td>0.27755 59982</td>
</tr>
<tr>
<td></td>
<td>0.82116 19131</td>
<td>0.38911 06684</td>
</tr>
<tr>
<td>3</td>
<td>0.16471 02869</td>
<td>0.12578 26743</td>
</tr>
<tr>
<td></td>
<td>0.54986 84991</td>
<td>0.30760 23676</td>
</tr>
<tr>
<td></td>
<td>0.90080 58292</td>
<td>0.23328 16246</td>
</tr>
<tr>
<td>4</td>
<td>0.10514 02826</td>
<td>0.06568 05199</td>
</tr>
<tr>
<td></td>
<td>0.37622 45144</td>
<td>0.19609 62654</td>
</tr>
<tr>
<td></td>
<td>0.69894 80124</td>
<td>0.25252 73457</td>
</tr>
<tr>
<td></td>
<td>0.98733 42493</td>
<td>0.15236 25356</td>
</tr>
<tr>
<td>5</td>
<td>0.07265 35129</td>
<td>0.03818 73467</td>
</tr>
<tr>
<td></td>
<td>0.26946 07913</td>
<td>0.12567 31527</td>
</tr>
<tr>
<td></td>
<td>0.53312 19512</td>
<td>0.19863 08015</td>
</tr>
<tr>
<td></td>
<td>0.78688 00558</td>
<td>0.19763 33763</td>
</tr>
<tr>
<td></td>
<td>0.95693 13076</td>
<td>0.10654 19894</td>
</tr>
<tr>
<td>6</td>
<td>0.05311 10354</td>
<td>0.02403 62680</td>
</tr>
<tr>
<td></td>
<td>0.20114 57477</td>
<td>0.08360 26285</td>
</tr>
<tr>
<td></td>
<td>0.41261 26738</td>
<td>0.14701 05789</td>
</tr>
<tr>
<td></td>
<td>0.64252 74355</td>
<td>0.17846 00808</td>
</tr>
<tr>
<td></td>
<td>0.84198 68221</td>
<td>0.15513 01778</td>
</tr>
<tr>
<td></td>
<td>0.96861 62852</td>
<td>0.07842 69326</td>
</tr>
<tr>
<td>7</td>
<td>0.04047 90635</td>
<td>0.01606 46414</td>
</tr>
<tr>
<td></td>
<td>0.15535 52844</td>
<td>0.05784 21902</td>
</tr>
<tr>
<td></td>
<td>0.32600 92219</td>
<td>0.10841 05888</td>
</tr>
<tr>
<td></td>
<td>0.52478 10495</td>
<td>0.14648 80937</td>
</tr>
<tr>
<td></td>
<td>0.71945 44081</td>
<td>0.15419 23470</td>
</tr>
<tr>
<td></td>
<td>0.87848 14120</td>
<td>0.12363 05295</td>
</tr>
<tr>
<td></td>
<td>0.97612 92156</td>
<td>0.06003 82760</td>
</tr>
<tr>
<td>8</td>
<td>0.08185 66030</td>
<td>0.01124 98760</td>
</tr>
<tr>
<td></td>
<td>0.12386 37516</td>
<td>0.04145 12327</td>
</tr>
</tbody>
</table>
Approximate Calculation of Definite Integrals

\[ x_k^{(n)} \quad A_k^{(n)} \]

\[ n = 8 \text{(contd.)} \]

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( A_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.26285</td>
<td>0.08098</td>
</tr>
<tr>
<td>0.43253</td>
<td>0.11690</td>
</tr>
<tr>
<td>0.61076</td>
<td>0.13666</td>
</tr>
<tr>
<td>0.77482</td>
<td>0.13177</td>
</tr>
<tr>
<td>0.90378</td>
<td>0.10024</td>
</tr>
<tr>
<td>0.98123</td>
<td>0.04739</td>
</tr>
</tbody>
</table>

B. In a manner similar to the last case we can construct formulas of the highest degree of precision for the segment \([0, 1]\) and the weight function \(x^{-\frac{1}{2}}\):

\[ \int_0^1 x^{-\frac{1}{2}} f(x) \, dx = \sum_{k=1}^{n} A_k f(x_k) + R(f). \quad (7.3.11) \]

The polynomials \( S_n(x) \) which are orthogonal on \([0, 1]\) with respect to \(x^{-\frac{1}{2}}\) are related to the Legendre polynomials \( P_k(x) \) by

\[ S_n(x) = P_{2n}(\sqrt{x}). \]

Thus the abscissas \( x_k \) in (7.3.11) are the squares of the positive roots \( \gamma_k \) of the Legendre polynomial \( P_{2n}(y) \):

\[ \gamma_k = \gamma_k^2 \quad (k = 1, 2, \ldots, n). \]

Let us write the coefficients of the Gauss formula (7.2.3) with \( 2n \) nodes as \( A_k^{(2n)} \) \((k = -n, \ldots, -1, +1, \ldots, n)\). The coefficients \( A_k \) in (7.3.11) are then

\[ A_k = 2 A_k^{(2n)} \quad (k = 1, 2, \ldots, n). \]

The remainder has the form

\[ R(f) = \frac{f^{(2n)}(\eta)}{(2n)!} \frac{2}{4n + 1} \left\{ \frac{2^{2n}[(2n)!]^2}{(4n)!} \right\}^2, \quad 0 < \eta < 1. \]

Values of the \( x_k \) and \( A_k \) in formula (7.3.11) are tabulated here for \( n = 1(1)8 \):
7.3. Integrals $\int_a^b (b - x)^\alpha (x - a)^\beta f(x) \, dx$

<table>
<thead>
<tr>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 3</td>
<td></td>
</tr>
<tr>
<td>0.05693</td>
<td>91160</td>
</tr>
<tr>
<td>0.43719</td>
<td>78527</td>
</tr>
<tr>
<td>0.86949</td>
<td>93948</td>
</tr>
<tr>
<td>n = 4</td>
<td></td>
</tr>
<tr>
<td>0.03364</td>
<td>82681</td>
</tr>
<tr>
<td>0.27618</td>
<td>43139</td>
</tr>
<tr>
<td>0.63467</td>
<td>74762</td>
</tr>
<tr>
<td>0.92215</td>
<td>66084</td>
</tr>
<tr>
<td>n = 5</td>
<td></td>
</tr>
<tr>
<td>0.02216</td>
<td>35688</td>
</tr>
<tr>
<td>0.18733</td>
<td>15676</td>
</tr>
<tr>
<td>0.46159</td>
<td>73614</td>
</tr>
<tr>
<td>0.74833</td>
<td>46283</td>
</tr>
<tr>
<td>0.94849</td>
<td>39262</td>
</tr>
<tr>
<td>n = 6</td>
<td></td>
</tr>
<tr>
<td>0.01568</td>
<td>34066</td>
</tr>
<tr>
<td>0.13530</td>
<td>00116</td>
</tr>
<tr>
<td>0.34494</td>
<td>23794</td>
</tr>
<tr>
<td>0.59275</td>
<td>01277</td>
</tr>
<tr>
<td>0.81742</td>
<td>80132</td>
</tr>
<tr>
<td>0.96846</td>
<td>12786</td>
</tr>
<tr>
<td>n = 7</td>
<td></td>
</tr>
<tr>
<td>0.01167</td>
<td>58719</td>
</tr>
<tr>
<td>0.10183</td>
<td>27040</td>
</tr>
<tr>
<td>0.26548</td>
<td>11572</td>
</tr>
<tr>
<td>0.47237</td>
<td>15370</td>
</tr>
<tr>
<td>0.68426</td>
<td>20156</td>
</tr>
<tr>
<td>0.86199</td>
<td>13331</td>
</tr>
<tr>
<td>0.97275</td>
<td>57512</td>
</tr>
<tr>
<td>n = 8</td>
<td></td>
</tr>
<tr>
<td>0.00902</td>
<td>73770</td>
</tr>
<tr>
<td>0.07930</td>
<td>05598</td>
</tr>
<tr>
<td>0.20977</td>
<td>93686</td>
</tr>
<tr>
<td>0.38177</td>
<td>10533</td>
</tr>
<tr>
<td>0.57063</td>
<td>58201</td>
</tr>
<tr>
<td>0.74931</td>
<td>73785</td>
</tr>
<tr>
<td>0.89222</td>
<td>19741</td>
</tr>
<tr>
<td>0.97891</td>
<td>42101</td>
</tr>
</tbody>
</table>

3. Application to multiple integrals.

One method often used in practice is to separate the variables, if possible, of the multiple integral and to apply quadrature formulas for functions of a single variable in turn to each of the variables separately. As an example consider the integral

$$ I = \int_\sigma \int f(x, \gamma) \, d\sigma $$
where the region $\sigma$ is a rectangle $a \leq x \leq b$, $c \leq y \leq d$. The integral $I$ can be written as two single integrals

$$ I = \int_a^b \left[ \int_c^d f(x, y) \, dy \right] \, dx. $$

Here we can replace the integral with respect to $y$ by a quadrature sum with $m$ nodes $y_i$ and coefficients $B_i$ ($i = 1, \ldots, m$) and the integral with respect to $x$ by a quadrature sum with $n$ nodes $x_j$ and coefficients $A_j$ ($j = 1, \ldots, n$). This leads to the following integration formula for $I$:

$$ I = \sum_{j=1}^{n} \sum_{i=1}^{m} A_j B_i f(x_j, y_i) $$

This formula requires us to evaluate the integrand $f(x, y)$ at $mn$ points which is a relatively large number compared to the individual numbers $m$ and $n$.

This method can also be applied to regions other than rectangles. In every case it leads to a relatively large number of points in the integration formula. The problem becomes even more acute when the above method is applied to triple and higher dimensional integrals. This method, however, does give useful formulas especially for two and three dimensions and they are especially valuable for relatively smooth functions so that formulas with extremely high accuracy do not have to be used.

We now consider certain special cases of this method.

4. Double integrals in polar coordinates.

Let us consider

$$ I = \iint_{\sigma} F(r, \phi) \, r \, dr \, d\phi, $$

and assume that the region of integration $\sigma$ is defined by the inequalities

$$ a \leq \phi \leq \beta, \quad 0 \leq r \leq R = R(\phi). $$

---

*Trans. note: The author’s discussion of methods for combining quadrature formulas for single integrals is, up to this point and in the remainder of this section, mostly descriptive in nature; he does not show for what class of functions the resulting formulas will be exact. Recent papers cited in the references at the end of this chapter by the following authors give some exact results of this nature: Hammer and Wymore; Hammer, Marlowe and Stroud; Peirce; Hetherington; and Secrest and Stroud.

Other formulas for multiple integrals, not of the type discussed in this section, which use fewer points for the same algebraic degree of precision are also known in a few cases. For references to such formulas see: A. H. Stroud, “A bibliography on approximate integration,” *Math. Comp.*, Vol. 15, 1961, pp. 52-80.*
If we introduce the parameter \( \rho \) by setting \( r = \rho R, \ 0 \leq \rho \leq 1 \), then the integral \( I \) can be written in the form

\[
I = \int_{\alpha}^{\beta} \left[ \int_{0}^{1} F(\rho R, \phi) \rho \, d\rho \right] R^2 \, d\phi.
\]

Hence we see that calculation of a double integral in polar coordinates reduces to a consideration of the integral

\[
\int_{0}^{1} f(x) \, x \, dx.
\]  

(7.3.12)

If we wish to construct a quadrature formula

\[
\int_{0}^{1} f(x) \, x \, dx = \sum_{k=1}^{n} A_k f(x_k) + R(f)
\]

of the highest degree of precision we must take its nodes as the roots of the polynomial \( \Pi_n(x) \) which is orthogonal on the segment \([0, 1]\) with respect to \( p(x) = x \) to all polynomials of degree \( \leq n - 1 \). The coefficients \( A_k \) can be calculated by the usual equations (7.1.3) or (7.1.4).

To find the \( x_k \) and \( A_k \) we can use the previously obtained results for the weight function \( (1 - z)\alpha (1 + z)\beta \). By making the change of variable \( x = \frac{1}{2} (1 - z), -1 \leq z \leq 1 \), (7.3.12) becomes

\[
\int_{0}^{1} f(x) \, x \, dx = \frac{1}{4} \int_{-1}^{1} F(z) (1 - z) \, dz
\]

(7.3.13)

\[
F(z) = f\left(\frac{1 - z}{2}\right).
\]

Under this transformation \( \Pi_n(x) \) is transformed into a polynomial of degree \( n \) in \( z \) which is orthogonal on the segment \([-1, 1]\) with respect to the weight \( 1 - z \) to all polynomials of degree \( \leq n - 1 \) and will differ from the Jacobi polynomial \( P_n^{(1,0)}(z) \) by only a constant factor

\[
\Pi_n(x) = c P_n^{(1,0)}(z).
\]

Hence we see that the nodes \( x_k \) of formula (7.3.7) are related to the roots \( z_k \) of \( P_n^{(1,0)}(z) \) by the relationship

\[
x_k = \frac{1 - z_k}{2} \quad (k = 1, \ldots, n).
\]

From (7.3.8), (7.3.2) and (7.3.4), for \( \alpha = 1, \quad \beta = 0 \), we have the following general expression for the coefficients \( A_k \):
Approximate Calculation of Definite Integrals

\[
A_k = \frac{1}{(1 - z_k^2) \left[P_{n,1,0}(z_k)\right]^2}.
\]

Values of the \(x_k\) and \(A_k\) are given below for \(n = 1(1)6^\circ\):

**Quadrature Formulas for the Integral** 

\[
\int_0^1 x f(x) \, dx.
\]

<table>
<thead>
<tr>
<th>(n)</th>
<th>(x_k^{(n)})</th>
<th>(A_k^{(n)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.66666 66666 67</td>
<td>0.50000 00000 00</td>
</tr>
<tr>
<td>2</td>
<td>0.35505 10257 22</td>
<td>0.18195 86182 56</td>
</tr>
<tr>
<td>3</td>
<td>0.21234 05382 39</td>
<td>0.06982 69799 01</td>
</tr>
<tr>
<td>4</td>
<td>0.13975 98643 44</td>
<td>0.03118 09709 50</td>
</tr>
<tr>
<td>5</td>
<td>0.09853 50857 99</td>
<td>0.01574 79145 22</td>
</tr>
<tr>
<td>6</td>
<td>0.07305 43286 80</td>
<td>0.00873 83018 14</td>
</tr>
</tbody>
</table>

5. Triple integrals in spherical coordinates.

To calculate

\[
I = \iiint f(r, \theta, \phi) r^2 \sin \theta \, dr \, d\theta \, d\phi
\]

we can reduce it to single integrals in each of the variables \(r, \theta, \phi\). For the integration with respect to \(r\) we will have an integral of the form

\[
\int_0^1 f(x) x^2 \, dx.
\]  

(7.3.14)

As in the last case for polar coordinates we can show that in the quadrature formula for (7.3.14) of the highest degree of precision

\[ \int_{a}^{b} (b-x)^{\alpha} (x-a)^{\beta} f(x) \, dx = \sum_{k=1}^{n} A_k f(x_k) + R(f) \]  

(7.3.15)

the nodes \( x_k \) must be related to the roots \( z_1, z_2, \ldots, z_n \) of the Jacobi polynomial \( P^{(2,0)}_n(z) \) by the relation

\[ x_k = \frac{1 - z_k}{2} \]

and the coefficients \( A_k \) must have the values

\[ A_k = \frac{1}{(1 - z_k^2) [P^{(2,0)}_n(z_k)]^2}. \]


Consider

\[ I = \iint_D f(x, y) \, dx \, dy. \]  

(7.3.16)

We will assume that \( f(x, y) \) is continuous and relatively smooth in \( D \).

Under certain assumptions about the region \( D \) the integral \( I \) can be reduced to two single integrals

\[ I = \int_{a}^{b} F(x) \, dx \]  

(7.3.17)

\[ F(x) = \int_{y_1(x)}^{y_2(x)} f(x, y) \, dy \]  

(7.3.18)

where \( y_1(x), y_2(x), a \) and \( b \) are known quantities. We will assume that the integral (7.3.18) can be calculated for all values of \( x \) for which we are interested and will concern ourselves with the problem of evaluating (7.3.17). The function \( F(x) \) depends both on the integrand \( f(x, y) \) and on the region \( D \).

It can be expected that among the quadrature formulas of highest degree of precision the Gauss formulas will not always give the best result since they are intended for use with a specific weight function and do not take into account the influence of \( D \) on the function \( F(x) \).

We now make some remarks about an appropriate weight function for (7.3.17). Construct a line through the region \( D \) which passes through the point \( x \) and which is parallel to the \( y \) axis. The part of this line which
lies in \( \sigma \) has length \( \gamma_2(x) - \gamma_1(x) \). The longer this line the greater will be the influence of a narrow strip of \( \sigma \) along this line on the formation of the double integral. Therefore to calculate the integral \( I \) we use the weight function

\[
p(x) = \gamma_2(x) - \gamma_1(x)
\]

(7.3.19)

and write \( I \) in the form

\[
I = \int_a^b [\gamma_2(x) - \gamma_1(x)] \Phi(x) \, dx
\]

(7.3.20)

\[
\Phi(x) = \frac{F(x)}{\gamma_2(x) - \gamma_1(x)}.
\]

In many cases the weight function (7.3.19) will account sufficiently well for the influence of the region on \( I \) and for sufficiently smooth functions \( f(x, y) \) will give good results. But this has the disadvantage that each region would have its own special class of quadrature formulas. However, the selection of the weight function for the integral \( I \) can be simplified by the following considerations. Consider the integral

\[
I_1 = \int_a^b p(x) f(x) \, dx
\]

and suppose that to evaluate it we wish to apply the quadrature formula of the highest degree of precision with \( n \) nodes

\[
I_1 = \int_a^b p(x) f(x) \, dx = \sum_{k=1}^n A_k f(x_k).
\]

(7.3.21)

The nodes of this formula are the zeros of the \( n \)th degree polynomial of the system of orthogonal polynomials for the weight function \( p(x) \). The accuracy of the quadrature formula (7.3.21) will, in general, depend on how closely the function \( f(x) \) can be approximated by a polynomial of degree \( 2n - 1 \).

Suppose now that \( p(x) \) can be represented as a product

\[
p(x) = \rho(x) q(x)
\]

where \( q(x) \) is positive throughout the interval \([a, b]\). Let us combine the function \( q(x) \) with the integrand \( f(x) \): \( q(x) f(x) = F(x) \) and consider the integral \( I_1 \) with weight function \( \rho(x) \)

\[
I_1 = \int_a^b \rho(x) F(x) \, dx.
\]
7.3. Integrals \( \int_a^b (b-x)^\alpha (x-a)^\beta f(x) \, dx \)

Using the roots \( x_k \) of the polynomial of degree \( n \) which belongs to the system of orthogonal polynomials with weight function \( \rho(x) \) we can construct the quadrature formula of the highest degree of precision

\[
I_1 = \int_a^b \rho(x) F(x) \, dx = \sum_{k=1}^{n} B_k F(x_k) \quad (7.3.22)
\]

If \( q(x) \) is a slowly varying function which has derivatives of high order or if it is an analytic function with singular points far from the segment \([a, b]\) then we can expect that the function \( f(x) \) and \( F(x) = q(x) f(x) \) can both be closely approximated by polynomials of degree \( 2n - 1 \). We can hope, therefore, that formulas (7.3.21) and (7.3.22), which both serve for calculating the integral \( I_1 \), will have about the same error and that only a small error is introduced in passing from (7.3.21) with weight function \( \rho(x) \) to (7.3.22) with weight function \( \rho(x) \).

In order to calculate the integral (7.3.17) this permits us to pass from the "natural" weight function \( \rho(x) = \gamma_2(x) - \gamma_1(x) \) to a simpler weight function. In many cases this can be done without a significant loss of accuracy. The simpler weight function can be chosen so that it can be used for many regions.

Suppose the interval of integration \([a, b]\) is finite and assume it is possible to select exponents \( \alpha \) and \( \beta \) so that the ratio

\[
q(x) = \frac{\gamma_2(x) - \gamma_1(x)}{(x-a)^\beta (b-x)^\alpha} \quad a \leq x \leq b
\]

is bounded from above and from below by positive numbers

\[
0 < m \leq q(x) \leq M < \infty.
\]

Then we can use the weight function \((x-a)^\beta (b-x)^\alpha\) to calculate (7.3.17):

\[
I = \int_a^b (x-a)^\beta (b-x)^\alpha \Psi(x) \, dx
\]

where

\[
\Psi(x) = (x-a)^{-\beta} (b-x)^{-\alpha} F(x).
\]

For example, if the region of integration has the form shown in Figure 3 where the boundary \( \lambda \) of the region has at the point \( A \) with coordinate \( x = a \) a tangent of the first order\(^{10}\) we can take \( \alpha = 0 \), \( \beta = \frac{1}{2} \) and use as the

\(^{10}\) Mme. H. Berthod-Zaborowski and H. Mineur, "Sur le calcul numérique des intégrales doubles," C. R. Acad. Sci. Paris, Vol. 229, 1949, pp. 919-21. Taking \( y \) as the independent variable then at the point \( A \) the boundary curve \( \lambda \) can be written in the form \( x = a + c_2 (y - y_0)^2 + c_3 (y - y_0)^3 + \cdots \); where \( c_2 \neq 0 \).
weight function

\[ p(x) = \sqrt{x - a}. \]

The integral

\[ I = \int_a^b x - a \Psi(x) \, dx, \quad \Psi(x) = (x - a)^{-\frac{1}{2}} F(x) \]

can be calculated by formula (7.3.9).

If the region \( \sigma \) has the form shown in Fig. 4 where the boundary \( \lambda \) has tangents of the first order at \( x = a \) and \( x = b \) then we can use the weight function

\[ p(x) = \sqrt{(x - a) (b - x)}. \]

To calculate the integral

\[ I = \int_a^b \sqrt{(x - a) (b - x)} \Psi(x) \, dx \]

\[ \Psi(x) = [(x - a) (b - x)]^{-\frac{1}{2}} F(x) \]

we can use (7.3.11).
7.4. The Integral $\int_{-\infty}^{+\infty} e^{-x^2} f(x) \, dx$

7.4. THE INTEGRAL $\int_{-\infty}^{+\infty} e^{-x^2} f(x) \, dx$.

The system of polynomials which are orthogonal on the entire real axis $-\infty < x < +\infty$ with respect to the weight function $e^{-x^2}$ is the system of Chebyshev-Hermite polynomials

$$H_n(x) = (-1)^n e^{-x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

A quadrature formula of the highest degree of precision

$$\int_{-\infty}^{+\infty} e^{-x^2} f(x) \, dx = \sum_{k=1}^{n} A_k f(x_k) + R(f) \quad (7.4.1)$$

must have as its nodes the roots of the polynomial $H_n(x)$:

$$H_n(x_k) = 0, \quad (k = 1, 2, \ldots, n).$$

The coefficients $A_k$ can be found from (7.1.3) by using the leading coefficients (2.4.4) of the normalized Chebyshev-Hermite polynomials (2.4.3):

$$A_k = -\frac{2^{n+1} n! \frac{1}{n^2}}{H'_n(x_k) H_{n+1}(x_k)}.$$

If we substitute $x = x_k$ in (2.4.2) we obtain $H_{n+1}(x_k) = -H'_n(x_k)$ and thus this expression for the $A_k$ can be written as

$$A_k = \frac{2^{n+1} n! \frac{1}{n^2}}{[H'_n(x_k)]^2} \quad (7.4.2)$$

To find the remainder in (7.4.1) we use the polynomial $\omega(x) = 2^{-n} H_n(x)$; then

$$\int_{-\infty}^{+\infty} p(x) \omega^2(x) \, dx = 2^{-2n} \int_{-\infty}^{+\infty} e^{-x^2} H_n^2(x) \, dx = 2^{-n} n! \frac{1}{n^2},$$

and by (7.1.5)

$$R(f) = \frac{n! \frac{1}{n^2}}{2^n} \frac{\int (2n)(\eta)}{(2n)!}. $$
In Appendix B we give values of the $x_k$ and $A_k$ in formula (7.4.1) for $n = 1(1)20$.

As an example$^{12}$, let us evaluate numerically the integral

$$
\int_{-\infty}^{+\infty} e^{-x^2} J_0(x) \, dx = \sqrt{\pi} e^{-1/8} I_0(1/8) = 1.5703011006678
$$

where $J_0(x)$ is the Bessel function of order zero and $I_0(x)$ is the modified Bessel function of order zero$^{13}$. Applying the quadrature formula (7.4.1) with 10 nodes we obtain

$$
\sum_{k=1}^{10} A_k J_0(x_k) = 1.5703011006676
$$

which differs by only two in the last place from the true value which was found from the series expansion for $I_0(x)$.

7.5. INTEGRALS OF THE FORM $\int_0^\infty x^\alpha e^{-x} f(x) \, dx$.

The system of polynomials which are orthogonal on the semi-infinite axis $0 \leq x < \infty$ with respect to the weight function $x^\alpha e^{-x}$ is the system of Chebyshev-Laguerre polynomials

$$
L_n^{(a)}(x) = (-1)^n x^{-a} e^{x} \frac{d^n}{dx^n} x^{a+n} e^{-x}.
$$

A quadrature formula of the highest degree of precision

$$
\int_0^\infty x^\alpha e^{-x} f(x) \, dx = \sum_{k=1}^{n} A_k f(x_k) + R(f) \quad (7.5.1)
$$

must have as its nodes the roots of the Laguerre polynomial $L_n^{(a)}(x)$.

The normalized Laguerre polynomials are given by (2.5.4) and their

---


$^{12}$ This example is from the paper by H. E. Salzer, R. Zucker, and R. Capuano cited in Appendix B.

leading coefficients by (2.5.5). Therefore the coefficients $A_k$ can be found from (7.1.3) to be

$$A_k = -\frac{n! \Gamma(\alpha + n + 1)}{L_n^{(a)_1}(x_k) L_{n+1}^{(a)}(x_k)}.$$ 

Using the relationship

$$L_n^{(a)}(x) = (x - \alpha - n - 1) L_n^{(a)}(x) - x L_{n+1}^{(a)}(x),$$ 

from the theory of Laguerre polynomials, we obtain

$$L_{n+1}^{(a)}(x_k) = -x_k L_{n}^{(a)}(x_k)$$

and therefore

$$A_k = \frac{n! \Gamma(\alpha + n + 1)}{x_k [L_{n}^{(a)}(x_k)]^2}.$$

Values of the $x_k$ and $A_k$ for $\alpha = 0$ for $n = 1 \ (1) 16 \ (4) 32$ are given in Appendix C.

1. Consider the integral

$$I = \int_0^\infty e^{-x} x^a e^{-x} f(x) \, dx = -2 - 1.2337.$$ 

Let us calculate the integral by using formula (7.5.1) for $\alpha = 0$ with 5 nodes. Using the $x_k$ and $A_k$ tabulated in Appendix C for $n = 5$ we obtain

$$I = A_1 f(x_1) + \cdots + A_5 f(x_5) = 1.2338.$$ 

2. We now calculate the integral

$$I = \int_0^\infty \frac{xdx}{e^x + e^{-x} - 1} = \int_0^\infty xe^{-x} [1 + e^{-2x} - e^{-x}]^{-1} dx = 1.17$$

by using the formula with two nodes for the weight function

$$p(x) = xe^{-x}$$

which corresponds to (7.5.1) with $\alpha = 1$. The second degree polynomial orthogonal with respect to $xe^{-x}$ is found from (2.5.2) to be

$$L_2^{(1)}(x) = x^2 - 6x + 6.$$
The roots of this polynomial are \( x_1 = 3 - \sqrt{3} \) and \( x_2 = 3 + \sqrt{3} \) and the coefficients can be calculated from (7.5.2):

\[
A_1 = \frac{3 + \sqrt{3}}{6}, \quad A_2 = \frac{3 - \sqrt{3}}{6}.
\]

We then obtain

\[
I = A_1 f(x_1) + A_2 f(x_2) = 1.20.
\]

REFERENCES

B. Bronwin, "On the determination of the coefficients in any series of sines and cosines of multiples of a variable angle from particular values of that series," *Philosophical Magazine*, (3) Vol. 34, 1849, pp. 260-68.


A. A. Markov, *Calculus of Finite Differences*, Moscow, 1911 (Russian).


8.1. MINIMIZATION OF THE REMAINDER OF QUADRATURE FORMULAS

In Chapter 7 we studied quadrature formulas of the highest algebraic degree of precision. It is reasonable to suppose that such formulas will give a small error provided that the integrand \( f(x) \) can be closely approximated by a polynomial of moderate degree, in particular if \( f(x) \) is an analytic function in a sufficiently wide region about the segment of integration \([a, b]\). Many years of experimentation has shown that these formulas give excellent precision in comparison with other types of quadrature formulas.

However these formulas are not universal, and in some practical cases they are known to give worse results than some of the elementary formulas: the midpoint formula, the trapezoidal formula, Simpson's formula, and others. This usually happens when the function \( f(x) \) has a low order of differentiability or is an analytic function with singular points close to the segment of integration.

In the theory of quadrature there arose the need for the construction of formulas for the integration of functions which belong to a predetermined class, in particular to a class of functions of low order of differentiability.

Let us briefly recall the comments we made on this problem in Section 5.1. Let us be given a class of functions \( F \). For each function \( f \in F \) the remainder \( R(f) \) of the quadrature is defined as

\[
R(f) = \int_a^b p(x) f(x) \, dx - \sum_{k=1}^n A_k f(x_k). \quad (8.1.1)
\]
A number which can be used to characterize the precision of the quadrature formula for all functions of \( F \) is

\[
R = \sup_f |R(f)| = \sup_f \left| \int_a^b p(x)f(x)\,dx - \sum_{k=1}^n A_k f(x_k) \right|. \quad (8.1.2)
\]

The value of \( R \) depends on the \( x_k \) and the \( A_k \), and we wish to select the nodes and coefficients so that \( R \) has the smallest possible value. The \( x_k \) and \( A_k \) are usually subjected to certain restraints which are related to the class \( F \) and to the way in which the functions of \( F \) are given. Two examples of such restraints are:

1. If the functions \( f \) are given in tabular form for a certain set of values of \( x \), then it would be desirable to restrict the choice of the \( x_k \) to values for which the function is tabulated.

2. In order to construct quadrature formulas with the least estimate of the remainder for the class of functions with continuous \( r \)th derivative for which \( |f^{(r)}| \leq M_r \), we must require that the quadrature formula be exact for all polynomials of degree \( \leq r - 1 \). This is the same as requiring

\[
\sum_{k=1}^n A_k x_k^m = \int_a^b p(x) x^m \,dx, \quad (m = 0, 1, \ldots, r - 1). \quad (8.1.3)
\]

In this chapter we assume that the segment of integration is finite. This assumption will be necessary for the particular cases which we will consider. With this assumption we can always consider that the segment \([a, b]\) has been transformed into the segment \([0, 1]\).

8.2. MINIMIZATION OF THE REMAINDER IN THE CLASS \( L_{q-r} \)

We will say that \( f(x) \) belongs to the class \( L_{q-r} \) \((q \geq 1)\) if \( f(x) \) has an absolutely continuous derivative of order \( r - 1 \) on \([0, 1]\) and \( f^{(r)}(x) \) is \( q \)th power summable on \([0, 1]\).

Each function \( f \in L_{q-r} \) can be represented in the form

\[
f(x) = \sum_{i=0}^{r-1} \frac{f^{(i)}(0)}{i!} x^i + \int_0^1 f^{(r)}(t) E(x - t) \frac{(x - t)^{r-1}}{(r-1)!} \,dt \quad (8.2.1)
\]

where the \( f^{(i)}(0) \) are numbers and \( f^{(r)}(t) \) is a measurable and \( q \)th power summable function on \([0, 1]\). The converse is also true: for any numbers \( f^{(i)}(0) \) and any \( f^{(r)}(t) \epsilon L_q \) the function defined by (8.2.1) belongs to \( L_{q-r} \).

Consider the integral \( \int_0^1 \rho(x)f(x)\,dx \), where \( f(x) \epsilon L_{q-r} \). At first it will be sufficient to assume that the weight function \( \rho(x) \) is measurable and summable on \([0, 1]\).
8.2. Minimization of the Remainder in the Class $L_q^{(r)}$

Suppose we use the quadrature formula

$$
\int_0^1 \rho(x) f(x) \, dx = \sum_{k=1}^{n} A_k f(x_k)
$$

(8.2.2)

to calculate this integral approximately. We wish to construct a formula which will be the "best" for all functions $f(x) \in L_q^{(r)}$ ($q \geq 1$) assuming that (8.2.2) is exact for all polynomials of degree < $r$. If we use the representation (8.2.1) for the functions of $L_q^{(r)}$, then the remainder $R(f)$ of the quadrature has the form:

$$
R(f) = \int_0^1 \rho(x) f(x) \, dx - \sum_{k=1}^{n} A_k f(x_k) = \int_0^1 f^{(r)}(t) K(t) \, dt
$$

(8.2.3)

$$
K(t) = \int_t^1 \rho(x) \frac{(x-t)^{r-1}}{(r-1)!} \, dx - \sum_{k=1}^{n} A_k E(x_k - t) \frac{(x_k - t)^{r-1}}{(r-1)!}.
$$

(8.2.4)

Consider now the class $F$ of functions $f(x)$ which satisfy the condition

$$
\left( \int_0^1 |f^{(r)}(t)|^q \, dt \right)^{\frac{1}{q}} \leq M_r.
$$

By Hölder's inequality we have

$$
|R(f)| \leq \left( \int_0^1 |f^{(r)}(t)|^q \, dt \right)^{\frac{1}{q}} \left( \int_0^1 |K(t)|^p \, dt \right)^{\frac{1}{p}} \leq M_r \left( \int_0^1 |K(t)|^p \, dt \right)^{\frac{1}{p}}
$$

for $\frac{1}{p} + \frac{1}{q} = 1$. The function

$$
f^{(r)}(t) = M_r \left( \int_0^1 |K(t)|^p \, dt \right)^{-\frac{1}{q}} |K(t)|^p \sign K(t)
$$

belongs to the class $F$ and, as is easily seen, for this function the above inequality becomes an equality. Therefore the right side will be an upper bound for $|R(f)|$ on the class $F$:

$$
R = \sup_F |R(f)| = M_r \left( \int_0^1 |K(t)|^p \, dt \right)^{\frac{1}{p}}.
$$

(8.2.5)

Thus we see that the dependence of $R$ on $x_k$ and $A_k$ occurs only in the term $\int_0^1 |K(t)|^p dt$. Our aim will be to select the $x_k$ and $A_k$ so that the integral $\int_0^1 |K(t)|^p dt$ will be a minimum. If such $x_k$ and $A_k$ exist then
they will furnish a least value for \( R \) for each \( M_r \) and the corresponding quadrature formula can be considered "the best" for the entire class \( \mathcal{L}_q(r) \).

The problem of minimizing \( \int_0^1 |K(t)|^p dt \) can be interpreted as the problem of best approximating the function \( \int_0^1 \rho(x) \frac{(x-t)^{n-1}}{(r-1)!} dx \) in the metric \( L_p \) (see Section 4.1) by means of functions of the form

\[
\sum_{k=1}^{n} A_k E(x_k - t) \frac{(x_k - t)^{r-1}}{(r-1)!}.
\]

For arbitrary \( \rho(x) \), \( r \) and \( n \) this problem can not be solved in closed form. We will restrict ourselves to certain special cases when the solution can be found by simple methods.

First of all we need to become familiar with certain facts from the theory of approximation of functions. Let us be given on the segment \( [0, 1] \) a certain function \( f \in L_p \). In addition, let us suppose that the functions \( \varphi_k \in L_p \) \((k = 1, 2, \ldots, n)\) are linearly independent on \([0, 1]\). This means that the equation

\[
\int_0^1 \left| \sum_{k=1}^{n} a_k \varphi_k \right|^p dx = 0
\]

is possible only when all the \( a_k \) are zero. This is equivalent to the statement that the equation \( \sum_{k=1}^{n} a_k \varphi_k(x) = 0 \) can be fulfilled on a set of points of measure greater than zero if and only if \( a_k = 0 \) \((k = 1, 2, \ldots, n)\).

The error \( \epsilon \) in the approximation of \( f \) by a linear combination \( s = \sum_{k=1}^{n} a_k \varphi_k \) is defined by

\[
\epsilon^p = \int_0^1 |f - s|^p dx = I.
\]

We now discuss the conditions under which \( \epsilon^p \) will be a minimum. From

\( R(f) \) is a linear functional defined for functions \( f^{(r)} \in L_q \). The integral \( \left( \int_0^1 |K|^p dt \right)^{1/p} \) is the norm of \( R(f) \) in the space \( L_q \). In the terminology of functional analysis our problem is to construct a quadrature formula (8.2.2) with the least norm for the remainder.
8.2. Minimization of the Remainder in the Class $L^p$)

A theorem of calculus we can assert that the values of $a_k$ which give a minimum for $l$ must satisfy the equations

$$
\frac{\partial I}{\partial a_i} = p \int_0^1 |f - s|^{p-1} \text{sign} (f - s) \phi_i \, dx = 0 \quad i = 1, 2, \ldots, n. \tag{8.2.6}
$$

We now show that the linear combination $s$ which satisfies (8.2.6) indeed gives the best approximation to $f$. Let us take any other linear combination $s^* = \sum_{k=1}^n a_k^* \phi_k$. We must show that $I \leq I^* = \int_0^1 |f - s^*|^p \, dx$. We have

$$
I = \int_0^1 |f - s|^p \, dx = \int_0^1 |f - s|^{p-1} (f - s) \text{sign} (f - s) \, dx =
= \int_0^1 |f - s|^{p-1} (f - (s - s^*) - s^*) \text{sign} (f - s) \, dx.
$$

By (8.2.6)

$$
I = \int_0^1 |f - s|^{p-1} (f - s^*) \text{sign} (f - s) \, dx. \tag{8.2.7}
$$

This integral cannot be made smaller if sign $(f - s)$ is replaced by sign $(f - s^*)$. Therefore

$$
I \leq \int_0^1 |f - s|^{p-1} |f - s^*| \, dx. \tag{8.2.8}
$$

Applying Hölder's inequality$^2$

$^2$See, for example, I. P. Natanson, Theory of Functions of a Real Variable, Ungar, New York, 1955, Chap. 7, Sec. 6. If $F \in L_p$ and $G \in L_q$, $\frac{1}{p} + \frac{1}{q} = 1$, then the product $FG$ is summable and

$$
\int_0^1 FG \, dx \leq \left( \int_0^1 |F|^p \, dx \right)^{\frac{1}{p}} \left( \int_0^1 |G|^q \, dx \right)^{\frac{1}{q}}. \tag{a}
$$

For the following presentation it is essential to note that equality can occur only when the following two conditions are satisfied:

1. $\frac{|F|^p}{\int_0^1 |F|^p \, dx} = \frac{|G|^q}{\int_0^1 |G|^q \, dx}$,

2. The signs of $F$ and $G$ coincide almost everywhere on $[0, 1]$. To apply (a) to (8.2.9) we take $F = |f - s^*|$ and $G = |f - s|^{p-1}$.
Approximate Calculation of Definite Integrals

\[ l \leq \left( \int_0^1 |f - s|^p \, dx \right)^{\frac{1}{p}} \left( \int_0^1 |f - s|^p \, dx \right)^{\frac{p-1}{p}} = I^* \frac{1}{I} \frac{p-1}{p}. \quad (8.2.9) \]

This gives \( \frac{1}{p} \leq I^* \frac{1}{p} \) and thus \( l \leq I^* \).

Finally we show that the linear combination \( s \), which minimizes \( l \), is unique. It is necessary to verify that if \( l = I^* \) then \( a_k = a_k^* \) \( (k = 1, 2, \ldots, n) \). This is clear if \( l = 0 \) because if \( f = s \) for almost all \( x \) then

\[ I^* = \int_0^1 |f - s|^p \, dx = \int_0^1 |s - s|^p \, dx = 0. \]

Therefore, for almost all \( x \), \( s = s^* \) and since the \( \phi_k \) are linearly independent \( a_k = a_k^* \). Thus we can suppose that \( |f - s| \) is positive on a set of points of measure greater than zero.

From the argument leading to (8.2.9) we see that \( l = I^* \) only if two conditions are satisfied:

1. For almost all \( x \) we must have

\[ |f - s|^{p-1} \text{sign} (f - s) = |f - s - s|^p. \quad (8.2.10) \]

This is necessary if (8.2.8) is to be an equality.

2. In (8.2.9) equality can only occur when almost everywhere

\[ \frac{|f - s|^p}{\int_0^1 |f - s|^p \, dx} = \frac{|f - s|^p}{\int_0^1 |f - s|^p \, dx}. \]

Since \( I = \int_0^1 |f - s|^p \, dx = I^* = \int_0^1 |f - s|^p \, dx \) then almost everywhere we must have

\[ |f - s| = |f - s^*|. \quad (8.2.11) \]

But \( |f - s| > 0 \) on a set of positive measure and from (8.2.10) and (8.2.11) it follows that on a set of positive measure

\[ f - s = f - s^* \quad \text{or} \quad s = s^*. \]

Since the \( \phi_k \) are linearly independent this is only possible when \( a_k = a_k^* \) \( (k = 1, 2, \ldots, n) \).

We now assume \( \rho (x) = 1 \) and let us consider the quadrature formula

\[ \int_0^1 f(x) \, dx = \sum_{k=1}^n A_k f(x_k) + R(f). \quad (8.2.12) \]

Now let \( f(x) \) be absolutely continuous and \( f''(x) \) be \( q^{th} \) power summable on [0, 1]. This corresponds to the case \( r = 1 \). We require that (8.2.12)
be exact for \( f(x) = 1 \) which imposes the condition \( \sum_{k=1}^{n} A_k = 1 \) on the coefficients. In the class \( L_q^{(1)} \) the remainder \( R(f) \) has the precise estimate

\[
R = \sup_f R(f) = M_1 \left( \int_0^1 |K(t)|^p \, dt \right)^{\frac{1}{p}}
\]

\[
M_1^q \geq \int_0^1 |f'(x)|^q \, dx
\]

\[
K(t) = 1 - t - \sum_{k=1}^{n} A_k E(x_k - t).
\]

The kernel \( K(t) \) of the remainder is a piece-wise linear function with leading coefficient equal to \(-1\), for which the nodes \( x_k \) are points of discontinuity. At the node \( x_k \) the function \( K(t) \) has a jump of \( A_k \). If the \( x_k \) lie inside the segment \([0, 1]\) then at \( t = 0 \) and \( t = 1 \) the kernel is zero. A typical graph of \( K(t) \) is illustrated in Fig. 5.

![Figure 5](image_url)

The problem of minimizing the integral \( \int_0^1 |K(t)|^p \, dt \) has the following geometric meaning: it is necessary to determine for what arrangement of points of discontinuity \( x_k \) \((k = 1, 2, \ldots, n)\) and for what values of the jumps \( A_k \) \((k = 1, 2, \ldots, n)\), subject to the restraint \( \sum A_k = 1 \), will the cross-hatched area in Fig. 5 have the least mean \( p^{th} \) power. The answer is easy to foresee: the minimum will be achieved when the area consists of \( 2n \) equal triangles.

The nodes \( x_k \) must be located at the points \( x_k = \frac{2k-1}{2n} \) \((k = 1, 2, \ldots, n)\). The coefficients \( A_k \) must all be equal and since their sum is unity \( A_k = \frac{1}{n} \) \((k = 1, 2, \ldots, n)\). This result can be easily verified by a calculation which we will not carry out.
The corresponding quadrature formula is

$$\int_0^1 f(x) \, dx = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{2k-1}{2n}\right) + R(f) \quad (8.2.13)$$

which is well known as the repeated midpoint formula. Its remainder in the class $L_q^{(1)}$ is

$$|R(f)| \leq \frac{M_1}{2n^{\frac{1}{p}}}, \quad M_1 = \left(\int_0^1 |f'|^q \, dt\right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Let us now take $r = 2$ and consider the class $L_q^{(2)}$ of functions with absolutely continuous first derivative for which $f^{(2)}(x)$ is $q^{th}$ power summable.

We require that the quadrature formula (8.2.12) be exact whenever $f(x)$ is a polynomial of degree zero or one. This is equivalent to the following two restraints on the $x_k$ and $A_k$:

$$\sum_{k=1}^{n} A_k = \int_0^1 1 \, dx = 1 \quad (8.2.14)$$
$$\sum_{k=1}^{n} A_k x_k = \int_0^1 x \, dx = \frac{1}{2}$$

Under these conditions the remainder $R(f)$ has the following precise estimate in the class $L_q^{(2)}$

$$|R(f)| \leq M_2 \left(\int_0^1 |K(t)|^p \, dt\right)^{\frac{1}{p}}, \quad M_2 \geq \left(\int_0^1 |f^{(2)}(x)|^q \, dx\right)^{\frac{1}{q}}$$

$$K(t) = \frac{(1 - t)^2}{2} - \sum_{k=1}^{n} A_k (x_k - t) (x_k - t). \quad (8.2.15)$$

For later use we tabulate the value of $K(t)$ on each of the segments $[0, x_1], [x_1, x_2], \ldots, [x_n, 1]$:

$$K(t) = \begin{cases} 
\frac{t^2}{2} & \text{for } 0 \leq t \leq x_1 \\
\frac{(1 - t)^2}{2} - \sum_{k=1}^{i} A_k (x_k - t) & \text{for } x_i \leq t \leq x_{i+1} \\
\frac{(1 - t)^2}{2} & \text{for } x_n \leq t \leq 1;
\end{cases}$$
8.2. Minimization of the Remainder in the Class $L^q(\tau)$

$K(t)$ is a continuous function of $t$ on $[0, 1]$. The first derivative $K'(t)$ has discontinuities of the first kind at the points $x_k$ and the size of the jumps of $K'(t)$ are

$$K'(x_k + 0) - K'(x_k - 0) = -A_k. \quad (8.2.16)$$

On each of the indicated segments $K(t)$ is a quadratic polynomial with leading coefficient $\frac{1}{2}t^2$. A typical graph of $K(t)$ is given in Fig. 6.

Let us now turn to the problem of minimizing the integral

$$U = \int_0^1 |K(t)|^p dt$$

with the restraints (8.2.14). We will assume that the minimum value exists and will use the method of Lagrangian multipliers to find it. The result obtained by this method will later be justified. Let us consider the function

$$G = U + \lambda_1 \left( \sum_{k=1}^n A_k - 1 \right) + \lambda_2 \left( \sum_{k=1}^n A_k x_k - \frac{1}{2} \right)$$

and let us set equal to zero the partial derivatives of this function with respect to the $x_i$ and $A_i$:

$$\frac{\partial G}{\partial x_i} = -A_ip \int_0^1 |K(t)|^{p-1} S(t) E(x_i - t) dt + \lambda_2 x_i = 0 \quad (8.2.17)$$

$$S(t) = \text{sign } K(t);$$

$$\frac{\partial G}{\partial A_i} = -p \int_0^1 |K(t)|^{p-1} S(t) E(x_i - t) (x_i - t) dt + \lambda_1 + \lambda_2 x_i = 0. \quad (8.2.18)$$
Here the $A_i$ are assumed to be different from zero because otherwise the quadrature sum would contain less than $n$ nodes. The term $A_i$ can then be cancelled from (8.2.17) to give:

$$
\int_0^{x_i} |K(t)|^{p-1} S(t) \, dt = \frac{\lambda_2}{p}.
$$

Since $i$ takes the values $1, 2, \ldots, n$ we have

$$
\int_0^{x_i} |K(t)|^{p-1} S(t) \, dt = \frac{\lambda_2}{p}
$$

$$
\int_{x_i}^{x_{i+1}} |K(t)|^{p-1} S(t) \, dt = 0, \quad i = 1, 2, \ldots, n - 1. \quad (8.2.19)
$$

By using (8.2.17) equation (8.2.18) can be reduced to the form

$$
p \int_0^{x_i} |K(t)|^{p-1} S(t) \, dt = 0.
$$

Hence we see that

$$
\int_0^{x_1} |K(t)|^{p-1} S(t) \, dt = \frac{-\lambda_1}{p}
$$

$$
\int_{x_i}^{x_{i+1}} |K(t)|^{p-1} S(t) \, dt = 0, \quad i = 1, 2, \ldots, n - 1. \quad (8.2.20)
$$

From (8.2.19) and (8.2.20) it follows that on each of the segments $[x_i, x_{i+1}]$ ($i = 1, 2, \ldots, n - 1$) the function $K(t)$ is a polynomial of second degree with leading coefficient $\frac{1}{2}a^2$ which deviates least from zero in the metric $L_p$. In order to find these polynomials let us first find the polynomial of the form $T_2(x) = x^2 + mx + r$ which deviates least from zero on the segment $[-1, +1]$

$$
\int_{-1}^{1} |T_2(x)|^p \, dx = \text{minimum}.
$$

We can see at once that for this polynomial $m = 0$. Indeed, if we replace $x$ by $-x$ in the above integral we find that $T_2(-x)$ also deviates least from zero. But then

$$
T_2(x) = T_2(-x)
$$

and consequently $m = 0$.

Let us now find the constant term $r$. 

8.2. Minimization of the Remainder in the Class $L^q(r)$

The following conditions must be satisfied by $T_2$

\[
\int_{-1}^{1} |T_2|^{p-1} \text{sign } T_2 \, dx = 0
\]

(8.2.21)

\[
\int_{-1}^{1} |T_2|^{p-1} x \text{ sign } T_2 \, dx = 0.
\]

The second of these equations is identically satisfied. From the first it follows that sign $T_2$ must change sign inside $[-1, +1]$. We suppose, therefore, that $r = -l^2$, $0 < l < 1$ and $T_2(x) = x^2 - l^2$, and write the first condition in the form

\[
- \int_0^l (l^2 - x^2)^{p-1} \, dx + \int_l^1 (x^2 - l^2)^{p-1} \, dx = 0.
\]

If we set $x = l\sqrt{t}$ then we can reduce this equation to the form

\[
\int_1^l t^{-\frac{1}{2}r} (t - 1)^{p-1} \, dt = \frac{\sqrt{\pi} \Gamma(p)}{\Gamma(p + \frac{1}{2})}. \tag{8.2.22}
\]

From this equation we can find $l$. As $l$ increases from 0 up to 1 the left side of (8.2.22) decreases from $\infty$ to 0 and hence this equation has one and only one solution.

In order to transform this result to the segment $[x_i, x_{i+1}]$ we write $a_i = \frac{1}{2}(x_i + x_{i+1})$ and $h_i = \frac{1}{2}(x_{i+1} - x_i)$. Then the second degree polynomial with leading coefficient $\frac{1}{2}t^2$ which deviates least from zero in the metric $L_p$ on $[x_i, x_{i+1}]$ is

\[
K(t) = \frac{h_i^2}{2} T_2 \left( t - \frac{a_i}{h_i} \right) = \frac{h_i}{2} \left( \frac{t - a_i}{h_i} \right)^2 - l^2, \quad x_i \leq t \leq x_{i+1}.
\]

At the point $t = x_i = a_i - h_i$ this polynomial has the value

\[
K(x_i) = \frac{h_i^2}{2} (1 - l^2).
\]

Similarly if we write $K(t)$ for the segment $[x_{i-1}, x_i]$ we see that its value at the point $t = x_i$ is

\[
K(x_i) = \frac{h_i^2}{2} (1 - l^2).
\]

Because $K(t)$ is continuous these values must be equal; thus

\[h_{i-1} = h_i \quad (i = 2, 3, \ldots, n).\]
The common value of the \( h_i \) we will denote by \( h \). Then for each of the \( x_i \) we have

\[
K(x_i) = \frac{h^2}{2}(1 - l^2).
\]

Now consider the segment \( 0 \leq t \leq x_1 \). Here \( K(t) = \frac{1}{2}t^2 \) and at \( t = x_1 \) we must have

\[
K(x_1) = \frac{x_1^2}{2} = \frac{h^2}{2}(1 - l^2) \quad \text{or} \quad x_1 = h\sqrt{1 - l^2}.
\]

Finally, a consideration of \( K(t) \) on the segment \( [x_n, 1] \) gives for its length

\[
1 - x_n = h\sqrt{1 - l^2}.
\]

Since the sum of the lengths of the segments \([0, x_1], [x_1, x_2], \ldots, [x_n, 1]\) is equal to 1 we must have

\[
2h\sqrt{1 - l^2} + (n - 1)2h = 1
\]

and

\[
h = \frac{1}{2} \left[ n - 1 + \sqrt{1 - l^2} \right]^{-1}
\]

\[
x_k = x_1 + 2h(k - 1) = h \left[ 2(k - 1) + \sqrt{1 - l^2} \right]. \quad (8.2.23)
\]

To calculate the coefficients \( A_k \) we use equation (8.2.16). On the segment \( x_i \leq t \leq x_{i+1} \)

\[
K(t) = \frac{1}{2}h^2 \left[ \left( \frac{t - a_i}{h} \right)^2 - l^2 \right],
\]

\[
K'(t) = t - a_i
\]

\[
K'(x_i + 0) = K'(a_i - h + 0) = -h
\]

\[
K'(x_{i+1} - 0) = K'(a_i + h - 0) = +h.
\]

Therefore

\[
A_i = 2h, \quad (i = 2, 3, \ldots, n - 1). \quad (8.2.24)
\]

A similar calculation for the nodes \( x_1 \) and \( x_n \) gives

\[
A_1 = A_n = (1 + \sqrt{1 - l^2})h. \quad (8.2.25)
\]

Let us find the value of \( \int_0^1 |K(t)|^p \, dt \). We have
Minimization of the Remainder in the Class $L_q^{(r)}$

\[
\int_0^1 |K(t)|^p dt = \int_0^{x_1} \left(\frac{t^2}{2}\right)^p dt + \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \left[\frac{h^2}{2} T_2\left(\frac{t-a_i}{h}\right)\right]^p dt + \int_{x_n}^1 \left[\frac{1}{2} (1-t)^2\right]^p dt.
\]

It is easily verified that

\[
\int_0^{x_1} \frac{t^{2p}}{2^p} dt = \int_{x_n}^1 \frac{(1-t)^{2p}}{2^p} dt = \frac{h^{2p+1}(1-l^2)^{p+\frac{1}{2}}}{(2p+1)2^p}
\]

\[
\int_{x_i}^{x_{i+1}} \left[\frac{h^2}{2} T_2\left(\frac{t-a_i}{h}\right)\right]^p dt = \frac{h^{2p+1}}{2^p} \int_{-1}^1 |T_2(x)|^p dx
\]

\[
l = \int_{-1}^1 |T_2(x)|^p dx = |T_2|^p x^{+1}_{-1} - p \int_{-1}^1 |T_2(x)|^{p-1} x T_2 \text{sign}(T_2) dx =
\]

\[
= 2(1-l^2)^p - 2p \int_{-1}^1 |T_2|^{p-1} x^2 \text{sign}(T_2) dx.
\]

If, in this last integral, we replace $x^2$ by $x^2 - l^2 + l^2 = T_2 + l^2$ we obtain

\[
l = 2(1-l^2)^p - 2p l - 2p l^2 \int_{-1}^1 |T_2|^{p-1} \text{sign}(T_2) dx.
\]

But, by (8.2.21), the integral on the right side of this equation is zero and consequently

\[
l = \frac{2(1-l^2)^p}{2p+1},
\]

\[
\int_0^1 |K(t)|^p dt = \frac{h^{2p+1}}{(2p+1)2^{p-1}} \left[(1-l^2)^{p+\frac{1}{2}} + (n-1)(1-l^2)^p\right] =
\]

\[
= \frac{h^{2p+1}(1-l^2)^p}{(2p+1)2^{p-1}} \left[\sqrt{1-l^2} + n - 1\right] =
\]

\[
= \frac{h^{2p}(1-l^2)^p}{(2p+1)2^p}.
\]
The remainder $R(f)$ in formula (8.2.12) with nodes (8.2.23) and coefficients (8.2.24) and (8.2.25) will have the following estimate for a function $f \in L_p^2$:

$$|R(f)| \leq M_2 \frac{h^2(1-l^2)}{2^{p+2}p+1}, \quad \quad M_2 = \left( \int^1_0 |f''(x)|^p \, dx \right)^{1/p} \tag{8.2.27}$$

We will now show that the nodes $x_k$ and coefficients $A_k$ indeed give the least value of the integral (8.2.26).

Let $x_k^*$ and $A_k^*$ be any other nodes and coefficients and $K^*(t)$ the corresponding kernel. We must show that

$$\int_0^1 |K^*(t)|^p \, dt \geq \frac{h^2p(1-l^2)^p}{(2p+1)2^p}.$$  

We have

$$\int_0^1 |K^*|^p \, dt = \int_0^1 \left( \frac{t^2}{2} \right)^p \, dt + \sum_{i=1}^{n-1} \int_{x_i^*}^{x_{i+1}^*} |K^*|^p \, dt +$$

$$+ \int_{x_n^*}^{x_1^*} \left( \frac{(1-t)^2}{2} \right)^p \, dt = \frac{x_{2p+1}^* + (1-x_{n}^*)^{2p+1}}{(2p+1)2^p} + \sum_{i=1}^{n-1} \int_{x_i^*}^{x_{i+1}^*} |K^*|^p \, dt.$$  

On each of the segments $[x_i^*, x_{i+1}^*]$ the kernel $K^*(t)$ is a certain quadratic polynomial with leading coefficient $\frac{1}{t^2}$. Let us replace $K^*(t)$ by the second degree polynomial with the same leading coefficient which deviates least from zero on $[x_i^*, x_{i+1}^*]$ in the metric $L_p$. If we denote $a_i = \frac{1}{2}(x_i^* + x_{i+1}^*)$ and $h_i = \frac{1}{2}(x_{i+1}^* - x_i^*)$ then such a polynomial will be $\frac{h_i^2}{2} \, T_2(t - \frac{a_i^*}{h_i^*})$. The last equation then becomes an inequality:

$$\int_0^1 |K^*|^p \, dt \geq \frac{x_{2p+1}^* + (1-x_{n}^*)^{2p+1}}{(2p+1)2^p} +$$

$$+ \sum_{i=1}^{n-1} \left( \frac{h_i^*}{2} \right)^p \int_{x_i^*}^{x_{i+1}^*} \left| T_2(t - \frac{a_i^*}{h_i^*}) \right|^p \, dt.$$  


8.2. Minimization of the Remainder in the Class $L_q^{(r)}$

Equality is possible only when $K^*(t)$ is the polynomial which deviates least from zero on each segment $[x_i^*, x_{i+1}^*]$. But in that case we will have $x_k^* = x_k$ and $A_k^* = A_k$.

Using our previous notation the integrals in the summation were shown to have the common value \(\frac{2(1 - l^2)^p}{2p + 1}\). Therefore

\[
\int_0^1 |K^*|^p dt \geq \frac{x_1^{*2p+1} + (1 - x_n^*)^2}{(2p + 1) 2p} + \frac{(1 - l^2)^p}{(2p + 1) 2^3 p} \sum_{i=1}^{n-1} (x_{i+1}^* - x_i^*)^{2p+1}. \tag{8.2.28}
\]

If in the sum $u_n = \sum_{i=1}^{n-1} (x_{i+1}^* - x_i^*)^{2p+1}$, we fix $x_1^*$ and $x_n^*$ then, as a function of $x_2^*, \ldots, x_{n-1}^*$, $u_n$ is a minimum when all the segments $[x_i^*, x_{i+1}^*]$ have the same length. This can be shown by means of induction. Suppose $n = 3$ and consider

\[ u_3 = (x_3^* - x_2^*)^{2p+1} + (x_2^* - x_1^*)^{2p+1}. \]

Then

\[ \frac{\partial u_3}{\partial x_2^*} = (2p + 1)[(x_2^* - x_1^*)^{2p} - (x_3^* - x_2^*)^{2p}] = 0 \]

shows that

\[ x_2^* - x_1^* = x_3^* - x_2^* \tag{8.2.29} \]

and because

\[ \frac{\partial^2 u_3}{\partial x_2^{2p}} = (2p + 1) 2p[(x_2^* - x_1^*)^{2p-1} + (x_3^* - x_2^*)^{2p-1}] > 0 \]

then (8.2.29) indeed gives a minimum.

Assuming that the assertion is true for $u_{n-1}$ we can verify it for $u_n$:

\[
\begin{align*}
\sum_{i=1}^{n-2} (x_{i+1}^* - x_i^*)^{2p+1} + (x_n^* - x_{n-1}^*)^{2p+1} &
\geq (n - 2) \left( \frac{x_{n-1}^* - x_1^*}{n - 2} \right)^{2p+1} + (x_n^* - x_{n-1}^*)^{2p+1} \\
&= \nu.
\end{align*}
\]
Let us find the minimum of \( v \) as a function of \( x_{n-1}^* \). From

\[
\frac{\partial v}{\partial x_{n-1}^*} = (2p + 1) \left[ \left( \frac{x_{n-1}^* - x_1^*}{n - 2} \right)^{2p} - \left( x_n^* - x_{n-1}^* \right)^{2p} \right] = 0
\]

it follows that the segment \([x_{n-1}^*, x_n^*]\) must have the same length as all the other segments \([x_i^*, x_{i+1}^*]\):

\[
x_n^* - x_{n-1}^* = \frac{x_{n-1}^* - x_1^*}{n - 2}.
\]  

(8.2.30)

Since

\[
\frac{\partial^2 v}{\partial x_{n-1}^*} = (2p + 1) 2p \left[ \left( \frac{x_{n-1}^* - x_1^*}{n - 2} \right)^{2p-1} + \left( x_n^* - x_{n-1}^* \right)^{2p-1} \right] > 0
\]

equation (8.2.30) indeed gives a minimum and therefore

\[
u_n \geq (n - 1) \left( \frac{x_n^* - x_1^*}{n - 1} \right)^{2p+1}.
\]

Substituting in (8.2.28) the minimum value for \( u_n \) we obtain

\[
\int_0^1 |K^*|^p \, dt \geq \frac{x_1^{2p+1} + (1 - x_n^*)^{2p+1}}{(2p + 1) 2p}
\]

\[
+ \frac{(1 - l^2)^p (x_n^* - x_1^*)^{2p+1}}{(2p + 1) 2^{3p} (n - 1)^{2p}} = w.
\]

By an argument similar to the preceding we can show that the minimum value of \( w \) is achieved for

\[
x_1^* = 1 - x_n^* = \frac{\sqrt{1 - l^2}}{2(n - 1 + \sqrt{1 - l^2})} = \frac{1}{2(n - 1 + \sqrt{1 - l^2})}
\]

and that

\[
\min w = \frac{(1 - l^2)^p h^{2p}}{2p(2p + 1)}.
\]

We finally obtain

\[
\int_0^1 |K^*|^p \, dt \geq \frac{(1 - l^2)^p h^{2p}}{2p(2p + 1)} = \int_0^1 |K|^p \, dt.
\]
8.3. MINIMIZATION OF THE REMAINDER IN THE CLASS $C_r$

In Section 5.3 we defined $C_r$ as the class of functions $f(x)$ which have a continuous derivative of order $r$ on $[0, 1]$. The characteristic representation for a function $f(x) \in C_r$ is given by

$$f(x) = \sum_{i=0}^{r-1} \frac{f^{(r)}(0)}{i!} x^i + \int_0^1 f^{(r)}(t) E(x - t) \frac{(x - t)^{r-1}}{(r-1)!} dt,$$

where the $f^{(i)}(0)$ are arbitrary real numbers and $f^{(r)}(t)$ is an arbitrary continuous function on $[0, 1]$.

A quadrature formula

$$\int_0^1 f(x) \, dx = \sum_{k=1}^{n} A_k f(x_k)$$

which has the least estimate of the remainder in $C_r$ must be exact whenever $f(x)$ is a polynomial of degree $<r$. Then the remainder in (8.3.2) can be represented in the form

$$R(f) = \int_0^1 f^{(r)}(t) K(t) \, dt$$
where

\[ K(t) = \frac{(1 - t)^r}{r!} - \sum_{k=1}^{n} A_k E(x_k - t) \frac{(x_k - t)^{r-1}}{(r - 1)!} \]

Consider the class \( F \), of functions \( f(x) \in C_r \), which satisfy the condition \( |f^{(r)}(x)| \leq M_r \). For functions of \( F \) we have

\[ |R(f)| \leq M_r \int_0^1 |K(t)| dt. \]

We can easily see that the right side of this inequality is an upper bound for \( |R(f)| \) on \( F \). This follows if we take a function \( f(t) \) for which

\[ f^{(r)}(t) = M_r \text{sign } K(t). \]

For such a function

\[ R(f) = M_r \int_0^1 |K(t)| dt. \]

Such a function does not belong to \( F \) because \( f^{(r)}(t) \) is not continuous, but this function, together with its first and second derivatives, can be approximated to any degree of precision in the metric \( L \) by means of a function of \( F \). Therefore in the above inequality for \( |R(f)| \) the right side can not be decreased:

\[ R = \sup_{F} |R(f)| = M_r \int_0^1 |K(t)| dt. \]  

(8.3.4)

We must minimize \( \int_0^1 |K(t)| dt \) subject to the restraining conditions

\[ \sum_{k=1}^{n} A_k x_k^i = \frac{1}{i + 1} \quad (i = 0, 1, \ldots, r - 1). \]  

(8.3.5)

As in the preceding section we will solve this problem for only the two simplest cases.

Let \( r = 1 \) and consider the class of functions with a continuous derivative on \([0, 1]\). In this case we must require that the quadrature formula will be exact whenever \( f(x) \) is a constant function. This is equivalent to requiring

\[ \sum_{k=1}^{n} A_k = 1. \]
The kernel $K(t)$ is

$$K(t) = 1 - t - \sum_{k=1}^{n} A_k E(x_k - t).$$

A typical graph of such a kernel is given in Fig. 5. The integral $\int_0^1 |K(t)| dt$ is numerically equal to the area which is shaded in the figure. This area will be the smallest when all of the $2n$ triangles have the same size. Therefore the formula which gives the least estimate of the remainder in the class $F$ is, for each $M_1$, the repeated midpoint formula (8.2.13).

The smallest value of the shaded area in Fig. 5 is $\frac{1}{4n}$ and hence the remainder $R(f)$ of formula (8.2.13) in the class $C_1$ has the estimate

$$|R(f)| \leq M_1 \frac{1}{4n}, \quad |f'(x)| \leq M_1.$$

Let us now consider the class of functions $C_2$ which have two continuous derivatives on $[0, 1]$. The nodes and coefficients must satisfy the two conditions (8.2.14) and hence the quadrature formula must be exact for any linear function.

The kernel of the remainder $K(t)$ is given by (8.2.15). We will obtain a representation for this kernel on the segment $[x_k, x_{k+1}]$. Let us assume that the minimum of $u = \int_0^1 |K(t)| dt$ exists. We construct the auxiliary function

$$G = u + \lambda_1 \left( \sum_{k=1}^{n} A_k - 1 \right) + \lambda_2 \left( \sum_{k=1}^{n} A_k x_k - \frac{1}{2} \right)$$

and set the partial derivatives of $G$ with respect to the $x_i$ and $A_i$ equal to zero:

$$\frac{\partial G}{\partial x_i} = -A_i \int_0^1 S(t) E(x_i - t) dt + \lambda_2 A_i = 0 \quad (8.3.6)$$

$$S(t) = \text{sign } K(t)$$

$$\frac{\partial G}{\partial A_i} = -\int_0^1 S(t) E(x_i - t) (x_i - t) dt + \lambda_1 + \lambda_2 x_i = 0 \quad (8.3.7)$$

$$\quad (i = 1, 2, \ldots, n).$$
From these equations we see that on each of the segments \([x_i, x_{i+1}]\)

\[
\int_{x_i}^{x_{i+1}} S(t) \, dt = 0, \quad \int_{x_i}^{x_{i+1}} tS(t) \, dt = 0 \quad (i = 1, 2, \ldots, n - 1).
\]

Thus on each of these segments the kernel \(K(t)\) is a second degree polynomial with leading coefficient \(\frac{1}{2} t^2\) which deviates least from zero on \([x_i, x_{i+1}]\) in the metric \(L\).

In Section 2.3 we showed that among all polynomials of degree \(n\) with leading coefficient equal to unity the polynomial which deviates least from zero on \([-1, 1]\) in the metric \(L\) is

\[
P_n(x) = \frac{1}{2^n} U_n(x) = \frac{\sin \left(\frac{(n + 1) \arccos x}{2}\right)}{2^n \sqrt{1 - x^2}}.
\]

For \(n = 2\) this is the polynomial

\[
P_2(x) = x^2 - \frac{1}{4}.
\]

Transforming the segment \([-1, 1]\) into the segment \([x_i, x_{i+1}]\) by the linear transformation

\[
t = a_i + h_ix, \quad a_i = \frac{1}{2}(x_i + x_{i+1}), \quad h_i = \frac{1}{2}(x_{i+1} - x_i)
\]

and making the leading coefficient equal to \(\frac{1}{2}\) we obtain

\[
K(t) = \frac{1}{2} h_i^2 P_2 \left( \frac{t - a_i}{h_i} \right), \quad x_i \leq t \leq x_{i+1}.
\]

If we start from this representation for \(K(t)\) and use an argument similar to that of the preceding section we can prove that this kernel indeed gives a minimum value for \(u\).

For the quadrature formula which provides the least estimate for the remainder in \(C_2\) we have proven:

1. The nodes and coefficients are

\[
x_k = \frac{\sqrt{3} + 4(k - 1)}{2} h, \quad h = \left[\sqrt{3} + 2(n - 1)\right]^{-1}
\]

\[
A_1 = A_n = \frac{2 + \sqrt{3}}{2} h, \quad A_k = 2h \quad (k = 2, \ldots, n - 1).
\]

2. These \(x_k\) and \(A_k\) minimize the integral \(\int_0^1 |K(t)| \, dt\) and they are unique.
3. The remainder $R(f)$ has the estimate

$$|R(f)| \leq M_2 \frac{h^2}{8}, \quad |f''(x)| \leq M_2 \quad \text{for} \quad x \in [0, 1].$$

4. The quadrature sum $\sum_{k=1}^{n} A_k f(x_k)$ is a Riemann sum and hence for any Riemann integrable function

$$\lim_{n \to \infty} \sum_{k=1}^{n} A_k f(x_k) = \int_{0}^{1} f(x) \, dx.$$ 

### 8.4. The Problem of Minimizing the Estimate of the Remainder for Quadrature with Fixed Nodes

We consider the problem of constructing quadrature formulas with given nodes and with minimal estimate of the remainder. We consider the case which occurs most often in applications: equally spaced nodes and a constant weight function. Let us assume that the segment of integration $[0, 1]$ is divided into $n$ equal parts of length $h = 1/n$.

The quadrature formula

$$\int_{0}^{1} f(x) \, dx = \sum_{k=0}^{n} A_k f\left(\frac{k}{n}\right) \quad (8.4.1)$$

has $n + 1$ coefficients $A_k$ which are to be determined. If we require that (8.4.1) be exact for all polynomials of degree $\leq n$ then, as we saw in Chapter 6, the coefficients $A_k$ are completely defined and the formulas are the Newton-Cotes formulas. Let us assume then that (8.4.1) is exact for polynomials of degree $r - 1 < n$. This imposes the following restraints on the $A_k$:

$$\sum_{k=0}^{n} A_k = 1 \quad (8.4.2)$$

$$\sum_{k=1}^{n} A_k k^i = \frac{n^i}{i + 1}, \quad (i = 1, 2, \ldots, r - 1).$$

If $f^{(r-1)}(x)$ is absolutely continuous on $[0, 1]$ then the remainder of the quadrature can be represented in the form:
Approximate Calculation of Definite Integrals

\[ R(f) = \int_0^1 f^{(r)}(t) K(t) \, dt \quad (8.4.3) \]

\[ K(t) = \frac{(1 - t)^r}{r!} - \sum_{k=1}^{n} A_k E \left( \frac{k}{n} - t \right) \left( \frac{1}{(r - 1)!} \right). \]

Among the \( n + 1 \) coefficients \( A_k \) there are \( n + 1 - r \) independent relations which are available for decreasing the estimate of the remainder of the formula (8.4.1).

In two cases we will find quadrature formulas which minimize the estimate of \( R(f) \).

Let us take first of all the functions of the class \( L_q^{(1)} \) for which

\[ \left( \int_0^1 |f'(t)|^q \, dt \right)^{\frac{1}{q}} \leq M_1 \]

If we assume that the formula is exact when \( f(x) \) is a constant function then the coefficients \( A_k \) must satisfy the first of the conditions (8.4.2) and we have the following estimate for the remainder

\[ |R(f)| \leq \left( \int_0^1 |f'|^q \, dt \right)^{\frac{1}{q}} \left( \int_0^1 |K|^p \, dt \right)^{\frac{1}{p}} \leq M_1 \left( \int_0^1 |K|^p \, dt \right)^{\frac{1}{p}} = \sup_{f} |R(f)|. \]

The integral \( \int_0^1 |K(t)|^p \, dt \) depends only on the \( A_k \) and these coefficients must be chosen to minimize this integral. The kernel \( K(t) \) is given by

\[ K(t) = 1 - t - \sum_{k=1}^{n} A_k E \left( \frac{k}{n} - t \right). \]

On each of the segments \( \left[ \frac{i - 1}{n}, \frac{i}{n} \right] \) \( K(t) \) is a linear function of \( t \):

\[ K(t) = 1 - t - \sum_{k=1}^{n} A_k. \]
8.4. Quadrature with Fixed Nodes

At the points \( t = i/n \) \((i = 1, 2, \ldots, n - 1)\) \(K(t)\) is discontinuous with a jump of \(A_i\) and at the ends of the segment \([0, 1]\) the kernel has the values \(A_0\) and \(-A_n\) respectively.

A typical graph of \(K(t)\) is illustrated in Fig. 7.

Figure 7.

We must determine the \(A_k\), subject to the restriction \(\sum_{k=0}^{n} A_k = 1\), so that the \(p^{th}\) power of the shaded area in the figure will have the least average value. A simple calculation shows that this will occur when the shaded area consists of \(2n\) equal right triangles.

Thus it immediately follows that

\[ A_0 = A_n = \frac{1}{2n}, \quad A_1 = A_2 = \cdots = A_{n-1} = \frac{1}{n}. \]

This is the well-known repeated trapezoidal rule:

\[
\int_0^1 f(x) \, dx = \frac{1}{n} \left[ \frac{1}{2} f(0) + f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) + \frac{1}{2} f(1) \right] + R(f)
\]

and its remainder \(R(f)\) has the estimate

\[
|R(f)| \leq \frac{M_1}{2n(p + 1)^{1/p}}, \quad M_1 = \left( \int_0^1 |f'(t)|^q \, dt \right)^{1/q}.
\]

Now we consider quadrature formulas with least estimate of the remainder in classes of functions of higher degrees of differentiability. We restrict ourselves exclusively to the class \(L_2(r)(r \geq 2)\). In this case the problem of determining the coefficients \(A_k\) has a simple solution.

We assume that the quadrature formula is exact for polynomials of degree \(< r\) which is equivalent to equations (8.4.2) being satisfied. The remainder has the representation (8.4.3). In the class of functions \(f(x)\) which satisfy
the remainder has the estimate

$$R(f) \leq M_r \left( \int_0^1 [K(t)]^2 \, dt \right)^{\frac{1}{2}} = \sup_f |R(f)|$$

The integral $I = \int_0^1 [K(t)]^2 \, dt$ is only a function of the $A_k$ and, as before, the problem is to minimize $I$. This problem is one of minimizing a second degree polynomial in the $A_k$ with the linear restraints (8.4.2). The integral $I$ does not depend on $A_0$ since $A_0$ only enters in the first of the equations (8.4.2). This equation is not needed to find the minimum of $I$ because it does not impose any restraint on the $A_k$ ($k = 1, 2, \ldots, n$) and we will use this equation to calculate $A_0$ when we have calculated the other $A_k$ ($k \geq 1$). The other restraints

$$\sum_{k=1}^n A_k k^i = \frac{n^i}{i + 1} \quad (i = 1, 2, \ldots, r - 1)$$

are independent and can be written as functions of any $r - 1$ of the $A_k$, for example as functions of $A_1, \ldots, A_{r-1}$.

In the integral $I$ the terms of second degree in the $A_k$ are obtained from the integral

$$\sigma(A_1, \ldots, A_n) = \frac{1}{(r - 1)!} \int_0^1 \left[ \sum_{k=1}^n A_k E \left( \frac{k}{n} - t \right) \left( \frac{k}{n} - t \right)^{r-1} \right]^2 \, dt.$$ 

The quadratic form $\sigma(A_1, \ldots, A_n)$ is positive definite since, clearly, $\sigma(A_1, \ldots, A_n) \geq 0$ and $\sigma(A_1, \ldots, A_n) = 0$ can only occur when for each $t \in [0, 1]$

$$\sum_{k=1}^n A_k E \left( \frac{k}{n} - t \right) \left( \frac{k}{n} - t \right)^{r-1} = 0$$

and this is possible only when $A_k = 0$ ($k = 1, 2, \ldots, n$).

From this it follows, by the usual algebraic argument, that the problem of minimizing

$$I = \int_0^1 [K(t)]^2 \, dt$$
subject to the restraints (8.4.2) has a unique solution. If we write the usual conditions for an extremum of $I$ then we obtain a system of linear equations which determine the $A_k$. Values of the $A_k$ and $I$ have been calculated by Sard and Meyers for $r = 2, m = 1(1) 20; r = 3, m = 2(1) 12; and r = 4, m = 2(1) 9$. We give these values in the following tables.

### $r = 2$

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>2</td>
<td>16</td>
<td>30</td>
<td>112</td>
<td>190</td>
<td>624</td>
<td>994</td>
<td>3 104</td>
<td>4 770</td>
</tr>
<tr>
<td>$A_0 \delta = A_m \delta$</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>11</td>
<td>15</td>
<td>41</td>
<td>56</td>
<td>153</td>
<td>209</td>
</tr>
<tr>
<td>$A_1 \delta = A_m - 1 \delta$</td>
<td>10</td>
<td>11</td>
<td>32</td>
<td>43</td>
<td>118</td>
<td>161</td>
<td>440</td>
<td>601</td>
<td></td>
</tr>
<tr>
<td>$A_2 \delta = A_m - 2 \delta$</td>
<td>26</td>
<td>37</td>
<td>100</td>
<td>137</td>
<td>374</td>
<td>511</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_3 \delta = A_m - 3 \delta$</td>
<td>106</td>
<td>143</td>
<td>392</td>
<td>535</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_4 \delta = A_m - 4 \delta$</td>
<td>386</td>
<td>529</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### $r = 3$

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>14 480</td>
<td>21 758</td>
<td>64 848</td>
<td>95 966</td>
<td>282 352</td>
<td>413 250</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_0 \delta = A_m \delta$</td>
<td>571</td>
<td>780</td>
<td>2 131</td>
<td>2 911</td>
<td>7 953</td>
<td>10 864</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_1 \delta = A_m - 1 \delta$</td>
<td>1 642</td>
<td>2 243</td>
<td>6 128</td>
<td>8 371</td>
<td>22 870</td>
<td>31 241</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_2 \delta = A_m - 2 \delta$</td>
<td>1 396</td>
<td>1 907</td>
<td>5 210</td>
<td>7 117</td>
<td>19 444</td>
<td>26 561</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_3 \delta = A_m - 3 \delta$</td>
<td>1 462</td>
<td>1 997</td>
<td>5 456</td>
<td>7 453</td>
<td>20 362</td>
<td>27 815</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_4 \delta = A_m - 4 \delta$</td>
<td>1 444</td>
<td>1 973</td>
<td>5 390</td>
<td>7 363</td>
<td>20 116</td>
<td>27 479</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_5 \delta = A_m - 5 \delta$</td>
<td>1 450</td>
<td>1 979</td>
<td>5 408</td>
<td>7 387</td>
<td>20 182</td>
<td>27 569</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_6 \delta = A_m - 6 \delta$</td>
<td>1 444</td>
<td>1 973</td>
<td>5 390</td>
<td>7 363</td>
<td>20 116</td>
<td>27 479</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_7 \delta = A_m - 7 \delta$</td>
<td>20 170</td>
<td>27 551</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### $r = 4$

<table>
<thead>
<tr>
<th>$m$</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>1 204 288</td>
<td>1 747 906</td>
<td>5 056 272</td>
<td>7 290 718</td>
<td>20 966 960</td>
</tr>
<tr>
<td>$A_0 \delta = A_m \delta$</td>
<td>29 681</td>
<td>40 545</td>
<td>110 771</td>
<td>151 316</td>
<td>413 403</td>
</tr>
<tr>
<td>$A_1 \delta = A_m - 1 \delta$</td>
<td>85 352</td>
<td>116 593</td>
<td>318 538</td>
<td>435 131</td>
<td>1 188 800</td>
</tr>
<tr>
<td>$A_2 \delta = A_m - 2 \delta$</td>
<td>72 566</td>
<td>99 127</td>
<td>270 820</td>
<td>369 947</td>
<td>1 010 714</td>
</tr>
<tr>
<td>$A_3 \delta = A_m - 3 \delta$</td>
<td>75 992</td>
<td>103 807</td>
<td>283 606</td>
<td>387 413</td>
<td>1 058 432</td>
</tr>
<tr>
<td>$A_4 \delta = A_m - 4 \delta$</td>
<td>75 074</td>
<td>102 553</td>
<td>280 180</td>
<td>382 733</td>
<td>1 045 646</td>
</tr>
<tr>
<td>$A_5 \delta = A_m - 5 \delta$</td>
<td>75 320</td>
<td>102 889</td>
<td>281 098</td>
<td>383 987</td>
<td>1 049 072</td>
</tr>
<tr>
<td>$A_6 \delta = A_m - 6 \delta$</td>
<td>75 254</td>
<td>102 799</td>
<td>280 852</td>
<td>383 651</td>
<td>1 048 154</td>
</tr>
<tr>
<td>$A_7 \delta = A_m - 7 \delta$</td>
<td>75 272</td>
<td>102 823</td>
<td>280 918</td>
<td>383 741</td>
<td>1 048 400</td>
</tr>
<tr>
<td>$A_8 \delta = A_m - 8 \delta$</td>
<td>75 266</td>
<td>102 817</td>
<td>280 900</td>
<td>383 717</td>
<td>1 048 334</td>
</tr>
<tr>
<td>$A_9 \delta = A_m - 9 \delta$</td>
<td>75 266</td>
<td>102 817</td>
<td>280 906</td>
<td>383 723</td>
<td>1 048 352</td>
</tr>
<tr>
<td>$A_{10} \delta = A_m - 10 \delta$</td>
<td>75 266</td>
<td>102 817</td>
<td>280 906</td>
<td>383 723</td>
<td>1 048 352</td>
</tr>
</tbody>
</table>

### $m^5 I$

<table>
<thead>
<tr>
<th>$m$</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>2 468</td>
<td>170 393</td>
<td>162 977</td>
<td>699 869</td>
<td>8 331</td>
</tr>
<tr>
<td>$A_0 \delta = A_m \delta$</td>
<td>94 085</td>
<td>6 169 080</td>
<td>5 618 080</td>
<td>23 023 320</td>
<td>262 087</td>
</tr>
</tbody>
</table>
**Approximate Calculation of Definite Integrals**

### Table for $r = 3$

<table>
<thead>
<tr>
<th>$m$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>6</td>
<td>24</td>
<td>240</td>
<td>1,560</td>
<td>980</td>
<td>607,152</td>
<td>643,104</td>
</tr>
<tr>
<td>$A_0 \delta = A_m \delta$</td>
<td>1</td>
<td>3</td>
<td>21</td>
<td>112</td>
<td>55</td>
<td>30,927</td>
<td>28,603</td>
</tr>
<tr>
<td>$A_1 \delta = A_{m-1} \delta$</td>
<td>4</td>
<td>9</td>
<td>76</td>
<td>379</td>
<td>192</td>
<td>106,573</td>
<td>99,124</td>
</tr>
<tr>
<td>$A_2 \delta = A_{m-2} \delta$</td>
<td>6</td>
<td>46</td>
<td>289</td>
<td>132</td>
<td>76,573</td>
<td>69,874</td>
<td></td>
</tr>
<tr>
<td>$A_3 \delta = A_{m-3} \delta$</td>
<td>7</td>
<td>8</td>
<td>172</td>
<td>89,503</td>
<td>85,684</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m^7 I$</td>
<td>1</td>
<td>11</td>
<td>11</td>
<td>73</td>
<td>11</td>
<td>184,081</td>
<td>3,961</td>
</tr>
<tr>
<td></td>
<td>1,890,890</td>
<td>8,960,12,600</td>
<td>69,888</td>
<td>10,850</td>
<td>124,899,840</td>
<td>3,617,460</td>
<td></td>
</tr>
</tbody>
</table>

### Table for $r = 4$

<table>
<thead>
<tr>
<th>$m$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>6</td>
<td>24</td>
<td>28,992</td>
<td>432,840</td>
<td>19,740,084</td>
</tr>
<tr>
<td>$A_0 \delta = A_m \delta$</td>
<td>1</td>
<td>3</td>
<td>2,349</td>
<td>29,392</td>
<td>1,082,811</td>
</tr>
<tr>
<td>$A_1 \delta = A_{m-1} \delta$</td>
<td>4</td>
<td>9</td>
<td>9,932</td>
<td>110,209</td>
<td>4,049,946</td>
</tr>
<tr>
<td>$A_2 \delta = A_{m-2} \delta$</td>
<td>6</td>
<td>46</td>
<td>4,830</td>
<td>76,819</td>
<td>2,225,043</td>
</tr>
<tr>
<td>$A_3 \delta = A_{m-3} \delta$</td>
<td>7</td>
<td>8</td>
<td>3,528,844</td>
<td>255,093,851</td>
<td>12,084,348</td>
</tr>
<tr>
<td>$m^9 I$</td>
<td>1</td>
<td>13</td>
<td>6,557</td>
<td>61,688</td>
<td>210,047</td>
</tr>
<tr>
<td></td>
<td>9,072</td>
<td>17,920</td>
<td>36,529,920</td>
<td>193,912,320</td>
<td>921,203,920</td>
</tr>
</tbody>
</table>

### Table for $r = 5$

<table>
<thead>
<tr>
<th>$m$</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>167,985,552</td>
<td>12,298,253,184</td>
<td>291,277,352,304</td>
</tr>
<tr>
<td>$A_0 \delta = A_m \delta$</td>
<td>8,013,897</td>
<td>509,110,987</td>
<td>10,764,281,184</td>
</tr>
<tr>
<td>$A_1 \delta = A_{m-1} \delta$</td>
<td>31,412,443</td>
<td>2,040,010,996</td>
<td>42,647,140,119</td>
</tr>
<tr>
<td>$A_2 \delta = A_{m-2} \delta$</td>
<td>18,665,443</td>
<td>1,105,566,730</td>
<td>24,253,340,709</td>
</tr>
<tr>
<td>$A_3 \delta = A_{m-3} \delta$</td>
<td>25,900,993</td>
<td>1,867,200,148</td>
<td>37,040,022,813</td>
</tr>
<tr>
<td>$A_4 \delta = A_{m-4} \delta$</td>
<td>1,254,475,462</td>
<td>30,983,891,327</td>
<td></td>
</tr>
<tr>
<td>$m^9 I$</td>
<td>56,097,271</td>
<td>2,876,254,589</td>
<td>18,892,720,083</td>
</tr>
<tr>
<td></td>
<td>207,342,167,040</td>
<td>11,621,849,258,880</td>
<td>72,495,696,573,440</td>
</tr>
</tbody>
</table>
REFERENCES


9.1. GENERAL THEOREMS

In applied problems it is sometimes necessary to construct quadrature formulas in which some of the nodes are given beforehand and the other nodes are free and may be chosen by any criterion we may desire.

Consider, for example, the boundary value problem on the segment \([a, b]\) for the second order differential equation

\[
L(y) + \lambda \rho(x) y = \frac{d}{dx} \left[ p(x) \frac{dy}{dx} \right] + (\lambda \rho(x) - q(x)) y = -f(x)
\]  
(9.1.1)

with the boundary conditions

\[
y(a) = 0, \quad y(b) = 0.
\]  
(9.1.2)

If we know Green's function for the operator \(L(y)\) under the conditions (9.1.2) then the solution of the boundary value problem can be reduced to the solution of the integral equation\(^1\)

\[
y(x) = F(x) + \lambda \int_a^b G(x, \xi) \rho(\xi) y(\xi) \, d\xi
\]  
(9.1.3)

\[
F(x) = \int_a^b G(x, \xi) f(\xi) \, d\xi.
\]

\(^1\)See, for example, V. I. Smirnov, Course of Higher Mathematics, Vol. 4 Gostekhizdat, Moscow, 1954, pp. 519–21 (Russian).
Suppose we wish to approximate the solution of this equation by applying a quadrature formula to the integrals in (9.1.3). It is natural to use the fact that the value of \( y(z) \) is known on the ends of the segment \([a, b]\) and to use a quadrature formula of the form

\[
\int_a^b f(x) \, dx = Af(a) + Bf(b) + \sum_{k=1}^n A_k f(x_k)
\]

which contains the two fixed nodes \( a \) and \( b \). The other nodes \( x_k \) (\( k = 1, 2, \ldots, n \)) are determined by some other method.

The above is a "two-point" boundary value problem. In other problems we may wish to use a quadrature formula which contains more than two fixed nodes.

Consider the quadrature formula

\[
\int_a^b p(x) f(x) \, dx = \sum_{k=1}^n A_k f(x_k) + \sum_{j=1}^m B_j f(a_j)
\]

(9.1.4)

in which the \( m \) nodes \( a_1, \ldots, a_m \) are fixed. It contains the \( 2n + m \) parameters \( x_k, A_k \) (\( k = 1, \ldots, n \)) and \( B_j \) (\( j = 1, \ldots, m \)). We will show how to choose these parameters so that (9.1.4) is exact for polynomials of as high degree as possible.

Let us introduce the two polynomials

\[
\Omega(x) = (x - a_1) \ldots (x - a_m)
\]

\[
\omega(x) = (x - x_1) \ldots (x - x_n).
\]

By counting the choices of the coefficients \( A_k \) and \( B_j \) we see that formula (9.1.4) can be made exact for polynomials of degree \( \leq n + m - 1 \). This can be accomplished by requiring that the formula be interpolatory. In order to make the formula exact for polynomials of higher degree we have at our disposal only the choice of the nodes \( x_k \).

**Theorem 1.** In order that formula (9.1.4) be exact for all polynomials of degree \( \leq 2n + m - 1 \) it is necessary and sufficient that (1) it be interpolatory, and (2) the polynomial \( \omega(x) \) be orthogonal on the segment \([a, b]\) with respect to the weight function \( p(x) \Omega(x) \) to every polynomial \( Q(x) \) of degree \( < n \).

**Proof.** The necessity of the first condition is obvious since if formula (9.1.4) is exact for all polynomials of degree \( \leq n + m - 1 \) then it must be interpolatory. The necessity of the second condition can be verified if we put \( f(x) = \Omega(x) \omega(x) Q(x) \). Then \( f(x) \) is a polynomial of degree \( \leq 2n + m - 1 \) and for it (9.1.4) must be exact. Since \( f(x) \) is zero at the points \( a_j \) and \( x_k \) the quadrature sum for this function is also zero and therefore
\[\int_a^b p(x) \Omega(x) \omega(x) Q(x) \, dx = 0. \quad (9.1.5)\]

Now let \(f(x)\) be an arbitrary polynomial of degree \(\leq 2n + m - 1\). It can be written in the form \(f(x) = \Omega(x) \omega(x) Q(x) + r(x)\) where \(Q(x)\) and \(r(x)\) are polynomials of degrees \(\leq n - 1\) and \(\leq n + m - 1\) respectively. Here it is clear that \(f(a_j) = r(a_j)\) \((j = 1, \ldots, m)\) and \(f(x_k) = r(x_k)\) \((k = 1, \ldots, n)\).

If the orthogonality condition (9.1.5) is satisfied and if formula (9.1.4) is interpolatory then the following relationship will be satisfied
\[
\int_a^b p(x) f(x) \, dx = \int_a^b p(x) \Omega(x) \omega(x) Q(x) \, dx + \int_a^b p(x) r(x) \, dx =
\]
\[
= \int_a^b p(x) r(x) \, dx = \sum_{k=1}^n A_k r(x_k) + \sum_{j=1}^m B_j f(a_j) =
\]
\[
= \sum_{k=1}^n A_k f(x_k) + \sum_{j=1}^m B_j f(a_j)
\]

This proves the theorem.

The construction of the quadrature formula (9.1.4) which is exact for all algebraic polynomials of degree \(\leq 2n + m - 1\) thus reduces to finding the polynomial of degree \(n\) which is orthogonal on \([a, b]\), with respect to the weight function \(p(x) \Omega(x)\), to all polynomials of degree \(< n\). The roots of \(\omega(x)\) must be real, distinct and lie inside the segment \([a, b]\). They must also be distinct from the fixed nodes \(a_j\) \((j = 1, \ldots, m)\).

Let us assume that the polynomial \(\omega(x)\) which satisfies the conditions of Theorem 1 exists. Then we can construct formula (9.1.4) so that it is exact for all polynomials of degree \(\leq 2n + m - 1\). We will make one more remark about the degree of precision of this formula. To do this we first need to construct a representation for the remainder. Let us construct the interpolating polynomial \(H(x)\) of degree \(\leq 2n + m - 1\) for \(f(x)\) on \([a, b]\) which satisfies the conditions
\[
H(a_j) = f(a_j) \quad (j = 1, \ldots, m)
\]
\[
H(x_k) = f(x_k), \quad H'(x_k) = f'(x_k) \quad (k = 1, \ldots, n).
\]

If \(f(x)\) has a derivative of order \(2n + m\) throughout the segment \([a, b]\) then the remainder of the interpolation \(r(x) = f(x) - H(x)\) can be represented as
\[
r(x) = \Omega(x) \omega^2(x) \frac{f^{(2n+m)}}{(2n + m)!} (\xi) \quad a < \xi < b.
\]
The remainder of the quadrature $R(f)$ satisfies $R(f) = R(H) + R(r)$. Since (9.1.4) is exact for all polynomials of degree $2n + m - 1$ then $R(H) = 0$. Also, at all the nodes $a_j$ and $x_k$ the remainder $r(x)$ is zero and thus the quadrature sum for $r(x)$ vanishes

$$\sum_{k=1}^{n} A_k r(x_k) + \sum_{j=1}^{m} B_j r(a_j) = 0.$$  

Consequently

$$R(f) = R(r) = \int_{a}^{b} p(x) r(x) \, dx = \int_{a}^{b} p(x) \Omega(x) \, \omega^2(x) \frac{f^{(2n+m)}(\xi)}{(2n+m)!} \, dx.$$  

Thus we see that if

$$I = \int_{a}^{b} p(x) \Omega(x) \, \omega^2(x) \, dx \neq 0$$  

then the degree of precision of (9.1.4) is $2n + m - 1$. This is true since if $f(x)$ is a polynomial of degree $2n + m$ then $f^{(2n+m)}(x)$ is a constant different from zero and for such a function

$$R(f) = \frac{f^{(2n+m)}}{(2n+m)!} \int_{a}^{b} p(x) \Omega(x) \, \omega^2(x) \, dx \neq 0.$$  

If $I = 0$ then the algebraic degree of precision of (9.1.4) will be greater than $2n + m - 1$. We could derive a criterion to determine the exact degree of precision in these exceptional cases but we do not choose to do so.

Since formula (9.1.4) is interpolatory the coefficients $A_k$ and $B_j$ have the following values:

$$A_k = \int_{a}^{b} p(x) \frac{\omega(x) \Omega(x)}{(x-x_k) \omega'(x_k) \Omega(x_k)} \, dx$$  \hspace{1cm} (9.1.6)  

$$B_j = \int_{a}^{b} p(x) \frac{\omega(x) \Omega(x)}{(x-a_j) \omega(a_j) \Omega'(a_j)} \, dx.$$  \hspace{1cm} (9.1.7)  

We can give for the coefficients $A_k$ a representation which is easier to use for computations than (9.1.6). Let us assume that there exists a unique system of polynomials $\pi_s(x)$ ($s = 1, 2, \ldots$) which form an orthonormal system with respect to the weight function $\rho(x) = p(x) \Omega(x)$ on $[a, b]$ where $\pi_s(x)$ has degree $s$. The polynomial $\pi_n(x)$ differs from $\omega(x)$ by only a constant factor so that
Approximate Calculation of Definite Integrals

\[ A_k = \frac{1}{\pi_n'(x_k) \Omega(x_k)} \int_a^b \rho(x) \frac{\pi_n(x)}{x-x_k} dx. \]

The integral in this expression was calculated in Section 7.1 in terms of a different notation. We obtained the following two expressions for this integral:

\[ \int_a^b \rho(x) \frac{\pi_n(x)}{x-x_k} dx = \frac{a_{n+1}}{a_n \pi_{n+1}(x_k)} = \frac{a_n}{a_{n-1} \pi_{n-1}(x_k)}, \]

where \( a_n \) is the leading coefficient of the polynomial \( \pi_n(x) \):

\[ \pi_n(x) = a_n x^n + \cdots \]

Therefore

\[ A_k = -\frac{a_{n+1}}{a_n \pi_n'(x_k) \pi_{n+1}(x_k) \Omega(x_k)} = \frac{a_n}{a_{n-1} \pi_n'(x_k) \pi_{n-1}(x_k) \Omega(x_k)}. \] (9.1.8)

If we compare (9.1.8) with the expressions (7.1.3) and (7.1.4) for the coefficients of the formula of the highest algebraic degree of precision then it is clear that the \( A_k \) in (9.1.4) differ only by the factor \( \frac{1}{\Omega(x_k)} \) from the corresponding coefficients in the quadrature formula with weight function \( \rho(x) = \rho(x) \Omega(x) \)

\[ \int_a^b \rho(x) f(x) dx = \sum_{k=1}^{n} A_k^* f(x_k) \]

which is exact for polynomials of degree \( \leq 2n - 1 \).

To construct formula (9.1.4) for each \( n \) we must construct the system of polynomials which are orthogonal on \([a, b]\) with respect to the weight function \( \rho(x) \Omega(x) \). In certain cases we can make use of a result on the representation of such a system of polynomials which are orthogonal with respect to a nonnegative function times a polynomial. We will formulate this result with the degree of generality which is required in the remainder of this chapter.

For simplicity of notation we assume that the orthogonal polynomials have leading coefficient of unity. We denote such a system by \( P_s^*(x) \) or \( \pi_s^*(x) \) to distinguish it from the corresponding system of orthonormal polynomials.

Together with the weight function \( \rho(x) \) we also consider the weight function \( \rho(x) = \rho(x) \Omega(x) \) where \( \Omega(x) = (x-a_1) \cdots (x-a_m) \) is any polynomial with distinct roots \( a_1, a_2, \ldots, a_m \).

We will assume that there exists a system of polynomials \( P_s^*(x) = \)
9.1. General Theorems

which are orthogonal on \([a, b]\) with respect to the weight function \(p(x)\) and that

\[
\int_a^b p(x) [P_s^*(x)]^2 dx \neq 0.
\]

This is equivalent to assuming that there exists a unique system of polynomials \(\pi_s^*(x) = x^s + \cdots (s = 0, 1, \ldots)\) which are orthogonal on \([a, b]\) with respect to the weight function \(\rho(x)\). We will show that \(\pi_s^*(x)\) can be expressed in terms of the \(P_s^*(x)\) as follows:

\[
\pi_s^*(x) = \frac{1}{\Delta \Omega(x)} \begin{vmatrix}
P_{n+m}^*(x) & P_{n+m}^*(a_1) & \cdots & P_{n+m}^*(a_m) \\
P_{n+m-1}^*(x) & P_{n+m-1}^*(a_1) & \cdots & P_{n+m-1}^*(a_m) \\
P_{n}^*(x) & P_{n}^*(a_1) & \cdots & P_{n}^*(a_m)
\end{vmatrix} = \frac{D_{n+m}(x)}{\Delta \Omega(x)}
\]

(9.1.9)

The product \(\Omega(x) \pi_n^*(x)\) is a polynomial of degree \(n + m\) with leading coefficient of unity. This polynomial can be expanded in terms of the polynomials \(P_s^*(x)\):

\[
\Omega(x) \pi_n^*(x) = P_{n+m}^*(x) + c_1 P_{n+m-1}^*(x) + c_2 P_{n+m-2}^*(x) + \cdots
\]

The orthogonality of \(\pi_n^*(x)\) with respect to the weight function \(p(x)\)\(\Omega(x)\) to all polynomials of degree less than \(n\) means that in this expression the terms involving \(P_s^*(x)\) for \(s < n - 1\) must be absent and therefore the expansion has the form

\[
\Omega(x) \pi_n^*(x) = P_{n+m}^*(x) + c_1 P_{n+m-1}^*(x) + \cdots + c_m P_{n}^*(x). \quad (9.1.10)
\]

When \(x\) is replaced by one of the numbers \(a_1, a_2, \ldots, a_m\) the left side of this equation is zero and therefore the coefficients \(c_1, \ldots, c_m\) must satisfy the system of equations

\[
P_{n+m}^*(a_1) + c_1 P_{n+m-1}^*(a_1) + \cdots + c_m P_{n}^*(a_1) = 0.
\]

\[
P_{n+m}^*(a_m) + c_1 P_{n+m-1}^*(a_m) + \cdots + c_m P_{n}^*(a_m) = 0.
\]

(9.1.11)

Thus the right side of (9.1.10) is divisible by \(\Omega(x)\) and hence we can write \(\pi_n^*(x)\) in the form

\[
\pi_n^*(x) = \Omega^{-1}(x) [P_{n+m}^*(x) + c_1 P_{n+m-1}^*(x) + \cdots + c_m P_{n}^*(x)].
\]
Since we assumed that there exists a unique sequence of polynomials \( \pi^*_n(x) \) the system (9.1.11) must have, for each \( n \), a unique solution for the unknown coefficients \( c_1, \ldots, c_m \). The determinant of the system coincides with \( \Delta \) and therefore \( \Delta \neq 0 \). The expression (9.1.9) for \( \pi^*_n(x) \) can be obtained in the following way. If we adjoin equation (9.1.10) to the system (9.1.11) then we obtain the system:

\[
\begin{align*}
-\Omega(x) \pi^*_n(x) + P^*_n(x) &+ c_1 P^*_{n+m-1}(x) + \cdots + c_m P^*_n(x) = 0 \\
P^*_{n+m}(a_1) + c_1 P^*_{n+m-1}(a_1) + \cdots + c_m P^*_n(a_1) &= 0 \\
\quad \cdots \\
P^*_{n+m}(a_m) + c_1 P^*_{n+m-1}(a_m) + \cdots + c_m P^*_n(a_m) &= 0.
\end{align*}
\]

This can be considered as a homogeneous system of \( n+1 \) equations in the \( m+1 \) quantities \( 1, c_1, \ldots, c_m \). By a well-known theorem of algebra we can assert that the determinant of this system must be zero:

\[
\begin{vmatrix}
-\Omega(x) \pi^*_n(x) & P^*_n(x) & P^*_{n+m-1}(x) & \cdots & P^*_n(x) \\
P^*_{n+m}(a_1) & P^*_n(a_1) & P^*_{n+m-1}(a_1) & \cdots & P^*_n(a_1) \\
P^*_{n+m}(a_m) & P^*_n(a_m) & P^*_{n+m-1}(a_m) & \cdots & P^*_n(a_m)
\end{vmatrix} = 0.
\]

This proves (9.1.9).

9.2. FORMULAS OF SPECIAL FORM

In the quadrature formulas considered in Chapter 7 all of the nodes and coefficients were chosen so that the formulas were exact for polynomials of the highest possible degree.

In attempting to generalize this idea A. A. Markov considered formulas in which all of the coefficients \( A_k \) but only part of the nodes \( x_k \) are chosen so that the formula has the greatest possible precision. The other nodes are fixed in some way. Markov studied this question for weight functions which do not change sign. Let us assume that the weight function \( p(x) \) in (9.1.4) is nonnegative: \( p(x) \geq 0 \). In order that \( \rho(x) = p(x) \Omega(x) \) does not change sign on \([a, b]\) we must also assume that \( \Omega(x) \) does not change sign in this interval and thus none of the fixed nodes can lie inside \([a, b]\).

If we do not allow quadrature formulas with nodes outside \([a, b]\) we must then limit ourselves to the cases studied by Markov:

1. \( m = 1 \) with a single fixed node \( a_1 = a \);

2Trans. note: These formulas are more often attributed to Radau; case (3) is also attributed to Lobatto. See the references at the end of this chapter.
9.2. Formulas of Special Form

(2) \( m = 1 \) with a single fixed node \( a_1 = b \) (this case reduces to (1) by the linear transformation \( x = a + b - t \); we will not consider this case separately);

(3) \( m = 2 \) with the two fixed nodes \( a_1 = a, a_2 = b \).

The assumption that \( \rho(x) \) has constant sign on \([a, b] \) means that the polynomial \( \omega(x) \) of degree \( n \) which is orthogonal on \([a, b] \) with respect to \( \rho(x) \) to all polynomials of degree \(<n \) exists for each \( n \). The roots \( x_k \) of this polynomial are real and distinct and all lie inside \([a, b] \). In each of the above cases the \( x_k \) are distinct from the fixed nodes which are situated at the ends of \([a, b] \).

Thus, for the cases considered by Markov, the quadrature formulas (9.1.4) which are exact for all polynomials of degree \( \leq 2n + m - 1 \) can be constructed for all \( n \). Since \( \rho(x) \Omega(x) \omega^2(x) \) does not change sign inside \([a, b] \) then \( \int_a^b \rho(x) \Omega(x) \omega^2(x) \, dx \neq 0 \) and the algebraic degree of precision of such formulas is \( 2n + m - 1 \).

Let us consider the first case: \( m = 1, a_1 = a \)

\[
\int_a^b \rho(x) f(x) \, dx = A f(a) + \sum_{k=1}^{n} A_k f(x_k) + R(f). \quad (9.2.1)
\]

The highest degree of precision which can be achieved in such a formula is \( 2n \).

Here \( \Omega(x) = x - a \). Let \( x_k \) be the roots of the \( n \)th degree polynomial \( \pi_n(x) \) which is orthogonal on \([a, b] \) with respect to \( \rho(x) = (x - a) \rho(x) \) to all polynomials of degree \(<n \). If \( P_s(x) (s = 0, 1, \ldots) \) is the orthogonal system of polynomials with respect to \( \rho(x) \) then by (9.1.9) \( \pi_n(x) \) can be written in the form

\[
\pi_n(x) = \frac{K_n}{x-a} \left| \begin{array}{cc} P_{n+1}(x) & P_{n+1}(a) \\ P_n(x) & P_n(a) \end{array} \right| = \frac{K_n}{x-a} [P_{n+1}(x) P_n(a) - P_n(x) P_{n+1}(a)].
\]

where \( K_n \) is a nonzero constant. Equation (9.1.8) gives a convenient method to compute the coefficients \( A_k \):

\[
A_k = \frac{\alpha_{n+1}}{\alpha_n (x_k - a) \pi_n^*(x_k) \pi_{n+1}(x_k)} = \frac{\alpha_n}{\alpha_{n-1} (x_k - a) \pi_n^*(x_k) \pi_{n-1}(x_k)}. \quad (9.2.2)
\]

Using (9.1.7) we find for \( A \)

\[
A = \pi_n^{-1}(a) \int_a^b \rho(x) \pi_n(x) \, dx. \quad (9.2.3)
\]
We can show that all the coefficients in formula (9.2.1) are positive. As an integrand let us take the polynomial of degree $2n - 1$:
\[ f(x) = (x - a) \left[ \frac{\omega(x)}{x - x_i} \right]^2. \]

This polynomial has the following values at the nodes of the formula:
\[
\begin{align*}
    f(a) &= 0, \quad f(x_k) = \begin{cases} 0 & \text{for } k \neq i, \\
    (x_i - a) [\omega'(x_i)]^2 & \text{for } k = i.
\end{cases}
\end{align*}
\]

Formula (9.2.1) must be exact for this function and thus
\[
\int_a^b p(x) (x - a) \left[ \frac{\omega(x)}{x - x_i} \right]^2 \, dx = A_i (x_i - a) [\omega'(x_i)]^2.
\]

Therefore
\[
A_i = \frac{1}{(x_i - a) [\omega'(x_i)]^2} \int_a^b p(x) (x - a) \left[ \frac{\omega(x)}{x - x_i} \right]^2 \, dx > 0.
\]

Similarly, if we take $f(x) = \omega^2(x)$ we obtain
\[
A = \omega^{-2}(a) \int_a^b p(x) \omega^2(x) \, dx > 0.
\]

If $f(x)$ has a continuous derivative of order $2n + 1$ then the remainder $R(f)$ in (9.2.1) can be represented in the form
\[
R(f) = f^{(2n+1)}(\xi) \int_a^b p(x) (x - a) \omega^2(x) \, dx,
\]

or, since $p(x) (x - a) \omega^2(x)$ does not change sign on $[a, b]$,
\[
R(f) = \frac{f^{(2n+1)}(\eta)}{(2n + 1)!} \int_a^b p(x) (x - a) \omega^2(x) \, dx, \quad a < \eta < b. \quad (9.2.4)
\]

We will now discuss in more detail the above theory for the weight function $p(x) = 1$.

We assume that the segment $[a, b]$ has been transformed into the segment $[-1, 1]$ and we consider the formula
\[
\int_{-1}^1 f(x) \, dx = A f(-1) + \sum_{k=1}^n A_k f(x_k) + R(f) \quad (9.2.5)
\]
which has degree of precision equal to $2n$. We have $\Omega(x) = 1 + x$ and the polynomial $\omega(x) = (x - x_1) \cdots (x - x_n)$ must be orthogonal on $[-1, 1]$ with
9.2. Formulas of Special Form

respect to \((1 + x)\) to all polynomials of lower degree. Therefore \(\omega(x)\) can differ from the Jacobi polynomial \(P_n^{(0,1)}(x)\) by only a constant factor:

\[
\omega(x) = \frac{2^nn! \Gamma(n + 2)}{\Gamma(2n + 2)} P_n^{(0,1)}(x).
\]

Thus the nodes \(x_k\) must be the roots of \(P_n^{(0,1)}(x)\). The coefficients \(A_k\) can easily be found if we use the remark following (9.1.8). Quadrature formula (7.3.2) for the Jacobi weight function \((1 - x)^a(1 + x)^b\) is exact for all polynomials of degree \(\leq 2n - 1\). The coefficients of this formula are given by (7.3.4). To find \(A_k\) in (9.2.5) we must multiply the corresponding coefficient (7.3.4), for \(a = 0, \beta = 1\), by

\[
\frac{1}{\Omega(x_k)} = \frac{1}{1 + x_k}.
\]

This gives

\[
A_k = \frac{4}{(1 + x_k)(1 - x_k^2) [P_n^{(0,1)}(x_k)]^2}.
\]

We can use (9.1.7) to calculate \(A\) by substituting \(p(x) = 1, \Omega(x) = 1 + x\) and \(a_j = -1\):

\[
A = \int_{-1}^{1} \frac{\omega(x)}{\omega(-1)} \, dx = \left[ P_n^{(0,1)}(-1) \right]^{-1} \int_{-1}^{1} P_n^{(0,1)}(x) \, dx.
\]

This last integral and the factor in front of it are easily found from known properties of Jacobi polynomials; these are \(\frac{2(-1)^n}{n + 1}\) and \((-1)^n(n + 1)\) respectively. Thus

\[
A = \frac{2}{(n + 1)^2}.
\]

The remainder \(R(f)\) can be computed from (9.2.4):

\[
R(f) = \int_{-1}^{1} (1 + x) \omega^2(x) \, dx, \quad -1 < \eta < 1.
\]

The integral in this expression can be found without difficulty

\[
\int_{-1}^{1} (1 + x) \omega^2(x) \, dx = \left[ \frac{2^n n! (n + 1)!}{(2n + 1)!} \right]^2 \int_{-1}^{1} (1 + x) [P_n^{(0,1)}(x)]^2 \, dx = \frac{2}{n + 1} \left[ \frac{2^n n! (n + 1)!}{(2n + 1)!} \right]^2.
\]

Therefore

\[
R(f) = \frac{2}{n + 1} \left[ \frac{2^n n! (n + 1)!}{(2n + 1)!} \right]^2 \frac{f^{(2n+1)}(\eta)}{(2n + 1)!}, \quad -1 < \eta < 1.
\]
The nodes and coefficients in formula (9.2.5) are given below for \( n = 1(1)6 \).^3

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( A_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 1 )</td>
<td>( n = 2 )</td>
</tr>
<tr>
<td>-1.00000000</td>
<td>0.50000000</td>
</tr>
<tr>
<td>0.33333333</td>
<td>1.50000000</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 2 )</td>
<td>( n = 4 )</td>
</tr>
<tr>
<td>-1.00000000</td>
<td>0.12500000</td>
</tr>
<tr>
<td>-0.5753189</td>
<td>0.6576886</td>
</tr>
<tr>
<td>0.1810663</td>
<td>0.7763870</td>
</tr>
<tr>
<td>0.8228241</td>
<td>0.4409244</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>( n = 4 )</td>
<td>( n = 5 )</td>
</tr>
<tr>
<td>-1.00000000</td>
<td>0.05555556</td>
</tr>
<tr>
<td>-0.8029298</td>
<td>0.3196408</td>
</tr>
<tr>
<td>-0.3909286</td>
<td>0.4853872</td>
</tr>
<tr>
<td>0.1240504</td>
<td>0.5209268</td>
</tr>
<tr>
<td>0.6039732</td>
<td>0.4169013</td>
</tr>
<tr>
<td>0.9203803</td>
<td>0.2015884</td>
</tr>
</tbody>
</table>

^3 This table was calculated at the Leningrad section of the Mathematical Institute of the Academy of Sciences of the U.S.S.R. by research assistants R. B. Akkerman and K. E. Chernin.

Now we consider case 3 where we are given two fixed nodes at the ends of the segment of integration: \( a_1 = a, a_2 = b \).

\[
\int_a^b p(x)f(x)\,dx = Af(a) + Bf(b) + \sum_{k=1}^{n} A_k f(x_k) + R(f). \quad (9.2.9)
\]

The highest degree of precision which can be achieved by such a formula is \( 2n + 1 \). Here \( \Omega(x) = (x-a)(x-b) \) and the \( x_k \) are the roots of the \( n^{th} \)
9.2. Formulas of Special Form

degree polynomial \( \pi_n(x) \) which is orthogonal on \([a, b]\) with respect to \( \rho(x) = (x - a) (x - b) \) \( p(x) \) to all polynomials of degree \(<n\).

The polynomials \( \pi_n(x) \) are related to the polynomials \( P_n(x) \) which are orthogonal with respect to \( p(x) \) by the following equation:

\[
\pi_n(x) = \frac{K_n}{(x - a)(x - b)} \begin{vmatrix} P_{n+2}(x) & P_{n+2}(a) & P_{n+2}(b) \\ P_{n+1}(x) & P_{n+1}(a) & P_{n+1}(b) \\ P_n(x) & P_n(a) & P_n(b) \end{vmatrix}
\]

The coefficients \( A, B, \) and \( A_k \) can be computed from (9.1.8) and (9.1.7):

\[
A_k = \frac{1}{a_n \pi_n^\prime(x_k) \pi_{n+1}(x_k) (x_k - a) (x_k - b)} \int_a^b (x - a) (x - b) \omega(x) \, dx
\]

\[
B = \frac{1}{\omega(a) \omega(b) (b - a)} \int_a^b p(x) (x - a) \omega(x) \, dx
\]

\[
1 \leq 2n - 1, \text{ then the coefficients } A_k \text{ in (9.2.8) differ from the coefficients } A_k^* \text{ by the factor } \frac{1}{(x_k - a)(x_k - b)}.
\]

It is easy to show that the \( A, B, \) and \( A_k \) are positive. To do this it suffices to apply formula (9.2.9) to the polynomials

\[(b - x) \omega^2(x), \quad (x - a) \omega^2(x), \quad \text{and} \quad (x - a)(x - b) \left[ \frac{\omega(x)}{x - x_i} \right]^2\]

The remainder \( R(f) \) of (9.2.9) can be represented in the form

\[
R(f) = \frac{f^{(2n+2)}(\eta)}{(2n + 2)!} \int_a^b p(x) (x - a) (x - b) \omega^2(x) \, dx
\]
Let us apply these results to the particular case of \( p(x) = 1 \) for the segment \([-1, 1]\):

\[
\int_{-1}^{1} f(x) \, dx = Af(-1) + Bf(1) + \sum_{k=1}^{n} A_k f(x_k) + R(f) \quad (9.2.12)
\]

\( \Omega(x) = 1 - x^2. \)

The polynomial \( \omega(x) \), which is orthogonal on \([-1, 1]\) with respect to \( 1 - x^2 \) to all polynomials of degree <\( n \), differs by only a constant factor from the Jacobi polynomial \( P_n^{(1,1)}(x) \):

\[
\omega(x) = \frac{2^n n! \, \Gamma(n + 3)}{\Gamma(2n + 3)} P_n^{(1,1)}(x).
\]

As in case 1 we can compute the coefficients and remainder:

\[
A_k = \frac{8(n + 1)}{(n + 2) \, (1 - x_k^2)^2 \, [P_n^{(1,1)}'(x_k)]^2}
\]

\[
A = B = \frac{2}{(n + 1) \, (n + 2)}
\]

\[
R(f) = \frac{8(n + 1)}{(2n + 3) \, (n + 2)} \left[ \frac{2^n n! \, (n + 2)!}{(2n + 2)!} \right]^2 \frac{f^{(2n+2)}(\eta)}{(2n + 2) !}, \quad -1 < \eta < 1.
\]

The nodes and coefficients in (9.1.12) are symmetric with respect to \( x = 0 \) and we tabulate below these values which correspond to \( 0 \leq x_k \leq 1 \) for \( n = 1(1)15. \)

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( A_k )</th>
<th>( n = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0.33333333</td>
<td>0.88333333</td>
</tr>
<tr>
<td>0.00000000</td>
<td>1.33333333</td>
<td>0.66666667</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( A_k )</th>
<th>( n = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0.16666667</td>
<td>0.44721360</td>
</tr>
<tr>
<td>0.44721360</td>
<td>0.83333333</td>
<td>0.37847496</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( A_k )</th>
<th>( n = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0.10000000</td>
<td>0.65465367</td>
</tr>
<tr>
<td>0.65465367</td>
<td>0.54444444</td>
<td>0.71111111</td>
</tr>
<tr>
<td>0.00000000</td>
<td>0.71111111</td>
<td>0.55485837</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( A_k )</th>
<th>( n = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0.06666667</td>
<td>0.76505532</td>
</tr>
<tr>
<td>0.76505532</td>
<td>0.37847496</td>
<td>0.28523152</td>
</tr>
</tbody>
</table>

4 This table was calculated at the Leningrad section of the Mathematical Institute of the Academy of Sciences of the U.S.S.R. by research assistant R. B. Akkerman.
### 9.2. Formulas of Special Form

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( n = 5 )</th>
<th>( A_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0.047619048</td>
<td></td>
</tr>
<tr>
<td>0.83022390</td>
<td>0.27682605</td>
<td></td>
</tr>
<tr>
<td>0.46884879</td>
<td>0.43174538</td>
<td></td>
</tr>
<tr>
<td>0.00000000</td>
<td>0.48761905</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( n = 6 )</th>
<th>( A_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0.035714286</td>
<td></td>
</tr>
<tr>
<td>0.87174015</td>
<td>0.21070423</td>
<td></td>
</tr>
<tr>
<td>0.59170018</td>
<td>0.34112268</td>
<td></td>
</tr>
<tr>
<td>0.20929922</td>
<td>0.41245881</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( n = 7 )</th>
<th>( A_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0.027777778</td>
<td></td>
</tr>
<tr>
<td>0.89975800</td>
<td>0.16549536</td>
<td></td>
</tr>
<tr>
<td>0.67718628</td>
<td>0.27453872</td>
<td></td>
</tr>
<tr>
<td>0.36311746</td>
<td>0.34642851</td>
<td></td>
</tr>
<tr>
<td>0.00000000</td>
<td>0.37151927</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( n = 8 )</th>
<th>( A_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0.022222222</td>
<td></td>
</tr>
<tr>
<td>0.91953891</td>
<td>0.13330599</td>
<td></td>
</tr>
<tr>
<td>0.78877886</td>
<td>0.22489834</td>
<td></td>
</tr>
<tr>
<td>0.47792495</td>
<td>0.29204268</td>
<td></td>
</tr>
<tr>
<td>0.16527896</td>
<td>0.32753976</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( n = 9 )</th>
<th>( A_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0.018181818</td>
<td></td>
</tr>
<tr>
<td>0.93400143</td>
<td>0.10961227</td>
<td></td>
</tr>
<tr>
<td>0.78448347</td>
<td>0.18716989</td>
<td></td>
</tr>
<tr>
<td>0.56528538</td>
<td>0.24804811</td>
<td></td>
</tr>
<tr>
<td>0.29575814</td>
<td>0.28687913</td>
<td></td>
</tr>
<tr>
<td>0.00000000</td>
<td>0.30021759</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( n = 10 )</th>
<th>( A_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0.015151515</td>
<td></td>
</tr>
<tr>
<td>0.94489927</td>
<td>0.091684521</td>
<td></td>
</tr>
<tr>
<td>0.81927932</td>
<td>0.15797471</td>
<td></td>
</tr>
<tr>
<td>0.63287615</td>
<td>0.21250842</td>
<td></td>
</tr>
<tr>
<td>0.39953094</td>
<td>0.25127560</td>
<td></td>
</tr>
<tr>
<td>0.18655293</td>
<td>0.27140524</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( n = 11 )</th>
<th>( A_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0.012820513</td>
<td></td>
</tr>
<tr>
<td>0.95330985</td>
<td>0.077801687</td>
<td></td>
</tr>
<tr>
<td>0.84634757</td>
<td>0.13498193</td>
<td></td>
</tr>
<tr>
<td>0.68618847</td>
<td>0.18364686</td>
<td></td>
</tr>
<tr>
<td>0.48290982</td>
<td>0.22076779</td>
<td></td>
</tr>
<tr>
<td>0.24928693</td>
<td>0.24401579</td>
<td></td>
</tr>
<tr>
<td>0.00000000</td>
<td>0.25193085</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x_k )</th>
<th>( n = 12 )</th>
<th>( A_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>0.010989011</td>
<td></td>
</tr>
<tr>
<td>0.95993505</td>
<td>0.066837283</td>
<td></td>
</tr>
<tr>
<td>0.86780105</td>
<td>0.11658665</td>
<td></td>
</tr>
<tr>
<td>0.72886860</td>
<td>0.16002185</td>
<td></td>
</tr>
<tr>
<td>0.55068940</td>
<td>0.19482615</td>
<td></td>
</tr>
<tr>
<td>0.34272401</td>
<td>0.21912625</td>
<td></td>
</tr>
<tr>
<td>0.11638187</td>
<td>0.23161279</td>
<td></td>
</tr>
</tbody>
</table>
Approximate Calculation of Definite Integrals

\[ x_k \quad A_k \]

\[ \begin{array}{c c}
1.00000000 & 0.0095238095 \\
0.96524592 & 0.058029922 \\
0.88508205 & 0.10166004 \\
0.76351967 & 0.14051171 \\
0.60625322 & 0.17278965 \\
0.42063805 & 0.19698723 \\
0.21535396 & 0.21197360 \\
0.00000000 & 0.21704810 \\
\end{array} \]

\[ n = 13 \]

\[ \begin{array}{c c}
1.00000000 & 0.0083333333 \\
0.96956804 & 0.050850369 \\
0.89920054 & 0.089393689 \\
0.79200828 & 0.12425539 \\
0.65238872 & 0.15402699 \\
0.48605941 & 0.17749190 \\
0.29983047 & 0.19369005 \\
0.10132627 & 0.20195830 \\
\end{array} \]

\[ n = 14 \]

\[ \begin{array}{c c}
1.00000000 & 0.0073529412 \\
0.97313217 & 0.044921950 \\
0.91088001 & 0.079198263 \\
0.81569624 & 0.11059290 \\
0.69102899 & 0.13798776 \\
0.54138540 & 0.16039465 \\
0.37217443 & 0.17700426 \\
0.18951198 & 0.18721635 \\
0.00000000 & 0.19066186 \\
\end{array} \]

9.3. REMARKS ON INTEGRALS WITH WEIGHT FUNCTIONS THAT CHANGE SIGN

The problem of constructing quadrature formulas with preassigned nodes is related to the problem of transforming weight functions which change sign into weight functions with constant sign.

Let us consider the integral

\[ \int_{a}^{b} p(x) f(x) \, dx \quad (9.3.1) \]

and assume that \( p(x) \) changes sign inside the segment \([a, b]\) at a finite number of points \( a_1, a_2, \ldots, a_m \).

We construct for \( f(x) \) the interpolating polynomial \( P(x) \) of degree \(<m\) based on the points \( a_j \):

\[^5\text{On each of the segments } [a, a_1], [a_1, a_2], \ldots, [a_m, b] \text{ the function } p(x) \text{ has constant sign and on adjacent segments it has opposite sign.}\]
9.3. Weight Functions that Change Sign

\[ P(a_j) = f(a_j) \quad (j = 1, 2, \ldots, m) \]

\[ f(x) = P(x) + r(x) \quad (9.3.2) \]

\[ P(x) = \sum_{j=1}^{m} \frac{\Omega(x)}{(x - a_j) \Omega'(a_j)} f(a_j). \]

The remainder \( r(x) \) of the interpolation can be represented in the form (see (3.2.9)):

\[ r(x) = (x - a_1) \cdots (x - a_m) f(a_1, \ldots, a_m, x) = \Omega(x) f(a_1, \ldots, a_m, x) \]

where \( f(a_1, \ldots, a_m, x) \) is the divided difference corresponding to the nodes \( a_1, \ldots, a_m, x \).

The integral (9.3.1) can be divided into two parts in the following way:

\[
\begin{align*}
\int_{a}^{b} p(x) f(x) \, dx &= \int_{a}^{b} p(x) P(x) \, dx + \int_{a}^{b} p(x) \Omega(x) f(a_1, \ldots, a_m, x) \, dx \\
&= \sum_{j=1}^{m} a_j f(a_j) + \int_{a}^{b} \rho(x) f(a_1, \ldots, a_m, x) \, dx \\
&= a_j = \int_{a}^{b} p(x) \frac{\Omega(x)}{(x - a_j) \Omega'(a_j)} \, dx. 
\end{align*}
\]

(9.3.3)

(9.3.4)

We will now be interested in the last integral in (9.3.3). The function \( \rho(x) = p(x) \Omega(x) \) in this integral does not change sign on \([a, b]\) because each of the factors changes sign at these points. We take \( \rho(x) \) as a new weight function. To calculate the integral

\[ \int_{a}^{b} \rho(x) f(a_1, \ldots, a_m, x) \, dx \]

we can use any of the methods which we employed for weight functions of constant sign. In particular we can construct for this integral a quadrature formula of the highest algebraic degree of precision. As in the preceding section let us denote by \( \pi_n(x) \) the \( n \)th degree polynomial of the orthogonal system belonging to the weight function \( \rho(x) = p(x) \Omega(x) \). Let us consider the quadrature formula with \( n \) nodes which is exact for polynomials of degree \( \leq 2n - 1 \):

\[
\int_{a}^{b} \rho(x) f(a_1, \ldots, a_m, x) \, dx = \sum_{k=1}^{n} \beta_k f(a_1, \ldots, a_m, x_k) 
\]

(9.3.5)
Approximate Calculation of Definite Integrals

\[ \pi_n(x_k) = 0 \quad (k = 1, 2, \ldots, n) \]

\[ \beta_k = \int_a^b \rho(x) \frac{\pi_n(x)}{(x-x_k)\pi_n'(x_k)} \, dx = \int_a^b p(x) \frac{\Omega(x)\omega(x)}{(x-x_k)\omega'(x_k)} \, dx \]

\[ \omega(x) = (x-x_1)\ldots(x-x_n). \]

We then obtain the following formula for the integral (9.3.1):

\[ \int_a^b p(x)f(x) \, dx = \sum_{j=1}^m a_j f(a_j) + \sum_{k=1}^n \beta_k f(a_1, \ldots, a_m, x_k). \quad (9.3.6) \]

It is easy to see that the algebraic degree of precision of this formula is 2n + m - 1.

To prove this let \( f(x) \) be any polynomial of degree \( \leq 2n + m - 1 \). In this case (9.3.3) is an identity whenever the terms on the right side of this equation are defined. The divided difference \( f(a_1, \ldots, a_m, x) \) is a polynomial of degree \( m \) less than the degree of \( f(x) \) and it does not exceed \( 2n - 1 \). Because (9.3.5) has degree of precision \( 2n - 1 \) then it will be exact for \( f(a_1, \ldots, a_m, x) \) and therefore (9.3.6) will also be exact.

On the other hand if \( f(x) \) is taken to be the polynomial

\[ f(x) = \Omega(x)\omega^2(x) \]

of degree \( 2n + m \) then (9.3.6) can not be exact. Indeed, for this function the interpolating polynomial \( P(x) \) is identically zero and from (9.3.2) we see that \( f(a_1, \ldots, a_m, x) = \omega^2(x) \). All the terms on the right side of (9.3.6) vanish and the integral on the left side is nonzero since \( p(x)\Omega(x) \) does not change sign on \([a, b]\):

\[ \int_a^b p(x)f(x) \, dx = \int_a^b p(x)\Omega(x)\omega^2(x) \, dx \neq 0. \]

We will now investigate the relationship between \( f(a_1, \ldots, a_m, x_k) \) and \( f(x) \). If the roots \( x_k \) \( (k = 1, \ldots, n) \) of the polynomial \( \pi_n(x) \), which are the nodes in (9.3.6), are different from the \( a_j \) \( (j = 1, \ldots, m) \) then the divided difference \( f(a_1, \ldots, a_m, x_k) \) is

\[ f(a_1, \ldots, a_m, x_k) = \frac{f(x_k) - P(x_k)}{\Omega(x_k)}. \quad (9.3.7) \]

In this case \( f(a_1, \ldots, a_m, x) \) depends only on the following values of \( f(x) \): \( f(x_k) \), \( f(a_j) \) \( (j = 1, \ldots, m) \).

If in (9.3.6) we substitute for \( f(a_1, \ldots, a_m, x_k) \) the values (9.3.7) then, by collecting the terms in \( f(x_k) \) and \( f(a_j) \), we see that (9.3.6) can be
9.3. Weight Functions that Change Sign

written in the form (9.1.4). In this case formula (9.3.6) is a particular case of (9.1.4) for which the \( a_j \) are the points at which the weight function \( p(x) \) changes sign.

If the node \( x_k \) coincides with one of the nodes \( a_j \) then \( \Omega(x_k) = 0 \) and (9.3.7) is meaningless. In this case (9.3.7) must be replaced by

\[
f(a_1, \ldots, a_m, x_k) = \frac{f'(x_k) - P'(x_k)}{\Omega'(x_k)}
\]

which can be obtained by applying l'Hospital's rule. The divided difference \( f(a_1, \ldots, a_m, x_k) \) then depends on \( f'(x_k) \) as well as on the \( f(a_j) \) \((j = 1, \ldots, m)\). In this case the quadrature formula (9.3.6) will contain, in addition to values of the integrand \( f(x) \), the value \( f'(x_k) \) where \( x_k \) is a point at which \( p(x) \) changes sign.

As an example, let us take \( m = 1 \) and assume that \( p(x) \) changes sign inside \([a, b]\) at only the point \( a_1 \). The interpolating polynomial \( P(x) \) will then be the constant function \( P(x) = f(a_1) \) and (9.3.3) becomes

\[
\int_a^b p(x) f(x) \, dx = f(a_1) \int_a^b p(x) \, dx + \int_a^b p(x)(x - a_1) f(a_1, x) \, dx
\]

(9.3.8)

\[
f(a_1, x) = \frac{f(x) - f(a_1)}{x - a_1}.
\]

If all the \( x_k \) are different from \( a_1 \) then formula (9.3.6) becomes

\[
\int_a^b p(x) f(x) \, dx = f(a_1) \int_a^b p(x) \, dx + \sum_{k=1}^n \beta_k \frac{f(x_k) - f(a_1)}{x_k - a_1}.
\]

If one of the \( x_k \), for example \( x_1 \), coincides with \( a_1 \) then (9.3.6) will have the form

\[
\int_a^b p(x) f(x) \, dx = f(a_1) \int_a^b p(x) \, dx + \beta_1 f'(a_1) + \sum_{k=2}^n \beta_k \frac{f(x_k) - f(a_1)}{x_k - a_1}.
\]

Let us obtain a formula of this form to calculate the integral

\[
\int_{-1}^1 x e^x \, dx = 2e^{-1} = 0.73576.
\]

Here we take \( p(x) = x \) and hence \( p(x) \) changes sign at \( x = 0 \). For this integral equation (9.3.8) is

\[
\int_{-1}^1 x e^x \, dx = e^0 \int_{-1}^1 x \, dx + \int_{-1}^1 x^2 f(0, x) \, dx = \int_{-1}^1 x^2 f(0, x) \, dx,
\]

\[
f(0, x) = \frac{e^x - 1}{x}.
\]
To calculate this last integral we will use the quadrature formula of the highest algebraic degree of precision with two nodes. The second degree polynomial \( \pi_2(x) \) which is orthogonal on \([-1, 1]\) with respect to \( \rho(x) = x^2 \) is \( \pi_2(x) = k(5x^2 - 3) \) which has roots

\[
x = \pm \frac{\sqrt{15}}{5} = \pm 0.7745967.
\]

The formula will then be

\[
\int_{-1}^{1} x^2 f(0, x) \, dx \approx \beta_1 f(0, x_1) + \beta_2 f(0, x_2).
\]

Since the weight function \( \rho(x) = x^2 \) is symmetric with respect to \( x = 0 \) it follows that \( \beta_1 \) and \( \beta_2 \) must be equal and thus

\[
\beta_1 + \beta_2 = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}
\]

\[
\beta_1 = \beta_2 = \frac{1}{3}
\]

\[
\int_{-1}^{1} x^2 f(0, x) \, dx = \frac{1}{3} \left[ \frac{e^{x_1} - 1}{x_1} + \frac{e^{x_2} - 1}{x_2} \right] \approx 0.73536.
\]

This result is exact to within 0.06% of the true value.

REFERENCES


10.1. DETERMINING THE NODES

Quadrature formulas with equal coefficients

\[ \int_a^b p(x)f(x)dx = c_n \sum_{k=1}^{n} f(x_k) \]  \hspace{1cm} (10.1.1)

are very convenient for computations and in particular for graphical calculations. These formulas have been the subject of many investigations and in this chapter we will develop their theory.

Formula (10.1.1) contains the \( n + 1 \) parameters \( c_n, x_1, \ldots, x_n \) and we can choose these parameters so that the formula will be exact for all possible polynomials of degree \( \leq n \).

The requirement that (10.1.1) be exact for \( f(x) \equiv 1 \) means that we must have

\[ \int_a^b p(x)dx = nc_n \]

which determines the coefficient \( c_n \):

\[ c_n = \frac{1}{n} \int_a^b p(x)dx. \]  \hspace{1cm} (10.1.2)

If we also require that (10.1.1) be exact for the monomials \( f(x) = x, x^2, \ldots, x^n \) then we obtain the following system of equations for the nodes \( x_k \):
Approximate Calculation of Definite Integrals

\[ x_1 + x_2 + \cdots + x_n = c_n^{-1} \int_a^b p(x) x \, dx \]

\[ x_1^2 + x_2^2 + \cdots + x_n^2 = c_n^{-1} \int_a^b p(x) x^2 \, dx \]

................................. (10.1.3)

\[ x_1^n + x_2^n + \cdots + x_n^n = c_n^{-1} \int_a^b p(x) x^n \, dx \]

Let \( \omega(x) \) be the polynomial of degree \( n \) which has the nodes \( x_1, \ldots, x_n \) for its roots

\[ \omega(x) = (x - x_1)(x - x_2) \cdots (x - x_n). \] (10.1.4)

Using equations (10.1.3) we can easily construct this polynomial. If we write \( \omega(x) \) in the form

\[ \omega(x) = x^n + A_1 x^{n-1} + A_2 x^{n-2} + \cdots + A_n \] (10.1.5)

then the coefficients \( A_1, \ldots, A_n \) are the well-known elementary symmetric functions of the roots. On the other hand the left sides of equations (10.1.3) are the sums of powers of the roots:

\[ s_k = x_1^k + x_2^k + \cdots + x_n^k \quad (k = 1, 2, \ldots, n). \]

The right sides of (10.1.3) are the values of these functions for the polynomial (10.1.4).

In the theory of equations the relationship between the elementary symmetric functions \( A_i \) \( (i = 1, \ldots, n) \) and the functions \( s_k \) \( (k = 1, \ldots, n) \) is well known. This is given by the following equations which are often called Newton's equations:

\[ \frac{\omega'(x)}{\omega(x)} = \sum_{i=1}^n \frac{1}{x - x_i}. \]

If \( |x| > |x_i| \) then the fraction \( \frac{1}{x - x_i} \) can be expanded in a power series in negative powers of \( x \):

\[ (x - x_i)^{-1} = \sum_{\nu=0}^{\infty} \frac{x_i^{\nu}}{x^{\nu+1}}. \]

Therefore if \( |x| > |x_i| \) \( (i = 1, \ldots, n) \) then the following expansion is valid:
10.1. Determining the Nodes

\[ s_1 + A_1 = 0 \]
\[ s_2 + A_1 s_1 + 2A_2 = 0 \]
\[ \ldots \]
\[ s_n + A_1 s_{n-1} + A_2 s_{n-2} + \cdots + nA_n = 0. \] \hspace{1cm} (10.1.6)

From these equations we can sequentially calculate the coefficients \( A_i \) \((i = 1, \ldots, n)\) from the values of \( s_k \) given by (10.1.3). From the \( A_i \) we can construct the polynomial \( \omega(x) \) and calculating the roots of this polynomial gives the quadrature formula (10.1.1). If (10.1.1) is to be useful the \( x_k \) should all be real, distinct and should belong to the segment of integration. The possibility of constructing formula (10.1.1) which is exact for all polynomials of degree \( \leq n \) is, therefore, determined by whether or not the roots of \( \omega(x) \) satisfy the above requirements where the coefficients \( A_k \) in \( \omega(x) \) are found from Newton's formulas.

We can construct another expression for the polynomial (10.1.4) by making use of a few results from the theory of analytic functions. Let us apply formula (10.1.1) to the fraction \( f(x) = \frac{1}{z-x} \) which is the kernel of the Cauchy integral and consider the remainder:

\[ R \left( \frac{1}{z-x} \right) = \int_a^b \frac{p(x)}{z-x} \, dx - c_n \sum_{k=1}^n \frac{1}{z-x_k} = \int_a^b \frac{p(x)}{z-x} \, dx - c_n \frac{\omega'(z)}{\omega(z)}. \]

We will find the expansion of the remainder in powers of \( z^{-1} \) for \( |z| \) large. Let \( \rho \) be a number so large that the segment of integration \([a, b]\) and all of the nodes \( x_k \) lie in the circle \( |z| \leq \rho \). Then for \( |z| > \rho \)

\[ \frac{\omega'(z)}{\omega(z)} = \sum_{i=1}^n \sum_{\nu=0}^{\infty} x_i^\nu \sum_{\nu=0}^{\infty} \frac{s_\nu}{x_\nu^{\nu+1}} \]

Multiplying both sides of this equation by \( \omega(x) \) and replacing \( \omega(x) \) by its representation (10.1.5) gives

\[ nx^{n-1} + (n-1) A_1 x^{n-2} + (n-2) A_2 x^{n-3} + \cdots + A_{n-1} = \]

\[ = (x^n + A_1 x^{n-1} + \cdots) \sum_{\nu=0}^{\infty} \frac{s_\nu}{x_\nu^{\nu+1}}. \]

Equating the coefficients of \( x^{n-2}, x^{n-3}, \ldots \), we then obtain Newton’s equations.
Approximate Calculation of Definite Integrals

\[ \frac{1}{z-x} = \sum_{\nu=0}^{\infty} \frac{x^\nu}{z^{\nu+1}} \]

and

\[ \int_a^b \frac{p(x)}{z-x} \, dx = \sum_{\nu=0}^{\infty} \frac{1}{z^{\nu+1}} \int_a^b p(x) x^\nu \, dx = \sum_{\nu=0}^{\infty} \frac{\mu_\nu}{z^{\nu+1}}. \]

Here \( \mu_\nu \) denotes the moment of order \( \nu \) of the weight function \( p(x) \). Similarly

\[ \frac{1}{z-x_k} = \sum_{\nu=0}^{\infty} \frac{x_k^\nu}{z^{\nu+1}} \]

and

\[
\frac{\omega'(z)}{\omega(z)} = \sum_{k=1}^{n} \frac{1}{z-x_k} = \sum_{\nu=0}^{\infty} \frac{s_\nu}{z^{\nu+1}}, \\
R \left( \frac{1}{z-x} \right) = \sum_{\nu=0}^{\infty} \frac{\mu_\nu - c_n s_\nu}{z^{\nu+1}}. \tag{10.1.7}
\]

Assuming that (10.1.1) is exact for the powers \( x, x^2, \ldots, x^n \) then by (10.1.2) and (10.1.3) we have

\[ \mu_\nu - c_n s_\nu = 0, \quad \nu = 0, 1, \ldots, n \]

and the smallest exponent of \( 1/z \) in the last expansion will be \( n + 2 \):

\[ \int_a^b \frac{p(x)}{z-x} \, dx - c_n \frac{\omega'(z)}{\omega(z)} = \sum_{\nu=n+1}^{\infty} \frac{\mu_\nu - c_n s_\nu}{z^{\nu+1}}. \]

Integrating with respect to \( z \) and applying a simple transformation we obtain:

\[ \omega(z) \exp \left( \sum_{\nu=n+1}^{\infty} \frac{s_\nu - c_n^{-1} \mu_\nu}{\nu z^\nu} \right) = A \exp \left( c_n^{-1} \int_a^b p(x) \ln (z-x) \, dx \right) \tag{10.1.8} \]

where \( A \) is a certain constant.

Since the expansion of \( \exp \left( \sum_{\nu=n+1}^{\infty} \frac{s_\nu - c_n^{-1} \mu_\nu}{\nu z^\nu} \right) \) in powers of \( 1/z \) differs from unity by only powers of \( 1/z \) greater than \( n \) it is clear that the integer part of the expansion of the right side of (10.1.8) in powers of \( 1/z \) must coincide with \( \omega(z) \) for large \( |z| \).
\[ \omega(z) = \text{integer part of } A \exp\left(c_n^{-1} \int_a^b p(x) \ln (z - x) \, dx \right). \quad (10.1.9) \]

The constant \( A \) could be found by using the fact that the leading term of \( \omega(z) \) is \( z^n \). We will not need to calculate this factor since it does not affect the roots of the right side of (10.1.9).

As mentioned above the formula (10.1.1) which is exact for all polynomials of degree \( \leq n \) is of interest only when the roots of \( \omega(x) \) are real, distinct and lie inside the segment \([a, b]\). The polynomial \( \omega(x) \) is completely defined by the weight function \( p(x) \) and we would like to know for what weight functions this polynomial has the properties we desire. The solution to this problem is not known in general. Below we discuss two weight functions for which the answer is known.

### 10.2. Uniqueness of the Quadrature Formulas of the Highest Algebraic Degree of Precision with Equal Coefficients

In Chapter 7 we discussed the quadrature formulas of the highest algebraic degree of precision for the weight function \( p(x) = (1 - x^2)^{-\frac{1}{2}} \) on \([ -1, 1]\). We obtained formula (7.3.2)

\[ \int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} f(x) \, dx = \frac{n}{\pi} \sum_{k=1}^{n} f \left( \cos \frac{2k - 1}{2n} \pi \right) \]

which is exact for all polynomials of degree \( \leq 2n - 1 \). In this formula the number of nodes \( n \) is arbitrary. It is remarkable that the coefficients in any one of these formulas are all equal.

We may ask whether these formulas are unique: does there exist on the segment \([ -1, 1]\) another weight function \( p(x) \) which is different from \((1 - x^2)^{-\frac{1}{2}}\) for which quadrature formulas of the highest algebraic degree of precision exist and which also have equal coefficients?

A negative answer to this question was first given by K. A. Posse and also later by N. Ia. Sonin.

Here we prove a more general theorem due to Ia. L. Geronimus from which the theorem of Posse easily follows.

Let us be given a weight function \( p(x) \) which is almost everywhere positive on the segment \([ -1, 1]\). Let us take the system of orthogonal polynomials \( \omega_n(x) = x^n + \beta_n x^{n-1} + \gamma_n x^{n-2} + \ldots \) \((n = 0, 1, 2, \ldots)\) which correspond to this weight function. Let \( x_k^{(n)} \) \((k = 1, \ldots, n)\) denote the...
Approximate Calculation of Definite Integrals

roots of \( \omega_n(x) \) and consider the quadrature formula with equal coefficients for which the nodes coincide with \( x_k^{(n)} \):

\[
\int_{-1}^{1} p(x)f(x) \, dx = c_n \sum_{k=1}^{n} f(x_k^{(n)}). \tag{10.2.1}
\]

**Theorem 1.** If for arbitrary values of \( n = 1, 2, \ldots \), there exists constants \( c_n \) such that formula (10.2.1) is exact for \( f(x) = 1, f(x) = x, f(x) = x^2 \), then \( p(x) \) coincides with the Chebyshev weight function \((1 - x^2)^{-\frac{1}{2}}\).

**Proof.** Without loss of generality we can assume

\[
\mu_0 = \int_{-1}^{1} p(x) \, dx = 1.
\]

The requirement that the quadrature formula be exact for \( f(x) = 1 \) then determines the constant \( c_n \):

\[
\int_{-1}^{1} p(x) \, dx = nc_n, \quad c_n = \frac{1}{n}.
\]

Assuming in turn that \( f(x) = x \) and \( f(x) = x^2 \) we obtain the following equations

\[
\mu_1 = \int_{-1}^{1} p(x) x \, dx = \frac{1}{n} \sum_{k=1}^{n} x_k^{(n)} = -\frac{1}{n} \beta_n, \quad n = 1, 2, \ldots
\]

\[
\mu_2 = \int_{-1}^{1} p(x) x^2 \, dx = \frac{1}{n} \sum_{k=1}^{n} [x_k^{(n)}]^2 = \frac{1}{n} \left\{ \left( \sum_{k=1}^{n} x_k^{(n)} \right)^2 - 2 \sum_{j<k} x_j^{(n)} x_k^{(n)} \right\} = \frac{1}{n} (\beta_n^2 - 2 \gamma_n), \quad n = 2, 3, \ldots
\]

Thus we can find the first two coefficients of \( \omega_n(x) \):

\[
\beta_n = -n\mu_1, \quad n = 1, 2, \ldots
\]

\[
\gamma_1 = 0
\]

\[
\gamma_n = \frac{1}{2} [\beta_n^2 - n\mu_2] = \frac{n}{2} [n\mu_1^2 - \mu_2], \quad n = 2, 3, \ldots
\]

\[\text{The requirement that (10.2.1) be exact for } f(x) = x^2 \text{ is only necessary for } n > 1.\]
In Section 2.1 we showed that there is a recursion relation between three consecutive polynomials of an orthogonal sequence. If we denote by \( P_n(x) \) the orthonormal polynomials for the weight function \( p(x) \) then the recursion relation is given by (2.1.10). The polynomial \( \omega_n(x) \) differs by only a constant multiple from the corresponding orthonormal polynomial \( P_n(x) \) of the same degree. Using the fact that the leading coefficient of \( \omega_n(x) \) is unity then the recursion relation for \( \omega_n(x) \) can be written in the form

\[
x \omega_0(x) = \omega_1(x) + \alpha_0
\]

\[
x \omega_n(x) = \omega_{n+1}(x) + \alpha_n \omega_n(x) + \lambda_n \omega_{n-1}(x)
\]

\[ n = 1, 2, \ldots \]

Knowing \( \beta_n \) and \( \gamma_n \) we can find the coefficients \( \alpha_n \) and \( \lambda_n \). Indeed, equating the coefficients of \( x^n \) on opposite sides of the last equation we find

\[
\beta_n = \beta_{n+1} + \alpha_n
\]

\[
\alpha_n = \beta_n - \beta_{n+1} = -n \mu_1 + (n + 1) \mu_1 = \mu_1.
\]

All the \( \alpha_n \) (\( n = 0, 1, \ldots \)) have the same value which for simplicity we denote by \( \alpha \):

\[
\alpha_n = \alpha \quad (n = 0, 1, \ldots).
\]

Equating the coefficients of \( x^{n-1} \) in the same way we obtain:

\[
\gamma_n = \gamma_{n+1} + \alpha_n \beta_n + \lambda_n
\]

\[
\lambda_n = \gamma_n - \gamma_{n+1} - \alpha_n \beta_n
\]

Introducing the quantity \( \sigma \) we can write \( \lambda_n \) as:

\[
\lambda_1 = \mu_2 - \mu_1 = \frac{\sigma^2}{2},
\]

\[
\lambda_n = \frac{1}{2} [\mu_2 - \mu_1] = \frac{\sigma^2}{4}, \quad n = 2, 3, \ldots.
\]

Thus the recursion relation for the polynomials \( \omega_n(x) \) is

\[
x \omega_0(x) = 1, \quad \omega_1(x) = x - \alpha
\]

\[
x \omega_n(x) = \omega_{n+1}(x) + \alpha \omega_n(x) + \frac{\sigma^2}{4} \omega_{n-1}(x), \quad n = 1, 2, \ldots
\]

We recall now (see Section 2.3) that the Chebyshev polynomial of the first kind \( T_n(x) = \cos(n \arccos x) = 2^n x^n + \cdots \) has the recursion relation
Approximate Calculation of Definite Integrals

\[ T_0(x) = 1, \quad T_1(x) = x \]

\[ xT_n(x) = \frac{1}{2} T_{n+1}(x) + \frac{1}{2} T_{n-1}(x). \]

If we reduce the leading coefficient of \( T_n(x) \) to unity we obtain the polynomial \( T^*(x) = 2^{-n+1} T_n(x), \quad T^*_0(x) = T_0(x) \). The recursion relation for \( T^*_n(x) \) is

\[ T^*_0(x) = 1, \quad T^*_1(x) = x \]

\[ xT^*_n(x) = T^*_{n+1}(x) + \frac{1}{4} T^*_{n-1}(x). \]

Finally if the variable \( x \) is replaced by \( \frac{x - \alpha}{\sigma} \) and we introduce the polynomials \( T^+_n(x) = \sigma^n T^*_n \left( \frac{x - \alpha}{\sigma} \right) \) then for these polynomials we obtain the recursion relation

\[ T^+_0(x) = 1, \quad T^+_1(x) = x - \alpha \]

\[ (x - \alpha)T^+_n(x) = T^+_{n+1}(x) + \frac{\sigma^2}{4} T^+_{n-1}(x). \]

These coincide with (10.2.2) and because these equations completely determine \( \omega_n(x) \) (\( n = 0, 1, \ldots \)) then

\[ \omega_n(x) = T^+_n(x) = \sigma^n T^* \left( \frac{x - \alpha}{\sigma} \right) = \frac{\sigma^n}{2^{n-1}} T_n \left( \frac{x - \alpha}{\sigma} \right) \quad n = 1, 2, \ldots. \]

The roots of the polynomial \( T_n(x) \) are \( \cos \frac{2k - 1}{2n} \pi \) (\( k = 1, 2, \ldots, n \)).

They lie inside the segment \([-1, 1]\) and as \( n \) increases they become dense in this segment. Hence it follows that the roots of \( \omega_n(x) \) lie in the segment \([\alpha - \sigma, \alpha + \sigma]\) and these also become dense in this segment.

On the other hand we showed in Chapter 2 that the roots of polynomials of an arbitrary orthogonal system corresponding to a positive weight function lie inside the segment of orthogonality. From the theory of orthogonal polynomials it is also known that the roots of a sequence of orthogonal polynomials become dense in the segment of orthogonality.\(^4\)

Therefore the roots of the polynomials \( \omega_n(x) \) belong to \([-1, 1]\) and form a dense set.

\(^4\)The more general theorem is known: If the segment of orthogonality is \([-1, 1]\) and if the function \( p(x) \) is summable and almost everywhere positive there, then the limiting distribution function of the zeros of the orthogonal polynomials coincides with the Chebyshev distribution function \( \mu(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{dt}{\sqrt{1 - t^2}} \).
10.3. Integrals with a Constant Weight Function

We must therefore have $\alpha = 0$ and $\sigma = 1$ and

$$
\omega_0(x) = T_0(x) = 1
$$

$$
\omega_n(x) = 2^{-n+1} T_n(x) \quad (n = 1, 2, \ldots)
$$

The polynomials $T_n(x)$ form an orthogonal system on $[-1, 1]$ with respect to the weight function $(1 - x^2)^{-\frac{1}{2}}$ and to complete the proof of the theorem there only remains to show that for a finite segment of integration and a given weight function the corresponding orthogonal polynomials are unique up to a constant multiple and up to their values on a set of points of measure zero.

Suppose that the $\omega_n(x)$ are orthogonal on $[-1, 1]$ with respect to both $p_1(x)$ and $p_2(x)$. If necessary we can multiply these weight functions by constants so that

$$
\int_{-1}^{1} p_1(x) \, dx = \int_{-1}^{1} p_2(x) \, dx = 1.
$$

By the orthogonality of $\omega_n(x)$ we must have

$$
\int_{-1}^{1} p_1(x) \omega_n(x) \, dx = \int_{-1}^{1} p_2(x) \omega_n(x) \, dx = 0, \quad (n = 1, 2, \ldots).
$$

Thus the difference $\phi(x) = p_1(x) - p_2(x)$ must satisfy

$$
\int_{-1}^{1} \phi(x) \omega_n(x) \, dx = 0, \quad (n = 0, 1, 2, \ldots)
$$

which is equivalent to

$$
\int_{-1}^{1} \phi(x) x^n \, dx = 0 \quad (n = 0, 1, 2, \ldots).
$$

It is known\(^5\) that the system of powers $x^n \ (n = 0, 1, 2, \ldots)$ is complete in $L$ and thus from the last equation it follows that $\phi(x)$ is equivalent to zero.

10.3. INTEGRALS WITH A CONSTANT WEIGHT FUNCTION

In this section we turn our attention to the much investigated case of a constant weight function. Let us assume that the segment of integration has been transformed into $[-1, 1]$ and consider the quadrature formula

\(^5\)See, for example, I. P. Natanson, *Constructive Theory of Functions*, Gostekhizdat, Moscow, Chap. 3, Sec. 1 (Russian).
Approximate Calculation of Definite Integrals

\[ \int_{-1}^{1} f(x) \, dx = c_n \sum_{k=1}^{n} f(x_k) \]  

(10.3.1)

The coefficient \( c_n \) and nodes \( x_k \) are to be chosen so that the formula is exact for all polynomials of degree \( \leq n \). The coefficient \( c_n \) is determined from the requirement that (10.3.1) be exact for \( f(x) = 1 \) and has the value

\[ c_n = \frac{2}{n} \]

Since

\[ \int_{-1}^{1} x^k \, dx = \frac{1 - (-1)^{k+1}}{k + 1} \]

the system of equations (10.3.1) which the nodes \( x_1, \ldots, x_n \) must satisfy is:

\begin{align*}
S_1 &= x_1 + x_2 + \cdots + x_n = 0 \\
S_2 &= x_1^2 + x_2^2 + \cdots + x_n^2 = \frac{n}{3} \\
S_3 &= x_1^3 + x_2^3 + \cdots + x_n^3 = 0 \\
S_4 &= x_1^4 + x_2^4 + \cdots + x_n^4 = \frac{n}{5} \\
&\vdots \\
S_n &= x_1^n + x_2^n + \cdots + x_n^n = \frac{n}{2} \left[ \frac{1 - (-1)^{n+1}}{n + 1} \right]  
\end{align*}

(10.3.2)

The coefficients of the polynomial \( \omega(x) = (x - x_1) \cdots (x - x_n) \) must be found from the system of equations (10.1.6) which is in this case:

\begin{align*}
A_1 &= 0 \\
\frac{n}{3} + 2A_2 &= 0 \\
A_3 &= 0 \\
\frac{n}{5} + \frac{n}{3} A_2 + 4A_4 &= 0 \\
A_5 &= 0 \\
\frac{n}{7} + \frac{n}{5} A_2 + \frac{n}{3} A_4 + 6A_6 &= 0 \\
A_7 &= 0 \\
&\vdots 
\end{align*}

(10.3.3)
Here all the \( A_k \) with odd subscripts are zero and the polynomial \( \omega(x) \) has the form

\[
\omega(x) = x^n + A_2x^{n-2} + A_4x^{n-4} + \cdots.
\]

The roots of \( \omega(x) \) are the nodes of the formula (10.3.1) and they are symmetrically located on \([-1, 1]\) with respect to the point \( x = 0 \). If \( n \) is odd then one of the nodes coincides with \( x = 0 \).

It should be noted that if \( n \) is an even number \( n = 2m \) then the \( x_k \) satisfy the equations

\[
x_1 + x_2 + \cdots + x_n = 0
\]

\[
x_1^n + x_2^n + \cdots + x_n^n = \frac{n}{n+1}.
\]

Since \( n + 1 = 2m + 1 \) is an odd number and since the \( x_k \) are symmetrically located with respect to \( x = 0 \) then the nodes will also satisfy

\[
x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1} = 0.
\]

In this case formula (10.3.1) will be exact for one higher degree, that is it will be exact for all polynomials of degree \( \leq n + 1 \).

We will now construct formula (10.3.1) for low values of \( n \).

For \( n = 1 \) we have \( \omega(x) = x \) and \( c_1 = 2 \)

\[
\int_{-1}^{1} f(x)dx = 2f(0).
\]

For \( n = 2 \) the coefficient is \( c_2 = 1 \) and the system of equations for \( A_1, A_2 \) is

\[
A_1 = 0
\]

\[
\frac{2}{3} + 2A_2 = 0.
\]

Thus

\[
\omega(x) = x^2 - \frac{1}{3}
\]

\[
x_1 = -\frac{\sqrt{3}}{3}, \quad x_2 = \frac{\sqrt{3}}{3}
\]

\[
\int_{-1}^{1} f(x)dx \approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).
\]
For $n = 3$ we have $c_3 = \frac{2}{3}$ and

\[ A_1 = 0 \]

\[ 1 + 2A_2 = 0 \]

\[ A_3 = 0 \]

\[ \omega(x) = x^3 - \frac{1}{2} x \]

and the formula is then

\[
\int_{-1}^{1} f(x)dx \approx \frac{2}{3} \left[ f\left(\frac{-\sqrt{2}}{2}\right) + f(0) + f\left(\frac{\sqrt{2}}{2}\right)\right].
\]

For $n = 4$ we have $c_4 = \frac{1}{2}$ and the following system of equations for the $A_k$:

\[ A_1 = 0 \]

\[ \frac{4}{3} + 2A_2 = 0 \]

\[ A_3 = 0 \]

\[ \frac{4}{5} + \frac{4}{3} A_2 + 4A_4 = 0 \]

Thus we obtain

\[ A_2 = -\frac{2}{3}, \quad A_4 = \frac{1}{45} \]

\[ \omega(x) = x^4 - \frac{2}{3} x^2 + \frac{1}{45} \]

which has the roots

\[ -x_1 = x_4 = \sqrt{\frac{5 + 2\sqrt{5}}{15}} \]

\[ -x_2 = x_3 = \sqrt{\frac{5 - 2\sqrt{5}}{15}}. \]
In a similar way we obtain the following polynomials:

\[ n = 5, \quad \omega(x) = x^5 - \frac{5}{6} x^3 + \frac{7}{72} x \]

\[ n = 6, \quad \omega(x) = x^6 - x^4 + \frac{1}{5} x^2 - \frac{1}{105} \]

\[ n = 7, \quad \omega(x) = x^7 - \frac{7}{6} x^5 + \frac{119}{360} x^3 - \frac{149}{6480} x \]

\[ n = 9, \quad \omega(x) = x^9 - \frac{3}{2} x^7 + \frac{27}{40} x^5 - \frac{57}{560} x^3 + \frac{53}{22400} x. \]

For \( n = 8 \) two of the roots of \( \omega(x) \) are complex and it is impossible to construct a Chebyshev formula (10.3.1) in this case with real roots. Here we tabulate the decimal values of the nodes in (10.3.1) for \( n = 1(1)7, 9. \)

<table>
<thead>
<tr>
<th>( n = 1 )</th>
<th>( n = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00000</td>
<td>0.26663</td>
</tr>
<tr>
<td></td>
<td>54015</td>
</tr>
<tr>
<td>0.00000</td>
<td>0.42251</td>
</tr>
<tr>
<td></td>
<td>86538</td>
</tr>
<tr>
<td>0.57735</td>
<td>0.86624</td>
</tr>
<tr>
<td></td>
<td>68181</td>
</tr>
<tr>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td></td>
<td>00000</td>
</tr>
<tr>
<td>0.00000</td>
<td>0.32391</td>
</tr>
<tr>
<td></td>
<td>18105</td>
</tr>
<tr>
<td>0.70710</td>
<td>0.52965</td>
</tr>
<tr>
<td></td>
<td>67753</td>
</tr>
<tr>
<td>0.57735</td>
<td>0.88386</td>
</tr>
<tr>
<td></td>
<td>17008</td>
</tr>
<tr>
<td>0.18759</td>
<td>0.00000</td>
</tr>
<tr>
<td></td>
<td>00000</td>
</tr>
<tr>
<td>0.79465</td>
<td>0.00000</td>
</tr>
<tr>
<td></td>
<td>00000</td>
</tr>
<tr>
<td>0.00000</td>
<td>0.16790</td>
</tr>
<tr>
<td></td>
<td>61842</td>
</tr>
<tr>
<td>0.83249</td>
<td>0.52876</td>
</tr>
<tr>
<td></td>
<td>17831</td>
</tr>
<tr>
<td>0.37454</td>
<td>0.60101</td>
</tr>
<tr>
<td></td>
<td>86554</td>
</tr>
<tr>
<td>0.18759</td>
<td>0.91158</td>
</tr>
<tr>
<td></td>
<td>93077</td>
</tr>
</tbody>
</table>

We could also calculate the nodes for the Chebyshev formulas for \( n > 9 \) but in every case it turns out that some of the roots of \( \omega(x) \) will be complex and it will be impossible to construct formula (10.3.1) with real nodes. The general question as to the existence of Chebyshev formulas for \( n > 9 \) with all real nodes remained unanswered until S. N. Bernstein proved that such formulas do not exist. The remainder of this chapter is devoted to a somewhat simplified presentation of his results.

We prove four preliminary lemmas.

Lemma 1. Let the formula

$$\int_{-1}^{1} f(x) dx = \frac{2}{n} \sum_{k=1}^{n} f(x_k)$$

(10.3.4)

be exact for all polynomials of degree \( \leq 2m - 1 \) where \( m < n \). Let \( \xi_m \) denote the largest root of the \( m \)th degree Legendre polynomial \( P_m(x) \). Then, assuming that the \( x_k \) are enumerated in order of size:

$$x_n > \xi_m.$$

Proof. Consider

$$f(x) = \frac{P_m^2(x)}{x \cdot \xi_m}.$$

The function \( P_m(x)/(x - \xi_m) \) is a polynomial of degree \( m - 1 \) and since \( P_m(x) \) is orthogonal on \([-1, 1]\) to all polynomials of lower degree then

$$\int_{-1}^{1} f(x) dx = 0.$$

On the other hand \( f(x) \) is a polynomial of degree \( 2m - 1 \) and equation (10.3.4) must be exact for this function. Therefore

$$\sum_{k=1}^{n} f(x_k) = 0.$$

The polynomial \( f(x) = P_m^2(x)/(x - \xi_m) \) has \( m \) distinct roots and therefore not all terms in the last sum can be zero. Thus this sum must contain positive and negative terms. But \( f(x) \) takes on positive values only for \( x > \xi_m \). Thus we can find a node \( x_k \) for which \( x_k > \xi_m \) and hence the largest node must also be greater than \( \xi_m \).

The following arguments are based on comparisons of (10.3.4) with Gauss quadrature formulas with \( m \) nodes

$$\int_{-1}^{1} f(x) dx = \sum_{i=1}^{m} A_i f(\xi_i)$$

(10.3.5)

$$P_m(\xi_i) = 0 \quad (i = 1, \ldots, m) \quad A_i = \frac{2}{(1 - \xi_i^2)[P_m'(\xi_i)]^2}.$$

Lemma 2. If formula (10.3.4) is exact for all polynomials of degree \( \leq 2m - 1 \) where \( m < n \) then
10.3. Integrals with a Constant Weight Function

\[ A_m > \frac{2}{n}. \]  

(10.3.6)

Proof. Let

\[ f(x) = \left[ \frac{P_m(x)}{(x - \xi_m)P_m'(\xi_m)} \right]^2. \]

Then \( f(\xi_m) = 1 \) and at the other \( \xi_i, i < m, f(\xi_i) = 0. \) Therefore for \( f(x) \) the quadrature sum (10.3.5) becomes:

\[ A_m f(\xi_m) = A_m. \]

The function \( f(x) \) is a polynomial of degree \( 2m - 2 \) and both (10.3.4) and (10.3.5) must be exact for this function. Therefore

\[ \frac{2}{n} \sum_{k=1}^{n} f(x_k) = A_m. \]

Because \( f(x) \geq 0 \) for all \( x \) it follows that

\[ \frac{2}{n} f(x_n) \leq A_m. \]  

(10.3.7)

Writing

\[ f(x) = [P_m'(\xi_m)]^{-2}(x - \xi_1)^2 \cdots (x - \xi_{m-1})^2 \]

we see that, for \( x \geq \xi_m, f(x) \) is an increasing function of \( x \) and since \( x_n > \xi_m \) we have \( f(x_n) > f(\xi_m) = 1. \) Combining this with (10.3.7) proves the lemma.

In order to estimate the coefficient

\[ A_m = \frac{2}{(1 - \xi_m^2)[P_m'(\xi_m)]^2} \]

in formula (10.3.5) we will obtain estimates for \( \xi_m \) and \( P_m'(\xi_m). \)

Lemma 3. For any value of \( m \) the largest root \( \xi_m \) of \( P_m(x) \) satisfies the inequality

\[ 1 - \xi_m < \frac{3}{m(m + 1)}. \]  

(10.3.8)

Proof. We begin with the differential equation satisfied by \( P_m(x): \)

\[ \frac{d}{dx} [(1 - x^2)P_m'(x)] + m(m + 1)P_m(x) = 0. \]
Integrating both terms in this equation between the limits \( \xi_m \) and 1 we obtain

\[
(1 - \xi_m^2)P_m'(\xi_m) = m(m + 1) \int_{\xi_m}^1 P_m(x)dx.
\]

Let us replace the polynomial \( P_m(x) \) in this integral by its expansion in terms of powers of \( x - \xi_m \)

\[
P_m(x) = \sum_{i=1}^{m} \frac{(x - \xi_m)^i}{i!} P_m^{(i)}(\xi_m).
\]

Carrying out the integration gives

\[
(1 - \xi_m^2)P_m'(\xi_m) = m(m + 1) \sum_{i=1}^{m} \frac{(1 - \xi_m)^{i+1}}{(i + 1)!} P_m^{(i)}(\xi_m)
\]

Between each pair of adjacent roots \( \xi_j, \xi_{j+1} \) of the polynomial \( P_m(x) \) there lies a root of \( P_m'(x) \). There are \( m - 1 \) such roots of \( P_m(x) \) and no others. The \( m - 2 \) roots of the second derivative of \( P_m(x) \) lie between adjacent roots of \( P_m'(x) \) and so forth. Thus for any \( i \) all the roots of \( P_m^{(i)}(x) \) lie in the interval \( [\xi_1, \xi_m] \) and none of these roots are greater than \( \xi_m \). Therefore \( P_m^{(i)}(\xi_m) > 0 \) and all terms on the right side of the last equation are positive. For a sufficiently precise estimate for \( \xi_m \) we can replace this sum by only its first two terms. Then dividing both sides by \( 1 - \xi_m \) we obtain the inequality

\[
(1 + \xi_m)P_m'(\xi_m) > m(m + 1) \times
\]

\[
\quad \times \left[ \frac{1}{2} (1 - \xi_m)P_m'(\xi_m) + \frac{1}{6} (1 - \xi_m)^2 P_m''(\xi_m) \right].
\]

The value of \( P_m''(\xi_m) \) is easily found from the equation

\[
(1 - x^2)P_m'(x) - 2xP_m'(x) + m(m + 1)P_m(x) = 0
\]

by substituting \( x = \xi_m \):

\[
P_m''(\xi_m) = \frac{2\xi_m}{1 - \xi_m^2} P_m'(\xi_m).
\]

Substituting this value in the inequality and cancelling the factor \( P_m'(\xi_m) \) gives:

\[
1 + \xi_m > m(m + 1) \left[ \frac{1}{2} (1 - \xi_m) + \frac{1}{3} \frac{\xi_m(1 - \xi_m)}{1 + \xi_m} \right].
\]
This inequality is made stronger if in the second term inside the brackets we replace \( 1 + \xi_m \) by the larger value 2:

\[
1 + \xi_m > m(m + 1) \left[ \frac{1}{2} (1 - \xi_m) + \frac{1}{6} \xi_m(1 - \xi_m) \right].
\]

Setting \( \lambda = m(m + 1) \) we can write this equation as

\[
\lambda \xi_m^2 + 2(3 + \lambda) \xi_m + 6 - 3\lambda > 0. \tag{10.3.10}
\]

Let us form the equations

\[
\lambda z^2 + 2(3 + \lambda) z + 6 - 3\lambda = 0.
\]

If \( \xi_m \) satisfies the inequality (10.3.10) then \( \xi_m \) must be larger than the positive value of \( z \):

\[
\xi_m > \frac{\sqrt{4\lambda^2 + 9 - 3 - \lambda}}{\lambda} > \frac{\sqrt{4\lambda^2 - 3 - \lambda}}{\lambda}.
\]

This gives

\[
\xi_m > 1 - \frac{3}{\lambda} = 1 - \frac{3}{m(m + 1)}
\]

\[
1 - \xi_m < \frac{3}{m(m + 1)}.
\]

This proves lemma 3.

**Lemma 4.** The value of the derivative \( P'_m(\xi_m) \) of the Legendre polynomial \( P_m(x) \) at the largest root \( x = \xi_m \) satisfies the inequality

\[
P'_m(\xi_m) > \frac{2}{3(1 - \xi_m)} \left[ 1 - \frac{\Gamma(m + 4)}{288 \Gamma(m - 2)} (1 - \xi_m)^3 \right]. \tag{10.3.11}
\]

**Proof.** Making use of Taylor’s series with two terms and the integral form of the remainder:

\[
P_m(x) = P'_m(\xi_m)(x - \xi_m) +
\]

\[
+ \frac{1}{2} P''_m(\xi_m)(x - \xi_m)^2 + \frac{1}{2} \int_\xi^x P_m^{(3)}(t)(x - t)^2 dt.
\]

For \( x = 1 \), using \( P_m(1) = 1 \), this becomes:
1 = P_m'(\xi_m)(1 - \xi_m) + \frac{1}{2} P_m''(\xi_m)(1 - \xi_m)^2 +
+ \frac{1}{2} \int_{\xi_m}^{1} P_m^{(3)}(t)(1 - t)^2 dt. \quad (10.3.12)

Consider \( P_m^{(3)}(t) \). In the proof of Lemma 3 we showed that all the roots of \( P_m''(x) \) are less than \( \xi_m \). Therefore \( P_m^{(3)}(x) \) is a monotonically increasing function on \([\xi_m, 1]\) and its greatest value is achieved for \( x = 1 \). The value of \( P_m^{(3)}(1) \) can be easily found using the differential equation

\[
(1 - x^2)P_m''(x) - 2xP_m'(x) + m(m + 1)P_m(x) = 0.
\]

Setting here \( x = 1 \) we find

\[
P_m^{(3)}(1) = \frac{m(m + 1)}{2}.
\]

Differentiating gives

\[
(1 - x^2)P_m^{(3)}(x) - 4xP_m''(x) + (m + 2)(m - 1)P_m'(x) = 0
\]

and again setting \( x = 1 \) gives

\[
P_m^{(1)}(1) = \frac{(m + 2)(m - 1)}{4} P_m^{(1)}(1) = \frac{(m + 2)(m + 1)m(m - 1)}{8}.
\]

Differentiating once more

\[
(1 - x^2)P_m^{(4)}(x) - 6xP_m^{(3)}(x) + (m + 3)(m - 2)P_m''(x) = 0
\]

and substituting \( x = 1 \):

\[
P_m^{(1)}(1) = \frac{(m + 3)(m - 2)}{6} P_m''(1) = \frac{(m + 3)(m + 2) \cdots (m - 2)}{48} = \frac{\Gamma(m + 4)}{48 \Gamma(m - 2)}.
\]

Substituting in (10.3.12) for \( P_m''(\xi_m) \) its expression (10.3.9) and for \( P_m^{(3)}(x) \) its upper bound on \([\xi_m, 1]\) leads to the inequality:

\[
P_m'(\xi_m)(1 - \xi_m) \left[ 1 + \frac{\xi_m}{1 + \xi_m} \right] + \frac{\Gamma(m + 4)}{48 \Gamma(m - 2)} \frac{(1 - \xi_m)^3}{3!} > 1.
\]

This, together with \( \frac{\xi_m}{1 + \xi_m} < \frac{1}{2} \), establishes (10.3.11).

We can now easily find an estimate for \( A_m = \frac{2}{(1 - \xi_m^2)[P_m'(\xi_m)]^2} \).
10.3. Integrals with a Constant Weight Function

Substituting for $P_n'(\xi_m)$ its smaller value from (10.3.11)

$$A_m < \frac{9(1-\xi_m)}{2(1+\xi_m)} \left[ 1 - \frac{\Gamma(m+4)}{288 \Gamma(m-2)} (1-\xi_m)^3 \right]^{-2}.$$ 

It will suffice to use a cruder inequality for $A_m$ for $m \geq 6$. As $m$ increases the value of $\xi_m$ also increases and since $\xi_6 = 0.93246\ldots$ we are justified in assuming $1 + \xi_m > 1.93$. We also replace $1 - \xi_m$ by the larger value $\frac{3}{m(m+1)}$. We now estimate the value inside the brackets.

$$\begin{align*}
(m + 3)(m - 2) &= m(m + 1) - 6 < m(m + 1) \\
(m + 2)(m - 1) &= m(m + 1) - 2 < m(m + 1) \\
\frac{\Gamma(m+4)}{\Gamma(m-2)} &= (m + 3)(m + 2)(m + 1)m(m - 1)(m - 2) < m^3(m + 1)^3
\end{align*}$$

$$1 - \frac{\Gamma(m+4)}{288 \Gamma(m-2)} (1-\xi_m)^3 > 1 - \frac{m^3(m + 1)^3}{288} \frac{3^3}{m^3(m + 1)^3} = \frac{29}{32}$$

$$A_m < \frac{27(32)^2}{2(1.93)(29)^2} \frac{1}{m(m + 1)} \approx \frac{8.517}{m(m + 1)}. \quad (10.3.13)$$

Theorem 2. For $n \geq 10$ there is no formula (10.3.4) with all real roots which is exact for all polynomials of degree $\leq n$.

Proof. Let us consider those values of $n$ for which formula (10.3.4) exists. Let us suppose that $n$ is an odd integer: $n = 2m - 1$. Then $m = \frac{1}{2}(n + 1)$ and $m(m + 1) = \frac{1}{4}(n + 1)(n + 3)$ and $A_m$ must satisfy the inequality $A_m < \frac{4(8.517)}{(n + 1)(n + 3)}$. By Lemma 2 we must have

$$\frac{4(8.517)}{(n + 1)(n + 3)} > \frac{2}{n}$$

or

$$n^2 - (13.034)n + 3 < 0$$

$$n < 13.$$ 

Thus formula (10.3.4) does not exist for $n \geq 13$. But for $n = 11$ it also
Approximate Calculation of Definite Integrals

does not exist because then \( m = 6, A_6 = 0.173 \ldots, \frac{2}{11} = 0.1818 \ldots \)
and the inequality \( \frac{2}{11} < A_6 \) is not satisfied.

Suppose now that \( n \) is even. Then (10.3.4) must be exact for polynomials of degree \( \leq n + 1 \). Set \( n + 1 = 2m - 1, m = \frac{1}{2} (n + 2) \). By (10.3.13) and (10.3.6) we must have

\[
\frac{4(8.517)}{(n + 2)(n + 4)} > \frac{2}{n}
\]

and hence

\( n < 11. \)

This means that for \( n > 10 \) formula (10.3.4) does not exist. For \( n = 10 \) it also does not exist because the inequality

\( A_6 = 0.173 \ldots > \frac{2}{10} = 0.2 \)

is clearly not valid.

REFERENCES


V. I. Krylov, "Mechanical quadratures with equal coefficients for the integrals \( \int_0^\infty e^{-x} f(x)dx \) and \( \int_{-\infty}^\infty e^{-x^2} f(x)dx \)," Dokl. Akad. Nauk SSSR, Vol. 2, 1958, pp. 187-92 (Russian).


10.3. Integrals with a Constant Weight Function


11.1. TWO APPROACHES TO THE PROBLEM

Let us consider a certain completely defined quadrature formula

\[ \int_a^b p(x)f(x)\,dx = \sum_{k=1}^{n} A_k f(x_k) \]  

(11.1.1)

where the weight function \( p(x) \), the coefficients \( A_k \) and the nodes \( x_k \) are fixed; \( f(x) \) is any function for which both sides of (11.1.1) are defined.

We will be interested in the remainder of formula (11.1.1)

\[ R(f) = \int_a^b p(x)f(x)\,dx - \sum_{k=1}^{n} A_k f(x_k) \]  

(11.1.2)

By increasing the precision of the approximate quadrature we mean the

addition of some quantity to the quadrature sum \( \sum_{k} A_k f(x_k) \) which will

decrease the size of the remainder.

The value of \( R(f) \) depends both on the quadrature formula, that is on \( p(x) \) and the \( x_k \) and \( A_k \), and also on the properties of the integrand \( f(x) \).

A method for increasing the precision of the formula must also depend on

these same factors. It is possible to construct such methods for a given class of quadrature formulas with similar properties or for a class of functions which possess certain common structural properties.

In the remainder of this section we discuss two methods for increasing the precision of quadrature formulas.
11.1. Two Approaches to the Problem

1. In most practical applications mechanical quadrature formulas are intended for use with integrands which possess some degree of smoothness. One might expect, for example, that Simpson’s formula

\[ \int_a^b f(x) \, dx = \frac{h}{3} [f_0 + f_n + 2(f_2 + f_4 + \cdots + f_{n-2}) + 4(f_1 + f_3 + \cdots + f_{n-1})] \]

\[ h = \frac{b - a}{n} \]

will give reasonably good results if \( f(x) \) is continuous on the entire segment \([a, b]\) and on each of the segments \([a, a + 2h], [a + 2h, a + 4h], \ldots\) it can be approximated reasonably well by a second or third degree polynomial.

Similarly it is to be expected that a quadrature formula of the highest algebraic degree of precision with \( n \) nodes will give an approximation which is close to the true value if \( f(x) \) can be closely approximated on the entire segment \([a, b]\) by an algebraic polynomial of degree \( 2n - 1 \).

It is not always possible to apply formula (11.1.1) to calculate an improper integral of an unbounded function since one or more values \( f(x_k) \) in the quadrature sum may be infinite. But even if it is possible to apply the formula the error might be very large. A large error can also be obtained in integrating a continuous function which has an unbounded derivative or in integrating an analytic function which has singular points close to the interval of integration.

In cases such as these the accuracy of the approximate integration can be appreciably improved if a preliminary transformation can be applied to the integrand which removes or weakens the singularities of \( f(x) \). This can be done if the integrand can be split into two parts

\[ f(x) = f_1(x) + f_2(x) \]

where \( f_1(x) \) is a function which contains “most” of the singularity of \( f(x) \) for which the integral \( \int_a^b p(x)f_1(x) \, dx \) can be evaluated exactly.

The function \( f_2(x) \) should be relatively smooth so that the integral \( \int_a^b p(x)f_2(x) \, dx \) can be closely approximated by a quadrature formula.

In Section 11.2 we discuss several methods for removing or weakening the singularities of \( f(x) \).

2. In most cases quadrature formulas can estimate an integral to any degree of precision provided that a sufficiently large number of nodes are used. The number of nodes which must be used to obtain a desired
accuracy can be determined in principle by employing the methods we have discussed for estimating the remainder \( R(f) \). Such estimates, however, are usually intended for a wide class of functions and do not take into account the individual properties of a particular integrand. Therefore, as a rule, these estimates are too large and only serve as a rough estimate for the number of nodes which are necessary.

To decide on the number of nodes to be used in a calculation one usually takes into account not only the estimate for the remainder but also other information such as experience derived from previous calculations, comparisons with similar integrals or a comparison of the results of integrations carried out by different methods. The value of \( n \) obtained in this way will often give the desired accuracy but we can not be absolutely certain that it will. We then have the problem of checking the result and if it is not sufficiently accurate of increasing the accuracy.

We will assume that \( f(x) \) is sufficiently smooth so that a large value for \( R(f) \) can only result from using an insufficiently exact quadrature formula.

To increase the precision of the formula we must find additional terms to add to the right side of (11.1.1) so that the new formula will be more precise than (11.1.1).

It is clear that these new terms must account for the principle part of the remainder \( R(f) \). There are many different ways in which the “principle part” of the remainder can be defined and we must determine a simple method to calculate the part which is appropriate to this problem. We discuss two such methods in the last two sections of this chapter.

Suppose that by some method a new term has been found for (11.1.1). If the correction provided by this term improves the accuracy to the desired degree then the computation is completed. If the desired accuracy is not achieved with the first term then the process is repeated and another term is found. It is usually impossible to determine beforehand how many steps will be necessary and therefore we must construct a sequence of principle parts for the remainder (11.1.2) for our initial formula.

### 11.2. WEAKENING THE SINGULARITY OF THE INTEGRAND

As we pointed out in the previous section we can improve the accuracy of an approximate integration by weakening the singularity of the integrand by splitting it into two parts \( f(x) = f_1(x) + f_2(x) \) where \( f_1(x) \) contains “most” of the singularity of \( f(x) \) such that the integral

\[
\int_a^b p(x) f_1(x) \, dx
\]

can be calculated exactly and where the integral of the
second part \( \int_a^b p(x) f_2(x) \, dx \) can be closely approximated by a quadrature formula.

The particular method used will depend on the character of the singularities of \( f(x) \) and on the weight function \( p(x) \). Let us consider some simple examples of such methods.

1. Suppose we are given the integral

\[
\int_a^b (x - x_1)^a \phi(x) \, dx
\]

where \( x_1 \) is a point in or close to the segment \([a, b]\). To be definite let us assume that \( x_1 \) belongs to \([a, b]\). We also assume that \( a \) is greater than \(-1 \) and is not an integer, that \( \phi(x) \) is continuous on \([a, b]\), that \( \phi(x) \) has derivatives up to a certain order \( m \) at \( x_1 \) and that \( \phi(x_1) \neq 0 \).

For \( a < 0 \) the above integral will be improper. If \( a > 0 \) then the integrand will not have derivatives of all orders at \( x_1 \). Thus quadrature formulas might give a large error for this integrand.

Let us split off from the Taylor series expansion of \( \phi(x) \) around the point \( x_1 \) the first \( k \) terms \((k < m)\) and write \( f(x) \) as

\[
f(x) = (x - x_1)^a \phi(x) = f_1(x) + f_2(x)
\]

where

\[
f_1(x) = (x - x_1)^a \left[ \phi(x_1) + \frac{\phi'(x_1)}{1!} (x - x_1) + \cdots + \frac{\phi^{(k-1)}(x_1)}{(k-1)!} (x - x_1)^{k-1} \right]
\]

\[
f_2(x) = (x - x_1)^a \times \left[ \phi(x) - \phi(x_1) - \frac{\phi'(x_1)}{1!} (x - x_1) - \cdots - \frac{\phi^{(k-1)}(x_1)}{(k-1)!} (x - x_1)^{k-1} \right]
\]

Thus the original integral will also be split into two parts

\[
\int_a^b (x - x_1)^a \phi(x) \, dx = \int_a^b f_1(x) \, dx + \int_a^b f_2(x) \, dx.
\]

The first of these integrals can be calculated exactly by elementary methods. At \( x_1 \) the function \( f_2(x) \) is differentiable \( k \) more times than the original function. Therefore the integral \( \int_a^b f_2(x) \, dx \) can be calculated with greater accuracy than (11.2.1) by a quadrature formula.
Approximate Calculation of Definite Integrals

As an example consider the integral

\[ \int_0^1 \sqrt{1 - x^2} \, dx = \frac{\pi}{4} \approx 0.785398163 \ldots \]

At the upper limit \( x = 1 \) the function \( \sqrt{1 - x^2} \) has an algebraic singularity. Let us remove the factor \( \sqrt{1 - x^2} \) and expand \( \sqrt{1 + x} \) in powers of \( x - 1 \) taking two terms in the expansion:

\[
\sqrt{1 + x} = \sqrt{2} \left( 1 - \frac{1 - x}{4} \right) + \left[ \sqrt{1 + x} - \sqrt{2} \left( 1 - \frac{1 - x}{4} \right) \right].
\]

The integral then splits into two integrals the first of which can be integrated exactly:

\[ I_1 = \int_0^1 \sqrt{2} \sqrt{1 - x} \left( 1 - \frac{1 - x}{4} \right) \, dx = \frac{17\sqrt{2}}{30} \approx 0.801388 \ldots. \]

The second integral

\[ I_2 = \int_0^1 \sqrt{1 - x} \left[ \sqrt{1 + x} - \sqrt{2} \left( \frac{3}{4} + \frac{1}{4} x \right) \right] \, dx \]

can be calculated by Simpson's formula (6.3.5) with three nodes:

\[ f_2(0) = 1 - \frac{3\sqrt{2}}{4} \approx -0.060660 \]

\[ 4f_2 \left( \frac{1}{2} \right) = 2\sqrt{3} - \frac{7}{2} = -0.035898 \]

\[ f_2(1) = 0 \]

\[ I_2 = \frac{1}{6} \left[ f_2(0) + 4f_2 \left( \frac{1}{2} \right) + f_2(1) \right] \approx -0.016035 \]

\[ \int_0^1 \sqrt{1 - x^2} \, dx = I_1 + I_2 \approx 0.785353. \]

This result is exact to four significant figures. Applying Simpson's formula with three and five nodes directly to the original integrand gives 0.637 and 0.744 respectively.

2. A similar transformation can be carried out when the integrand has singularities at several points. Suppose the integral has the form

\[ \int_a^b f(x) \, dx = \int_a^b (x - x_1)^{a_1}(x - x_2)^{a_2} \cdots (x - x_n)^{a_n} \phi(x) \, dx. \] (11.2.2)
11.2. Weakening the Singularity of the Integrand

We combine all but the first factor

\[(x - x_2)^a \cdot \cdots \cdot (x - x_n)^a \cdot n \cdot \phi(x)\]

and expand this function in a Taylor series in powers of \(x - x_1\). Taking the first \(k\) terms of this expansion we split the integral as before into two parts

\[f(x) = f_1(x) + [f(x) - f_1(x)]\]

where \(f_1(x)\) is a sum of powers and at the point \(x_1\) \(f(x) - f_1(x)\) has derivatives of higher order than \(f(x)\). In a similar way we can expand around the other points \(x_2, \ldots, x_n\) and obtain

\[f(x) = f_k(x) + [f(x) - f_k(x)], \quad k = 2, \ldots, n.\]

We can then split the original integral into two parts

\[\int_a^b f(x) \, dx = \int_a^b [f_1(x) + f_2(x) + \cdots + f_n(x)] \, dx +\]

\[+ \int_a^b [f(x) - f_1(x) - f_2(x) - \cdots - f_n(x)] \, dx\]

where the first integral is easily calculated exactly. The function in the second integral has higher order derivatives than \(f(x)\) and a quadrature formula applied to this integral will give a more accurate result than when applied to (11.2.2).

3. Taylor's formula can be used to weaken the singularity of the integrand any time that the integral has the form

\[\int_a^b \psi(x) \phi(x) \, dx\]

where \(\psi(x)\) has a singularity at a point \(x_1\) provided that the integrals

\[\int_a^b \psi(x)(x - x_1)^d \, dx\]

can be calculated exactly and that the function \(\phi(x)\) is differentiable several times at the point \(x_1\). An example is the integral

\[\int_a^b (x - x_1)^a \ln^p |x - x_1| \phi(x) \, dx\]

where \(a\) is a real number greater than \(-1\) and \(p\) is an integer.

4. Consider the integral

\[\int_a^b \psi[\phi(x)] \, dx\]  \hspace{1cm} (11.2.3)
where \( \psi(t) \) has a singularity at \( t = 0 \) and \( \phi(x) \) is a continuously differentiable function which is zero at \( x = x_1 \) and such that \( \phi'(x_1) \neq 0 \).

To weaken the singularity of the integrand we can split it into two parts

\[
\psi[\phi(x)] = \psi[A(x-x_1)] + [\psi[\phi(x)] - \psi[A(x-x_1)]]
\]

and if the first integral \( \int_a^b \psi[A(x-x_1)] \, dx \) can be calculated exactly then a quadrature formula applied to the second integral will give a more exact result than when it is applied to (11.2.3).

As an example consider the integral

\[
\int_0^{\pi/2} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2 \approx -1.089045.
\]

This integrand has a logarithmic singularity at \( x = 0 \). We remove from \( \sin x \) the first term of its expansion in powers of \( x \) and write the integral in the following way:

\[
\int_0^{\pi/2} \ln \sin x \, dx = \int_0^{\pi/2} \ln x \, dx + \int_0^{\pi/2} \frac{\ln \sin x}{x} \, dx = I_1 + I_2
\]

\[
I_1 = \int_0^{\pi/2} \ln x \, dx = \frac{\pi}{2} \left( \ln \frac{\pi}{2} - 1 \right) \approx -0.861451.
\]

The function \( y(x) = \frac{\ln \sin x}{x} \) has no singular points in \([0, \frac{\pi}{2}]\). To calculate \( I_2 \) we use Simpson's formula with 3 nodes:

\[
I_2 \approx \frac{\pi}{12} \left[ y(0) + 4y\left(\frac{\pi}{4}\right) + y\left(\frac{\pi}{2}\right) \right] \approx -0.228189.
\]

Thus

\[
\int_0^{\pi/2} \ln \sin x \, dx = I_1 + I_2 \approx -1.089640.
\]

11.3. EULER'S METHOD FOR EXPANDING THE REMAINDER

We now consider the problem of increasing the precision of a quadrature formula by removing the principle part of the remainder. The most appropriate way of doing this depends on the properties of the remainder and there are many different methods which may be used.
11.3. Euler’s Method for Expanding the Remainder

In this chapter we discuss two of these methods, the first of which is closely related to the Euler-Maclaurin sum formula.

The simplest type of Euler’s formula serves to increase the accuracy of the simple one-point formula. It is another form of the method for expanding an arbitrary function in Bernoulli polynomials.

Let \( f(x) \) have \( \nu \) continuous derivatives on the finite segment \([a, b]\). In Chapter 1 we established the representation (1.4.2) which expresses \( f(x) \) in terms of Bernoulli polynomials and the periodic functions \( B^*_\nu(x) \).

This representation can be written in the form

\[
\int_a^b f(t) \, dt = (b - a) f(x) - (b - a) \sum_{k=0}^{\nu} \frac{B_k}{k!} f^{(k)}(a) - (b - a) \sum_{k=1}^{\nu} \frac{B_{k-1}}{k!} f^{(k-1)}(a) + \int_a^b f^{(\nu)}(t) \sum_{k=0}^{\nu} \frac{B^*_\nu}{k!} f^{(k)}(t) - B^*_\nu \int_a^b f(t) \, dt.
\]

The first term on the right side of this equation \((b - a) f(x)\) gives an approximate value for the integral \( \int_a^b f(t) \, dt \) and is a one-point formula which uses the point \( x \). The approximate equation

\[
\int_a^b f(t) \, dt \approx (b - a) f(x)
\]

will be exact when \( f(t) \) is a constant function.

If we adjoin to the term \((b - a) f(x)\) the second term on the right side we obtain

\[
\int_a^b f(t) \, dt \approx (b - a) f(x) - (b - a) B_1 \left(\frac{x - a}{b - a}\right) [f(b) - f(a)]
\]

which is exact for any linear function. If we add a third term then the resulting equation is exact for any quadratic polynomial and so forth. Adding one term at a time from (11.3.1) increases the algebraic degree of precision of the formula each time by one. We can expect, at least in certain cases, that each new term will increase the accuracy of the approximate integration.

The integral on the right side of (11.3.1) is the remainder term in the final quadrature formula. Below we will investigate this integral further.
Equation (11.3.1) is more valuable than we have indicated for from it we can construct, in principle, a method for increasing the precision of any quadrature formula for use with a constant weight function. Let us consider an arbitrary quadrature formula of the form

\[ \int_a^b f(t) \, dt \approx (b - a) \sum_{k=1}^n A_k f(x_k). \tag{11.3.2} \]

Let us assume that this formula is exact if \( f(t) \) is a constant, that is \( \sum_{k=1}^n A_k = 1 \). Then it is obvious that (11.3.2) is a linear combination of \( n \) elementary one point formulas

\[ \int_a^b f(t) \, dt \approx (b - a) f(x_k) \quad (k = 1, 2, \ldots, n). \]

Therefore a linear combination with coefficients \( A_k \) of \( n \) equations (11.3.1) with \( x = x_1, x = x_2, \ldots, x = x_n \) gives a new equation which will increase the accuracy of the formula (11.3.2) to an arbitrarily high degree.

One can see that similar equations can also be constructed for quadrature formulas for the approximate evaluation of an integral \( \int_a^b p(t) f(t) \, dt \) with any summable weight function \( p(t) \). Such formulas are formally very simple to derive by using the theorem on the expansion of a function in Bernoulli polynomials together with special forms of integral representations for the remainder of the quadrature formulas. But it is not clear in which cases the formulas obtained in this way will actually increase the accuracy of the quadrature formulas and in which cases they will give a worse result.

We will begin our discussion from an intuitive point of view and will derive a method to increase the accuracy which will be very generally applicable. Our discussion will also clarify the conditions under which formulas of Euler's type are to be preferred over other methods.

We assume again that the segment \([a, b]\) is finite and that \( f(x) \) has a continuous derivative of order \( m + s \) on \([a, b]\) where \( m \) and \( s \) are positive integers which will enter into the following discussion.

We will consider the remainder \( R(f) \) of the quadrature formula

\[ \int_a^b p(x) f(x) \, dx = \sum_{k=1}^n A_k f(x_k) + R(f) \quad \tag{11.3.3} \]

which we assume is exact for all polynomials of degree \( \leq m - 1 \).
The function $f(x)$ can be represented by the Taylor series:

$$f(x) = \sum_{i=0}^{m-1} \frac{f^{(i)}(a)}{i!} (x - a)^i + \int_a^x f^{(m)}(t) \frac{(x - t)^{m-1}}{(m - 1)!} \, dt =$$

$$= P_{m-1}(x) + \int_a^b f^{(m)}(t) E(x - t) \frac{(x - t)^{m-1}}{(m - 1)!} \, dt.$$  

Now, since $R(P_{m-1}) = 0$, the remainder $R(f)$ will be:

$$R(f) = \int_a^b p(x) \int_a^b f^{(m)}(t) E(x - t) \frac{(x - t)^{m-1}}{(m - 1)!} \, dt \, dx -$$

$$- \sum_{k=1}^n A_k \int_a^b f^{(m)}(t) E(x_k - t) \frac{(x_k - t)^{m-1}}{(m - 1)!} \, dt.$$  

The assumptions of the continuity of $f^{(m)}(x)$, the summability of $p(x)$, and the finiteness of $[a, b]$ allow us to change the order of this double integral. This allows us to construct a representation for $R(f)$ which will be useful for analyzing the remainder and especially for selecting its “principal part”:

$$R(f) = \int_a^b f^{(m)}(t) K(t) \, dt$$  

(11.3.4)

where the kernel $K(t)$ is given by

$$K(t) = p(x) \frac{(x - t)^{m-1}}{(m - 1)!} \, dx - \sum_{k=1}^n A_k E(x_k - t) \frac{(x_k - t)^{m-1}}{(m - 1)!}. \quad (11.3.5)$$

When $K(t)$ is a “slowly varying” function the part of $K(t)$ which most influences the numerical value of $R(f)$ is the average value of the kernel. The principle part of $R(f)$ can then be separated by writing

$$K(t) = C_0 + [K(t) - C_0]$$

where $C_0 = (b - a)^{-1} \int_a^b K(t) \, dt$.

Then

$$R(f) = C_0 \int_a^b f^{(m)}(t) \, dt + \int_a^b f^{(m)} [K(t) - C_0] \, dt =$$

$$= C_0 [f^{(m-1)}(b) - f^{(m-1)}(a)] + \int_a^b f^{(m+1)}(t) L_1(t) \, dt.$$
where

\[ L_1(t) = \int_a^t [C_0 - K(x)] \, dx. \]

If the new kernel \( L_1(t) \) is again a "slowly varying" function we can again separate the principle part from the integral

\[ \int_a^b f^{(m+1)}(t) L_1(t) \, dt \]

and so forth.

After performing this operation \( s \) times the original quadrature formula (11.3.3) will be transformed into an equation of Euler's form which can be used to increase the accuracy of (11.3.3) provided that the functions \( L_0 = K, L_1, L_2, \ldots \) do not have large variation:

\[
\int_a^b p(x) f(x) \, dx = \sum_{k=1}^{n} A_k f(x_k) + C_0 [f^{(m-1)}(b) - f^{(m-1)}(a)] + \cdots + C_{s-1} [f^{(m+s-2)}(b) - f^{(m+s-2)}(a)] + R_s(f) \quad (11.3.6)
\]

\[
C_i = (b - a)^{-1} \int_a^b L_i(t) \, dt,
\]

\[
L_0(t) = K(t)
\]

\[
L_{i+1}(t) = \int_a^t [C_i - L_i(x)] \, dx \quad (11.3.6*)
\]

\[
R_s(f) = \int_a^b f^{(m+s)}(t) L_s(t) \, dt.
\]

Equations (11.3.6*) give a method for sequentially calculating the \( C_i \) and \( L_i(t) \). However, we can find a representation for \( C_i \) and \( L_i(t) \) directly from the kernel \( K(t) \). To do this we return to the initial quadrature formula (11.3.3) with the integral representation for the remainder

\[
R(f) = \int_a^b f^{(m)}(t) K(t) \, dt.
\]

Replacing \( f^{(m)}(t) \) by its expansion in terms of Bernoulli polynomials
11.3. Euler's Method for Expanding the Remainder

\[ f^{(m)}(t) = (b-a)^{-1} \int_a^b f^{(m)}(x) \, dx + \]

\[ + \sum_{i=1}^{s-1} \frac{(b-a)^{i-1}}{i!} B_i \left( \frac{t-a}{b-a} \right) [f^{(m+i-1)}(b) - f^{(m+i-1)}(a)] - \]

\[ - \frac{(b-a)^{s-1}}{s!} \int_a^b f^{(m+s)}(x) \left[ B_s^* \left( \frac{t-x}{b-a} \right) - B_s^* \left( \frac{t-a}{b-a} \right) \right] \, dx \]

and integrating we obtain

\[ \int_a^b p(x) f(x) \, dx = \sum_{k=1}^{\infty} A_k f(x_k) + \]

\[ + (b-a)^{-1} \int_a^b K(t) \, dt [f^{(m-1)}(b) - f^{(m-1)}(a)] + \]

\[ + \sum_{i=1}^{s-1} \frac{(b-a)^{i-1}}{i!} \int_a^b K(t) B_i \left( \frac{t-a}{b-a} \right) \, dt \times \]

\[ \times [f^{(m+i-1)}(b) - f^{(m+i-1)}(a)] - \frac{(b-a)^{s-1}}{s!} \int_a^b K(t) \times \]

\[ \times \int_a^b f^{(m+s)}(x) \left[ B_s^* \left( \frac{t-x}{b-a} \right) - B_s^* \left( \frac{t-a}{b-a} \right) \right] \, dx \]

which must coincide with (11.3.6) for any function \( f(x) \) which has a continuous derivative of order \( m+s \) on \([a, b]\).

This can happen only when the coefficients of the terms \([f^{(m+i-1)}(b) - f^{(m+i-1)}(a)]\) are equal and when \( R_s(f) \) in (11.3.6) coincides with the last term in the previous equation. Thus we have shown that

\[ C_i = \frac{(b-a)^{i-1}}{i!} \int_a^b K(t) B_i \left( \frac{t-a}{b-a} \right) \, dt \quad \text{(11.3.7)} \]

\[ L_s(t) = -\frac{(b-a)^{s-1}}{s!} \int_a^b K(x) \left[ B_s^* \left( \frac{x-t}{b-a} \right) - B_s^* \left( \frac{x-a}{b-a} \right) \right] \, dx. \quad \text{(11.3.8)} \]

There is a simple interpretation for the \( C_i \) and \( L_s(t) \). Comparing (11.3.7) with the integral representation for the remainder \( R(f) \) given by (11.3.4)
we see that $C_i$ is the remainder when the quadrature formula is applied to a function which has for its $m$th derivative $\frac{(b - a)^{i-1}}{i!} \cdot B_i \left( \frac{t - a}{b - a} \right)$.

Recalling the rule (1.2.6) for differentiating a Bernoulli polynomial we see that the polynomial

$$\frac{(b - a)^{m+i-1}}{(m + i)!} B_{m+i} \left( \frac{t - a}{b - a} \right)$$

has this property. Thus

$$C_i = \frac{(b - a)^{m+i-1}}{(m + i)!} R \left[ B_{m+i} \left( \frac{t - a}{b - a} \right) \right] =$$

$$= \frac{(b - a)^{m+i-1}}{(m + i)!} \left\{ \int_a^b p(t) B_{m+i} \left( \frac{t - a}{b - a} \right) dt -$$

$$- \sum_{k=1}^{n} A_k B_{m+i} \left( \frac{x_k - a}{b - a} \right) \right\}. \quad (11.3.9)$$

This equation provides a simple method for calculating the $C_i$.

Similarly we obtain for $L_s(t)$ the expression

$$L_s(t) = - \frac{(b - a)^{m+s-1}}{(m + s)!} R_x \left[ B_{m+s}^* \left( \frac{x - t}{b - a} \right) -$$

$$- B_{m+s}^* \left( \frac{x - a}{b - a} \right) \right]. \quad (11.3.10)$$

where $R_x$ indicates the remainder when the quadrature formula is applied with respect to the variable $x$.

Now we construct some special cases of Euler's formula. We begin by obtaining the Euler-Maclaurin\(^1\) formula for increasing the accuracy of the trapezoidal rule.

Consider the simple trapezoidal formula

$$\int_a^b f(x) dx = \frac{b - a}{2} [f(a) + f(b)] + R(f) \quad (11.3.11)$$

which is exact for linear polynomials and for which we must take $m = 2$. To construct (11.3.6) we must first compute the coefficients $C_i$. The easiest method in this case is to use (11.3.9).

\(^1\)For other Euler-Maclaurin formulas see J. F. Steffensen, *Interpolation*, Chap. 18.
11.3. Euler's Method for Expanding the Remainder

The polynomials $B_n(x)$, $n = 2, 3, \ldots$, have the property that $B_n(0) = B_n(1)$ so that

$$\int_a^b B_{i+2} \left( \frac{t-a}{b-a} \right) dt = \frac{b-a}{i+3} \left[ B_{i+3}(1) - B_{i+3}(0) \right] = 0$$

$$C_i = -\frac{(b-a)^{i+2}}{2(i+2)!} \left[ B_{i+2}(0) + B_{i+2}(1) \right] =$$

$$= -\frac{(b-a)^{i+2}}{(i+2)!} \left[ \frac{1 + (-1)^{i+2}}{2} \right] B_{i+2}.$$  \hspace{1cm} (11.3.12)

All the odd order Bernoulli numbers, except $B_1$, are zero so that $C_1 = C_3 = C_5 = \ldots = 0$. The coefficients $C_i$ with even subscript $i = 2j$ are

$$C_{2j} = -\frac{(b-a)^{2j+2}}{(2j+2)!} \frac{B_{2j+2}}{}.$$  \hspace{1cm} (11.3.12)

The first few $C_{2j}$ are:

$$C_0 = -\frac{(b-a)^2}{12}, \quad C_2 = \frac{(b-a)^4}{720}, \quad C_4 = -\frac{(b-a)^6}{30240},$$

$$C_6 = \frac{(b-a)^8}{1209600}, \quad C_8 = -\frac{(b-a)^{10}}{47900160}.$$  \hspace{1cm} (11.3.12)

To construct the remainder $R_s(f)$ in (11.3.6) we will calculate the kernel $L_s(t)$ from (11.3.10):

$$L_s(t) = -\frac{(b-a)^{s+1}}{(s+2)!} \left\{ \int_a^b \left[ B^*_s+2 \left( \frac{x-t}{b-a} \right) - B^*_s+2 \left( \frac{x-a}{b-a} \right) \right] dx -$$

$$- \frac{b-a}{2} \left[ B^*_s+2 \left( \frac{a-t}{b-a} \right) - B^*_s+2(0) +$$

$$+ B^*_s+2 \left( \frac{b-t}{b-a} \right) - B^*_s+2(1) \right] \right\}.$$  \hspace{1cm} (11.3.10)

The period of $B^*_s+2 \left( \frac{x-t}{b-a} \right)$ is $b-a$ so that

$$\int_a^b B^*_s+2 \left( \frac{x-t}{b-a} \right) dx = \int_a^b B^*_s+2 \left( \frac{x-a}{b-a} \right) dx$$

and the integral in the expression for $L_s(t)$ is zero. Also
Approximate Calculation of Definite Integrals

\[ BS+2 (b - t) = BS+2 (b - a), \quad BS+2(0) = BS+2(1) = BS+2 \]

and defining \( y_k^s(x) = B_k^s(x) - B_k \) we obtain

\[
L_s(t) = \frac{(b - a)s+2}{(s + 2)!} \left[ BS+2 \left( \frac{b - t}{b - a} \right) - BS+2 \right] = \frac{(b - a)s+2}{(s + 2)!} y_{s+2}^s \left( \frac{b - t}{b - a} \right).
\]

We can now write equation (11.3.6) for the trapezoidal rule. Since the \( C_i \) are zero for all odd \( i \) we have

\[
\int_a^b f(x)dx = \frac{(b - a)}{2} \left[ f(a) + f(b) \right] - \sum_{k=1}^{\nu-1} \frac{(b - a)^{2k}}{(2k)!} B_{2k} \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] + \rho_{2\nu}(f)
\]

where the remainder \( \rho_{2\nu}(f) \) is either

\[
\rho_{2\nu}(f) = \frac{(b - a)^{2\nu-1}}{(2\nu - 1)!} \int_a^b f^{(2\nu-1)}(t)y_{2\nu-1}^s \left( \frac{b - t}{b - a} \right) dt
\]

or

\[
\rho_{2\nu}(f) = \frac{(b - a)^{2\nu}}{(2\nu)!} \int_a^b f^{(2\nu)}(t)y_{2\nu}^s \left( \frac{b - t}{b - a} \right) dt
\]

depending on whether \( f(t) \) has a continuous derivative of order \( 2\nu - 1 \) or \( 2\nu \).

In the following discussion we assume that \( f(t) \) has a continuous derivative of order \( 2\nu \) so that \( \rho_{2\nu}(f) \) satisfies the second equation of (11.3.14) and will transform this equation into a somewhat simpler form. We make the transformation \( t = a + (b - a)u, \ 0 < u < 1 \). Using the relationships

\[
y_{2\nu}^s \left( \frac{b - t}{b - a} \right) = y_{2\nu}(1 - u) = B_{2\nu}(1 - u) - B_{2\nu} = B_{2\nu}(u) - B_{2\nu} = y_{2\nu}(u)
\]

we obtain

\[
\rho_{2\nu}(f) = \frac{(b - a)^{2\nu+1}}{(2\nu)!} \int_0^1 f^{(2\nu)}(a + (b - a)u)y_{2\nu}(u)du
\]
11.3. Euler's Method for Expanding the Remainder

In order to obtain an equation for increasing the precision of the repeated trapezoidal formula (6.3.4) we divide the segment \([a, b]\) into any number \(n\) of equal parts of length \(h = \frac{b - a}{n}\) and apply (11.3.13) to the subsegment \([a + ph, a + (p + 1)h]\):

\[
\int_{a+ph}^{a+(p+1)h} f(x)dx = \frac{h}{2} \left\{ f[a + ph] + f[a + (p + 1)h] \right\} -
\]

\[
- \sum_{k=1}^{\nu-1} \frac{h^{2k}}{(2k)!} B_{2k} \left[ f^{(2k-1)}[a + (p + 1)h] - f^{(2k-1)}[a + ph] \right] + \rho_{2\nu}^{(p)}(f)
\]

\[
\rho_{2\nu}^{(p)}(f) = \frac{h^{2\nu+1}}{(2\nu)!} \int_0^1 f^{(2\nu)}(a + ph + hu) \gamma_{2\nu}(u)du.
\]

By adding these equations for \(p = 0, 1, \ldots, n - 1\) we obtain

\[
\int_a^b f(x)dx = T_n - \sum_{k=1}^{\nu-1} \frac{h^{2k}}{(2k)!} B_{2k} \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] + \rho_{2\nu}(f) =
\]

\[
= T_n - \frac{h^2}{12} [f'(b) - f'(a)] + \frac{h^4}{720} [f^{(3)}(b) - f^{(3)}(a)] -
\]

\[
- \frac{h^6}{30240} [f^{(5)}(b) - f^{(5)}(a)] + \frac{h^8}{1209600} [f^{(7)}(b) - f^{(7)}(a)] -
\]

\[
- \frac{h^{10}}{47900160} [f^{(9)}(b) - f^{(9)}(a)] + \cdots + \rho_{2\nu}(f)
\]  
(11.3.16)

where

\[
T_n = h \left[ \frac{1}{2} f(a) + f(a + h) + \cdots + f(a + (n - 1)h) + \frac{1}{2} f(b) \right]
\]

and

\[
\rho_{2\nu}(f) = \frac{h^{2\nu+1}}{(2\nu)!} \int_0^1 \gamma_{2\nu}(u) \sum_{p=0}^{n-1} f^{(2\nu)}(a + ph + hu)du =
\]

\[
= \frac{h^{2\nu+1}}{(2\nu)!} \int_0^1 \left[ B_{2\nu}(u) - B_{2\nu} \right] \sum_{p=0}^{n-1} f^{(2\nu)}(a + ph + hu)du.
\]
Approximate Calculation of Definite Integrals

Equation (11.3.16) is the well known Euler-Maclaurin sum formula relating the integral \( \int_a^b f(x)dx \) to the sum of integrand values at equally spaced points:

\[
S_n = f(a) + f(a + h) + \cdots + f(b) = h^{-1} T_n + \frac{1}{2} [f(a) + f(b)].
\]

From (11.3.16) we can calculate \( S_n \) if we know the value of the integral or the value of the integral in terms of \( S_n \). We are only interested in this second application.

If \( \nu \) increases without bound then the terms in the summation in (11.3.16) become the infinite series

\[
\sum_{k=1}^{\infty} \frac{h^{2k}}{(2k)!} B_{2k} \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right]
\]

(11.3.17)

We recall that, for large integers \( k \), the Bernoulli numbers \( B_{2k} \) grow very rapidly and are approximately equal to

\[
B_{2k} = \frac{2}{(2\pi)^{2k}} \frac{(-1)^{k-1}}{(2k)!} (2n)^{-2k}
\]

Therefore the series (11.3.17) converges for only a very small subclass of the functions we have been considering. In spite of this shortcoming the Euler-Maclaurin formulas are often used because for the first few values of \( \nu \) the remainder decreases and the first few corrections applied to \( T_n \) significantly increase the accuracy of the trapezoidal formula.

We now prove three simple theorems about the remainder \( \rho_{2\nu}(f) \).

**Theorem 1.** If \( f^{(2\nu)}(x) \) is continuous on \([a, b]\) then there exists a point \( \xi \in [a, b] \) for which

\[
\rho_{2\nu}(f) = -\frac{nh^{2\nu+1}}{(2\nu)!} B_{2\nu} f^{(2\nu)}(\xi).
\]

(11.3.18)

**Proof.** In Section 1.2 we showed that the function \( \gamma_{2\nu}(u) \) does not change sign on the interval \([0, 1]\):

\[
(-1)^{\nu} \gamma_{2\nu}(u) > 0 \quad \text{for} \quad 0 < u < 1.
\]

Consider the integral

\[
I_\nu = \int_0^1 (-1)^{\nu} \gamma_{2\nu}(u) \sum_{p=0}^{n-1} f^{(2\nu)}(a + ph + hu) du.
\]

Let \( m \) and \( M \) denote the smallest and greatest values of \( f^{(2\nu)}(x) \) on
11.3. Euler's Method for Expanding the Remainder

Then it is clear that

\[ (-1)^{\nu} n M \int_0^1 \gamma_{2\nu}(u) \, du \leq I_\nu \leq (-1)^{\nu} n M \int_0^1 \gamma_{2\nu}(u) \, du \]

and since

\[ \int_0^1 \gamma_{2\nu}(u) \, du = \int_0^1 [B_{2\nu}(u) - B_{2\nu}] \, du = \]

\[ = \frac{1}{2\nu + 1} [B_{2\nu+1}(1) - B_{2\nu+1}(0)] - B_{2\nu} = -B_{2\nu} \]

then

\[ I_\nu = (-1)^{\nu + 1} n P B_{2\nu} \]

where \( m \leq P \leq M \). From the continuity of \( f^{(2\nu)}(x) \) there must be a point \( \xi \in [a, b] \) for which \( f^{(2\nu)}(\xi) = P \) so that

\[ I_\nu = (-1)^{\nu + 1} n B_{2\nu} f^{(2\nu)}(\xi) \]

Since \( \rho_{2\nu}(f) = (-1)^{\nu} \frac{h^{2\nu+1}}{(2\nu)!} I_\nu \) the theorem is proved.

Theorem 2. If \( f^{(2\nu)}(x) \) is continuous and does not change sign on \([a, b]\) then \( \rho_{2\nu}(f) \) can be written in the form

\[ \rho_{2\nu}(f) = -\theta(2 - 2^{-2\nu+1}) \frac{h^{2\nu} B_{2\nu}}{(2\nu)!} [f^{(2\nu-1)}(b) - f^{(2\nu-1)}(a)] \]

\[ 0 < \theta < 1. \quad (11.3.19) \]

Proof. In Section 1.2 we showed that \( (-1)^{k} \gamma_{2k}(x) \) does not change sign on \([0, 1]\) and that this function increases for \( 0 \leq x \leq \frac{1}{2} \) and decreases for \( \frac{1}{2} \leq x \leq 1 \) with its largest value at \( x = \frac{1}{2} \) where

\[ (-1)^{k} \gamma_{2k}(\frac{1}{2}) = -(-1)^{k}(2 - 2^{-2k+1}) B_{2k}. \]

Therefore \( \rho_{2\nu}(f) \) has the same sign as

\[ \gamma_{2\nu}(\frac{1}{2}) \frac{h^{2\nu+1}}{(2\nu)!} \int_0^1 \sum_{p=0}^{n-1} f^{(2\nu)}(a + ph + hu) \, du \]

and, in absolute value, is less than this quantity. Therefore
\[ \rho_{2\nu}(f) = \theta \gamma_{2\nu} \left( \frac{1}{2} \right) h^{2\nu+1} \frac{B_{2\nu+1}}{(2\nu)!} \int_0^1 \sum_{p=0}^{n-1} f^{(2\nu)}(a + ph + hu) du, \quad 0 < \theta < 1. \]

But
\[ \gamma_{2k} \left( \frac{1}{2} \right) = -(2 - 2^{-2k+1}) B_{2k} \]

and
\[
\int_0^1 \sum_{p=0}^{n-1} f^{(2\nu)}(a + ph + hu) du = h^{-1} \sum_{p=0}^{n-1} \{ f^{(2\nu-1)}[a + h(p + 1)] - f^{(2\nu-1)}[a + hp] \} =
\]
\[ = h^{-1} [ f^{(2\nu-1)}(b) - f^{(2\nu-1)}(a) ] \]
which then establishes (11.3.19).

From this second theorem we see, provided \( f^{(2\nu)}(x) \) satisfies the necessary assumptions, that \( \rho_{2\nu}(f) \) has the same sign as the first neglected term in (11.3.16) and is smaller, in absolute value, than twice this term.

It turns out that under certain assumptions on \( f(x) \) the remainder \( \rho_{2\nu}(f) \) in the Euler-Maclaurin formula (11.3.16) has an estimate similar to the estimate for the partial sum of an alternating series.

**Theorem 3.** If \( f(x) \) has a continuous derivative of order \( 2\nu + 2 \) on \([a, b]\) and for each \( x \in [a, b] \) either
\[ f^{(2\nu)}(x) \geq 0 \quad \text{and} \quad f^{(2\nu+2)}(x) \geq 0 \]
or
\[ f^{(2\nu)}(x) \leq 0 \quad \text{and} \quad f^{(2\nu+2)}(x) \leq 0 \]
then \( \rho_{2\nu}(f) \) has the same sign as
\[ -\frac{h^{2\nu} B_{2\nu}}{(2\nu)!} [ f^{(2\nu-1)}(b) - f^{(2\nu-1)}(a) ] \]
and is less, in absolute value, than this term.

**Proof.** The remainders \( \rho_{2\nu}(f) \) and \( \rho_{2\nu+2}(f) \) satisfy the relationship
\[ \rho_{2\nu}(f) = -\frac{h^{2\nu} B_{2\nu}}{(2\nu)!} [ f^{(2\nu-1)}(b) - f^{(2\nu-1)}(a) ] + \rho_{2\nu+2}(f) \]
11.3. Euler's Method for Expanding the Remainder

which can be written as

\[
\frac{h^{2\nu+1}}{(2\nu)!} \int_0^1 \gamma_{2\nu}(u) \sum_{p=0}^{n-1} f^{(2\nu)}(a + ph + hu) du + \\
+ \frac{h^{2\nu+3}}{(2\nu + 2)!} \int_0^1 [-\gamma_{2\nu+2}(u)] \sum_{p=0}^{n-1} f^{(2\nu+2)}(a + ph + hu) du = \\
= -\frac{h^{2\nu} B_{2\nu} [f^{(2\nu-1)}(b) - f^{(2\nu-1)}(a)]}{(2\nu)!}.
\]

In Section 1.2 we showed that \(\gamma_{2\nu}(u)\) and \(-\gamma_{2\nu+2}(u)\) have the same sign on \([0, 1]\). If \(f^{(2\nu)}(x)\) and \(f^{(2\nu+2)}(x)\) also have the same sign then both terms on the left side of the last equation must also have the same sign. Therefore each of these terms must have the same sign as the right side and can not be larger, in absolute value, than this term.

Let us apply the Euler-Maclaurin formula (11.3.16) to approximate the integral

\[
\int_0^1 \frac{dx}{1 + x} = \ln 2.
\]

Here \(a = 0, b = 1\) and we will divide \([0, 1]\) into 10 equal parts so that \(n = 10, h = 0.1\). In the formula we will use two terms in addition to \(T_2\) and thus \(\nu = 3\):

\[
\int_0^1 \frac{dx}{1 + x} = T_n - \frac{h^2}{12} [f'(1) - f'(0)] + \frac{h^4}{720} [f^{(3)}(1) - f^{(3)}(0)].
\]

\[
T_n = (0.1) \left[ \frac{1}{2} \cdot \frac{1}{1} + \frac{1}{1.1} + \ldots + \frac{1}{1.9} + \frac{1}{2} \cdot \frac{1}{2} \right] = 0.693771403.
\]

\[
-\frac{h^2}{12} [f'(1) - f'(0)] = -\frac{1}{1200} \left[ 1 - \frac{1}{4} \right] = -0.000625.
\]

\[
\frac{h^4}{720} [f^{(3)}(1) - f^{(3)}(0)] = \frac{6 \times 10^{-4}}{720} \left[ 1 - \frac{1}{24} \right] = 0.000000781.
\]

Therefore the approximate value of the integral is

\[
0.693771403 - 0.000625 + 0.000000781 = 0.693147184.
\]

We now estimate the error in this approximation. We note that the derivatives \(f^{(6)}(x) = \frac{6!}{(1 + x)^7}\) and \(f^{(8)}(x) = \frac{8!}{(1 + x)^9}\) are both positive.
throughout \([0, 1]\) and therefore we can apply Theorem 3. The first neglected term in the formula is

\[
- \frac{h^6}{30240} [f^{(5)}(1) - f^{(5)}(0)] = - \frac{120 \times 10^{-6}}{30240} \left[1 - \frac{1}{64}\right] < -0.000000004.
\]

Therefore

\[
0.693147180 < \int_0^1 \frac{dx}{1 + x} < 0.693147184.
\]

We will now construct an Euler type formula for increasing the accuracy of the repeated Simpson's rule (6.3.10). To do this we first construct equation (11.3.6) for

\[
\int_a^{a+2h} f(x)dx = \frac{h}{3} [f(a) + 4f(a + h) + f(a + 2h)] + R(f). \quad (11.3.20)
\]

This formula is exact for all cubic polynomials and thus we must take \(m = 4\).

We again use (11.3.9) to calculate the coefficients \(C_i:\)

\[
C_i = \frac{(2h)^{i+3}}{(i + 4)!} \left\{ \int_a^{a+2h} B_{i+4} \left( \frac{t - a}{2h} \right) dt - \frac{h}{3} \left[ B_{i+4}(0) + 4B_{i+4} (\frac{1}{2}) + B_{i+4} (1) \right] \right\} = \frac{(2h)^{i+4}}{3(i + 4)!} (1 + 2^{-i-2}) B_{i+4}.
\]

Since \(B_{2k+1} = 0\) \((k = 1, 2, \ldots)\) then only the \(C_i\) with even subscripts will be nonzero.

To find \(R_s(f)\) we calculate \(L_s(t):\)

\[
L_s(t) = - \frac{(2h)^s+3}{(s + 4)!} \left\{ \int_a^{a+2h} \left[ B^*_s + 4 \left( \frac{x - t}{2h} \right) \right] - B_{s+4} \left( \frac{x - a}{2h} \right) \right\} dx - \frac{h}{3} \left[ B^*_s + 4 \left( \frac{a - t}{2h} \right) - B_{s+4}(0) + 4 \left[ B^*_s + 4 \left( \frac{a + h - t}{2h} \right) - B_{s+4} (\frac{1}{2}) \right] + B^*_s + 4 \left( 1 + \frac{a - t}{2h} \right) - B_{s+4}(1) \right\}.
\]
11.3. Euler's Method for Expanding the Remainder

The integral in this expression is zero and replacing the function $B^*_s + 4(x)$ by the function $\gamma^*_s + 4(x) = B^*_s + 4(x) - B_s + 4$ we obtain

$$L_s(t) = \frac{(2h)^{s+4}}{3(s+4)!} \left\{ \left( \gamma^*_s + 4 \left( \frac{2h + a - t}{2h} \right) + 2 \left[ \gamma^*_s + 4 \left( \frac{h + a - t}{2h} \right) - \gamma_s + 4 \left( \frac{1}{2} \right) \right] \right\}.$$  

Substituting $t = a + 2hu$ ($0 \leq u \leq 1$) we can write the remainder $R_s(f)$ in the form

$$R_s(f) = \frac{(2h)^{s+5}}{3(s+4)!} \int_0^1 f(s+4)(a + 2hu) \times$$

$$\times \left\{ \gamma_s + 4(1 - u) + 2 \left[ \gamma^*_s + 4 \left( \frac{1}{2} - u \right) - \gamma_s + 4 \left( \frac{1}{2} \right) \right] \right\} \, du.$$  

Thus Simpson's formula (11.3.20) can be written as

$$\int_a^{a+2h} f(x) \, dx = \frac{h}{3} [f(a) + 4f(a + h) + f(a + 2h)] +$$

$$+ \sum_{k=2}^{\nu-1} \frac{(2h)^{2k}}{(2k)!} (1 - 2^{-2k+2}) B_{2k} \times$$

$$\times [f^{(2k-1)}(a + 2hu) - f^{(2k-1)}(a)] + \rho_{2\nu}(f). \quad (11.3.20*)$$  

where the remainder $\rho_{2\nu}(f)$ can be written as either

$$\rho_{2\nu}(f) = \frac{(2h)^{2\nu}}{3(2\nu - 1)!} \int_0^1 f^{(2\nu-1)}(a + 2hu) \times$$

$$\times \left\{ \gamma_{2\nu-1}(1 - u) + 2 \gamma^*_{2\nu-1} \left( \frac{1}{2} - u \right) \right\} \, du.$$  

or

$$\rho_{2\nu}(f) = \frac{(2h)^{2\nu+1}}{3(2\nu)!} \int_0^1 f^{(2\nu)}(a + 2hu) \times$$

$$\times \left\{ \gamma_{2\nu}(u) + 2 \left[ \gamma^*_{2\nu} \left( \frac{1}{2} - u \right) - \gamma_{2\nu} \left( \frac{1}{2} \right) \right] \right\} \, du.$$  

depending on whether $f(x)$ has a continuous derivative of order $2\nu - 1$ or $2\nu$. Below we will assume that $f^{(2\nu)}(x)$ exists and is continuous and will, therefore, use the second expression for the remainder.
Let us now consider the repeated Simpson's rule (6.3.10). We divide
the segment of integration \([a, b]\) into an even number \(n\) of equal sub-
segments and apply (11.3.20*) to the double segment \([a + 2hp, a + 2h(p + 1)]\). Writing these equations for \(p = 0, 1, \ldots, \frac{n}{2} - 1\) and
adding we obtain

\[
\int_a^b f(x)dx = U_n + \sum_{k=2}^{n-1} \frac{(2h)^{2k}}{3(2k)!} (1 - 2^{-2k+2}) B_{2k} \times
\]

\[
\times [f^{(2k-1)}(b) - f^{(2k-1)}(a)] + \rho_{2\nu}(f) =
\]

\[
= U_n - \frac{h^4}{180} [f^{(3)}(b) - f^{(3)}(a)] + \frac{h^6}{1512} [f^{(5)}(b) - f^{(5)}(a)] -
\]

\[
- \frac{h^8}{14400} [f^{(7)}(b) - f^{(7)}(a)] + \cdots + \rho_{2\nu}(f) \quad (11.3.21)
\]

where

\[
U_n = \frac{h}{3} [f(a) + f(b) + 2[f(a + 2h) + \cdots + f(a + (n - 2)h)] +
\]

\[
+ 4[f(a + h) + \cdots + f(a + (n - 1)h)]
\]

\[
\rho_{2\nu}(f) = \frac{(2h)^{2\nu+1}}{3(2\nu)!} \int_0^1 \left\{ \gamma_{2\nu}(u) + 2 \left[ \gamma^*_{2\nu} \left( \frac{1}{2} - u \right) - \gamma_{2\nu} \left( \frac{1}{2} \right) \right] \right\} \times
\]

\[
\times \sum_{p=0}^{\frac{n}{2}-1} f^{(2\nu)}(a + 2ph + 2hu)du. \quad (11.3.22)
\]

In order to study the remainder \(\rho_{2\nu}(f)\) it will be necessary to investi-
gate the kernel

\[
F(u) = \gamma_{2\nu}(u) + 2 \left[ \gamma^*_{2\nu} \left( \frac{1}{2} - u \right) - \gamma_{2\nu} \left( \frac{1}{2} \right) \right].
\]

To do this we need the following Lemma.

**Lemma.** For each \(\nu \geq 1\) the function

\[
\phi_{2\nu+1}(x) = \gamma_{2\nu+1}(x) - 2\gamma_{2\nu+1} \left( \frac{1}{2} - x \right)
\]

has no zeros inside the segment \([0, \frac{1}{2}]\) and the sign of this function is
11.3. Euler's Method for Expanding the Remainder

given by

\[ (-1)^\nu \phi_{2\nu+1}(x) > 0, \quad 0 < x < \frac{1}{2}. \]

**Proof.** Assume \( \nu \geq 1 \). Since \( \gamma_{2\nu+1}(0) = \gamma_{2\nu+1}\left(\frac{1}{2}\right) = 0 \) then it is clear that \( x = 0 \) and \( x = \frac{1}{2} \) are zeros of \( \phi_{2\nu+1}(x) \). Let us suppose that the point \( a \left(0 < a < \frac{1}{2}\right) \) was also a zero of \( \phi_{2\nu+1}(x) \). Then inside each of the segments \([0, a] \) and \([a, \frac{1}{2}] \) the derivative \( \phi'_{2\nu+1}(x) \) will have at least one zero. Therefore the second derivative \( \phi''_{2\nu+1}(x) \) will have at least one zero inside \([0, \frac{1}{2}] \). But

\[
\phi''_{2\nu+1}(x) = (2\nu + 1)(2\nu) \left[ \gamma_{2\nu-1}(x) - 2\gamma_{2\nu-1}\left(\frac{1}{2} - x\right) \right] = (2\nu + 1)(2\nu) \phi_{2\nu-1}(x).
\]

Thus, from the assumption that \( \phi_{2\nu+1}(x) \) has a zero inside \([0, \frac{1}{2}] \) it follows that \( \phi_{2\nu-1}(x) \) also has a zero inside this segment. From this it follows that \( \phi_3(x) \) would have a zero inside \([0, \frac{1}{2}] \). However, we can easily verify that \( \phi_3(x) = 3x^2 \left(x - \frac{1}{2}\right) \) and this function has no zeros inside \([0, \frac{1}{2}] \). To determine the sign of \( \phi_{2\nu+1}(x) \) it is sufficient to determine the sign of \( \phi_{2\nu+1}\left(\frac{1}{4}\right) \):

\[
\phi_{2\nu+1}\left(\frac{1}{4}\right) = -\gamma_{2\nu+1}\left(\frac{1}{4}\right)
\]

and in Section 1.2 we showed that

\[ (-1)^{\nu+1} \gamma_{2\nu+1}(x) > 0 \quad \text{for} \quad 0 < x < \frac{1}{2}. \]

Therefore

\[ (-1)^\nu \phi_{2\nu+1}(x) > 0, \quad 0 < x < \frac{1}{2}. \]
This proves the lemma.

Let us now consider the function $(-1)^{\nu-1} F(u)$ for $0 \leq u \leq \frac{1}{2}$. We have

$$(-1)^{\nu-1} F'(u) = 2\nu (-1)^{\nu-1} \left[ \gamma_{2\nu-1}(u) - 2\gamma_{2\nu-1}\left(\frac{1}{2} - u\right) \right] = 2\nu (-1)^{\nu-1} \phi_{2\nu-1}(u).$$

By the Lemma we see that $(-1)^{\nu-1} F'(u) > 0$ which means that $(-1)^{\nu-1} F(u)$ is a monotonically increasing function for $0 \leq u \leq \frac{1}{2}$.

Since $F(0) = 0$ it follows that $(-1)^{\nu-1} F(u) > 0$ for $0 < u \leq \frac{1}{2}$.

In order to see how $F(u)$ behaves on $\frac{1}{2} \leq u \leq 1$ it will be sufficient to show that $F(u)$ is symmetric with respect to $u = \frac{1}{2}$: $F(1 - u) = F(u)$.

Indeed

$$F(1 - u) = \gamma_{2\nu}(1 - u) + 2 \left[ \gamma_{2\nu}^{*}\left(u - \frac{1}{2}\right) - \gamma_{2\nu}\left(\frac{1}{2}\right) \right]$$

$$\gamma_{2\nu}(1 - u) = \gamma_{2\nu}(u)$$

$$\gamma_{2\nu}^{*}\left(u - \frac{1}{2}\right) = \gamma_{2\nu}^{*}\left(u + \frac{1}{2}\right) = \gamma_{2\nu}^{*}\left(\frac{1}{2} - u\right)$$

from which we see that $F(1 - u) = F(u)$.

Thus it follows that $(-1)^{\nu-1} F(u)$ is a positive monotonically decreasing function on $\frac{1}{2} \leq u < 1$ and that this function has a relative maximum at $u = \frac{1}{2}$:

$$\max_{[0, 1]} (-1)^{\nu-1} F(u) = -(-1)^{\nu-1} \gamma_{2\nu}\left(\frac{1}{2}\right) = (-1)^{\nu-1} 2^{-2} (1 - 2^{-2\nu}) B_{2\nu} \quad (11.3.23)$$

These properties of the kernel $F(u)$ permit us to prove three theorems about the remainder $\rho_{2\nu}(f)$ of (11.3.22) analogous to the theorems about the remainder of (11.3.16).

We omit the proofs of these theorems because the proofs exactly follow the proofs of the preceding theorems.
11.3. Euler's Method for Expanding the Remainder

Theorem 4. If \( f(x) \) has a continuous derivative of order 2\( \nu \) on \([a, b]\) then there exists a point \( \xi \in [a, b] \) for which the remainder of (11.3.21) satisfies

\[
\rho_{2\nu}(f) = \frac{n h (2h)^{2\nu}}{3(2\nu)!} (1 - 2^{-2\nu+2}) B_{2\nu} f^{(2\nu)}(\xi).
\] (11.3.24)

Comparing this representation for the remainder of (11.3.21) with the representation (11.3.18) for the remainder of the Euler-Maclaurin formula (11.3.16) we see that if \( f^{(2\nu)}(x) \) does not change sign on \([a, b]\) then the remainders of these two quadrature formulas have opposite signs. Hence we have the following useful rule:

If the derivative \( f^{(2\nu)}(x) \) does not change sign on \([a, b]\) then the exact value of \( \int_a^b f(x) dx \) lies between the approximate values obtained from (11.3.16) and (11.3.21) by neglecting the remainder terms \( \rho_{2\nu}(f) \) in these equations.

Theorem 5. If the derivative \( f^{(2\nu)}(x) \) does not change sign on \([a, b]\) then the remainder \( \rho_{2\nu}(f) \) of formula (11.3.21) can be written in the form

\[
\rho_{2\nu}(f) = 2 \theta \frac{(2h)^{2\nu}}{3(2\nu)!} (1 - 2^{-2\nu}) B_{2\nu} [f^{(2\nu-1)}(b) - f^{(2\nu-1)}(a)]
\] (11.3.25)

0 < \( \theta \) < 1.

From (11.3.25) we see that the remainder \( \rho_{2\nu}(f) \) has the same sign as the first neglected term of (11.3.21).

Theorem 6. If the function \( f(x) \) has a continuous derivative of order 2\( \nu + 2 \) and for all \( x \) in \([a, b]\) either

\[
f^{(2\nu)}(x) \geq 0 \quad \text{and} \quad f^{(2\nu+2)}(x) \geq 0
\]
or

\[
f^{(2\nu)}(x) \leq 0 \quad \text{and} \quad f^{(2\nu+2)}(x) \leq 0
\]

then the remainder \( \rho_{2\nu}(f) \) of (11.3.21) has the same sign as the first neglected term

\[
\frac{(2h)^{2\nu}}{3(2\nu)!} (1 - 2^{-2\nu+2}) B_{2\nu} [f^{(2\nu-1)}(b) - f^{(2\nu-1)}(a)]
\]

and is not greater, in absolute value, than this term.

We now give the series in (11.3.6) for increasing the precision of certain other special quadrature formulas.

1. The Newton-Cotes formula with 4 nodes:
\[
\int_a^{a+3h} f(x)\,dx = \frac{3h}{8} [f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)] +
\]
\[
+ \sum_{k=2}^{\infty} \frac{(3h)^{2k}}{8(2k)!} (1 - 3^{-2k+2}) B_{2k} [f^{(2k-1)}(a + 3h) - f^{(2k-1)}(a)] =
\]
\[
= \frac{3h}{8} [f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)] -
\]
\[
- \frac{h^4}{80} [f^{(3)}(a + 3h) - f^{(3)}(a)] + \frac{h^6}{336} [f^{(5)}(a + 3h) - f^{(5)}(a)] -
\]
\[
- \frac{13h^8}{19200} [f^{(7)}(a + 3h) - f^{(7)}(a)] + \ldots .
\]

(11.3.26)

2. The quadrature formula of the highest degree of precision \(2n-1\) for the segment \([-1, 1]\) and the weight function \((1 - x)^\alpha (1 + x)^\beta (\alpha, \beta > -1)\); the nodes are the zeros of the Jacobi polynomial \(P_n^{(\alpha, \beta)}(x)\):

\[
\int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta f(x)\,dx = \sum_{k=1}^{n} A_k f(x_k) +
\]
\[
+ C_0 [f^{(2n-1)}(1) - f^{(2n-1)}(-1)] + C_1 [f^{(2n)}(1) - f^{(2n)}(-1)] + \ldots
\]
\[
C_0 = \frac{2^{\alpha+\beta+2n}n! \Gamma(\alpha + \beta + n + 1) \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{(2n)! (\alpha + \beta + 2n + 1) \Gamma(\alpha + \beta + 2n + 1)^2}
\]
\[
C_1 = \frac{\beta - \alpha}{\alpha + \beta + 2n} \left[ \frac{\alpha + \beta}{\alpha + \beta + 2n + 2} + 2n \right] \times
\]
\[
\times \frac{n! 2^{\alpha+\beta+2n} \Gamma(\alpha + \beta + n + 1) \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{(2n + 1)! \Gamma(\alpha + \beta + 2n + 1) \Gamma(\alpha + \beta + 2n + 2)}.
\]

For the special ultraspherical case, \(\alpha = \beta\), the \(C_i\) with odd subscripts are zero:

\[
C_0 = \frac{2^{2\alpha n}n! \Gamma(2\alpha + n + 1) \left[ \frac{2^n \Gamma(\alpha + n + 1)}{\Gamma(2\alpha + 2n + 1)} \right]^2}{(2\alpha + 2n + 1) \Gamma(2\alpha + 2n + 1)}
\]
\[
C_2 = \frac{2^{2\alpha n}n! \Gamma(2\alpha + n + 1)}{(2\alpha + 2n + 1) \Gamma(2\alpha + 2n + 1)} \left[ \frac{2^n \Gamma(\alpha + n + 1)}{\Gamma(2\alpha + 2n + 1)} \right]^2 \times
\]
\[
\times \frac{2n^2 + 2 (2\alpha + 1) n + 2\alpha - 1}{(2\alpha + 2n - 1) (2\alpha + 2n + 1) (2\alpha + 2n + 3)} + \frac{n (n - 1)}{(2\alpha + 2n - 1) (2\alpha + 2n + 1)} -
\]
\[
- \frac{(n + 1) (2\alpha + 2n)}{3 (2\alpha + 2n + 1)}.
\]
3. For the Gauss formulas \( \alpha = \beta = 0 \) and the nodes are the zeros of the \( n \)th degree Legendre polynomial:

\[
\int_{-1}^{1} f(x) dx = \sum_{k=1}^{n} A_k f(x_k) + \frac{1}{(2n + 1)!} \left[ \frac{2^n (n!)^2}{(2n)!} \right]^2 \times \\
\times \left[ f^{(2n-1)}(1) - f^{(2n-1)}(-1) \right] + \\
+ \frac{1}{(2n + 2)!} \left[ \frac{2^n (n!)^2}{(2n)!} \right]^2 \left[ -n \left( \frac{4n^2 + 5n - 2}{3(2n - 1)(2n + 3)} \right) \right] \times \\
\times \left[ f^{(2n+1)}(1) - f^{(2n+1)}(-1) \right] + \ldots.
\]

To apply formulas of Euler's type it is necessary to find the values of the derivative of the integrand at the ends of the segment \([a, b]\) and in many cases this may be difficult to do. We can construct other formulas for increasing the precision of quadrature formulas in which the corrective terms are expressed in terms of differences or values of the integrand in place of its derivatives. There can be a wide variety of such formulas since the derivatives can be replaced by finite differences in many different ways. As an illustration we show one example of how this may be done.

Suppose we wish to calculate derivatives of \( f(x) \) at the point \( a \). To do this we interpolate on \( f(x) \) using its values at certain points. The form of the interpolating polynomial depends on the choice of the nodes and we will assume that we can only use the values of \( f(x) \) at the equally spaced points \( a + kh \ (k = 0, 1, \ldots) \) which belong to the segment \([a, b]\).

We will use Newton's representation of the interpolating polynomial with the nodes \( x_0 = a, x_1 = a + h, x_2 = a + 2h, \ldots \):

\[
f(x) = f(a) + (x - a) f(a, a + h) + \\
+ (x - a)(x - a - h) f(a, a + h, a + 2h) + \cdots + r(x)
\]

where \( r(x) \) is the remainder of the interpolation. For equally spaced nodes the divided differences are easily expressed in terms of differences:

\[
f(a, a + h, \ldots, a + kh) = h^k k! \Delta^k f_0, \quad f_0 = f(a).
\]

Making the change of variable \( x = a + ht \) we obtain the well known Newton-Gregory formula which is useful for interpolating near the beginning of a table

\[
f(a + ht) = f_0 + \frac{t}{1!} \Delta f_0 + \frac{t(t - 1)}{2!} \Delta^2 f_0 + \\
+ \frac{t(t - 1)(t - 2)}{3!} \Delta^3 f_0 + \ldots.
\]
Taking derivatives of both sides of this equation and setting $t = 0$ gives

\[
hf'(a) = \Delta f_0 - \frac{1}{2} \Delta^2 f_0 + \frac{1}{3} \Delta^3 f_0 - \frac{1}{4} \Delta^4 f_0 + \frac{1}{5} \Delta^5 f_0 - \ldots
\]

\[
h^2 f''(a) = \Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12} \Delta^4 f_0 - \frac{5}{6} \Delta^5 f_0 + \ldots
\]

\[
h^3 f'''(a) = \Delta^3 f_0 - \frac{3}{2} \Delta^4 f_0 + \frac{7}{4} \Delta^5 f_0 - \ldots
\]

\[
h^4 f^{(4)}(a) = \Delta^4 f_0 - 2\Delta^5 f_0 + \ldots
\]

\[
h^5 f^{(5)}(a) = \Delta^5 f_0 - \ldots
\]

\[
\Delta^k f^{(k)}(a) = \Delta^k f_0 - \ldots
\]

In a similar way we can find the values of the derivatives $f^{(k)}(b)$ ($k = 1, 2, \ldots$) by interpolating on $f(x)$ close to $b = a + nh$. We use the same representation for the interpolating polynomial and set

\[
x_0 = a + nh, \ x_1 = a + (n - 1)h, \ x_2 = a + (n - 2)h, \ldots:
\]

\[
f(x) = f(a + nh) + (x - a - nh)f(a + nh, a + (n - 1)h) + \]

\[
+ (x - a - nh)(x - a - (n - 1)h)f(a + nh, a + (n - 1)h, a + (n - 2)h) + \]

\[
+ \cdots + r(x)
\]

or

\[
f(a + th) = f_n + \frac{t}{1!} \Delta f_{n-1} + \frac{t(t + 1)}{2!} \Delta^2 f_{n-2} + \]

\[
+ \frac{t(t + 1)(t + 2)}{3!} \Delta^3 f_{n-3} + \ldots.
\]

Thus we obtain

\[
hf'(b) = \Delta f_{n-1} + \frac{1}{2} \Delta^2 f_{n-2} + \frac{1}{3} \Delta^3 f_{n-3} + \frac{1}{4} \Delta^4 f_{n-4} + \frac{1}{5} \Delta^5 f_{n-5} + \ldots
\]

\[
h^2 f''(b) = \Delta^2 f_{n-2} + \Delta^3 f_{n-3} + \frac{11}{12} \Delta^4 f_{n-4} + \frac{5}{6} \Delta^5 f_{n-5} + \ldots
\]

\[
h^3 f'''(b) = \Delta^3 f_{n-3} + \frac{3}{2} \Delta^4 f_{n-4} + \frac{7}{4} \Delta^5 f_{n-5} + \ldots
\]

\[
h^4 f^{(4)}(b) = \Delta^4 f_{n-4} + 2\Delta^5 f_{n-5} + \ldots
\]

\[
h^5 f^{(5)}(b) = \Delta^5 f_{n-5} + \ldots
\]

\[
\Delta^k f^{(k)}(b) = \Delta^k f_{n-k} + \ldots
\]

\[
\Delta^k f^{(k)}(b) = \Delta^k f_0 - \ldots
\]
Suppose we wish to replace the derivatives in the Euler-Maclaurin formula (11.3.16) by finite differences. If we substitute the above expressions for the derivatives we obtain the Gregory formula:

\[
\int_a^{a+nh} f(x) \, dx = T_n - \frac{h}{12} (\Delta f_{n-1} - \Delta f_0) - \frac{h}{24} (\Delta^2 f_{n-2} + \Delta^2 f_0) - \frac{19h}{720} (\Delta^3 f_{n-3} - \Delta^3 f_0) - \frac{3h}{160} (\Delta^4 f_{n-4} + \Delta^4 f_0) - \frac{63h}{60480} (\Delta^5 f_{n-5} - \Delta^5 f_0) - \frac{275h}{24192} (\Delta^6 f_{n-6} + \Delta^6 f_0) - \cdots - C_k h [\Delta^k f_{n-k} + (-1)^k \Delta^k f_0] + R_1(f)
\]

(11.3.27)

where it can be shown that

\[
C_k = \frac{(-1)^k}{(k+1)!} \int_0^1 x(x-1) \cdots (x-k) \, dx.
\]

11.4. INCREASING THE PRECISION WHEN THE INTEGRAL REPRESENTATION OF THE REMAINDER CONTAINS A SHORT PRINCIPLE SUBINTERVAL

As in the preceding section we will consider a mechanical quadrature formula for an arbitrary weight function

\[
\int_a^b p(x) f(x) \, dx = \sum_{k=1}^n A_k f(x_k) + R(f).
\]

(11.4.1)

If the algebraic degree of precision of (11.4.1) is \(m - 1\) and if \(f(x)\) has a continuous derivative of order \(m\) on \([a, b]\) then, as shown in the preceding section, \(R(f)\) can in many cases be written in the form

\[
R(f) = \int_a^b f^{(m)}(x) K(x) \, dx
\]

(11.4.2)

where the kernel \(K(x)\) is independent of \(f(x)\).

Let us assume that in \([a, b]\) there exists a subinterval \([\alpha, \beta]\) outside of which \(K(x)\) has a negligibly small value or that \(K(x)\) rapidly becomes small away from \([\alpha, \beta]\). Then the value of the integral (11.4.2) will be mainly due to its value on \([\alpha, \beta]\). In addition let \(f^{(m)}(x)\) have "small variation" on \([\alpha, \beta]\) or, what is essentially the same, let \([\alpha, \beta]\) have a relatively small length. In order to remove the principle part of the remainder \(R(f)\) let us assume that it suffices to write \(f^{(m)}(x)\) as the
Approximate Calculation of Definite Integrals

The sum of two terms

\[ f^{(m)}(x) = f^{(m)}(a_0) + [f^{(m)}(x) - f^{(m)}(a_0)] \]

where \( a_0 \) is a point in the subsegment \([a, \beta]\):

\[ R(f) = f^{(m)}(a_0) \int_a^b K(x) \, dx + \int_a^b [f^{(m)}(x) - f^{(m)}(a_0)] K(x) \, dx. \]

The choice of the point \( a_0 \) is still arbitrary. When the kernel does not change sign it is natural to take \( a_0 \) as the point of the \( x \)-axis about which \( K(x) \) is concentrated:

\[ a_0 = \frac{\int_a^b xK(x) \, dx}{\int_a^b K(x) \, dx}. \]

Let us suppose that \( f(x) \) has a continuous derivative of order \( m + 2s \). We transform the last expression for \( R(f) \) by expanding \( f^{(m)}(x) - f^{(m)}(a_0) \) in a Taylor series with two terms

\[ f^{(m)}(x) - f^{(m)}(a_0) = f^{(m+1)}(a_0)(x - a_0) + \int_a^x f^{(m+2)}(t)(x - t) \, dt = \]

\[ = f^{(m+1)}(a_0)(x - a_0) + \int_a^b f^{(m+2)}(t) \times \]

\[ \times [E(x - t) - E(a_0 - t)] (x - t) \, dt. \]

We substitute this expression for \( f^{(m)}(x) - f^{(m)}(a_0) \) into the expression for \( R(f) \) and integrate. By our choice of \( a_0 \), \( \int_a^b K(t)(t - a_0) \, dt = 0 \) and

\[ R(f) = C_0 f^{(m)}(a_0) + \int_a^b f^{(m+2)}(t)K_1(t) \, dt \]

\[ C_0 = \int_a^b K(x) \, dx, \quad K_1(t) = \int_a^b K(x) [E(x - t) - E(a_0 - t)] (x - t) \, dx. \]

If we perform this transformation \( s \) times we obtain the following formula which is sometimes useful for sequentially increasing the precision of a quadrature formula:
\[ \int_a^b p(x) f(x) \, dx = \sum_{k=1}^n A_k f(x_k) + C_0 f^{(m)}(x_0) + C_1 f^{(m+2)}(x_1) + \cdots + C_{s-1} f^{(m+2s-2)}(x_{s-1}) + \int_a^b f^{(m+2s)}(x) K_s(x) \, dx \] (11.4.3)

\[ K_0(x) = K(x), \quad K_{i+1}(x) = \int_a^b K_i(t) [E(t - x) - E(\alpha_i - x)] (t - x) \, dt \]

\[ C_i = \int_a^b K_i(x) \, dx, \quad \alpha_i = C_i^{-1} \int_a^b xK_i(x) \, dx. \]

The above expression for \( K_{i+1}(x) \) can be written as

\[ K_{i+1}(x) = \begin{cases} \int_a^x K_i(t)(x - t) \, dt & \text{if } a \leq x < \alpha_i \\ \int_x^b K_i(t)(t - x) \, dt & \text{if } \alpha_i < x \leq b. \end{cases} \]

Thus we see that if \( K_i(x) \) does not change sign on \( a \leq x < b \) then \( K_{i+1}(x) \) also does not change sign on this interval and \( K_{i+1}(x) \) has the same sign as \( K_i(x) \). In particular if the initial kernel \( K(x) \) of the remainder (11.4.2) is positive throughout \([a, b]\) then all the kernels \( K_i(x) \), \( i = 1, 2, \ldots \), will also be positive throughout \([a, b]\).

Let us now consider the quadrature formula with \( n \) nodes with the highest algebraic degree of precision \( 2n - 1 \) for the weight function \((1 - x)^p (1 + x)^q\), \( p, q > -1 \):

\[ \int_{-1}^1 (1 - x)^p (1 + x)^q f(x) \, dx = \sum_{k=1}^n A_k f(x_k) + R(f). \] (11.4.4)

The nodes of this formula are the zeros of the Jacobi polynomial \( P_n^{(p, q)}(x) \). We will assume that these nodes are innumerated in increasing order: \(-1 < x_1 < \cdots < x_n < 1\).

First we show that when \( p \) or \( q \) is large there is a principle subinterval \([\alpha, \beta]\) for the integral representation of \( R(f) \). We will show this by constructing an electrostatic analogy for the roots of \( P_n^{(p, q)}(x) \).
It is known that \( P_{n}^{(p, q)}(x) \) satisfies the differential equation\(^2\)

\[
\frac{d^2}{dx^2} P_{n}^{(p, q)}(x) + \left( \frac{p + 1}{x - 1} + \frac{q + 1}{x + 1} \right) \frac{d}{dx} P_{n}^{(p, q)}(x) + \frac{n(p + q + n + 1)}{1 - x^2} P_n^{(p, q)}(x) = 0.
\]

Substituting \( x = x_k \) makes the third term on the left side vanish and since \( \frac{d}{dx} P_{n}^{(p, q)}(x_k) \neq 0 \) we can divide by this term to obtain

\[
\sum_{i \neq k} \frac{2}{x_k - x_i} + \frac{p + 1}{x_k - 1} + \frac{q + 1}{x_k + 1} = 0 \quad (k = 1, 2, \ldots, n).
\]

This equation has a simple physical interpretation.

Consider a planar electrostatic field in which particles with like charges are repelled with a force proportional to their charge and inversely proportional to the distance between them. If two particles with charges \( m_1 \) and \( m_2 \) lie on the \( x \)-axis at the points \( x_1 \) and \( x_2 \) then the force which one particle exerts on the other is

\[
\frac{\lambda m_1 m_2}{x_2 - x_1}.
\]

Let particles with charges of \( p + 1 \) and \( q + 1 \) be fixed at the respective points \( x = +1 \) and \( x = -1 \). In addition we place \( n \) particles of charge 2 inside the segment \([-1, 1]\) and assume that these are only free to move along the \( x \)-axis. Let \( x_k \) \((k = 1, 2, \ldots, n)\) denote the coordinates of the free particles. If the free particles are at equilibrium then the force on each free particle is zero. Thus for the particle at \( x_k \)

\[
\sum_{i \neq k} \frac{4\lambda}{x_k - x_i} + \frac{2\lambda(p + 1)}{x_k - 1} + \frac{2\lambda(q + 1)}{x_k + 1} = 0 \quad (k = 1, 2, \ldots, n).
\]

Thus the equations of equilibrium differ from \( (11.4.5) \) by only the multiple \( 2\lambda \) and the position of these particles will coincide with the zeros of the Jacobi polynomial \( P_n^{(p, q)}(x) \).

When the charges \( p + 1 \) and \( q + 1 \) of the fixed particles are large they will strongly repel the free particles and will force them to concentrate in a “small” subinterval so that \([x_1, x_n]\) will have a relatively small length. When \( p \) is significantly larger than \( q \) the interval \([x_1, x_n]\) will be close to \(-1\); conversely if \( q \) is significantly larger than \( p \) the interval \([x_1, x_n]\) will be close to \(+1\).

11.4. Short Principle Subinterval

The remainder \( R(f) \) of the quadrature formula (11.4.4) has a representation of the form (11.4.2). In general the kernel is given by (11.3.5) which in the present case is

\[
K(x) = \int_x^1 (1-t)^p (1+t)^q \frac{(t-x)^2}{(2n-1)!} dt - \sum_{k=1}^n A_k E(x_k - x) \frac{(x_k - x)^{2n-1}}{(2n-1)!}.
\]

In particular for a point \( x \) outside \([x_1, x_n]\):

\[
K(x) = \int_{-1}^x (1-t)^p (1+t)^q \frac{(x-t)^{2n-1}}{(2n-1)!} dt \quad \text{for} \quad -1 \leq x \leq x_1
\]

\[
K(x) = \int_x^1 (1-t)^p (1+t)^q \frac{(t-x)^{2n-1}}{(2n-1)!} dt \quad \text{for} \quad x_n \leq x \leq 1.
\]

Consider, for example, the case \( x_n \leq x < 1 \). The factor \( 1+t \) lies between the limits \( 1+x_n \leq 1+t \leq 2 \). Therefore \( K(x) \) is greater than

\[
(1+x_n)^q \int_x^1 (1-t)^p \frac{(t-x)^{2n-1}}{(2n-1)!} dt = (1+x_n)^q \int_0^1 (1-u)^p u^{2n-1} du = (1+x_n)^q C (1-x)^p+2n
\]

and less than

\[
2^q \int_x^1 (1-t)^p \frac{(t-x)^{2n-1}}{(2n-1)!} dt = 2^q C (1-x)^p+2n.
\]

As \( x \) increases from \( x_n \) up to 1 the kernel \( K(x) \) approaches zero as \( (1-x)^p+2n \). If \( p \) is large \( K(x) \) will approach zero very rapidly. If \( p \) is not large but if \( q \) is large then \( x_n \) will be close to unity and \( (1-x)^p+2n \) will again be a small number.

From this discussion we can expect that the method outlined above can be used to increase the accuracy of formula (11.4.4) when \( p \) or \( q \) is large.

Formula (11.4.3) for the quadrature formula (11.4.4) is:

\[
\int_{-1}^1 (1-x)^p (1+x)^q f(x) dx = \sum_{k=1}^n A_k f(x_k) + C_0 f^{(2n)}(a_0) +
\]

\[
+ C_1 f^{(2n+2)}(a_1) + \cdots + C_{s-1} f^{(2n+2s-2)}(a_{s-1}) +
\]

\[
+ \int_{-1}^1 f^{(2n+2s)}(x) K_s(x) dx.
\]
The coefficient \( C_0 = \int_{-1}^{1} K(x) \, dx \) of the first corrective term is the remainder when the quadrature formula is applied to a function for which \( f^{(2n)}(x) = 1 \). We can take this function to be

\[
 f(x) = \frac{1}{(2n)! a_n^2} \left[ P_n^{(p,q)}(x) \right]^2
\]

where \( a_n \) is the leading coefficient of \( P_n^{(p,q)}(x) \). Since \( P_n^{(p,q)}(x_k) = 0 \) then

\[
 C_0 = R(f) = \frac{1}{(2n)! a_n^2} \left\{ \int_{-1}^{1} (1 - x)^p(1 + x)^q [P_n^{(p,q)}(x)]^2 \, dx - \sum_{k=1}^{n} A_k [P_n^{(p,q)}(x_k)]^2 \right\} = \frac{1}{(2n)! a_n^2} \int_{-1}^{1} (1 - x)^p(1 + x)^q [P_n^{(p,q)}(x)]^2 \, dx.
\]

The value of \( a_n \) is given by (2.2.2) and the value of the integral is (2.2.5) and thus

\[
 C_0 = \frac{2^p + q + 2n + 1}{(2n)! \Gamma(p + n + 1) \Gamma(q + n + 1)} \frac{\Gamma(p + q + n + 1)}{\Gamma(p + q + 2n + 1) \Gamma(p + q + 2n + 2)}.
\]

We now calculate \( \alpha_0 \). We note that the integral in the expression \( \alpha_0 = C_0^{-1} \int_{-1}^{1} x K(x) \, dx \) is the remainder when formula (11.4.4) is applied to a function \( f(x) \) which has a derivative of order \( 2n \) equal to \( x \). Writing

\[
 P_n^{(p,q)}(x) = a_n x^n + b_n x^{n-1} + \ldots
\]

we see that we can take \( f(x) \) to be

\[
 f(x) = \frac{1}{(2n + 1)! a_n^2} \left[ x - \frac{2b_n}{a_n^2} \right] [P_n^{(p,q)}(x)]^2.
\]

Since \( f(x_k) = 0 \) we have

\[
 \int_{-1}^{1} x K(x) \, dx = R(f) =
\]
\[
\frac{1}{(2n + 1)! a_n^2} \int_{-1}^{1} (1 - x)^p (1 + x)^q x [P_n^{(p, q)}(x)]^2 \, dx - \frac{2b_n}{a_n} \int_{-1}^{1} (1 - x)^p (1 + x)^q [P_n^{(p, q)}(x)]^2 \, dx \right\}.
\]

(11.4.7)

We have found that the second integral inside the brackets is \((2n)! a_n^2 C_0\). The coefficients \(a_n\) and \(b_n\) of the Jacobi polynomial are known to have the values

\[
a_n = \frac{\Gamma(p + q + 2n + 1)}{2^n n! \Gamma(p + q + n + 1)} \quad \text{and} \quad b_n = \frac{n(p - q)\Gamma(p + q + 2n)}{2^n n! \Gamma(p + q + n + 1)}.
\]

Therefore

\[
\frac{b_n}{a_n} = \frac{n(p - q)}{p + q + 2n}.
\]

We now calculate the first integral inside the brackets in (11.4.7). The following recursion relation is known for Jacobi polynomials

\[
(p + q + 2n)(p + q + 2n + 1)(p + q + 2n + 2)x P_n^{(p, q)}(x) = 2(n + 1)(p + q + n + 1)(p + q + 2n)P_{n+1}^{(p, q)}(x) + (q^2 - p^2)(p + q + 2n + 1)P_n^{(p, q)}(x) + 2(p + n)(q + n)(p + q + 2n + 2)P_{n-1}^{(p, q)}(x).
\]

We multiply both sides of this equation by \((1 - x)^p (1 + x)^q P_n^{(p, q)}(x)\) and integrate over \([-1, 1]\) and thus obtain

\[
\int_{-1}^{1} (1 - x)^p (1 + x)^q x [P_n^{(p, q)}(x)]^2 \, dx = \frac{q^2 - p^2}{(p + q + 2n)(p + q + 2n + 2)} \int_{-1}^{1} (1 - x)^p (1 + x)^q [P_n^{(p, q)}(x)]^2 \, dx = \frac{(q^2 - p^2)(2n)! a_n C_0}{(p + q + 2n)(p + q + 2n + 2)}.
\]

---


Thus we obtain
\[ \int_{-1}^{1} x K(x) \, dx = \frac{q - p}{2n + 1} \left[ \frac{p + q}{(p + q + 2n)(p + q + 2n + 2)} + \frac{2n}{p + q + 2n} \right] C_0. \]

Rewriting this expression gives
\[ \alpha_0 = \frac{q - p}{2n + 1} \left[ \frac{n}{p + q + 2n} + \frac{n + 1}{p + q + 2n + 2} \right]. \]

In the special ultraspherical case \( p = q \) and the \( \alpha_k \) (\( k = 0, 1, \ldots \)) will be zero and formula (11.4.6) will be
\[
\int_{-1}^{1} (1 - x)^p f(x) \, dx = \sum_{k=1}^{n} A_k f(x_k) + \\
\frac{2^{2p+2n+1} n! [\Gamma(p + n + 1)]^2 \Gamma(2p + n + 1)}{(2n)! \Gamma(2p + 2n + 1) \Gamma(2p + 2n + 2)} \times \\
\times \left\{ f^{(2n)}(0) + \frac{1}{2(2n + 1)(2n + 2)} \times \\
\times \left[ \frac{(n + 1)(n + 2)}{2p + 2n + 3} + \frac{n(n - 1)}{2p + 2n - 1} \right] f^{(2n+2)}(0) + \ldots \right\}. 
\]

Suppose we wish to approximate the integral
\[
\int_{-1}^{1} (1 - x^2)^2 e^x \, dx = 8 e - 56 e^{-1} \approx 1.145006. 
\]

Here \( p = q = 2 \) and let us take \( n = 1 \) so that the formula has the form
\[
\int_{-1}^{1} (1 - x^2)^2 f(x) \, dx = A_1 f(x_1) + \frac{8}{105} \left[ f^{(2)}(0) + \frac{1}{36} f^{(4)}(0) + \ldots \right].
\]

Since the weight function is symmetric about the origin \( x_1 = 0 \) and
\[ A_1 = \int_{-1}^{1} (1 - x^2)^2 \, dx = \frac{16}{15} \approx 1.066667. \]

The formula with only the first term
\[ A_1 f(x_1) = \frac{16}{15} (1) \approx 1.066667. \]
11.4. Short Principle Subinterval

gives a very poor result. The first two terms give

\[ A_1 f(x_1) + \frac{8}{105} f^{(2)}(0) = \frac{16}{15} + \frac{8}{105} = \frac{8}{7} \approx 1.142857 \]

and three terms give

\[ A_1 f(x_1) + \frac{8}{105} f^{(2)}(0) + \frac{2}{945} f^{(4)}(0) = \frac{8}{7} + \frac{2}{945} = \frac{1082}{945} \approx 1.144974 \]

which differs from the exact value in only the sixth significant figure.

The method of removing from the integral representation of the remainder several successive "principal parts" which we have discussed in this section in connection with increasing the accuracy of mechanical quadrature formulas is closely related to a problem in the constructive theory of functions which is usually called the problem of interpolation by derivatives of successive orders or the problem of Abel-Goncharov.5

Let \( f(x) \) be a function with \( n + 1 \) derivatives defined on \([a, \beta]\) and consider the \( n + 1 \) points \( \xi_0, \xi_1, \ldots, \xi_n \). We wish to find a polynomial \( P(x) \) of degree \( \leq n \) which satisfies the conditions

\[ P^{(i)}(\xi_i) = f^{(i)}(\xi_i) \quad (i = 0, 1, \ldots, n). \quad (11.4.8) \]

It is easy to find an explicit expression for \( P(x) \). From the last condition (11.4.8) we can take

\[ P^{(n)}(x) = f^{(n)}(\xi_n). \]

Integrating this equation between the limits \( \xi_{n-1} \) and \( x \) and using the second from the last condition (11.4.8):

\[ P^{(n-1)}(x) = f^{(n-1)}(\xi_{n-1}) + f^{(n)}(\xi_n) \int_{\xi_{n-1}}^x dt_n. \]

Continuing in this way we obtain after \( n \) steps

\[ P(x) = f(\xi_0) + f'(\xi_1) \int_{\xi_0}^x dt_1 + f''(\xi_2) \int_{\xi_0}^{\xi_1}dt_2 dt_1 + \cdots + \]

\[ + f^{(n)}(\xi_n) \int_{\xi_0}^{\xi_1} \cdots \int_{\xi_{n-1}}^x dt_n \cdots dt_2 dt_1. \quad (11.4.9) \]

Introducing the notation

\[ ^5 \text{See V. L. Goncharov, The Theory of Interpolation and Approximation of Functions, Moscow, 1954, pp. 84-87 (Russian) and M. A. Evgrafov, The Interpolation Problem of Abel-Goncharov, Moscow, 1954 (Russian) which contains a bibliography on this subject.} \]
\[ L_0(x) = 1, \quad L_i(x) = \int_{\xi_0}^{x} \int_{\xi_1}^{t_1} \cdots \int_{\xi_{i-1}}^{t_{i-1}} dt_i \cdots dt_2 dt_1 \]

we can write \( P(x) \) in the form

\[ P(x) = \sum_{i=0}^{m} f^{(i)}(\xi_i) L_i(x). \quad (11.4.10) \]

Consider the remainder of the interpolation

\[ r(x) = f(x) - P(x). \]

Under certain assumptions on the function \( f(x) \) we can construct another representation for the remainder which is better suited for studying and estimating \( r(x) \).

Let the point \( x \) and the nodes \( \xi_k \ (k = 0, 1, \ldots, n) \) belong to the segment \([a, \beta]\). If \( f(x) \) has a continuous derivative of order \( n + 1 \) on \([a, \beta]\) then the remainder of the interpolation \( r(x) \) can be represented in the form:

\[ r(x) = \int_{\xi_0}^{x} \int_{\xi_1}^{t_1} \cdots \int_{\xi_n}^{t_n} f^{(n+1)}(t_{n+1}) dt_{n+1} \cdots dt_2 dt_1. \quad (11.4.11) \]

The validity of this representation follows from the fact that at the nodes \( \xi_i \) the remainder \( r(x) \) must satisfy

\[ r(\xi_0) = 0, \quad r'(\xi_1) = 0, \quad \ldots, \quad r^{(n)}(\xi_n) = 0 \]

and in addition that

\[ r^{(n+1)}(x) = f^{(n+1)}(x). \]

We now return to the expression for the remainder of formula (11.4.1):

\[ R(f) = \int_{a}^{b} f^{(m)}(x) K(x) \, dx. \quad (11.4.12) \]

In order to remove the principle part of the remainder \( R(f) \) suppose we have selected, by some means, a point \( \xi_0 \) so that we can split \( f^{(m)}(x) \) into two parts

\[ f^{(m)}(x) = f^{(m)}(\xi_0) + [f^{(m)}(x) - f^{(m)}(\xi_0)] = \]

\[ = f^{(m)}(\xi_0) + \int_{\xi_0}^{x} f^{(m+1)}(t) \, dt = \]

\[ = f^{(m)}(\xi_0) + \int_{a}^{b} f^{(m+1)}(t) [E(x - t) - E(\xi_0 - t)] \, dt. \]
11.4. Short Principle Subinterval

Previously we denoted the selected point by \( \alpha_0 \) and choose it so that the remainder was concentrated around it. Now we will not say how \( \xi_0 \) is selected and assume that it is arbitrary.

Substituting the above expression for \( f^{(m)}(x) \) into (11.4.12) we obtain

\[
R(f) = D_0 f^{(m)}(\xi_0) + \int_a^b f^{(m+1)}(x) N_1(x) \, dx
\]

where

\[
D_0 = \int_a^b K(x) \, dx,
N_1(x) = \int_a^b K(t) [E(t - x) - E(\xi_0 - x)] \, dt.
\]

In order to remove the second principle part from \( R(f) \) let us select a second point \( \xi_1 \) and expand \( f^{(m+1)}(x) \) into two parts

\[
f^{(m+1)}(x) = f^{(m+1)}(\xi_1) + [f^{(m+1)}(x) - f^{(m+1)}(\xi_1)]
\]

and so forth. After carrying out this transformation \( s \) times we obtain

\[
R(f) = D_0 f^{(m)}(\xi_0) + D_1 f^{(m+1)}(\xi_1) + \cdots + D_{s-1} f^{(m+s-1)}(\xi_{s-1}) + \int_a^b f^{(m+s)}(x) N_s(x) \, dx
\]  

(11.4.13)

\[
N_0(x) = K(x), \quad N_{i+1}(x) = \int_a^b N_i(t) [E(t - x) - E(\xi_i - x)] \, dt
\]

\[
D_i = \int_a^b N_i(x) \, dx.
\]

This expansion for the remainder is clearly similar to the expansion for \( R(f) \) in equation (11.4.3). The only difference between the two expansion is that the points \( \alpha_0, \alpha_1, \ldots \) were chosen in a definite way and the points \( \xi_0, \xi_1, \ldots \) are arbitrary. But if we select the \( \xi_i \) so that \( \xi_0 = \alpha_0, \xi_2 = \alpha_1, \ldots \), then it is clear that the expansion (11.4.13) for \( R(f) \) will coincide with the expansion (11.4.3).

Equation (11.4.13) can be obtained in another way which is closely connected with the above mentioned problem of Abel-Goncharov. Taking the nodes \( \xi_0, \xi_1, \ldots, \xi_{s-1} \) we interpolate on \( f^{(m)}(x) \) by a sequence of its derivatives

\[
f^{(m)}(x) = \sum_{i=0}^{s-1} f^{(m+i)}(\xi_i) L_i(x) + r(x).
\]  

(11.4.14)

If we substitute this expansion for \( f^{(m)}(x) \) into (11.4.12) we obtain the representation
Approximate Calculation of Definite Integrals

\[ R(f) = \sum_{i=0}^{s-1} f^{(m+i)}(\xi_i) \int_a^b K(x) L_i(x) \, dx + \int_a^b K(x) r(x) \, dx. \]  

(11.4.15)

This representation must clearly coincide with (11.4.13) for any function which has a continuous derivative of order \( m + s \). Therefore

\[ D_i = \int_a^b K(x) L_i(x) \, dx \]

\[ \int_a^b f^{(m+s)}(x) N_s(x) \, dx = \int_a^b K(x) r(x) \, dx. \]  

(11.4.16)

This relationship between interpolation on \( f^{(m)}(x) \) and the expansion of the remainder of quadrature formulas in "principal parts" is useful in the theory of quadrature formulas in the following way.

If \( f(x) \) has continuous derivatives of all orders then in (11.4.13) we can increase \( s \) without limit. Then the sum \( \sum_{i=0}^{s-1} D_i f^{(m+i)}(\xi_i) \) can be replaced by the infinite series

\[ R(f) = D_0 f^{(m)}(\xi_0) + D_1 f^{(m+1)}(\xi_1) + \ldots. \]  

(11.4.17)

This series will converge to \( R(f) \) if and only if

\[ \lim_{s \to \infty} \int_a^b f^{(m+s)}(x) N_s(x) \, dx = 0. \]

From (11.4.16) this is equivalent to

\[ \lim_{s \to \infty} \int_a^b K(x) r(x) \, dx = 0. \]

Thus the possibility of expanding the remainder \( R(f) \) in a series (11.4.16) of "principal parts" is related to the convergence of the Abel-Goncharov interpolation (11.4.14) for the function \( f^{(m)}(x) \).

In particular if \([a, b]\) is finite and if \( r(x) \) converges to zero uniformly with respect to \( x \) then the expansion

\[ R(f) = D_0 f^{(m)}(\xi_0) + D_1 f^{(m+1)}(\xi_1) + \ldots \]

is certainly possible.
11.4. Short Principle Subinterval

For a discussion of the conditions under which the Abel-Goncharov interpolation converges the reader is referred to the book by M. A. Evgrafov.

REFERENCES


J. F. Steffensen, Interpolation, Williams and Wilkins, Baltimore, 1927.
12.1. INTRODUCTION

In this chapter we consider a sequence of quadrature formulas with \( n \) nodes \((n = 1, 2, 3, \ldots)\). Such a sequence is defined by two triangular matrices: a matrix of nodes

\[
X = \begin{bmatrix}
    x^{(1)}_1 \\
    x^{(2)}_1 & x^{(2)}_2 \\
    \vdots & \vdots \\
    x^{(n)}_1 & x^{(n)}_2 & \ldots & x^{(n)}_n
\end{bmatrix}
\]  

(12.1.1)

and a matrix of coefficients.

\[
A = \begin{bmatrix}
    A^{(1)}_1 \\
    A^{(2)}_1 & A^{(2)}_2 \\
    \vdots & \vdots \\
    A^{(n)}_1 & A^{(n)}_2 & \ldots & A^{(n)}_n
\end{bmatrix}
\]  

(12.1.2)

Consider the quadrature formula which corresponds to the \( n \)th row of these matrices:

\[
\int_a^b p(x)f(x)\,dx = \sum_{k=1}^{n} A^{(n)}_k f(x^{(n)}_k) + R_n(f) = Q_n(f) + R_n(f). 
\]  

(12.1.3)

We will say that the sequence of quadrature formulas, defined by the
12.2. Interpolatory Quadrature Formulas

matrices $X$ and $A$, converges if

$$\lim_{n \to \infty} Q_n (f) = \lim_{n \to \infty} \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) = \int_{a}^{b} p(x) f(x) \, dx.$$  \hspace{1cm} (12.1.4)

Whether or not the process converges depends on the properties of the integrand $f(x)$ and also on the properties of the quadrature formulas. A study of the convergence of the process consists of studying what relationship between the integrand $f(x)$ and the matrices $X$ and $A$ will lead to a convergent process.

There are two basic problems:

1. Given the matrices $X$ and $A$ determine for what class of functions $F$ equation (12.1.4) will hold.

2. Given a class of functions $F$ determine the properties that the matrices $X$ and $A$ must have to assure convergence of the process for all $f(x) \in F$.

In the rest of this chapter we discuss the solutions to these problems for certain particular cases of practical importance in the theory of quadrature formulas.

We limit our discussion to finite segments of integration and will not be concerned with the harder problem of convergence of quadrature formulas for integrals with infinite limits.

12.2. CONVERGENCE OF INTERPOLATORY QUADRATURE FORMULAS FOR ANALYTIC FUNCTIONS

In order to simplify the proofs in this section and to make them more general we will write the integral $\int_{a}^{b} p(x) f(x) \, dx$ as a Stieltjes integral.

Suppose that we are given a certain function $\sigma(x)$ of bounded variation on the segment $[a, b]$ and consider the integral $\int_{a}^{b} f(x) \, d\sigma(x)$. Let us take $n$ points $x_k^{(n)} (k = 1, \ldots, n)$ on the segment $[a, b]$ and construct the interpolatory quadrature formula

$$\int_{a}^{b} f(x) \, d\sigma(x) = \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) + R_n (f)$$  \hspace{1cm} (12.2.1)

$$\omega_n (x) = \prod_{k=1}^{n} (x - x_k^{(n)}), \quad A_k^{(n)} = \int_{a}^{b} \frac{\omega_n (x)}{(x - x_k^{(n)}) \omega_n (x_k^{(n)})} \, d\sigma(x).$$

A sequence of such formulas is completely defined by the matrix of their nodes (12.1.1).
It is remarkable that we can give an effective and simple criterion to
decide whether or not the interpolatory quadrature process converges for
analytic functions. Such a criterion can be formulated by means of a
function which can be interpreted as the limiting distribution function of
the nodes \( x_k^{(n)} \) (\( k = 1, \ldots, n \)) as \( n \to \infty \).

The nodes \( x_k^{(n)} \) are assumed to lie on the segment \([a, b]\) and a distribu-
tion function for these nodes will be defined on this segment.

Consider a unit mass of arbitrary form distributed on the segment \([a, b]\). If \( x \) is any point of this segment then for the value of \( \mu(x) \) at the point \( x \)
we take the mass which lies to the left of \( x \). In particular, since there is
no mass to the left of \( a \), \( \mu(a) = 0 \) and \( \mu(b) = 1 \).

Thus it is clear that the function \( \mu(x) \) must have the following prop-
erties:

1. \( \mu(a) = 0 \);
2. \( \mu(x) \) is a monotone nondecreasing function of \( x \) which is continuous
from the left at each point inside \([a, b]\);
3. \( \mu(b) = 1 \).

These properties follow from the definition and each function possess-
ing these properties will be called a distribution function for the segment
\([a, b]\).

Let us be given a sequence of distribution functions \( \mu_n(x) \), \( n = 1, 2, \ldots \).
We will say that this sequence converges fundamentally to a function
\( \mu(x) \) if \( \mu_n(x) \to \mu(x) \) at each point of continuity\(^1\) of \( \mu(x) \).

We now consider the \( n \)th row of the matrix \( X \)
\[
\begin{align*}
x_1^{(n)}, & \ x_2^{(n)}, \ldots, x_n^{(n)} 
\end{align*}
\]
and assume that the nodes \( x_k^{(n)} \) are enumerated in increasing order. We
assign to each of these nodes a mass of \( \frac{1}{n} \). To this row of the matrix \( X \)
there then corresponds a distribution function \( \mu_n(x) \).

If there exists a function \( \mu(x) \), which possesses the above three prop-
erties, to which the sequence \( \mu_n(x) \) converges fundamentally:
\[
\mu_n(x) \to \mu(x) \quad \text{as} \quad n \to \infty \quad \text{fund.}
\]
then we call \( \mu(x) \) the limiting distribution function for the matrix \( X \).

We will only be concerned with cases for which \( \mu(x) \) exists.\(^2\)

Let \( r_n(x) \) denote the remainder of the interpolation for \( f(x) \) using its
values at the nodes \( x_k^{(n)} \) (\( k = 1, \ldots, n \)):

\(^1\) Note that at the end points we have \( \mu_n(a) = \mu(a) = 0 \) and \( \mu_n(b) = \mu(b) = 1 \) and
thus the sequence will always converge at the ends of the segment.

\(^2\) From known theorems concerning distribution functions we could consider \( X \)
to be an arbitrary matrix.
12.2. Interpolatory Quadrature Formulas

\[ r_n(x) = f(x) - \sum_{k=1}^{n} \frac{\omega_n(x)}{(x - x_k^{(n)})\omega_n(x_k^{(n)})} f(x_k^{(n)}) = f(x) - L_n(x). \]

The remainder \( R_n(f) \) of the quadrature (12.2.1) is the integral of \( r_n(x) \):

\[ R_n(f) = \int_a^b r_n(x) \, d\sigma(x). \]

The convergence of the quadrature process is closely related to the convergence of the interpolation and in particular if \( r_n(x) \to 0 \) uniformly with respect to \( x \in [a, b] \) as \( n \to \infty \) then \( R_n(f) \to 0 \), that is the quadrature process will also converge. To investigate the convergence of the quadrature process (12.2.1) we will first study convergence of the interpolation.

We assume that \( f(z) \) is an analytic function of \( z \) and that it is holomorphic in a certain simply connected region \( B \) of the complex plane and that the segment \([a, b]\) of the real line is contained in the interior of \( B \). Let \( l \) denote a simple closed rectifiable curve which is contained in \( B \) and which encloses \([a, b]\).

By (3.2.11) the remainder of the interpolation can be represented as the contour integral

\[ r_n(x) = \frac{1}{2\pi i} \int_l \frac{\omega_n(x)f(z)}{\omega_n(z)(z - x)} \, dz \quad (12.2.2) \]

where \( x \) is any point inside \( l \). Let \( \mu(x) \) be the limiting distribution function of the nodes of the matrix \( X \). The logarithmic potential

\[ u(z) = \int_a^b \log \frac{1}{|z - t|} \, d\mu(t) \quad (12.2.3) \]

is very useful for studying \( r_n(x) \) as \( n \to \infty \). The function \( u(z) \) is harmonic and is holomorphic everywhere in the complex plane except at the point at infinity and on the segment \([a, b]\). As \( z \) approaches the point at infinity \( u(z) \) approaches \(-\infty\).

Consider the curve

\[ u(z) = C. \]

If \( C \) is a large negative number then this curve encloses the segment \([a, b]\) and will be "close" to a circle with a large radius. We call this curve \( l_C \) and denote by \( B_C \) the part of the plane which lies inside it. As \( C \) increases in the positive direction \( B_C \) will become smaller. We denote by \( \lambda \) the least upper bound of the values of \( C \) for which \([a, b]\) lies inside \( B_C \). For each \( C < \lambda \) the curve \( l_C \) will enclose \([a, b]\).
We will denote by $\chi$ the open region of the $z$ plane for which $u(z) < \lambda$. The complement of $\chi$ will be denoted by $\beta$.

**Theorem 1.** If $f(z)$ is an analytic function holomorphic in a certain domain $D$ which contains $\beta$ in its interior then

$$r_n(x) \to 0 \quad \text{as} \quad n \to \infty$$

uniformly with respect to $x \in \beta$.

**Proof.** Since $\beta$ lies in the interior of $D$ there exists a number $C' < \lambda$ for which $B_{C'} \cup l_{C'}$ also lies inside $D$.

Take an arbitrary number $C''$ between $C'$ and $\lambda$: $C' < C'' < \lambda$. The curve $l_{C''}$ is enclosed by $l_{C'}$ and contains $\beta$ and thus $[a, b]$ in its interior.

We take $l_{C'}$ as the contour of integration in the integral representation of $r_n(x)$. We also assume that $x$ lies on $l_{C''}$.

Let $M$ denote the largest value of $|f(z)|$ on $l_{C'}$ and $\delta$ the distance between $l_{C'}$ and $l_{C''}$. The following estimate is valid:

$$|r_n(x)| \leq \frac{M}{2\pi \delta} \int_{l_{C'}} \frac{\omega_n(x)}{|\omega_n(z)|} ds.$$

Consider $|\omega_n(z)|^{-1}$:

$$|\omega_n(z)|^{-1} = \exp \sum_{k=1}^{n} \ln \frac{1}{|z - x_k^{(n)}|}.$$

As above, we assign to each node $x_k^{(n)}$ ($k = 1, \ldots, n$) a mass of $\frac{1}{n}$ and introduce the corresponding distribution function $\mu_n(x)$. Clearly

$$\int_a^b \ln \frac{1}{|z - t|} d\mu_n(t) = \frac{1}{n} \sum_{k=1}^{n} \ln \frac{1}{|z - x_k^{(n)}|}$$

and therefore

$$|\omega_n(z)|^{-1} = \exp n \int_a^b \ln \frac{1}{|z - t|} d\mu_n(t).$$

As $n \to \infty$ $\mu_n(t)$ converges fundamentally to the limiting distribution function of the nodes $\mu(t)$. The point $z$ lies outside $[a, b]$ and $\ln \frac{1}{|z - t|}$ is a continuous function of $t \in [a, b]$. According to the theorem on pas-
sage to the limit for Stieltjes integrals, which is often called Helly's second theorem, we have

\[ \int_a^b \ln \frac{1}{|z - t|} \, d\mu_n(t) \to \int_a^b \ln \frac{1}{|z - t|} \, d\mu(t) = C^* \quad \text{as } n \to \infty. \]

Here \( z \) plays the role of a parameter and the convergence will be uniform with respect to \( z \in L_C^* \). This can be shown by the usual proof of Helly's theorem. Thus there exists a number \( n' \) such that for \( n > n' \) and for any \( z \in L_C^* \) we have

\[ C^* - \frac{1}{3} (C'' - C^*) < \int_a^b \ln \frac{1}{|z - t|} \, d\mu_n(t) < C^* + \frac{1}{3} (C'' - C^*). \]

Similarly for \( x \in L_C'' \) we have

\[ \int_a^b \ln \frac{1}{|x - t|} \, d\mu_n(t) \to \int_a^b \ln \frac{1}{|x - t|} \, d\mu(t) = C'' \quad \text{as } n \to \infty \]

uniformly with respect to \( x \).

Therefore there exists an \( n'' \) such that for \( n > n'' \) and for each \( x \in L_C'' \)

\[ C'' - \frac{1}{3} (C'' - C^*) < \int_a^b \ln \frac{1}{|x - t|} \, d\mu_n(t) < C'' + \frac{1}{3} (C'' - C^*). \]

Taking \( n_0 = \max (n', n'') \) we can assert that for \( n > n_0 \) and any \( z \in L_C^* \), \( x \in L_C'' \) we have

\[ \int_a^b \ln \frac{1}{|z - t|} \, d\mu_n(t) - \int_a^b \ln \frac{1}{|x - t|} \, d\mu_n(t) < \]

\[ < \left[ C^* + \frac{1}{3} (C'' - C^*) \right] - \left[ C'' - \frac{1}{3} (C'' - C^*) \right] = -\frac{1}{3} (C'' - C^*). \]

Hence we obtain the estimate

\[ \frac{1}{\omega_n(x)} < \exp \left[ -\frac{n}{3} (C'' - C^*) \right]. \]

\[ ^3 \text{V. I. Glivenko, The Stieltjes Integral, Moscow, 1936, Sec. 14 (Russian). Also see I. P. Natanson, Theory of Functions of a Real Variable, Ungar, New York, 1955, Chap. 8, Sec. 7, where Helly's theorem is proved with slightly different assumptions than } \mu_n \to \mu. \text{ But it is easy to see that with slight modifications this proof also holds for the case } \mu_n \to \mu. \]
This gives
\[ |r_n(x)| < \frac{Ms}{2\pi\delta} \exp \left[ -\frac{n}{3} (C'' - C') \right], \quad n > n_0, \quad x \in l_{C''} \quad (12.2.3^*) \]
where \( s \) is the length of \( l_{C''} \).

From (12.2.3*) it follows that as \( n \to \infty \), \( r_n(x) \to 0 \) uniformly on \( l_{C''} \).
Since \( r_n(x) \) is an analytic function which is holomorphic in \( l_{C''} \cup B_{C''} \), then \( r_n(x) \) will converge uniformly to zero in the entire region \( l_{C''} \cup B_{C''} \). In particular this will be true on \( \beta \) which lies inside \( l_{C''} \). This completes the proof of Theorem 1.

Theorem 1 immediately leads to the following theorem on convergence of the interpolatory quadrature process.

Theorem 2. Let \([a, b]\) be a finite segment. If \( f(x) \) is an analytic function which is holomorphic in a certain region containing the set \( \beta \) in its interior then for any function \( \sigma(x) \) the interpolatory quadrature process defined by (12.2.1) converges:
\[ R_n(f) = \int_a^b r_n(x) d\sigma(x) \to 0 \quad \text{as} \quad n \to \infty. \]

We now discuss the case when the limit function \( \mu(x) \) corresponds to a uniform distribution of a unit mass on \([a, b]\):
\[ \mu(x) = \frac{(x-a)}{(b-a)} \quad a < x < b \quad (12.2.4) \]
This case corresponds to a sequence of quadrature formulas with equally spaced nodes, one instance of which are the Newton-Cotes formulas.

For simplicity of notation we assume that the segment \([a, b]\) has been transformed into \([0,1]\):
\[ \mu(x) = x. \]
In this case the logarithmic potential (12.2.3) is
\[ u(z) = \int_0^1 \ln \frac{1}{|z-t|} \ dt. \]
Since
\[ \int_0^1 \ln |z-t| \ dt = \Re \int_0^1 \ln (z-t) \ dt = \Re \{ (1-z) \ln (1-z) + z \ln z - 1 \} \]
then
\[ u(z) = \Re \{ 1-z \ln z - (1-z) \ln (1-z) \} = \\
= 1 - x \ln \sqrt{x^2 + y^2} - (1-x) \ln \sqrt{(1-x)^2 + y^2} + y \arctan \frac{y}{x-x^2-y^2}. \]
Curves for \( u(z) = C \) are depicted in Fig. 8. The set \( \beta \) consists of the curve which passes through the ends of the segment \([0, 1]\) and the region inside this curve.

![Figure 8.](image)

The greatest horizontal dimension of \( \beta \) is unity and the greatest vertical dimension is about 0.5 and is obtained by the section of \( \beta \) by the line \( x = 0.5 \).

From this discussion we see that we can guarantee convergence of the Newton-Cotes quadrature process with equally-spaced nodes only when \( f(z) \) is an analytic function which is holomorphic in a sufficiently wide region about the segment \([0, 1]\) which contains the indicated region \( \beta \) in its interior.

We consider two more questions on the theory of convergence of interpolatory quadrature formulas which, in a certain sense, are extreme cases of Theorem 2.

It can be expected that if \( f(z) \) is a function which is holomorphic in a very wide region about the segment \([a, b]\) then the quadrature process will certainly converge for any function \( \sigma(x) \) and for any choice of nodes \( x_k^{(n)} \) \((k = 1, \ldots, n)\). We now determine the smallest region in which \( f(z) \) must be holomorphic in order that the quadrature process will converge for any \( \sigma(x) \) and any choice of \( x_k^{(n)} \).

First of all we study the related interpolation problem.

Construct a circle of radius \( b - a \) around each of the points \( a \) and \( b \) and let \( \chi \) denote the closed region inside the union of these two circles.

**Theorem 3.** If \( f(z) \) is holomorphic in the region \( \chi \) then for any choice of nodes on \([a, b]\) the corresponding interpolation process converges

\[
L_n(x) \to f(x) \quad \text{as} \quad n \to \infty
\]

uniformly with respect to all \( x \in [a, b] \).
Proof. Let $x$ and $t$ be two arbitrary points of the segment $[a, b]$ and $z$ be an arbitrary point in the complex plane. If $z$ is not in $X$ then 
\[ |\frac{x-t}{z-t}| < 1. \]
If $z$ belongs to $X$ then we can find points $x$ and $t$ in $[a, b]$ for which 
\[ |\frac{x-t}{z-t}| \geq 1. \]
We now prove the first part of the theorem. Since $f(z)$ is assumed holomorphic in the closed region $X$ then it will also be holomorphic in some larger region. Therefore there exists a closed curve $l$ which encloses $X$ so that $f(z)$ is holomorphic on this curve and in its interior. The remainder of the interpolation has the representation 
\[ r_n(x) = \frac{1}{2\pi i} \int_l \frac{\omega_n(x) f(z)}{\omega_n(z) z-x} \, dz. \]
Since $x$ and the $x_k^{(n)}$ lie on $[a, b]$ and $z$ lies on $l$ we have 
\[ |x-x_k^{(n)}| \leq q < 1. \]
Then there exists a number $q < 1$ which is independent of $x$ and the $x_k^{(n)}$ and $z$ for which 
\[ |x-x_k^{(n)}| \leq q < 1. \]
Thus for each $x \in [a, b]$ and each $z \in l$ 
\[ \left| \frac{\omega_n(x)}{\omega_n(z)} \right| \leq q^n \]
and then 
\[ |r_n(x)| \leq \frac{q^n}{2\pi} \max_{z \in l} \int_l \frac{|f(z)|}{|z-x|} \, ds = A_0 q^n. \]
Hence it follows that $r_n(x) \to 0$ uniformly with respect to $x$.

We now show that the region $X$ can not be made smaller. To do this it suffices to show that for any $a \in X$ we can always find a function $f(z)$ which is holomorphic everywhere in $X$ except at $a$ and also a system of nodes $x_k^{(n)}$ for which the interpolation process for $f(z)$ will diverge at a certain point $x \in [a, b]$. 

The domain of holomorphy $\chi$ is the smallest which will guarantee convergence of the interpolation for any matrix of nodes $x_k^{(n)}$. 

Approximate Calculation of Definite Integrals
Suppose \( a \) is any point of \( \chi \). We can assume that \( a \) does not lie on \([a, b]\). Consider the function \( f(z) = \frac{1}{z - a} \). This function is holomorphic on the entire plane except at \( z = a \). For the line \( l \) in the integral representation for \( r_n(x) \) we take the boundary \( \Gamma \) of \( \chi \) together with a small circle \( \gamma \) drawn around \( a \) and connected to \( \Gamma \) by a cut.

The integrals along the sides of the cut cancel so we obtain

\[
r_n(x) = r_n\left(\frac{1}{z-a}; x\right) = \frac{\omega_n(x)}{2\pi i} \int_{\Gamma + \gamma} \frac{dz}{\omega_n(z) (z-a) (z-x)}.
\]

The integral over \( \Gamma \) is zero since at infinity the integrand has a zero of multiplicity greater than two. The integral over \( \gamma \) in a clockwise direction is the residue of the integrand at \( a \) multiplied by \(-2\pi i\)

\[
r_n\left(\frac{1}{z-a}; x\right) = \frac{\omega_n(x)}{\omega_n(a) (x-a)}.
\]

Since \( a \in \chi \) there exists points \( x \) and \( t \) of \([a, b]\) for which \( \left|\frac{x-t}{a-t}\right| \geq 1 \). Let us fix the value of \( x \) and take all the nodes of the interpolation to coincide with \( t \). The interpolation must coincide with the value of \( f(z) = \frac{1}{z-a} \) and its derivatives up to order \( n-1 \) at the point \( t \). The interpolating polynomial will be the truncated Taylor series for \( \frac{1}{z-a} \) close to \( t \).

Then

\[
\omega_n(z) = (z-t)^n
\]

\[
r_n\left(\frac{1}{z-a}; x\right) = \left(\frac{x-t}{a-t}\right)^n \frac{1}{x-a}.
\]

Since \( \left|\frac{x-t}{a-t}\right| \geq 1 \) the remainder will not approach zero as \( n \to \infty \) and the interpolation for \( \frac{1}{z-a} \) will not converge at \( x \).

The above example is a divergent Hermite interpolation process which uses a single node of multiplicity \( n \). But it is clear that if the \( x_k^{(n)} (k = 1, \ldots, n) \) are taken very close to \( t \) and if the \( x_k^{(n)} \) approach \( t \) sufficiently fast as \( n \to \infty \) then we can construct an example of a divergent interpolation process with distinct nodes. This completes the proof of the second part of the theorem.

From this result it is not difficult to prove the following theorem.
Theorem 4. If \( f(z) \) is holomorphic in the region \( \chi \) then the interpolatory quadrature process defined by (12.2.1) is convergent

\[
R_n(f) \to 0 \quad \text{as} \quad n \to \infty
\]

for any nodes \( x_k^{(n)} \) and any function \( \sigma(x) \) of bounded variation on \([a, b]\).

The region of holomorphy \( \chi \) is the smallest which will guarantee convergence of the quadrature process (12.2.1) for arbitrary \( x_k^{(n)} \) and \( \sigma(x) \).

Proof. If \( f(z) \) is holomorphic in \( \chi \) then for arbitrary nodes \( x_k^{(n)} \) the remainder of the interpolation approaches zero uniformly with respect to \( x \) as \( n \to \infty \). Therefore

\[
R_n(f) = \int_a^b r_n(x) \, d\sigma(x)
\]

also approaches zero as \( n \to \infty \) for any \( \sigma(x) \) of bounded variation.

To prove the second part of the theorem it suffices to show that if we remove a single point from \( \chi \) then we can find functions \( f(z) \) and \( \sigma(x) \) and nodes \( x_k^{(n)} \) for which \( R_n(f) \) will not approach zero.

Let \( \sigma(x) \) be a function which is piece-wise constant with a unit jump at \( x \). Then

\[
R_n(f) = \int_a^b r_n(x) \, d\sigma(x) = r_n(x).
\]

Convergence of the quadrature process for this \( \sigma(x) \) is equivalent to convergence of the interpolation at \( x \). But from the last theorem we see that for any point \( \alpha \in \chi \) we can find, for the function \( f(z) = \frac{1}{z - \alpha} \), a point \( x \) and nodes \( x_k^{(n)} \) of \([a, b]\) for which \( r_n(x) \) diverges. This completes the proof.

In the remainder of this section we will assume that the segment of integration \([a, b]\) has been transformed into \([-1, 1]\). We will call the function

\[
\mu(x) = \frac{1}{n} \int_{-1}^{x} \frac{dt}{\sqrt{1 - t^2}} \quad (12.2.5)
\]

the Chebyshev distribution function. Let us suppose that the matrix \( X \) has a limiting distribution function for its nodes and that this function is (12.2.5). This will happen, for example, when the \( x_k^{(n)} \) \((k = 1, \ldots, n; \quad n = 1, 2, \ldots) \) are the roots of the sequence of orthogonal polynomials \( P_n(x) \) which are orthogonal on \([-1, 1]\) with respect to an arbitrary summable almost everywhere positive weight function \( p(x) \). Such nodes correspond to integration formulas of the highest algebraic degree of precision.
Consider the logarithmic potential

\[ u(z) = \frac{1}{\pi} \int_{-1}^{1} \ln \frac{1}{|z-t|} \frac{dt}{\sqrt{1-t^2}}. \]  

(12.2.6)

This function is the real part of

\[ F(z) = \frac{1}{\pi} \int_{-1}^{1} \ln \frac{1}{z-t} \frac{dt}{\sqrt{1-t^2}}. \]  

(12.2.7)

In the complex plane \( z \) we draw a cut along the real axis from the point 1 to \(-\infty\) and choose that branch of the logarithm for which \( \arg(z-t) = 0 \) for real \( z \) greater than \( t \)

\[ F'(z) = -\frac{1}{\pi} \int_{-1}^{1} \frac{dt}{(z-t)\sqrt{1-t^2}}. \]

This integral is easily seen to be

\[ F'(z) = -\frac{1}{\sqrt{z^2-1}}, \]

where we choose that branch of the root which has a positive value for \( z > 1 \)

\[ F(z) = \ln \frac{K}{z + \sqrt{z^2-1}}. \]

The constant \( K \) is found from the condition that for large \( z \) (12.2.7) can be represented as

\[ F(z) = \ln \frac{1}{z} + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots. \]

This gives \( K = 2 \) and thus

\[ F(z) = \ln \frac{2}{z + \sqrt{z^2-1}} \]

\[ u(z) = \ln \frac{2}{|z + \sqrt{z^2-1}|}. \]  

(12.2.8)

The curve

\[ u(z) = C \]

for \( C < \ln 2 \) is a closed curve enclosing the segment \([-1, 1]\). For \( C = \ln 2 \) the curve coincides with the segment \([-1, 1]\). The set \( \beta \) consists only of the interval of integration \([-1, 1]\). Thus we have:
Theorem 5. If the matrix $X$ has for its limiting distribution function the Chebyshev function (12.2.5) then

1. The corresponding interpolation process converges uniformly with respect to $x \in [-1, 1]$ for each function which is analytic on $[-1, 1]$;

2. The quadrature process defined by (12.2.1) on $[-1, 1]$ converges for any function $f(x)$ which is analytic on $[-1, 1]$ for an arbitrary function $\sigma(x)$ which has bounded variation on $[-1, 1]$.

It is interesting to note that the converse to this theorem is also true. We will now prove the converse for the interpolation process.

Theorem 6. If the matrix $X$ has the property that the interpolation process converges for all points of the segment $[-1, 1]$ for each analytic function on $[-1, 1]$ then $X$ has a limiting distribution function which is the Chebyshev function (12.2.5).

To prove this theorem it will be necessary to become acquainted with certain properties of the logarithmic potential. Consider the sequence of distribution functions which correspond to the rows of the matrix $X$: $\mu_1(x)$, $\mu_2(x), \ldots$. By Helly’s theorem we can always select from this sequence a subsequence which converges fundamentally to a certain distribution function which we denote by $\mu(x)$. In the following we assume that the index $n$ runs through the integers for which $\mu_n(x) \rightarrow \mu(x)$ fundamentally.

The theorem will be proved if we establish that

$$\mu(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{dt}{\sqrt{1 - t^2}}.$$  

For $x$ in $[-1, 1]$ the integral

$$u(x) = \int_{-1}^{1} \ln \frac{1}{|x - t|} d\mu(t)$$  

is improper and we will define it as follows. We define the function $\ln N x$ by

$$\ln N x = \begin{cases} 
\ln x & \text{for } \ln x \leq N \\
N & \text{for } \ln x > N.
\end{cases}$$

Then $\ln N \frac{1}{|x - t|}$ is bounded and continuous for $t \in [-1, 1]$. The integral

---

4 L. Kalmár, "Az interpolációtól," Mathematikai es physikai lapok, Vol. 32, 1926, p. 120, where a more general theorem is proved.

8 V. I. Glivenko, The Stieltjes Integral, Moscow, 1936, Sec. 13; or I. P. Natanson, Theory of Functions of a Real Variable, Ungar, New York, 1955, Chap. 8, Sec. 4.
12.2. Interpolatory Quadrature Formulas

\[ \int_{-1}^{1} \ln \frac{1}{|x-t|} \, d\mu(t) \] is a nondecreasing function of \( N \) and we set

\[ u(x) = \int_{-1}^{1} \ln \frac{1}{|x-t|} \, d\mu(t) = \lim_{N \to \infty} \int_{-1}^{1} \ln \frac{1}{|x-t|} \, d\mu(t). \]

**Lemma 1.** If \( \mu(t) \) has a derivative at the point \( x \in [-1, 1] \) then the integral (12.2.9) is finite at this point.

**Proof.** Let \( x \) belong to the interior of \([-1, 1]\). For large \( N \) we take the interval \( x - \delta \leq t \leq x + \delta \) close to \( x \) where \( \delta = e^{-N} \). Then

\[ u(x) = \int_{-1}^{1} \ln \frac{1}{|x-t|} \, d\mu(t) = \lim_{N \to \infty} \left\{ \int_{-1}^{x-\delta} \ln \frac{1}{x-t} \, d\mu(t) + \int_{x+\delta}^{1} \ln \frac{1}{t-x} \, d\mu(t) + N \left[ \mu(x+\delta) - \mu(x-\delta) \right] \right\}. \]

After integrating the term in brackets by parts and using \( \ln \delta = -N \), \( \mu(1) = 1 \), \( \mu(-1) = 0 \):

\[ u(x) = \lim_{N \to \infty} \left\{ \ln \frac{1}{1-x} + \int_{-1}^{x-\delta} \frac{\mu(t)}{x-t} \, dt + \int_{x+\delta}^{1} \frac{\mu(t)}{x-t} \, dt \right\} = \ln \frac{1}{1-x} + \text{princ. value} \int_{-1}^{1} \frac{\mu(t)}{x-t} \, dt. \]

Since \( \mu(t) \) has a derivative at \( x \) then princ. value \( \int_{-1}^{1} \frac{\mu(t)}{x-t} \, dt \) exists and is finite.

In a similar way we can verify the assertion of the lemma when \( x \) is an end point of \([-1, 1]\).

Since the derivative \( \mu'(t) \) exists almost everywhere the integral (12.2.9) is finite almost everywhere. This completes the proof of Lemma 1.

Consider the logarithmic potential \( u(z) = \int_{-1}^{1} \ln \frac{1}{|z-t|} \, d\mu(t) \). Let \( x \) be a point of \([-1, 1]\) and assume that the point \( z = x + iy \) approaches \( x \) in the vertical direction.

**Lemma 2.** For any \( x \in [-1, 1] \)

\[ \lim_{y \to 0} \int_{-1}^{1} \ln \frac{1}{|z-t|} \, d\mu(t) = \int_{-1}^{1} \ln \frac{1}{|x-t|} \, d\mu(t) \] (12.2.10)

regardless of whether the integral on the right side is finite or infinite.
Proof. Consider a small segment \( x - \epsilon \leq t \leq x + \epsilon \) close to \( x \) and let 
\( E_\epsilon \) be the part of \([-1, 1]\) which remains after we remove the segment 
\([x - \epsilon, x + \epsilon]\) from \([-1, 1]\). For small values of \( \epsilon \) we have \( |z - t| < 1 \) for 
each \( t \in [x - \epsilon, x + \epsilon] \). Then \( \ln \frac{1}{|z - t|} > 0 \) and since \( \mu(t) \) is nondecreasing then we must have 
\[
\int_{E_\epsilon} \ln \frac{1}{|z - t|} \, d\mu(t) \leq \int_{-1}^{1} \ln \frac{1}{|z - t|} \, d\mu(t).
\]
Passing to the limit as \( \epsilon \to 0 \) in the integral over \( E_\epsilon \) and using 
\( \ln \frac{1}{|z - t|} < \ln \frac{1}{|x - t|} \) we obtain 
\[
\int_{E_\epsilon} \ln \frac{1}{|x - t|} \, d\mu(t) \leq \liminf_{\epsilon \to 0} \int_{-1}^{1} \ln \frac{1}{|z - t|} \, d\mu(t) \leq \limsup_{\epsilon \to 0} \int_{-1}^{1} \ln \frac{1}{|z - t|} \, d\mu(t) \leq \int_{-1}^{1} \ln \frac{1}{|x - t|} \, d\mu(t).
\]
But as \( \epsilon \to 0 \)
\[
\int_{E_\epsilon} \ln \frac{1}{|x - t|} \, d\mu(t) \to \int_{-1}^{1} \ln \frac{1}{|x - t|} \, d\mu(t)
\]
independent of whether this last integral is finite or infinite. Therefore 
\[
\liminf_{\epsilon \to 0} \int_{-1}^{1} \ln \frac{1}{|z - t|} \, d\mu(t) \quad \text{and} \quad \limsup_{\epsilon \to 0} \int_{-1}^{1} \ln \frac{1}{|z - t|} \, d\mu(t)
\]
must both coincide with \( \int_{-1}^{1} \ln \frac{1}{|x - t|} \, d\mu(t) \).
This proves lemma 2.

We now study \( u_n(x) = \int_{-1}^{1} \ln \frac{1}{|x - t|} \, d\mu_n(t) \). If \( x = x_k^{(n)} \) we define 
\( u_n(x) = \infty \).

Lemma 3. Almost everywhere on \(-1 \leq x \leq 1\)
\[
\liminf_{n \to \infty} \int_{-1}^{1} \ln \frac{1}{|x - t|} \, d\mu_n(t) = \int_{-1}^{1} \ln \frac{1}{|x - t|} \, d\mu(t). \quad (12.2.11)
\]

Proof. By the definition of \( \ln_N x \) we have 
\[
\int_{-1}^{1} \ln \frac{1}{|x - t|} \, d\mu_n(t) \leq \int_{-1}^{1} \ln_N \frac{1}{|x - t|} \, d\mu_n(t).
\]
12.2. Interpolatory Quadrature Formulas

Passing to the limit as $n \to \infty$ gives

$$\liminf_{n \to \infty} \int_{-1}^{1} \ln \left| \frac{1}{x-t} \right| d\mu_n(t) \geq \int_{-1}^{1} \frac{1}{|x-t|} d\mu(t)$$

and since this inequality is true for each $N$

$$\liminf_{n \to \infty} \int_{-1}^{1} \ln \left| \frac{1}{x-t} \right| d\mu_n(t) \geq \int_{-1}^{1} \ln \left| \frac{1}{x-t} \right| d\mu(t). \quad (12.2.12)$$

We now find an upper bound for the left side of (12.2.12). The function $\int_{-1}^{1} \ln \left| \frac{1}{x-t} \right| d\mu_n(t)$ is bounded from below by $\ln \frac{1}{2}$ and for any $a, \beta \in [-1, 1]$ we have by Fatou's theorem:

$$\beta \liminf_{n \to \infty} \int_{-1}^{1} \ln \left| \frac{1}{x-t} \right| d\mu_n(t) dx \leq \liminf_{n \to \infty} \int_{a}^{\beta} \int_{-1}^{1} \ln \left| \frac{1}{x-t} \right| d\mu_n(t) dx = \liminf_{n \to \infty} \int_{-1}^{1} \beta \int_{a}^{\beta} \ln \left| \frac{1}{x-t} \right| dx d\mu_n(t).$$

In this last integral we can pass to the limit under the integral sign because $\int_{a}^{\beta} \ln \left| \frac{1}{x-t} \right| dx$ is a continuous function of $t$:

$$\int_{a}^{\beta} \liminf_{n \to \infty} \int_{-1}^{1} \ln \left| \frac{1}{x-t} \right| d\mu_n(t) dx \leq \int_{-1}^{1} \beta \int_{a}^{\beta} \ln \left| \frac{1}{x-t} \right| dx d\mu(t).$$

Since $\ln \left| \frac{1}{x-t} \right| \geq \ln \frac{1}{2}$ from Fubini's theorem we can change the order of integration on the right side of this inequality

$$\int_{a}^{\beta} \liminf_{n \to \infty} \int_{-1}^{1} \ln \left| \frac{1}{x-t} \right| d\mu_n(t) dx \leq \int_{a}^{\beta} \int_{-1}^{1} \ln \left| \frac{1}{x-t} \right| d\mu(t) dx.$$

This inequality is valid for each $a, \beta \in [-1, 1]$ and thus we have almost everywhere on $-1 \leq x \leq 1$

$$\liminf_{n \to \infty} \int_{-1}^{1} \ln \left| \frac{1}{x-t} \right| d\mu_n(t) \leq \int_{-1}^{1} \ln \left| \frac{1}{x-t} \right| d\mu(t). \quad (12.2.12^*)$$

Combining (12.2.12) and (12.2.12*), completes the proof of Lemma 3.
Let $E$ denote the set of points of $[-1, 1]$ for which (12.2.11) holds and for which \[ \int_{-1}^{1} \ln \frac{1}{|x-t|} \, d\mu(t) \] is finite. The set $E$ differs from $[-1, 1]$ by only a set of measure zero.

**Lemma 4.** If the matrix $X$ has the property that for the above indicated values of $n$ the remainder of the interpolation $r_n(x)$ strives to zero for any \( x \in [-1, 1] \) and for any function of the form $f(x) = \frac{1}{x-a}$ where $a$ lies outside $[-1, 1]$ then the potential (12.2.9) is a constant on $E$.

**Proof.** Suppose the converse is true. Then we can show that there exists a function $f(x) = \frac{1}{x-a}$ for which the remainder will not tend to zero.

Let $x_1$ and $x_2$ be two points of $E$ for which (12.2.9) has different values. We can assume that $u(x_1) < u(x_2)$.

Let $\delta = u(x_2) - u(x_1)$ and take $\epsilon < \delta$. Consider a straight line passing through $x_2$ parallel to the imaginary axis. As $z$ approaches $x_2$ along this line $u(z)$ will approach $u(x_2)$ by Lemma 2. As $z$ approaches $x_2$ we also note that $\ln \frac{1}{|z-t|}$ increases for each $t$. Since $\mu(t)$ is a nondecreasing function then $u(z)$ will also increase as $z$ approaches $x_2$.

Thus on this line there exists a point $z_2 \neq x_2$ for which $u(x_2) - \frac{1}{3} \epsilon < u(z_2) < u(x_2)$.

We fix $z_2$ and construct the function

\[ f(x) = \frac{1}{x-z_2}. \]

The remainder of the interpolation for this function is

\[ r_n(x) = r_n\left(\frac{1}{x-z_2}; x\right) = \frac{\omega_n(x)}{\omega_n(z_2) (x-z_2)} \]

so that at $x = x_1$

\[ r_n(x_1) = \frac{\omega_n(x_1)}{\omega_n(z_2) (x_1-z_2)} \]
12.2. Interpolatory Quadrature Formulas

\[ |r_n(x_1)| = \frac{1}{|x_1 - z_2|} \left| \frac{\omega_n(x_1)}{\omega_n(z_2)} \right| = \]

\[ = \exp n \left[ \int_{-1}^{1} \ln \frac{1}{|z_2 - t|} \, d\mu_n(t) - \int_{-1}^{1} \ln \frac{1}{|x_1 - t|} \, d\mu_n(t) \right] = \]

\[ = \frac{1}{|x_1 - z_2|} \exp n [u_n(z_2) - u_n(x_1)]. \]

By Lemma 3 there exists an infinite sequence of values of \( n \) for which

\[ |u_n(x_1) - u(x_1)| < \frac{1}{3} \epsilon. \]

Therefore there exists such a sequence of numbers \( n \) for which

\[ u_n(z_2) - u_n(x_1) = u(z_2) - u(x_1) - [u(z_2) - u_n(z_2)] - [u_n(x_1) - u(x_1)] > \]

\[ > \delta - \frac{1}{3} \epsilon - \frac{1}{3} \epsilon - \frac{1}{3} \epsilon = \delta - \epsilon > 0. \]

For these values of \( n \)

\[ |r_n(x_1)| > \frac{1}{|x_1 - z_2|} \exp n (\delta - \epsilon) \]

and the interpolation thus diverges at \( x_1 \) for \( f(x) = \frac{1}{x - z_2} \). This proves Lemma 4.

In the statement of Theorem 6 we assumed that the interpolation converges on \([-1, 1]\) for each function which is analytic on \([-1, 1]\). Then this will be true for a function of the form \( f(x) = \frac{1}{x - a} \) for \( a \in [-1, 1] \) and we have seen that \( u(x) \) is then a constant on \( E \) and therefore is a constant almost everywhere on \([-1, 1]\).

In order to complete the proof of Theorem 6 we still have to prove the following lemma.

**Lemma 5.** If the logarithmic potential (12.2.9) is almost everywhere constant on \([-1, 1]\) then \( \mu(t) \) is the Chebyshev distribution function

\[ \mu(t) = \frac{1}{n} \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}}. \]

**Proof.** Consider the potential

\[ u(z) = \int_{-1}^{1} \ln \frac{1}{|z - t|} \, d\mu(t) \quad (12.2.13) \]
defined in the $z$ plane. In order to use results from the theory of the Poisson integral and the theory of trigonometric series we pass from the $z$ plane to a circle.

In the $z$ plane we make a cut along the segment $[-1, 1]$ and distinguish the two sides of the cut. We transform (12.2.13) into an integral along the contour $\lambda$ consisting of both sides of the cut. To do this it is sufficient to represent (12.2.13) in the form

$$u(z) = \frac{1}{2} \int_{-1}^{1} \ln \frac{1}{|z - t|} \, d\mu(t) - \frac{1}{2} \int_{-1}^{1} \ln \frac{1}{|z - t|} \, d\mu(t)$$

and introduce the function $\nu(t)$ defined on $\lambda$ by

$$\nu(t) = \begin{cases} \frac{1}{2} \mu(t) & \text{on the top of the cut} \\ 1 - \frac{1}{2} \mu(t) & \text{on the bottom of the cut}. \end{cases} \quad (12.2.14)$$

Then

$$u(z) = \int_{\lambda} \ln \frac{1}{|z - t|} \, d\nu(t). \quad (12.2.15)$$

The integration is carried out along the top of the cut from $-1$ to $1$ and in the opposite direction along the bottom.

In the plane $\zeta = \rho e^{i\phi}$ consider the circle $|\zeta| \leq 1$. This circle is transformed onto the $z$ plane with the cut along $[-1, 1]$ by

$$z = \frac{1}{2}(\zeta + \zeta^{-1}).$$

The point $\tau = e^{i\psi}$ of the circumference corresponds to the point $t = \frac{1}{2}(\tau + \tau^{-1}) = \frac{1}{2}(e^{-i\psi} + e^{-i\psi}) = \cos \psi$. As $\psi$ varies from $-\pi$ to $\pi$ we pass around the contour $\lambda$ in the above indicated direction. The function $\nu(t)$, defined on $\lambda$, corresponds to the function

$$\nu(t) = \nu(\cos \psi) = F(\psi) \quad -\pi \leq \psi \leq \pi$$

of the polar angle $\psi$.

The contour integral (12.2.15) corresponds to the following integral over the circumference of the circle in the $\zeta$ plane:

$$u(z) = \int_{-\pi}^{\pi} \ln \left| \frac{2\zeta}{1 - 2\zeta \cos \psi + \zeta^2} \right| \, dF(\psi) =$$

$$= \ln 2 |\zeta| + \int_{-\pi}^{\pi} \ln \frac{1}{|\zeta - e^{i\psi}| \, |\zeta - e^{-i\psi}|} \, dF(\psi) =$$

$$= \ln 2 |\zeta| + I(\zeta).$$
The integral \( I(\zeta) \) splits into the sum of two logarithmic potentials which are harmonic in the circle \( |\zeta| < 1 \):

\[
I(\zeta) = \int_{-\pi}^{\pi} \ln \frac{1}{|\zeta - e^{i\psi}|} \, dF(\psi) + \int_{-\pi}^{\pi} \ln \frac{1}{|\zeta - e^{-i\psi}|} \, dF(\psi) = I_1(\zeta) + I_2(\zeta).
\]

Because of the similarity of \( I_1(\zeta) \) and \( I_2(\zeta) \) it will suffice to only study \( I_1(\zeta) \). We can see that \( I_1(\zeta) \) can be represented as a Poisson-Lebesgue integral. Let \( E \) be any measurable set on \([-\pi, \pi]\) with measure \( mE \leq \delta \):

\[
\int_E I_1(\rho e^{i\phi}) \, d\phi = \int_E \int_{-\pi}^{\pi} \ln \frac{1}{\rho e^{i\phi} - e^{i\psi}} \, dF(\psi) \, d\phi
\]

\[
= \int_{-\pi}^{\pi} \left[ \int_E \ln \frac{1}{\rho e^{i\phi} - e^{i\psi}} \, d\phi \right] \, dF(\psi).
\]

Here it was possible to change the order of integration by Fubini's theorem since \( \ln \frac{1}{|\zeta - e^{i\psi}|} \) is bounded from below by \( \ln \frac{1}{2} \). The inside integral has the upper bound

\[
\left| \int_E \ln \frac{1}{\rho e^{i\phi} - e^{i\psi}} \, d\phi \right| \leq \int_E \ln \frac{1}{|\sin (\phi - \psi)|} \, d\phi \leq \int_{\delta/2}^{\pi/2} \ln \frac{1}{|\sin x|} \, dx.
\]

The function \( v(\zeta) \) can be represented as a Poisson-Lebesgue integral if it is harmonic in the circle \( |\zeta| < 1 \) and if there exists on \([-\pi, \pi]\) a summable function \( f(\psi) \) for which

\[
v(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi) \, d\psi - \frac{1 - \rho^2}{1 - 2\rho \cos (\psi - \phi) + \rho^2} \, d\psi.
\]

The following theorem is known: A necessary and sufficient condition that a function \( v(\zeta) \) which is harmonic in the circle \( |\zeta| < 1 \) can be represented as a Poisson-Lebesgue integral is that the family of functions \( F_\rho(\alpha) = \int_0^\alpha v(\rho e^{i\phi}) \, d\phi \) be uniformly absolutely continuous in \( \alpha \), that is for all \( \rho < 1 \) and each \( \epsilon > 0 \) there exists a number \( \delta(\epsilon) > 0 \) such that for each set \( E \) of measure \( mE < \delta(\epsilon) \) we have

\[
\left| \int_E v(\rho e^{i\phi}) \, d\phi \right| < \epsilon.
\]

It is also known that as the point \( \zeta \) approaches a point \( \psi = \psi_0 \) of the circumference by any path not tangent to the circle then for almost all values of \( \psi_0 \), \( v(\zeta) \rightarrow f(\psi_0) \). See, for example, I. I. Privalov, *Boundary Properties of Analytic Functions*, Gostekhizdat, Moscow, 1950, Chap. 1, Sec. 3 (Russian).
Thus for each \( \epsilon > 0 \) we can find a \( \delta (\epsilon) \) for which

\[
\left| \int_E I_1 (\rho e^{i\phi}) d\phi \right| < \epsilon \quad \rho < 1.
\]

Hence it follows that \( I_1 (\zeta) \) can be represented as a Poisson-Lebesgue integral.

Therefore \( I_1 (\zeta) \) and also \( I (\zeta) \) can be represented as a Poisson-Lebesgue integral.

As the point \( z = x + iy \) approaches the segment \([-1, 1]\) along a line parallel to the imaginary axis, \( u(z) \) tends to a constant for almost all \( x \). When we transform to the circle \( |\zeta| < 1 \) the indicated line transforms into a line which is orthogonal to the circumference \( |\zeta| = 1 \). As we approach the boundary along this curve \( u(z) \) strives almost everywhere to a constant value and since \( \ln 2 |\zeta| \) tends to \( \ln 2 \) then on the circumference \( I (\zeta) \) will in the limit be almost everywhere constant. Since \( I (\zeta) \) can be represented as a Poisson-Lebesgue integral \( I (\zeta) \) is a constant everywhere in the circle. But since \( I (0) = 0 \) then

\[
I (\zeta) = 0
\]

everywhere in the circle.

It is easy to see that the functions \( \ln \frac{1}{|\zeta - e^{i\psi}|} \) and \( \ln \frac{1}{|\zeta - e^{-i\psi}|} \), \( \zeta = \rho e^{i\phi} \), have the following expansions in powers of \( \rho \) for \( \rho < 1 \):

\[
\ln \frac{1}{|\zeta - e^{i\psi}|} = \sum_{k=1}^{\infty} \frac{1}{k} \rho^k \cos k(\phi - \psi)
\]

\[
\ln \frac{1}{|\zeta - e^{-i\psi}|} = \sum_{k=1}^{\infty} \frac{1}{k} \rho^k \cos k(\phi + \psi).
\]

Therefore

\[
I (\zeta) = \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \rho^k \left[ \cos k(\phi - \psi) + \cos k(\phi + \psi) \right] dF(\psi) =
\]

\[
= 2 \sum_{k=1}^{\infty} \frac{1}{k} \rho^k \cos k\phi \int_{-\pi}^{\pi} \cos k\psi dF(\psi) = 0.
\]

Hence

\[
\int_{-\pi}^{\pi} \cos k\psi dF(\psi) = F(\psi) \cos k\psi \left[ \int_{-\pi}^{\pi} + k \int_{-\pi}^{\pi} F(\psi) \sin k\psi d\psi =
\]

\[
= (-1)^k \left[ F(\pi) - F(-\pi) \right] + k\pi b_k = 0.
\]
12.2. Interpolatory Quadrature Formulas

Here $b_k$ is the coefficient of $\sin k\psi$ in the Fourier expansion of $F(\psi)$. From $F(\pi) = 1$ and $F(-\pi) = 0$ we have

$$b_k = \frac{(-1)^k}{k\pi}.$$

From the definitions of $\nu(t)$ and $F(\psi)$ we see that

$$F(\psi) = \begin{cases} \frac{1}{2} \mu \cos \psi & -\pi \leq \psi \leq 0 \\ 1 - \frac{1}{2} \mu \cos \psi & 0 \leq \psi \leq \pi. \end{cases}$$

The even part of $F(\psi)$ is

$$\frac{1}{2} \left[ F(\psi) + F(-\psi) \right] = \frac{1}{2}$$

and hence the coefficients $a_k$ of $\cos k\psi$ in the Fourier expansion of $F(\psi)$ are

$$a_0 = \frac{1}{2}, \quad a_k = 0, \quad k = 1, 2, \ldots.$$ 

Thus we obtain

$$F(\psi) = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k\pi} \sin k\psi = \frac{1}{2} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dx}{1 - x^2}.$$ 

If we return to the $z$ plane we have $t = \cos \psi$ from which follows

$$\mu(t) = 1 - \frac{1}{\pi} \operatorname{Arc} \cos t = \frac{1}{\pi} \int_{-1}^{t} \frac{dx}{\sqrt{1 - x^2}}.$$ 

This proves Lemma 5 and completes the proof of Theorem 6.

From this result it is not difficult to establish the corresponding theorem for quadrature formulas.

**Theorem 7.** If the interpolatory quadrature process defined by (12.2.1) for the segment $[-1, 1]$ converges for each function $\sigma(x)$ of bounded variation and for any analytic function $f(x)$ on $[-1, 1]$ then the matrix of nodes $X$ has a limiting distribution function which is the Chebyshev function (12.2.5).

**Proof.** Consider the remainder of the quadrature

$$R_n(f) = \int_{-1}^{1} r_n(x) d\sigma(x)$$

where $r_n(x)$ is the remainder of the interpolation. Take an arbitrary point $x$ on $[-1, 1]$. As the function $\sigma(x)$ we take a piece-wise constant func-
tion which has a unit jump at \( x \). For such a \( \sigma(x) \)
\[
R_n(f) = r_n(x)
\]
and the convergence of the quadrature process is equivalent to con-
vergence of the interpolation. Then the proof is completed by using
Theorem 6.

12.3. CONVERGENCE OF THE GENERAL QUADRATURE PROCESS

In this section we study the general quadrature process (12.1.3) de-
defined by the matrix of nodes (12.1.1) and the matrix of coefficients
(12.1.2). The weight function \( p(x) \) can be any summable function. We
assume that we are given a certain class \( F \) of functions \( f \). We wish to
determine what conditions \( X \) and \( A \) must satisfy in order that the quadra-
ture process will converge for each \( f \in F \). This problem has been studied
for many classes \( F \). We consider here only the simplest and most im-
portant of these results.

In the remainder of this section we assume that the segment of inte-
gration is finite.

Theorem 8. In order that the quadrature process (12.1.3) converge for
each continuous function \( f \) on \([a, b]\) the following two conditions are
necessary and sufficient:
1. The process converge for each polynomial;
2. There exists a number \( K \) for which

\[
\sum_{k=1}^{n} |A_k^{(n)}| \leq K
\]

for \( n = 1, 2, \ldots \).

Proof. If in the class of continuous functions on \([a, b]\) we define a
norm by \( \|f\| = \max_{[a, b]} |f(x)| \) then this class can be considered as the
Banach space \( C \). The quadrature sum \( Q_n(f) = \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) \) and the
integral \( I(f) = \int_{a}^{b} p(x) f(x) \, dx \) are two linear functionals defined on \( C \).
The values \( Q_n(f) \) and \( I(f) \) belong to the set of real numbers which is
also a Banach space.

7The sufficiency of this condition was proved by V. A. Steklov, the necessity
by G. Polya.
12.3. The General Quadrature Process

We can then apply Theorem 1 of Section 4.3 which gives conditions for the convergence of a sequence of linear operators. A necessary and sufficient condition that such a sequence converge is that 1) it converge on a set of elements dense in the space where the operators are defined and 2) that the norms of the operators have a common bound.

From the theorem of Weierstrass it is known that we can uniformly approximate each continuous function on \([a, b]\) by means of polynomials and thus the class of polynomials is a set of functions which is dense in \(C\). This establishes the first condition of the theorem.

The norm of the functional \(Q_n(f)\) is

\[
\|Q_n\| = \sup_{|f| \leq 1} \left| \sum_{k=1}^{n} A_k^{(n)} f(x_k) \right| = \sum_{k=1}^{n} |A_k^{(n)}|.
\]

Thus (12.3.1) is the condition that the functionals have a common bound. This completes the proof.

The following two theorems are simple corollaries to Theorem 8.

**Theorem 9.** If all the coefficients \(A_k^{(n)}\) are nonnegative then in order that the quadrature process converge for each continuous function it is necessary and sufficient that it converge for each polynomial.

**Proof.** The necessity of the condition is obvious. If the process converges for each polynomial then for \(f(x) \equiv 1\)

\[
Q_n(1) \rightarrow \int_{a}^{b} p(x) \, dx \quad \text{as } n \rightarrow \infty.
\]

Therefore the values of \(Q_n(1), n = 1, 2, \ldots,\) are bounded:

\[
Q_n(1) \leq K.
\]

But

\[
\sum_{k=1}^{n} |A_k^{(n)}| = \sum_{k=1}^{n} A_k^{(n)} = Q_n(1) \leq K
\]

and thus by Theorem 8 the quadrature process converges for each continuous function.

**Theorem 10.** For an interpolatory quadrature process to converge for any continuous function it is necessary and sufficient that

\[
\sum_{k=1}^{n} |A_k^{(n)}| \leq K < \infty.
\]
The second condition of Theorem 8 coincides with the condition of Theorem 10. The first condition of Theorem 8 is fulfilled since if \( f(x) \) is a polynomial of degree \( m \) then for any \( n > m \), \( Q_n(f) = \int_a^b p(x) f(x) \, dx \).

This establishes the theorem.

We now discuss conditions for convergence of the quadrature process in classes of differentiable functions.

As above we enumerate the nodes in increasing order and introduce the piece-wise constant functions \( F_{n,0}(x) \) for the nodes and coefficients

\[
F_{n,0}(x) = \sum_{k=1}^{n} A_k^{(n)} E(x - x_k^{(n)}).
\]

We also consider the primitive functions of any order \( r \) of the functions \( F_{n,0}(x) \) defined by the initial conditions \( F_{n,0}^{(j)}(a) = 0 \) \( (j = 0, 1, \ldots, r - 1) \):

\[
F_{n,r}(x) = \int_a^x F_{n,0}(t) \frac{(x - t)^{r-1}}{(r-1)!} \, dt = \sum_{k=1}^{n} A_k^{(n)} E(x - x_k^{(n)}) \frac{(x - x_k^{(n)})^r}{r!}.
\] (12.3.2)

**Theorem 11.** In order that the quadrature process (12.1.3) converge as \( n \to \infty \) for each function \( f \in C_r[a, b] \) it is necessary and sufficient that the following conditions be fulfilled:

1. The process converge for each polynomial;
2. The total variation of the primitive functions \( F_{n,r}(x) \) of order \( r \) have a common bound for \( n = 1, 2, \ldots \):

\[
\text{Var}_{[a,b]} F_{n,r}(t) \leq M.
\]

**Proof.** If \( f \in C_r[a, b] \), \( r \geq 1 \), then expanding \( f \) in a Taylor series about the point \( b \) we obtain the representation

\[
f(x) = \sum_{i=0}^{r-1} \frac{f^{(i)}(b)}{i!} (x - b)^i + \int_b^x f^{(r)}(t) \frac{(x - t)^{r-1}}{(r-1)!} \, dt = \sum_{i=0}^{r-1} \frac{f^{(i)}(b)}{i!} (x - b)^i + (-1)^r \int_a^b f^{(r)}(t) E(t - x) \frac{(t - x)^{r-1}}{(r-1)!} \, dt.
\]

Conversely, for any numbers \( f^{(i)}(b) \) and any continuous function \( f^{(r)}(t) \) on \([a, b]\) the function \( f(x) \) defined by this equation belongs to \( C_r[a, b] \).

The remainder \( R_n(f) \) is
12.3. The General Quadrature Process

$$R_n(f) = \sum_{i=0}^{r-1} \frac{f^{(i)}(b)}{i!} R_n[(x-b)^i] +$$

$$+ (-1)^r \int_a^b f^{(r)}(t) \left[ \int_a^t p(x) E(t-x) \frac{(t-x)^{n-1}}{(r-1)!} \, dx - \sum_{k=1}^{n} A^{(n)}_k E(t-x_k^{(n)}) \frac{(t-x_k^{(n)})^{n-1}}{(r-1)!} \right] dt =$$

$$= \sum_{i=0}^{r-1} \frac{f^{(i)}(b)}{i!} R_n[(x-b)^i] +$$

$$+ (-1)^r \int_a^b f^{(r)}(t) \left[ \int_a^t p(x) \frac{(t-x)^{n-1}}{(r-1)!} \, dx - F_{n,r-1}(t) \right] dt.$$  

(12.3.3)

Because the parameters $f^{(i)}(b)$ ($i = 0, 1, \ldots, r-1$) and $f^{(r)}(t)$ are independent, convergence of the quadrature process is equivalent to

$$R_n[(x-b)^i] \to 0 \quad (i = 0, 1, \ldots, r-1) \quad (12.3.4)$$

$$R^*_n(f^{(r)}) \to 0 \quad (12.3.5)$$

where

$$R^*_n(f^{(r)}) = \int_a^b f^{(r)}(t) \left[ \int_a^t p(x) \frac{(x-t)^{n-1}}{(r-1)!} \, dx - F_{n,r-1}(t) \right] dt.$$  

Condition (12.3.4) means that the quadrature process must converge for each polynomial of degree $\leq r-1$.

Condition (12.3.5) must be satisfied for any continuous function $f^{(r)}(t)$. Introducing the norm $\|f^{(r)}\| = \max |f^{(r)}(t)|$ for the class of functions $f^{(r)}(t)$ this class becomes the Banach space $\mathbb{C}$. By Theorem 1 of Section 4.3 condition (12.3.5) is equivalent to the two requirements:

1. The functional $R^*_n(f^{(r)})$ must tend to zero on a set of elements dense in $\mathbb{C}$. For this set we can take the set of polynomials. But the requirement that $R^*_n(f^{(r)}) \to 0$ as $n \to \infty$ when $f^{(r)}(t)$ is a polynomial together with (12.3.4) is the same as the condition that the quadrature process converge for polynomials.

2. The norm of the functionals $R^*_n$ ($n = 1, 2, \ldots$) must have a common bound:

$$\|R^*_n\| = \int_a^b \left| \int_a^t p(x) \frac{(t-x)^{n-1}}{(r-1)!} \, dx - F_{n,r-1}(t) \right| dt \leq L \quad (n = 1, 2, \ldots)$$
Since \( \int_a^b \int_a^t p(x) \frac{(t-x)^{r-1}}{(r-1)!} \, dx \, dt \) is independent of \( n \) then the boundedness of \( \| R_n^* \| \) is equivalent to

\[
\int_a^b |F_{n,r-1}(t)| \, dt \leq M \quad (n = 1, 2, \ldots). 
\]

Since \( \frac{d}{dt} F_{n,r}(t) = F_{n,r-1}(t) \) this last inequality is equivalent to

\[
\text{Var} \, F_{n,r}(t) \leq M \quad (n = 1, 2, \ldots). 
\]

It can also be shown that the above discussion is also valid for \( r = 0 \). This proves Theorem 11.

We mention a particular case of this theorem for the class of functions with a continuous derivative on \([a, b]\), that is the case \( r = 1 \). The function \( F_{n,0}(t) \) is the piece-wise constant function which has the values:

\[
F_{n,0}(t) = \sum_{k=1}^{n} A_k^{(n)} E(t-x_k^{(n)}) = \begin{cases} 
0 & \text{for } a \leq t < x_1^{(n)} \\
A_1^{(n)} & \text{for } x_1^{(n)} < t < x_2^{(n)} \\
A_1^{(n)} + A_2^{(n)} & \text{for } x_2^{(n)} < t < x_3^{(n)} \\
\cdots & \\
A_1^{(n)} + \cdots + A_n^{(n)} & \text{for } x_n^{(n)} < t \leq b.
\end{cases}
\]

Hence

\[
\text{Var} \, F_{n,1}(t) = \int_a^b |F_{n,0}(t)| \, dt = |A_1^{(n)}|(x_2^{(n)} - x_1^{(n)}) + \\
|A_1^{(n)} + A_2^{(n)}|(x_3^{(n)} - x_2^{(n)}) + \cdots + \\
|A_1^{(n)} + \cdots + A_n^{(n)}|(b - x_n^{(n)}).
\]

Therefore we have:

**Theorem 12.** In order that the quadrature process (12.1.9) converge for any function with a continuous derivative the following conditions are necessary and sufficient:

1. The process converge for each polynomial;
2. There exists a number \( M \) for which

\[
|A_1^{(n)}|(x_2^{(n)} - x_1^{(n)}) + |A_1^{(n)} + A_2^{(n)}|(x_3^{(n)} - x_2^{(n)}) + \\
\cdots + |A_1^{(n)} + \cdots + A_n^{(n)}|(b - x_n^{(n)}) \leq M 
\]

for \( n = 1, 2, \ldots \).
We will say that \( f \) belongs to the class \( A_r(a, b) \) if \( f^{(r)} \) is an absolutely continuous function.

If \( f \in A_r[a, b] \) then we can expand it in a Taylor series:

\[
f(x) = \sum_{i=0}^{r} \frac{f^{(i)}(b)}{i!} (x - b)^i + \int_b^x f^{(r+1)}(t) \frac{(x - t)^r}{r!} \, dt = \]

\[
= \sum_{i=0}^{r} \frac{f^{(i)}(b)}{i!} (x - b)^i + (-1)^{r+1} \int_a^b f^{(r+1)}(t) E(t - x) \frac{(t - x)^r}{r!} \, dt.
\]

Here \( f^{(i)}(b) \) \( (i = 0, 1, \ldots, r) \) are arbitrary numbers and \( f^{(r+1)}(t) \) is an arbitrary summable function on \([a, b]\).

**Theorem 13.** The following conditions are necessary and sufficient for the quadrature process to converge for each \( f \in A_r[a, b] \):

1. The process converge for each polynomial;
2. The primitive functions \( F_{n,r}(t) \) of order \( r \) for \( F_{n,0}(t) \) have a common bound

\[
|F_{n,r}(t)| \leq M, \quad a < x < b, \quad n = 1, 2, \ldots
\]

**Proof.** If \( f \in A_r[a, b] \) then from (12.3.7) the remainder \( R_n(f) \) can be expressed as

\[
R_n(f) = \sum_{i=0}^{r} \frac{f^{(i)}(b)}{i!} R_n[(x - b)^i] +
\]

\[
+ (-1)^{r+1} \int_a^b f^{(r+1)}(t) \left[ \int_a^t p(x) \frac{(t - x)^r}{r!} \, dx - F_{n,r}(t) \right] \, dt.
\]

Thus the convergence of the quadrature process in \( A_r[a, b] \) is equivalent to

\[
R_n[(x - b)^i] \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad (i = 0, 1, \ldots, r)
\]

and

\[
R_n^*(f^{(n+1)}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\]

\[
R_n^*(f^{(n+1)}) = \int_a^b f^{(r+1)}(t) \left[ \int_a^t p(x) \frac{(t - x)^r}{r!} \, dx - F_{n,r}(t) \right] \, dt.
\]

The rest of the argument is very similar to the argument used in proving Theorem 8. We introduce the norm
Approximate Calculation of Definite Integrals

\[ \| f^{(r+1)} \| = \int_a^b | f^{(r+1)}(t) | \, dt. \]

Thus the space of functions \( f^{(r+1)} \) coincides with the Banach space \( L \) and we can apply Theorem 1 of Section 4.3 to obtain a condition that \( R_n^* (f^{(r+1)}) \to 0 \). The set of polynomials is dense in \( L \). The requirement that \( R_n^* \) converge for each polynomial together with (12.3.9) is the same as the requirement that the quadrature process converge for each polynomial. By (4.2.6) the norm of \( R_n^* (f^{(r+1)}) \) is

\[ \| R_n^* \| = (b-a) \max_t \left| \int_a^t p(x) \frac{(t-x)^r}{r!} \, dx - F_{n,r}(t) \right|. \]

The integral \( \int_a^t p(x) \frac{(t-x)^r}{r!} \, dx \) is independent of \( n \) and thus the condition that \( \| R_n^* \| \) be bounded for \( n = 1, 2, \ldots \) is equivalent to the condition that

\[ | F_{n,r}(t) | \leq M, \quad a \leq t \leq b, \quad n = 1, 2, \ldots. \]

This completes the proof.

We now mention the particular case \( r = 0 \) for which \( A_0[a, b] \) is the class of absolutely continuous functions on \( [a, b] \). The function \( F_{n,0}(t) \) is the piece-wise constant function which has the values

\[ 0, \ A_1^{(n)}, \ A_1^{(n)} + A_2^{(n)}, \ldots, A_1^{(n)} + \cdots + A_n^{(n)} \]

on the segments

\[ [a, x_1^{(n)}], \ [x_1^{(n)}, x_2^{(n)}], \ldots, \ [x_n^{(n)}, b] \]

respectively. Thus we obtain as a corollary to the last theorem:

**Theorem 14.** The following conditions are necessary and sufficient for the quadrature process (12.1.3) to converge for each absolutely continuous function \( f \) on \( [a, b] \):

1. The process converge for each polynomial;
2. The partial sums of the quadrature coefficients

\[ A_1^{(n)}, \ A_1^{(n)} + A_2^{(n)}, \ldots, A_1^{(n)} + \cdots + A_n^{(n)}, \quad n = 1, 2, \ldots \]

have a common bound:

\[ \left| \sum_{k=1}^i A_k^{(n)} \right| \leq M < \infty, \quad i = 1, 2, \ldots, n, \quad n = 1, 2, \ldots \quad (12.3.11) \]

---

We now study convergence in one more class of functions. We will say that \( f \) belongs to the class \( V_r[a, b] \) if \( f^{(r)} \) is a function of bounded variation on \([a, b]\). The characteristic representation of a function in this class can also be obtained from the Taylor series:

\[
f(x) = \sum_{i=0}^{r} \frac{f^{(i)}(b)}{i!} (x - b)^i + \int_{b}^{x} \frac{(x - t)^r}{r!} df^{(r)}(t) = \sum_{i=0}^{r} \frac{f^{(i)}(b)}{i!} (x - b)^i + (-1)^{r+1} \int_{a}^{b} E(t-x) \frac{(t-x)^r}{r!} df^{(r)}(t). (12.3.12)
\]

The parameters \( f^{(i)}(b) \) are any numbers and \( f^{(r)}(t) \) is any function of bounded variation on \([a, b]\).

**Theorem 15.** In order that the quadrature process converge for each \( f \in V_r[a, b] \) for \( r \geq 1 \) it is necessary and sufficient that:

1. The process converge for all polynomials of degree \( \leq r \);
2. The primitive functions \( F_{n,r}(x) \) of order \( r \) for \( F_{n,0}(x) \) have a common bound

\[
|F_{n,r}(x)| \leq M < \infty, \quad a \leq x \leq b, \quad n = 1, 2, \ldots; \quad (12.3.13)
\]

3. For all \( t \in [a, b] \)

\[
F_{n,r}(t) \longrightarrow \int_{a}^{t} p(x) \frac{(t-x)^r}{r!} dx \quad \text{as} \quad n \to \infty.
\]

**Proof.** If \( f \in V_r[a, b] \) then using (12.3.12) the remainder \( R_n(f) \) can be represented in the form:

\[
R_n(f) = \sum_{i=0}^{r} \frac{f^{(i)}(b)}{i!} R_n[(x - b)^i] + \left(\int_{a}^{b} p(x) \frac{(t-x)^r}{r!} dx - F_{n,r}(t)\right) df^{(r)}(t).
\]

Since the parameters \( f^{(i)}(b) \) \( (i = 0, 1, \ldots, r) \) and \( f^{(r)}(t) \) are independent then the condition that the quadrature process converge for all functions of \( V_r[a, b] \) is equivalent to

\[
\lim_{n \to \infty} R_n[(x - b)^i] = 0, \quad i = 0, 1, \ldots, r \quad (12.3.14)
\]

and

\[
\lim_{n \to \infty} R_n^{*}(f^{(r)}) = 0 \quad (12.3.15)
\]
where

\[ R^*_n(f^{(r)}) = \int_a^b \left[ \int_a^t p(x) \frac{(t-x)^r}{r!} \, dx - F_{n,r}(t) \right] \, df^{(r)}(t). \]

The first of these conditions means that the process must converge for all polynomials of degree \( \leq r \). The functional \( R^*_n \) is defined on the linear space of functions of bounded variation. Without loss of generality we may assume that \( f^{(r)}(a) = 0 \). Then as a norm we take \( \|f^{(r)}\| = \text{var} f^{(r)} \).

The set of functions then becomes the Banach space \( V \). If \( R^*_n(f^{(r)}) \to 0 \) as \( n \to \infty \) for each \( f^{(r)} \) of \( V \) then by Theorem 1 of Section 4.3 the norms of the functionals \( R^*_n \) must have a common bound

\[ \| R^*_n \| \leq N \quad n = 1, 2, \ldots. \] (12.3.16)

But

\[ \| R^*_n \| = \max_t \left[ \int_a^t p(x) \frac{(t-x)^r}{r!} \, dx - F_{n,r}(t) \right] \]

and since the integral in this expression is independent of \( n \), condition (12.3.16) is equivalent to the second condition of the theorem.

To show the necessity of the third condition let \( x \) be an arbitrary point of \([a, b]\) and take \( f^{(r)} \) to be a piece-wise constant function with a jump of unity at \( x \). Then

\[ R^*_n(f^{(r)}) = \int_a^x p(u) \frac{(x-u)^r}{r!} \, du - F_{n,r}(x). \]

Such a function \( f^{(r)} \) determines the function \( f \) up to a polynomial of degree \( r - 1 \). If the quadrature process converges for this function then

\[ R^*_n(f^{(r)}) \to 0 \quad \text{as } n \to \infty. \]

This proves the necessity of the third condition.

We must still prove the sufficiency of all three conditions. The condition (12.3.14) is equivalent to the first condition of the theorem. There remains to be shown that the second and third conditions imply (12.3.15). But these conditions imply that

\[ \Phi_n(t) = \int_a^t p(x) \frac{(t-x)^r}{r!} \, dx - F_{n,r}(t) \]

is bounded in absolute value by a certain number for all \( t \in [a, b] \) and all \( n = 1, 2, \ldots \) and that for all \( t \in [a, b] \), \( \Phi_n(t) \to 0 \) as \( n \to \infty \). If we trans-
form the Stieltjes integral in $\mathbb{R}^n_+$ into a Lebesgue integral we can see that (12.3.15) will be satisfied.

REFERENCES


*The following theorem is known: If the functions $f_n(x)$ are measurable on $[a, b]$ and if $|f_n(x)| \leq N < \infty$ for all $n$ and if $f_n(x) \to f(x)$ almost everywhere on $[a, b]$ then

$$\int_a^b f_n(x) dx \to \int_a^b f(x) dx.$$
Part Three

APPROXIMATE CALCULATION OF INDEFINITE INTEGRALS
13.1. PRELIMINARY REMARKS

The problem of calculating an integral with variable limits has been studied considerably less than the problem of calculating a definite integral which we discussed in Part 2.

We mention here several examples of integrals with variable limits which occur in applications. We consider cases in which only one of the limits of integration is variable and the other is fixed.

The simplest integral of this kind occurs in the problem of finding a primitive function. If we are given a function \( f(x) \) which is continuous on the segment \([x_0, X] \) then any primitive of this function can be represented by the following formula:

\[
y(x) = y_0 + \int_{x_0}^{x} f(t) \, dt \quad x \in [x_0, X]
\]

and thus calculating \( y(x) \) is equivalent to finding the value of the integral \( \int_{x_0}^{x} f(t) \, dt \).

A more complicated example is the following integral which occurs in many applied problems:

\[
y(x) = \int_{a}^{x} K(x - t) f(t) \, dt.
\]

Here \( K(x - t) \) can be considered as a weight function whose value on \( a \leq t \leq x \) depends only on the distance \( x - t \) from the upper limit \( x \); \( f(x) \) is an arbitrary function of a certain class.
Another example is the Volterra integral equation

\[ f(x) = \phi(x) + \int_a^x K(x, t)f(t)\,dt \]

and certain other problems which involve the integral

\[ y(x) = \int_a^x K(x, t)f(t)\,dt \quad (13.1.3) \]

where the weight function \( K(x, t) \) is an arbitrary function of \( x \) and \( t \).

Methods for calculating the above integrals must take into account the properties of the weight function. For example, a computational scheme constructed for (13.1.3) can also be applied, in principle, to the calculation of the more special integral (13.1.1). Such a method, however, cannot be expected to be the very best for (13.1.1) since, for example, we might be able to use to advantage the fact that the weight function in (13.1.1) does not change sign. Thus we should develop separate methods for each of the above integrals.

In this book we will be exclusively concerned with the problem of calculating the integral (13.1.1).

Suppose it is necessary to calculate the value of (13.1.1) for a given set of values of the argument \( x: x_k \) \((k = 0, 1, 2, \ldots)\). We assume that the calculations have been carried up to step \( n \) and that we have constructed\(^1\) the following table of values of \( y(x_n) = y_n \). We wish to find \( y_{n+1} \). To do this we can use any of the previously calculated values \( y_k, k \leq n \), and any values of \( f(t) \) which are available for use.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 )</td>
<td>( y_0 )</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( y_1 )</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( x_n )</td>
<td>( y_n )</td>
</tr>
<tr>
<td>( x_{n+1} )</td>
<td></td>
</tr>
</tbody>
</table>

If \( f(t) \) is given by a table of its values at the nodes \( x_k \) we will be restricted in our possible choice of values of \( f(t) \) and any computational method will belong to the field of discrete analysis. One possible solution to the problem in this case is presented in Chapter 14.

For the present we assume that to compute \( y_{n+1} \) we may use values of \( f(t) \) at any points we wish and we assume only that the number of these points is fixed. In this case the points may be selected to reduce the

\(^1\)We do not consider, in this book, the problem of constructing the values of \( y(x) \) near the beginning or near the end of the table; we only consider the problem of continuing the table.
error in computing $y_{n+1}$. As in the problem of computing a definite integral it is often desirable to construct formulas of the highest algebraic degree of precision. Formulas of this type will be discussed in Chapters 15 and 16.

The construction of quadrature formulas of the highest algebraic degree of precision for definite integrals is related to the problem of calculating an integral to within a certain precision with the smallest number of integrand values and thus with the least amount of work. In indefinite integration an additional way to reduce the computational work is to use each value of $f(t)$ to calculate not just one value of $y(x)$ but for many steps in the computation.

In Chapters 15 and 16 we discuss this problem of constructing methods which use values of $f(x_k)$ and $y_k$ for calculating several values of $y(x)$. We discuss two methods in detail and do not attempt to treat all aspects of the problem.

The problem of calculating an indefinite integral has another special feature. One usually calculates $y(x)$ for a large number of values of $x$ by the repeated application of some particular method. Each step produces an approximate value for $y(x)$. As a rule, the error will accumulate and increase from step to step. The rate of growth of the error depends on the computational method and for some methods the error can grow very rapidly and in only a few steps produce an undesirably large error.

We can illustrate these remarks with a simple example of a method which gives good accuracy for a small number of steps but which is totally unsuitable when the number of steps is large.

In order to compute $y(x_{n+1})$ suppose we desire to use the two preceding values of $y(x)$ and also the values of its derivative $y'(x) = f(x)$ at these points: $y(x_n)$, $y(x_{n-1})$, $f(x_n)$, $f(x_{n-1})$. Then it is natural to construct an interpolating polynomial using these values of the function and its derivative. This will be the Hermite interpolating polynomial with the two double nodes $x_n$ and $x_{n-1}$. As can be verified from (3.3.8) this polynomial will be

$$y(x_{n+1}) = -4y(x_n) + 5y(x_{n-1}) + h[4f(x_n) + 2f(x_{n-1})] + r_n(x).$$

If we neglect the remainder $r_n(x)$ we obtain the approximate formula

$$y_{n+1} = -4y_n + 5y_{n-1} + h(4f_n + f_{n-1}) \quad (13.1.4)$$

which is exact for all algebraic polynomials of degree $\leq 3$. To use this formula we must know the first two values of $y(x)$: $y_0$ and $y_1$. Let us use (13.1.4) to evaluate the integral

$$y(x) = \int_0^x e^t \, dt = e^x - 1$$
on the segment \([0, 1]\). At first we take \(h = 0.2\) and assume that \(y(0) = 0\) and \(y(0.2) = 0.22140\) are known and from these values calculate the following table which gives the approximate values of \(y(x)\) together with the errors in these values.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x))</th>
<th>(y_{\text{approx}})</th>
<th>(y - y_{\text{approx}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.00000</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>1.22140</td>
<td>0.22140</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>1.49182</td>
<td>0.49152 + 0.00030</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>1.82212</td>
<td>0.82294 - 0.00082</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>2.22554</td>
<td>1.22026 + 0.00528</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>2.71828</td>
<td>1.74294 - 0.02466</td>
<td></td>
</tr>
</tbody>
</table>

This table shows that the error grows very rapidly as we go farther down the table. The number of significant figures in the calculation shows that the large error is not due to rounding but to other causes.

We can easily see that the rapid rate of growth of the error is not due to the large interval size and that it cannot be corrected by decreasing \(h\). In fact let us try to obtain a more exact value of the integral by decreasing the step size to \(h = 0.1\).

Here again we assume that we know the first two values of \(y(x)\): \(y(0) = 0\), \(y(0.1) = 0.10517\). The new table is as follows.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x))</th>
<th>(y_{\text{approx}})</th>
<th>(y - y_{\text{approx}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.00000</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>1.10517</td>
<td>0.10517</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>1.22140</td>
<td>0.22139 + 0.00001</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>1.34986</td>
<td>0.34988 - 0.00002</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>1.49182</td>
<td>0.49165 + 0.00017</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>1.64872</td>
<td>0.64950 - 0.00078</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>1.82212</td>
<td>0.81810 + 0.00402</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>2.01375</td>
<td>1.08610 - 0.02235</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>2.22554</td>
<td>1.11602 + 0.10952</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>2.45960</td>
<td>2.01039 - 0.55079</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>2.71828</td>
<td>-1.08251 + 2.75079</td>
<td></td>
</tr>
</tbody>
</table>

This smaller interval size gives a smaller error for only the single value \(y(0.4)\). The error grows so rapidly that at the end of the table the error exceeds the size of the function.

It is easy to see that the rapid rate of growth of the error in this example depends entirely on the unsuitable form of the computational method. To calculate the integral in

\[
y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(t) \, dt
\]

let us use the simple trapezoidal formula
13.2. The Error of the Computation

\[ y_{n+1} = y_n + \frac{h}{2} (f_n + f_{n+1}) \]  

(13.1.5)

which is exact when \( f(x) \) is any linear function. The algebraic degree of precision of this formula is thus less than that of (13.1.4) and one might expect the values of \( y_n \) obtained using (13.1.5) to be less exact than those obtained using (13.1.4). The table below shows that this is indeed true at the beginning of the table. However, the error grows at a much slower rate and the value of \( y(1.0) \) is much more exact than that obtained in the previous case.

<table>
<thead>
<tr>
<th>x</th>
<th>( y_{\text{approx}} )</th>
<th>( y - y_{\text{approx}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00000</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.10526</td>
<td>-0.00009</td>
</tr>
<tr>
<td>0.2</td>
<td>0.22159</td>
<td>-0.00019</td>
</tr>
<tr>
<td>0.3</td>
<td>0.35015</td>
<td>-0.00029</td>
</tr>
<tr>
<td>0.4</td>
<td>0.49223</td>
<td>-0.00041</td>
</tr>
<tr>
<td>0.5</td>
<td>0.64926</td>
<td>-0.00054</td>
</tr>
<tr>
<td>0.6</td>
<td>0.82280</td>
<td>-0.00068</td>
</tr>
<tr>
<td>0.7</td>
<td>1.01459</td>
<td>-0.00084</td>
</tr>
<tr>
<td>0.8</td>
<td>1.22656</td>
<td>-0.00102</td>
</tr>
<tr>
<td>0.9</td>
<td>1.46082</td>
<td>-0.00122</td>
</tr>
<tr>
<td>1.0</td>
<td>1.71971</td>
<td>-0.00143</td>
</tr>
</tbody>
</table>

Thus it is clear that (13.1.5) is the better of the two formulas for a large number of intervals.

13.2. THE ERROR OF THE COMPUTATION

We denote the exact value of the function

\[ y(x) = y_0 + \int_{x_0}^{x} f(t) \, dt \]

at the nodes \( x_k \) by \( y(x_k) \) \((k = 0, 1, \ldots)\). The approximate values of \( y(x_k) \) which are calculated by some computational method we will denote by \( y_k \).

To calculate \( y_{n+1} \) let us assume that we use several preceding values of \( y(x) \), \( y_n, y_{n-1}, \ldots, y_{n-p} \), and \( m = m(n) \) values of \( f(x) \) at the points \( \xi_{n,j} \) \((j = 1, \ldots, m)\). Thus we assume that the computational formula has the following form:

\[ Y_{n+1} = A \cdot y_n + B \cdot f_n \]

The coefficients \( A_{n,i} \) and \( B_{n,j} \) in this equation may depend on \( n \) so that the computational formula would be changed at each step. The step size \( h \) may also change from step to step. Equation (13.2.1) is an equation in finite differences for the \( y_k \) and in our discussion it is only necessary that this equation has a certain fixed order \( p + 1 \).
Approximate Calculation of Indefinite Integrals

\[ y_{n+1} = \sum_{i=0}^{p} A_{n,i} y_{n-i} + \sum_{j=1}^{m} B_{n,j} f(\xi_{n,j}). \quad (13.2.1) \]

If in this equation we substitute the exact values \( y(x_k) \) in place of the approximate values \( y_k \) then the equation will be an approximation and it is necessary to add an auxiliary term in order to make it exact

\[ y(x_{n+1}) = \sum_{i=0}^{p} A_{n,i} y(x_{n-i}) + \sum_{j=1}^{m} B_{n,j} f(\xi_{n,j}) + r_n. \quad (13.2.2) \]

We will call \( r_n \) the error in formula (13.2.1).

In the form in which (13.2.1) is written we have assumed that the computation is carried out using exact (unrounded) numbers. This, however, will happen only very rarely. This formula must be modified to indicate the method used for rounding. If the operation of rounding is indicated by enclosing the quantity to be rounded in curly brackets then the computational formula is more exactly written as

\[ y_{n+1} = \left\{ \sum_{i=0}^{p} A_{n,i} y_{n-i} + \sum_{j=1}^{m} B_{n,j} f(\xi_{n,j}) \right\}_n \quad (13.2.3) \]

where the subscript \( n \) outside the brackets indicates that the rule for rounding can be changed at each step.

To use (13.2.3) we must know the initial values \( y_0, y_1, \ldots, y_p \) and we assume that these are given. We will now construct a difference equation for the error

\[ \epsilon_k = y(x_k) - y_k. \]

If we denote by \(-\alpha_n\) the rounding error which we indicated by brackets in (13.2.3) then (13.2.3) becomes

\[ y_{n+1} = \sum_{i=0}^{p} A_{n,i} y_{n-i} + \sum_{j=1}^{m} B_{n,j} f(\xi_{n,j}) - \alpha_n. \quad (13.2.4) \]

Subtracting (13.2.4) from (13.2.2) gives

\[ \epsilon_{n+1} = \sum_{i=0}^{p} A_{n,i} \epsilon_{n-i} + r_n + \alpha_n. \quad (13.2.5) \]

If the initial values of the error \( \epsilon_k \) \((k = 0, 1, \ldots, p)\) corresponding to the approximate values \( y_k \) \((k = 0, 1, \ldots, p)\) formed at the start of the computation of the table are known then all following values of \( \epsilon_k \) \((k > p)\) can be sequentially found from equation (13.2.5).
13.2. The Error of the Computation

The errors $\epsilon_n$ ($n > p$) depend first of all on the values $\epsilon_0, \ldots, \epsilon_p$, secondly on the rounding errors $\alpha_k$ ($k < n$) and finally on the errors of the formula (13.2.1) $r_k$ ($k < n$).

To analyze the error it will be useful to determine how each of the above three factors separately affect $\epsilon_n$. To do this we will write $\epsilon_n$ as a sum of three terms which correspond to the errors from each of the three sources:

$$\epsilon_n = E_n + E'_n + E''_n. \tag{13.2.6}$$

Here $E_n$ is the solution of the homogeneous equation

$$E_{n+1} = \sum_{i=0}^{p} A_{n,i} E_{n-i} \tag{13.2.7}$$

subject to the initial conditions

$$E_k = \epsilon_k \quad k = 0, 1, \ldots, p. \tag{13.2.8}$$

The term $E'_n$ satisfies the nonhomogeneous equation

$$E'_{n+1} = \sum_{i=0}^{p} A_{n,i} E'_{n-i} + \alpha_n \tag{13.2.9}$$

and has the initial conditions

$$E'_k = 0, \quad k = 0, 1, \ldots, p. \tag{13.2.10}$$

The term $E''_n$ is the solution of the nonhomogeneous equation

$$E''_{n+1} = \sum_{i=0}^{p} A_{n,i} E''_{n-i} + r_n \tag{13.2.11}$$

also with the initial conditions

$$E''_k = 0, \quad k = 0, 1, \ldots, p. \tag{13.2.12}$$

Here $E_n$ is the part of the error $\epsilon_n$ due to the errors $\epsilon_0, \ldots, \epsilon_p$ in the initial values, $E'_n$ is the part of $\epsilon_n$ due to rounding, and $E''_n$ is the part of $\epsilon_n$ due to the error $r_n$ in formula (13.2.1).

A simple expression for $E_n$ in terms of $\epsilon_k$ ($k \leq p$) which will suffice for our purposes can be constructed in the following way. Denote by $E_n^i$ the solution of the homogeneous equation (13.2.7) which satisfies the conditions

$$E_n^i = \begin{cases} 0 & k \neq i \\ 1 & k = i \end{cases}, \quad i, k = 0, 1, \ldots, p.$$
Approximate Calculation of Indefinite Integrals

Then clearly

$$E_n = E_n^0 \varepsilon_0 + E_n^1 \varepsilon_1 + \cdots + E_n^p \varepsilon_p.$$  \hspace{1cm} (13.2.13)

Hence we can easily obtain an estimate for $E_n$. We will seldom know the exact values of the errors $\varepsilon_k$ ($k \leq p$) but we will know that their absolute values do not exceed a certain number $\varepsilon$:

$$|\varepsilon_k| \leq \varepsilon \quad k \leq p.$$  \hspace{1cm} (13.2.14)

If we assume that the initial errors $\varepsilon_k$ can have arbitrary values subject to (13.2.14) then from (13.2.13) we obtain the following estimate

$$|E_n| \leq \varepsilon \sum_{k=0}^{p} |E_n^k|.$$  \hspace{1cm} (13.2.15)

Equation (13.2.13) or (13.2.15) permits us to determine how precisely we must calculate the initial values $\gamma_k$ ($k \leq p$) in order that $E_n$ does not exceed a predetermined value.

Now we consider the second part of the error $E'_n$. It must be found from equation (13.2.9) with the initial conditions (13.2.10). We see at once that $E'_n$ is a linear combination of $\alpha_p, \alpha_{p+1}, \ldots, \alpha_{n-1}$:

$$E'_n = \sum_{k=p}^{n-1} E_{n, k} \alpha_k.$$  \hspace{1cm} (13.2.16)

The coefficient $E_{n, k}$ is the influence on $E'_n$ of a rounding error of a unit in the right side of (13.2.3) for $n = k$. The $E_{n, k}$ are Green's functions or functions of influence for the above problem.

In the theory of difference equations\(^3\) an explicit expression for $E_{n, k}$ is obtained in terms of the solutions of the homogeneous equation (13.2.9). We do not give it here because of its complexity.

For our purpose it is useful to note that $E_{n, k}$ is the solution of the equation

$$E_{n+1} = \sum_{i=0}^{p} A_n, i E_{n-i} + \delta_{n, k}.$$  \hspace{1cm} (13.2.17)

which satisfies the initial conditions

$$E_i = 0, \quad i = 0, 1, \ldots, p$$

\(^3\)See A. A. Markov, Calculus of Finite Differences, Part II, Sec. 19, Moscow, 1911 (Russian) or A. O. Gel'fond, Calculus of Finite Differences, Part 3, Sec. 3, Moscow, 1936 (Russian).
13.2. The Error of the Computation

where $\delta_{n,k}$ is the Kronecker symbol

$$
\delta_{n,k} = \begin{cases} 
0 & n \neq k \\
1 & n = k.
\end{cases}
$$

From the sum (13.2.6) we can determine the number of significant figures which must be used in order that $E_n^*$ will not exceed a given value.

Suppose we know that for all steps of the computation the errors $\alpha_n$ do not exceed $\alpha$:

$$
|\alpha_n| \leq \alpha.
$$

Then from (13.2.16) we obtain the inequality

$$
|E_n^*| \leq \alpha \sum_{k=p}^{n-1} |E_{n,k}|.
$$

(13.2.18)

The quantities $E_n$ and $E_n^*$ depend on the precision of the initial values $\gamma_0, \gamma_1, \ldots, \gamma_p$ and on the number of significant figures carried in the calculations. These quantities can be made as small in absolute value as we desire for each $n \leq N$.

We turn, finally, to the last part of the error $E_n^{**}$. The difference equation (13.2.11) for $E_n^{**}$ is obtained from (13.2.9) by replacing the constant term $\alpha_n$ by $r_n$. The initial conditions for both $E_n^*$ and $E_n^{**}$ are the same. Therefore an equation similar to (13.2.16) is valid for $E_n^{**}$ with $\alpha_n$ replaced by $r_n$:

$$
E_n^{**} = \sum_{k=p}^{n-1} E_{n,k} r_k.
$$

(13.2.19)

The error $E_n^{**}$ depends entirely on the form of the computational formula (13.2.1) or to be more precise on the remainders $r_k$, the coefficients $A_{n,i}$ and on the number of steps $n$.

In the next section, where we study the convergence of computational formulas, the sum $\sum_{k=p}^{n-1} E_{n,k} r_k$ will be discussed in more detail.

As an example let us analyze the error of equation (13.1.4) which we used in the last section to evaluate the integral

$$
\gamma(x) = \int_0^x e^t \, dt.
$$
The expression (13.2.5) for \( e_n \) which corresponds to equation (13.1.4) is

\[
e_{n+1} = -4e_n + 5e_{n-1} + r_n + \alpha_n.
\]

This is a nonhomogeneous difference equation of the second order with constant coefficients and constant term \( r_n + \alpha_n \). To solve this equation the initial values \( e_0 \) and \( e_1 \) of the error must be known.

Let us find the first part of the error \( E_n \) which depends on \( e_0 \) and \( e_1 \). The homogeneous equation for \( E_n \) is

\[
E_{n+1} = -4E_n + 5E_{n-1}.
\]

The solution of this equation for the initial conditions \( E_0 = e_0 \) and \( E_1 = e_1 \) is

\[
E_n = \frac{1}{6} (e_1 + 5e_0) + \frac{(-1)^n}{6} (e_0 - e_1) 5^n.
\]

If \( e_0 - e_1 \neq 0 \), \( E_n \) grows very rapidly as \( n \) increases. For \( n = 10 \), that is for only 9 steps in the computation, the coefficient of \( e_0 - e_1 \) is \( 5^{10}/6 = 1.5 \times 10^6 \) which will cause the loss of 6 significant figures in the computations.

Now we investigate \( E'_n \), the error due to rounding. The nonhomogeneous equation (13.2.9) for \( E'_n \) is

\[
E'_{n+1} = -4E'_n + 5E'_{n-1} + \alpha_n.
\]

The solution of this equation for the initial conditions \( E'_0 = 0 \), \( E'_1 = 0 \) can be easily found:

\[
E' = \frac{1}{6} \sum_{i=0}^{n-1} [1 - (-5)^{n-i-1}] a_{i+1} =
\]

\[
= \frac{1}{6} \{[1 - (-5)^{n-1}]a_1 + [1 - (-5)^{n-2}]a_2 + \cdots \}.
\]

As with \( E_n \), we see that \( E'_n \) can grow rapidly as \( n \) increases and it can become large even in a small number of steps. A similar remark holds for \( E''_n \).

The rapidity with which the error grows for formula (13.1.4) is illustrated by the computations of the previous section.

Let us again consider the general problem of studying the error \( e_n \). The behavior of \( e_n \) as \( n \) increases naturally depends on the coefficients \( A_{n,i} \).

Let us consider the special case when all the coefficients \( A_{n,i} \) are
positive. Suppose also that
\[ \sum_{i=0}^{p} A_{n,i} = 1. \]  (13.2.20)

This means that in a calculation without rounding the formula will be exact when \( f(t) = 0 \) and \( y(x) \) is a constant.

With these assumptions we can find a very simple and effective estimate for \( \epsilon_n \). Let us suppose that the initial errors \( \epsilon_0, \ldots, \epsilon_p \) do not exceed \( \epsilon \) in absolute value:
\[ |\epsilon_i| \leq \epsilon \quad i = 0, \ldots, p. \]

We can show that for any \( n \) the following inequality is valid:
\[ |\epsilon_n| \leq \epsilon + \sum_{k=p}^{n-1} |\alpha_k + r_k|. \]  (13.2.21)

For \( n = p + 1 \) we easily verify
\[
|\epsilon_{p+1}| = \left| \sum_{i=0}^{p} A_{p,i} \epsilon_i + \alpha_p + r_p \right| \leq \\
\leq \sum_{i=0}^{p} A_{p,i} \epsilon + |\alpha_p + r_p| = \epsilon + |\alpha_p + r_p|.
\]

Assuming that the inequality is true for all \( \epsilon_i, i \leq n \), we can show that it is also true for \( \epsilon_{n+1} \). We have
\[
|\epsilon_{n+1}| \leq \sum_{i=0}^{p} A_{n,i} |\epsilon_{n-i}| + |\alpha_n + r_n|.
\]

Substituting for the \( |\epsilon_{n-i}| \) the larger value \( \epsilon + \sum_{i=0}^{n-1} |\alpha_i + r_i| \) we obtain
\[
|\epsilon_{n+1}| \leq \epsilon + \sum_{i=p}^{n-1} |\alpha_i + r_i| + |\alpha_n + r_n|
\]
which proves the assertion.

From (13.2.21) we see that in a computational formula with nonnegative coefficients \( A_{n,i} \) the errors \( \epsilon_n \) will "grow slowly" as a function of \( n \). In this respect this type of formula is very well behaved.
13.3. CONVERGENCE AND STABILITY OF THE COMPUTATIONAL PROCESS

First of all we will clarify certain concepts concerning the problem of convergence of a computational process. To simplify the discussion we will assume that the formula is of a certain special form which is most often used in practical problems. We assume that the segment \([x_0, X]\) on which the function \(y(x)\) is to be calculated is finite and that values of \(y(x)\) are to be found at a set of equally spaced points

\[x_k = x_0 + kh, \quad k = 0, 1, \ldots, N\]

\[x_0 + Nh \leq X < x_0 + (N + 1)h\]

which we denote by \(S_h\).

Suppose that the coefficients of the computational formula do not depend on \(n\):

\[y(x_{n+1}) = \sum_{i=0}^{p} A_i y(x_i) + \sum_{j=1}^{m} B_{n,j} f(\xi_{n,j}) + r_n.\]  \hspace{1cm} (13.3.1)

The computational method is thus obtained by neglecting the term \(r_n\) and rounding the sum to a certain number of significant figures

\[y_{n+1} = \left\{ \sum_{i=0}^{p} A_i y_n - i + \sum_{j=1}^{m} B_{n,j} f(\xi_{n,j}) \right\} / n.\]  \hspace{1cm} (13.3.2)

If \(y_0, \ldots, y_p\) are known we can find from (13.3.2) the approximate values \(y_n\) corresponding to the values \(y(x_n)\) on the set \(S_h\).

We define the distance \(\rho(y, y_n)\) between \(y(x)\) and the function \(y_n\) \((n = 0, 1, \ldots, N)\) which is defined on \(S_h\) to be the largest absolute value of the error \(\epsilon_n = y(x_n) - y_n\):

\[\rho(y, y_n) = \max_n |\epsilon_n| = \max_n |y(x_n) - y_n|.\]

We will say that the computational process converges if, as \(h \to 0\), we have

\[\rho(y, y_n) \to 0.\]  \hspace{1cm} (13.3.3)

The error \(\epsilon_n\) depends on the errors \(\epsilon_0, \epsilon_1, \ldots, \epsilon_p\) of the initial values \(y_k\) \((k = 0, 1, \ldots, p)\), the rounding error \(\alpha_n\) and the remainder \(r_n\) of formula (13.3.1). As in the preceding section we split the error \(\epsilon_n\) into three parts and discuss how each of these parts influences \(\epsilon_n\):

\[\epsilon_n = E_n + E'_n + E''_n.\]
13.3. Convergence and Stability

In the preceding section we discussed the conditions which \( E_n \), \( E'_n \) and \( E''_n \) must satisfy.

Since each of the quantities \( \epsilon_i \) \((i \leq p)\), \( \alpha_n \) and \( r_n \) are independent then in order that \( \rho(y, y_n) \to 0 \) as \( h \to 0 \) we must require that the following three conditions be satisfied:

\[
\max_n |E_n| \to 0, \quad \max_n |E'_n| \to 0, \quad \max_n |E''_n| \to 0. \tag{13.3.4}
\]

The errors \( E_n \) and \( E'_n \) depend respectively on the \( \epsilon_i \) \((i \leq p)\) and \( \alpha_n \). Thus it is clear that for any fixed \( h \) the precision of the initial values \( y_i \) \((i \leq n)\) and the rounding errors can be made as small as we desire so that \( \max_n |E_n| \) and \( \max_n |E'_n| \) can be made arbitrarily small. Therefore it is only a technical problem to obtain conditions which must be satisfied if the first two conditions (13.3.4) are to be fulfilled. As \( h \) decreases we must determine how the accuracy of the initial values \( y_i \) \((i \leq n)\) must be increased and how the number of significant figures must be increased so that the error in \( \epsilon_n \) due to these quantities will tend to zero. Such an investigation gives a criterion for testing the practical suitability of the computational formula and thus will be very valuable. If it turns out that as \( h \) decreases the accuracy of these quantities must rapidly increase then such a computational formula must be rejected as being unsuitable in most cases.

With these remarks in mind we must prefer computational formulas for which the precision of the initial values \( y_i \) \((i \leq p)\) and the number of significant figures must increase the slowest as \( h \to 0 \). This can also be expressed in another way. Consider for example \( E_n \). Suppose that the initial values \( y_i \) \((i \leq p)\) have certain errors \( \epsilon_i \). In the computation of the succeeding values \( y_i \) \((i > p)\) the error will grow from step to step. The rate of growth clearly depends on the choice of the computational formula. The computational formulas which are of most interest are those for which the rate of growth is minimal. In the theory of the approximate solution of differential equations methods which have the minimal rate of growth of the error are called stable. Thus we will say that the formula is stable with respect to the errors in the initial values if the rate of growth of \( E_n \) is minimal. In a similar way we can define stability with respect to the rounding errors \( \alpha_n \), that is the errors in the right side of (13.3.2).

We now discuss the error \( E_n \) in more detail. The homogeneous equation for \( E_n \) for formula (13.3.2) will be an equation with constant coefficients

\[
E_{n+1} = \sum_{i=0}^{p} A_i E_{n-i}. \tag{13.3.5}
\]
As we saw in the last section the solution of this equation, which satisfies the initial conditions $E_i = \epsilon_i (i \leq p)$, can be written in the form

$$ E_n = E_0^n + E_n^1 \epsilon_1 + \cdots + E_n^p \epsilon_p $$  \hspace{1cm} (13.3.6)

where $E_n^i$ is the solution of (13.3.5) for the initial conditions

$$ E_{k}^i = \begin{cases} 0 & k \neq i \\ 1 & k = i \end{cases} \quad i, k = 0, 1, \ldots, p. $$

Thus the rate of growth of $E_n$ is related to the rate of growth of $E_n^i$. If we assume that the initial errors are bounded in absolute value by $\epsilon$

$$ |\epsilon_i| \leq \epsilon, \quad i = 0, 1, \ldots, p $$  \hspace{1cm} (13.3.7)

then the following estimate will be valid for $E_n$

$$ |E_n| \leq \epsilon \sum_{i=0}^{p} |E_n^i|. $$  \hspace{1cm} (13.3.8)

We will assume that formula (13.3.1) is exact (that is $r_n = 0$) when $f(x) = 0$ and $y(x)$ is a constant. This will be true in most practical cases. Then the coefficients $A_i$ must satisfy

$$ \sum_{i=0}^{p} A_i = 1. $$  \hspace{1cm} (13.3.9)

This says that $E_n = 1$ is a solution of the homogeneous equation (13.3.5). This solution is the sum of all the $E_n^i$:

$$ 1 = E_n^0 + E_n^1 + \cdots + E_n^p. $$

Thus for each $n$ we have the inequality

$$ \sum_{i=0}^{p} |E_n^i| \geq 1. $$

It is possible to give examples for which $\sum_{i=0}^{p} |E_n^i|$ will grow without bound as $n \to \infty$ and it can also turn out then that $E_n$ will be unbounded.

The most well behaved formulas with respect to the rate of growth of $E_n$ are clearly those for which the sum $\sum_{i=0}^{p} E_n^i$ is bounded\footnote{We will only need to know that this sum is bounded. We will not discuss the problem of finding a bound.} for $n > p$.\footnote{We will only need to know that this sum is bounded. We will not discuss the problem of finding a bound.}
Thus we are led to the following definition:

Equation (13.3.2) is said to be stable with respect to the initial values \( y_i \) \((i \leq p)\) if there exists a number \( M \) such that for any \( n \) the following inequality is satisfied

\[
|E_n| \leq M \varepsilon \tag{13.3.10}
\]

where \( |\varepsilon_i| \leq \varepsilon \), \( i = 0, 1, \ldots, p \).

We note that the boundedness of \( E_n \) \((n = 0, 1, \ldots)\) together with the condition \( |\varepsilon_i| \leq \varepsilon \) is equivalent to the boundedness of all the \( E_i^n, i = 0, 1, \ldots, p \). In fact if all the \( E_i^n \) are bounded then from (13.3.6) it follows that \( E_n \) is also bounded.

Let us take an arbitrary \( k \leq p \) and assume that all the \( \varepsilon_k (k \neq i, k \leq p) \) are zero. Then

\[
E_n = E_i^n \varepsilon_i
\]

and if \( E_n \) is bounded then \( E_i^n \) is also bounded.

The most general solution of (13.3.5) is determined by the algebraic equation

\[
\lambda^{p+1} = \sum_{i=0}^{p} A_i \lambda^{p-i}.
\]

Let \( \lambda_1, \lambda_2, \ldots, \lambda_m \) denote the distinct roots of this equation and let \( k_1, k_2, \ldots, k_m \) be the multiplicities of these roots. Then the functions

\[
\lambda_i^j n^i \quad (j = 0, 1, \ldots, k_i - 1; \quad i = 1, 2, \ldots, m) \tag{13.3.11}
\]

form a complete system of linearly independent solutions.

The solutions \( E_i^n \) \((i = 0, 1, \ldots, p)\) are obtained from (13.3.11) by a transformation with a nonsingular matrix and therefore the boundedness of all the \( E_i^n \) for \( i = 0, 1, \ldots \) is equivalent to the boundedness of the solutions (13.3.11). This occurs if and only if there are no \( \lambda_i \) greater than 1 in modulus and if \( |\lambda_i| = 1 \) then \( k_i = 1 \). Thus we have established:

**Theorem 1.** In order that equation (13.3.2) be stable with respect to the errors in the initial values \( y_i \) \((i \leq p)\) it is necessary and sufficient that

1. The roots of the equation \( \lambda^{p+1} = \sum_{i=0}^{p} A_i \lambda^{p-i} \) do not exceed unity in modulus.
2. Any root of modulus unity must be simple.

We now study \( E_n^\varepsilon \) which is the error due to the effect of the rounding errors \( a_p, \ldots, a_{n-1} \). The error \( E_n^\varepsilon \) satisfies equation (13.2.16):
Approximate Calculation of Indefinite Integrals

\[ E_n' = \sum_{k=p}^{n-1} E_{n,k} \alpha_k. \]

The coefficients \( E_{n,k} \), as functions of \( n \), must satisfy (13.2.17) which in the present case is

\[ E_{n+1}' = \sum_{i=0}^{p} A_i E_{n-i}' + \delta_{n,k} \]  
(13.3.12)

with initial values

\[ E_{i,k} = 0, \quad i = 0, 1, \ldots, p. \]
(13.3.13)

We can establish a simple relationship between \( E_{n,k} \) and the solution \( E_n^p \) which we discussed above. For \( n < k \) equation (13.3.12) will be homogeneous and in view of the zero initial conditions \( E_{n,k} \) will be zero for each \( n \leq k \). In addition \( E_{k+1,k} = 1 \) which can be seen from (13.3.12) by putting \( n = k \). When \( n > k \) equation (13.3.12) will also be homogeneous.

Let us consider \( E_{n,k} \) for \( n \geq k - p + 1 \). From the above discussion we can assume that \( E_{n,k} \) has the initial values

\[ E_{k-p+1,k} = 0, \ldots, E_{k,k} = 0, \quad E_{k+1,k} = 1 \]
and that it satisfies the homogeneous equation

\[ E_{n+1}' = \sum_{i=0}^{p} A_i E_{n-i}'. \]  
(13.3.14)

But we at once see that these same conditions are also satisfied by \( E_{n+p-k-1}^p \) and, since the solution is unique for fixed initial conditions, \( E_{n,k} \) and \( E_{n+p-k-1}^p \) must coincide.

Thus we obtain

\[ E_n' = \sum_{k=p}^{n-1} \alpha_k E_{n+p-k-1}^p. \]  
(13.3.15)

We will assume that \( \alpha \) is an upper bound for the rounding errors \( \alpha_n \) for all \( n, |\alpha_n| \leq \alpha \). Then

\[ |E_n'| \leq \alpha \sum_{k=p}^{n-1} |E_{n+p-k-1}^p| \]

\[ \max_n |E_n'| \leq \alpha \sum_{k=p}^{N-1} |E_{N+p-k-1}^p| = \alpha \sum_{k=p}^{N-1} |E_k^p|. \]  
(13.3.16)
13.3. Convergence and Stability

If we assume that the errors $a_n$ can have any values subject to the condition $|a_n| \leq \alpha$ then the above estimate cannot be improved and equality is achieved for $n = N$ when $a_k = \alpha \operatorname{sign} E_{N+p-k-1}^P$. Because $E_P^P = 1$ then for each $N \geq p + 1$ we have $\sum_{k=p}^{N-1} |E_k^P| \geq 1$. As $h$ tends to zero $N$ grows without bound. The value of $\sum_{k=p}^{N-1} |E_k^P|$ will depend on the behavior of the solutions $E_k^P$ as $k \to \infty$.

Let us consider the particular solutions of the homogeneous equation (13.3.14)

$$E_n^P, \quad E_{n+1}^P, \quad \ldots, \quad E_{n+p}^P.$$  \hspace{1cm} (13.3.17)

Their initial values for $n = 0, 1, \ldots, p$ form the following matrix

$$
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & \cdots & 0 & 1 & E_{p+1}^P \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & E_{p+1}^P & \cdots & E_{2p-2}^P & E_{2p-1}^P & E_{2p}^P
\end{bmatrix}
$$

The determinant of this matrix is different from zero and therefore the solutions of (13.3.17) are linearly independent. Thus these solutions are obtained from the $E_n^i$ $(i = 0, 1, \ldots, p)$ by a nonsingular linear transformation.

Therefore the boundedness of $E_n^i$ $(i = 0, 1, \ldots, p)$ is equivalent to the boundedness of the solutions (13.3.17).

From the assumption (13.3.9) we saw that

$$\sum_{i=0}^{p} |E_n^i| \geq 1, \quad n = 0, 1, \ldots$$

and in this case the slowest rate of growth of $\sum_{k=p}^{N-1} |E_k^P|$, as $N \to \infty$, will occur when all the terms $E_k^P$ in this sum are bounded by a certain number. Then $\sum_{k=p}^{N-1} |E_k^P|$ will be of the order of magnitude $O(N)$. Thus we are led to the following definition:

Equation (13.3.2) is said to be stable with respect to the rounding errors $a_n$ if there exists a number $M_1$, which is independent of $h$, with
the property that for each $N > p$ we have

$$|E'_n| \leq M_1 N \alpha, \quad (n = p + 1, \ldots, N - 1) \quad (13.3.18)$$

where $|\alpha_n| \leq \alpha$.

A simple theorem which gives a sufficient condition for stability is:

**Theorem 2.** In order that (13.3.2) be stable with respect to the rounding error it is sufficient that the following two conditions be fulfilled:

1. The equation $\lambda^{p+1} = \sum_{i=0}^{p} A_i \lambda^{p-i}$ has no roots of modulus greater than unity.
2. Any roots of modulus unity are simple.

**Proof.** If the conditions of the theorem are satisfied then the solutions (13.3.11) will be bounded for $n \geq 0$. These solutions are a complete system of solutions and the $E^p_n$ are linear combinations of them. Thus there exists a number $M_1$ which, for $n \geq 0$, satisfies

$$|E^p_n| \leq M_1.$$

Combining this with (13.3.16) establishes the theorem.

We now study $E''_n$ which is the part of the error due to the error $r_n$ in (13.3.1). The error $\varepsilon_n$ will coincide with $E'_n$ if the computations are carried out using exact initial values $y_k = y(x_k)$ ($k = 0, 1, \ldots, p$) and if no rounding needs to be performed.

We will say that formula (13.3.1) provides a convergent computational process if

$$\max_n |E'_n| \to 0 \quad \text{as} \quad h \to 0. \quad (13.3.19)$$

Since $E_n, k = E^p_{n+p-k-1}$ equation (13.2.19) can be written as

$$E''_n = \sum_{k=p}^{n-1} r_k E^p_{n+p-k-1}. \quad (13.3.20)$$

This gives an explicit expression for $E''_n$ in terms of the errors $r_k$ in the computational formula.

To estimate $E''_n$ suppose that $r$ is an upper bound for the absolute values of the errors $r_n$ on the entire segment $[x_0, X]$, so that for any $n$ ($0 \leq n \leq N$)

$$|r_n| \leq r. \quad (13.3.21)$$

Then we have the following estimate for $E''_n$.
13.3. Convergence and Stability

\[
|E_n''| \leq r \sum_{k=p}^{n-1} |E_{n+p-k-1}^P| = r \sum_{k=p}^{n-1} |E_k^P|. \tag{13.3.22}
\]

Hence

\[
\max_{n} |E_n''| \leq r \sum_{k=p}^{N-1} |E_k^P|. \tag{13.3.23}
\]

The terms \( r \) and \( \sum_{k=p}^{N-1} |E_k^P| \) on the right of this equation usually depend on the interval size \( h \) and if we know how they depend on \( h \) we can often predict the behavior of \( \max_n |E_n''| \) as \( n \to \infty \). In particular we can state:

**Theorem 3.** If, as \( h \to 0 \),

\[
r \sum_{k=p}^{N-1} |E_k^P| \to 0
\]

then formula (13.3.2) provides a convergent computational process.

Let us assume that (13.3.2) is stable with respect to the initial values and also with respect to the rounding errors. Thus we assume that the roots of \( \lambda^{p+1} = \sum_{i=0}^{p} A_i \lambda^{p-i} \) do not exceed unity in modulus and that any roots with modulus equal to unity are simple.

Then we showed that there exists a number \( M_1 \) which for all \( n \geq 0 \) satisfies \( |E_n^P| \leq M_1 \). From this and from (13.3.23) we have the following estimate

\[
\max_n |E_n''| \leq rM_1(N - p) \leq rM_1N. \tag{13.3.24}
\]

Thus we have established:

**Theorem 4.** If the equation

\[
\lambda^{p+1} = \sum_{i=0}^{p} A_i \lambda^{p-i}
\]

has no roots greater than unity in modulus and if the roots of modulus
equal to unity are simple then formula (13.3.2) provides a convergent computational process providing that

\[
\frac{r}{h} \to 0 \quad \text{as} \quad h \to 0.
\]

Let us consider the case which we discussed at the end of the last section in which the coefficients \(A_k\) are positive numbers 

\[A_k > 0\]

which satisfy the condition

\[
\sum_{k=0}^{p} A_k = 1.
\]

In this case the error \(\varepsilon_n\) satisfies inequality (13.2.21):

\[
|\varepsilon_n| \leq \varepsilon + \sum_{k=p}^{n-1} |\alpha_k + r_k| \quad n > p
\]

where \(\varepsilon \geq |\varepsilon_i|, \ i = 0, 1, \ldots, p\).

Thus it is easy to obtain an estimate for the summands \(E_n, E'_n\) and \(E''_n\) of \(\varepsilon_n\). We note that if \(\alpha_k = 0\) and \(r_k = 0\) \((k > p)\) then \(\varepsilon_n\) must coincide with \(E_n\) and therefore we have

\[
|E_n| \leq \varepsilon, \quad n > p. \quad (13.3.25)
\]

Similarly

\[
|E'_n| \leq \sum_{k=p}^{n-1} |\alpha_k|, \quad n > p \quad (13.3.26)
\]

\[
|E''_n| \leq \sum_{k=p}^{n-1} |r_k|, \quad n > p. \quad (13.3.27)
\]

If \(|\alpha_k| \leq \alpha\) and \(|r_k| \leq r\) for \(p < n \leq N\), then \(E'_n\) and \(E''_n\) satisfy the estimates

\[
|E'_n| \leq (n - p)\alpha \leq Na \quad (13.3.28)
\]

\[
|E''_n| \leq (n - p)r \leq Nr. \quad (13.3.29)
\]

These inequalities permit us to state:
Theorem 5. If the coefficients $A_k$ ($k = 0, 1, \ldots, p$) are all positive and satisfy the condition $\sum_{k=0}^{p} A_k = 1$ then equation (13.3.2) is stable with respect to both the errors in the initial values and the rounding errors. If, in addition,

$$\frac{r}{h} \to 0$$

as $h \to 0$ then formula (13.3.2) provides a convergent computational process.
14.1. ONE METHOD FOR SOLVING THE PROBLEM

Suppose it is necessary to calculate the value of the integral
\[ y(x) = y_0 + \int_{x_0}^{x} f(t) dt \]  
(14.1.1)
for equally spaced points \( x_n = x_0 + nh \) on the segment \( x_0 \leq x \leq X \) where \( f(x) \) is only known for a set of equally spaced points which includes the \( x_n \). This problem has been widely investigated and many methods for its solution are known. The relationship between this problem and Cauchy's problem for ordinary differential equations has also received much attention. If we are given the equation \( y' = f(x, y) \) and we wish to find the solution which satisfies the condition \( y(x_0) = y_0 \) then this problem can be replaced by the equivalent problem of finding the solution of the integral equation
\[ y(x) = y_0 + \int_{x_0}^{x} f(t, y(t)) dt. \]  
(14.1.2)

Thus we can also apply methods for the numerical calculation of an indefinite integral to the solution of first order differential equations. In this chapter we consider one possible method for computing the function (14.1.1). This method leads to a simple computational scheme.

\footnote{These problems are different in the following respect. In order to compute the integral (14.1.1) we assume that \( f(t) \) is known at all points of the segment \([x_0, X]\) and to find each value of \( y(x) \) we can use any values of \( f(t) \). In the integral (14.1.2) we will know the values of the function \( f(x, y) \) for tabular points preceding \( x \), but the values of \( f(x, y) \) for points following \( x \) will not be known.}
14.1. One Method for Solving the Problem

and, as a rule, gives good accuracy if the function is sufficiently smooth on the segment of integration and close to this segment.

Suppose that the computation has been carried up to \( x_n = x_0 + nh \). To find the next value \( y(x_{n+1}) \) of the function (14.1.1) we will use only the immediately preceding value of \( y(x) \):

\[
y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(t) \, dt.
\]

(14.1.3)

To compute the integral in (14.1.3) we construct an interpolating polynomial for \( f(x) \) on the segment \([x_n, x_{n+1}]\). We will use the nodes closest to this segment to construct the interpolating polynomial and will take the same number of nodes on each side of this segment.

We apply Newton's interpolation formula (3.2.6) using the nodes \( x_n, x_n + h, x_n - h, x_n + 2h, x_n - 2h, \ldots \) to obtain

\[
f(x) = f(x_n) + (x - x_n)f(x_n, x_n + h) + (x - x_n)x
\]

\[
\times (x - x_n - h)f(x_n, x_n + h, x_n - h) +
\]

\[
(x - x_n)(x - x_n - h)f(x_n, x_n + h, x_n - h)\times
\]

\[
(x - x_n + h)f(x_n, x_n + h, x_n - h, x_n + 2h) + \ldots.
\]

Introducing the new variable \( u, x_n = x_0 + uh \), and expressing the divided differences in terms of finite differences gives

\[
f(x_n + uh) = f_n + \frac{u}{1!} \Delta f_n + \frac{u(u - 1)}{2!} \Delta^2 f_{n-1} +
\]

\[
+ \frac{(u + 1)u(u - 1)}{3!} \Delta^3 f_{n-1} +
\]

\[
+ \frac{(u + 1)u(u - 1)(u - 2)}{4!} \Delta^4 f_{n-2} + \ldots
\]

To put this equation in a form which is symmetric with respect to \( x_n + \frac{1}{2}h \) we transform the differences of even order using the identities

\[
f_n = \frac{1}{2} [f_{n+1} + f_n] - \frac{1}{2} [f_{n+1} - f_n] = \frac{1}{2} [f_{n+1} + f_n] - \frac{1}{2} \Delta f_n
\]

\[
\Delta^2 f_{n-1} = \frac{1}{2} [\Delta^2 f_n + \Delta^2 f_{n-1}] - \frac{1}{2} [\Delta^2 f_n - \Delta^2 f_{n-1}] =
\]

\[
= \frac{1}{2} [\Delta^2 f_n + \Delta^2 f_{n-1}] - \frac{1}{2} \Delta^3 f_{n-1}
\]

..........................................................
This gives:

\[
f(x_n + uh) = \frac{f_n + f_{n+1}}{2} + \frac{u - \frac{1}{2}}{1!} \Delta f + \frac{u(u - 1)}{2!} \frac{\Delta^2 f_{n-1} + \Delta^2 f_n}{2} + \\
\left(\frac{u - \frac{1}{2}}{3!}\right) u(u - 1) \Delta^3 f_{n-1} + \cdots + \\
\frac{(u + k - 1) \cdots (u - k)}{(2k)!} \frac{\Delta^{2k} f_{n-k} + \Delta^{2k} f_{n-k+1}}{2} + \\
\left(\frac{u - \frac{1}{2}}{2k + 1}\right) (u + k - 1) \cdots (u - k) \Delta^{2k+1} f_{n-k} + r(x).
\] (14.1.4)

Substituting this representation for \( f(t) \) in the integral

\[
\int_{x_n}^{x_n+h} f(t) \, dt = h \int_0^1 f(x_n + uh) \, du
\]

leads to the following expression for \( \gamma(x_n+1) \):

\[
\gamma(x_n+1) = \gamma(x_n) + h \left[ \frac{f_n + f_{n+1}}{2} - \frac{1}{12} \frac{\Delta^2 f_{n-1} + \Delta^2 f_n}{2} + \\
+ \frac{11}{720} \frac{\Delta^4 f_{n-2} + \Delta^4 f_{n-1}}{2} - \frac{191}{60480} \frac{\Delta^6 f_{n-3} + \Delta^6 f_{n-2}}{2} + \\
+ \cdots + C_k \frac{\Delta^{2k} f_{n-k} + \Delta^{2k} f_{n-k+1}}{2} \right] + R_{n,k}
\] (14.1.5)

where

\[
C_k = \frac{1}{(2k)!} \int_0^1 (u + k - 1) \cdots (u - k) \, du
\]

A computational formula is thus obtained by selecting some value of \( k \) and neglecting the remainder \( R_{n,k} \).

Let us consider an example. Suppose we wish to calculate the value of the following integral on the segment \([0, 1] \):

In the theory of interpolation this equation is called Bessel's formula.
14.1. One Method for Solving the Problem

\[ y(x) = \int_0^x J_1(t) \, dt = 1 - J_0(x) \]

where \( J_0(t) \) and \( J_1(t) \) are Bessel functions of the first kind. We use formula (14.1.5) with \( h = 0.2 \) and with differences up to and including those of the fourth order

\[ y_{n+1} = y_n + 0.2 \left[ \frac{f_n + f_{n+1}}{2} - \frac{1}{12} \frac{\Delta^2 f_{n-1} + \Delta^2 f_n}{2} + \frac{11}{720} \frac{\Delta^4 f_{n-2} + \Delta^4 f_{n-1}}{2} \right] \]

\[ y_0 = 0, \quad f(x) = J_1(x). \]

The table of differences of \( J_1(x) \) which are necessary to use this formula is given below:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( J_1(x) )</th>
<th>( \Delta J_1 \cdot 10^7 )</th>
<th>( \Delta^2 J_1 \cdot 10^7 )</th>
<th>( \Delta^3 J_1 \cdot 10^7 )</th>
<th>( \Delta^4 J_1 \cdot 10^7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_2 = -0.4 )</td>
<td>-0.1960266</td>
<td>965258</td>
<td>+29750</td>
<td>-29750</td>
<td>0000</td>
</tr>
<tr>
<td>( x_1 = -0.2 )</td>
<td>-0.0995008</td>
<td>995008</td>
<td>00000</td>
<td>-29750</td>
<td>986</td>
</tr>
<tr>
<td>( x_0 = 0.0 )</td>
<td>0.0000000</td>
<td>995008</td>
<td>-29750</td>
<td>-28764</td>
<td>1944</td>
</tr>
<tr>
<td>( x_1 = 0.2 )</td>
<td>0.0995008</td>
<td>965258</td>
<td>-58514</td>
<td>-26820</td>
<td>2830</td>
</tr>
<tr>
<td>( x_2 = 0.4 )</td>
<td>0.1960266</td>
<td>906744</td>
<td>-85334</td>
<td>-23990</td>
<td>3613</td>
</tr>
<tr>
<td>( x_3 = 0.6 )</td>
<td>0.2867010</td>
<td>821410</td>
<td>-109324</td>
<td>-20377</td>
<td>4279</td>
</tr>
<tr>
<td>( x_4 = 0.8 )</td>
<td>0.3688420</td>
<td>712086</td>
<td>-129701</td>
<td>-16098</td>
<td></td>
</tr>
<tr>
<td>( x_5 = 1.0 )</td>
<td>0.4400506</td>
<td>582385</td>
<td>-145799</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_6 = 1.2 )</td>
<td>0.4982891</td>
<td>436586</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_7 = 1.4 )</td>
<td>0.5419477</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From this table we can calculate the values of the integral \( y(x) \). The computation for \( y(0.2) \) is:

\[
y(0.2) = y(0) + h \left[ \frac{f(0) + f(0.2)}{2} - \frac{1}{12} \frac{\Delta^2 f(-0.2) + \Delta^2 f(0)}{2} + \frac{11}{720} \frac{\Delta^4 f(-0.4) + \Delta^4 f(-0.2)}{2} \right] = 0 + 0.2 \left[ 0 + 0.0995008 - \frac{1}{12} \frac{0 - 0.0029750}{2} + \frac{11}{720} \frac{0 + 0.0000986}{2} \right] = 0.0099750.
\]

The calculated values of \( y(x) \) are tabulated below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \int_0^x J_1(t) , dt )</th>
<th>( x )</th>
<th>( \int_0^x J_1(t) , dt )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0000000</td>
<td>0.6</td>
<td>0.0879951</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0099750</td>
<td>0.8</td>
<td>0.1537126</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0396017</td>
<td>1.0</td>
<td>0.2348023</td>
</tr>
</tbody>
</table>
All of these values are exact to the seven decimal places which are given, except \( y(0.4) \) which has an error of one in the last place.\(^3\)

### 14.2. THE REMAINDER

The order of the highest finite difference in (14.1.4) is \( 2k + 1 \) and to use this formula we must know the value of \( f(x) \) at the \( 2k + 2 \) points \( x_n - kh, \ldots, x_n + (k + 1)h \). If we assume that \( f(x) \) has a continuous derivative of order \( 2k + 2 \) on \( [x_n - kh, x_n + (k + 1)h] \) then the remainder \( r(x) \) of the interpolation (14.1.4) can be found from Theorem 4 of Chapter 3:

\[
r(x) = \frac{[x - x_n + kh][x - x_n + (k - 1)h] \cdots [x - x_n - (k + 1)h]}{(2k + 2)!} f^{(2k+2)}(\eta) = h^{2k+2} \frac{(u + k)(u + k - 1) \cdots (u - k - 1) f^{(2k+2)}(\eta)}{(2k + 2)!}
\]

\( x_n - kh < \eta < x_n + (k + 1)h. \)

Thus we obtain the following expression for \( R_{n, k} \) in formula (14.1.5):

\[
R_{n, k} = h \int_0^1 r(x_n + uh)du = \frac{h^{2k+3}}{(2k + 2)!} \int_0^1 (u + k)(u + k - 1) \cdots (u - k - 1) f^{(2k+2)}(\eta)du.
\]

Since the factor \( (u + k) \cdots (u - k - 1) \) does not change sign on \([0, 1]\) the mean value theorem can be applied to the last integral and thus we can make the following assertion:

If \( f(x) \) has a continuous derivative of order \( 2k + 2 \) on \( [x_n - kh, x_n + (k + 1)h] \) then the remainder \( R_{n, k} \) of (14.1.5) has the representation

\[
R_{n, k} = h^{2k+3} \frac{f^{(2k+2)}(\xi)}{(2k + 2)!} \int_0^1 (u + k)(u + k - 1) \cdots (u - k - 1)du \quad (14.2.1)
\]

where \( \xi \) is an interior point of the segment \( [x_n - kh, x_n + (k + 1)h] \).

\(^3\)See, for example, G. N. Watson, A Treatise on the Theory of Bessel Functions, Macmillan, New York, 1944, p. 666.
15.1. GENERAL ASPECTS OF THE PROBLEM

Here, as in the preceding chapter, we will consider the problem of computing the indefinite integral

\[ y(x) = y_0 + \int_{x_0}^{x} f(t) \, dt \]  

(15.1.1)

for equally spaced values of the argument \( x_k = x_0 + kh \) (\( k = 0, 1, \ldots \)). Here, however, we assume that we may use in the computational formula any nodes for which \( f(x) \) is defined.

The largest part of the work in computing the integral (15.1.1) by means of a formula of the form (13.2.1) is usually in calculating the values of the function \( f(x) \). There are two ways in which we can reduce this part of the work. We can choose the nodes to achieve a high degree of precision in the formula or we can choose the nodes so that they are used for not just one step in the calculation but for several steps so that for each successive step it is necessary to calculate only a few additional values of \( f(x) \).

In the following discussion we will use a combination of these methods to construct formulas. To calculate the value of \( y_{n+1} \) we again use only the preceding value of \( y(x) \):

\[ y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(t) \, dt \]
and thus the problem reduces to the computation of the integral in this expression.

If the coefficients of the formula are to be independent of \( n \) we must assume that the nodes are situated with period \( h \) on the \( x \)-axis. We will say that a set of points \( \alpha + kh \), for distinct integers \( k \), are similar to the point \( \alpha \).

To calculate the above integral we assume that we will use \( m \) nodes \( \alpha, \beta, \ldots, \lambda \) on the segment \([x_n, x_{n+1}]\): \( x_n \leq \alpha < \beta < \cdots < \lambda < x_{n+1} \). In addition to these basic nodes we will also use the following:

- \( a \) nodes \( \alpha + p_i h \) (\( i = 1, \ldots, a \)) similar to \( \alpha \)
- \( b \) nodes \( \beta + q_i h \) (\( i = 1, \ldots, b \)) similar to \( \beta \)
- \( l \) nodes \( \lambda + t_i h \) (\( i = 1, \ldots, l \)) similar to \( \lambda \).

The way in which these additional nodes are situated among the points \( x_k \) will depend on the numbers \( p_i, q_i, \ldots, t_i \) which we assume can be any integers different from zero. We denote the total number of nodes by \( N + 1 \):

\[
m + a + b + \cdots + l = N + 1.\]

Let us consider a formula of the form

\[
\int_{x_n}^{x_{n+1}} f(t) \, dt = A_0 f(\alpha) + \sum_{i=1}^{a} A_i f(\alpha + p_i h) +
\]

\[
+ \cdots + L_0 f(\lambda) + \sum_{i=1}^{l} L_i f(\lambda + t_i h).
\] (15.1.2)

If we assume that the numbers \( p_i, \ldots, t_i \) are given then we must still determine the nodes \( \alpha, \ldots, \lambda \) and the coefficients \( A_i, \ldots, L_i \) (\( i = 0, 1, \ldots \)). We wish to choose these quantities so that (15.1.2) has the highest possible algebraic degree of precision.

For each choice of the \( \alpha, \ldots, \lambda, p_i, \ldots, t_i \) we can always construct a formula which is exact for all polynomials of degree \( \leq N \). We can do this by constructing the Lagrange interpolating polynomial for \( f(x) \) using the nodes \( \alpha, \alpha + p_i h, \ldots, \lambda, \lambda + t_i h \) and taking as the coefficients in (15.1.2) the integrals of the coefficients of this interpolating polynomial. In this way the coefficients \( A_i, \ldots, L_i \) are completely determined. Thus to increase the precision of the formula we have only at our disposal the choice of the nodes \( \alpha, \ldots, \lambda \). Below we will show that for any \( p_i, \ldots, t_i \) formula (15.1.2) can be made exact for all polynomials of degree \( m + N \) by a suitable choice of \( \alpha, \ldots, \lambda \) and that this is the highest possible degree of precision.
15.1. General Aspects of the Problem

From the nodes of the formula we construct the following polynomials:

\[ \omega(x) = (x - \alpha) \cdots (x - \lambda) \]

\[ \omega_{\alpha}(x) = \prod_{i=1}^{a} (x - \alpha - p_i h), \ldots, \omega_{\lambda}(x) = \prod_{i=1}^{l} (x - \lambda - t_i h) \]

(15.1.3)

\[ \Omega(x) = \omega_{\alpha}(x) \cdots \omega_{\lambda}(x). \]

Theorem 1. No matter how we choose the nodes \( \alpha, \ldots, \lambda \) and the integers \( p_i, \ldots, t_i \) the formula (15.1.2) cannot be exact for all polynomials of degree \( m + N + 1 \).

Proof. It is sufficient to consider the polynomial \( f(x) = \Omega(x)\omega^2(x) \). The degree of this polynomial is \( m + N + 1 \). Since all the nodes of the formula are roots of \( \Omega(x)\omega^2(x) \) the quadrature sum on the right side of (15.1.2) is zero. The integral \( \int_{x_n}^{x_{n+1}} \Omega(x)\omega^2(x) \, dx \), however, is different from zero since the polynomial \( \Omega(x)\omega^2(x) \) does not change sign on the segment of integration and it is not identically zero. Therefore (15.1.2) cannot be exact for \( f(x) = \Omega(x)\omega^2(x) \).

The algebraic degree of precision of (15.1.2) is always less than \( m + N + 1 \) and the greatest it can be is \( m + N \).

Theorem 2. In order that formula (15.1.2) be exact for all polynomials of degree \( \leq m + N \) it is necessary and sufficient that the following two conditions be fulfilled:

1. The formula must be interpolatory
2. For any polynomial \( Q(x) \) of degree less than \( m \) we must have

\[ \int_{x_n}^{x_{n+1}} \Omega(x)\omega(x)Q(x) \, dx = 0. \]

(15.1.4)

Proof. The necessity of the first condition is evident. To verify the necessity of the second condition let us take an arbitrary polynomial \( Q(x) \) of degree less than \( m \) and set \( f(x) = \Omega(x)\omega(x)Q(x) \). This is a polynomial of degree at most \( m + N \) and for it equation (15.1.2) must be exact. But the quadrature sum for \( f(x) \) is zero; hence equation (15.1.4) must be satisfied.

Suppose now that both conditions of the theorem are fulfilled and let \( f(x) \) be any polynomial of degree \( \leq m + N \). Dividing \( f(x) \) by \( \Omega(x)\omega(x) \) we can represent \( f(x) \) in the form \( f(x) = \Omega(x)\omega(x)Q(x) + r(x) \) where \( Q(x) \) and \( r(x) \) are polynomials of degree less than \( m \) and \( N + 1 \) respectively. Since the polynomial \( \Omega(x)\omega(x) \) is zero at all the nodes in the
Approximate Calculation of Indefinite Integrals

formula then at these nodes the polynomials \( f(x) \) and \( r(x) \) must have the same values. Using the fact that the degree of \( r(x) \) is not greater than \( N \) and the fact that formula (15.1.2) is interpolatory the following equations must be satisfied:

\[
\int_{x_n}^{x_{n+1}} f(x) \, dx = \int_{x_n}^{x_{n+1}} \Omega(x)\omega(x)Q(x) \, dx + \int_{x_n}^{x_{n+1}} r(x) \, dx =
\]

\[
= A_0 r(\alpha) + \sum_{i=1}^{a} A_i r(\alpha + p_i h) + \cdots =
\]

\[
= A_0 f(\alpha) + \sum_{i=1}^{a} A_i f(\alpha + p_i h) + \cdots.
\]

This establishes the sufficiency of the conditions and completes the proof.

Theorem 2 reduces the question of the existence of quadrature formulas (15.1.2) which have the highest algebraic degree of precision \( m + N \) to the question of the existence of nodes \( \alpha, \ldots, \lambda \) for which the corresponding polynomial \( \Omega(x)\omega(x) \) satisfies the orthogonality condition (15.1.4).

Theorem 3. For any integers \( p_i, \ldots, t_i \) we can find nodes \( \alpha, \ldots, \lambda \) so that the corresponding quadrature formula (15.1.2) will have the highest algebraic degree of precision \( m + N \).

Proof. Let us take any system of nodes \( \alpha, \beta, \ldots, \lambda \) which satisfy the inequalities

\[
x_n \leq \alpha \leq \beta \leq \cdots \leq \lambda \leq x_{n+1}
\]

and construct for these nodes the polynomials \( \Omega(x) \) and \( \omega(x) \). The polynomial \( \Omega(x) \) does not change sign on the segment \([x_n, x_{n+1}]\). We will consider \( \Omega(x) \) as a weight function and investigate the system of polynomials \( P_h(x) \) which are orthogonal on \([x_n, x_{n+1}]\) with respect to \( \Omega(x) \). Let \( P_m(x) \) be the \( m \)-th degree polynomial of this system and let us assume that its leading coefficient is unity:

\[
P_m(x) = x^m + p_1x^{m-1} + p_2x^{m-2} + \cdots.
\]

Any polynomial \( Q(x) \) of degree \(<m \) satisfies

\[
\int_{x_n}^{x_{n+1}} \Omega(x)P_m(x)Q(x) \, dx = 0.
\]

The roots of \( P_m(x) \) are all real and simple and they all lie inside the
15.1. General Aspects of the Problem

We denote the roots of $P_m(x)$ by $\xi_1, \ldots, \xi_m$ and assume that they are enumerated in increasing order $x_n < \xi_1 < \ldots < \xi_m < x_{n+1}$. If it turns out that $\xi_1 = \alpha, \xi_2 = \beta, \ldots, \xi_m = \lambda$ then $P_m(x)$ coincides with $\omega(x)$ and then $\Omega(x)$ and $\omega(x)$ will satisfy (15.1.4) and the corresponding formula (15.1.2) will have the highest algebraic degree of precision $m + N$.

If the $\xi_1, \ldots, \xi_m$ do not coincide with the $\alpha, \ldots, \lambda$ let us construct a system of linear equations for the coefficients $p_k$ ($k = 1, \ldots, m$). The orthogonality property (15.1.6) is equivalent to the equations

\[ \int_{x_n}^{x_{n+1}} \Omega(x)P_m(x)x^i dx = 0, \quad i = 0, 1, \ldots, m - 1 \quad (15.1.7) \]

or if we replace $P_m(x)$ by its expansion in powers of $x$:

\[ c_{m+i} + c_{m+i-1} + c_{m+i-2} + \cdots + c_ip_m = 0, \quad i = 0, 1, \ldots, m - 1 \]

where

\[ c_k = \int_{x_n}^{x_{n+1}} \Omega(x)x^k dx. \]

Since $\Omega(x)$ is a polynomial in $\alpha, \ldots, \lambda$ then the numbers $c_k$ will also be polynomials in $\alpha, \ldots, \lambda$.

The determinant of the system (15.1.7)

\[
\begin{vmatrix}
    c_0 & c_1 & \cdots & c_{m-1} \\
    c_1 & c_2 & \cdots & c_m \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{m-1} & c_m & \cdots & c_{2m-2}
\end{vmatrix}
\]

is the determinant of a positive-definite quadratic form

\[
\sigma(z_1, \ldots, z_m) = \int_{x_n}^{x_{n+1}} \Omega(x) \left( \sum_{i=1}^{m} x^{i-1}z_i \right)^2 dx
\]

and it is known to be different from zero for each set of $\alpha, \ldots, \lambda$ which satisfies (15.1.5). The coefficients $p_k$ ($k = 1, \ldots, m$) will be rational continuous functions of $\alpha, \ldots, \lambda$.

The roots $\xi_1, \xi_2, \ldots, \xi_m$ of $P_m(x)$ depend continuously on the coefficients $p_k$ and will therefore be continuous functions of $\alpha, \ldots, \lambda$:

\[ \xi_1 = \phi_1 (\alpha, \ldots, \lambda) \]

\[ \ldots \ldots \ldots \ldots \]

\[ \xi_m = \phi_m (\alpha, \ldots, \lambda). \quad (15.1.8) \]
These equations can be interpreted in a geometric manner. Consider an $m$-dimensional Euclidean space of points $(x_1, x_2, \ldots, x_m)$ which we denote by $E_m$. Equations (15.1.8) can be interpreted as a transformation of the point $(\alpha, \ldots, \lambda)$ of $E_m$ into another point $(\xi_1, \ldots, \xi_m)$ of $E_m$. Condition (15.1.5), to which $\alpha, \ldots, \lambda$ are subjected, defines an $m$-dimensional closed simplex\(^1\) in $E_m$. Since the roots $\xi_k$ satisfy the inequalities $x_n < \xi_1 < \cdots < \xi_m < x_{n+1}$ then equations (15.1.8) define a single-valued, continuous transformation of this simplex onto itself. By the Brouwer fixed-point theorem\(^2\) it is known that there exists an invariant point of this transformation and consequently there exists values $\alpha, \ldots, \lambda$ for which $\xi_1 = \alpha, \ldots, \xi_m = \lambda$ and $P_m(x) = \omega(x)$. Therefore there certainly exists nodes $\alpha, \ldots, \lambda$ which satisfy the inequalities $x_n < \alpha < \cdots < \lambda < x_{n+1}$ for which (15.1.4) is fulfilled. This completes the proof of Theorem 3.

It is not known, in general, whether the points $\alpha, \ldots, \lambda$ will be unique.

We now find a representation for the remainder of (15.1.2). Let $[a', b']$ be the segment which contains $[x_n, x_{n+1}]$ and all the nodes of formula (15.1.2).

**Theorem 4.** If $f(x)$ has a continuous derivative of order $m + N + 1$ on $[a', b']$ and if formula (15.1.2) has degree of precision $m + N$, then there exists a point $\xi$ in $[a', b']$ with the property that the remainder $R(f)$ of formula (15.1.2) satisfies

$$R(f) = \frac{f^{(m+N+1)}(\xi)}{(m + N + 1)!} \int_{x_n}^{x_{n+1}} \Omega(x) \omega^2(x) \, dx. \quad (15.1.9)$$

**Proof.** Let us construct an interpolating polynomial for $f(x)$ in the following way. Suppose that at each of the basic nodes $\alpha, \ldots, \lambda$ we are given both the value of $f(x)$ and the value of its derivative $f'(x)$ and at each node of the form $\alpha + p_i h, \ldots, \lambda + t_i h$ we are given only the value of the function $f(x)$. We will have a total of $m + N + 1$ known values. The interpolating polynomial based on these values will be denoted by $H(x)$ and will have degree $\leq m + N$:

$$f(x) = H(x) + r(x).$$

---

\(^1\)An $m$-dimensional simplex is the generalization of a triangle for two dimensions and a tetrahedron for three dimensions and has $m + 1$ vertices, which do not lie in any $(m - 1)$-dimensional subspace, and is bounded by $m + 1$ $(m - 1)$-dimensional faces.

\(^2\)Brouwer has proved the following theorem: If we are given any single-valued, continuous transformation of an $m$-dimensional simplex onto itself then this transformation has at least one invariant point; L. E. J. Brouwer, "Über Abbildung von Mannigfaltigkeiten," Math. Annalen, Vol. 71, 1912, pp. 97-115 or V. V. Nemytskii, "Method of fixed points," Uspehi Mat. Nauk, Vol. 1, 1936, p. 153.
By the results of Section 3.3 the remainder \( r(x) \) can be represented in the form

\[
r(x) = \frac{f^{(m+N+1)}(\eta)}{(m + N + 1)!} \Omega(x)\omega^2(x)
\]

where \( \eta \) is a point inside the segment which contains the nodes of the interpolation and the point \( x \).

It is clear that \( R(f) = R(H) + R(r) \). But \( H(x) \) has degree \( \leq m + N \) so that \( R(H) = 0 \) and hence \( R(f) = R(r) \). The quadrature sum for \( r(x) \) is zero since \( r(x) \) is zero at all the nodes of the formula. Thus \( R(r) \) coincides with the integral of \( r(x) \):

\[
R(f) = R(r) = \int_{x_n}^{x_{n+1}} r(x) \, dx = \int_{x_n}^{x_{n+1}} \frac{f^{(m+N+1)}(\eta)}{(m + N + 1)!} \Omega(x)\omega^2(x) \, dx.
\]

Since \( \Omega(x)\omega^2(x) \) does not change sign on \([x_n, x_{n+1}]\) the assertion of the theorem immediately follows.

15.2. FORMULAS OF SPECIAL FORM

Here we consider formulas for calculating the indefinite integral

\[
y(x) = y_0 + \int_{x_0}^{x} f(t) \, dt
\]

which use one, two or three values of the integrand \( f(x) \) on each step or, in other words, formulas which contain one, two or three basic nodes. We will give numerical values for the nodes and coefficients in these formulas.\(^3\)

All of these formulas can be constructed by a standard method and we describe this method in detail for only one case and in the other cases we only give the final results.

1. We begin with the case of one value of \( f(x) \) on each step. These formulas reduce to formulas studied by Gauss and obtained by him in another problem in a different way.

On the segment \([x_n, x_n + h]\) we take the basic node \( \alpha_n = x_n + qh \), \( 0 \leq q < 1 \). The nodes are then situated as in Fig. 9.

In order to construct a formula of the form (15.1.2) we use \( k \) nodes preceding and following \( \alpha_n \) which are similar to \( \alpha_n \). The formula then contains \( 2k + 1 \) nodes. We are free to choose only the parameter \( q \) and the

---

\(^3\)The values of the coefficients and nodes of the formulas given in this section were computed by Junior Research Assistant M. A. Filippov of the Leningrad Division of Mat. In-Ta Akad. Nauk SSSR.
highest degree of precision of the formula is $2k + 1$. In order to achieve this precision the formula must be interpolatory and it must satisfy the orthogonality condition (15.1.4) which in this case is

$$\int_{x_n}^{x_{n+h}} \Omega(x)w(x) \, dx = 0 \quad (15.2.1)$$

$$\Omega(x)w(x) = (x - \alpha_n)(x - \alpha_{n-1})(x - \alpha_{n+1}) \cdots (x - \alpha_{n-k})(x - \alpha_{n+k}) =
\quad = [x - x_n - qh][(x - x_n - qh)^2 - h^2] \cdots [(x - x_n - qh)^2 - k^2h^2].$$

It is easy to show that (15.2.1) has the solution $q = \frac{1}{2}$ and that this solution is unique for $0 \leq q \leq 1$.

We transform the integral (15.2.1) by the transformation $x = x_n + hq + ht$ to obtain

$$\int_{x_n}^{x_{n+h}} \Omega(x)w(x) \, dx = h^{2k+2} \int_{-q}^{1-q} \pi(t) \, dt$$

$$\pi(t) = t(t^2 - 1^2) \cdots (t^2 - k^2).$$

Thus (15.2.1) is equivalent to

$$\phi(q) = \int_{-q}^{1-q} \pi(t) \, dt = 0. \quad (15.2.2)$$

Since $\pi(t)$ is an odd function of $t$ then $\phi(\frac{1}{2}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \pi(t) \, dt = 0$ and $q = \frac{1}{2}$ is a root of (15.2.2). The derivative of $\phi(q)$ is

$$\phi'(q) = \pi(1 - q) - \pi(-q)$$

and since $\pi(1 - q)$ and $\pi(-q)$ have opposite signs for $0 < q < 1$ then $\phi'(q)$ does not change sign on the interval $0 < q < 1$. Therefore the root $q = \frac{1}{2}$ is unique for $0 \leq q \leq 1$ and
To interpolate for $f(x)$ on $[x_n, x_n + h]$ with respect to its values at the nodes $a_m$ ($m = n - k, \ldots, n + k$) we use Newton’s interpolation formula (3.2.6) substituting the nodes in the order $a_n, a_n + h, a_n - h, a_n + 2h, a_n - 2h, \ldots$.

We obtain

\[ f(x) = f(a_n) + \frac{x - a_n}{1! h} \Delta f(a_n) + \frac{(x - a_n)(x - a_n - h)}{2! h^2} \Delta^2 f(a_n - h) + \ldots + \frac{(x - a_n + kh) \ldots (x - a_n - kh)}{(2k + 1)! h^{2k+1}} \Delta^{2k+1} f(a_n - kh) + r(x). \]

Substituting this expression for $f(x)$ into the equation

\[ y_{n+1} = y_n + \int_{x_n}^{x_n+h} f(t) \, dt \]

gives

\[ y_{n+1} = y_n + h \left[ f(a_n) + \frac{1}{24} \Delta^2 f(a_n - h) - \frac{17}{5760} \Delta^4 f(a_n - 2h) + \frac{367}{967680} \Delta^6 f(a_n - 3h) - \frac{27859}{464486400} \Delta^8 f(a_n - 4h) + \frac{1295803}{122624409600} \Delta^{10} f(a_n - 5h) + \ldots + c_k \Delta^{2k} f(a_n - kh) \right] + R_{n,k}. \]

\[ c_k = \frac{1}{(2k)!} \int_{-\frac{1}{2}}^{\frac{1}{2}} t^2(t^2 - 1^2) \ldots (t^2 - (k - 1)^2) \, dt. \]  

(15.2.3)
Approximate Calculation of Indefinite Integrals

If \( f(x) \) has a continuous derivative of order \( 2k + 2 \) on \([a_n - kh, a_n + kh] \) then, by Theorem 4 of Section 3.2, the remainder of the interpolation can be represented as

\[
r(x) = \frac{(x - a_n + kh) \cdots (x - a_n - kh)}{(2k + 2)!} f^{(2k+2)}(\eta)
\]

for some interior point \( \eta \) of this segment. To find the remainder \( R_{n,k} \) we integrate this expression and make the substitution

\[
x = a_n + \xi h = x_n + \frac{1}{2} h + \xi h
\]

to obtain

\[
R_{n,k} = \frac{f^{(2n+2)}(\xi)}{(2k + 2)!} \int \frac{1}{2} t^2 (t^2 - 1^2) \cdots (t^2 - k^2) \, dt.
\] (15.2.4)

2. We now consider some of the simplest computational formulas which require two values of \( f(x) \) on each step. We use the two basic nodes \( a_n, \beta_n \) on the segment \([x_n, x_n + h]\). The nodes are situated as in Fig. 10.

In addition to \( a_n \) and \( \beta_n \) we will also use \( k \) nodes on each side of \([x_n, x_n + h]\) so that the total number of nodes is \( 2k + 2 \). The highest degree of precision which can be achieved is \( 2k + 3 \).

In the present case \( \omega(x) = (x - a_n)(x - \beta_n) \) and \( \Omega(x)\omega(x) \) will contain \( 2k + 2 \) factors of the following form:

\[
\Omega(x)\omega(x) = (x - a_n)(x - \beta_n)(x - \alpha_{n+1}) \times
\]

\[
\times (x - \beta_{n-1})(x - \beta_{n+1})(x - \alpha_{n-1}) \ldots.
\]

The orthogonality condition (15.1.4) which \( \Omega(x)\omega(x) \) must satisfy reduces to the two equations

\[
\int_{x_n}^{x_{n+1}} \Omega(x)\omega(x) \, dx = 0 \quad \int_{x_n}^{x_{n+1}} x\Omega(x)\omega(x) \, dx = 0.
\] (15.2.5)

In the first case, \( k = 1 \), we have four nodes: \( a_n, \beta_n, a_{n+1}, \beta_{n-1} \). In order to simplify the problem we make the transformation
which transforms the points \( \ldots, x_{n-1}, x_n, x_{n+1}, x_{n+2}, \ldots \) into the points \( \ldots, -3, -1, 1, 3, \ldots \) and the midpoint of \([x_n, x_n + h]\) transforms into \(z = 0\). The points which \( x = \alpha_n \) and \( x = \beta_n \) transform into we will denote by \( p \) and \( q \). In terms of \( z \) we have

\[
\omega(x) = (x - \alpha_n)(x - \beta_n) = \frac{h^2}{4} (z - p)(z - q)
\]

\[
\Omega(x)\omega(x) = (x - \alpha_n)(x - \beta_n)(x - \alpha_{n+1})(x - \beta_{n-1}) = \frac{h^4}{16} (z - q + 2)(z - p)(z - q)(z - p - 2).
\]

In terms of \( z \) the orthogonality conditions (15.2.5) are

\[
\int_{-1}^{1} (z - q + 2)(z - p)(z - q)(z - p - 2) \, dz = 0
\]

or after integrating and collecting terms

\[
p^2(1 - 6q + 3q^2) + 2p(3q^2 - 4q - 1) + \left(q^2 + 2q - \frac{17}{5}\right) = 0
\]

\[
(p + q)\left[\frac{1}{5} + \frac{1}{3} (pq + q - p - 2)\right] = 0. \tag{15.2.6}
\]

From the second of these equations \( p \) can have the values

\[
p_1 = \frac{q - \frac{7}{5}}{1 - q}, \quad p_2 = -q.
\]

Since \( p \) and \( q \) must satisfy the condition

\[-1 < p < q < 1\]

we see that the solution \( p_1 \) must be rejected since it does not satisfy \(-1 < p_1\).

From the second solution \( p = p_2 = -q \) the first of the equations (15.2.6) gives

\[
3q^4 - 12q^3 + 10q^2 + 4q - \frac{17}{5} = 0.
\]
Now \( q \) must lie in the segment \((0, 1)\) and it is possible to show that this equation has only one root of this form

\[
q = 0.53332 \ 38475.
\]

The basic nodes \( \alpha_n \) and \( \beta_n \) are then

\[
\alpha_n = x_n + \frac{1}{2}(1 - q)h = x_n + (0.23333 \ 80763)h
\]

\[
\beta_n = x_n + \frac{1}{2}(1 + q)h = x_n + (0.76666 \ 19237)h.
\]

Let us construct the interpolating polynomial for \( f(x) \) using its values at the nodes \( \beta_{n-1}, \alpha_n, \beta_n, \alpha_{n+1} \):

\[
f(x) = \frac{(x - \alpha_n)(x - \beta_n)(x - \alpha_{n+1})}{(\beta_{n-1} - \alpha_n)(\beta_{n-1} - \beta_n)(\beta_{n-1} - \alpha_{n+1})} f(\beta_{n-1}) + \cdots + r(x) = P(x) + r(x).
\]

Then

\[
\gamma_{n+1} = \gamma_n + \int_{x_n}^{x_{n+1}} f(t) \, dt = \gamma_n + \int_{x_n}^{x_{n+1}} P(t) \, dt + R_n.
\]

Computing the integral of \( P(t) \) leads to the following formula:

\[
\gamma_{n+1} = \gamma_n + (0.48690 \ 23179)h[f(\alpha_n) + f(\beta_n)] + \]

\[
+ (0.01309 \ 76821)h[f(\beta_{n-1}) + f(\alpha_{n+1})] + R_n. \quad (15.2.7)
\]

The remainder \( R_n \) can be found from the representation (15.1.9). Here we must use \( m = 2, N + 1 = 4 \) and

\[
\Omega(x) \omega^2(x) = (x - \beta_{n-1})(x - \alpha_n)^2(x - \beta_n)^2(x - \alpha_{n+1}).
\]

This leads to

\[
R_n = -\frac{14.732017}{4838400} h^7 f^{(6)}(\xi) = -0.00000305h^7 f^{(6)}(\xi) \quad (15.2.8)
\]

\[
\beta_{n-1} < \xi < \alpha_{n+1}.
\]

We now consider the case \( k = 2 \). In addition to two basic nodes in the interval \([x_n, x_n + h] \) we also use two nodes in each of the adjoining intervals \([x_n - h, x_n] \) and \([x_n + h, x_n + 2h] \). The nodes are depicted in Fig. 10.

The highest algebraic degree of precision is 7.

The nodes of this quadrature formula are
15.2. Formulas of Special Form

\[ \alpha_n = x_n + (0.23896\ 17210)h, \quad \beta_n = x_n + (0.76103\ 82790)h. \]

The formula is

\[ y_{n+1} = y_n + (0.48309\ 24404)h[f(\alpha_n) + f(\beta_n)] + 
+ (0.01737\ 14226)h[f(\beta_n-1) + f(\alpha_n+1)] - 
- (0.00046\ 38630)h[f(\alpha_n-1) + f(\beta_n+1)] + R_n. \quad (15.2.9) \]

The estimate for the remainder is

\[ |R_n| \leq 0.00000008h^9M_8 \]

\[ M_8 = \max_x |f^{(8)}(x)|, \quad \alpha_{n-1} < x < \beta_{n+1}. \]

In all the cases which we consider below the nodes are situated symmetrically with respect to the middle of the segment \([x_n, x_n + h].\) We will not derive any nonsymmetric formulas of the highest degree of precision.

The case \(k = 3.\) In addition to two basic nodes in \([x_n, x_n + h]\) we also use two nodes in \([x_n - h, x_n]\) and in \([x_n + h, x_n + 2h]\) and one node in \([x_n - 2h, x_n - h]\) and in \([x_n + 2h, x_n + 3h].\) The nodes are situated as shown in Fig. 11.

The highest degree of precision is 9. The formula which achieves this precision is

\[ y_{n+1} = y_n + (0.48259\ 37250)h[f(\alpha_n) + f(\beta_n)] + 
+ (0.01797\ 22221)h[f(\beta_{n-1}) + f(\alpha_{n+1})] - 
- (0.00057\ 82647)h[f(\alpha_{n-1}) + f(\beta_{n+1})] + 
+ (0.00001\ 23177)h[f(\beta_{n-2}) + f(\alpha_{n+2})] + R_n \quad (15.2.10) \]

where the nodes are

\[ \alpha_n = x_n + (0.23963\ 00931)h, \quad \beta_n = x_n + (0.76036\ 99069)h. \]

The remainder satisfies the estimate

\[ |R_n| \leq 0.000000003h^{11}M_{10} \]

\[ M_{10} = \max_x |f^{(10)}(x)|, \quad \beta_{n-2} < x < \alpha_{n+2}. \]
The case $k = 4$. We use the nodes shown in Fig. 12.

\[ \begin{array}{cccccccc}
\alpha_{n-2} & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} & \alpha_n & \beta_n & \alpha_{n+1} & \beta_{n+1} \\
X_{n-2} & & X_{n-1} & & X_n & & X_{n+1} & \\
\end{array} \]

\[
\begin{align*}
& X_{n-1} X_n X_{n+1} X_{n+2}
\end{align*}
\]

Figure 12.

The highest algebraic degree of precision is 11 and is achieved by the formula

\[
\begin{align*}
\gamma_{n+1} &= \gamma_n + (0.47911 \ 31668)h[f(\alpha_n) + f(\beta_n)] + \\
&+ (0.02153 \ 22932)h[f(\beta_{n-1}) + f(\alpha_{n+1})] - \\
&- (0.00136 \ 32927)h[f(\alpha_{n-1}) + f(\beta_{n+1})] + \\
&+ (0.00012 \ 36065)h[f(\beta_{n-2}) + f(\alpha_{n+2})] - \\
&- (0.00000 \ 57738)h[f(\alpha_{n-2}) + f(\beta_{n+2})] + R_n \quad (15.2.11)
\end{align*}
\]

where the nodes are

\[
\begin{align*}
\alpha_n &= x_n + (0.24346 \ 00865)h, & \beta_n &= x_n + (0.75653 \ 99135)h.
\end{align*}
\]

The remainder satisfies

\[
|R_n| \leq 0.000000000011h^{13}M_{12}
\]

\[
M_{12} = \max_{x} |f^{(12)}(x)|, \quad \alpha_{n-2} < x < \beta_{n+2}.
\]

3. Finally we give three formulas which use three values of $f(x)$ on each step.

Using three basic nodes and one additional node on each adjacent interval as depicted in Fig. 13 a formula of degree 7 can be constructed.

\[
\begin{align*}
& \gamma_{n-1} \gamma_n \alpha_n \beta_n \gamma_n \alpha_{n+1} \beta_{n+1} \\
& X_{n-1} X_n X_{n+1} X_{n+2}
\end{align*}
\]

Figure 13.

The formula is

\[
\begin{align*}
\gamma_{n+1} &= \gamma_n + (0.40010 \ 36566)h[f(\beta_n)] + \\
&+ (0.29348 \ 93491)h[f(\alpha_n) + f(\gamma_n)] + \\
&+ (0.00645 \ 88226)h[f(\gamma_{n-1}) + f(\alpha_{n+1})] + R_n \quad (15.2.12)
\end{align*}
\]

\[
\begin{align*}
\alpha_n &= x_n + (0.13518 \ 35561)h \\
\beta_n &= x_n + (0.5)h \\
\gamma_n &= x_n + (0.86481 \ 64439)h
\end{align*}
\]
15.2. Formulas of Special Form

\[ |R_n| \leq 0.0000000024h^9M_8 \]

\[ M_8 = \max_x |f^{(8)}(x)|, \quad \gamma_{n-1} < x < \alpha_{n+1}. \]

With the nodes shown in Fig. 14 a formula of degree 9 can be constructed.

The formula is

\[ Y_{n+1} = Y_n + (0.38762 \ 75418)hf(\beta_n) + \]
\[ + (0.29781 \ 27562)h[f(\alpha_n) + f(Y_n)] + \]
\[ + (0.00848 \ 08932)h[f(Y_{n-1}) + f(\alpha_{n+1})] - \]
\[ - (0.00010 \ 74203)h[f(\beta_{n-1}) + f(\beta_{n+1})] + R_n \quad (15.2.13) \]

\[ \alpha_n = x_n + (0.14145 \ 83289)h \]
\[ \beta_n = x_n + (0.5)h \]
\[ \gamma_n = x_n + (0.85854 \ 16711)h \]

\[ |R_n| \leq 0.000000000002h^{11}M_{10} \]

\[ M_{10} = \max_x |f^{(10)}(x)|, \quad \beta_{n-1} < x < \beta_{n+1}. \]

The last formula we give uses nodes situated as in Fig. 15 and has degree 11.

The formula is

\[ Y_{n+1} = Y_n + (0.38134 \ 28493)hf(\beta_n) + \]
\[ + (0.29986 \ 68413)h[f(\alpha_n) + f(Y_n)] + \]
\[ + (0.00967 \ 80471)h[f(Y_{n-1}) + f(\alpha_{n+1})] - \]
\[ - (0.00022 \ 28947)h[f(\beta_{n-1}) + f(\beta_{n+1})] + \]
\[ + (0.00000 \ 65816)h[f(\alpha_{n-1}) + f(Y_{n+1})] + R_n \quad (15.2.14) \]
Approximate Calculation of Indefinite Integrals

\[ \alpha_n = x_n + (0.14469 \ 85558)h \]
\[ \beta_n = x_n + (0.5)h \]
\[ \gamma_n = x_n + (0.85530 \ 14442)h \]

\[ |R_n| \leq 0.0000000000003h^{13}M_{12} \]
\[ M_{12} = \max_x |f^{(12)}(x)|, \quad \alpha_{n-1} < x < \gamma_{n+1}. \]

Example. Let us calculate the elliptic integral of the first kind

\[ y(x) = \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} \]

for \( k^2 = 0.5 \).

For this calculation we use (15.2.7) with step size \( h = 0.1 \). This formula contains 4 nodes and its degree of precision is 5. For each additional step in the calculation we must compute two new values of the integrand.

As a comparison the above integral was also calculated by formula (14.1.5):

\[ y_{n+1} = y_n + h \left[ \frac{f_n + f_{n+1}}{2} - \frac{1}{12} \frac{\Delta^2 f_{n-1} + \Delta^2 f_n}{2} + \cdots \right]. \]

Here the step size was taken to be \( h = 0.05 \) so that for each step of length 0.1 two new values of \( f(x) \) would also be required. Two forms of this formula were used:

1. with four nodes \( x_{n-1}, x_n, x_{n+1}, x_{n+2} \)
2. with six nodes \( x_{n-2}, x_{n-1}, \ldots, x_{n+3} \).

In the first case formula (14.1.5) contains the same number of nodes as (15.2.7) and in the second case the formula has the same algebraic degree of precision as (15.2.7). The exact values of the integrals were taken from the table of Legendre.\(^4\)

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact value of ( y(x) )</th>
<th>Formula (15.2.7)</th>
<th>Error ( \times 10^{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00000 00000</td>
<td>0.00000 00000</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.10025 11947</td>
<td>0.10025 11947</td>
<td>-1</td>
</tr>
<tr>
<td>0.2</td>
<td>0.20203 89251</td>
<td>0.20203 89251</td>
<td>-3</td>
</tr>
<tr>
<td>0.3</td>
<td>0.30705 49305</td>
<td>0.30705 49312</td>
<td>-7</td>
</tr>
<tr>
<td>0.4</td>
<td>0.41734 51597</td>
<td>0.41734 51612</td>
<td>-15</td>
</tr>
<tr>
<td>0.5</td>
<td>0.53562 72210</td>
<td>0.53562 73702</td>
<td>-42</td>
</tr>
<tr>
<td>0.6</td>
<td>0.66584 78254</td>
<td>0.66584 78390</td>
<td>-136</td>
</tr>
<tr>
<td>0.7</td>
<td>0.81448 92840</td>
<td>0.81448 93476</td>
<td>-636</td>
</tr>
<tr>
<td>0.8</td>
<td>0.99390 71263</td>
<td>0.99390 73932</td>
<td>-1669</td>
</tr>
</tbody>
</table>

### 15.2. Formulas of Special Form

<table>
<thead>
<tr>
<th>$x$</th>
<th>(14.1.5) with 4 nodes</th>
<th>Error $\times 10^{10}$</th>
<th>(14.1.5) with 6 nodes</th>
<th>Error $\times 10^{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00000 00000</td>
<td></td>
<td>0.00000 00000</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>0.05003 12182</td>
<td></td>
<td>0.05003 12881</td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>0.10025 10523</td>
<td>1423</td>
<td>0.10025 11965</td>
<td>-19</td>
</tr>
<tr>
<td>0.15</td>
<td>0.15085 26647</td>
<td></td>
<td>0.15085 28927</td>
<td></td>
</tr>
<tr>
<td>0.20</td>
<td>0.20203 86018</td>
<td>3230</td>
<td>0.20203 89297</td>
<td>-49</td>
</tr>
<tr>
<td>0.25</td>
<td>0.25402 62515</td>
<td></td>
<td>0.25402 67037</td>
<td></td>
</tr>
<tr>
<td>0.30</td>
<td>0.30705 43281</td>
<td>6024</td>
<td>0.30705 49416</td>
<td>-111</td>
</tr>
<tr>
<td>0.35</td>
<td>0.36139 09430</td>
<td></td>
<td>0.36139 17730</td>
<td></td>
</tr>
<tr>
<td>0.40</td>
<td>0.41734 40530</td>
<td>11067</td>
<td>0.41734 51861</td>
<td>-264</td>
</tr>
<tr>
<td>0.45</td>
<td>0.47527 55059</td>
<td></td>
<td>0.47527 70768</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>0.53562 05749</td>
<td>21579</td>
<td>0.53562 28057</td>
<td>-729</td>
</tr>
<tr>
<td>0.55</td>
<td>0.59891 61532</td>
<td></td>
<td>0.59891 94255</td>
<td></td>
</tr>
<tr>
<td>0.60</td>
<td>0.66584 30644</td>
<td>47610</td>
<td>0.66584 80773</td>
<td>-2519</td>
</tr>
<tr>
<td>0.65</td>
<td>0.73729 25074</td>
<td></td>
<td>0.73729 06471</td>
<td></td>
</tr>
<tr>
<td>0.70</td>
<td>0.81447 62323</td>
<td>130517</td>
<td>0.81447 05609</td>
<td>-12769</td>
</tr>
<tr>
<td>0.75</td>
<td>0.89912 20667</td>
<td></td>
<td>0.89920 04674</td>
<td></td>
</tr>
<tr>
<td>0.80</td>
<td>0.99385 27113</td>
<td>544150</td>
<td>0.99392 09917</td>
<td>-138654</td>
</tr>
</tbody>
</table>
16.1. INTRODUCTION

In the last two chapters we have discussed separate problems on the approximate evaluation of indefinite integrals and in both cases we used only one preceding value of the integral in order to approximate the next value. Computational methods of this type are always stable with respect to the errors of the initial values and the rounding errors providing that they are exact for \( f(x) = 0 \) and \( y(x) = 1 \). In such cases the formula must have the form

\[
y_{n+1} = y_{n-k} + \sum_{j=1}^{m} B_{n,j} f(\xi_{n,j}) + r_n
\]

and thus is a formula with positive coefficients \( A_i \) and therefore is stable with respect to the errors in the initial values and the rounding errors as we showed in Section 13.3.

It is a more difficult problem to derive formulas which use more than one preceding value of \( y(x) \) since formulas of the highest algebraic degree of precision of this type which are stable cannot always be found.

In this chapter we discuss one method for constructing stable formulas of the highest degree of precision.

Suppose we wish to calculate the integral

\[
y(x) = y_0 + \int_{x_0}^{x} f(t) \, dt
\]

320
16.1. Introduction

for an arbitrary set of points

\[ x_k \quad (k = 0, 1, \ldots, N; \quad x_k < x_{k+1}) \]

on the segment \([x_0, X]\). We assume that the calculation has been carried up to the point \(x_n\) and that we have computed \(y(x_n)\). In order to compute \(y(x_{n+1})\) we can use any of the previously computed values \(y_k \quad (k \leq n)\) of \(y(x)\) and any values of \(y'(x) = f(x)\). Here we assume that the derivative \(y'(x)\) is known everywhere on \([x_0, X]\) and that we may use any nodes whatsoever on this segment at which to evaluate \(f(x)\).

This is a problem of interpolation to find the value of \(y(x)\) at one of the fixed nodes \(x_{n+1}\) in terms of values of the same function and of its derivative \(y'(x) = f(x)\). We will see below that it will be useful to divide the nodes in the formula into three classes.

Let \(y(z)\) be any function which is defined and differentiable on a certain segment \([a, b]\). On this segment we take \(r + s + u\) distinct points

\[
\xi_1, \quad \xi_2, \quad \ldots, \quad \xi_r
\]

\[
\xi_{r+1}, \quad \xi_{r+2}, \quad \ldots, \quad \xi_{r+s}
\]

\[
\xi_{r+s+1}, \quad \xi_{r+s+2}, \quad \ldots, \quad \xi_{r+s+u}
\]

(16.1.1)

At the first \(r\) of these nodes we assume that we know the value of \(y(z)\):

\[ y(\xi_1), \ldots, y(\xi_r) \]

At the next \(s\) nodes we assume that we know the value of both the function and its derivative:

\[ y(\xi_j), \quad y'(\xi_j) \quad j = r + 1, \ldots, r + s. \]

At the last \(u\) nodes we assume that we know only the value of the derivative

\[ y'(\xi_j) \quad j = r + s + 1, \ldots, r + s + u. \]

We will call the nodes written in the first line of (16.1.1) the simple nodes, those in the second line the double nodes and those in the last line the auxiliary nodes. In addition we let \(x\) denote a point of \([a, b]\) which does not coincide with any of the simple or double nodes but which may coincide with an auxiliary node.

We select certain \(r + s\) numbers \(a_j \quad (j = 1, 2, \ldots, r + s)\) and \(s + u\) numbers \(\beta_j \quad (j = r + 1, \ldots, r + s + u)\) which will be defined later and consider the expression

\[ y(x) = \sum_{j=1}^{r+s} a_j y(\xi_j) + \sum_{j=r+1}^{r+s+u} \beta_j y'(\xi_j) + R. \]  

(16.1.2)
Approximate Calculation of Indefinite Integrals

Neglecting the remainder $R$ gives an approximate expression for $y(x)$:

$$y(x) \approx \sum_{j=1}^{r+s} a_j y(\xi_j) + \sum_{j=r+1}^{r+s+u} \beta_j y'(\xi_j). \quad (16.1.3)$$

The degree of precision of this equation is defined in the usual way: we say that (16.1.3) has algebraic degree of precision $m$ if it is exact for all monomials $y(z) = z^k$ ($k = 0, 1, \ldots, m$) and is not exact for $y(z) = z^{m+1}$. We will determine what the highest degree of precision of (16.1.3) may be and under what conditions it is achieved.

**Theorem 1.** For any $a_j, \beta_j$ and any disposition of points $\xi_j$ and $x$ the degree of precision $m$ of (16.1.3) is always less than $r + 2s + 2u$:

$$m < r + 2s + 2u.$$  

**Proof.** It suffices to show that there always exists a polynomial of degree not exceeding $r + 2s + 2u$ for which equation (16.1.3) is not exact.

We assume, at first, that none of the auxiliary nodes coincide with $x$ and consider the polynomial

$$y(z) = (z - \xi_1) \cdots (z - \xi_r)(z - \xi_{r+1})^2 \cdots (z - \xi_{r+s+u})^2 = A(z) \quad (16.1.4)$$

of degree $r + 2s + 2u$.

It is obvious that $A(\xi_j) = 0$ for $j = 1, \ldots, r + s + u$ and $A'(\xi_j) = 0$ for $j \geq r + 1$. Thus the right side of (16.1.3) is zero for this function. The left side $y(x) = A(x) \neq 0$ because $x \neq \xi_j$ and (16.1.3) can not be exact.

Now assume that one of the auxiliary nodes, for example $\xi_{r+s+u}$, coincides with $x$. We introduce the polynomial $B(z)$ of degree $r + 2s + 2u - 2$:

$$B(z) = \frac{A(z)}{(z - \xi_{r+s+u})^2}.$$  

If $B'(x) \neq 0$ we put

$$y(z) = B(z) \left[ z - x - \frac{B(x)}{B'(x)} \right].$$

This is a polynomial of degree $r + 2s + 2u - 1$ for which the right side of (16.1.3) is zero. The left side is $y(x) = -B^2(x)/B'(x) \neq 0$ and (16.1.3) is not satisfied. In this case the degree of precision of (16.1.3) is less than $r + 2s + 2u - 1$.

If $B'(x) = 0$ then (16.1.3) is not satisfied for $y(z) = B(z)$. In this case the degree of precision is less than $r + 2s + 2u - 2$. This proves Theorem 1.
This theorem shows that the greatest possible degree of precision of (16.1.3) is \( r + 2s + 2u - 1 \). Later we show that this degree of precision can be achieved by an appropriate choice of the coefficients \( \alpha_j \) and \( \beta_j \) and auxiliary nodes \( \xi_j(j > r + s) \).

From the proof of Theorem 1 we see that when one of the nodes \( \xi_j(j > r + s) \) coincides with \( x \) the degree of precision of (16.1.3) is less than \( r + 2s + 2u - 1 \) and cannot achieve its highest value. Therefore we will always assume that all the \( \xi_j \) are different from the point \( x \).

### 16.2. Conditions Under Which the Highest Degree of Precision Is Achieved

If we require that (16.1.3) be exact for the monomials \( x^k, k = 0, 1, \ldots, r + 2s + 2u - 1 \), we obtain the following system of \( r + 2s + 2u \) equations

\[
\sum_{j=1}^{r+s} a_j \xi_j^k + \sum_{j=r+1}^{r+s+u} \beta_j k \xi_j^{k-1} = x^k \quad (k = 0, 1, \ldots, r + 2s + 2u - 1). \tag{16.2.1}
\]

This system can be studied by comparing it with the related interpolation problem.

At the simple nodes let us be given the values of the function

\[
y(\xi_j) \quad j = 1, \ldots, r. \tag{16.2.2}
\]

At all the double and auxiliary nodes let us be given both the value of the function and also its derivative

\[
y(\xi_j), \quad y'(\xi_j) \quad j = r + 1, \ldots, r + s + u. \tag{16.2.3}
\]

Using these values consider the problem of interpolating for the value \( y(x) \). Taking certain numbers \( a_j' \) (\( j = 1, \ldots, r + s + u \)) and \( \beta_j' \) (\( j = r + 1, \ldots, r + s + u \)) we construct the approximate equation

\[
y(x) = \sum_{j=1}^{r+s+u} a_j' y(\xi_j) + \sum_{j=r+1}^{r+s+u} \beta_j' y'(\xi_j). \tag{16.2.4}
\]

Here it is possible to determine the \( r + 2s + 2u \) coefficients \( a_j' \) and \( \beta_j' \) if we require that (16.2.4) be exact for the monomials \( y(x) = x^k, k = 0, 1, \ldots, r + 2s + 2u - 1 \). This gives a system of \( r + 2s + 2u \) linear equations for the \( a_j', \beta_j' \):

\[
\sum_{j=1}^{r+s+u} a_j' \xi_j^k + \sum_{j=r+1}^{r+s+u} \beta_j' k \xi_j^{k-1} = x^k \quad (k = 0, 1, \ldots, r + 2s + 2u - 1). \tag{16.2.5}
\]
The determinant of this system is

\[
\Delta = \begin{vmatrix}
1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\xi_1 & \xi_2 & \cdots & \xi_{r+s+u} & 1 & \cdots & 1 \\
\xi_1^2 & \xi_2^2 & \cdots & \xi_{r+s+u}^2 & 2\xi_{r+1} & \cdots & 2\xi_{r+s+u} \\
& & & & & & \\
\end{vmatrix}
\]

where we have written the coefficients of the \( \alpha_j \) in the first \( r + s + u \) columns and the coefficients of the \( \beta_j \) in the last \( s + u \) columns. This determinant is different from zero\(^1\) if \( \xi_i \neq \xi_j \) (\( i, j = 1, \ldots, r + s + u; i \neq j \)). In the above case this will be true.

Thus the system (16.2.5) has a unique solution provided that the points \( \xi_j \) are all distinct. The relationship between the systems (16.2.1) and (16.2.5) is given by the following two assertions which are easily verified.

1. Suppose the system (16.2.1) has a solution. Then this solution is unique and the unknowns \( \alpha_j \) and \( \beta_j \) in (16.2.5) satisfy the following relationships:

\[
\begin{align*}
\alpha_j &= \alpha_j & j = 1, \ldots, r + s \\
\alpha_{r+s+1} &= 0, \ldots, \alpha_{r+s+u} &= 0 \\
\beta_j &= \beta_j & j = r + 1, \ldots, r + s.
\end{align*}
\]

Indeed, suppose that the numbers \( \alpha_j \) (\( j = 1, \ldots, r + s \)) and \( \beta_j \) (\( j = r + 1, \ldots, r + s + u \)) are a solution of the system (16.2.1). Together with these numbers we also take \( u \) numbers \( \alpha_{r+s+1} = 0, \ldots, \alpha_{r+s+u} = 0 \). The resulting system of \( r + 2s + 2u \) numbers will clearly satisfy (16.2.5) and since (16.2.5) has a unique solution the relationships (16.2.6) must be valid.

2. If the numbers \( \alpha_j \) and \( \beta_j \) are a solution of the system (16.2.5) and if

\[
\begin{align*}
\alpha_{r+s+1} &= 0, \ldots, \alpha_{r+s+u} &= 0
\end{align*}
\]

then the system (16.2.1) has a solution and the following relationships

\[
\begin{align*}
\alpha_{r+s+1} &= \xi_{r+s+1}, \ldots, \alpha_{r+s+u} &= \xi_{r+s+u} \\
\beta_j &= \beta_j & j = r + 1, \ldots, r + s.
\end{align*}
\]

\(^1\)We can easily see this if we take the Vandermonial determinant of order \( r + 2s + 2u \) in the parameters \( \xi_1, \ldots, \xi_{r+2s+2u} \)

\[
\mathcal{W}(\xi_1, \ldots, \xi_{r+2s+2u}) = \prod_{i > j} (\xi_i - \xi_j)
\]

and calculate the mixed derivative with respect to \( \xi_{r+s+u+1}, \ldots, \xi_{r+2s+2u} \) and then set \( \xi_{r+s+u+1} = \xi_{r+1}, \ldots, \xi_{r+2s+2u} = \xi_{r+s+u} \). After performing these operations we obtain \( \Delta \). But this is the same as striking out the factors \( (\xi_{r+s+u+j} - \xi_{r+j}), j = 1, \ldots, s + u \), from the product (\(^*\)). What remains is a new product which is clearly different from zero.
are valid:

\[ a_j = a_j' \quad j = 1, \ldots, r + s \]  
\[ \beta_j = \beta_j' \quad j = r + 1, \ldots, r + s + u \]  

(16.2.8)

Here we need only note that if the conditions (16.2.7) are fulfilled then the equations (16.2.5) coincide with (16.2.1).

The solution of the system (16.2.5) is easily constructed from the theory of interpolation. Let \( y(z) \) be a polynomial of degree \( \leq r + 2s + 2u - 1 \). When the \( a_j' \) and \( \beta_j' \) satisfy equations (16.2.5) then (16.2.4) must be exact for any value of \( x \). This is an interpolation formula for the value \( y(x) \) in terms of the values of this polynomial at the points \( \xi_j \) (\( j \leq r + s + u \)) and the values of its derivative at the points \( \xi_j \) (\( r + 1 \leq j \leq r + s + u \)). This is an interpolation with \( r \) simple nodes \( \xi_j \) (\( j \leq r \)) and \( s + u \) double nodes \( \xi_j \) (\( r < j \leq r + s + u \)). The interpolating polynomial can be represented by Hermite's formula (3.3.8) which in the present case is

\[
y(x) = \sum_{j=1}^{r} \frac{A_j(x)}{(x - \xi_j)A_j'(\xi_j)} y(\xi_j) + \sum_{j=r+1}^{r+s+u} \frac{A_j(x)}{A_j(\xi_j)} \left[ 1 - (x - \xi_j) \frac{A_j'(\xi_j)}{A_j(\xi_j)} \right] y(\xi_j) + \sum_{j=r+1}^{r+s+u} \frac{A_j(x)}{A_j(\xi_j)} (x - \xi_j) y'(\xi_j) \\
A_j(x) = A(x)/(x - \xi_j)^2.
\]

(16.2.9)

The right sides of (16.2.4) and (16.2.9) must coincide and since the values \( y(\xi_j) \) and \( y'(\xi_j) \) are arbitrary the coefficients \( a_j' \) and \( \beta_j' \) must be equal to the corresponding coefficients of (16.2.9).

The conditions (16.2.7) which must hold for (16.2.1) to be solvable are

\[
\frac{A_j(x)}{A_j(\xi_j)} \left[ 1 - \frac{A_j'(\xi_j)}{A_j(\xi_j)} (x - \xi_j) \right] = 0 \quad j = r + s + 1, \ldots, r + s + u.
\]

(16.2.10)

Since \( A_j(x)/A_j(\xi_j) \neq 0 \) the expression in brackets must be zero and dividing this expression by \( x - \xi_j \) gives

\[
\sum_{k=r+s+1}^{r+s+u} \frac{2}{\xi_j - \xi_k} + \sum_{k=1}^{r} \frac{2}{\xi_j - \xi_k} + \sum_{k=r+1}^{r+s} \frac{2}{\xi_j - \xi_k} + \frac{1}{\xi_j - x} = 0.
\]

Here in the first sum the symbol * indicates that the term for \( k = j \) is omitted.

The results of this section can be summarized in the following theorem.
Theorem 2. The following two conditions are necessary and sufficient for (16.1.3) to have the highest degree of precision $r + 2s + 2u - 1$:

1. The points $\xi_j$ and $x$ must satisfy the system of $u$ equations (16.2.10).

2. The coefficients $a_j$ and $\beta_j$ must have the values

$$a_j = \frac{A(x)}{(x - \xi_j) A'(\xi_j)} \quad j = 1, \ldots, r$$

$$a_j = \frac{A_j(x)}{A_j(\xi_j)} \left[ 1 - (x - \xi_j) \frac{A_j'(\xi_j)}{A_j(\xi_j)} \right] \quad j = r + 1, \ldots, r + x \quad (16.2.11)$$

$$\beta_j = \frac{A_j(x)}{A_j(\xi_j)} (x - \xi_j) \quad j = r + 1, \ldots, r + s + u.$$ 

16.3. THE NUMBER OF INTERPOLATING POLYNOMIALS OF THE HIGHEST DEGREE OF PRECISION

Equations (16.2.10) can be studied in an intuitive way by using an electrostatic analogy similar to the analogy of Section 11.4. We take two points $z_1$ and $z_2$ in the complex plane and place at these points particles with charges $e_1$ and $e_2$. We assume that these particles exert on each other a force which is inversely proportional to the distance between them and directly proportional to the size of their charges.

Assuming that the coefficient of proportionality is unity then the force which $z_1$ exerts on $z_2$ is

$$\frac{e_1 e_2}{z_2 - z_1}.$$

Suppose that in the plane we take $r + s + 1$ points $x, \xi_1, \ldots, \xi_{r+s}$. At each of the points $x, \xi_1, \ldots, \xi_r$ we put particles with unit charge and at each of the points $\xi_{r+1}, \ldots, \xi_{r+s}$ particles with charge two and we fix these particles at these points. Together with these particles we take $u$ free particles of charge 2 at the points $\xi_{r+s+1}, \ldots, \xi_{r+s+u}$.

When the free particles are at equilibrium the sum of the forces on each free particle must be zero

$$\sum_{k=r+s+1}^{r+s+u} \frac{4}{\xi_j - \xi_k} + \sum_{k=1}^{r} \frac{2}{\xi_j - \xi_k} + \sum_{k=r+1}^{r+s} \frac{4}{\xi_j - \xi_k} + \frac{2}{\xi_j - x} = 0, \quad j = r + s + 1, \ldots, r + s + u$$

These equations differ only by the multiple of 2 from equations
16.4. Minimizing the Remainder

(16.2.10) which are the conditions for which (16.1.3) has the highest degree of precision.

From this analogy the following assertions concerning (16.2.10) are evident.\(^2\)

1. If \(x, \xi_1, \ldots, \xi_{r+s}\) are any complex numbers and if \(\xi_{r+s+1}, \ldots, \xi_{r+s+u}\) satisfy (16.2.10) then the points \(\xi_{r+s+1}, \ldots, \xi_{r+s+u}\) lie in the smallest convex polygon containing \(x, \xi_1, \ldots, \xi_{r+s}\). In particular when \(x, \xi_1, \ldots, \xi_{r+s}\) are real numbers the points \(\xi_{r+s+1}, \ldots, \xi_{r+s+u}\) lie inside the smallest segment containing \(x\) and \(\xi_j\) (\(j = 1, \ldots, r + s\)).

2. Let \(x, \xi_1, \ldots, \xi_{r+s}\) be real and distinct. These points divide the real axis into \(r + s\) adjacent intervals. Suppose that we have indicated beforehand how many of the auxiliary nodes should belong to each of these intervals. The number of such ways in which these auxiliary nodes may be arranged is \(\frac{(r + s + u - 1)!}{u!(r + s - 1)!}\). For each of these arrangements of the auxiliary nodes there exists a solution of the system (16.2.10).

3. If we consider solutions which differ only by permutations of the nodes \(\xi_{r+s+1}, \ldots, \xi_{r+s+u}\) as a single solution then for each arrangement of these nodes among the points \(x, \xi_1, \ldots, \xi_{r+s}\) there will be one and only one solution for (16.2.10).

These results can be expressed in the following theorem.

**Theorem 3.** For any set of real and distinct points \(x, \xi_1, \ldots, \xi_{r+s}\) the auxiliary nodes \(\xi_{r+s+1}, \ldots, \xi_{r+s+u}\) can be selected in \(\frac{(r + s + u - 1)!}{u!(r + s - 1)!}\) ways which will make (16.1.3) have the highest algebraic degree of precision \(m = r + 2s + 2u - 1\). For each partitioning of the auxiliary nodes among the \(r + s\) intervals formed by \(x, \xi_1, \ldots, \xi_{r+s}\) there exists one and only one solution of this type.

16.4. THE REMAINDER OF THE INTERPOLATION AND MINIMIZATION OF ITS ESTIMATE

Consider the remainder of the interpolation formula (16.1.3)

\[
R(y) = y(x) - \sum_{j=1}^{r+s} a_j y(\xi_j) - \sum_{j=r+1}^{r+s+u} \beta_j y'(\xi_j) \tag{16.4.1}
\]

\(^2\)These can be proved by an arithmetic argument similar to that used by T. Stieltjes in an analogous case concerning the existence of polynomial solutions to differential equations (Collected Works, Groningen, 1914, v. 1, p. 434–439). The relationship between the interpolation problem and differential equations is studied in Section 16.6.
for which we assume (16.2.10) is satisfied so that it has the highest algebraic degree of precision \( r + 2s + 2u - 1 \). Then the interpolation (16.1.3) coincides with (16.2.4) and they have the same precision. Equation (16.2.4) has \( r \) simple nodes and \( s + u \) double nodes. Assuming that \( y(z) \) has a continuous derivative of order \( r + 2s + 2u \) on the segment \([a, b]\) then, by Theorem 6 of Chapter 3, we can find a point \( \xi \in [a, b] \) for which

\[
R(y) = \frac{A(x)}{(r + 2s + 2u)!} y^{(r+2s+2u)}(\xi)
\]

(16.4.2)

\[A(x) = (x - \xi_1) \cdots (x - \xi_r)(x - \xi_{r+1})^2 \cdots (x - \xi_{r+s+u})^2.\]  

(16.4.3)

This remainder then coincides with the remainder (16.4.1).

In the class of functions defined by the inequality

\[|f^{(r+2s+2u)}(z)| \leq M \quad z \in [a, b]\]

the remainder has the precise estimate

\[|R(y)| \leq \frac{|A(x)|}{(r + 2s + 2u)!} M.\]  

(16.4.4)

The highest degree of precision \( r + 2s + 2u - 1 \) can be achieved by \( \frac{(r + s + u - 1)!}{u!(r + s - 1)!} \) different choices of the auxiliary nodes and it is natural to ask which of these formulas will minimize the right side of (16.4.4).

The only term in (16.4.4) which depends on the nodes is \(|A(x)|\) and we will determine which choice of nodes makes this a minimum. The problem formulated in Section 16.1 was that of calculating \( y(x) \) at the node \( x_{n+1} \). The simple and double nodes \( \xi_j \) \((j = 1, \ldots, r + s)\) must be points of the fixed set \( x_0, x_1, x_2, \ldots, x_n \).

We must therefore select the simple nodes and indicate how the auxiliary nodes are distributed among them. We assume that the auxiliary nodes are enumerated in increasing order.

From the discussion of the previous section we can solve the problem of minimizing \(|A(x)|\) by an intuitive argument. We prefer this line of reasoning since it is much shorter than a constructive arithmetic proof.

Assume that the simple and double nodes have been chosen and that we wish to determine the distribution of the auxiliary nodes. Consider the factors

\[(x_{n+1} - \xi_{r+s+1}), \ldots, (x_{n+1} - \xi_{r+s+u})\]  

(16.4.5)

of \( A(x_{n+1}) \). These are distances between \( x_{n+1} \) and the nodes \( \xi_{r+s+1}, \ldots, \xi_{r+s+u} \). These distances will be a minimum when all the auxiliary
nodes lie in the segment adjacent to $x_{n+1}$. This is true for any choice of the simple and double nodes.

Now consider the factors

$$(x_{n+1} - \xi_1), \ldots, (x_{n+1} - \xi_r), (x_{n+1} - \xi_{r+1})^2, \ldots, (x_{n+1} - \xi_{r+s})^2.\)

Since the terms containing the double nodes are raised to the second power it is clear that the double nodes must be the points of the set $x_k$ which are closest to $x_{n+1}$; the double nodes $\xi_j$ ($j = r + 1, \ldots, r + s$) must coincide with the points $x_n, x_{n-1}, \ldots, x_{n-s+1}$. The simple nodes then must be the points $x_{n-s}, \ldots, x_{n-s-r+1}$.

Thus we can state:

The estimate of the remainder (16.4.4) is a minimum when the nodes are chosen in the following way:

1. The auxiliary nodes are situated in the segment $[x_n, x_{n+1}]$;
2. The double nodes are taken at the points $x_n, x_{n-1}, \ldots, x_{n-s+1}$;
3. The simple nodes are taken at the points $x_{n-s}, \ldots, x_{n-s-r+1}$.

### 16.5. Conditions for Which the Coefficients $\alpha_j$ Are Positive

Formulas with positive coefficients $\alpha_j$ play an important role in the theory of indefinite integration because, as we showed in Section 13.3, such formulas are stable with respect to the errors in the initial values and the rounding errors. In this section we will see which formulas of the form (16.1.3) of the highest degree of precision have positive $\alpha_j$. We take $x = x_{n+1}$ and assume that the $\xi_j$ ($j = 1, \ldots, r + s + u$) are chosen as described at the end of the previous section.

Let $\xi_j$ ($1 \leq j \leq r$) be one of the simple nodes. The coefficient $\alpha_j$ which corresponds to this node is, by (16.2.11):

$$\alpha_j = \frac{A(x_{n+1})}{(x_{n+1} - \xi_j)A'(\xi_j)}.$$

The term $A(x_{n+1})$ is positive since all its factors $(x_{n+1} - \xi_j)$ are positive and thus $\alpha_j$ has the same sign as $A'(\xi_j)$:

$$A'(\xi_j) = (\xi_j - \xi_1) \cdots (\xi_j - \xi_{j-1})(\xi_j - \xi_{j+1}) \cdots (\xi_j - \xi_r) \times$$

$$\times (\xi_j - \xi_{r+1})^2 \cdots (\xi_j - \xi_{r+s+u})^2.$$

Thus for two adjacent simple nodes the values of $A'(\xi_j)$ will have opposite signs.

Therefore the $\alpha_j$ for the simple nodes can all be positive only in the two cases $r = 0$, in which there are no simple nodes, and $r = 1$, in which there is one simple node.
Approximate Calculation of Indefinite Integrals

We now consider a double node \( \xi_j \) \((r < j \leq r + s)\). By (16.2.11) the corresponding \( \alpha_j \) is

\[
\alpha_j = \frac{A_j(x_{n+1})(\xi_j - x_{n+1})}{A_j(\xi_j)} \left[ \frac{1}{\xi_j - x_{n+1}} + \frac{A_j'(\xi_j)}{A_j(\xi_j)} \right]
\]

\[
A_j(z) = \frac{A(z)}{(z - \xi_j)^2}.
\]

Since \( \xi_j \) lies to the left of \( x_{n+1} \) and the simple nodes all lie to the left of all the double nodes then the term outside the brackets is negative. Then \( \alpha_j \) will be positive if the following inequality is satisfied:

\[
\frac{1}{\xi_j - x_{n+1}} + \frac{A_j'(\xi_j)}{A_j(\xi_j)} = \frac{1}{\xi_j - x_{n+1}} + \sum_{k=1}^{r} \frac{1}{\xi_j - \xi_k} + \sum_{k=r+1}^{r+s+u} \frac{2}{\xi_j - \xi_k} < 0.
\]

This equation has a simple physical interpretation if we use the electrostatic analogy of Section 16.3. At \( \xi_j \) is a particle with charge 2 and the left side of (16.5.1) is the resultant of all the repulsive forces which act on this particle from all the other particles of this system.

Inequality (16.5.1) states that this resultant must be directed towards the left.

The point \( x_{n+1} \) and the auxiliary nodes always lie to the right of \( \xi_j \) and the particles situated at these points exert a leftward directed force on \( \xi_j \). Thus it is clear that we can make the following assertion concerning the existence of formulas with positive coefficients for all the double nodes:

For any \( r \) and \( s \) there exists a number \( u_0 \) with the property that if \( u > u_0 \) then all the coefficients \( \alpha_j \) \((j = r + 1, \ldots, r + s)\) will be positive.

Suppose we calculate \( \gamma(x) \) for a set of equally spaced points \( x_k = x_0 + kh \) \((k = 0, 1, \ldots)\). Let us take one double node \( (s = 1) \) at \( \xi_2 = x_n \) and one simple node \( (r = 1) \) at \( \xi_1 = x_{n-1} \). We have shown that the coefficient \( \alpha_1 \) for the simple node is positive. The condition that \( \alpha_2 \) be positive is

\[
\frac{1}{x_n - x_{n+1}} + \frac{1}{x_n - x_{n-1}} + \sum_{j=3}^{u+2} \frac{2}{x_n - \xi_j} = \sum_{j=3}^{u+2} \frac{2}{x_n - \xi_j} < 0.
\]

Since all the \( \xi_j > x_n \) this inequality is satisfied for all \( u > 1 \).

Now let us take two double nodes \( (s = 2) \) at \( x_n \) and \( x_{n-1} \) and no simple nodes \( (r = 0) \). The conditions that \( \alpha_1 \) and \( \alpha_2 \) be positive are:
16.6. Connection with a Differential Equation

\[
\frac{1}{x_{n-1} - x_{n+1}} + \frac{2}{x_{n-1} - x_n} + \sum_{j=3}^{u+2} \frac{2}{x_{n-1} - \xi_j} =
\]

\[
= -\frac{1}{2h} - \frac{2}{h} + \sum_{j=3}^{u+2} \frac{2}{x_{n-1} - \xi_j} < 0
\]

\[
\frac{1}{x_n - x_{n+1}} + \frac{2}{x_n - x_{n-1}} + \sum_{j=3}^{u+2} \frac{2}{x_n - \xi_j} =
\]

\[
= -\frac{1}{h} + \frac{2}{h} + \sum_{j=3}^{u+2} \frac{2}{x_n - \xi_j} < 0.
\]

These are also satisfied for all \( u \geq 1 \).

16.6. CONNECTION WITH THE EXISTENCE OF A POLYNOMIAL SOLUTION TO A CERTAIN DIFFERENTIAL EQUATION

In this section we show that equation (16.2.10) is equivalent to the existence of a polynomial solution to a certain differential equation.

Equation (16.2.10) was obtained from the equation which preceded it which can be written in the form

\[
A_j(\xi_j) + (\xi_j - x)A'_j(\xi_j) = \left\{ \frac{d}{dz} [(z - x)A_j(z)] \right\}_{z=\xi_j} = 0
\]

or since \( A_j(z) = A(z)/(z - \xi_j)^2 \)

\[
\frac{d}{dz} \left[ \frac{(z - x)A(z)}{(z - \xi_j)^2} \right]_{z=\xi_j} = 0 \quad (j = r + s + 1, \ldots, r + s + u). \quad (16.6.1)
\]

It will be convenient to write this in another form. We introduce the polynomials \( \sigma(z) \) and \( \Pi_u(z) \) corresponding to the double and auxiliary nodes:

\[
\sigma(z) = (z - \xi_{r+1}) \cdots (z - \xi_{r+s})
\]

\[
\Pi_u(z) = (z - \xi_{r+s+1}) \cdots (z - \xi_{r+s+u}).
\]

We also form

\[
p(z) = (z - x)(z - \xi_1) \cdots (z - \xi_r)
\]

\[
(z - x)A(z) = p(z)\sigma^2(z)\Pi_u^2(z)
\]
so that (16.6.1) can be written as
\[
\frac{d}{dz} \left[ \frac{p(z) \sigma^2(z) \Pi''_u(z)}{(z - \xi_j)^2} \right]_{z=\xi_j} = 0
\]
or
\[
\Pi'_u(\xi_j) \left\{ \frac{d}{dz} \left[ p(z) \sigma^2(z) \frac{d\Pi_u(z)}{dz} \right] \right\}_{z=\xi_j} = 0.
\]
Since \( z = \xi_j \) \((j > r + s)\) is a simple root of \( \Pi_u(z) \) then \( \Pi'_u(\xi_j) \neq 0 \) and we must have
\[
\left\{ \frac{d}{dz} \left[ p(z) \sigma^2(z) \frac{d\Pi_u(z)}{dz} \right] \right\}_{z=\xi_j} = 0 \quad (j > r + s). \tag{16.6.2}
\]
This says that each root of \( \Pi_u(z) \) is also a root of \( (p \sigma^2 \Pi'_u)' \). Because the roots of \( \Pi_u(z) \) are simple the polynomial \( (p \sigma^2 \Pi'_u)' \) must be divisible by \( \Pi_u(z) \).

The polynomial \( (p \sigma^2 \Pi'_u)' \) must also be divisible by \( \sigma(z) \) and since the roots of \( \sigma(z) \) are distinct from the roots of \( \Pi_u(z) \) then \( (p \sigma^2 \Pi'_u)' \) must be divisible by \( \sigma(z) \Pi_u(z) \). Since the degree of \( (p \sigma^2 \Pi'_u)' \) is \( r + 2s + u - 1 \) then there is a polynomial \( \rho(z) \) of degree \( r + s - 1 \) for which
\[
[p(z) \sigma^2(z) \Pi'_u(z)]' = \rho(z) \sigma(z) \Pi_u(z). \tag{16.6.3}
\]
This equation can be considered as a second order differential equation with respect to \( \Pi_u(z) \) and we can make the following assertion:

If equation (16.2.10) is satisfied then there exists a polynomial \( \rho(z) \) of degree \( r + s - 1 \) which will make \( \Pi_u(z) \) the solution of the differential equation (16.6.3).

The proof of the converse assertion requires certain preliminary remarks on the analytic properties of the solution of (16.6.3).

If we perform the differentiation in (16.6.3) and divide both sides by \( p(z) \sigma^2(z) \) we obtain
\[
\Pi''_u(z) + \left( \sum_{k=1}^{r} \frac{1}{z - \xi_k} + \sum_{k=r+1}^{r+s} \frac{2}{z - \xi_k} + \frac{1}{z - x} \right) \Pi'_u(z) + \\
+ \left( \sum_{k=1}^{r+s} \frac{a_k}{z - \xi_k} + \frac{a_u}{z - x} \right) \Pi_u(z) = 0 \tag{16.6.4}
\]
\[
\sum_{k=0}^{r+s} a_k = 0.
\]
The singular points of this equation are $x, \xi_1, \ldots, \xi_{r+s}, \infty$. These are regular singular points and we consider any one of the points $\xi_k$ or $x$.

The analytic construction of the canonical solution of (16.6.4) in a neighborhood of this point depends on the roots of an algebraic equation which is either $a(a - 1) + a = a^2 = 0$ or $a(a - 1) + 2a = a(a + 1) = 0$. These have for solutions the double root $a = 0$ or the roots $a = 0$, $a = -1$. In both cases one of the canonical solutions will be holomorphic at the singular point and different from zero there; the other solution is unbounded in a neighborhood of this point.

We will now assume that $\rho(z)$ has the property that (16.6.3) has a polynomial of degree $u$ as a solution. For all the singular points $x$ and $\xi_k$ this will be a holomorphic canonical solution and therefore is known to be different from zero at each of these points. The roots of $\Pi_u(z)$ are distinct from $x$ and $\xi_k$ and thus they are simple because (16.6.4) can have no multiple roots other than at the singular points. Let the roots of $\Pi_u(z)$ be $\xi_{r+s+1}, \ldots, \xi_{r+s+u}$.

If in (16.6.4) we set $z$ equal to one of the roots $\xi_j$ ($j > r + s$) then the term in $\Pi_u(z)$ vanishes. Then dividing by $\Pi'_u(\xi_j)$ gives

$$\frac{\Pi''(\xi_j)}{\Pi'_u(\xi_j)} + \sum_{k=1}^{r} \frac{1}{\xi_j - \xi_k} + \sum_{k=r+1}^{r+s} \frac{2}{\xi_j - \xi_k} + \frac{1}{\xi_j - x} = 0$$

$(j = r + s + 1, \ldots, r + s + u)$. This equation coincides with (16.2.10). Hence we can make the assertion:

If $\rho(z)$ is a polynomial of degree $r + s - 1$ for which (16.6.3) has a polynomial of degree $u$ as a solution then the roots $\xi_{r+s+1}, \ldots, \xi_{r+s+u}$ of this solution satisfy the system of equations (16.2.10).

The nodes of the interpolation (16.1.3) which has the highest algebraic degree of precision can be determined by finding the solution of (16.6.3) which is a polynomial of degree $u$ and then finding its roots.

16.7. SOME PARTICULAR FORMULAS

In this section we tabulate certain formulas of the highest degree of precision for low values of $r$, $s$ and $u$. The nodes are chosen to minimize the remainder in the manner described at the end of Section 16.4. The coefficients of the formulas were calculated using (16.2.11). The

---

3 See, for example, V. I. Smirnov, Course of Higher Mathematics, Gostekhizdat, Moscow, 1949, Vol. 3, part 2, sec. 98 (Russian).

4 The coefficients and nodes in these formulas were calculated by research assistant K. E. Chernin of the Leningrad section of the Mathematical Institute of the Academy of Sciences of the U.S.S.R. These values are exact to within a unit in the last place.
nodes were found by means of the differential equation (16.6.3). The set of points \( x_n \) are assumed to be equally spaced with an interval \( h \).

1. \( r = 1, s = 0 \). Here we use the value of \( y(x) \) at the point \( x_n \) and \( u \) values of the derivative at auxiliary nodes between \( x_n \) and \( x_{n+1} \). The auxiliary nodes are chosen so that the formula has the highest algebraic degree of precision. This is equivalent to representing \( y(x_{n+1}) \) in terms of \( y(x_n) \) by

\[
y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(t) \, dt
\]

and calculating this integral by a Gauss quadrature formula

\[
y_{n+1} = y_n + h[ B_1 f(x_1 + t_1 h) + \cdots + B_u f(x_u + t_u h) ]
\]

where the \( B_k \) and \( t_k \) are the Gauss coefficients and nodes for the segment \([0, 1]\).

2. \( r = 0, s = 1 \). We use the value of \( y(x) \) and \( f(x) \) at the point \( x_n \) and, in addition, \( u \) values of \( f(x) \) at auxiliary nodes between \( x_n \) and \( x_{n+1} \). The highest degree of precision is \( 2u + 1 \). The formula corresponds to the Markov (or Radau) formula with one fixed node at \( x_n \) and \( u \) nodes between \( x_n \) and \( x_{n+1} \). Values of the coefficients and nodes for \( u = 1, 2, \ldots, 6 \) are given in Section 9.2 for the segment \([-1, 1]\).

3. \( r = 1, s = 1 \). We use the value of \( y(x) \) at \( x_{n-1} \), the values of \( y(x) \) and \( f(x) \) at \( x_n \) and the value of \( f(x) \) at \( u \) auxiliary nodes between \( x_n \) and \( x_{n+1} \):

\[
y_{n+1} = A_{-1} y_{n-1} + A_0 y_0 + h \left[ B_0 f(x_n) + \sum_{j=1}^{u} B_j f(x_n + t_j h) \right] + R.
\]

The degree of precision is \( 2u + 2 \) and the remainder has the estimate

\[
R = \theta\frac{2h^{2u+3}}{(2u+3)!} \left[ \frac{u!(u+1)!}{(2u+1)!} \right]^2 f^{(2u+2)}(\xi)
\]

\( 0 < \theta < 1, \quad x_{n-1} < \xi < x_{n+1} \).

16.7. Some Particular Formulas

The nodes and coefficients for $u = 1, 2, 3, 4$ are tabulated below.

$u = 1$
\[
\begin{align*}
A_{-1} &= 0.02943725 \\
A_0 &= 0.97056275 \\
t_1 &= 0.7071068
\end{align*}
\]
\[
B_0 = 0.3431458 \\
B_1 = 0.6862915
\]

$u = 2$
\[
\begin{align*}
A_{-1} &= 0.001113587 \\
A_0 &= 0.998886413 \\
t_1 &= 0.3879073 \\
t_2 &= 0.8593118
\end{align*}
\]
\[
B_0 = 0.1334818 \\
B_1 = 0.5221058 \\
B_2 = 0.3455260
\]

$u = 3$
\[
\begin{align*}
A_{-1} &= 0.00004136036 \\
A_0 &= 0.999958639648 \\
t_1 &= 0.2312666 \\
t_2 &= 0.6124982 \\
t_3 &= 0.9177954
\end{align*}
\]
\[
B_0 = 0.07095688 \\
B_1 = 0.3458379 \\
B_2 = 0.3776724 \\
B_3 = 0.2055739
\]

$u = 4$
\[
\begin{align*}
A_{-1} &= 0.000001479556 \\
A_0 &= 0.999998520444 \\
t_1 &= 0.1507625 \\
t_2 &= 0.4352756 \\
t_3 &= 0.7366581 \\
t_4 &= 0.9462337
\end{align*}
\]
\[
B_0 = 0.04407358 \\
B_1 = 0.2361168 \\
B_2 = 0.3128314 \\
B_3 = 0.2713300 \\
B_4 = 0.1356498
\]

4. $r = 0$, $s = 2$. We use the formula
\[
y_{n+1} = A_{-1}y_{n-1} + A_0y_n + h \left[ B_{-1}f(x_{n-1}) + B_0f(x_0) + \sum_{j=1}^{u} B_j f(x_n + t_j h) \right] + R.
\]
for which the highest degree of precision is $2u + 3$. The remainder has the estimate
\[
R = \theta \frac{4h^{2u+4}}{(2u+4)!} \left[ \frac{u(u+1)}{(2u+1)!} \right]^2 f^{(2u+3)}(\xi)
\]
\[
0 < \theta < 1, \quad x_{n-1} < \xi < x_{n+1}.
\]

The nodes and coefficients for $u = 1, 2, 3, 4$ are tabulated below.

$u = 1$
\[
\begin{align*}
A_{-1} &= 0.16250915 \\
A_0 &= 0.83749085 \\
t_1 &= 0.74031242
\end{align*}
\]
\[
B_{-1} = 0.044532584 \\
B_0 = 0.49218941 \\
B_1 = 0.62578716
\]
Approximate Calculation of Indefinite Integrals

\( u = 2 \)

\[
\begin{align*}
A_{-1} &= 0.007766326 \\
A_0 &= 0.99223367 \\
t_1 &= 0.4207573 \\
t_2 &= 0.8717520
\end{align*}
\]

\( B_{-1} = 0.001560689 \quad B_0 = 0.1640716 \quad B_1 = 0.5242954 \quad B_2 = 0.3178386 \)

\( u = 3 \)

\[
\begin{align*}
A_{-1} &= 0.0008626295 \\
A_0 &= 0.9996373705 \\
t_1 &= 0.2515111 \\
t_2 &= 0.6333509 \\
t_3 &= 0.9235139
\end{align*}
\]

\( B_{-1} = 0.00005699653 \quad B_0 = 0.08143433 \quad B_1 = 0.3609307 \quad B_2 = 0.3658920 \quad B_3 = 0.1920487 \)

\( u = 4 \)

\[
\begin{align*}
A_{-1} &= 0.00001576632 \\
A_0 &= 0.99998423368 \\
t_1 &= 0.1627293 \\
t_2 &= 0.4540978 \\
t_3 &= 0.7493776 \\
t_4 &= 0.9492874
\end{align*}
\]

\( B_{-1} = 0.000002030488 \quad B_0 = 0.04885024 \quad B_1 = 0.2491361 \quad B_2 = 0.3124621 \quad B_3 = 0.2613448 \quad B_4 = 0.1282206 \)
APPENDIX A

GAUSSIAN QUADRATURE FORMULAS FOR CONSTANT WEIGHT FUNCTION

Here we give values of the $A_k^{(n)}$ and $x_k^{(n)}$ which make the approximation

$$\int_{-1}^{1} f(x) \, dx \approx \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)})$$

exact whenever $f(x)$ is a polynomial of degree $\leq 2n - 1$. These formulas are discussed in Section 7.2. The $A_k^{(n)}$ and $x_k^{(n)}$ are symmetric with respect to $x = 0$:

$$A_k^{(n)} = A_{n-k+1}^{(n)}, \quad x_k^{(n)} = -x_{n-k+1}^{(n)}$$

and the tables give only the values corresponding to $0 \leq x_k^{(n)} \leq 1$.

The tabulated values are taken from


where the $A_k^{(n)}$ and $x_k^{(n)}$ are given to 20 decimal places. Values for $n = 2, 4, 8, 16, 24, 32, 40, 48, 64, 80, 96$ of the same accuracy are given in the two tables


These three tables are the most extensive values of the $A_k^{(n)}$ and $x_k^{(n)}$ which are known and are believed to be accurate to within a few units in the last significant figures.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_k^{(n)}$</th>
<th>$x_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.00000</td>
<td>0.57735 02691 89625 76451</td>
</tr>
<tr>
<td>3</td>
<td>0.55555 55555 55555 55555 55555 55555 55555 55555 55555 55555</td>
<td>0.77459 66692 41483 37704</td>
</tr>
<tr>
<td></td>
<td>0.88888 88888 88888 88888 88888 88888 88888 88888 88888 88888</td>
<td>0.00000 00000 00000 00000</td>
</tr>
<tr>
<td>4</td>
<td>0.34785 48451 37453 85737</td>
<td>0.86113 63115 94052 57522</td>
</tr>
<tr>
<td></td>
<td>0.65214 51548 62546 14263</td>
<td>0.33998 10435 84856 26480</td>
</tr>
</tbody>
</table>
### APPENDIX A (Continued)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.90617 98459 38663 99280</td>
<td>0.23692 68850 56189 08751</td>
</tr>
<tr>
<td></td>
<td>0.53846 93101 05683 09104</td>
<td>0.47862 86704 99366 46804</td>
</tr>
<tr>
<td></td>
<td>0.00000 00000 00000 00000</td>
<td>0.56888 88888 88888 88889</td>
</tr>
<tr>
<td>6</td>
<td>0.93246 95142 03152 02781</td>
<td>0.17132 44923 79170 34504</td>
</tr>
<tr>
<td></td>
<td>0.66120 98864 66264 51366</td>
<td>0.36076 15730 48138 60757</td>
</tr>
<tr>
<td></td>
<td>0.23861 91860 83196 90863</td>
<td>0.46791 39345 72691 04739</td>
</tr>
<tr>
<td>7</td>
<td>0.94910 79123 42758 52453</td>
<td>0.12948 49661 68869 69327</td>
</tr>
<tr>
<td></td>
<td>0.74153 11855 99394 43986</td>
<td>0.27970 53914 89276 66790</td>
</tr>
<tr>
<td></td>
<td>0.40584 51513 77397 16691</td>
<td>0.38183 00505 05118 94495</td>
</tr>
<tr>
<td></td>
<td>0.00000 00000 00000 00000</td>
<td>0.41795 91836 73469 38776</td>
</tr>
<tr>
<td>8</td>
<td>0.96028 98564 97536 23168</td>
<td>0.10122 85362 90376 25915</td>
</tr>
<tr>
<td></td>
<td>0.79666 64774 13626 73959</td>
<td>0.22288 10344 53374 47054</td>
</tr>
<tr>
<td></td>
<td>0.52553 24099 16328 98582</td>
<td>0.31370 66458 77887 28734</td>
</tr>
<tr>
<td></td>
<td>0.18343 46424 95649 80494</td>
<td>0.36268 37383 78361 98297</td>
</tr>
<tr>
<td>9</td>
<td>0.96816 02395 07626 08984</td>
<td>0.08127 43883 61574 41197</td>
</tr>
<tr>
<td></td>
<td>0.83603 11073 26635 79430</td>
<td>0.18064 81606 94857 40406</td>
</tr>
<tr>
<td></td>
<td>0.61387 14327 00590 39731</td>
<td>0.26061 06964 02935 46232</td>
</tr>
<tr>
<td></td>
<td>0.32425 34234 03808 92904</td>
<td>0.31234 70770 40002 84007</td>
</tr>
<tr>
<td></td>
<td>0.00000 00000 00000 00000</td>
<td>0.33023 93550 01259 76316</td>
</tr>
<tr>
<td>10</td>
<td>0.97390 65285 17171 72008</td>
<td>0.06667 13443 08688 13759</td>
</tr>
<tr>
<td></td>
<td>0.86506 33666 88984 51073</td>
<td>0.14945 13491 50580 59315</td>
</tr>
<tr>
<td></td>
<td>0.67940 95682 99024 40623</td>
<td>0.21908 63625 15982 04400</td>
</tr>
<tr>
<td></td>
<td>0.43339 53941 29247 19080</td>
<td>0.26926 67193 09996 35509</td>
</tr>
<tr>
<td></td>
<td>0.14887 43389 81631 21089</td>
<td>0.29552 42247 14752 87017</td>
</tr>
<tr>
<td>11</td>
<td>0.97822 86581 46056 99280</td>
<td>0.05566 85671 16173 66648</td>
</tr>
<tr>
<td></td>
<td>0.88706 25997 68095 29908</td>
<td>0.12558 03694 64904 62464</td>
</tr>
<tr>
<td></td>
<td>0.73015 20055 74049 32409</td>
<td>0.18629 02109 27734 25143</td>
</tr>
<tr>
<td></td>
<td>0.51909 61292 06811 81593</td>
<td>0.23319 37645 91990 47992</td>
</tr>
<tr>
<td></td>
<td>0.26954 31559 52344 97238</td>
<td>0.26280 45445 10246 66218</td>
</tr>
<tr>
<td></td>
<td>0.00000 00000 00000 00000</td>
<td>0.27292 50867 77900 63071</td>
</tr>
<tr>
<td>12</td>
<td>0.98156 06342 46719 25069</td>
<td>0.04717 53363 86511 82719</td>
</tr>
<tr>
<td></td>
<td>0.90411 72563 70474 85668</td>
<td>0.10693 93259 95318 43096</td>
</tr>
<tr>
<td></td>
<td>0.76990 26741 94304 68704</td>
<td>0.16007 83285 43346 22633</td>
</tr>
<tr>
<td></td>
<td>0.58731 79542 86617 44730</td>
<td>0.20316 74267 23065 92175</td>
</tr>
<tr>
<td></td>
<td>0.36783 14989 98180 19375</td>
<td>0.23349 25365 38354 80876</td>
</tr>
<tr>
<td></td>
<td>0.12523 34085 11468 91547</td>
<td>0.24914 70458 13402 78500</td>
</tr>
</tbody>
</table>
## Gaussian-Legendre Quadrature Formulas

### APPENDIX A (Continued)

<table>
<thead>
<tr>
<th>$n = 13$</th>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.98418</td>
<td>30547</td>
<td>18588</td>
</tr>
<tr>
<td>0.91759</td>
<td>83992</td>
<td>22977</td>
</tr>
<tr>
<td>0.80157</td>
<td>80907</td>
<td>33309</td>
</tr>
<tr>
<td>0.64234</td>
<td>93394</td>
<td>40340</td>
</tr>
<tr>
<td>0.44849</td>
<td>27510</td>
<td>36446</td>
</tr>
<tr>
<td>0.23045</td>
<td>83159</td>
<td>55134</td>
</tr>
<tr>
<td>0.00000</td>
<td>00000</td>
<td>00000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n = 14$</th>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.98628</td>
<td>30868</td>
<td>96812</td>
</tr>
<tr>
<td>0.92843</td>
<td>48836</td>
<td>63573</td>
</tr>
<tr>
<td>0.82720</td>
<td>13150</td>
<td>69764</td>
</tr>
<tr>
<td>0.68729</td>
<td>29048</td>
<td>11685</td>
</tr>
<tr>
<td>0.51524</td>
<td>86363</td>
<td>58154</td>
</tr>
<tr>
<td>0.31911</td>
<td>23689</td>
<td>27889</td>
</tr>
<tr>
<td>0.10805</td>
<td>49487</td>
<td>07343</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n = 15$</th>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.98799</td>
<td>25180</td>
<td>20485</td>
</tr>
<tr>
<td>0.93727</td>
<td>33924</td>
<td>00705</td>
</tr>
<tr>
<td>0.84820</td>
<td>65834</td>
<td>10427</td>
</tr>
<tr>
<td>0.72441</td>
<td>77313</td>
<td>60170</td>
</tr>
<tr>
<td>0.57097</td>
<td>21726</td>
<td>08538</td>
</tr>
<tr>
<td>0.39415</td>
<td>13470</td>
<td>77563</td>
</tr>
<tr>
<td>0.20119</td>
<td>40939</td>
<td>97434</td>
</tr>
<tr>
<td>0.00000</td>
<td>00000</td>
<td>00000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n = 16$</th>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.98940</td>
<td>09349</td>
<td>91649</td>
</tr>
<tr>
<td>0.94457</td>
<td>50230</td>
<td>73232</td>
</tr>
<tr>
<td>0.86563</td>
<td>12023</td>
<td>87831</td>
</tr>
<tr>
<td>0.75540</td>
<td>44083</td>
<td>55003</td>
</tr>
<tr>
<td>0.61787</td>
<td>62444</td>
<td>02643</td>
</tr>
<tr>
<td>0.45801</td>
<td>67776</td>
<td>57227</td>
</tr>
<tr>
<td>0.28160</td>
<td>35507</td>
<td>79258</td>
</tr>
<tr>
<td>0.09501</td>
<td>25098</td>
<td>37637</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n = 20$</th>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99312</td>
<td>85991</td>
<td>85094</td>
</tr>
<tr>
<td>0.96397</td>
<td>19272</td>
<td>77913</td>
</tr>
<tr>
<td>0.91223</td>
<td>44282</td>
<td>51325</td>
</tr>
<tr>
<td>0.89111</td>
<td>69718</td>
<td>22218</td>
</tr>
<tr>
<td>0.74633</td>
<td>19064</td>
<td>60150</td>
</tr>
<tr>
<td>0.63605</td>
<td>36807</td>
<td>26515</td>
</tr>
<tr>
<td>0.51086</td>
<td>70019</td>
<td>50827</td>
</tr>
<tr>
<td>0.37370</td>
<td>60887</td>
<td>15419</td>
</tr>
<tr>
<td>0.22778</td>
<td>58511</td>
<td>41645</td>
</tr>
<tr>
<td>0.07652</td>
<td>65211</td>
<td>33497</td>
</tr>
</tbody>
</table>

### Notes
- $A_k^{(n)}$ represents the weights for each point $x_k^{(n)}$.
### APPENDIX A (Continued)

<table>
<thead>
<tr>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99518, 72199, 97021, 36018</td>
<td>0.01234, 12297, 99987, 19955</td>
</tr>
<tr>
<td>0.97472, 85559, 71309, 49820</td>
<td>0.02853, 13886, 28933, 66318</td>
</tr>
<tr>
<td>0.93827, 45520, 02732, 75852</td>
<td>0.04427, 74388, 17419, 80617</td>
</tr>
<tr>
<td>0.88641, 55270, 04401, 03421</td>
<td>0.05929, 85849, 15436, 78075</td>
</tr>
<tr>
<td>0.82000, 19859, 73902, 92195</td>
<td>0.07334, 64814, 11080, 30573</td>
</tr>
<tr>
<td>0.74012, 41915, 78554, 36424</td>
<td>0.08619, 01615, 31953, 27592</td>
</tr>
<tr>
<td>0.64809, 36519, 36975, 56925</td>
<td>0.09761, 86521, 04113, 88827</td>
</tr>
<tr>
<td>0.54542, 14713, 88839, 53566</td>
<td>0.10744, 42701, 15965, 63478</td>
</tr>
<tr>
<td>0.43379, 35076, 26045, 13849</td>
<td>0.11550, 56680, 53725, 60135</td>
</tr>
<tr>
<td>0.31504, 26796, 96163, 37439</td>
<td>0.12167, 04729, 27803, 39120</td>
</tr>
<tr>
<td>0.19111, 88674, 73616, 30916</td>
<td>0.12583, 74563, 46828, 29612</td>
</tr>
<tr>
<td>0.06405, 68928, 62605, 62609</td>
<td>0.12793, 81953, 46752, 15697</td>
</tr>
</tbody>
</table>

**n = 24**

<table>
<thead>
<tr>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99644, 24975, 73954, 44995</td>
<td>0.00912, 42825, 93094, 51774</td>
</tr>
<tr>
<td>0.98130, 31653, 70872, 75369</td>
<td>0.02113, 21125, 92771, 25975</td>
</tr>
<tr>
<td>0.95425, 92806, 28938, 19725</td>
<td>0.03290, 14277, 82304, 37998</td>
</tr>
<tr>
<td>0.91563, 30263, 92132, 07387</td>
<td>0.04427, 29347, 59004, 22784</td>
</tr>
<tr>
<td>0.86589, 25225, 74395, 04894</td>
<td>0.05510, 73456, 75716, 74543</td>
</tr>
<tr>
<td>0.80564, 13709, 17179, 17145</td>
<td>0.06527, 29239, 66999, 59579</td>
</tr>
<tr>
<td>0.73561, 08780, 13631, 72703</td>
<td>0.07464, 62142, 34568, 77902</td>
</tr>
<tr>
<td>0.65665, 10940, 38864, 96122</td>
<td>0.08311, 34172, 28901, 21839</td>
</tr>
<tr>
<td>0.56972, 04718, 11401, 71931</td>
<td>0.09057, 17443, 98302, 84094</td>
</tr>
<tr>
<td>0.47587, 42249, 55118, 26103</td>
<td>0.09693, 06579, 97929, 91585</td>
</tr>
<tr>
<td>0.37625, 15160, 89078, 71022</td>
<td>0.10211, 29675, 78060, 76981</td>
</tr>
<tr>
<td>0.27206, 16276, 35178, 07768</td>
<td>0.10605, 57659, 22846, 41791</td>
</tr>
<tr>
<td>0.16456, 98281, 33380, 77128</td>
<td>0.10871, 11922, 58294, 13525</td>
</tr>
<tr>
<td>0.05507, 92898, 84034, 27043</td>
<td>0.11004, 70130, 16475, 9628</td>
</tr>
</tbody>
</table>

**n = 28**

<table>
<thead>
<tr>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99726, 38618, 49481, 56354</td>
<td>0.00701, 86100, 09470, 09660</td>
</tr>
<tr>
<td>0.98561, 15115, 45268, 33540</td>
<td>0.01627, 43947, 30905, 67061</td>
</tr>
<tr>
<td>0.96476, 22555, 87506, 43077</td>
<td>0.02539, 20653, 09262, 05945</td>
</tr>
<tr>
<td>0.93490, 60759, 37739, 68917</td>
<td>0.03427, 38629, 13021, 43310</td>
</tr>
<tr>
<td>0.89632, 11557, 66052, 12397</td>
<td>0.04283, 58980, 22226, 68066</td>
</tr>
<tr>
<td>0.84936, 76137, 32569, 97013</td>
<td>0.05099, 80592, 62376, 17620</td>
</tr>
<tr>
<td>0.79448, 37399, 67942, 40696</td>
<td>0.05868, 40934, 78535, 54714</td>
</tr>
<tr>
<td>0.73218, 21187, 40289, 68039</td>
<td>0.06582, 22227, 76361, 84684</td>
</tr>
<tr>
<td>0.66304, 42669, 30215, 20098</td>
<td>0.07234, 57941, 08848, 50623</td>
</tr>
<tr>
<td>0.58771, 57572, 40762, 32904</td>
<td>0.07819, 38957, 87070, 30647</td>
</tr>
<tr>
<td>0.50689, 99089, 32229, 39002</td>
<td>0.08331, 19242, 26946, 75522</td>
</tr>
<tr>
<td>0.42135, 12761, 30635, 34536</td>
<td>0.08765, 20930, 04403, 81114</td>
</tr>
<tr>
<td>0.33186, 86022, 82127, 64978</td>
<td>0.09117, 38786, 95763, 88471</td>
</tr>
<tr>
<td>0.23928, 73622, 52137, 07454</td>
<td>0.09384, 43990, 80804, 55654</td>
</tr>
<tr>
<td>0.14447, 19615, 82796, 49349</td>
<td>0.09563, 87200, 79274, 85942</td>
</tr>
<tr>
<td>0.04830, 76656, 87738, 31623</td>
<td>0.09654, 00885, 14727, 80057</td>
</tr>
</tbody>
</table>
### Gaussian-Legendre Quadrature Formulas

#### APPENDIX A (Continued)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.99783</td>
<td>0.4624</td>
<td>0.84085</td>
</tr>
<tr>
<td>0.98858</td>
<td>0.64789</td>
<td>0.02212</td>
</tr>
<tr>
<td>0.97202</td>
<td>0.76910</td>
<td>0.94997</td>
</tr>
<tr>
<td>0.94827</td>
<td>0.29843</td>
<td>0.99507</td>
</tr>
<tr>
<td>0.91749</td>
<td>0.77745</td>
<td>0.15659</td>
</tr>
<tr>
<td>0.87992</td>
<td>0.98008</td>
<td>0.90397</td>
</tr>
<tr>
<td>0.83584</td>
<td>0.71669</td>
<td>0.92475</td>
</tr>
<tr>
<td>0.78557</td>
<td>0.62301</td>
<td>0.32206</td>
</tr>
<tr>
<td>0.72948</td>
<td>0.91715</td>
<td>0.93556</td>
</tr>
<tr>
<td>0.66800</td>
<td>1.2365</td>
<td>0.85621</td>
</tr>
<tr>
<td>0.60156</td>
<td>0.76581</td>
<td>0.35980</td>
</tr>
<tr>
<td>0.53068</td>
<td>0.02859</td>
<td>0.26245</td>
</tr>
<tr>
<td>0.45586</td>
<td>0.39444</td>
<td>0.3342</td>
</tr>
<tr>
<td>0.37767</td>
<td>0.25471</td>
<td>0.19689</td>
</tr>
<tr>
<td>0.29668</td>
<td>0.49953</td>
<td>0.44028</td>
</tr>
<tr>
<td>0.21350</td>
<td>0.08923</td>
<td>0.16865</td>
</tr>
<tr>
<td>0.12873</td>
<td>0.61038</td>
<td>0.9384</td>
</tr>
<tr>
<td>0.04301</td>
<td>0.81984</td>
<td>0.73708</td>
</tr>
<tr>
<td>40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.99823</td>
<td>0.77907</td>
<td>0.10559</td>
</tr>
<tr>
<td>0.99072</td>
<td>0.62836</td>
<td>0.99457</td>
</tr>
<tr>
<td>0.97725</td>
<td>0.99499</td>
<td>0.83774</td>
</tr>
<tr>
<td>0.95791</td>
<td>0.68192</td>
<td>0.13791</td>
</tr>
<tr>
<td>0.93281</td>
<td>0.28082</td>
<td>0.78676</td>
</tr>
<tr>
<td>0.90209</td>
<td>0.88069</td>
<td>0.68874</td>
</tr>
<tr>
<td>0.86595</td>
<td>0.95032</td>
<td>0.12259</td>
</tr>
<tr>
<td>0.82461</td>
<td>0.22308</td>
<td>0.33311</td>
</tr>
<tr>
<td>0.77830</td>
<td>0.56514</td>
<td>0.26519</td>
</tr>
<tr>
<td>0.72731</td>
<td>0.82551</td>
<td>0.89927</td>
</tr>
<tr>
<td>0.67195</td>
<td>0.66846</td>
<td>14179</td>
</tr>
<tr>
<td>0.61255</td>
<td>0.38896</td>
<td>0.67980</td>
</tr>
<tr>
<td>0.54946</td>
<td>0.71250</td>
<td>0.91528</td>
</tr>
<tr>
<td>0.48307</td>
<td>0.58016</td>
<td>0.86178</td>
</tr>
<tr>
<td>0.41377</td>
<td>0.92043</td>
<td>0.71605</td>
</tr>
<tr>
<td>0.34199</td>
<td>0.40908</td>
<td>25758</td>
</tr>
<tr>
<td>0.26815</td>
<td>0.21850</td>
<td>0.07253</td>
</tr>
<tr>
<td>0.19269</td>
<td>0.75007</td>
<td>01371</td>
</tr>
<tr>
<td>0.11608</td>
<td>0.40706</td>
<td>07525</td>
</tr>
<tr>
<td>0.03877</td>
<td>0.24175</td>
<td>0.06050</td>
</tr>
</tbody>
</table>
### APPENDIX A (Continued)

<table>
<thead>
<tr>
<th>( x_k^{(n)} )</th>
<th>( A_k^{(n)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99877</td>
<td>0.00315</td>
</tr>
<tr>
<td>0.99353</td>
<td>0.00732</td>
</tr>
<tr>
<td>0.98412</td>
<td>0.01147</td>
</tr>
<tr>
<td>0.97059</td>
<td>0.01557</td>
</tr>
<tr>
<td>0.95298</td>
<td>0.01961</td>
</tr>
<tr>
<td>0.93138</td>
<td>0.02357</td>
</tr>
<tr>
<td>0.90587</td>
<td>0.02742</td>
</tr>
<tr>
<td>0.87657</td>
<td>0.03116</td>
</tr>
<tr>
<td>0.84358</td>
<td>0.03477</td>
</tr>
<tr>
<td>0.80706</td>
<td>0.03824</td>
</tr>
<tr>
<td>0.76715</td>
<td>0.04154</td>
</tr>
<tr>
<td>0.72403</td>
<td>0.04467</td>
</tr>
<tr>
<td>0.67787</td>
<td>0.04761</td>
</tr>
<tr>
<td>0.62886</td>
<td>0.05035</td>
</tr>
<tr>
<td>0.57722</td>
<td>0.05289</td>
</tr>
<tr>
<td>0.52316</td>
<td>0.05519</td>
</tr>
<tr>
<td>0.46690</td>
<td>0.05727</td>
</tr>
<tr>
<td>0.40868</td>
<td>0.05911</td>
</tr>
<tr>
<td>0.34875</td>
<td>0.06070</td>
</tr>
<tr>
<td>0.28736</td>
<td>0.06203</td>
</tr>
<tr>
<td>0.22476</td>
<td>0.06311</td>
</tr>
<tr>
<td>0.16122</td>
<td>0.06392</td>
</tr>
<tr>
<td>0.09700</td>
<td>0.06446</td>
</tr>
<tr>
<td>0.03238</td>
<td>0.06473</td>
</tr>
</tbody>
</table>
Here we give values of the $A_k^{(n)}$ and $x_k^{(n)}$ which make the approximation

$$\int_{-\infty}^{\infty} e^{-x^2} f(x) \, dx = \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)})$$

exact whenever $f(x)$ is a polynomial of degree $\leq 2n - 1$. These formulas are discussed in Section 7.4. The $A_k^{(n)}$ and $x_k^{(n)}$ are symmetric with respect to $x = 0$ and the tables give only the values corresponding to $0 \leq x_k^{(n)}$.

We give here values for $n = 1(1)20$ given by


These are the most extensive values of these quadrature formulas which are known. A number in parenthesis before a value of a coefficient is the power of 10 by which the tabulated value must be multiplied; for example, $(-1)0.8131\ldots$ means that the coefficient is $0.08131\ldots$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00000</td>
<td>1.77245</td>
</tr>
<tr>
<td></td>
<td>0.00000</td>
<td>38509</td>
</tr>
<tr>
<td></td>
<td>0.00000</td>
<td>055</td>
</tr>
<tr>
<td>2</td>
<td>0.70710</td>
<td>0.88622</td>
</tr>
<tr>
<td></td>
<td>67811</td>
<td>69254</td>
</tr>
<tr>
<td></td>
<td>86548</td>
<td>528</td>
</tr>
<tr>
<td>3</td>
<td>0.00000</td>
<td>1.18163</td>
</tr>
<tr>
<td></td>
<td>0.00000</td>
<td>59006</td>
</tr>
<tr>
<td></td>
<td>0.00000</td>
<td>087</td>
</tr>
<tr>
<td></td>
<td>1.22474</td>
<td>89751</td>
</tr>
<tr>
<td></td>
<td>48713</td>
<td>509</td>
</tr>
<tr>
<td>4</td>
<td>0.52464</td>
<td>0.80491</td>
</tr>
<tr>
<td></td>
<td>76232</td>
<td>40900</td>
</tr>
<tr>
<td></td>
<td>75290</td>
<td>055</td>
</tr>
<tr>
<td></td>
<td>1.65068</td>
<td>(-1)0.81312</td>
</tr>
<tr>
<td></td>
<td>01238</td>
<td>83544</td>
</tr>
<tr>
<td></td>
<td>85785</td>
<td>725</td>
</tr>
<tr>
<td>5</td>
<td>0.00000</td>
<td>0.94530</td>
</tr>
<tr>
<td></td>
<td>0.00000</td>
<td>87204</td>
</tr>
<tr>
<td></td>
<td>0.95857</td>
<td>829</td>
</tr>
<tr>
<td></td>
<td>24646</td>
<td>522</td>
</tr>
<tr>
<td></td>
<td>13819</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.20208</td>
<td>(-1)0.19953</td>
</tr>
<tr>
<td></td>
<td>28704</td>
<td>24205</td>
</tr>
<tr>
<td></td>
<td>56086</td>
<td>905</td>
</tr>
<tr>
<td>6</td>
<td>0.43607</td>
<td>0.72462</td>
</tr>
<tr>
<td></td>
<td>74119</td>
<td>244</td>
</tr>
<tr>
<td></td>
<td>27617</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.33584</td>
<td>229</td>
</tr>
<tr>
<td></td>
<td>90740</td>
<td></td>
</tr>
<tr>
<td></td>
<td>13697</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.35060</td>
<td>(-2)0.45300</td>
</tr>
<tr>
<td></td>
<td>49736</td>
<td>509</td>
</tr>
<tr>
<td></td>
<td>74492</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.00000</td>
<td>0.81026</td>
</tr>
<tr>
<td></td>
<td>0.00000</td>
<td>46175</td>
</tr>
<tr>
<td></td>
<td>0.81628</td>
<td>568</td>
</tr>
<tr>
<td></td>
<td>78828</td>
<td></td>
</tr>
<tr>
<td></td>
<td>58965</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.67355</td>
<td>(-1)0.54515</td>
</tr>
<tr>
<td></td>
<td>16287</td>
<td>913</td>
</tr>
<tr>
<td></td>
<td>67471</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.65196</td>
<td>(-3)0.97178</td>
</tr>
<tr>
<td></td>
<td>13568</td>
<td>995</td>
</tr>
<tr>
<td></td>
<td>35233</td>
<td></td>
</tr>
</tbody>
</table>
**APPENDIX B (Continued)**

<table>
<thead>
<tr>
<th>(x_k^{(n)})</th>
<th>(A_k^{(n)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 8</td>
<td></td>
</tr>
<tr>
<td>0.38118</td>
<td>0.66114</td>
</tr>
<tr>
<td>1.15719</td>
<td>0.20780</td>
</tr>
<tr>
<td>1.98165</td>
<td>(-1)0.17077</td>
</tr>
<tr>
<td>2.93063</td>
<td>(-3)0.19960</td>
</tr>
<tr>
<td>0.00000</td>
<td>0.72023</td>
</tr>
<tr>
<td>0.72355</td>
<td>0.43265</td>
</tr>
<tr>
<td>1.46855</td>
<td>(-1)0.88474</td>
</tr>
<tr>
<td>2.26658</td>
<td>(-2)0.49436</td>
</tr>
<tr>
<td>3.19099</td>
<td>(-4)0.39606</td>
</tr>
<tr>
<td>n = 9</td>
<td></td>
</tr>
<tr>
<td>0.00000</td>
<td>0.65475</td>
</tr>
<tr>
<td>0.65680</td>
<td>0.42985</td>
</tr>
<tr>
<td>1.32655</td>
<td>0.11722</td>
</tr>
<tr>
<td>2.02594</td>
<td>(-1)0.11911</td>
</tr>
<tr>
<td>2.78329</td>
<td>(-3)0.34681</td>
</tr>
<tr>
<td>3.43615</td>
<td>(-5)0.76404</td>
</tr>
<tr>
<td>n = 10</td>
<td></td>
</tr>
<tr>
<td>0.34290</td>
<td>0.61086</td>
</tr>
<tr>
<td>1.03661</td>
<td>0.24013</td>
</tr>
<tr>
<td>1.75668</td>
<td>(-1)0.33874</td>
</tr>
<tr>
<td>2.53273</td>
<td>(-2)0.13436</td>
</tr>
<tr>
<td>3.43615</td>
<td>(-5)0.76404</td>
</tr>
<tr>
<td>n = 11</td>
<td></td>
</tr>
<tr>
<td>0.00000</td>
<td>0.65475</td>
</tr>
<tr>
<td>0.65680</td>
<td>0.42985</td>
</tr>
<tr>
<td>1.32655</td>
<td>0.11722</td>
</tr>
<tr>
<td>2.02594</td>
<td>(-1)0.11911</td>
</tr>
<tr>
<td>2.78329</td>
<td>(-3)0.34681</td>
</tr>
<tr>
<td>3.43615</td>
<td>(-5)0.76404</td>
</tr>
<tr>
<td>n = 12</td>
<td></td>
</tr>
<tr>
<td>0.31424</td>
<td>0.57013</td>
</tr>
<tr>
<td>0.94778</td>
<td>0.26049</td>
</tr>
<tr>
<td>1.59768</td>
<td>(-1)0.51607</td>
</tr>
<tr>
<td>2.27950</td>
<td>(-2)0.39053</td>
</tr>
<tr>
<td>3.02063</td>
<td>(-4)0.85736</td>
</tr>
<tr>
<td>3.88972</td>
<td>(-6)0.26585</td>
</tr>
<tr>
<td>n = 13</td>
<td></td>
</tr>
<tr>
<td>0.00000</td>
<td>0.60439</td>
</tr>
<tr>
<td>0.60576</td>
<td>0.42161</td>
</tr>
<tr>
<td>1.22005</td>
<td>0.14032</td>
</tr>
<tr>
<td>1.85810</td>
<td>(-1)0.20862</td>
</tr>
<tr>
<td>2.51973</td>
<td>(-2)0.12074</td>
</tr>
<tr>
<td>3.24660</td>
<td>(-4)0.20430</td>
</tr>
<tr>
<td>4.10133</td>
<td>(-7)0.48257</td>
</tr>
<tr>
<td>n = 14</td>
<td></td>
</tr>
<tr>
<td>0.29174</td>
<td>0.53640</td>
</tr>
<tr>
<td>0.87871</td>
<td>0.27310</td>
</tr>
<tr>
<td>1.47668</td>
<td>(-1)0.68505</td>
</tr>
<tr>
<td>2.09518</td>
<td>(-2)0.78500</td>
</tr>
<tr>
<td>2.74847</td>
<td>(-3)0.35509</td>
</tr>
<tr>
<td>3.46265</td>
<td>(-5)0.47164</td>
</tr>
<tr>
<td>4.30444</td>
<td>(-8)0.86285</td>
</tr>
</tbody>
</table>
### APPENDIX B (Continued)

<table>
<thead>
<tr>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 15$</td>
<td></td>
</tr>
<tr>
<td>0.00000</td>
<td>0.56410 03087 264</td>
</tr>
<tr>
<td>0.55506</td>
<td>0.41202 86874 989</td>
</tr>
<tr>
<td>1.13611</td>
<td>0.15848 89157 959</td>
</tr>
<tr>
<td>1.71999</td>
<td>(-1)0.30780 0387 255</td>
</tr>
<tr>
<td>2.32573</td>
<td>(-2)0.27780 68842 913</td>
</tr>
<tr>
<td>2.96716</td>
<td>(-3)0.10000 44412 325</td>
</tr>
<tr>
<td>3.66995</td>
<td>(-5)0.10591 15547 711</td>
</tr>
<tr>
<td>4.49999</td>
<td>(-8)0.15224 75804 254</td>
</tr>
<tr>
<td>$n = 16$</td>
<td></td>
</tr>
<tr>
<td>0.27348</td>
<td>0.50792 94790 166</td>
</tr>
<tr>
<td>0.82295</td>
<td>0.28064 74585 285</td>
</tr>
<tr>
<td>1.38025</td>
<td>(-1)0.83810 04139 899</td>
</tr>
<tr>
<td>1.95178</td>
<td>(-1)0.12880 31153 551</td>
</tr>
<tr>
<td>2.54620</td>
<td>(-3)0.93228 40086 242</td>
</tr>
<tr>
<td>3.17699</td>
<td>(-4)0.27118 60092 538</td>
</tr>
<tr>
<td>3.86944</td>
<td>(-6)0.23209 80844 865</td>
</tr>
<tr>
<td>4.68873</td>
<td>(-9)0.26548 07474 011</td>
</tr>
<tr>
<td>$n = 17$</td>
<td></td>
</tr>
<tr>
<td>0.00000</td>
<td>0.53091 79376 249</td>
</tr>
<tr>
<td>0.53163</td>
<td>0.40182 64694 704</td>
</tr>
<tr>
<td>1.06764</td>
<td>0.17264 82976 701</td>
</tr>
<tr>
<td>1.61292</td>
<td>(-1)0.40920 03414 976</td>
</tr>
<tr>
<td>2.17350</td>
<td>(-2)0.50673 49957 628</td>
</tr>
<tr>
<td>2.75776</td>
<td>(-3)0.29864 32866 978</td>
</tr>
<tr>
<td>3.37893</td>
<td>(-5)0.71122 89140 021</td>
</tr>
<tr>
<td>4.06194</td>
<td>(-7)0.49770 78981 631</td>
</tr>
<tr>
<td>4.87134</td>
<td>(-10)0.45805 78930 799</td>
</tr>
<tr>
<td>$n = 18$</td>
<td></td>
</tr>
<tr>
<td>0.25826</td>
<td>0.48349 56947 255</td>
</tr>
<tr>
<td>0.77668</td>
<td>0.28480 72856 700</td>
</tr>
<tr>
<td>1.30092</td>
<td>(-1)0.97301 74764 132</td>
</tr>
<tr>
<td>1.83553</td>
<td>(-1)0.18640 40238 754</td>
</tr>
<tr>
<td>2.38629</td>
<td>(-2)0.18885 22630 268</td>
</tr>
<tr>
<td>2.96137</td>
<td>(-4)0.91811 26867 929</td>
</tr>
<tr>
<td>3.57376</td>
<td>(-5)0.18106 54481 093</td>
</tr>
<tr>
<td>4.24811</td>
<td>(-7)0.10467 20579 579</td>
</tr>
<tr>
<td>5.04836</td>
<td>(-11)0.78281 99772 116</td>
</tr>
<tr>
<td>$n = 19$</td>
<td></td>
</tr>
<tr>
<td>0.00000</td>
<td>0.50297 48882 762</td>
</tr>
<tr>
<td>0.50352</td>
<td>0.39160 89886 130</td>
</tr>
<tr>
<td>1.01036</td>
<td>0.18363 27013 070</td>
</tr>
<tr>
<td>1.52417</td>
<td>(-1)0.50810 38690 905</td>
</tr>
<tr>
<td>2.04923</td>
<td>(-2)0.79888 66777 723</td>
</tr>
<tr>
<td>2.59113</td>
<td>(-3)0.67087 75214 072</td>
</tr>
<tr>
<td>3.15784</td>
<td>(-4)0.27209 19776 316</td>
</tr>
<tr>
<td>3.76218</td>
<td>(-6)0.44882 43147 223</td>
</tr>
<tr>
<td>4.42853</td>
<td>(-8)0.21630 51009 864</td>
</tr>
<tr>
<td>5.22027</td>
<td>(-11)0.13262 97094 499</td>
</tr>
<tr>
<td>$x_k^{(n)}$</td>
<td>$A_k^{(n)}$</td>
</tr>
<tr>
<td>------------</td>
<td>------------</td>
</tr>
<tr>
<td>0.24534 07083 009</td>
<td>0.46224 36696 006</td>
</tr>
<tr>
<td>0.73747 37285 454</td>
<td>0.28667 55053 628</td>
</tr>
<tr>
<td>1.23407 62153 953</td>
<td>0.10901 72060 200</td>
</tr>
<tr>
<td>1.73853 77121 166</td>
<td>(-1)0.24810 52088 746</td>
</tr>
<tr>
<td>2.25497 40020 893</td>
<td>(-2)0.32437 73342 238</td>
</tr>
<tr>
<td>2.78880 60584 281</td>
<td>(-3)0.22833 86360 163</td>
</tr>
<tr>
<td>3.34785 45673 832</td>
<td>(-5)0.78025 56478 532</td>
</tr>
<tr>
<td>3.94476 40401 156</td>
<td>(-6)0.10860 69370 769</td>
</tr>
<tr>
<td>4.60368 24495 507</td>
<td>(-9)0.43993 40992 273</td>
</tr>
<tr>
<td>5.38748 08900 112</td>
<td>(-12)0.22293 93645 534</td>
</tr>
</tbody>
</table>
APPENDIX C

GAUSSIAN-LAGUERRE QUADRATURE FORMULAS

Here we give values of the $A_k^{(n)}$ and $x_k^{(n)}$ which make the approximation

$$\int_0^{\infty} x^a e^{-x} f(x) \, dx = \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)})$$

exact whenever $f(x)$ is a polynomial of degree $\leq 2n-1$. These formulas are discussed in Section 7.5.

We give the values for $\alpha = 0$, $n = 4 \ (4) \ 32$ tabulated by


and also the values for $\alpha = 0$, $n = 1 \ (1) \ 15$ tabulated by


except for the three cases $n = 4, 8, 12$ where we give the more accurate values given by Rabinowitz and Weiss. Rabinowitz and Weiss also give values of the $A_k^{(n)}$ and $x_k^{(n)}$ for $\alpha = 1 \ (1) \ 5$, $n = 4 \ (4) \ 16$ which we have not included here.

$$
x_k^{(n)} \quad A_k^{(n)}
\begin{array}{ccc}
\hline
n & 1 & 2 \\
\hline
1 & 1.00000 & 0.00000 \\
2 & 0.58578 & 0.85355 \\
3 & 0.41577 & 0.71109 \\
4 & 0.32254 & 0.75942 \\
5 & 0.26356 & 0.60315 \\
\hline
\end{array}
\begin{array}{ccc}
\hline
n & 1 & 2 \\
\hline
1 & 1.00000 & 0.00000 \\
2 & 0.58578 & 0.33905 \\
3 & 0.41577 & 0.71109 \\
4 & 0.32254 & 0.75942 \\
5 & 0.26356 & 0.60315 \\
\hline
\end{array}
\begin{array}{ccc}
\hline
n & 1 & 2 \\
\hline
1 & 0.85355 & 0.14644 \\
2 & 0.71109 & 0.27851 \\
3 & 0.60315 & 0.10389 \\
4 & 0.58355 & 0.33905 \\
5 & 0.52175 & 0.56105 \\
\hline
\end{array}
\begin{array}{ccc}
\hline
n & 1 & 2 \\
\hline
1 & 0.14644 & 0.66094 \\
2 & 0.71109 & 0.77335 \\
3 & 0.60315 & 0.25650 \\
4 & 0.58355 & 0.27851 \\
5 & 0.52175 & 0.56105 \\
\hline
\end{array}
\begin{array}{ccc}
\hline
n & 1 & 2 \\
\hline
1 & 0.33905 & 0.10389 \\
2 & 0.71109 & 0.27851 \\
3 & 0.60315 & 0.10389 \\
4 & 0.58355 & 0.33905 \\
5 & 0.52175 & 0.56105 \\
\hline
\end{array}
\begin{array}{ccc}
\hline
n & 1 & 2 \\
\hline
1 & 0.14644 & 0.66094 \\
2 & 0.71109 & 0.77335 \\
3 & 0.60315 & 0.25650 \\
4 & 0.58355 & 0.27851 \\
5 & 0.52175 & 0.56105 \\
\hline
\end{array}
\begin{array}{ccc}
\hline
n & 1 & 2 \\
\hline
1 & 0.33905 & 0.10389 \\
2 & 0.71109 & 0.27851 \\
3 & 0.60315 & 0.10389 \\
4 & 0.58355 & 0.33905 \\
5 & 0.52175 & 0.56105 \\
\hline
\end{array}$
## APPENDIX C (Continued)

<table>
<thead>
<tr>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 6$</td>
<td></td>
</tr>
<tr>
<td>0.22284 66041 79</td>
<td>0.45896 46739 50</td>
</tr>
<tr>
<td>1.18893 21016 73</td>
<td>0.41700 08307 72</td>
</tr>
<tr>
<td>2.99273 63260 59</td>
<td>0.11337 33820 74</td>
</tr>
<tr>
<td>5.77514 35691 05</td>
<td>$(-1)$ 0.10399 19745 31</td>
</tr>
<tr>
<td>9.83746 74183 83</td>
<td>$(-3)$ 0.26101 72028 15</td>
</tr>
<tr>
<td>15.98287 39806 02</td>
<td>$(-6)$ 0.89854 79064 30</td>
</tr>
<tr>
<td>$n = 7$</td>
<td></td>
</tr>
<tr>
<td>0.19304 36765 60</td>
<td>0.40931 89517 01</td>
</tr>
<tr>
<td>1.02666 48953 39</td>
<td>0.42183 12778 62</td>
</tr>
<tr>
<td>2.55787 67449 51</td>
<td>0.14712 63486 58</td>
</tr>
<tr>
<td>4.90035 30845 26</td>
<td>$(-1)$ 0.20633 51466 87</td>
</tr>
<tr>
<td>8.18215 34445 63</td>
<td>$(-2)$ 0.10740 10143 28</td>
</tr>
<tr>
<td>12.73418 02917 98</td>
<td>$(-4)$ 0.15865 46434 86</td>
</tr>
<tr>
<td>19.39572 78622 63</td>
<td>$(-7)$ 0.31703 15479 00</td>
</tr>
<tr>
<td>$n = 8$</td>
<td></td>
</tr>
<tr>
<td>0.17027 96323 05101 000</td>
<td>0.36918 85893 41635 530</td>
</tr>
<tr>
<td>0.90370 17767 99379 912</td>
<td>0.41878 67808 14342 956</td>
</tr>
<tr>
<td>2.25108 66298 66130 69</td>
<td>0.17579 49866 87171 806</td>
</tr>
<tr>
<td>4.26670 01702 87658 79</td>
<td>$(-1)$ 0.33343 49226 12156 515</td>
</tr>
<tr>
<td>7.04590 54023 93465 70</td>
<td>$(-2)$ 0.27945 36235 22567 252</td>
</tr>
<tr>
<td>10.75851 60101 80995 2</td>
<td>$(-4)$ 0.90765 08773 35821 310</td>
</tr>
<tr>
<td>15.74067 86412 78004 6</td>
<td>$(-6)$ 0.84857 46716 27253 154</td>
</tr>
<tr>
<td>22.86313 17368 89264 1</td>
<td>$(-8)$ 0.10480 01174 87151 038</td>
</tr>
<tr>
<td>$n = 9$</td>
<td></td>
</tr>
<tr>
<td>0.15232 22277 32</td>
<td>0.33612 64217 98</td>
</tr>
<tr>
<td>0.80722 00227 42</td>
<td>0.41121 39804 24</td>
</tr>
<tr>
<td>2.00513 51556 19</td>
<td>0.19928 75253 71</td>
</tr>
<tr>
<td>3.78347 39733 31</td>
<td>$(-1)$ 0.47460 56276 57</td>
</tr>
<tr>
<td>6.20495 67778 77</td>
<td>$(-2)$ 0.55996 26610 79</td>
</tr>
<tr>
<td>9.37298 52516 88</td>
<td>$(-3)$ 0.30524 97670 93</td>
</tr>
<tr>
<td>13.46623 69110 92</td>
<td>$(-5)$ 0.65921 23026 08</td>
</tr>
<tr>
<td>18.83359 77889 92</td>
<td>$(-7)$ 0.41107 69380 35</td>
</tr>
<tr>
<td>26.37407 18909 27</td>
<td>$(-10)$ 0.32908 74030 35</td>
</tr>
<tr>
<td>$n = 10$</td>
<td></td>
</tr>
<tr>
<td>0.13779 34705 40</td>
<td>0.30844 11157 65</td>
</tr>
<tr>
<td>0.72945 45495 03</td>
<td>0.40111 99291 55</td>
</tr>
<tr>
<td>1.80834 29017 40</td>
<td>0.21806 82876 12</td>
</tr>
<tr>
<td>3.40143 36978 55</td>
<td>$(-1)$ 0.62087 45609 87</td>
</tr>
<tr>
<td>5.55249 61400 64</td>
<td>$(-2)$ 0.95015 16975 18</td>
</tr>
<tr>
<td>8.33015 27467 64</td>
<td>$(-3)$ 0.75300 83885 88</td>
</tr>
<tr>
<td>11.84378 58379 00</td>
<td>$(-4)$ 0.28259 23349 60</td>
</tr>
<tr>
<td>16.27925 78313 78</td>
<td>$(-6)$ 0.42493 13984 96</td>
</tr>
<tr>
<td>21.99658 58119 81</td>
<td>$(-8)$ 0.18395 64823 98</td>
</tr>
<tr>
<td>29.92069 70122 74</td>
<td>$(-12)$ 0.99118 27219 61</td>
</tr>
</tbody>
</table>
### Gaussian-Laguerre Quadrature Formulas

**APPENDIX C (Continued)**

<table>
<thead>
<tr>
<th>(x_k^{(n)})</th>
<th>(A_k^{(n)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n = 11)</td>
<td></td>
</tr>
<tr>
<td>0.12579</td>
<td>0.28493</td>
</tr>
<tr>
<td>0.66541</td>
<td>0.38972</td>
</tr>
<tr>
<td>1.64715</td>
<td>0.23278</td>
</tr>
<tr>
<td>3.09113</td>
<td>(-1) 0.76564</td>
</tr>
<tr>
<td>5.02928</td>
<td>(-1) 0.14393</td>
</tr>
<tr>
<td>7.50988</td>
<td>(-2) 0.15188</td>
</tr>
<tr>
<td>10.60595</td>
<td>(-4) 0.85131</td>
</tr>
<tr>
<td>14.43161</td>
<td>(-5) 0.22924</td>
</tr>
<tr>
<td>19.17885</td>
<td>(-7) 0.24863</td>
</tr>
<tr>
<td>25.21770</td>
<td>(-10) 0.77126</td>
</tr>
<tr>
<td>33.49719</td>
<td>(-13) 0.28837</td>
</tr>
<tr>
<td>(n = 12)</td>
<td></td>
</tr>
<tr>
<td>0.11572</td>
<td>0.26473</td>
</tr>
<tr>
<td>0.61175</td>
<td>0.37775</td>
</tr>
<tr>
<td>1.51261</td>
<td>0.24408</td>
</tr>
<tr>
<td>2.83375</td>
<td>(-1) 0.90449</td>
</tr>
<tr>
<td>4.59922</td>
<td>(-1) 0.20102</td>
</tr>
<tr>
<td>6.84452</td>
<td>(-2) 0.26639</td>
</tr>
<tr>
<td>9.62135</td>
<td>(-3) 0.20323</td>
</tr>
<tr>
<td>13.00605</td>
<td>(-5) 0.83650</td>
</tr>
<tr>
<td>17.11685</td>
<td>(-6) 0.16684</td>
</tr>
<tr>
<td>22.15109</td>
<td>(-8) 0.13423</td>
</tr>
<tr>
<td>28.48796</td>
<td>(-11) 0.30616</td>
</tr>
<tr>
<td>37.09912</td>
<td>(-15) 0.81480</td>
</tr>
<tr>
<td>(n = 13)</td>
<td></td>
</tr>
<tr>
<td>0.10714</td>
<td>0.24718</td>
</tr>
<tr>
<td>0.56613</td>
<td>0.36568</td>
</tr>
<tr>
<td>1.39856</td>
<td>0.25256</td>
</tr>
<tr>
<td>2.61659</td>
<td>0.10347</td>
</tr>
<tr>
<td>4.23884</td>
<td>(-1) 0.26432</td>
</tr>
<tr>
<td>6.29225</td>
<td>(-2) 0.42203</td>
</tr>
<tr>
<td>8.81500</td>
<td>(-3) 0.41188</td>
</tr>
<tr>
<td>11.86140</td>
<td>(-4) 0.23515</td>
</tr>
<tr>
<td>15.51076</td>
<td>(-6) 0.73173</td>
</tr>
<tr>
<td>19.88463</td>
<td>(-7) 0.11088</td>
</tr>
<tr>
<td>25.18526</td>
<td>(-10) 0.67708</td>
</tr>
<tr>
<td>31.80038</td>
<td>(-12) 0.11599</td>
</tr>
<tr>
<td>40.72300</td>
<td>(-16) 0.22450</td>
</tr>
<tr>
<td>(n = 14)</td>
<td></td>
</tr>
<tr>
<td>0.09974</td>
<td>0.23181</td>
</tr>
<tr>
<td>0.52685</td>
<td>0.35378</td>
</tr>
<tr>
<td>1.30062</td>
<td>0.25873</td>
</tr>
<tr>
<td>2.43080</td>
<td>0.11548</td>
</tr>
<tr>
<td>3.92210</td>
<td>(-1) 0.33192</td>
</tr>
<tr>
<td>5.82553</td>
<td>(-2) 0.61928</td>
</tr>
<tr>
<td>8.14024</td>
<td>(-3) 0.73989</td>
</tr>
<tr>
<td>10.91649</td>
<td>(-4) 0.54907</td>
</tr>
</tbody>
</table>

*(contd.*)*
APPENDIX C (Continued)

<table>
<thead>
<tr>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 14</td>
<td></td>
</tr>
<tr>
<td>14.21080</td>
<td>50111.61</td>
</tr>
<tr>
<td>18.10489</td>
<td>22202.18</td>
</tr>
<tr>
<td>22.72388</td>
<td>16282.69</td>
</tr>
<tr>
<td>28.27298</td>
<td>17232.48</td>
</tr>
<tr>
<td>35.14944</td>
<td>36605.92</td>
</tr>
<tr>
<td>44.36608</td>
<td>17111.17</td>
</tr>
<tr>
<td>n = 15</td>
<td></td>
</tr>
<tr>
<td>0.09330</td>
<td>78120.17</td>
</tr>
<tr>
<td>0.49269</td>
<td>17403.02</td>
</tr>
<tr>
<td>1.21559</td>
<td>54120.71</td>
</tr>
<tr>
<td>2.26994</td>
<td>95262.04</td>
</tr>
<tr>
<td>3.66762</td>
<td>27217.51</td>
</tr>
<tr>
<td>5.42538</td>
<td>66274.14</td>
</tr>
<tr>
<td>7.56591</td>
<td>62266.13</td>
</tr>
<tr>
<td>10.12022</td>
<td>56580.19</td>
</tr>
<tr>
<td>13.13028</td>
<td>24871.76</td>
</tr>
<tr>
<td>16.65440</td>
<td>77083.30</td>
</tr>
<tr>
<td>20.77647</td>
<td>88994.49</td>
</tr>
<tr>
<td>25.62389</td>
<td>42267.29</td>
</tr>
<tr>
<td>31.40751</td>
<td>91697.54</td>
</tr>
<tr>
<td>38.53068</td>
<td>33064.86</td>
</tr>
<tr>
<td>48.02608</td>
<td>55726.86</td>
</tr>
<tr>
<td>n = 16</td>
<td></td>
</tr>
<tr>
<td>0.08764</td>
<td>94104.7892</td>
</tr>
<tr>
<td>0.46269</td>
<td>63289.1508</td>
</tr>
<tr>
<td>1.14105</td>
<td>77748.3122</td>
</tr>
<tr>
<td>2.12928</td>
<td>36450.9838</td>
</tr>
<tr>
<td>3.43708</td>
<td>66338.9320</td>
</tr>
<tr>
<td>5.07801</td>
<td>86145.4976</td>
</tr>
<tr>
<td>7.07033</td>
<td>85350.4823</td>
</tr>
<tr>
<td>9.43831</td>
<td>43363.9193</td>
</tr>
<tr>
<td>12.21422</td>
<td>33688.6615</td>
</tr>
<tr>
<td>15.44152</td>
<td>73687.8161</td>
</tr>
<tr>
<td>19.18015</td>
<td>68567.5313</td>
</tr>
<tr>
<td>23.51590</td>
<td>56939.9190</td>
</tr>
<tr>
<td>28.57872</td>
<td>97428.8214</td>
</tr>
<tr>
<td>34.58339</td>
<td>87022.8662</td>
</tr>
<tr>
<td>41.94045</td>
<td>26476.8832</td>
</tr>
<tr>
<td>51.70116</td>
<td>03395.4331</td>
</tr>
<tr>
<td>n = 20</td>
<td></td>
</tr>
<tr>
<td>0.07053</td>
<td>98996.9198</td>
</tr>
<tr>
<td>0.37212</td>
<td>68180.0161</td>
</tr>
<tr>
<td>0.91658</td>
<td>21024.8327</td>
</tr>
<tr>
<td>1.70730</td>
<td>65310.2834</td>
</tr>
<tr>
<td>2.74919</td>
<td>92553.0943</td>
</tr>
<tr>
<td>4.04892</td>
<td>53138.5089</td>
</tr>
<tr>
<td>5.61517</td>
<td>49708.6161</td>
</tr>
<tr>
<td>7.45901</td>
<td>74536.7106</td>
</tr>
<tr>
<td>9.59439</td>
<td>28695.8109</td>
</tr>
</tbody>
</table>

(contd.)
### Gaussian-Laguerre Quadrature Formulas

**APPENDIX C (Continued)**

<table>
<thead>
<tr>
<th>$x_k^{(n)}$</th>
<th>$A_k^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.03880 25469 64316 3</td>
<td>(-4) 0.15401 44086 52249 157</td>
</tr>
<tr>
<td>14.81429 34426 30740 0</td>
<td>(-5) 0.10864 86366 51798 235</td>
</tr>
<tr>
<td>17.94889 55205 19376 0</td>
<td>(-7) 0.53301 20909 55671 475</td>
</tr>
<tr>
<td>21.47878 82402 85011 0</td>
<td>(-8) 0.17579 81179 05058 200</td>
</tr>
<tr>
<td>25.45170 27931 86905 5</td>
<td>(-10) 0.37255 02402 51223 087</td>
</tr>
<tr>
<td>29.93255 46817 00612 0</td>
<td>(-12) 0.47675 29251 57819 052</td>
</tr>
<tr>
<td>35.01343 42404 79000 0</td>
<td>(-14) 0.33728 44243 36243 841</td>
</tr>
<tr>
<td>40.83305 70567 28571 1</td>
<td>(-16) 0.11550 14339 50039 883</td>
</tr>
<tr>
<td>47.61999 40473 46502 1</td>
<td>(-19) 0.15395 22140 58234 355</td>
</tr>
<tr>
<td>55.81079 57500 68898 9</td>
<td>(-23) 0.52864 42725 56915 783</td>
</tr>
<tr>
<td>66.52441 65256 15753 8</td>
<td>(-27) 0.16564 56612 49902 330</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n = 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.05901 98521 81507 9770$</td>
</tr>
<tr>
<td>$0.31123 91461 92483 727$</td>
</tr>
<tr>
<td>$0.76609 69055 45936 464$</td>
</tr>
<tr>
<td>$1.42559 75908 38613 09$</td>
</tr>
<tr>
<td>$2.29256 20586 32190 29$</td>
</tr>
<tr>
<td>$3.37077 42642 08997 72$</td>
</tr>
<tr>
<td>$4.66508 37034 67170 79$</td>
</tr>
<tr>
<td>$6.18158 51187 36756 41$</td>
</tr>
<tr>
<td>$7.92753 92471 72152 18$</td>
</tr>
<tr>
<td>$9.91209 80150 77706 02$</td>
</tr>
<tr>
<td>$12.14610 27117 29765 6$</td>
</tr>
<tr>
<td>$14.64273 22895 96674 3$</td>
</tr>
<tr>
<td>$17.41799 26465 08978 7$</td>
</tr>
<tr>
<td>$20.49146 00826 16424 7$</td>
</tr>
<tr>
<td>$23.88732 98481 69733 2$</td>
</tr>
<tr>
<td>$27.63593 71743 32717 4$</td>
</tr>
<tr>
<td>$31.77604 13523 74723 3$</td>
</tr>
<tr>
<td>$36.35840 58016 51621 7$</td>
</tr>
<tr>
<td>$41.45170 48480 70767 0$</td>
</tr>
<tr>
<td>$47.15310 64451 56323 0$</td>
</tr>
<tr>
<td>$53.60857 45446 95099 8$</td>
</tr>
<tr>
<td>$61.05853 14472 87616 6$</td>
</tr>
<tr>
<td>$69.96224 00351 05030 4$</td>
</tr>
<tr>
<td>$81.49827 92339 48885 4$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n = 24$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.05073 46248 49873 8876$</td>
</tr>
<tr>
<td>$0.26748 72686 40741 084$</td>
</tr>
<tr>
<td>$0.56813 66283 54971 519$</td>
</tr>
<tr>
<td>$1.22397 18083 84007 72$</td>
</tr>
<tr>
<td>$1.96676 76124 73777 70$</td>
</tr>
<tr>
<td>$2.88888 33260 30321 89$</td>
</tr>
<tr>
<td>$3.99331 16592 50114 14$</td>
</tr>
<tr>
<td>$5.28373 60628 43442 56$</td>
</tr>
<tr>
<td>$6.76460 34042 43505 15$</td>
</tr>
<tr>
<td>$8.44121 63282 71324 49$</td>
</tr>
<tr>
<td>$10.31985 04629 93260 1$</td>
</tr>
<tr>
<td>$12.40790 34144 60671 7$</td>
</tr>
</tbody>
</table>

(continues..)
### APPENDIX C (Continued)

\[ x^{(n)}_k \]
\[ A^{(n)}_k \]

\[ n = 28 \]

<table>
<thead>
<tr>
<th>( x^{(n)}_k )</th>
<th>( A^{(n)}_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.71408 51641 35748 8</td>
<td>((-6)) 0.98470 12256 24928 887</td>
</tr>
<tr>
<td>17.24866 34156 08056 3</td>
<td>((-7)) 0.85640 75852 67304 245</td>
</tr>
<tr>
<td>20.02378 33299 51712 7</td>
<td>((-8)) 0.58368 38763 13834 429</td>
</tr>
<tr>
<td>23.05389 01350 30296 0</td>
<td>((-9)) 0.30756 38877 88728 887</td>
</tr>
<tr>
<td>26.35629 73744 01317 6</td>
<td>((-10)) 0.12235 90952 72442 282</td>
</tr>
<tr>
<td>29.95196 68335 96182 1</td>
<td>((-12)) 0.36821 23707 72576 3</td>
</tr>
<tr>
<td>33.86660 55165 84459 2</td>
<td>((-14)) 0.79987 90575 96890 965</td>
</tr>
<tr>
<td>38.13225 44101 94646 8</td>
<td>((-15)) 0.12249 22500 32408 341</td>
</tr>
<tr>
<td>42.78967 23707 72576 3</td>
<td>((-17)) 0.12711 24295 03067 374</td>
</tr>
<tr>
<td>47.89207 16336 22743 7</td>
<td>((-20)) 0.84885 93367 68654 320</td>
</tr>
<tr>
<td>53.51129 79596 64294 2</td>
<td>((-22)) 0.34024 55379 42551 185</td>
</tr>
<tr>
<td>59.74879 60846 41240 8</td>
<td>((-25)) 0.74201 56588 86748 513</td>
</tr>
<tr>
<td>66.75697 72839 06469 6</td>
<td>((-28)) 0.23521 32296 69848 005</td>
</tr>
<tr>
<td>74.87677 81523 39161 8</td>
<td>((-31)) 0.98080 33066 14955 132</td>
</tr>
<tr>
<td>84.31783 71072 27043 1</td>
<td>((-35)) 0.39203 41967 98794 720</td>
</tr>
<tr>
<td>96.58242 06275 27319 1</td>
<td>((-40)) 0.19590 33359 72881 043</td>
</tr>
</tbody>
</table>

\[ n = 32 \]

<table>
<thead>
<tr>
<th>( x^{(n)}_k )</th>
<th>( A^{(n)}_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04448 93658 3267 0184</td>
<td>0.10921 83419 52384 971</td>
</tr>
<tr>
<td>0.23452 61095 19618 537</td>
<td>0.21044 31079 38813 234</td>
</tr>
<tr>
<td>0.57688 46293 01886 426</td>
<td>0.28521 32296 69848 005</td>
</tr>
<tr>
<td>1.07244 87538 17817 63</td>
<td>0.19590 33359 72881 043</td>
</tr>
<tr>
<td>1.72240 87764 44645 44</td>
<td>0.12998 37862 86071 761</td>
</tr>
<tr>
<td>2.52833 67064 25794 88</td>
<td>((-1)) 0.70578 62386 57174 415</td>
</tr>
<tr>
<td>3.49221 32730 21994 49</td>
<td>((-1)) 0.31760 91250 91750 703</td>
</tr>
<tr>
<td>4.61645 67697 49767 39</td>
<td>((-1)) 0.11918 21483 48385 571</td>
</tr>
<tr>
<td>5.90395 85041 74243 95</td>
<td>((-2)) 0.37388 16294 61152 479</td>
</tr>
<tr>
<td>7.35812 67331 86241 11</td>
<td>((-3)) 0.98080 33066 14955 132</td>
</tr>
<tr>
<td>8.98294 09242 12596 10</td>
<td>((-3)) 0.21486 49188 01364 888</td>
</tr>
<tr>
<td>10.78301 86325 39972 1</td>
<td>((-4)) 0.39203 41967 98794 720</td>
</tr>
<tr>
<td>12.76369 79767 42725 1</td>
<td>((-5)) 0.59345 46121 86863 288</td>
</tr>
<tr>
<td>14.93113 97555 22557 3</td>
<td>((-6)) 0.74164 04578 66755 222</td>
</tr>
<tr>
<td>17.29245 48367 15314 8</td>
<td>((-7)) 0.76045 67879 12078 148</td>
</tr>
<tr>
<td>19.85586 09403 36054 7</td>
<td>((-8)) 0.63506 2226 62580 764</td>
</tr>
<tr>
<td>22.63088 90131 96774 5</td>
<td>((-9)) 0.42813 82971 04092 888</td>
</tr>
<tr>
<td>25.62863 60224 59247 8</td>
<td>((-10)) 0.23058 99491 89133 608</td>
</tr>
<tr>
<td>28.86210 18163 23474 7</td>
<td>((-12)) 0.97993 79288 72709 406</td>
</tr>
<tr>
<td>32.34662 91539 64737 0</td>
<td>((-13)) 0.32278 01657 72926 646</td>
</tr>
<tr>
<td>36.10049 48057 51973 8</td>
<td>((-15)) 0.81718 23443 42071 943</td>
</tr>
<tr>
<td>40.14571 97715 39441 5</td>
<td>((-16)) 0.15421 33833 39382 337</td>
</tr>
<tr>
<td>44.50920 79957 54938 0</td>
<td>((-18)) 0.21197 92290 16361 861</td>
</tr>
<tr>
<td>49.22439 49873 08639 2</td>
<td>((-20)) 0.20544 29673 78804 543</td>
</tr>
<tr>
<td>54.33372 13333 69097 3</td>
<td>((-22)) 0.13469 82586 63739 516</td>
</tr>
<tr>
<td>59.89250 91621 34018 2</td>
<td>((-25)) 0.56612 94130 39735 937</td>
</tr>
<tr>
<td>65.97537 72879 35052 8</td>
<td>((-27)) 0.14185 60545 46303 691</td>
</tr>
<tr>
<td>72.68762 80906 62708 6</td>
<td>((-30)) 0.19133 75494 45422 431</td>
</tr>
<tr>
<td>80.18744 69779 15321 3</td>
<td>((-33)) 0.11922 48760 09822 236</td>
</tr>
<tr>
<td>88.73534 04178 92398 7</td>
<td>((-37)) 0.26715 11219 24013 699</td>
</tr>
<tr>
<td>96.82954 28682 88972 6</td>
<td>((-41)) 0.13386 16942 10625 628</td>
</tr>
<tr>
<td>111.75139 80979 37695</td>
<td>((-47)) 0.45105 36193 89897 424</td>
</tr>
</tbody>
</table>
INDEX

Absolutely continuous functions, 269
- convergence of quadrature formulas for, 269-270
Akkerman, R. B., 170, 172
Analytic function
- convergence of quadrature formulas for, 243-264
- remainder of interpolation for, 45
Banach space, 51
- $C$, 51
- $L_p$, 52
- $V$, 53
Bernoulli numbers, $B_n$, 3-5, 7
- asymptotic value for, 6
Bernoulli polynomials, $B_n(x)$, 6-17
- expansion of an arbitrary function in, 15-17
Bernstein, S. N., 198
Bessel functions, 301-302
Bessel's interpolation formula, 300
Best quadrature formulas, see Quadrature formulas with least estimate of the remainder
Bounded variation, functions of, 53
- see also Classes of functions
Bouzitat, J., 178
Bronwin, B., 132
Brouwer, L. E. J., 308
Capuano, R., 130, 343
Cauchy
- integral, 46
- kernel, 46
Characteristic representation of a class of functions, 75
- of the class $A_r$, 269
- of the class $C_r$, 76, 266
- of the class $L_{q(r)}$, 134
- of the class $V_r$, 271
Chebyshev, P. L., 28, 198
Chebyshev distribution function, 186, 252-264
Chebyshev polynomials of first kind, 26, 114
- leading coefficient, 27
- normalizing factor, 27
- property of deviating least from zero, 28
- recursion relation, 27
Chebyshev polynomials of second kind, 29, 115
- leading coefficient, 29
- minimal property, 30-33
- normalizing factor, 29
- recursion relation, 29
Chernin, K. E., 170, 333
Christoffel, E. B., 132
Christoffel-Darboux relationship, 22-23, 103
Classes of functions
- $A_r$, 269
- $C$, 51
- $C_r$, 75
- $L_p$, 52
- $L_{q(r)}$, 134
- $V$, 53
- $V_r$, 271
- see also Characteristic representation; Convergence of quadrature formulas
Complete space, 51
- system of functions, 67
Continuous functions, 51-52
- convergence of quadrature formulas for, 264-266
- see also Classes of functions
Convergence
- of distribution functions, 244
- of linear operators, 59-61

353
Convergence of quadrature formulas, 242-243
in the class $A_r$, 269-270
in the class $C_r$, 264-266
in the class $C_r^*$, 266-268
in the class $V_r$, 271-273
of "best" quadratures
in the class $L_{q^2}$, 149
in the class $C_2$, 153
of highest algebraic degree of precision, 106
of interpolatory quadratures for analytic functions, 243-264

Davids, N., 108
Davis, P., 337
Distribution function(s), 244
convergence of a sequence of, 244
for roots of orthogonal polynomials, 252
see also Chebyshev distribution function
Divided differences, 38
relation to finite differences, 40
relation to $n^{th}$ derivative, 40, 41
$E(x)$, 76
Electrostatic analogy
for nodes in indefinite integration, 326-327
for roots of Jacobi polynomials, 231-232
Euler's method for expanding the remainder, 206-229
for formulas of highest degree of precision for Jacobi weight functions, 296-227
for Simpson's rule, 220-225
for three-eighths rule, 226
for trapezoidal rule, 214-219
Euler-Maclaurin sum formula, 216
Evgrafov, M. A., 237, 241
Filippov, M. A., 309
Finite differences, 37
Fishman, H., 124
Fixed nodes, see Quadrature formulas with preassigned nodes
Fixed-point theorem, 308
Functional, 55
Gauss, C. F., 132
Gawlik, H. J., 337
Gelfond, A. O., 17, 124
Geronimus, Ia. L., 132, 198, 273
Glivenko, V. I., 257, 254
Goncharov, V. L., 35, 49, 237
Greenwood, R. E., 130
Hammer, P. C., 132
Hardy, G. H., 17
Hermite, C., 48, 49
Hermite (Chebyshev-Hermite) polynomials, 33, 129
leading coefficient, 33
normalizing factor, 33
recursion relation, 33
Rodriguez formula, 33
Hetherington, R. G., 132
Hölder's inequality, 135, 137
Indefinite integration, 277-281
convergence of, 294-297
error of, 281-287
due to initial values, 283-284, 289
due to rounding, 283, 285, 289
due to formula, 283, 285, 289
nodes in, see Nodes
of tabular functions, 298-302
remainder of, 302
stability
with respect to initial values, 291
with respect to rounding, 293-294
Indefinite integration using one previous value of the integral, 303-319
highest degree of precision, 305-306
existence of formulas of, 306-309
specific formulas, 312-318
Indefinite integration using several previous values of the integral, 320-336
highest degree of precision, 322-323
conditions for, 323-326
conditions for positive coefficients, 329-331
differential equation for nodes, 331-333
number of formulas, 326-327
remainder of, 327-329
tables of formulas, 333-336
Integral equation
approximate solution of, 110-111
equivalent to differential equation, 160
of Volterra, 278
Integral representation of remainder, 209
with short principle subinterval, 229-241
Interpolation by successive derivatives, see Interpolation problem of Abel-Goncharov
Interpolation problem of Abel-Goncharov, 237-241
Interpolation with multiple nodes, 45
Hermite's form, 49
remainder, 49
Interpolation with simple nodes, 42
Lagrange's form, 43
Newton's form, 43
remainder, 43-45
Interpolatory quadrature formulas, 80, 100

Jackson, D., 35
Jacobi polynomials, 23, 112-113
leading coefficient, 24
normalizing factor, 25
Rodriguez formula, 23
Johnson, W. W., 83, 98

Kalmár, L., 254
Kantorovich, L. V., 62, 241
Kernel of remainder of quadratures, 77
for Newton-Cotes formulas, 89, 92
Kneschke, A., 78
Kopal, Z., 83, 98
Korkin, A. N., 36
Krylov, A. N., 241
Krylov, V. I., 198, 241, 273, 334
Kuz'min, R. O., 84, 99, 198, 273

Laguerre (Chebyshev-Laguerre) polynomials, 34, 130-131
leading coefficient, 34
normalizing factor, 35
Rodriguez formula, 34
Legendre polynomials, 26, 108
leading coefficients, 26
normalizing factor, 26
Rodriguez formula, 26
Levenson, A., 108
Linear normed (vector) space, 51
Lobatto, 166
Logarithmic potential, 245
constant almost everywhere, 259
for Chebyshev distribution function, 253-263
for Newton-Cotes formulas, 248
Lowan, A. N., 108
Lozinskii, S. M., 270, 273
Lyusternik, L. A., 62

Markov, A. A. (Markoff), 132, 166, 178, 284
Marlowe, O. J., 132
Mehler, F. G., 114, 132
Meyers, L. F., 157, 159
Midpoint quadrature formula, 140, 151
Mikeladze, Sh. E., 273
Miller, J. J., 130
Mineur, H., 127, 178
Minimization of the remainder in the class \( C^n \), 149-153
in the class \( L^r \), 134-153
with fixed nodes, 153-158
Minkowski inequality, 52

Natanson, I. P., 36, 247, 273
Nemytskii, V. V., 308
Newton-Cotes formulas, 82-98
coefficients for \( n = 1 - 10 \), 83
convergence of, for analytic functions, 248-249
estimates for coefficients, 86
remainder, 89, 91
see also Trapezoidal rule; Simpson's rule; Three-eighths rule
Newton's equations, 180
Nikol'skii, S. M., 82, 99, 159

Nodes
in indefinite integration
auxiliary nodes, 321
basic nodes, 304
double nodes, 321
simple nodes, 321
in quadrature formulas, 66
Norm, 51
in space \( C \), 51
in space \( L^p \), 52
in space \( V \), 53
of an operator, 56

Operator, 54
continuous, 55
linear, 55
norm of, 56
Orthogonal polynomials, 18-35
distribution of roots of, 21
see also Orthonormal polynomials; Jacobi polynomials; etc.
Orthonormal polynomials, 21
recursion relation for, 21-22
see also Orthogonal polynomials

Peirce, W. H., 132
Periodic Bernoulli functions, \( B^*_n(x) \), 13-15
trigonometric Fourier series for, 15
Poisson-Lebesgue integral, 261
Polya, G., 264, 273
Posse, K. A., 132, 198
Precision, degree of, 68
highest, 68-69, 100-104
for formulas with preassigned
nodes, 161-162
in indefinite integration, 305-308,
322-326
methods to increase, 200-202
Privalov, I. I., 261

Quadrature formulas, 66
choice of nodes and coefficients, 66-72
convergence of, see Convergence for indefinite integration, see Infinite integration
Quadrature formulas (Contd.)

increasing precision of, 200-202

see also Euler's method; singular integrand; Integral representation of remainder

nodes in, see Nodes
tables of, see Tables

with positive coefficients, 72, 104

Quadrature formulas of highest degree of precision for algebraic polynomials, 69, 100-107

coefficients, 103-104

constant weight function, convergence of, 106-107

Hermite weight function, 129-130

Jacobi weight functions, 111-121

Laguerre weight function, 130-132

nodes, 101

remainder, 104-105

see also Tables

Quadrature formulas of highest degree of precision for trigonometric polynomials, 73-74

Quadrature formulas with equal coefficients, 71-72, 179-199

on infinite intervals, 198-199

with Chebyshev weight function, 114-115, 183-187

with constant weight function, 187-199

table, 191

Quadrature formulas with least estimate of the remainder, 70-71, 133-134

convergence of, see Convergence in the class $C_1$, 151

in the class $C_2$, 152-153

in the class $L^1_q(2)$, 139-140

in the class $L^2_q(r)$, 140-149

with fixed nodes

in the class $L^1_q(2)$, 154-155

in the class $L^2_q(r)$ ($r \geq 2$), 155-158

Quadrature formulas with preassigned nodes, 160-178

coefficients for fixed nodes, 163

coefficients for free nodes, 163-164

fixed nodes at end points of the interval, 166-167

both end points, 170-174

one end point, 167-170

free nodes

choice of, 161

orthogonal polynomial for, 164-166

highest degree of precision, 161-163

remainder, 163

Quadrature formulas with sign-changing weight functions, 174-178

Quadrature sum, 66

Rabinowitz, P., 178, 337, 347

Radau, R., 166, 178

Radon, J., 78

Remainder in interpolation, 43-45, 49

Lagrange form, 44

representation as contour integral, 45

Remainder in quadrature, 74-77, 81-82

expansion of, see Euler's method

see also Integral representation;

Newton-Cotes formulas; etc.

Remez, E. Ia., 49, 78

Robinson, G., 49, 99

Salzer, H. E., 130, 131, 199, 347

Sard, A., 78, 157, 159

Secret, D., 132

Sequence of distribution functions, 244

of linear operators, 59

convergence of, 59-61

of quadrature formulas, 242-243

Shaïdaeva, T. A., 159

Siga $x$, 30

Simplex, $m$-dimensional, 308

Simpson's rule, 94

Euler's method for expanding remainder, 220-225

remainder, 96

Singular, integrand, 201

weakening singularity of, 202-206

Smirnov, V. N., 241

Sobolev, V. I., 62

Sonin, N. Ia., 132, 199

Steffensen, J. F., 17, 99, 241

Steklov, V. A., 264, 273

Stieltjes, T. J., 132, 327

Stroud, A. H., 122, 132

Struble, G. W., 178

Structural formula, 75

see also Characteristic representation

Szegö, G., 36, 113, 232

Tables of quadrature formulas

Chebyshev formulas, 191

for indefinite integrals, 315-318, 335-336

Newton-Cotes formulas, 83

of highest degree of precision

$$\int_{-1}^{1} f(x) \, dx, \quad 337-342$$

$$\int_{-\infty}^{\infty} e^{-x^2} f(x) \, dx, \quad 343-346$$

$$\int_{0}^{\infty} e^{-x} f(x) \, dx, \quad 347-352$$

$$\int_{0}^{1} \sqrt{x} f(x) \, dx, \quad 119-120$$
\[
\int_0^1 f(x)/\sqrt{x} \, dx, \quad 120-121
\]
\[
\int_0^1 x f(x) \, dx, \quad 124
\]

with least estimate of remainder with fixed nodes, 157-158
with one fixed node, 170
with two fixed nodes, 172-174

Three-eighths rule, 96
Euler's method for expanding remainder of, 226
remainder, 98

Trapezoidal rule, 92, 155
Euler's method for expanding remainder of, 212-219
remainder, 93-94

Vandermonde determinant, 42, 324
Vector space, 51

Weierstrass, theorem of, 265
Weight function, 18
nonnegative, 18-21
which changes sign, 174-178

Weiss, G., 347
Whittaker, E. T., 49, 99
Wilf, H. S., 199
Wymore, A. W., 132

Zolotarev, E. I., 36
Zucker, R., 130, 131, 343, 347
CATALOG OF DOVER BOOKS

Astronomy

BURNHAM’S CELESTIAL HANDBOOK, Robert Burnham, Jr. Thorough guide to the stars beyond our solar system. Exhaustive treatment. Alphabetical by constellation: Andromeda to Cetus in Vol. 1; Chamaeleion to Orion in Vol. 2; and Pavo to Vulpecula in Vol. 3. Hundreds of illustrations. Index in Vol. 3. 2,000pp. 6% x 9%.
Vol. II: 0-486-23568-8
Vol. III: 0-486-23673-0

0-486-24491-1

THE EXTRATERRESTRIAL LIFE DEBATE, 1750-1900, Michael J. Crowe. First detailed, scholarly study in English of the many ideas that developed from 1750 to 1900 regarding the existence of intelligent extraterrestrial life. Examines ideas of Kant, Herschel, Voltaire, Percival Lowell, many other scientists and thinkers. 16 illustrations. 704pp. 5% x 8%.
0-486-40675-X

THEORIES OF THE WORLD FROM ANTIQUITY TO THE COPERNICAN REVOLUTION, Michael J. Crowe. Newly revised edition of an accessible, enlightening book recreates the change from an earth-centered to a sun-centered conception of the solar system. 242pp. 5% x 8%.
0-486-41444-2

A HISTORY OF ASTRONOMY, A. Pannekoek. Well-balanced, carefully reasoned study covers such topics as Ptolemaic theory, work of Copernicus, Kepler, Newton, Eddington’s work on stars, much more. Illustrated. References. 521pp. 5% x 8%.
0-486-65994-1

A COMPLETE MANUAL OF AMATEUR ASTRONOMY: TOOLS AND TECHNIQUES FOR ASTRONOMICAL OBSERVATIONS, P. Clay Sherrod with Thomas L. Koed. Concise, highly readable book discusses: selecting, setting up and maintaining a telescope; amateur studies of the sun; lunar topography and occultations; observations of Mars, Jupiter, Saturn, the minor planets and the stars; an introduction to photoelectric photometry; more. 1981 ed. 124 figures. 25 halftones. 37 tables. 335pp. 6% x 9%.
0-486-40675-X

AMATEUR ASTRONOMER’S HANDBOOK, J. B. Sidgwick. Timeless, comprehensive coverage of telescopes, mirrors, lenses, mountings, telescope drives, micrometers, spectrosopes, more. 189 illustrations. 576pp. 5% x 8%. (Available in U.S. only.)
0-486-24034-7

0-486-69424-0
CATALOG OF DOVER BOOKS

Chemistry

THE SCEPTICAL CHYMIST: THE CLASSIC 1661 TEXT, Robert Boyle. Boyle defines the term "element," asserting that all natural phenomena can be explained by the motion and organization of primary particles. 1911 ed. viii+232pp. 5% x 8%. 0-486-42825-7

RADIOACTIVE SUBSTANCES, Marie Curie. Here is the celebrated scientist's doctoral thesis, the prelude to her receipt of the 1903 Nobel Prize. Curie discusses establishing atomic character of radioactivity found in compounds of uranium and thorium; extraction from pitchblende of polonium and radium; isolation of pure radium chloride; determination of atomic weight of radium; plus electric, photographic, luminous, heat, color effects of radioactivity. ii+94pp. 5% x 8%. 0-486-42550-9


CATALYSIS IN CHEMISTRY AND ENZYMEOLOGY, William P. Jencks. Exceptionally clear coverage of mechanisms for catalysis, forces in aqueous solution, carbonyl- and acyl-group reactions, practical kinetics, more. 864pp. 5% x 8%. 0-486-65460-5

ELEMENTS OF CHEMISTRY, Antoine Lavoisier. Monumental classic by founder of modern chemistry in remarkable reprint of rare 1790 Kerr translation. A must for every student of chemistry or the history of science. 539pp. 5% x 8%. 0-486-64624-6

THE HISTORICAL BACKGROUND OF CHEMISTRY, Henry M. Leicester. Evolution of ideas, not individual biography. Concentrates on formulation of a coherent set of chemical laws. 260pp. 5% x 8%. 0-486-61053-5

A SHORT HISTORY OF CHEMISTRY, J. R. Partington. Classic exposition explores origins of chemistry, alchemy, early medical chemistry, nature of atmosphere, theory of valency, laws and structure of atomic theory, much more. 428pp. 5% x 8%. (Available in U.S. only) 0-486-65977-1


FROM ALCHEMY TO CHEMISTRY, John Read. Broad, humanistic treatment focuses on great figures of chemistry and ideas that revolutionized the science. 50 illustrations. 240pp. 5% x 8%. 0-486-28690-8
CATALOG OF DOVER BOOKS

Engineering

DE RE METALLICA, Georgius Agricola. The famous Hoover translation of greatest treatise on technological chemistry, engineering, geology, mining of early modern times (1556). All 289 original woodcuts. 638pp. 6% x 11. 0-486-60006-8

FUNDAMENTALS OF ASTRODYNAMICS, Roger Bate et al. Modern approach developed by U.S. Air Force Academy. Designed as a first course. Problems, exercises. Numerous illustrations. 455pp. 5% x 8%. 0-486-60061-0

DYNAMICS OF FLUIDS IN POROUS MEDIA, Jacob Bear. For advanced students of ground water hydrology, soil mechanics and physics, drainage and irrigation engineering and more. 335 illustrations. Exercises, with answers. 784pp. 6% x 9%. 0-486-65675-6


MECHANICS, J. P. Den Hartog. A classic introductory text or refresher. Hundreds of applications and design problems illuminate fundamentals of trusses, loaded beams and cables, etc. 334 answered problems. 462pp. 5% x 8%. 0-486-60754-2


STRENGTH OF MATERIALS, J. P. Den Hartog. Full, clear treatment of basic material (tension, torsion, bending, etc.) plus advanced material on engineering methods, applications. 350 answered problems. 323pp. 5% x 8%. 0-486-60755-0

A HISTORY OF MECHANICS, René Dugas. Monumental study of mechanical principles from antiquity to quantum mechanics. Contributions of ancient Greeks, Galileo, Leonardo, Kepler, Lagrange, many others. 671pp. 5% x 8%. 0-486-65632-2

STABILITY THEORY AND ITS APPLICATIONS TO STRUCTURAL MECHANICS, Clive L. Dym. Self-contained text focuses on Koiter postbuckling analyses, with mathematical notions of stability of motion. Basing minimum energy principles for static stability upon dynamic concepts of stability of motion, it develops asymptotic buckling and postbuckling analyses from potential energy considerations, with applications to columns, plates, and arches. 1974 ed. 208pp. 5% x 8%. 0-486-42541-X

METAL FATIGUE, N. E. Frost, K. J. Marsh, and L. P. Pook. Definitive, clearly written, and well-illustrated volume addresses all aspects of the subject, from the historical development of understanding metal fatigue to vital concepts of the cyclic stress that causes a crack to grow. Includes 7 appendixes. 544pp. 5% x 8%. 0-486-40927-9
ROCKETS, Robert Goddard. Two of the most significant publications in the history of rocketry and jet propulsion: "A Method of Reaching Extreme Altitudes" (1919) and "Liquid Propellant Rocket Development" (1936). 128pp. 5% x 8%. 0-486-42537-1

STATISTICAL MECHANICS: PRINCIPLES AND APPLICATIONS, Terrell L. Hill. Standard text covers fundamentals of statistical mechanics, applications to fluctuation theory, imperfect gases, distribution functions, more. 448pp. 5% x 8%. 0-486-65390-0

ENGINEERING AND TECHNOLOGY 1650–1750: ILLUSTRATIONS AND TEXTS FROM ORIGINAL SOURCES, Martin Jensen. Highly readable text with more than 200 contemporary drawings and detailed engravings of engineering projects dealing with surveying, leveling, materials, hand tools, lifting equipment, transport and erection, piling, bailing, water supply, hydraulic engineering, and more. Among the specific projects outlined—transporting a 50-ton stone to the Louvre, erecting an obelisk, building timber locks, and dredging canals. 207pp. 8'k x 11'/. 0-486-42232-1

THE VARIATIONAL PRINCIPLES OF MECHANICS, Cornelius Lanczos. Graduate level coverage of calculus of variations, equations of motion, relativistic mechanics, more. First inexpensive paperbound edition of classic treatise. Index. Bibliography. 418pp. 5% x 8%. 0-486-65067-7

PROTECTION OF ELECTRONIC CIRCUITS FROM OVERVOLTAGES, Ronald B. Standler. Five-part treatment presents practical rules and strategies for circuits designed to protect electronic systems from damage by transient overvoltages. 1989 ed. xxiv+434pp. 6% x 9%. 0-486-42552-5


INTRODUCTION TO SPACE DYNAMICS, William Tyrrell Thomson. Comprehensive, classic introduction to space-flight engineering for advanced undergraduate and graduate students. Includes vector algebra, kinematics, transformation of coordinates. Bibliography. Index. 352pp. 5% x 8%. 0-486-65113-4

HISTORY OF STRENGTH OF MATERIALS, Stephen P. Timoshenko. Excellent historical survey of the strength of materials with many references to the theories of elasticity and structure. 245 figures. 452pp. 5% x 8%. 0-486-61187-6

ANALYTICAL FRACTURE MECHANICS, David J. Unger. Self-contained text supplements standard fracture mechanics texts by focusing on analytical methods for determining crack-tip stress and strain fields. 336pp. 6% x 9%. 0-486-41737-9

STATISTICAL MECHANICS OF ELASTICITY, J. H. Weiner. Advanced, self-contained treatment illustrates general principles and elastic behavior of solids. Part 1, based on classical mechanics, studies thermoelastic behavior of crystalline and polymeric solids. Part 2, based on quantum mechanics, focuses on interatomic force laws, behavior of solids, and thermally activated processes. For students of physics and chemistry and for polymer physicists. 1983 ed. 96 figures. 496pp. 5% x 8%. 0-486-42260-7
CATALOG OF DOVER BOOKS

Mathematics

FUNCTIONAL ANALYSIS (Second Corrected Edition), George Bachman and Lawrence Narici. Excellent treatment of subject geared toward students with background in linear algebra, advanced calculus, physics and engineering. Text covers introduction to inner-product spaces, normed, metric spaces, and topological spaces; complete orthonormal sets, the Hahn-Banach Theorem and its consequences, and many other related subjects. 1966 ed. 544pp. 6% x 9%. 0-486-40251-7


AN INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS, Earl A. Coddington. A thorough and systematic first course in elementary differential equations for undergraduates in mathematics and science, with many exercises and problems (with answers). Index. 304pp. 5% x 8%. 0-486-65942-9

FOURIER SERIES AND ORTHOGONAL FUNCTIONS, Harry F. Davis. An incisive text combining theory and practical example to introduce Fourier series, orthogonal functions and applications of the Fourier method to boundary-value problems. 570 exercises. Answers and notes. 416pp. 5% x 8%. 0-486-65973-9

COMPUTABILITY AND UNSOLVABILITY, Martin Davis. Classic graduate-level introduction to theory of computability, usually referred to as theory of recurrent functions. New preface and appendix. 288pp. 5% x 8%. 0-486-61471-9

ASYMPTOTIC METHODS IN ANALYSIS, N. G. de Bruijn. An inexpensive, comprehensive guide to asymptotic methods—the pioneering work that teaches by explaining worked examples in detail. Index. 224pp. 5% x 8% 0-486-64221-6

APPLIED COMPLEX VARIABLES, John W. Dettman. Step-by-step coverage of fundamentals of analytic function theory—plus lucid exposition of five important applications: Potential Theory; Ordinary Differential Equations; Fourier Transforms; Laplace Transforms; Asymptotic Expansions. 66 figures. Exercises at chapter ends. 512pp. 5% x 8%. 0-486-64670-X

INTRODUCTION TO LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS, John W. Dettman. Excellent text covers complex numbers, determinants, orthonormal bases, Laplace transforms, much more. Exercises with solutions. Undergraduate level. 416pp. 5% x 8%. 0-486-65191-6

RIEMANN'S ZETA FUNCTION, H. M. Edwards. Superb, high-level study of landmark 1859 publication entitled "On the Number of Primes Less Than a Given Magnitude" traces developments in mathematical theory that it inspired. xiv+315pp. 5% x 8%. 0-486-41740-9
CATALOG OF DOVER BOOKS

CALCULUS OF VARIATIONS WITH APPLICATIONS, George M. Ewing. Applications-oriented introduction to variational theory develops insight and promotes understanding of specialized books, research papers. Suitable for advanced undergraduate/graduate students as primary, supplementary text. 352pp. 5% x 8%. 0-486-64856-7

COMPLEX VARIABLES, Francis J. Flanigan. Unusual approach, delaying complex algebra till harmonic functions have been analyzed from real variable viewpoint. Includes problems with answers. 364pp. 5% x 8%. 0-486-61388-7

AN INTRODUCTION TO THE CALCULUS OF VARIATIONS, Charles Fox. Graduate-level text covers variations of an integral, isoperimetrical problems, least action, special relativity, approximations, more. References. 279pp. 5% x 8%. 0-486-65499-0

COUNTEREXAMPLES IN ANALYSIS, Bernard R. Gelbaum and John M. H. Olmsted. These counterexamples deal mostly with the part of analysis known as "real variables." The first half covers the real number system, and the second half encompasses higher dimensions. 1962 edition. xxiv+198pp. 5% x 8%. 0-486-42875-3


INTRODUCTION TO DIFFERENCE EQUATIONS, Samuel Goldberg. Exceptionally clear exposition of important discipline with applications to sociology, psychology, economics. Many illustrative examples; over 250 problems. 260pp. 5% x 8%. 0-486-65084-7

NUMERICAL METHODS FOR SCIENTISTS AND ENGINEERS, Richard Hamming. Classic text stresses frequency approach in coverage of algorithms, polynomial approximation, Fourier approximation, exponential approximation, other topics. Revised and enlarged 2nd edition. 721pp. 5% x 8%. 0-486-65241-6

INTRODUCTION TO NUMERICAL ANALYSIS (2nd Edition), F. B. Hildebrand. Classic, fundamental treatment covers computation, approximation, interpolation, numerical differentiation and integration, other topics. 150 new problems. 669pp. 5% x 8%. 0-486-65363-3

THREE PEARLS OF NUMBER THEORY, A. Y. Khinchin. Three compelling puzzles require proof of a basic law governing the world of numbers. Challenges concern van der Waerden’s theorem, the Landau-Schnirelmann hypothesis and Mann’s theorem, and a solution to Waring’s problem. Solutions included. 64pp. 5% x 8%. 0-486-40026-3

THE PHILOSOPHY OF MATHEMATICS: AN INTRODUCTORY ESSAY, Stephan Körner. Surveys the views of Plato, Aristotle, Leibniz & Kant concerning propositions and theories of applied and pure mathematics. Introduction. Two appendices. Index. 198pp. 5% x 8%. 0-486-25048-2
INTRODUCTORY REAL ANALYSIS, A.N. Kolmogorov, S. V. Fomin. Translated by Richard A. Silverman. Self-contained, evenly paced introduction to real and functional analysis. Some 350 problems. 403pp. 5% x 8%.

APPLIED ANALYSIS, Cornelius Lanczos. Classic work on analysis and design of finite processes for approximating solution of analytical problems. Algebraic equations, matrices, harmonic analysis, quadrature methods, much more. 559pp. 5% x 8%.

AN INTRODUCTION TO ALGEBRAIC STRUCTURES, Joseph Landin. Superb self-contained text covers "abstract algebra": sets and numbers, theory of groups, theory of rings, much more. Numerous well-chosen examples, exercises. 247pp. 5% x 8%.

QUALITATIVE THEORY OF DIFFERENTIAL EQUATIONS, V. V. Nemytskii and V.V. Stepanov. Classic graduate-level text by two prominent Soviet mathematicians covers classical differential equations as well as topological dynamics and ergodic theory. Bibliographies. 523pp. 5% x 8%.

THEORY OF MATRICES, Sam Perlis. Outstanding text covering rank, nonsingularity and inverses in connection with the development of canonical matrices under the relation of equivalence, and without the intervention of determinants. Includes exercises. 237pp. 5% x 8%.

INTRODUCTION TO ANALYSIS, Maxwell Rosenlicht. Unusually clear, accessible coverage of set theory, real number system, metric spaces, continuous functions, Riemann integration, multiple integrals, more. Wide range of problems. Undergraduate level. Bibliography. 254pp. 5% x 8%.

MODERN NONLINEAR EQUATIONS, Thomas L. Saaty. Emphasizes practical solution of problems; covers seven types of equations. "... a welcome contribution to the existing literature..."—Math Reviews. 490pp. 5% x 8%.

MATRICES AND LINEAR ALGEBRA, Hans Schneider and George Phillip Barker. Basic textbook covers theory of matrices and its applications to systems of linear equations and related topics such as determinants, eigenvalues and differential equations. Numerous exercises. 432pp. 5% x 8%.

LINEAR ALGEBRA, Georgi E. Shilov. Determinants, linear spaces, matrix algebras, similar topics. For advanced undergraduates, graduates. Silverman translation. 387pp. 5% x 8%.

ELEMENTS OF REAL ANALYSIS, David A. Sprecher. Classic text covers fundamental concepts, real number system, point sets, functions of a real variable, Fourier series, much more. Over 500 exercises. 352pp. 5% x 8%.

SET THEORY AND LOGIC, Robert R. Stoll. Lucid introduction to unified theory of mathematical concepts. Set theory and logic seen as tools for conceptual understanding of real number system. 496pp. 5% x 8%.
CATALOG OF DOVER BOOKS

TENSOR CALCULUS, J.L. Synge and A. Schild. Widely used introductory text covers spaces and tensors, basic operations in Riemannian space, non-Riemannian spaces, etc. 324pp. 5% x 8%. 0-486-63612-7


FOURIER SERIES, Georgi P. Tolstov. Translated by Richard A. Silverman. A valuable addition to the literature on the subject, moving clearly from subject to subject and theorem to theorem. 107 problems, answers. 336pp. 5% x 8%. 0-486-63317-9

INTRODUCTION TO MATHEMATICAL THINKING, Friedrich Waismann. Examinations of arithmetic, geometry, and theory of integers; rational and natural numbers; complete induction; limit and point of accumulation; remarkable curves; complex and hypercomplex numbers, more. 1959 ed. 27 figures. xii+260pp. 5% x 8%. 0-486-63317-9


CALCULUS OF VARIATIONS, Robert Weinstock. Basic introduction covering isoperimetric problems, theory of elasticity, quantum mechanics, electrostatics, etc. Exercises throughout. 326pp. 5% x 8%. 0-486-65251-3

THE CONTINUUM: A CRITICAL EXAMINATION OF THE FOUNDATION OF ANALYSIS, Hermann Weyl. Classic of 20th-century foundational research deals with the conceptual problem posed by the continuum. 156pp. 5% x 8%. 0-486-67982-9

CHALLENGING MATHEMATICAL PROBLEMS WITH ELEMENTARY SOLUTIONS, A. M. Yaglom and I. M. Yaglom. Over 170 challenging problems on probability theory, combinatorial analysis, points and lines, topology, convex polygons, many other topics. Solutions. Total of 445pp. 5% x 8%. Two-vol. set.

INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS, E. C. Zachmanoglou and Dale W. Thoe. Essentials of partial differential equations applied to common problems in engineering and the physical sciences. Problems and answers. 416pp. 5% x 8%. 0-486-65532-6

THE THEORY OF GROUPS, Hans J. Zassenhaus. Well-written graduate-level text acquaints reader with group-theoretic methods and demonstrates their usefulness in mathematics. Axioms, the calculus of complexes, homomorphic mapping, p-group theory, more. 276pp. 5% x 8%. 0-486-40922-8
Math–Decision Theory, Statistics, Probability

ELEMENTARY DECISION THEORY, Herman Chernoff and Lincoln E. Moses. Clear introduction to statistics and statistical theory covers data processing, probability and random variables, testing hypotheses, much more. Exercises. 364pp. 5% x 8%. 0-486-65218-1


SOME THEORY OF SAMPLING, William Edwards Deming. Analysis of the problems, theory and design of sampling techniques for social scientists, industrial managers and others who find statistics important at work. 61 tables. 90 figures. xvii +602pp. 5% x 8%. 0-486-64684-X

LINEAR PROGRAMMING AND ECONOMIC ANALYSIS, Robert Dorfman, Paul A. Samuelson and Robert M. Solow. First comprehensive treatment of linear programming in standard economic analysis. Game theory, modern welfare economics, Leontief input-output, more. 525pp. 5% x 8%. 0-486-65491-5

PROBABILITY: AN INTRODUCTION, Samuel Goldberg. Excellent basic text covers set theory, probability theory for finite sample spaces, binomial theorem, much more. 360 problems. Bibliographies. 322pp. 5% x 8%. 0-486-65252-1

GAMES AND DECISIONS: INTRODUCTION AND CRITICAL SURVEY, R. Duncan Luce and Howard Raiffa. Superb nontechnical introduction to game theory, primarily applied to social sciences. Utility theory, zero-sum games, n-person games, decision-making, much more. Bibliography. 509pp. 5% x 8%. 0-486-65943-7

INTRODUCTION TO THE THEORY OF GAMES, J. C. C. McKinsey. This comprehensive overview of the mathematical theory of games illustrates applications to situations involving conflicts of interest, including economic, social, political, and military contexts. Appropriate for advanced undergraduate and graduate courses; advanced calculus a prerequisite. 1952 ed. x+372pp. 5% x 8%. 0-486-42811-7

FIFTY CHALLENGING PROBLEMS IN PROBABILITY WITH SOLUTIONS, Frederick Mosteller. Remarkable puzzlers, graded in difficulty, illustrate elementary and advanced aspects of probability. Detailed solutions. 88pp. 5% x 8%. 65355-2

PROBABILITY THEORY: A CONCISE COURSE, Y. A. Rozanov. Highly readable, self-contained introduction covers combination of events, dependent events, Bernoulli trials, etc. 148pp. 5% x 8%. 0-486-63544-9

STATISTICAL METHOD FROM THE VIEWPOINT OF QUALITY CONTROL, Walter A. Shewhart. Important text explains regulation of variables, uses of statistical control to achieve quality control in industry, agriculture, other areas. 192pp. 5% x 8%. 0-486-65232-7
Math—Geometry and Topology

ELEMENTARY CONCEPTS OF TOPOLOGY, Paul Alexandroff. Elegant, intuitive approach to topology from set-theoretic topology to Betti groups; how concepts of topology are useful in math and physics. 25 figures. 57pp. 5% x 8%. 0-486-60747-X

COMBINATORIAL TOPOLOGY, P. S. Alexandrov. Clearly written, well-organized, three-part text begins by dealing with certain classic problems without using the formal techniques of homology theory and advances to the central concept, the Betti groups. Numerous detailed examples. 654pp. 5% x 8%. 0-486-40179-0

EXPERIMENTS IN TOPOLOGY, Stephen Barr. Classic, lively explanation of one of the byways of mathematics. Klein bottles, Moebius strips, projective planes, map coloring, problem of the Koenigsberg bridges, much more, described with clarity and wit. 43 figures. 210pp. 5% x 8%. 0-486-25933-1

THE GEOMETRY OF RENÉ DESCARTES, René Descartes. The great work founded analytical geometry. Original French text, Descartes's own diagrams, together with definitive Smith-Latham translation. 244pp. 5% x 8%. 0-486-60068-8

EUCLIDEAN GEOMETRY AND TRANSFORMATIONS, Clayton W. Dodge. This introduction to Euclidean geometry emphasizes transformations, particularly isometries and similarities. Suitable for undergraduate courses, it includes numerous examples, many with detailed answers. 1972 ed. viii+296pp. 6% x 9%. 0-486-43476-1

PRACTICAL CONIC SECTIONS: THE GEOMETRIC PROPERTIES OF ELLIPSES, PARABOLAS AND HYPERBOLAS, J. W. Downs. This text shows how to create ellipses, parabolas, and hyperbolae. It also presents historical background on their ancient origins and describes the reflective properties and roles of curves in design applications. 1993 ed. 98 figures. xii+100pp. 6% x 9%. 0-486-42876-1


SPACE AND GEOMETRY: IN THE LIGHT OF PHYSIOLOGICAL, PSYCHOLOGICAL AND PHYSICAL INQUIRY, Ernst Mach. Three essays by an eminent philosopher and scientist explore the nature, origin, and development of our concepts of space, with a distinctness and precision suitable for undergraduate students and other readers. 1906 ed. vi+148pp. 5% x 8%. 0-486-43909-7

GEOMETRY OF COMPLEX NUMBERS, Hans Schwerdtfeger. Illuminating, widely praised book on analytic geometry of circles, the Moebius transformation, and two-dimensional non-Euclidean geometries. 200pp. 5% x 8%. 0-486-63830-8

DIFFERENTIAL GEOMETRY, Heinrich W. Guggenheimer. Local differential geometry as an application of advanced calculus and linear algebra. Curvature, transformation groups, surfaces, more. Exercises. 62 figures. 378pp. 5% x 8%. 0-486-63433-7
CATALOG OF DOVER BOOKS

History of Math

THE WORKS OF ARCHIMEDES, Archimedes (T. L. Heath, ed.). Topics include the famous problems of the ratio of the areas of a cylinder and an inscribed sphere; the measurement of a circle; the properties of conoids, spheroids, and spirals; and the quadrature of the parabola. Informative introduction. clxxxvi+326pp. 5% x 8%. 0-486-42084-1

A SHORT ACCOUNT OF THE HISTORY OF MATHEMATICS, W. W. Rouse Ball. One of clearest, most authoritative surveys from the Egyptians and Phoenicians through 19th-century figures such as Grassman, Galois, Riemann. Fourth edition. 522pp. 5% x 8%. 0-486-20630-0

THE HISTORY OF THE CALCULUS AND ITS CONCEPTUAL DEVELOPMENT, Carl B. Boyer. Origins in antiquity, medieval contributions, work of Newton, Leibniz, rigorous formulation. Treatment is verbal. 346pp. 5% x 8%. 0-486-60509-4

THE HISTORICAL ROOTS OF ELEMENTARY MATHEMATICS, Lucas N. H. Bunt, Phillip S. Jones, and Jack D. Bedient. Fundamental underpinnings of modern arithmetic, algebra, geometry and number systems derived from ancient civilizations. 320pp. 5% x 8%. 0-486-25563-8

A HISTORY OF MATHEMATICAL NOTATIONS, Florian Cajori. This classic study notes the first appearance of a mathematical symbol and its origin, the competition it encountered, its spread among writers in different countries, its rise to popularity, its eventual decline or ultimate survival. Original 1929 two-volume edition presented here in one volume. xxviii+820pp. 5% x 8%. 0-486-67766-4

GAMES, GODS & GAMBLING: A HISTORY OF PROBABILITY AND STATISTICAL IDEAS, F. N. David. Episodes from the lives of Galileo, Fermat, Pascal, and others illustrate this fascinating account of the roots of mathematics. Features thought-provoking references to classics, archaeology, biography, poetry. 1962 edition. 304pp. 5% x 8%. (Available in U.S. only.) 0-486-40023-9

OF MEN AND NUMBERS: THE STORY OF THE GREAT MATHEMATICIANS, Jane Muir. Fascinating accounts of the lives and accomplishments of history’s greatest mathematical minds—Pythagoras, Descartes, Euler, Pascal, Cantor, many more. Anecdotal, illuminating. 30 diagrams. Bibliography. 256pp. 5% x 8%. 0-486-28973-7


A CONCISE HISTORY OF MATHEMATICS, Dirk J. Struik. The best brief history of mathematics. Stresses origins and covers every major figure from ancient Near East to 19th century. 41 illustrations. 195pp. 5% x 8%. 0-486-60255-9
Physics

OPTICAL RESONANCE AND TWO-LEVEL ATOMS, L. Allen and J. H. Eberly. Clear, comprehensive introduction to basic principles behind all quantum optical resonance phenomena. 53 illustrations. Preface. Index. 256pp. 5% x 8%. 0-486-65533-4

QUANTUM THEORY, David Bohm. This advanced undergraduate-level text presents the quantum theory in terms of qualitative and imaginative concepts, followed by specific applications worked out in mathematical detail. Preface. Index. 655pp. 5% x 8%. 0-486-65969-0

ATOMIC PHYSICS (8th EDITION), Max Born. Nobel laureate's lucid treatment of kinetic theory of gases, elementary particles, nuclear atom, wave-corpuscles, atomic structure and spectral lines, much more. Over 40 appendices, bibliography. 495pp. 5% x 8%. 0-486-65984-4

A SOPHISTICATE'S PRIMER OF RELATIVITY, P. W. Bridgman. Geared toward readers already acquainted with special relativity, this book transcends the view of theory as a working tool to answer natural questions: What is a frame of reference? What is a "law of nature"? What is the role of the "observer"? Extensive treatment, written in terms accessible to those without a scientific background. 1983 ed. xlviii+172pp. 5% x 8%. 0-486-42549-5

AN INTRODUCTION TO HAMILTONIAN OPTICS, H. A. Buchdahl. Detailed account of the Hamiltonian treatment of aberration theory in geometrical optics. Many classes of optical systems defined in terms of the symmetries they possess. Problems with detailed solutions. 1970 edition. xv + 360pp. 5% x 8%. 0-486-67597-1

PRIMER OF QUANTUM MECHANICS, Marvin Chester. Introductory text examines the classical quantum bead on a track: its state and representations; operator eigenvalues; harmonic oscillator and bound bead in a symmetric force field; and bead in a spherical shell. Other topics include spin, matrices, and the structure of quantum mechanics; the simplest atom; indistinguishable particles; and stationary-state perturbation theory. 1992 ed. xiv+314pp. 6% x 9%. 0-486-42878-8

LECTURES ON QUANTUM MECHANICS, Paul A. M. Dirac. Four concise, brilliant lectures on mathematical methods in quantum mechanics from Nobel Prize-winning quantum pioneer build on idea of visualizing quantum theory through the use of classical mechanics. 96pp. 5% x 8%. 0-486-41713-1


CATALOG OF DOVER BOOKS

HYDRODYNAMIC AND HYDROMAGNETIC STABILITY, S. Chandrasekhar. Lucid examination of the Rayleigh-Benard problem; clear coverage of the theory of instabilities causing convection. 704pp. 5% x 8%. 0-486-64071-X

INVESTIGATIONS ON THE THEORY OF THE BROWNIAN MOVEMENT, Albert Einstein. Five papers (1905–8) investigating dynamics of Brownian motion and evolving elementary theory. Notes by R. Fürth. 122pp. 5% x 8%. 0-486-60304-0

THE PHYSICS OF WAVES, William C. Elmore and Mark A. Heald. Unique overview of classical wave theory. Acoustics, optics, electromagnetic radiation, more. Ideal as classroom text or for self-study. Problems. 477pp. 5% x 8%. 0-486-64926-1

GRAVITY, George Gamow. Distinguished physicist and teacher takes reader-friendly look at three scientists whose work unlocked many of the mysteries behind the laws of physics: Galileo, Newton, and Einstein. Most of the book focuses on Newton's ideas, with a concluding chapter on post-Einsteinian speculations concerning the relationship between gravity and other physical phenomena. 160pp. 5% x 8%. 0-486-42563-0

PHYSICAL PRINCIPLES OF THE QUANTUM THEORY, Werner Heisenberg. Nobel Laureate discusses quantum theory, uncertainty, wave mechanics, work of Dirac, Schroedinger, Compton, Wilson, Einstein, etc. 184pp. 5% x 8%. 0-486-60113-7

ATOMIC SPECTRA AND ATOMIC STRUCTURE, Gerhard Herzberg. One of best introductions; especially for specialist in other fields. Treatment is physical rather than mathematical. 80 illustrations. 257pp. 5% x 8%. 0-486-60115-3

AN INTRODUCTION TO STATISTICAL THERMODYNAMICS, Terrell L. Hill. Excellent basic text offers wide-ranging coverage of quantum statistical mechanics, systems of interacting molecules, quantum statistics, more. 523pp. 5% x 8%. 0-486-65242-4

THEORETICAL PHYSICS, Georg Joos, with Ira M. Freeman. Classic overview covers essential math, mechanics, electromagnetic theory, thermodynamics, quantum mechanics, nuclear physics, other topics. First paperback edition. xxiii + 885pp. 5% x 8%. 0-486-65227-0

PROBLEMS AND SOLUTIONS IN QUANTUM CHEMISTRY AND PHYSICS, Charles S. Johnson, Jr. and Lee G. Pedersen. Unusually varied problems, detailed solutions in coverage of quantum mechanics, wave mechanics, angular momentum, molecular spectroscopy, more. 280 problems plus 139 supplementary exercises. 430pp. 6% x 9%. 0-486-65236-X


WHAT IS RELATIVITY? L. D. Landau and G. B. Rumer. Written by a Nobel Prize physicist and his distinguished colleague, this compelling book explains the special theory of relativity to readers with no scientific background, using such familiar objects as trains, rulers, and clocks. 1960 ed. vi+72pp. 5% x 8%. 0-486-42806-0

QUANTUM MECHANICS: PRINCIPLES AND FORMALISM, Roy McWeeny. Graduate student-oriented volume develops subject as fundamental discipline, opening with review of origins of Schrödinger's equations and vector spaces. Focusing on main principles of quantum mechanics and their immediate consequences, it concludes with final generalizations covering alternative "languages" or representations. 1972 ed. 15 figures. xi+155pp. 5% x 8%. 0-486-42829-X

INTRODUCTION TO QUANTUM MECHANICS With Applications to Chemistry, Linus Pauling & E. Bright Wilson, Jr. Classic undergraduate text by Nobel Prize winner applies quantum mechanics to chemical and physical problems. Numerous tables and figures enhance the text. Chapter bibliographies. Appendices. Index. 468pp. 5% x 8%. 0-486-64871-0

METHODS OF THERMODYNAMICS, Howard Reiss. Outstanding text focuses on physical technique of thermodynamics, typical problem areas of understanding, and significance and use of thermodynamic potential. 1965 edition. 238pp. 5% x 8%. 0-486-69445-3

THE ELECTROMAGNETIC FIELD, Albert Shadowitz. Comprehensive undergraduate text covers basics of electric and magnetic fields, builds up to electromagnetic theory. Also related topics, including relativity. Over 900 problems. 768pp. 5% x 8%. 0-486-65660-8

GREAT EXPERIMENTS IN PHYSICS: FIRSTHAND ACCOUNTS FROM GALILEO TO EINSTEIN, Morris H. Shamos (ed.). 25 crucial discoveries: Newton's laws of motion, Chadwick's study of the neutron, Hertz on electromagnetic waves, more. Original accounts clearly annotated. 370pp. 5% x 8%. 0-486-25346-5

EINSTEIN'S LEGACY, Julian Schwinger. A Nobel Laureate relates fascinating story of Einstein and development of relativity theory in well-illustrated, nontechnical volume. Subjects include meaning of time, paradoxes of space travel, gravity and its effect on light, non-Euclidean geometry and curving of space-time, impact of radio astronomy and space-age discoveries, and more. 189 b/w illustrations. xiv+250pp. 8% x 9%. 0-486-41974-6


Paperbound unless otherwise indicated. Available at your book dealer, online at www.doverpublications.com, or by writing to Dept. GI, Dover Publications, Inc., 31 East 2nd Street, Mineola, NY 11501. For current price information or for free catalogues (please indicate field of interest), write to Dover Publications or log on to www.doverpublications.com and see every Dover book in print. Dover publishes more than 500 books each year on science, elementary and advanced mathematics, biology, music, art, literary history, social sciences, and other areas.
CATALOG OF DOVER BOOKS

TENSOR CALCULUS, J.L. Synge and A. Schild. Widely used introductory text covers spaces and tensors, basic operations in Riemannian space, non-Riemannian spaces, etc. 324pp. 5% x 8%. 0-486-63612-7


FOURIER SERIES, Georgi P. Tolstov. Translated by Richard A. Silverman. A valuable addition to the literature on the subject, moving clearly from subject to subject and theorem to theorem. 107 problems, answers. 336pp. 5% x 8%. 0-486-63317-9

INTRODUCTION TO MATHEMATICAL THINKING, Friedrich Waismann. Examinations of arithmetic, geometry, and theory of integers; rational and natural numbers; complete induction; limit and point of accumulation; remarkable curves; complex and hypercomplex numbers, more. 1959 ed. 27 figures. xii+260pp. 5% x 8%. 0-486-63317-9


CALCULUS OF VARIATIONS, Robert Weinstock. Basic introduction covering isoperimetric problems, theory of elasticity, quantum mechanics, electrostatics, etc. Exercises throughout. 326pp. 5% x 8%. 0-486-63069-2

THE CONTINUUM: A CRITICAL EXAMINATION OF THE FOUNDATION OF ANALYSIS, Hermann Weyl. Classic of 20th-century foundational research deals with the conceptual problem posed by the continuum. 156pp. 5% x 8%. 0-486-67982-9


Paperbound unless otherwise indicated. Available at your book dealer, online at www.doverpublications.com, or by writing to Dept. GI, Dover Publications, Inc., 31 East 2nd Street, Mineola, NY 11501. For current price information or for free catalogues (please indicate field of interest), write to Dover Publications or log on to www.doverpublications.com and see every Dover book in print. Dover publishes more than 500 books each year on science, elementary and advanced mathematics, biology, music, art, literary history, social sciences, and other areas.
INTRODUCTORY REAL ANALYSIS, A. N. Kolmogorov and S. V. Fomin. (0-486-61226-0)

SPECIAL FUNCTIONS AND THEIR APPLICATIONS, N. N. Lebedev. (0-486-60624-4)

CHANCE, LUCK AND STATISTICS, Horace C. Levinson. (0-486-41997-5)

TENSORS, DIFFERENTIAL FORMS, AND VARIATIONAL PRINCIPLES, David Lovelock and Hanno Rund. (0-486-65840-6)

SURVEY OF MATRIX THEORY AND MATRIX INEQUALITIES, Marvin Marcus and Henryk Minc. (0-486-67102-X)

ABSTRACT ALGEBRA AND SOLUTION BY RADICALS, John E. and Margaret W. Maxfield. (0-486-67121-6)

FUNDAMENTAL CONCEPTS OF ALGEBRA, Bruce E. Meserve. (0-486-61470-0)

FUNDAMENTAL CONCEPTS OF GEOMETRY, Bruce E. Meserve. (0-486-63415-9)

FIFTY CHALLENGING PROBLEMS IN PROBABILITY WITH SOLUTIONS, Frederick Mosteller. (0-486-65355-2)

NUMBER THEORY AND ITS HISTORY, Oystein Ore. (0-486-65620-9)

MATRICES AND TRANSFORMATIONS, Anthony J. Pettofrezzo. (0-486-63634-8)

THE UMBRAL CALCULUS, Steven Roman. (0-486-44139-3)

PROBABILITY THEORY: A CONCISE COURSE, Y. A. Rozanov. (0-486-63544-9)

LINEAR ALGEBRA, Georgi E. Shilov. (0-486-63518-X)

ESSENTIAL CALCULUS WITH APPLICATIONS, Richard A. Silverman. (0-486-66097-4)

A CONCISE HISTORY OF MATHEMATICS, Dirk J. Struik. (0-486-60255-9)

PROBLEMS IN PROBABILITY THEORY, MATHEMATICAL STATISTICS AND THEORY OF RANDOM FUNCTIONS, A. A. Sveshnikov. (0-486-63717-4)

TENSOR CALCULUS, J. L. Synge and A. Schild. (0-486-63612-7)

MODERN ALGEBRA: TWO VOLUMES BOUND AS ONE, B.L. Van der Waerden. (0-486-63544-9)

CALCULUS OF VARIATIONS WITH APPLICATIONS TO PHYSICS AND ENGINEERING, Robert Weinstock. (0-486-63069-2)

INTRODUCTION TO VECTOR AND TENSOR ANALYSIS, Robert C. Wrede. (0-486-61879-X)

DISTRIBUTION THEORY AND TRANSFORM ANALYSIS, A. H. Zemanian. (0-486-65479-6)

Paperbound unless otherwise indicated. Available at your book dealer, online at www.doverpublications.com, or by writing to Dept. 23, Dover Publications, Inc., 31 East 2nd Street, Mineola, NY 11501. For current price information or for free catalogs (please indicate field of interest), write to Dover Publications or log on to www.doverpublications.com and see every Dover book in print. Each year Dover publishes over 500 books on fine art, music, crafts and needlework, antiques, languages, literature, children's books, chess, cookery, nature, anthropology, science, mathematics, and other areas.

Manufactured in the U.S.A.
Approximate Calculation of Integrals
V. I. Krylov
Translated by Arthur H. Stroud

A systematic introduction to the principal ideas and results of the contemporary theory of approximate integration, this volume approaches its subject from the viewpoint of functional analysis. In addition, it offers a useful reference for practical computations. Its primary focus lies in the problem of approximate integration of functions of a single variable, rather than the more difficult problem of approximate integration of functions of more than one variable.

The three-part treatment begins with concepts and theorems encountered in the theory of quadrature. The second part is devoted to the problem of calculation of definite integrals. This section considers three basic topics: the theory of the construction of mechanical quadrature formulas for sufficiently smooth integrand functions, the problem of increasing the precision of quadratures, and the convergence of the quadrature process. The final part explores methods for the calculation of indefinite integrals, and the text concludes with helpful appendixes.


ALSO AVAILABLE
THEORY OF APPROXIMATION OF FUNCTIONS OF A REAL VARIABLE, A. F. Timan. 631pp. 5⅞ x 8½. 0-486-67830-X
METHODS OF MATHEMATICS APPLIED TO CALCULUS, PROBABILITY, AND STATISTICS, Richard W. Hamming. 878pp. 6⅝ x 9⅛. 0-486-43945-3

For current price information write to Dover Publications, or log on to www.doverpublications.com—and see every Dover book in print.

$19.95 USA
$29.95 CANADA